**KASDI MERBAH UNIVERSITY OUARGLA**

**Faculty of Mathematics and Sciences Material**


N° d'ordre :
N° de série :


**Department of:**
**Mathematics**


## Doctorate

**Path: Mathematics**

**Speciality: Analysis**


**Present by: Ibtissam TALHA**


**Theme:**

# Particular recurrent linear sequences with applications in difference and differential equations

**Represented in : 14 /06 / 2023**

**Limb from jury:**

| | | | |
|---|---|---|---|
| Mabrouk MEFLAH | Pr | Kasdi Merbah University-Ouargla | President |
| Salim BADIDJA | MCA | Kasdi Merbah University-Ouargla | Supervisor |
| Abdelhamid TELLAB | MCA | Mohamed Boudiaf University-M'Sila | Examiner |
| Abderachid SAADI | MCA | Mohamed Boudiaf University-M'Sila | Examiner |
| Brahim TELLAB | MCA | Kasdi Merbah University-Ouargla | Examiner |
| Mohammed KOUIDRI | MCA | Kasdi Merbah University-Ouargla | Examiner |

# *Dedication*

My modest effort dedicated to my wonderful father and mother, who gave me life, hope, and an upbringing with a thirst for knowledge. To everyone who assisted me in my research journey: my brothers and sisters. To those who helped me along the walk to success in our scientific trip, to my companion: Dr. Bochra AZZAOUI.

Finally, thank you to everyone who helped me along the way and had a part in finishing this research.

# *Acknowledgement*

Firstly, I thank Allah for this gift and for providing me with the strength to complete this difficult task. This is due to His kindness and grace rather than my work and devotion.

Secondly, I would like to thank the supervising professor, Dr.Salim BADIDJA, for his efforts during the supervisor period.

We would like to thank the academics and members of the discussion committee : Pr. Mabrouk MEFLAH, Dr. Abdelhamid TELLAB, Dr. Abderachid SAADI, Dr. Mohammed KOUIDRI, Dr. Brahim TELLAB.

Finally, I'd want to thank everyone who assisted me in completing this academic work.

## الملخص

يهدف عملنا في هاته الأطروحة إلى إجراء دراسة نوعيةحول بعض المتتاليات التراجعية الخطية الخاصةوتطبيقاتها في ميدان جمل معادلات الفروق والمعادلات التفاضلية. حيث قسمنا دراستنا إلى جزأين :
في الجزء الأول قمنا بِعرض حلول جمل معادلات الفروق من الرتب العليا في شكلها المرتبط بمتتالياتفيبوناشي وتريبوناشي، بعدها درسنا استقرار ودورانية هاته الحلول.
في الجزء الثاني توصلنا إلى إيجاد معادلات تفاضلية إنطلاقا من كثيرات حدود فيبوناشيوتريبوناشي.
يُذكرُ أنّنا أرفقنا محاور دراستنا بأمثلة عددية من أجل تأكيد النتائج التي توصّلنا إليها.

**الكلمات المفتاحية**

جمل معادلات فروق، أعداد فيبوناشي، أعداد تريبوناشي، معادلة تفاضلية، استقرار.



**Abstract**

Our work in this thesis aims to conduct a qualitative study on some particular recurrent linear sequences and how they can be used in systems of difference equations and differential equations. We divided our study into two parts: In the first pat, we showed the solutions to the systems of higher-order difference equations. Formally, they are related to the Fibonacci and Tribonacci sequences. The stability and periodicity of these solutions were then investigated.
In the second part, we studied about differential equations derived from the Fibonacci and Tribonacci polynomials. We also added numerical examples to the axes of our research to validate our results.

**Key words**

System of differences equations, Fibonacci numbers, Tribonacci numbers, Differential equation, Stability.



## Résumé

Notre travail dans cette thèse vise à mener une étude qualitative sur certaines séquences récurrente linéaires particulières et leurs applications dans le domaine des systèmes d'équations aux différences et d'équations différentielles. Nous avons divisé notre étude en deux parties :
Dans la première partie, nous avons présenté les solutions des systèmes des équations aux différences d'ordres supérieurs sous leur forme liée aux suites de Fibonacci et de Tribonacci.
Puis nous avons étudié la stabilité et la pèriodicitè de ces solutions.
Dans la deuxième partie, nous avons trouvé des équations différentielles basées sur les polynômes de Fibonacci et Tribonacci.
Nous avons également joint les axes de notre étude avec des exemples numériques pour confirmer nos résultats.

**Les mots-clés**

Système d'équations aux différences, Nombres de Fibonacci, Nombres de Tribonacci, équations différentielle, la stabilité.

# Contents

# List of Figures

# Notations

$E$ : Category of commutative fields of characteristic zero.

$E_0$ : Sequence with values in A.

$A$ : a commutative ring with unity.

$A[[x]]$ : Formal series with coefficients in A.

$LRC$ : Linear recurrent sequence with constant coefficients.

$LRP$ : Linear recurrent sequence with polynomial coefficients.

$G_n$ : General term of generalized Fibonacci sequence.

$R(x)$ : The radius of convergence.

## Acronyms

FRR : Fibonacci Recurrent Relation.

GFS : Generalized Fibonacci Sequence.

PMI : Principle of Mathematical Induction.

TRR : Tribonacci Recurrent Relation.

# Introduction

The theories of difference equations are particularly intriguing, since they have become a significant and popular subject among academics and scientists in a variety of fields in recent years. This is owing to advancements in computer hardware and computer science.

Difference equations are essential in mathematical models. It is also recognized as the cornerstone of applied mathematics since it accurately portrays challenges in our daily lives and in a wide range of areas, including numerical analysis, control theory, biology, economics, computer science, finite mathematics, and others (some models are listed in the cited reference: [7]).

[18] Many mathematical models of physical and biological issues have taken the form of differential equations; it should also be emphasized that many of the mathematical approaches employed in this subject try to replace differential equations with suitable difference equations. Several findings were also obtained demonstrating that the theory of difference equations explains natural occurrences that are distinct from the comparable

conclusions of differential equations. a computer environment in which differential equations are solved using the approximation method and a number of computer programs. Then, all computer observations are shown by analytical techniques.

[35] The system of difference equations is a system of equations including unknowns that are also sequences. The first portion of the difference equations consists of a linear difference equation. This section is somewhat comprehensible and accessible because to its reliance on linearity's principles, which give straightforward methods for solving this sort of problem.

Nonlinear difference equations are the second component. This is a challenging problem since there is no apparent technique or procedure for addressing it.

The development of computers substantially facilitated the solution of this sort of problem and the identification of a number of its features. However, all computer observations and predictions had to be satisfied analytically, and that is precisely what we did in this study.

[18] In order to comprehend the physical, engineering, and biological sciences, differential equations have been employed since the time of Newton. In addition to their contribution to the study of mathematical analysis, its applications have grown to include the economic and social sciences.

A differential equation is an equation including at least one function and its derivatives. Their applications have developed and grown in importance in several scientific domains and their applications.

Leonardo Fibonacci, one of the most important medieval European mathematicians, was known by the moniker Fibonacci, which means "son of Bonacci" (see: [34]).

Figure 1: Leonardo Fibonacci

He was educated in the Algerian city of Bejaia, where he mastered the numbering system as well as the techniques of Indo-Arabic arithmetic. Then, he moved to many countries, such as Egypt, Syria, Greece, France, and others, to study the different arithmetic systems.

He authored many books in which he provided solutions to several mathematical problems. The Fibonacci sequence $\{F_n\}_{n\geq 0}$ or Fibonacci numbers $F_n$ are one of his most famous works. This is the only sequence in which

the term is equal to the sum of the previous two terms. The first terms of the sequence are given as : $0, 1, 1, 2, 3, 5, 8, \ldots$.
He has many works, most notably his famous book (Liber Abaci), which he published in 1202, through which he contributed to the publication of Arabic numerals in Europe. The book (Libre Quardratorum) in the year 1225, or what is known as the "Book of Squares". In 1228, he published the book "Parctice Geometry," in which he presented a solution to many mathematical problems, and among the works attributed to him were the Fibonacci numbers or the Fibonacci sequence.
The ratio: $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ (with $\alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2}$) is known as Binet's formula; it expresses a Fibonacci sequence of degree $n$, and it is clear from it that the ratio between two consecutive periods of the sequence when $n$ moves to infinity turns into the number $1.61803398875\ldots$ which is called the "golden ratio."
The Fibonacci sequences have intriguing and significant mathematical features and linkages that may be discovered in today's biology, physics, chemistry, electrical engineering, music, and other domains. We'll look at some basic natural Fibonacci number instances later (for example, see: [34]).
The well-known extensions of the Fibonacci sequence, the Tribonacci and Tribonacci-Lucas sequences, are third-order recurrent relations.
Fenberg first investigated the Tribonacci sequences in 1963 (see: [19]).
The Tribonacci and Tribonacci-Lucas sequences have a wide range of intriguing features and uses in research. Several writers gave Binet formulae

and summation methods for creating functions. Tribonacci sequences offer fascinating and important mathematical properties and linkages that may be found in biology, physics, chemistry, electrical engineering, music, and other professions today (for example see: [40]).

The Tribonacci polynomials were introduced in 1973 by Hoggat and Bicknell.

We arranged our work in this thesis into four chapters, the contents of which are as follows :

In the first chapter, we offered some definitions and outcomes for the most significant topics that we will be studying in the next chapters. First, a few reminders about algebra $r(K)$. Then, we discussed several recurrent linear sequences and polynomials (Fibonacci and Tribonacci sequences and polynomials). We then discussed various results and characteristics of difference equations and their applications in our everyday lives. In addition, we present differential equations definitions.

In the second chapter, we provided the solutions for various systems of higher-order difference equations in terms of Fibonacci and Tribonacci sequences.

In the third chapter, we determined the equilibrium point and investigated its stability and periodicity solutions for various higher-order difference equations systems.

Then, we have included numerical examples to back up our results.

In the fourth chapter, we perform a qualitative research on the applications of several linear recurrent sequences in differential equations, beginning with the Fibonacci polynomial and studying its development. Then, we create a differential equation in which the unknown is the generating function of Fibonacci sequence $\{F_n\}_{n\geq 0}$, which represents the same Fibonacci polynomial that we began with. We completed the investigation using the Tribonacci polynomial and got a new differential equation.

# Chapter 1

# Preliminaries

## 1.1 Some particular sequences and polynomials

### 1.1.1 Fibonacci sequence and polynomial

**Definition 1.1** *[34] The Fibonacci sequence $\{F_n\}_{n\geq 0}$ is defined by the following recurrent relation*

$$\begin{cases} F_{n+2} = F_{n+1} + F_n, \quad n \geq 0 \\ \\ F_0 = 0; \quad F_1 = 1, \end{cases} \tag{1.1}$$

*$F_0 = 1$ and $1, 1, 2, 3, 5, 8, 13, \ldots$ are the initial terms of the Fibonacci sequence.*

**Theorem 1.1** *[34] Binet's formula is given as follows*

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

*with $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.*

**Proof.** [34] We prove Binet's formula using the generating function, about which we will give explanations in the coming pages.
We use the method of finding the coefficients of the Taylor series that correspond directly to the Fibonnaci sequence.
By Definition 1.1, the general coefficients of the Taylor series are given as

$$\begin{aligned} F(x) &= F_0 + F_1x + F_2x^2 + F_3x^3 + \ldots = x + x^2 + 2x^3 + \ldots \\ xF(x) &= F_0x + F_1x^2 + F_2x^3 + \ldots = x^2 + x^3 + \ldots \\ x^2F(x) &= F_0x^2 + F_1x^3 + F_2x^4 + F_3x^5 + \ldots = x^3 + x^4 + 2x^5 + \ldots \end{aligned}$$

Then,

$$F(x) - xF(x) - x^2F(x) = F_0 + (F_1 - F_0)x = F_0 = x.$$

So,

$$F(x) = \frac{x}{1 - x - x^2}$$

where the roots of the following quadratic equation: $1 - x - x^2$ are given as $x_1 = \frac{-1+\sqrt{5}}{2}$ and $x_2 = \frac{-1-\sqrt{5}}{2}$. ■

**Definition 1.2** *(Geneating function) [34] Let the series $a_0, a_1, a_2, \ldots$ consist of real numbers, subsequently the function*

$$g(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots$$

*is given the sequence $a_n$'s generating function.*
*We may alternatively define the generating function for the finite series $a_0, a_1, \ldots, a_n$*

*and* $a_i = 0$ *for* $i > n$. *Then*

$$g(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

*is the generating function for the finite sequence* $a_0, a_1, \ldots, a_n$.

**Theorem 1.2** *[34] The generating function of the Fibonacci recurrent relation is given as follows*

$$g(x) = F_0 + F_1 x + F_2 x^2 + \ldots + F_n x^n + \ldots = \sum_{0}^{\infty} F_n x^n.$$

**Proof.** [34] We are known that the Fibonacci sequence $\{F_n\}_{n\geq 0}$ has the following definition

$$\begin{cases} F_n = F_{n-1} + F_{n-2}, \quad n \geq 0 \\ \\ F_0 = 0; \quad F_1 = 1, \end{cases} \tag{1.2}$$

respectively, we get $xg(x)$ and $x^2 g(x)$ :

$$\begin{aligned} xg(x) &= F_1 x^2 + F_2 x^3 + F_3 x^4 + \ldots + F_{n-1} x^n + \ldots \\ x^2 g(x) &= F_1 x^3 + F_2 x^4 + F_3 x^5 + \ldots + F_{n-2} x^n + \ldots \\ g(x) - xg(x) - x^2 g(x) &= F_1 x + (F_2 - F_1) x^2 + (F_3 - F_2 - F_1) x^3 + \ldots \end{aligned}$$

and by using the Definition 1.1, we get

$$g(x) = \frac{x}{1 - x - x^2}$$

on the other hand, we have

$$\begin{aligned} g(x) = \frac{x}{1-x-x^2} &= \frac{1}{\sqrt{5}}\left[\frac{1}{1-\alpha x} - \frac{1}{1-\beta x}\right] \\ &= \sqrt{5}g(x) = \frac{1}{1-\alpha x} - \frac{1}{1-\beta x} \\ &= \sum_{0}^{+\infty} \alpha^n x^n - \sum_{0}^{+\infty} \beta^n x^n \\ &= \sum_{0}^{+\infty} (\alpha^n - \beta^n) x^n. \end{aligned}$$

So

$$g(x) = \sum_{0}^{+\infty} \frac{(\alpha^n - \beta^n)}{\sqrt{5}} x^n = \sum_{0}^{+\infty} \frac{(\alpha^n - \beta^n)}{\alpha - \beta} x^n = \sum_{0}^{\infty} F_n x^n.$$

■

**Definition 1.3** *(GFS) [5] A generalized Fibonacci sequence* $(G_n)$ *is any series that follows the same recurrent relation as the Fibonacci numbers but with modified beginning conditions*

$$\begin{cases} G_n = G_{n-1} + G_{n-2}, \quad n \geq 1 \\ \\ G_1 = a; \quad G_2 = b. \end{cases} \tag{1.3}$$

*The first terms of* $(GFS)$ *are given as*

$$a, b, a+b, a+2b, 2a+3b, \ldots$$

**Theorem 1.3** *[34] Let* $(G_n)$ *be the* $nth$ *term of the generalized Fibonacci sequence. Hence*

$$G_n = aF_{n-2} + bF_{n-1}, \quad n \geq 3. \tag{1.4}$$

**Proof.** [34] By using the principle of mathematical induction (PMI), we have

$$G_3 = aF_1 + bF_2 = a + b. \tag{1.5}$$

Consequently, the relation holds when $n = 3$.

Let $k$ be any number and suppose that the formula (1.4) holds for all integers $i$, which $3 \leq i \leq k$. That is

$$G_i = aF_{i-2} + bF_{i-1}.$$

Then

$$\begin{aligned} G_{k+1} &= G_k + G_{k-1} \\ &= (aF_{k-2} + bF_{k-1}) + (aF_{k-3} + bF_{k-2}) \\ &= a\,(F_{k-2} + F_{k-3}) + b\,(F_{k-1} + F_{k-2}) \\ &= aF_{k-1} + bF_k. \end{aligned}$$

Hence, the formula (1.4) is true for all integer $n \geq 3$,

and we note that the formula (1.4) remains true for all $n \geq 1$. ■

**Definition 1.4** *( Generating function of GFS) [34]*

*Let $G_n = G_{n-1} + G_{n-2}$, which $G_1 = a$ and $G_2 = b$. Then, the following formula*

$$g(x) = a + bx + (a+b)x^2 + (a+2b)x^3 + (2a+3b)x^4 + \dots$$

*is referred to as the generating function of the generalized Fibonacci sequence (GFS).*

**Definition 1.5** *[34] The golden ratio is defined as an irrational number*

$$\frac{1+\sqrt{5}}{2} = 1.618033\ldots$$

*[?] It was and is of interest to mathematicians, physicists, philosophers, architects, painters, and even musicians.*
*The golden ratio is also known as the golden mean, Fibonacci number, ...*
*It is often referred to as the greek letter $\phi$ after the mathematician who researched its characteristics, phidias.*
*Pharaohs and ancient Greeks used the golden ratio in the art of architecture and the construction of pyramids.*
*The height of the biggest pyramid is $484.4$ feet, or about $5813$ inches, these three numbers are consecutive in the Fibonacci sequence.*
*Leonardo da Vinci draw the Mona Lisa using the golden ratio and thought it to be the epitome of beauty.*

**Theorem 1.4** *[34] When $n$ is sufficiently enough, it is seen that the ratio approaches $1.618033\ldots$ in the Fibonacci sequence. That is*

$$\lim_{n\to+\infty} \frac{F_{n+1}}{F_n}.$$

**Proof.** According to the recurrent relation of Fibonacci sequence, we have

$$F_{n+1} = F_n + F_{n-1}.$$

Hence

$$\begin{aligned} \frac{F_{n+1}}{F_n} &= 1 + \frac{F_{n-1}}{F_n} \\ &= 1 + \frac{1}{\frac{F_n}{F_{n-1}}} \end{aligned}$$

when $n \mapsto +\infty$, we fined the characteristic equation

$$x^2 - x - 1 = 0.$$

Since the limit is positive, only the positive root is importance. Hence

$$\alpha = \lim_{n \to +\infty} \frac{F_{n+1}}{F_n} = 1.618033\ldots.$$

■ A Fibonacci-like recurrence relation is used to create several polynomial classes. Belgian mathematician Eugene Charles Catalan (1814-1894) and German mathematician E. Jacobsthal conducted research on Fibonacci polynomials in 1883.

**Definition 1.6** *[34] The sequence of Fibonacci polynomials $f_n(x)$ is defined as follows*

$$\begin{cases} f_n(x) = x f_{n-1}(x) + f_{n-2}(x), \quad n \geq 3 \\ \\ f_1(x) = 1; \quad f_2(x) = x. \end{cases} \tag{1.6}$$

*The Fibonacci polynomials are given as*

| $n$ | *The terms of Fibonacci polynomials* $f_n(x)$ |
|---|---|
| 1 | $1$ |
| 2 | $x$ |
| 3 | $x^2 + 1$ |
| 4 | $x^3 + 2x$ |
| 5 | $x^4 + 3x^2 + 1$ |
| 6 | $x^5 + 4x^3 + 3x$ |

**Proposition 1.1** *[34] Fibonacci polynomials can be constructed using the binomials expansions of $(x+1)^n$ as follows*

*for all $n \geq 0$, we gave the next table*

| $n$ | *Expansion of* $(x+1)^n$ |
|---|---|
| 0 | $1$ |
| 1 | $x+1$ |
| 2 | $x^2+2x+1$ |
| 3 | $x^3+3x^2+3x+1$ |
| 4 | $x^4+4x^3+6x^2+4x+1$ |
| 5 | $x^5+5x^4+10x^3+10x^2+5x+1$ |

*Noting that the sum of the elements along the diagonal starting at row $n$ is $f_{n+1}(x)$, the sum of the elements beginning at row 3 is $f_4(x)$.*

**Theorem 1.5** *(Binet's formula) [34]*

*The characteristic equation of Fibonacci polynomials is given as*

$$t^2 - xt - 1 = 0$$

*this equation has two real roots $\alpha(x)$ and $\beta(x)$, which*
$\alpha(x) = \frac{x+\sqrt{x^2+4}}{2}$ *and* $\beta(x) = \frac{x-\sqrt{x^2+4}}{2}$.
*Then, the formula for the Fibonacci polynomials by Binet is*

$$f_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)}.$$

**Theorem 1.6 (Generating function)** *[34]*

*The generating function of the Fibonacci polynomial $f_n(x)$ is given as*

$$g(t) = \sum_{n=0}^{\infty} f_n(x)t^n.$$

**Proof.** We have

$$g(t) = \sum_{0}^{\infty} f_n(x)t^n = t + xt^2 + (x^2+1)t^3 + (x^3+2x)t^4 + \dots$$

So,

$$\begin{aligned} xtg(t) &= \sum_{0}^{\infty} xf_n(x)t^{n+1} = xt^2 + x^2t^3 + x^3t^4 + xt^4 + x^4t^5 + 2x^2t^5 + \dots \\ t^2g(t) &= \sum_{0}^{\infty} f_n(x)t^{n+2} = t^3 + xt^4 + x^2t^5 + t^5 + x^3t^6 + 2xt^6 + \dots \end{aligned}$$

and by subtracting in the following equation: $g(t) - xtg(t) - t^2g(t)$, we get

$$g(t) - xtg(t) - t^2g(t) = (1 - xt - t^2)g(t) = f_0(x) - tf_1(x) - xtf_0(x) = t.$$

Hence, $g(t) = \frac{t}{1-xt-t^2}$ generates $f_n(x)$. ∎

**Definition 1.7** *[34] The sequence of generalized Fibonacci polynomials $f_n(x)$ is defined by the recurrent relation shown below*

$$\begin{cases} f_n(x) = xf_{n-1}(x) + f_{n-2}(x), \quad n \geq 3 \\ \\ f_1(x) = S; \quad f_2(x) = Sx. \end{cases} \tag{1.7}$$

*If $S = 1$. Then, we obtained the classical Fibonacci polynomial sequence.*

**Proposition 1.2** *(Binet's formula) [34] $R_1$ and $R_2$ are two roots of the characteristic equation*

$$t^2 - xt - 1 = 0.$$

*The following expression provides the nth generalized Fibonacci polynomials*

$$f_n(x) = S\frac{R_1^n - R_2^n}{R_1 - R_2}$$

*which* $R_1 = \frac{x+\sqrt{x^2+4}}{2}$ *and* $R_2 = \frac{x-\sqrt{x^2+4}}{2}$.

### 1.1.2 Tribonacci sequence and polynomials

**Definition 1.8** *The Tribonacci sequence $\{T_n\}_{n\geq 0}$ is defined by the following thrid order linear recurrent relation*

$$\begin{cases} T_n = T_{n-1} + T_{n-2} + T_{n-3}, \quad n \geq 4, \\ \\ T_0 = 0; \quad T_1 = T_2 = 1, \end{cases} \tag{1.8}$$

*where the first terms of the Tribonacci sequence are given as :*$1, 1, 2, 4, 7, 13, 24, \ldots$

**Remark 1.1** *[40] We can extend the definition of Tribonacci sequence to the set of negative numbers as follows*

$$T_{-n} = -\,T_{-(n-1)} -\, T_{-(n-2)} + T_{-(n-3)}, \quad n \geq 1. \tag{1.9}$$

*and we have*

$$T_{-n} = 2T_{-(n-3)} - T_{-(n-4)}, \quad n \geq 1. \tag{1.10}$$

**Proof.** From Remark 1.1, the definition of the Tribonacci sequence for the set of negative numbers is given as follows

$$T_{-n} = -\, T_{-(n-1)} - \, T_{-(n-2)} + T_{-(n-3)}, \quad n \geq 1. \tag{1.11}$$

Then

$$\begin{aligned} T_{-n} &= -T_{-(n-1)} - T_{-(n-2)} + T_{-(n-3)} - T_{-(n-4)} + T_{-(n-4)} \\ &= T_{-(n-1)} - T_{-(n-2)} + T_{-(n-3)} - T_{-(n-4)} + \left(T_{-(n-1)} + T_{-(n-2)} + T_{-(n-3)}\right) \\ &= 2T_{-(n-3)} - T_{-(n-4)}. \end{aligned}$$

■

**Theorem 1.7** *[34] Binet's formula for the $n_{th}$ Tribonacci numbers is given for all $n \geq 0$ as follows*

$$T_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)},$$

*where $\alpha$, $\beta$, and $\gamma$ are the roots of the following cubic equation*

$$x^3 - x^2 - x - 1 = 0, \tag{1.12}$$

*such that*

$$\begin{aligned} \alpha &= \frac{1+A+B}{3} &= \frac{1+\sqrt[3]{19+3\sqrt{33}}+\sqrt[3]{19-3\sqrt{33}}}{3}, \\ \beta &= \frac{1+\omega A+\omega^2 B}{3} &= \frac{1+\omega\sqrt[3]{19+3\sqrt{33}}+\omega^2\sqrt[3]{19-3\sqrt{33}}}{3}, \\ \gamma &= \frac{1+\omega^2 A+\omega B}{3} &= \frac{1+\omega^2\sqrt[3]{19+3\sqrt{33}}+\omega\sqrt[3]{19-3\sqrt{33}}}{3}, \end{aligned}$$

*where*

$$A = \sqrt[3]{19 + 3\sqrt{33}}, \quad B = \sqrt[3]{19 - 3\sqrt{33}},$$

*which* $\omega = \frac{-1+i\sqrt{3}}{2} = exp(\frac{2\pi i}{3})$ *is a primitive cube root of one. Then, the following identities are true*

$$\alpha + \beta + \gamma = 1,$$
$$\alpha\beta + \alpha\gamma + \beta\gamma = -1,$$
$$\alpha\beta\gamma = 1.$$

**Corollary 1.1** *[40] The generating function of Tribonacci numbers $T_n$ is given as*

$$g(t) = \sum_{n=0}^{\infty} T_n t^n = \frac{t}{1 - t - t^2 - t^3}.$$

**Proof.** [40] Let $g(t) = \sum_{n=0}^{\infty} T_n t^n$, which $g(t)$ be the generating function of Tribonacci numbers. Hence

$$\begin{aligned} g(t) &= T_0 + T_1 t + T_2 t^2 + \ldots + T_n t^n + \ldots, \\ tg(t) &= T_0 t + T_1 t^2 + T_2 t^3 + \ldots + T_{n-1} t^n + \ldots, \\ t^2 g(t) &= T_0 t^2 + T_1 t^3 + T_2 t^4 + \ldots + T_{n-2} t^n + \ldots, \\ t^3 g(t) &= T_0 t^3 + T_1 t^4 + T_2 t^5 + \ldots + T_{n-3} t^n + \ldots \end{aligned}$$

Then, we obtained

$$(1 - t - t^2 - t^3)g(t) = T_0 + T_1 t + T_2 t^2 - T_0 t - T_1 t^2 - T_0 t^2.$$

For (1.8) and here the coefficients of $t^n$ are equal to zero for $n \geq 3$. Therefore, the generating function of Tribonacci numbers is given as

$$g(t) = \frac{t}{1 - t - t^2 - t^3}.$$

■

**Definition 1.9** *(GTS) [34] The generalized Tribonacci sequence* $\{T_n(a,b,c;r,s,t)\}_{n\geq 0}$ *is defined as follows*

$$\begin{cases} T_n = rT_{n-1} + sT_{n-2} + tT_{n-3}, \quad n \geq 4, \\ \\ T_0 = a; \quad T_1 = b; \quad T_2 = c, \end{cases} \tag{1.13}$$

*where* $T_0, T_1$ *and* $T_2$ *are arbitrary integers and* $r, s, t$ *are real numbers.*

*The sequence* $\{T_n\}_{n\geq 0}$ *may be extended to negative subscripts by declaring*

$$T_{-n} = -\frac{s}{t}T_{-(n-1)} - \frac{r}{t}T_{-(n-2)} + \frac{1}{t}T_{-(n-3)}. \tag{1.14}$$

**Definition 1.10** *[34] The Tribonacci polynomials sequence* $\{T_n(x)\}_{n\geq 0}$ *is defined as*

$$\begin{cases} T_n(x) = x^2T_{n-1}(x) + xT_{n-2}(x) + T_{n-3}(x), \quad n \geq 3, \\ \\ T_0(x) = 0; \quad T_1(x) = 1 \quad T_2(x) = x^2, \end{cases} \tag{1.15}$$

*since* $T_n = t_n(1)$*.*

*The sequence of Tribonacci polynomials can be defined in another way as follows*

$$\begin{cases} t_n(1) = x^2t_{n-1}(1) + xt_{n-2}(1) + t_{n-3}(1), \quad n \geq 3, \\ \\ t_0(1) = 0; \quad t_1(1) = 1; \quad t_2(1) = 1. \end{cases} \tag{1.16}$$

**Remark 1.2** *[34]The definition of Tribonacci polynomials may be extended to the set of negative integers as follows*

$$t_{-n}(x) = x^2 t_{-(n-1)}(x) + x t_{-(n-2)}(x) + t_{-(n-3)}(x), \tag{1.17}$$

*where* $t_0(x) = 0, t_{-1}(x) = 1$ *and* $t_{-2}(x) = 1.$

The initial numbers of the Tribonacci polynomials are given as follows

| $n$ | The terms of Tribonacci polynomials $t_n(x)$ |
|---|---|
| 0 | 0 |
| 1 | 1 |
| 2 | $x^2$ |
| 3 | $x^4 + x$ |
| 4 | $x^6 + 2x^3 + 1$ |
| 5 | $x^8 + 3x^5 + 3x^2$ |
| 6 | $x^{10} + 4x^7 + 6x^4 + 2x$ |

**Theorem 1.8** *[34] The Binet's formula of the* $n_{th}$ *Tribonacci polynomial is given as*

*for all* $n \geq 0$

$$\begin{aligned} T_n(x) = &\frac{\alpha_1^{n+1}(x)}{(\alpha_1(x) - \alpha_2(x))(\alpha_1(x) - \alpha_3(x))} + \frac{\alpha_2^{n+1}(x)}{(\alpha_2(x) - \alpha_1(x))(\alpha_2(x) - \alpha_1(x))} \\ &+ \frac{\alpha_3^{n+1}(x)}{(\alpha_3(x) - \alpha_1(x))(\alpha_3(x) - \alpha_2(x))}, \end{aligned} \tag{1.18}$$

*where* $\alpha_1$, $\alpha_2$, *and* $\alpha_3$ *are the roots of the following quadratic equation*

$$\lambda^3 - x^2\lambda^2 - x\lambda - 1 = 0,$$

*with*

$$\begin{aligned}\alpha_1(x) &= \frac{x^2}{3} + A(x) + B(x)\\ \alpha_2(x) &= \frac{x^2}{3} + \omega A(x) + \omega^2 B(x)\\ \alpha_3(x) &= \frac{x^2}{3} + \omega^2 A(x) + \omega B(x),\end{aligned}$$

*where*

$$A(x) = \sqrt[3]{\frac{x^6}{27} + \frac{x^3}{6} + \frac{1}{2} + \sqrt{\frac{x^6}{36} + \frac{7x^3}{54} + \frac{1}{4}}},$$
$$B(x) = \sqrt[3]{\frac{x^6}{27} + \frac{x^3}{6} + \frac{1}{2} - \sqrt{\frac{x^6}{36} + \frac{7x^3}{54} + \frac{1}{4}}},$$

*which* $\omega = \frac{-1+i\sqrt{3}}{2}$ *is a basic unity cube root, and in the other hand* $\alpha_1(x), \alpha_2(x)$ *and* $\alpha_3(x)$ *satisfied the following identities*

$$\alpha_1(x) + \alpha_2(x) + \alpha_3(x) = x^2,$$
$$\alpha_1(x)\alpha_2(x) + \alpha_1(x)\alpha_3(x)\alpha_2(x)\alpha_3(x) = -x,$$
$$\alpha_1(x)\alpha_2(x)\alpha_3(x) = 1.$$

**Theorem 1.9** *[34] The generating function of the Tribonacci polynomials is given as*

$$G(t) = \sum_{n=0}^{\infty} T_n(x)t^n = \frac{t}{1 - x^2t - xt^2 - t^3}. \tag{1.19}$$

### 1.1.3 Simple models about difference equations

**Definition 1.11** *[7]* **(Compound interest)** *Compound interest relates to loans and deposits that are paid over long periods. The interest is added to the first sum at equal intervals called " the conversion period", when the new amount is used to calculate the next transfer period.*

**(i)** *$a$ denotes the part of the year that the conversion period accupies which the conversion period for one month occurs by $a = \frac{1}{2}$*

**(ii)** *The annual interest rate is denoted by $\gamma\%$. Then the equal interest earned for the period is equal to $a\gamma\%$ of the total amount deposited at the beginning of period which*

$$\left\{\begin{array}{c}\text{amount on}\\ \text{deposit}\\ \text{after } n+1\\ \text{conversion}\\ \text{periods}\end{array}\right\} = \left\{\begin{array}{c}\text{amount on}\\ \text{deposit}\\ \text{after } n\\ \text{conversion}\\ \text{periods}\end{array}\right\} + \frac{a\gamma}{100}\left\{\begin{array}{c}\text{amount on}\\ \text{deposit}\\ \text{after } n\\ \text{conversion}\\ \text{periods}\end{array}\right\} \tag{1.20}$$

*and to model the last statement in the form of a difference equation, the following steps are suggested :*

*For each $n$, $I_n$ denotes the amount deposited after $n$ conversion periods. Then*

$$I_{n+1} = I_n + \frac{a\gamma}{100} I_n = I_n \left(1 + \frac{a\gamma}{100}\right). \tag{1.21}$$

*The last formula 1.21 is a linear difference equation of the first order.*

*$I_n$ is a geometric sequence representing the compound interest, its expression is written as follows*

$$I_n = \left(1 + \frac{a\gamma}{100}\right)^n I_0. \tag{1.22}$$

**Definition 1.12** *[7]* (**Loan repayments)** *To find the equation that expresses the loan repayment, the researchers made a simple modification to the above* *statement. The scheme attached below is usually used to repayment of home or car loans. So that the repayment is made at regular intervals and in equal amounts in order to reduce loans and pay interest on the outstanding amounts. The researchers also noted that the debt increases and this is due to the interest imposed because of the debts that always remain after the last repayment.*

$$\left\{\begin{array}{c}\text{debt after}\\ n+1 \text{ payments}\end{array}\right\}=\left\{\begin{array}{c}\text{debt after}\\ n \text{ payments}\end{array}\right\}+\left\{\begin{array}{c}\text{interest}\\ \text{on this debt}\end{array}\right\}-\left\{\text{payment}\right\}. \tag{1.23}$$

*Let us model this statement in the form of the difference equation, they displayed the following :*

*Let $A_0$ be the initial debt which is quick, and for every $n$ let $A_n$ be the debt owed after payment $n$, and let $P$ the payment after each period conversion, Then*

$$A_{n+1} = A_n + \frac{a\gamma}{100}A_n - P = A_n\left(1+\frac{a\gamma}{100}\right) - P$$

*It's a difficult equation to solve. Hence, these two examples included a simple model that was embodied in presenting some of the rules stipulated by the bank regulations and then translating them into mathematical symbols.*

**Difference equations in population theory**

[7] Since many times, mankind has been striving to know how the population growth, as well as the factors that effect its growth. There is doubt that this important type of study contributes greatly to the management of the wildlife. It also helps in studying bacterial growth.

Several types of animals reproduce during a short and well-defined season, and the same thing happens with humans. Therefore, the time is measured separately using positive integers that indicate the breeding seasons.
Therefore, the best way to express population growth is to write appropriate difference equation.
Firstly, here are some simple population models, to discuss some of the variables that most effect reproduction processes.

**Linear first order difference equation: Exponential growth**

[7] Several types of organisms in nature compete with other species in order to obtain food, and sometimes some species resort to prey on other to meet their needs.
In order to study the growth of these organisms, researchers in scientific laboratories studied each species separately, there are large groups in which individuals reproduce but die after a period of time. Accordingly, the growth process is linked to the average behavior of the members of the population.
For this reason, the following assumptions were made

**(i)** All members of population have the same chance of having the same number of offspring.

**(ii)** Every one in population has the same chances of dying or surviving.

**(iii)** In each breeding season, the ratio of females to males rest the same.

**(iiii)** It was assumed that the age differences between members of the population are neglected.

**(iiiii)** The study population was also isolated.

Assume that all members of the populaion have the same average offspring $\alpha$ each season.

- $\alpha$ denotes the percapita birth rate.

- Whereas $\beta$ denote for the individuals mortality rate before reproduction.

$N_k$ is the number of individuals in the population at the beginning of the breeding season $k$.

$$N_{k+1} = N_k - \beta N_k + \alpha N_k$$

this means that

$$N_{k+1} = (1 - \beta + \alpha)N_k. \tag{1.24}$$

It is the equation of geometric progression. Then

$$N_k = (1 - \beta + \alpha)N_0, \quad k = 0, 1, \dots \tag{1.25}$$

To avoid this contradiction, resort to rounding $N_k$ to its nearest integer.
On the other hand, it is clear that the behavior of the model depends on the growth rate that is given by the relation

$$r = \alpha - \beta \tag{1.26}$$

- If $r$ is less than zero, the population is heading towards extinction.

- But, if $r$ is greater than zero, then the population is constantly increasing.

What is noticeable in the above study is that the model does not care about the age structure, where the hypothesis was presented that the offspring directly enters the reproductive cycle regardless of age.
Now, let us review two examples in which two models of population are presented that fulfill the above-mentioned assumptions.

**Fibonacci numbers and second-order difference equations**

[7] In this population, the presence of a pair of rabbits is assumed, since this couple gives birth to a new pair every month, and the new pair is capable of reproducing after two months. This procedure is repeated frequently, At the conclusion of each month. Then

$$\left\{ \begin{array}{c} \text{Current number} \\ \text{per mont } k+1 \end{array} \right\} = \left\{ \begin{array}{c} \text{Current number} \\ \text{per mont } k \end{array} \right\} + \left\{ \begin{array}{c} \text{Number of birth} \\ \text{per mont } k+1 \end{array} \right\} \tag{1.27}$$

(This is assuming that the mortality rate is negligible).
Since rabbits are able to reproduce only two months after their birth, they give birth to one pair of rabbits per month. So

$$\left\{ \begin{array}{c} \text{Number of birth} \\ \text{per month} k+1 \end{array} \right\} = \left\{ \begin{array}{c} \text{Current number} \\ \text{per month} k-1 \end{array} \right\} \tag{1.28}$$

$N_k$ denotes the number of pairs at the end of each month, and based on the previous two equations (1.27) and (1.28)

$$N_{k+1} = N_k + N_{k-1}, \quad k = 1, 2, \ldots \tag{1.29}$$

it is an equation of the second order called the Fibonacci equation.

**Constrained growth: Non-linear difference equations**

[7] Exponential growth remains the best model for expressing population growth, this is because linear difference equations are not generally suitable us a model for population growth because it expects unlimited growth in the event that the population is increasing, and this is contrary to what is observed in nature.

To avoid rejecting the last model, the researchers made modification to it in order to approximate the behavior observed in nature. Another study proved that as the population increases, the number of deaths increases while births decrease, this is due to several factors most notably crowding, competition for food ...

Carrying capacity is defined as the number of people when the birth rate equals the death rate.

The linear difference equation for population growth is given as follows

$$N_{k+1} = N_k + rN_k,$$

where $r$ represents the growth rate.

The previous study was summarized in the following writing

$$N_{k+1} = N_k + R(N_k)N_k, \tag{1.30}$$

where $R(N_k)$ indicates the related population growth rate. The equation (1.30) is a nonlinear difference equation because the unknown function appears as a mediator of the nonlinear function $xR(x)$ and because the unknown function $R$ meets the following conditions.

**(i)** a) As a result of congestion, $R(N_k)$ decreases and $N_k$ increases until it, becomes equal to the carrying capacity.

**(ii)** Given that $N_k$ is much smaller than $k$ it is noticeable that the growth is exponential in the population. So that is

$$R(N_k) \longrightarrow r \quad asN_k \longrightarrow 0$$

$r$ is constant called the unrestricted growth rate, where $r$ and $N$ are constants to be determined empirically.

Start with the simplest linear function that fulfills conditions $a$ and $b$, it can be chosen as follows

$$R(N_k) = -\frac{r}{k}N_k + r$$

substituting this relationship into equation (1.30) yields the following discrete logistic equation

$$N_{k+1} = N_k + rN_k\left(1 - \frac{N_k}{K}\right) \tag{1.31}$$

it is the most widely used equation for the dynamics of population.

**The dual epidemic model: System of difference equations**

[7] Measles is a highly contagious disease, caused by a virus that spreads the rough active contact between individuals. It is a disease that effects children greatly.

An epidemic of measles is observed in Britain and U.S.A every two or three

years. Hence, the spread of the measles epidemic leads to the emergence of system of difference equations.
Now let's see the development of measles in the body of one child:
A child who has not yet contracted measles is called susceptible, immediately after a child becomes ill for the first time, there comes a latent period that is not contagious, in which no symptoms of the disease appear, and this period lasts from 5 to 7 days.
After that, the turn comes to the contagious period, which is a period that lasts about one week, during which the child is called a contagious child.
After this period has passed, the child recovers completely and acquires immunity against this disease so that it cannot be infected with it in the future.
To simplify this, we have the following hypotheses :

**(i)** The duration of borth the latent and contagious period is one week.

**(ii)** All interactions between children accurat the end of the week. that is people exposed to measles remain constant during the week.
The researchers then modeled the prevalence of the disease and used one week as the time period. Hance

$$I_k = \left\{ \begin{array}{c} \text{Number of} \\ \text{infections per week } k \end{array} \right\}, \tag{1.32}$$

and

$$S_k = \left\{ \begin{array}{c} \text{Number of} \\ \text{people exposed to measles per week } k \end{array} \right\}. \tag{1.33}$$

The researchers also developed an equation that expresses the number of infections, this equation is given as follows

$$I_{k+1} = \left\{ \begin{array}{c} \text{Number of} \\ \text{infections} \\ \text{per week } k+1 \end{array} \right\} = \left\{ \begin{array}{c} \text{Number of people exposed to} \\ \text{measles who contracted} \\ \text{befor the week } k \end{array} \right\}. \tag{1.34}$$

Since there are new borns, this is an important factor in counting the number of people infected with measles. Then

$$S_{k+1} = \left\{ \begin{array}{c} \text{Number of} \\ \text{people} \\ \text{exposed} \\ \text{two measles} \\ \text{per week } k \end{array} \right\} - \left\{ \begin{array}{c} \text{Number of} \\ \text{people exposed} \\ \text{to measles who} \\ \text{contracted befor} \\ \text{the week } k \end{array} \right\} + \left\{ \begin{array}{c} \text{Number} \\ \text{of births} \\ \text{per} \\ \text{week } k+1 \end{array} \right\},$$

So, to find the number of people exposed to measles per week $fS_k$, suppose that one infectious agent infects a fixed number of the total number of people exposed to mea sles $I_k$.

To finally get the following system

$$\left\{ \begin{array}{c} \text{Number of people exposed to measles} \\ \text{who contracted befor the week } k \end{array} \right\} = fS_kI_k, \tag{1.35}$$

which $B$ and $f$ are costant coefficients of the of the model, which $f$ represents the number of people exposed to measles by a single infective.

# Chapter 2

# Form of solutions to several difference equations systems utilizing specific sequences

## 2.1 Difference equations

### 2.1.1 Nonlinear difference equations

Let $f$ be a continously differentiable function which

$$f : I^{k+1} \longrightarrow I, \quad I \subseteq \mathbb{R}.$$

**Definition 2.1** *[35] The (k+1)-order nonlinear difference equation is given as*

$$x_{n+1} = f(x_n, x_{n-1}, x_{n-2}, ..., x_{n-k}), \tag{2.1}$$

*which $n, k \in \mathbb{N}_0$ such that the initial conditions $(x_{-k}, x_{-k+1}, \ldots, x_0) \in I^{k+1}$.*

**Definition 2.2** *[35] The point $\overline{x} \in I$ is known as the equilibrium point of an equation (2.1), if it satisfied*

$$\overline{x} = f(\overline{x}, \overline{x} \ldots, \overline{x}).$$

*In another way, if we have*

$$x_n = \overline{x}, \quad \forall n \geq -k,$$

*with $n, k \in \mathbb{N}_0$.*

**Definition 2.3** *[35] The equation*

$$y_{n+1} = p_0 y_n + p_1 y_{n-1} + \cdots + p_k y_{n-k} \tag{2.2}$$

*is said to be a linear difference equation related to the equation (2.1), with*

$$p_i = \frac{\partial f}{\partial u_i}(\overline{x}, \overline{x} ..., \overline{x}), \quad i = 0, \ldots, k$$

*,*

and

$$p(\lambda) = \lambda^{k+1} - p_0 \lambda^k - p_1 \lambda^{k-1} - \cdots - p_k, \tag{2.3}$$

where $p(\lambda)$ is the characteristic polynomial associated to (2.2).

**Theorem 2.1** *($Rouche^{'}s$ Theorem) [23], [35] Let $f(z)$ and $g(z)$ two holomorphes functions in the open $\Omega$ of complex plane $\mathbb{C}$. If*

$$|g(z)| < |f(z)|, \quad \forall z \in \partial K.$$

*Then, the number of zeros of $f(z) + g(z)$ in $K$ equals to the number of zeros of $f(z)$ in $K$.*

## 2.2 Systems of nonlinear difference equations

**Definition 2.4** *[21] Let $f$ and $g$ be two continuously differentiable functions given as*

$$f : I^{k+1} \times J^{k+1} \longrightarrow I, \quad g : I^{k+1} \times J^{k+1} \longrightarrow J, \quad I, J \subseteq \mathbb{R},$$

*and we have the following difference equations systems*

$$\begin{cases} x_{n+1} = f(x_n, x_{n-1}, x_{n-2}, \dots, x_{n-k}, y_n, y_{n-1}, y_{n-2}, \dots, y_{n-k}), \\ y_{n+1} = g(x_n, x_{n-1}, x_{n-2}, \dots, x_{n-k}, y_n, y_{n-1}, y_{n-2}, \dots, y_{n-k}) \end{cases} \tag{2.4}$$

*which $n, k \in \mathbb{N}_0$, and we have the following initial conditions*
$(x_{-k}, x_{-k+1}, x_{-k+2}, \dots, x_0) \in I^{k+1}$ and $(y_{-k}, y_{-k+1}, y_{-k+2}, \dots, y_0) \in J^{k+1}$.
*We define the function*

$$H : I^{k+1} \times J^{k+1} \longrightarrow I^{k+1} \times J^{k+1},$$

*as*

$$H(\chi) = (f_0(\chi), f_1(\chi), f_2(\chi), \dots, f_k(\chi), g_0(\chi), g_1(\chi), g_2(\chi), \dots, g_k(\chi)),$$

*with*

$$\chi = (u_0, u_1, u_2, \dots, u_k, v_0, v_1, v_2, \dots, v_k)^T$$

$$f_0(\chi) = f(\chi), \quad f_1(\chi) = u_0, \quad f_2(\chi) = u_1, \quad f_3(\chi) = u_2, \quad \dots \quad , f_k(\chi) = u_{k-1},$$

*and*

$$g_0(\chi) = g(\chi), \quad g_1(\chi) = v_0, \quad g_2(\chi) = v_1, \quad g_3(\chi) = v_2, \quad \dots \quad , g_k(\chi) = v_{k-1}.$$

*Let*

$$\chi_n = (x_n, x_{n-1}, x_{n-2}, x_{n-3}, ..., x_{n-k}, y_n, y_{n-1}, y_{n-2}, y_{n-3}, ..., y_{n-k})^T.$$

*As a result, the system (2.4) is equivalent to the system shown below*

$$\chi_{n+1} = H(\chi_n), \quad n = 0, 1, 2, \ldots \tag{2.5}$$

*which*

$$\begin{cases} x_{n+1} = f(x_n, x_{n-1}, x_{n-2}, x_{n-3}, \dots, x_{n-p}, y_n, y_{n-1}, y_{n-2}, y_{n-3} \dots, y_{n-p}) \\ x_n = x_n \\ x_{n-1} = x_{n-1} \\ x_{n-2} = x_{n-2} \\ \quad \dots\dots\dots \\ \quad \dots\dots\dots \\ \quad \dots\dots\dots \\ x_{n-(p-1)} = x_{n-(p-1)} \\ y_{n+1} = g(x_n, x_{n-1}, x_{n-2}, x_{n-3}, \dots, x_{n-p}, y_n, y_{n-1}, y_{n-2}, y_{n-3} \dots, y_{n-p}) \\ y_n = y_n \\ y_{n-1} = y_{n-1} \\ y_{n-2} = y_{n-2} \\ \quad \dots\dots\dots \\ \quad \dots\dots\dots \\ \quad \dots\dots\dots \\ y_{n-(p-1)} = y_{n-(p-1)}. \end{cases}$$

**Definition 2.5** *(Equilibrium point) [35]*

**(i)** *The point* $(\overline{x}, \overline{y})$ *is known as the equilibrium point of system* (2.4)*, if we* *have*

$$\begin{cases} \overline{x} = f(\overline{x}, \overline{x}, \ldots, \overline{x}, \overline{y}, \overline{y}, \ldots, \overline{y}) \\ \overline{y} = g(\overline{x}, \overline{x}, \ldots, \overline{x}, \overline{y}, \overline{y}, \ldots, \overline{y}). \end{cases}$$

**(ii)** *The point* $\overline{\chi} = (\overline{x}, \overline{x}, \ldots, \overline{x}, \overline{y}, \overline{y}, \ldots, \overline{y}) \in I^k \times J^k$ *is the equilibrium point of system* (2.5), *if we have*

$$\chi = H(\chi).$$

**Definition 2.6** *[25]*

**(i)** *If there exists a* $n_0 \geq -k$. *Then, the solution* $(x_n, y_n)_{n \geq -k}$ *is said to be eventually periodic with period* $p$, *which*

$$x_{n+p} = x_n, \quad n \geq n_0.$$

*(ii) if* $n_0 = -k$. *Then, we say that the solution* $(x_n, y_n)_{n \geq -k}$ *is periodic with period* $p \in \mathbb{N}$.

**Definition 2.7** *(Stability) [21]Let* $\|.\|$ *be the Euclidean norm and* $\overline{\chi}$ *be the equilibrium point of system (2.5).*

**(i)** *The equilibrium point* $\overline{\chi}$ *is said to be locally stable if for any* $\epsilon > 0$, *there exists* $\delta > 0$, *such that*

$$\|\chi_0 - \overline{\chi}\| < \delta \Longrightarrow \|\chi_n - \overline{\chi}\| < \epsilon, \quad \forall n \geq 0.$$

**(ii)** *The equilibrium point* $\overline{\chi}$ *is said to be locally asymptotically stable when it is stable and there exists* $\gamma > 0$, *such that*

$$\|\chi_0 - \overline{\chi}\| < \gamma \Longrightarrow \|\chi_n - \overline{\chi}\| \longrightarrow 0, \quad n \longrightarrow +\infty.$$

**(iii)** *It is said that the equilibrium point* $\overline{\chi}$ *is a global attractor if for each* $\chi_0$ *(respectively for each* $\chi_0$ *in* $G \subseteq I^k \times J^k$*)*

$$\|\chi_n - \overline{\chi}\| \longrightarrow 0, \quad n \longrightarrow +\infty.$$

**(iiii)** *The equilibrium point* $\overline{\chi}$ *is said to be globally asymptotically stable (or globally asymptotically stable relative to* $G$*) if it is asymptotically stable, and if for every* $\chi_0 \in G$

$$\|\chi_n - \overline{\chi}\| \longrightarrow 0, \quad n \longrightarrow +\infty.$$

**(iiiii)** *If the equilibrium point* $\overline{\chi}$ *is not stable, Then, it is said to be unstable.*

**<u>Remark</u> 2.1** *[21] It is clear that* $(\overline{x}, \overline{y}) \in I \times J$ *is equilibrium point of system* (2.4) *if and only if* $\overline{\chi} = (\overline{x}, \overline{x}, \ldots, \overline{y}, \overline{y}, \ldots) \in I^{k+1} \times J^{k+1}$ *is equilibrium point of system* (2.5).

**<u>Definition</u> 2.8** *[21] The following system stated that the linear system related to system* (2.5) *around the equilibrium point* $\overline{\chi} = (\overline{x}, \overline{x}, \ldots, \overline{y}, \overline{y}, \ldots)$ *is given as follows*

$$\chi_{n+1} = A\chi_n, \quad n = 0, 1, \ldots$$

*where $A$ is the Jacobian matrix of the function $H$ at the equilibrium point $\overline{\chi}$*

$$A = \begin{bmatrix} \frac{\partial f_0}{\partial u_0}(\overline{\chi}) & \dots & \frac{\partial f_0}{\partial u_k}(\overline{\chi}) & \frac{\partial f_0}{\partial v_0}(\overline{\chi}) & \dots & \frac{\partial f_0}{\partial v_k}(\overline{\chi}) \\ \frac{\partial f_1}{\partial u_0}(\overline{\chi}) & \dots & \frac{\partial f_1}{\partial u_k}(\overline{\chi}) & \frac{\partial f_1}{\partial v_0}(\overline{\chi}) & \dots & \frac{\partial f_1}{\partial v_k}(\overline{\chi}) \\ \vdots & \dots & \ddots & \dots & \ddots & \vdots \\ \vdots & \dots & \ddots & \dots & \ddots & \vdots \\ \frac{\partial f_k}{\partial u_0}(\overline{\chi}) & \dots & \frac{\partial f_k}{\partial u_k}(\overline{\chi}) & \frac{\partial f_k}{\partial v_0}(\overline{\chi}) & \dots & \frac{\partial f_k}{\partial v_k}(\overline{\chi}) \\ \frac{\partial g_0}{\partial u_0}(\overline{\chi}) & \dots & \frac{\partial g_0}{\partial u_k}(\overline{\chi}) & \frac{\partial g_0}{\partial v_0}(\overline{\chi}) & \dots & \frac{\partial g_0}{\partial v_k}(\overline{\chi}) \\ \vdots & \dots & \ddots & \dots & \ddots & \vdots \\ \vdots & \dots & \ddots & \dots & \ddots & \vdots \\ \frac{\partial g_k}{\partial u_0}(\overline{\chi}) & \dots & \frac{\partial g_k}{\partial u_k}(\overline{\chi}) & \frac{\partial g_k}{\partial v_0}(\overline{\chi}) & \dots & \frac{\partial g_k}{\partial v_k}(\overline{\chi}) \end{bmatrix}$$

**Theorem 2.2** *[21]*

**(i)** *If all eigenvalues of the Jacobian matrix $A$ satisfy the condition $|\lambda| < 1$, then the equilibrium point $\overline{\chi}$ of system (2.5) is asymptotically stable.*

**(ii)** *If at least one eigenvalue of the Jacobian matrix $A$ has an absolute value larger than one, then the equilibrium point $\overline{\chi}$ of system (2.5) is unstable. (For more details see the proof in [33]).*

## 2.3 Form of the solutions of certain systems of difference equations

### 2.3.1 The first system

In this part, we provide the solutions to the extended system of difference equations whose solutions are related to (GFS).

Consider the following difference equation system

$$\begin{cases} x_{n+1} = \frac{1}{1+y_{(n+1)-k}}, \\ y_{n+1} = \frac{1}{1+x_{(n+1)-k}}, \end{cases} \quad n \in \mathbb{N}_0, \quad k = 1, 2, \dots, \tag{2.6}$$

where the initial conditions of the negative index terms

$x_{-k}, x_{-(k-1)}, x_{-(k-2)}, \dots, x_0, y_{-k}, y_{-(k-1)}, y_{-(k-2)}, \dots, y_0 \in \mathbb{R} - F$.
with $F = \left\{-\frac{F_{n+1}}{F_n}; n = 1, 2, \dots\right\}$.
The following theorem specifies the form of system (2.6) solutions.

**Theorem 2.3** *[24] The solutions of system* (2.6) *are given as follows*

**(i)** *From* $i = 1, 2, \dots, k$, *we have*

$$\begin{cases} x_{2kn+i} = \dfrac{F_{2n+1} + F_{2n}y_{i-k}}{F_{2n+2} + F_{2n+1}y_{i-k}}, \\ y_{2kn+i} = \dfrac{F_{2n+1} + F_{2n}x_{i-k}}{F_{2n+2} + F_{2n+1}x_{i-k}}. \end{cases} \quad n \in \mathbb{N}_0, \tag{2.7}$$

**(ii)** *From* $i = (k+1), (k+2), \dots, 2k$, *we have*

$$\begin{cases} x_{2kn+i} = \dfrac{F_{2n+2} + F_{2n+1}x_{i-2k}}{F_{2n+3} + F_{2n+2}x_{i-2k}}, \\ y_{2kn+i} = \dfrac{F_{2n+2} + F_{2n+1}y_{i-2k}}{F_{2n+3} + F_{2n+2}y_{i-2k}}. \end{cases} \quad n \in \mathbb{N}_0, \tag{2.8}$$

**Proof.** [24] From (2.6) we have

for $n = 0$ and when $i = 1, 2, \dots, k$, we have

$$x_1 = \frac{1}{1+y_{1-k}}, \quad x_2 = \frac{1}{1+y_{2-k}}, \quad x_3 = \frac{1}{1+y_{3-k}}, \dots,$$
$$y_1 = \frac{1}{1+x_{1-k}}, \quad y_2 = \frac{1}{1+x_{2-k}}, \quad y_3 = \frac{1}{1+x_{3-k}}, \dots,$$

On the other hand, for $n = 0$ and when $i = (k+1), (k+2), \ldots, 2k$ we get

$$x_{k+1} = \frac{1 + x_{-k+1}}{2 + x_{-k+1}}, \quad x_{k+2} = \frac{1 + x_{-k+2}}{2 + x_{-k+2}}, \quad \frac{1 + x_{-k+3}}{2 + x_{-k+3}}, \ldots,$$
$$y_{k+1} = \frac{1 + y_{-k+1}}{2 + y_{-k+1}}, \quad y_{k+2} = \frac{1 + y_{-k+2}}{2 + y_{-k+2}}, \quad \frac{1 + y_{-k+3}}{2 + y_{-k+3}}, \ldots,$$

So, the result holds for $n = 0$.

Now suppose that $n \geq 1$ and that our assumption holds for $n - 1$. That is,

$$x_{2k(n-1)+i} = \frac{F_{2n-1} + F_{2n-2}y_{i-k}}{F_{2n} + F_{2n-1}y_{i-k}}, \quad i = 1, 2, \ldots, k, \tag{2.9}$$

$$y_{2k(n-1)+i} = \frac{F_{2n-1} + F_{2n-2}x_{i-k}}{F_{2n} + F_{2n-1}x_{i-k}}, \quad i = 1, 2, \ldots, k, \tag{2.10}$$

$$x_{2k(n-1)+i} = \frac{F_{2n} + F_{2n-1}x_{i-2k}}{F_{2n+1} + F_{2n}x_{i-2k}}, \quad i = (k+1), (k+2), \ldots, 2k, \tag{2.11}$$

$$y_{2k(n-1)+i} = \frac{F_{2n} + F_{2n-1}y_{i-2k}}{F_{2n+1} + F_{2n}y_{i-2k}}, \quad i = (k+1), (k+2), \ldots, 2k. \tag{2.12}$$

For $i = 1, 2, 3, \ldots, k$, it follows from (2.6), (2.9), and (2.10) that

$$\begin{aligned}
x_{2kn+i} &= \frac{1}{1 + y_{2kn+i-k}}, \\
&= \frac{1}{1 + \frac{1}{1+x_{2kn+i-2k}}}, \\
&= \frac{1 + x_{2k(n-1)+i}}{2 + x_{2k(n-1)+i}}, \\
&= \frac{1 + \frac{F_{2n-1}+F_{2n-2}y_{i-k}}{F_{2n}+F_{2n-1}y_{i-k}}}{2 + \frac{F_{2n-1}+F_{2n-2}y_{i-k}}{F_{2n}+F_{2n-1}y_{i-k}}}, \\
&= \frac{F_{2n} + F_{2n-1}y_{i-k} + F_{2n-1} + F_{2n-2}y_{i-k}}{2F_{2n} + 2F_{2n-1}y_{i-k} + F_{2n-1} + F_{2n-2}y_{i-k}}, \\
&= \frac{F_{2n+1} + F_{2n}y_{i-k}}{F_{2n+2} + F_{2n+1}y_{i-k}},
\end{aligned}$$

and

$$\begin{aligned} y_{2kn+i} &= \frac{1}{1+x_{2kn+i-k}}, \\ &= \frac{1}{1+\frac{1}{1+y_{2kn+i-2k}}}, \\ &= \frac{1+y_{2k(n-1)+i}}{2+y_{2k(n-1)+i}}, \\ &= \frac{1+\frac{F_{2n-1}+F_{2n-2}x_{i-k}}{F_{2n}+F_{2n-1}x_{i-k}}}{2+\frac{F_{2n-1}+F_{2n-2}x_{i-k}}{F_{2n}+F_{2n-1}x_{i-k}}}, \\ &= \frac{F_{2n}+F_{2n-1}x_{i-k}+F_{2n-1}+F_{2n-2}x_{i-k}}{2F_{2n}+2F_{2n-1}x_{i-k}+F_{2n-1}+F_{2n-2}x_{i-k}}, \\ &= \frac{F_{2n+1}+F_{2n}x_{i-k}}{F_{2n+2}+F_{2n+1}x_{i-k}}. \end{aligned}$$

Similarly, for $i=(k+1),(k+2),\ldots,2k$ and from (2.6), (2.11) and (2.12), we get

$$\begin{aligned} x_{2kn+i} &= \frac{1}{1+x_{2kn+i-k}}, \\ &= \frac{1}{1+\frac{1}{1+x_{2kn+i-2k}}}, \\ &= \frac{1+x_{2k(n-1)+i}}{2+x_{2k(n-1)+i}}, \\ &= \frac{1+\frac{F_{2n}+F_{2n-1}x_{i-2k}}{F_{2n+1}+F_{2n}x_{i-2k}}}{2+\frac{F_{2n}+F_{2n-1}x_{i-2k}}{F_{2n+1}+F_{2n}x_{i-2k}}}, \\ &= \frac{F_{2n+1}+F_{2n}x_{i-2k}+F_{2n}+F_{2n-1}x_{i-2k}}{2F_{2n+1}+2F_{2n}x_{i-2k}+F_{2n}+F_{2n-1}x_{i-2k}}, \\ &= \frac{F_{2n+2}+F_{2n+1}x_{i-2k}}{F_{2n+3}+F_{2n+2}x_{i-2k}}, \end{aligned}$$

and

$$
\begin{aligned}
y_{2kn+i} &= \frac{1}{1+y_{2kn+i-k}}, \\
&= \frac{1}{1+\frac{1}{1+y_{2kn+i-2k}}}, \\
&= \frac{1+y_{2k(n-1)+i}}{2+y_{2k(n-1)+i}}, \\
&= \frac{1+\frac{F_{2n}+F_{2n-1}y_{i-2k}}{F_{2n+1}+F_{2n}y_{i-2k}}}{2+\frac{F_{2n}+F_{2n-1}y_{i-2k}}{F_{2n+1}+F_{2n}y_{i-2k}}}, \\
&= \frac{F_{2n+1}+F_{2n}y_{i-2k}+F_{2n}+F_{2n-1}y_{i-2k}}{2F_{2n+1}+2F_{2n}y_{i-2k}+F_{2n}+F_{2n-1}y_{i-2k}}, \\
&= \frac{F_{2n+2}+F_{2n+1}y_{i-2k}}{F_{2n+3}+F_{2n+2}y_{i-2k}}.
\end{aligned}
$$

This completes the proof. ■

**Theorem 2.4** *The solutions of system (2.6) are described by*

**(i)** *From $i = 1, 2, \ldots, k$, we have*

$$\begin{cases} x_{2kn+i} = \dfrac{\frac{(b-a)G_{2n+2}-aG_{2n+1}}{b^2-a^2-ab} + \frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab} y_{i-k}}{\frac{bG_{2n+3}-aG_{2n+4}}{b^2-a^2-ab} + \frac{(b-a)G_{2n+2}-aG_{2n+1}}{b^2-a^2-ab} y_{i-k}}, \\ y_{2kn+i} = \dfrac{\frac{(b-a)G_{2n+2}-aG_{2n+1}}{b^2-a^2-ab} + \frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab} x_{i-k}}{\frac{bG_{2n+3}-aG_{2n+4}}{b^2-a^2-ab} + \frac{(b-a)G_{2n+2}-aG_{2n+1}}{b^2-a^2-ab} x_{i-k}}. \end{cases} \quad n \in \mathbb{N}_0. \qquad (2.13)$$

**(ii)** *From $i = (k+1), (k+2), \ldots, 2k$, we have*

$$\begin{cases} x_{2kn+i} = \dfrac{\frac{bG_{2n+3}-aG_{2n+4}}{b^2-a^2-ab} + \frac{(b-a)G_{2n+2}-aG_{2n+1}}{b^2-a^2-ab} x_{i-2k}}{\frac{(b-a)G_{2n+4}-aG_{2n+3}}{b^2-a^2-ab} + \frac{bG_{2n+3}-aG_{2n+4}}{b^2-a^2-ab} x_{i-2k}}, \\ y_{2kn+i} = \dfrac{\frac{bG_{2n+3}-aG_{2n+4}}{b^2-a^2-ab} + \frac{(b-a)G_{2n+2}-aG_{2n+1}}{b^2-a^2-ab} y_{i-2k}}{\frac{(b-a)G_{2n+4}-aG_{2n+3}}{b^2-a^2-ab} + \frac{bG_{2n+3}-aG_{2n+4}}{b^2-a^2-ab} y_{i-2k}} \end{cases} \quad n \in \mathbb{N}_0. \qquad (2.14)$$

To satisfied this theorm, we require the following remark

**Remark 2.2** *we have*

$$\begin{cases} G_{2n+2} = bF_{2n+1} + aF_{2n}, \\ G_{2n+1} = bF_{2n} + aF_{2n-1}. \end{cases}$$

*By [34], we can obtained the following formulas*

$$F_{2n} = \frac{\begin{vmatrix} b & G_{2n+2} \\ a & G_{2n+1} \end{vmatrix}}{\begin{vmatrix} b & a \\ a & b-a \end{vmatrix}} = \frac{bG_{2n+1} - aG_{2n+2}}{b^2 - a^2 - ab}$$

,

$$F_{2n+1}=\frac{\begin{vmatrix} G_{2n+2} & a \\ G_{2n+1} & b-a \end{vmatrix}}{\begin{vmatrix} b & a \\ a & b-a \end{vmatrix}}=\frac{(b-a)G_{2n+2}-aG_{2n+1}}{b^2-a^2-ab}$$

. *Similarly, we obtained*

$$F_{2n+2}=\frac{\begin{vmatrix} b & G_{2n+4} \\ a & G_{2n+3} \end{vmatrix}}{\begin{vmatrix} b & a \\ a & b-a \end{vmatrix}}=\frac{bG_{2n+3}-aG_{2n+4}}{b^2-a^2-ab},$$

$$F_{2n+3}=\frac{\begin{vmatrix} G_{2n+4} & a \\ G_{2n+3} & b-a \end{vmatrix}}{\begin{vmatrix} b & a \\ a & b-a \end{vmatrix}}=\frac{(b-a)G_{2n+4}-aG_{2n+3}}{b^2-a^2-ab}$$

.

**Proof.** Now, let us return to the proof of theorem 2.4

from (2.13) we have

for $n=0$ and $i=1,2,\ldots,k$

$$x_i=\frac{\frac{(b-a)b-a^2}{b^2-a^2-ab}+\frac{ab-ab}{b^2-a^2-ab}y_{i-k}}{\frac{b(a+b)-a(a+2b)}{b^2-a^2-ab}+\frac{(b-a)b-a^2}{b^2-a^2-ab}y_{i-k}}=\frac{1}{1+y_{i-k}}.$$

So,

$$x_1=\frac{1}{1+y_{1-k}},\quad x_2=\frac{1}{1+y_{2-k}},\quad x_3=\frac{1}{1+y_{3-k}},\ldots$$

Similarly, we have

$$y_i = \frac{\frac{(b-a)b-a^2}{b^2-a^2-ab} + \frac{ab-ab}{b^2-a^2-ab}x_{i-k}}{\frac{b(a+b)-a(a+2b)}{b^2-a^2-ab} + \frac{(b-a)b-a^2}{b^2-a^2-ab}x_{i-k}} = \frac{1}{1+x_{i-k}}.$$

Then,

$$y_1 = \frac{1}{1+x_{1-k}}, \quad y_2 = \frac{1}{1+x_{2-k}}, \quad y_3 = \frac{1}{1+x_{3-k}}, \ldots$$

Now, for $n = 0$, with $i = (k+1), (k+2), ..., 2k$

$$x_i = \frac{\frac{b(a+b)-a(a+2b)}{b^2-a^2-ab} + \frac{(b-a)b-a^2}{b^2-a^2-ab}x_{i-2k}}{\frac{(b-a)(a+2b)-a(a+b)}{b^2-a^2-ab} + \frac{b(a+b)-a(a+2b)}{b^2-a^2-ab}x_{i-2k}} = \frac{1+x_{i-2k}}{2+x_{i-2k}}.$$

So,

$$x_{k+1} = \frac{1+x_{-k+1}}{2+x_{-k+1}}, \quad x_{k+2} = \frac{1+x_{-k+2}}{2+x_{-k+2}}, \quad x_{k+3} = \frac{1+x_{-k+3}}{2+x_{-k+3}}, \ldots,$$

and

$$y_i = \frac{\frac{b(a+b)-a(a+2b)}{b^2-a^2-ab} + \frac{(b-a)b-a^2}{b^2-a^2-ab}y_{i-2k}}{\frac{(b-a)(a+2b)-a(a+b)}{b^2-a^2-ab} + \frac{b(a+b)-a(a+2b)}{b^2-a^2-ab}y_{i-2k}} = \frac{1+y_{i-2k}}{2+y_{i-2k}}.$$

Then,

$$y_{k+1} = \frac{1+y_{-k+1}}{2+y_{-k+1}}, \quad y_{k+2} = \frac{1+y_{-k+2}}{2+y_{-k+2}}, \quad y_{k+3} = \frac{1+y_{-k+3}}{2+y_{-k+3}}, \ldots$$

As a consequence, the results holds for $n = 0$.

Now, we assume that $n > 1$ and that our assumptions remains true for $n-1$.

That is

**(i)** For $i = 1, 2, \ldots, k$

$$\begin{cases} x_{2k(n-1)+i} = \dfrac{\frac{(b-a)G_{2n}-aG_{2n-1}}{b^2-a^2-ab} + \frac{bG_{2n-1}-aG_{2n}}{b^2-a^2-ab}y_{i-k}}{\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab} + \frac{(b-a)G_{2n}-aG_{2n-1}}{b^2-a^2-ab}y_{i-k}}, \\ y_{2k(n-1)+i} = \dfrac{\frac{(b-a)G_{2n}-aG_{2n-1}}{b^2-a^2-ab} + \frac{bG_{2n-1}-aG_{2n}}{b^2-a^2-ab}x_{i-k}}{\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab} + \frac{(b-a)G_{2n}-aG_{2n-1}}{b^2-a^2-ab}x_{i-k}}. \end{cases} \quad n \in \mathbb{N}_0 \qquad (2.15)$$

**(ii)** For $i = (k+1), (k+2), \ldots, 2k$

$$\begin{cases} x_{2k(n-1)+i} = \dfrac{\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab} + \frac{(b-a)G_{2n}-aG_{2n-1}}{b^2-a^2-ab}x_{i-2k}}{\frac{(b-a)G_{2n+2}-aG_{2n+1}}{b^2-a^2-ab} + \frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}x_{i-2k}}, \\ y_{2k(n-1)+i} = \dfrac{\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab} + \frac{(b-a)G_{2n}-aG_{2n-1}}{b^2-a^2-ab}y_{i-2k}}{\frac{(b-a)G_{2n+2}-aG_{2n+1}}{b^2-a^2-ab} + \frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}y_{i-2k}}. \end{cases} \quad n \in \mathbb{N}_0 \quad (2.16)$$

Then, for $i = 1, 2, 3, \ldots, k$, it follows from (2.6) and (2.15) that is

$$\begin{aligned} x_{2kn+i} &= \frac{1}{1+y_{2kn+i-k}}, \\ &= \frac{1}{1+\frac{1}{1+x_{2kn+i-2k}}}, \\ &= \frac{1+x_{2k(n-1)+i}}{2+x_{2k(n-1)+i}}, \\ &= \frac{1+\frac{F_{2n-1}+F_{2n-2}y_{i-k}}{F_{2n}+F_{2n-1}y_{i-k}}}{2+\frac{F_{2n-1}+F_{2n-2}y_{i-k}}{F_{2n}+F_{2n-1}y_{i-k}}}, \\ &= \frac{1+\frac{\frac{(b-a)G_{2n}-aG_{2n-1}}{b^2-a^2-ab}+\frac{bG_{2n-1}-aG_{2n}}{b^2-a^2-ab}y_{i-k}}{\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}+\frac{(b-a)G_{2n}-aG_{2n-1}}{b^2-a^2-ab}y_{i-k}}}{2+\frac{\frac{(b-a)G_{2n}-aG_{2n-1}}{b^2-a^2-ab}+\frac{bG_{2n-1}-aG_{2n}}{b^2-a^2-ab}y_{i-k}}{\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}+\frac{(b-a)G_{2n}-aG_{2n-1}}{b^2-a^2-ab}y_{i-k}}}, \\ &= \frac{\frac{\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}+\frac{(b-a)G_{2n}-aG_{2n-1}}{b^2-a^2-ab}y_{i-k}+\frac{(b-a)G_{2n}-aG_{2n-1}}{b^2-a^2-ab}+\frac{bG_{2n-1}-aG_{2n}}{b^2-a^2-ab}y_{i-k}}{\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}+\frac{(b-a)G_{2n}-aG_{2n-1}}{b^2-a^2-ab}y_{i-k}}}{\frac{2\times\left(\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}+\frac{(b-a)G_{2n}-aG_{2n-1}}{b^2-a^2-ab}y_{i-k}\right)+\frac{(b-a)G_{2n}-aG_{2n-1}}{b^2-a^2-ab}+\frac{bG_{2n-1}-aG_{2n}}{b^2-a^2-ab}y_{i-k}}{\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}+\frac{(b-a)G_{2n}-aG_{2n-1}}{b^2-a^2-ab}y_{i-k}}}, \\ &= \frac{\frac{\frac{((b-a)G_{2n}-aG_{2n-1}-aG_{2n+2}+bG_{2n+1})+(bG_{2n}-2aG_{2n}+(b-a)G_{2n-1})y_{i-k}}{b^2-a^2-ab}}{\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}+\frac{(b-a)G_{2n}-aG_{2n-1}}{b^2-a^2-ab}y_{i-k}}}{\frac{\frac{(2(b-a)G_{2n}+2bG_{2n+1}-2aG_{2n+2}-2aG_{2n-1})+(bG_{2n}-2aG_{2n}+(b-a)G_{2n-1})y_{i-k}}{b^2-a^2-ab}}{\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}+\frac{(b-a)G_{2n}-aG_{2n-1}}{b^2-a^2-ab}y_{i-k}}}, \\ &= \frac{\frac{(b-a)G_{2n+2}-aG_{2n+1}}{b^2-a^2-ab}+\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}y_{i-k}}{\frac{bG_{2n+3}-aG_{2n+4}}{b^2-a^2-ab}+\frac{(b-a)G_{2n+2}-aG_{2n+1}}{b^2-a^2-ab}y_{i-k}}, \end{aligned}$$

and

$$\begin{aligned}
y_{2kn+i} &= \frac{1}{1+x_{2kn+i-k}}, \\
&= \frac{1}{1+\frac{1}{1+y_{2kn+i-2k}}}, \\
&= \frac{1+y_{2k(n-1)+i}}{2+y_{2k(n-1)+i}}, \\
&= \frac{1+\frac{F_{2n-1}+F_{2n-2}x_{i-k}}{F_{2n}+F_{2n-1}x_{i-k}}}{2+\frac{F_{2n-1}+F_{2n-2}x_{i-k}}{F_{2n}+F_{2n-1}x_{i-k}}}, \\
&= \frac{1+\frac{\frac{(b-a)G_{2n}-aG_{2n-1}}{b^2-a^2-ab}+\frac{bG_{2n-1}-aG_{2n}}{b^2-a^2-ab}x_{i-k}}{\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}+\frac{(b-a)G_{2n}-aG_{2n-1}}{b^2-a^2-ab}x_{i-k}}}{2+\frac{\frac{(b-a)G_{2n}-aG_{2n-1}}{b^2-a^2-ab}+\frac{bG_{2n-1}-aG_{2n}}{b^2-a^2-ab}x_{i-k}}{\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}+\frac{(b-a)G_{2n}-aG_{2n-1}}{b^2-a^2-ab}x_{i-k}}}, \\
&= \frac{\frac{\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}+\frac{(b-a)G_{2n}-aG_{2n-1}}{b^2-a^2-ab}x_{i-k}+\frac{(b-a)G_{2n}-aG_{2n-1}}{b^2-a^2-ab}+\frac{bG_{2n-1}-aG_{2n}}{b^2-a^2-ab}x_{i-k}}{\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}+\frac{(b-a)G_{2n}-aG_{2n-1}}{b^2-a^2-ab}x_{i-k}}}{\frac{2\times\left(\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}+\frac{(b-a)G_{2n}-aG_{2n-1}}{b^2-a^2-ab}x_{i-k}\right)+\frac{(b-a)G_{2n}-aG_{2n-1}}{b^2-a^2-ab}+\frac{bG_{2n-1}-aG_{2n}}{b^2-a^2-ab}x_{i-k}}{\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}+\frac{(b-a)G_{2n}-aG_{2n-1}}{b^2-a^2-ab}x_{i-k}}}, \\
&= \frac{\frac{\frac{\left((b-a)G_{2n}-aG_{2n-1}-aG_{2n+2}+bG_{2n+1}\right)+\left(bG_{2n}-2aG_{2n}+(b-a)G_{2n-1}\right)x_{i-k}}{b^2-a^2-ab}}{\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}+\frac{(b-a)G_{2n}-aG_{2n-1}}{b^2-a^2-ab}x_{i-k}}}{\frac{\frac{\left(2(b-a)G_{2n}+2bG_{2n+1}-2aG_{2n+2}-2aG_{2n-1}\right)+\left(bG_{2n}-2aG_{2n}+(b-a)G_{2n-1}\right)x_{i-k}}{b^2-a^2-ab}}{\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}+\frac{(b-a)G_{2n}-aG_{2n-1}}{b^2-a^2-ab}x_{i-k}}}, \\
&= \frac{\frac{(b-a)G_{2n+2}-aG_{2n+1}}{b^2-a^2-ab}+\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}x_{i-k}}{\frac{bG_{2n+3}-aG_{2n+4}}{b^2-a^2-ab}+\frac{(b-a)G_{2n+2}-aG_{2n+1}}{b^2-a^2-ab}x_{i-k}}.
\end{aligned}$$

Similarly, for $i=(k+1),(k+2),\ldots,2k$.

From (2.6) and (2.16), we get

$$\begin{aligned}
x_{2kn+i} &= \frac{1}{1+x_{2kn+i-k}}, \\
&= \frac{1}{1+\frac{1}{1+x_{2kn+i-2k}}}, \\
&= \frac{1+x_{2k(n-1)+i}}{2+x_{2k(n-1)+i}}, \\
&= \frac{1+\frac{F_{2n}+F_{2n-1}x_{i-2k}}{F_{2n+1}+F_{2n}x_{i-2k}}}{2+\frac{F_{2n}+F_{2n-1}x_{i-2k}}{F_{2n+1}+F_{2n}x_{i-2k}}}, \\
&= \frac{1+\frac{\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}+\frac{(b-a)G_{2n}-aG_{2n-1}}{b^2-a^2-ab}x_{i-2k}}{\frac{(b-a)G_{2n+2}-aG_{2n+1}}{b^2-a^2-ab}+\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}x_{i-2k}}}{2+\frac{\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}+\frac{(b-a)G_{2n}-aG_{2n-1}}{b^2-a^2-ab}x_{i-2k}}{\frac{(b-a)G_{2n+2}-aG_{2n+1}}{b^2-a^2-ab}+\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}x_{i-2k}}}, \\
&= \frac{\frac{\frac{(b-a)G_{2n+2}-aG_{2n+1}}{b^2-a^2-ab}+\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}x_{i-2k}+\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}+\frac{(b-a)G_{2n}-aG_{2n-1}}{b^2-a^2-ab}x_{i-2k}}{\frac{(b-a)G_{2n+2}-aG_{2n+1}}{b^2-a^2-ab}+\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}x_{i-2k}}}{\frac{2\times\left(\frac{(b-a)G_{2n+2}-aG_{2n+1}}{b^2-a^2-ab}+\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}x_{i-2k}\right)+\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}+\frac{(b-a)G_{2n}-aG_{2n-1}}{b^2-a^2-ab}x_{i-2k}}{\frac{(b-a)G_{2n+2}-aG_{2n+1}}{b^2-a^2-ab}+\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}x_{i-2k}}}, \\
&= \frac{\frac{\frac{((b-a)G_{2n+2}-aG_{2n+1}-aG_{2n+2}+bG_{2n+1})+(bG_{2n+1}-aG_{2n+2}+(b-a)G_{2n}-aG_{2n-1})x_{i-2k}}{b^2-a^2-ab}}{\frac{(b-a)G_{2n+2}-aG_{2n+1}}{b^2-a^2-ab}+\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}x_{i-2k}}}{\frac{\frac{(2(b-a)G_{2n+2}+2aG_{2n+1}+bG_{2n+1}-aG_{2n+2})+(bG_{2n+1}-aG_{2n+2}+(b-a)G_{2n}-aG_{2n-1})x_{i-2k}}{b^2-a^2-ab}}{\frac{(b-a)G_{2n+2}-aG_{2n+1}}{b^2-a^2-ab}+\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}x_{i-2k}}}, \\
&= \frac{\frac{bG_{2n+3}-aG_{2n+4}}{b^2-a^2-ab}+\frac{(b-a)G_{2n+2}-aG_{2n+1}}{b^2-a^2-ab}x_{i-2k}}{\frac{(b-a)G_{2n+4}-aG_{2n+3}}{b^2-a^2-ab}+\frac{bG_{2n+3}-aG_{2n+4}}{b^2-a^2-ab}x_{i-2k}},
\end{aligned}$$

and

$$\begin{aligned}
y_{2kn+i} &= \frac{1}{1+y_{2kn+i-k}}, \\
&= \frac{1}{1+\frac{1}{1+y_{2kn+i-2k}}}, \\
&= \frac{1+y_{2k(n-1)+i}}{2+y_{2k(n-1)+i}}, \\
&= \frac{1+\frac{F_{2n}+F_{2n-1}y_{i-2k}}{F_{2n+1}+F_{2n}y_{i-2k}}}{2+\frac{F_{2n}+F_{2n-1}y_{i-2k}}{F_{2n+1}+F_{2n}y_{i-2k}}}, \\
&= \frac{1+\frac{\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}+\frac{(b-a)G_{2n}-aG_{2n-1}}{b^2-a^2-ab}y_{i-2k}}{\frac{(b-a)G_{2n+2}-aG_{2n+1}}{b^2-a^2-ab}+\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}y_{i-2k}}}{2+\frac{\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}+\frac{(b-a)G_{2n}-aG_{2n-1}}{b^2-a^2-ab}y_{i-2k}}{\frac{(b-a)G_{2n+2}-aG_{2n+1}}{b^2-a^2-ab}+\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}y_{i-2k}}}, \\
&= \frac{\frac{\frac{(b-a)G_{2n+2}-aG_{2n+1}}{b^2-a^2-ab}+\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}y_{i-2k}+\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}+\frac{(b-a)G_{2n}-aG_{2n-1}}{b^2-a^2-ab}y_{i-2k}}{\frac{(b-a)G_{2n+2}-aG_{2n+1}}{b^2-a^2-ab}+\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}y_{i-2k}}}{\frac{2\times\left(\frac{(b-a)G_{2n+2}-aG_{2n+1}}{b^2-a^2-ab}+\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}y_{i-2k}\right)+\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}+\frac{(b-a)G_{2n}-aG_{2n-1}}{b^2-a^2-ab}y_{i-2k}}{\frac{(b-a)G_{2n+2}-aG_{2n+1}}{b^2-a^2-ab}+\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}y_{i-2k}}}, \\
&= \frac{\frac{\frac{\left((b-a)G_{2n+2}-aG_{2n+1}-aG_{2n+2}+bG_{2n+1}\right)+\left(bG_{2n+1}-aG_{2n+2}+(b-a)G_{2n}-aG_{2n-1}\right)y_{i-2k}}{b^2-a^2-ab}}{\frac{(b-a)G_{2n+2}-aG_{2n+1}}{b^2-a^2-ab}+\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}y_{i-2k}}}{\frac{\frac{\left(2(b-a)G_{2n+2}+2aG_{2n+1}+bG_{2n+1}-aG_{2n+2}\right)+\left(bG_{2n+1}-aG_{2n+2}+(b-a)G_{2n}-aG_{2n-1}\right)y_{i-2k}}{b^2-a^2-ab}}{\frac{(b-a)G_{2n+2}-aG_{2n+1}}{b^2-a^2-ab}+\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}y_{i-2k}}}, \\
&= \frac{\frac{bG_{2n+3}-aG_{2n+4}}{b^2-a^2-ab}+\frac{(b-a)G_{2n+2}-aG_{2n+1}}{b^2-a^2-ab}y_{i-2k}}{\frac{(b-a)G_{2n+4}-aG_{2n+3}}{b^2-a^2-ab}+\frac{bG_{2n+3}-aG_{2n+4}}{b^2-a^2-ab}y_{i-2k}}.
\end{aligned}$$

This completes the proof. ■

### 2.3.2 The second system

Now, we discuss the form of solutions related to the Tribonacci sequence of the following system of rational difference equations

$$\begin{cases} x_{n+1} = \frac{1}{y_{n-(p-1)}(x_{n-p}+1)+1}, \\ \\ y_{n+1} = \frac{1}{x_{n-(p-1)}(y_{n-p}+1)+1}, \end{cases} \quad n \in \mathbb{N}_0, \quad p \geq 1 \tag{2.17}$$

where the initial conditions of the negative index terms :

$x_{-p}, x_{-(p-1)}, x_{-(p-2)}, \ldots, x_0, y_{-p}, y_{-(p-1)}, y_{-(p-2)}, \ldots, y_0 \in \mathbb{R} - F$, with

$$F = \cup \{(x_{-p}, \ldots, x_0, y_{-p}, \ldots, y_0) : A_n = 0, B_n = 0, C_n = 0, D_n = 0\},$$

such that

$$\begin{aligned} A_n &= T_{2n-(2p-1)}x_{i-(p+1)}y_{i-p} + (T_{2n-2p} + T_{2n-(2p-1)})y_{i-p} + T_{2n-(2p-2)}, \\ B_n &= T_{2n-(2p-1)}y_{i-(p+1)}x_{i-p} + (T_{2n-2p} + T_{2n-(2p-1)})x_{i-p} + T_{2n-(2p-2)}, \\ C_n &= T_{2n-(2p-2)}y_{i-(p+1)}x_{i-p} + (T_{2n-(2p-1)} + T_{2n-(2p-2)})x_{i-p} + T_{2n-(2p-3)}, \\ D_n &= T_{2n-(2p-2)}x_{i-(p+1)}y_{i-p} + (T_{2n-(2p-1)} + T_{2n-(2p-2)})y_{i-p} + T_{2n-(2p-3)}. \end{aligned}$$

The following theorem describes the form of solutions of system (2.17).

**Theorem 2.5** *[31] Let $\{x_n, y_n\}_{n\geq -p}$ be a solutions of system (2.17). Then, for $n = 0, 1, 2, \ldots$, and $p \geq 1$, the form of $\{x_n, y_n\}_{n\geq -p}$ are given as*

**(1)** *For $i = 1, 2, \ldots, p$*

$$x_{2pn-i} = \frac{T_{2n-2p}x_{i-(p+1)}y_{i-p} + (T_{2n-(2p-2)} - T_{2n-(2p-1)})y_{i-p} + T_{2n-(2p-1)}}{T_{2n-(2p-1)}x_{i-(p+1)}y_{i-p} + (T_{2n-2p} + T_{2n-(2p-1)})y_{i-p} + T_{2n-(2p-2)}}. \tag{2.18}$$

**(2)** *For* $i = 1, 2, \ldots, p$

$$y_{2pn-i} = \frac{T_{2n-2p} y_{i-(p+1)} x_{i-p} + (T_{2n-(2p-2)} - T_{2n-(2p-1)}) x_{i-p} + T_{2n-(2p-1)}}{T_{2n-(2p-1)} y_{i-(p+1)} x_{i-p} + (T_{2n-2p} + T_{2n-(2p-1)}) x_{i-p} + T_{2n-(2p-2)}}. \tag{2.19}$$

**(3)** *For* $i = (p+1), (p+2), \ldots, 2p$

$$x_{2pn-(i-1)} = \frac{T_{2n-(2p-1)} y_{i-(p+1)} x_{i-p} + (T_{2n-(2p-3)} - T_{2n-(2p-2)}) x_{i-p} + T_{2n-(2p-2)}}{T_{2n-(2p-2)} y_{i-(p+1)} x_{i-p} + (T_{2n-(2p-1)} + T_{2n-(2p-2)}) x_{i-p} + T_{2n-(2p-3)}}. \tag{2.20}$$

**(4)** *For* $i = (p+1), (p+2), \ldots, 2p$

$$y_{2pn-(i-1)} = \frac{T_{2n-(2p-1)} x_{i-(p+1)} y_{i-p} + (T_{2n-(2p-3)} - T_{2n-(2p-2)}) y_{i-p} + T_{2n-(2p-2)}}{T_{2n-(2p-2)} x_{i-(p+1)} y_{i-p} + (T_{2n-(2p-1)} + T_{2n-(2p-2)}) y_{i-p} + T_{2n-(2p-3)}}, \tag{2.21}$$

*where* $T_n$ *is the nth Tribonacci numbers.*

**Proof.** By induction

For $k = 0$, the results holds.

Now we assuming that $k > 0$ and our assumption remains true for $k-1$. Then,

**(1)** For $i = 1, 2, \ldots, p$

$$x_{2p(k-1)-i} = \frac{T_{2k-(2p+2)} x_{i-(p+1)} y_{i-p} + (T_{2k-2p} - T_{2k-(2p+1)}) y_{i-p} + T_{2k-(2p+1)}}{T_{2k-(2p+1)} x_{i-(p+1)} y_{i-p} + (T_{2k-(2p+2)} + T_{2k-(2p+1)}) y_{i-p} + T_{2k-2p}}. \tag{2.22}$$

**(2)** For $i = 1, 2, \dots, p$

$$y_{2p(k-1)-i} = \frac{T_{2k-(2p+2)}y_{i-(p+1)}x_{i-p} + (T_{2k-2p} - T_{2k-(2p+1)})x_{i-p} + T_{2k-(2p+1)}}{T_{2k-(2p+1)}y_{i-(p+1)}x_{i-p} + (T_{2k-(2p+2)} + T_{2k-(2p+1)})x_{i-p} + T_{2k-2p}}. \tag{2.23}$$

**(3)** For $i = (p+1), (p+2), \dots, 2p$

$$x_{2p(k-1)-(i-1)} = \frac{T_{2k-(2p+1)}y_{i-(p+1)}x_{i-p} + (T_{2k-(2p-1)} - T_{2k-2p})x_{i-p} + T_{2k-2p}}{T_{2k-2p}y_{i-(p+1)}x_{i-p} + (T_{2k-(2p+1)} + T_{2k-2p})x_{i-p} + T_{2k-(2p-1)}}. \tag{2.24}$$

**(4)** For $i = (p+1), (p+2), \dots, 2p$

$$y_{2p(k-1)-(i-1)} = \frac{T_{2k-(2p+1)}x_{i-(p+1)}y_{i-p} + (T_{2k-(2p-1)} - T_{2k-2p})y_{i-p} + T_{2k-2p}}{T_{2k-2p}x_{i-(p+1)}y_{i-p} + (T_{2k-(2p+1)} + T_{2k-2p})y_{i-p} + T_{2k-(2p-1)}}. \tag{2.25}$$

On the other hand and for $i = 1, 2, 3, \dots, p$, it follows from (2.17), (2.22) and (2.23) that

$$\begin{aligned}
x_{2pk-i} &= \frac{1}{y_{2p(k-1)-(i-1)}(x_{2p(k-1)-i}+1)+1}, \\
&= \frac{1}{\frac{T_{2k-(2p+1)}x_{i-(p+1)}y_{i-p}+(T_{2k-(2p-1)}-T_{2k-2p})y_{i-p}+T_{2k-2p}}{T_{2k-2p}x_{i-(p+1)}y_{i-p}+(T_{2k-(2p+1)}+T_{2k-2p})y_{i-p}+T_{2k-(2p-1)}} \times \left(x_{2p(k-1)-i}+1\right)+1}, \\
&= \frac{1}{y_{2p(k-1)-(i-1)} \times \left(\frac{T_{2k-(2p+2)}x_{i-(p+1)}y_{i-p}+(T_{2k-2p}-T_{2k-(2p+1)})y_{i-p}+T_{2k-(2p+1)}}{T_{2k-(2p+1)}x_{i-(p+1)}y_{i-p}+(T_{2k-(2p+2)}+T_{2k-(2p+1)})y_{i-p}+T_{2k-2p}}+1\right)+1}, \\
&= \frac{T_{2k-2p}x_{i-(p+1)}y_{i-p} + (T_{2k-(2p-2)} - T_{2k-(2p-1)})y_{i-p} + T_{2k-(2p-1)}}{T_{2k-(2p-1)}x_{i-(p+1)}y_{i-p} + (T_{2k-2p} + T_{2k-(2p-1)})y_{i-p} + T_{2k-(2p-2)}}.
\end{aligned}$$

Then, for $i = 1, 2, \ldots, p$ we have

$$x_{2pk-i} = \frac{T_{2k-2p}x_{i-(p+1)}y_{i-p} + (T_{2k-(2p-2)} - T_{2k-(2p-1)})y_{i-p} + T_{2k-(2p-1)}}{T_{2k-(2p-1)}x_{i-(p+1)}y_{i-p} + (T_{2k-2p} + T_{2k-(2p-1)})y_{i-p} + T_{2k-(2p-2)}},$$

and

$$\begin{aligned} &y_{2pk-i} \\ &= \frac{1}{x_{2p(k-1)-(i-1)}(y_{2p(k-1)-i} + 1) + 1}, \\ &= \frac{1}{\frac{T_{2k-(2p+1)}y_{i-(p+1)}x_{i-p} + (T_{2k-(2p-1)} - T_{2k-2p})x_{i-p} + T_{2k-2p}}{T_{2k-2p}y_{i-(p+1)}x_{i-p} + (T_{2k-(2p+1)} + T_{2k-2p})x_{i-p} + T_{2k-(2p-1)}}, \times \left(y_{2p(k-1)-i} + 1\right) + 1}, \\ &= \frac{1}{x_{2p(k-1)-(i-1)} \times \left(\frac{T_{2k-(2p+2)}y_{i-(p+1)}x_{i-p} + (T_{2k-2p} - T_{2k-(2p+1)})x_{i-p} + T_{2k-(2p+1)}}{T_{2k-(2p+1)}y_{i-(p+1)}x_{i-p} + (T_{2k-(2p+2)} + T_{2k-(2p+1)})x_{i-p} + T_{2k-2p}} + 1\right) + 1}, \\ &= \frac{T_{2k-2p}y_{i-(p+1)}x_{i-p} + (T_{2k-(2p-2)} - T_{2k-(2p-1)})x_{i-p} + T_{2k-(2p-1)}}{T_{2k-(2p-1)}y_{i-(p+1)}x_{i-p} + (T_{2k-2p} + T_{2k-(2p-1)})x_{i-p} + T_{2k-(2p-2)}}. \end{aligned}$$

So, for $i = 1, 2, \ldots, p$ we have

$$y_{2pk-i} = \frac{T_{2k-2p}y_{i-(p+1)}x_{i-p} + (T_{2k-(2p-2)} - T_{2k-(2p-1)})x_{i-p} + T_{2k-(2p-1)}}{T_{2k-(2p-1)}y_{i-(p+1)}x_{i-p} + (T_{2k-2p} + T_{2k-(2p-1)})x_{i-p} + T_{2k-(2p-2)}}.$$

Similarly, for $i = (p+1), (p+2), \ldots, 2p$ and from (2.17), (2.24) and (2.25) we obtaind

$$\begin{aligned} &x_{2pk-(i-1)} \\ &= \frac{1}{y_{2pk-i}(x_{2p(k-1)-(i-1)} + 1) + 1}, \\ &= \frac{1}{\frac{T_{2k-2p}y_{i-(p+1)}x_{i-p} + (T_{2k-(2p-2)} - T_{2k-(2p-1)})x_{i-p} + T_{2k-(2p-1)}}{T_{2k-(2p-1)}y_{i-(p+1)}x_{i-p} + (T_{2k-2p} + T_{2k-(2p-1)})x_{i-p} + T_{2k-(2p-2)}} \times \left(x_{2p(k-1)-(i-1)} + 1\right) + 1}, \\ &= \frac{1}{y_{2pk-i} \times \left(\frac{T_{2k-(2p+1)}y_{i-(p+1)}x_{i-p} + (T_{2k-(2p-1)} - T_{2k-2p})x_{i-p} + T_{2k-2p}}{T_{2k-2p}y_{i-(p+1)}x_{i-p} + (T_{2k-(2p+1)} + T_{2k-2p})x_{i-p} + T_{2k-(2p-1)}} + 1\right) + 1}, \\ &= \frac{T_{2k-(2p-1)}y_{i-(p+1)}x_{i-p} + (T_{2k-(2p-3)} - T_{2k-(2p-2)})x_{i-p} + T_{2k-(2p-2)}}{T_{2k-(2p-2)}y_{i-(p+1)}x_{i-p} + (T_{2k-(2p-1)} + T_{2k-(2p-2)})x_{i-p} + T_{2k-(2p-3)}}. \end{aligned}$$

Then, for $i = (p+1), (p+2), \ldots, 2p$ we have

$$
\begin{aligned}
&x_{2pk-(i-1)} \\
&= \frac{T_{2k-(2p-1)} y_{i-(p+1)} x_{i-p} + (T_{2k-(2p-3)} - T_{2k-(2p-2)}) x_{i-p} + T_{2k-(2p-2)}}{T_{2k-(2p-2)} y_{i-(p+1)} x_{i-p} + (T_{2k-(2p-1)} + T_{2k-(2p-2)}) x_{i-p} + T_{2k-(2p-3)}},
\end{aligned}
$$

and

$$
\begin{aligned}
&y_{2pk-(i-1)} \\
&= \frac{1}{x_{2pk-i}(y_{2p(k-1)-(i-1)} + 1) + 1}, \\
&= \frac{1}{\frac{T_{2k-2p} x_{i-(p+1)} y_{i-p} + (T_{2k-(2p-2)} - T_{2k-(2p-1)}) y_{i-p} + T_{2k-(2p-1)}}{T_{2k-(2p-1)} x_{i-(p+1)} y_{i-p} + (T_{2k-2p} + T_{2k-(2p-1)}) y_{i-p} + T_{2k-(2p-2)}} \times (+1) + 1}, \\
&= \frac{1}{x_{2pk-i} \times \left( \frac{T_{2k-(2p+1)} x_{i-(p+1)} y_{i-p} + (T_{2k-(2p-1)} - T_{2k-2p}) y_{i-p} + T_{2k-2p}}{T_{2k-2p} x_{i-(p+1)} y_{i-p} + (T_{2k-(2p+1)} + T_{2k-2p}) y_{i-p} + T_{2k-(2p-1)}} + 1 \right) + 1}, \\
&= \frac{T_{2k-(2p-1)} x_{i-(p+1)} y_{i-p} + (T_{2k-(2p-3)} - T_{2k-(2p-2)}) y_{i-p} + T_{2k-(2p-2)}}{T_{2k-(2p-2)} x_{i-(p+1)} y_{i-p} + (T_{2k-(2p-1)} + T_{2k-(2p-2)}) y_{i-p} + T_{2k-(2p-3)}}.
\end{aligned}
$$

So, for $i = (p+1), (p+2), \ldots, 2p$ we have

$$
\begin{aligned}
&y_{2pk-(i-1)} \\
&= \frac{T_{2k-(2p-1)} x_{i-(p+1)} y_{i-p} + (T_{2k-(2p-3)} - T_{2k-(2p-2)}) y_{i-p} + T_{2k-(2p-2)}}{T_{2k-(2p-2)} x_{i-(p+1)} y_{i-p} + (T_{2k-(2p-1)} + T_{2k-(2p-2)}) y_{i-p} + T_{2k-(2p-3)}}.
\end{aligned}
$$

■

# Chapter 3

# On the global stability and periodicity of certain higher-order difference equations systems

## 3.1 Introduction

We started our study by investigating the stability and periodicity of solutions to certain systems of difference equations of orders $2$, $3$, $4$, and $\dots$ Then we were able to generalise this investigation to any order $p \geq 1$.
The research has already been referenced in various publications for $p = 1$.

## 3.2 Linearized stability

### 3.2.1 The system $x_{n+1} = \frac{1}{1+y_{(n+1)-k}}, y_{n+1} = \frac{1}{1+x_{(n+1)-k}}$

Firstly, we have the following system

$$\begin{cases} x_{n+1} = \frac{1}{1+y_{(n+1)-k}}, \\ \\ y_{n+1} = \frac{1}{1+x_{(n+1)-k}}, \end{cases} \quad n \in \mathbb{N}_0, \quad k = 1, 2, \ldots, \tag{3.1}$$

where the initial conditions of the negative index terms $x_0, x_{-1}, x_{-2}, \ldots, x_{-(k-1)}, y_0, y_{-1}, y_{-2}, \ldots, y_{-(k-1)}$ are real numbers and we have the following conditions

$$x_{-(k-1)}, x_{-(k-2)}, \ldots, x_0, y_{-(k-1)}, y_{-(k-2)}, \ldots, y_0 \notin \left\{ -\frac{F_{n+1}}{F_n}; n = 1, 2, \ldots, \right\}.$$

**Theorem 3.1** *[24] The equilibrium point $\chi$ is locally asymptotically stable.*

To prove Theorem 3.1, we need the following lemma.

**Lemma 3.1** *[24] The system (3.1) admits just one positive solution that represents the equilibrium point in the set $I^k \times J^k$, which is given by*

$$\chi := \left( \frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2} \right).$$

**Proof.** [24] Clearly, the system

$$\begin{cases} \overline{x} = \frac{1}{1+\overline{y}}, \\ \\ \overline{y} = \frac{1}{1+\overline{x}}, \end{cases} \tag{3.2}$$

admits just one positive solution in $I \times J$ which is given as

$$(\overline{x}, \overline{y}) = \left( \frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2} \right).$$

Now, let go back to the proof of Theorem 3.1.

According to the equilibrium point

$$\overline{\chi} = \left( \frac{-1+\sqrt{5}}{2}, \ldots, \frac{-1+\sqrt{5}}{2}; \frac{-1+\sqrt{5}}{2} \ldots, \frac{-1+\sqrt{5}}{2} \right) \in I^k \times J^k.$$

From Definition 2.4, we get the following linearized system

$$\chi_{n+1} = A\chi_n, \quad n \in \mathbb{N},$$

which in

$$\overline{\chi}_n = \left(x_n, x_{n-1}, x_{n-2}, \ldots, x_{n-(k-1)}; y_n, y_{n-1}, y_{n-2}, \ldots, y_{n-(k-1)}\right)^T, \tag{3.3}$$

such that $A$ is $2k \times 2k$ Jacobian matrix given as follows

$$A = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & \frac{-1+\sqrt{5}}{2} \\ 1 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & \frac{-1+\sqrt{5}}{2} & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \ldots & \ddots & \ldots & \ddots & \vdots & \ldots & \vdots \\ \vdots & \ldots & \ddots & \ldots & \ddots & \vdots & \ldots & \vdots \\ \vdots & \ldots & \ddots & \ldots & \ddots & \vdots & \ldots & \vdots \\ \vdots & \ldots & \ddots & \ldots & \ddots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 1 & 0 \end{bmatrix}$$

We may get the eigenvalues of the matrix $A$ using the characteristic polynomial

$$P(\lambda) = \det(A - \lambda I_{2k}) = \lambda^{2k} - \left(\frac{-1+\sqrt{5}}{2}\right)^2 = 0.$$

Consider the following two functions

$$a(\lambda) = \lambda^{2k}, \quad b(\lambda) = \left(\frac{-1+\sqrt{5}}{2}\right)^2 < 1,$$

we have

$$|b(\lambda)| < |a(\lambda)|, \forall \lambda : |\lambda| = 1.$$

Thus, by $Rouche^{'}s$ Theorem, all zeros of $P(\lambda) = a(\lambda) - b(\lambda) = 0$, lie in the open unit disk ($|\lambda| < 1$). Then, by Theorem 2.2 we get that $\chi$ is locally asymptotically stable. ■

**Theorem 3.2** *[24] The equilibrium point $\chi$ is globally asymptotically stable.*

When we have proven Theorem 3.2, we must apply the following lemma

**Lemma 3.2** *[24] The Binet's formula is given by*

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \in \mathbb{N}$$

*where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. So, we have*

$$\lim_{n \longrightarrow +\infty} \frac{F_{2n}}{F_{2n+1}} = \lim_{n \longrightarrow +\infty} \frac{\alpha^{2n} \times \frac{1-\left(\frac{\beta}{\alpha}\right)^{2n}}{\alpha-\beta}}{\alpha^{2n+1} \times \frac{1-\left(\frac{\beta}{\alpha}\right)^{2n+1}}{\alpha-\beta}} = \frac{1}{\alpha}. \tag{3.4}$$

*Similarly, we get*

$$\lim_{n \longrightarrow +\infty} \frac{F_{2n+2}}{F_{2n+1}} = \alpha. \tag{3.5}$$

Now, return to the proof of Theorem 3.2

**Proof.** Let $\{x_n, y_n\}_{n\geq-(k-1)}$ be a solution of system (3.1). By Definition 2.7 we need only to prove that $\chi$ is global attractor, that is

$$\lim_{n\longrightarrow+\infty}(x_n, x_{n-1}, x_{n-2}, \ldots, x_{n-(k-1)}; y_n, y_{n-1}, y_{n-2}, \ldots, _{n-(k-1)}) = \overline{\chi}$$

which

$$\lim_{n\longrightarrow+\infty}(x_n, y_n) = \chi.$$

Now, we demonstrate that for $i = 1, 2, ..., 2k$, we obtain

$$\lim_{n\longrightarrow+\infty} x_{2kn+i} = \lim_{n\longrightarrow+\infty} y_{2kn+i} = \frac{-1+\sqrt{5}}{2}.$$

Indeed, for $i = 1, 2, ..., k$, Theorem 3.2 yields

$$\begin{aligned}
\lim_{n\longrightarrow+\infty} x_{2kn+i} &= \lim_{n\longrightarrow+\infty} \frac{\frac{(b-a)G_{2n+2}-aG_{2n+1}}{b^2-a^2-ab} + \frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab} y_{i-k}}{\frac{bG_{2n+3}-aG_{2n+4}}{b^2-a^2-ab} + \frac{(b-a)G_{2n+2}-aG_{2n+1}}{b^2-a^2-ab} y_{i-k}}, \\
&= \lim_{n\longrightarrow+\infty} \frac{1 + \frac{\frac{bG_{2n+1}-aG_{2n+2}}{b^2-a^2-ab}}{\frac{(b-a)G_{2n+2}-aG_{2n+1}}{b^2-a^2-ab}} y_{i-k}}{\frac{\frac{bG_{2n+3}-aG_{2n+4}}{b^2-a^2-ab}}{\frac{(b-a)G_{2n+2}-aG_{2n+1}}{b^2-a^2-ab}} + y_{i-k}}, \\
&= \frac{1+\frac{1}{\alpha}y_{i-k}}{\alpha + y_{i-k}} = \frac{1}{\alpha} = \frac{-1+\sqrt{5}}{2}.
\end{aligned}$$

Similarly, we obtained

$$\lim_{n\longrightarrow+\infty} y_{2kn+i} = \frac{1+\frac{1}{\alpha}y_{i-k}}{\alpha + y_{i-k}} = \frac{1}{\alpha} = \frac{-1+\sqrt{5}}{2}.$$

Similarly, we get, for $i = (k+1), (k+2), \ldots, 2k$

$$\lim_{n\longrightarrow+\infty} x_{2kn+i} = \frac{-1+\sqrt{5}}{2}, \quad \lim_{n\longrightarrow+\infty} y_{2kn+i} = \frac{-1+\sqrt{5}}{2}.$$

■

**Example 3.1** *We present the following numerical examples to satisfied the results of this section. As a result, GFS generalization affects the form of solutions to systems of difference equations.*

*The sequences* $(x_n)_{n\geq -(k-1)}$ *and* $(y_n)_{n\geq -(k-1)}$ *of solutions to system* (3.1) *with specified beginning circumstances converge to the equilibrium point* $\left(\frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right)$*, as shown in figures 3.1 and 3.2.*

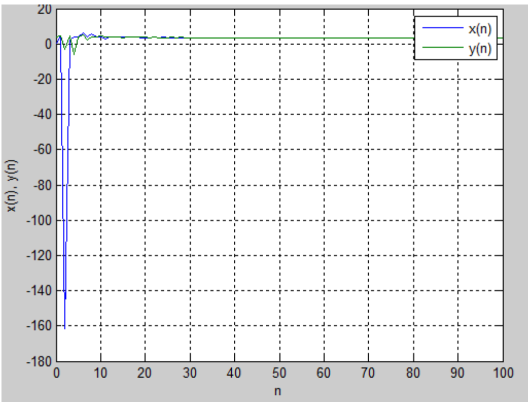


Figure 3.1: (a) Plot of $(x_n, y_n)_{n\geq 1}$ for the system (3.1) which $k = 3$.

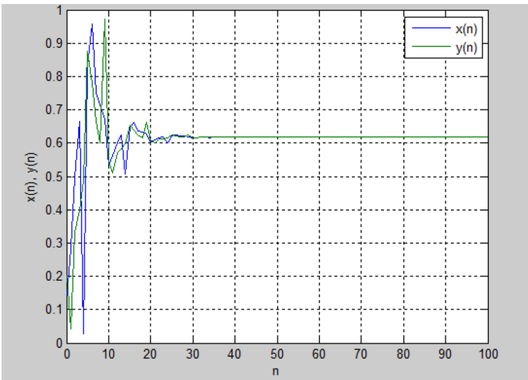


Figure 3.2: (b) Plot of $(x_n, y_n)_{n\geq 1}$ for the system (3.1) which $k = 4$.

**(i)** *In figure 3.1, we assume that* $x_{-3} = 5.6, x_{-2} = 21, x_{-1} = 0.6, x_0 =$

$17, y_{-3} = 1.1, y_{-2} = 18, y_{-1} = 0.03, y_0 = 1.6.$

**(ii)** *In figure 3.2, we assume that* $x_{-4} = 4, x_{-3} = 22, x_{-2} = 2, x_{-1} = 1.5, x_0 = 1.05, y_{-4} = 6, y_{-3} = 2.5, y_{-2} = 1, y_{-1} = 0.5, y_0 = 35.$

*We can see from the preceding figures that the sequences* $\{x_n, y_n\}_{n\geq -(k-1)}$ *of systems (3.6) converge to the following equilibrium point.*

$$\overline{\chi} = \left(\frac{-1+\sqrt{5}}{2}, \ldots, \frac{-1+\sqrt{5}}{2}; \frac{-1+\sqrt{5}}{2} \ldots, \frac{-1+\sqrt{5}}{2}\right) \in I^k \times J^k.$$

### 3.2.2 The system: $x_{n+1} = \frac{1}{y_{n-(p-1)}(x_{n-p}+1)+1}, y_{n+1} = \frac{1}{x_{n-(p-1)}(y_{n-p}+1)+1}$

We have the following system of difference equations

$$\begin{cases} x_{n+1} = \frac{1}{y_{n-(p-1)}(x_{n-p}+1)+1}, \\ \\ y_{n+1} = \frac{1}{x_{n-(p-1)}(y_{n-p}+1)+1}. \end{cases} \quad n \in \mathbb{N}_0, \quad p \geq 1, \tag{3.6}$$

#### Existence of equilibrium point

We consider the following system

$$\begin{cases} \bar{x} = \frac{1}{\bar{y}(\bar{x}+1)+1}, \\ \\ \bar{y} = \frac{1}{\bar{x}(\bar{y}+1)+1}. \end{cases} \tag{3.7}$$

From system (3.7),and by subtracting the second equation from the first equation and doing some calculations we obtained the following formula

$$\bar{x} - \bar{y} = \frac{1}{1 + \bar{x}\bar{y} + \bar{y}} - \frac{1}{1 + \bar{y}\bar{x} + \bar{x}},$$

and by some operations we get the following result

$$(\bar{x}-\bar{y})[(1+\bar{x}\bar{y}+\bar{y})(1+\bar{y}\bar{x}+\bar{x})-1]=0,$$

if $(1+\bar{x}\bar{y}+\bar{y})(1+\bar{y}\bar{x}+\bar{x})=1$. Then, the system (3.7) cannot be satisfied car

$$\bar{x}=\frac{1}{1+\bar{x}\bar{y}+\bar{y}}=\frac{1+\bar{y}\bar{x}+\bar{x}}{(1+\bar{x}\bar{y}+\bar{y})(1+\bar{y}\bar{x}+\bar{x})},$$

if

$$(1+\bar{x}\bar{y}+\bar{y})(1+\bar{y}\bar{x}+\bar{x})=1,$$

we get

$$\bar{x}=(1+\bar{y}\bar{x}+\bar{x}).$$

this is a contradiction, because $\bar{x}\neq(1+\bar{y}\bar{x}+\bar{x})$.

So

$$\bar{x}=\bar{y}.$$

Then, the system (3.7) can be written as

$$\begin{cases}\bar{x}^3+\bar{x}^2+\bar{x}-1=0,\\ \bar{y}^3+\bar{y}^2+\bar{y}-1=0,\end{cases} \tag{3.8}$$

where the characteristic equation : $\bar{x}^3+\bar{x}^2+\bar{x}-1=0$ has three roots $a$, $b$ and $c$, such that

$$\begin{cases}a=\frac{1+\sqrt[3]{19+3\sqrt{33}}+\sqrt[3]{19-3\sqrt{33}}}{3},\\ b=\frac{1+\omega\sqrt[3]{19+3\sqrt{33}}+\omega^2\sqrt[3]{19-3\sqrt{33}}}{3},\\ c=\frac{1+\omega\sqrt[3]{19+3\sqrt{33}}+\omega\sqrt[3]{19-3\sqrt{33}}}{3},\end{cases}$$

and $\omega = \frac{-1+i\sqrt{3}}{2}$ is a primitive cube root of unity.

As a result $\overline{\chi} = (a, a, \dots, a, \dots, a) \in I^{p+1} \times J^{p+1}$ is the unique real positive equilibrium point of system (3.6).

**Theorem 3.3** *[31] The equilibrium point of the system (3.6) is locally asymptotically stable.*

**Proof.** [31]From Definition 2.4

Let $f$ and $g$ be two continuously differentiable functions, and consider $I = J = (0, +\infty)$

$$f : I^{p+1} \times J^{p+1} \longrightarrow I, \quad g : I^{p+1} \times J^{p+1} \longrightarrow J,$$

where f and g be two continuously differentiable functions defined by

$$f(x_n, x_{n-1}, \dots, x_{n-p}, y_n, y_{n-1}, \dots, y_{n-p}) = \frac{1}{y_{n-(p-1)}(x_{n-p}+1)+1},$$
$$g(x_n, x_{n-1}, \dots, x_{n-p}, y_n, y_{n-1}, \dots, y_{n-p}) = \frac{1}{x_{n-(p-1)}(y_{n-p}+1)+1},$$

and we have the following transformation

$$(x_n, x_{n-1}, \dots, x_{n-(p-1)}, y_n, y_{n-1}, \dots, y_{n-(p-1)}) = (f_1, f_2, \dots, f_p, g_1, g_2, \dots, g_p),$$

with

$$
\begin{cases}
f(x_n, x_{n-1}, \dots, x_{n-p}, y_n, y_{n-1}, \dots, y_{n-p}) = \frac{1}{y_{n-(p-1)}(x_{n-p}+1)+1}, \\
f_1(x_n, x_{n-1}, \dots, x_{n-p}, y_n, y_{n-1}, \dots, y_{n-p}) = x_n, \\
f_2(x_n, x_{n-1}, \dots, x_{n-p}, y_n, y_{n-1}, \dots, y_{n-p}) = x_{n-1}, \\
f_3(x_n, x_{n-1}, \dots, x_{n-p}, y_n, y_{n-1}, \dots, y_{n-p}) = x_{n-2}, \\
\quad \dots\dots\dots \\
\quad \dots\dots\dots \\
\quad \dots\dots\dots \\
f_p(x_n, x_{n-1}, \dots, x_{n-p}, y_n, y_{n-1}, \dots, y_{n-p}) = x_{n-(p-1)}, \\
g(x_n, x_{n-1}, \dots, x_{n-p}, y_n, y_{n-1}, \dots, y_{n-p}) = \frac{1}{x_{n-(p-1)}(y_{n-p}+1)+1}, \\
g_1(x_n, x_{n-1}, \dots, x_{n-p}, y_n, y_{n-1}, \dots, y_{n-p}) = y_n, \\
g_2(x_n, x_{n-1}, \dots, x_{n-p}, y_n, y_{n-1}, \dots, y_{n-p}) = y_{n-1}, \\
g_3(x_n, x_{n-1}, \dots, x_{n-p}, y_n, y_{n-1}, \dots, y_{n-p}) = y_{n-2}, \\
\quad \dots\dots\dots \\
\quad \dots\dots\dots \\
\quad \dots\dots\dots \\
g_p(x_n, x_{n-1}, \dots, x_{n-p}, y_n, y_{n-1}, \dots, y_{n-p}) = y_{n-(p-1)}.
\end{cases}
$$

The linearised system associated to the nonlinear system (3.6) about the positive equilibrium point $\overline{\chi} = (a, \dots, a, a, \dots, a) \in I^{p+1} \times J^{p+1}$ is given as

$$\chi_{n+1} = M\chi_n, \tag{3.9}$$

which

$$\chi_n = (x_n, x_{n-1}, x_{n-2}, x_{n-3}, \ldots, x_{n-p}, y_n, y_{n-1}, y_{n-2}, y_{n-3} \ldots, y_{n-p})^T,$$

and $M$ is $2p \times 2p$ Jacobian matrix given as

$$M = \begin{bmatrix} 0 & \ldots & 0 & \frac{-a}{(a(a+1)+1)^2} & 0 & \ldots & \frac{-(a+1)}{(a(a+1)+1)^2} & 0 \\ 1 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \ldots & \ddots & \ddots & \ddots & \ldots & \vdots & \vdots \\ \vdots & \ldots & \ddots & \ddots & \ddots & \ldots & \vdots & \vdots \\ 0 & \ldots & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & \ldots & \frac{-(a+1)}{(a(a+1)+1)^2} & 0 & 0 & \ldots & 0 & \frac{-a}{(a(a+1)+1)^2} \\ 0 & \ldots & 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \ldots & \ddots & \ddots & \ddots & \ldots & \vdots & \vdots \\ \vdots & \ldots & \ddots & \ddots & \ddots & \ldots & \vdots & \vdots \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 1 & 0 \end{bmatrix}$$

The characteristic equation of the Jacobian matrix $M$ is provided as follows

$$(\lambda^2 + (a-1)\lambda + a^3)(\lambda^2 - (a-1)\lambda + a^3) = 0.$$

Numerically, we get

$$|\lambda_1| = |\lambda_2| = |\lambda_3| = \ldots = |\lambda_{2p}| \simeq 0.40089 < 1.$$

Consequently and from Definition 2.7, the equilibrium point $\chi$ is locally asymptotically stable. ■

**Theorem 3.4** *[31] the equilibrium point $\chi$ of system (3.6) is globally asymptotically stable.*

**Proof.** Let $\{x_n, y_n\}_{n\geq -p}$ be a solution of system (3.6).
By Definition 2.7, requires just that we demonstrate that $\chi$ is a global attractor, that is

$$\lim_{n\to+\infty}(x_n, y_n) = \chi.$$

From Theorem 2.5, we have

$$\begin{aligned}
&\lim_{n\longrightarrow+\infty} x_{2pn-i} \\
&= \frac{T_{2n-2p}x_{i-(p+1)}y_{i-p} + (T_{2n-(2p-2)} - T_{2n-(2p-1)})y_{i-p} + T_{2n-(2p-1)}}{T_{2n-(2p-1)}x_{i-(p+1)}y_{i-p} + (T_{2n-2p} + T_{2n-(2p-1)})y_{i-p} + T_{2n-(2p-2)}}, \\
&= \lim_{n\longrightarrow+\infty} \frac{T_{2n-2p}\Big(x_{i-(p+1)}y_{i-p} + (\frac{T_{2n-(2p-2)}}{T_{2n-2p}} - \frac{T_{2n-(2p-1)}}{T_{2n-2p}})y_{i-p} + \frac{T_{2n-(2p-1)}}{T_{2n-2p}}\Big)}{T_{2n-(2p-1)}\Big(x_{i-(p+1)}y_{i-p} + (\frac{T_{2n-2p}}{T_{2n-(2p-1)}} + 1)y_{i-p} + \frac{T_{2n-(2p-2)}}{T_{2n-(2p-1)}}\Big)}, \\
&= \left(\frac{x_{i-(p+1)}y_{i-p} + (\alpha^2-\alpha)y_{i-p} + \alpha}{x_{i-(p+1)}y_{i-p} + (\frac{1}{\alpha}+1)y_{i-p} + \alpha}\right)\left(\lim_{n\longrightarrow+\infty}\frac{T_{2n-2p}}{T_{2n-(2p-1)}}\right), \\
&= \lim_{n\longrightarrow+\infty}\frac{T_{2n-2p}}{T_{2n-(2p-1)}}, \\
&= \frac{1}{\alpha}, \\
&= a,
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{n\longrightarrow+\infty} x_{2pn-(i-1)} \\
&= \lim_{n\longrightarrow+\infty} \frac{T_{2n-(2p-1)}y_{i-(p+1)}x_{i-p} + (T_{2n-(2p-3)} - T_{2n-(2p-2)})x_{i-p} + T_{2n-(2p-2)}}{T_{2n-(2p-2)}y_{i-(p+1)}x_{i-p} + (T_{2n-(2p-1)} + T_{2n-(2p-2)})x_{i-p} + T_{2n-(2p-3)}}, \\
&= \lim_{n\longrightarrow+\infty} \frac{T_{2n-(2p-1)}\left(y_{i-(p+1)}x_{i-p} + (\frac{T_{2n-(2p-3)}}{T_{2n-(2p-1)}} - \frac{T_{2n-(2p-2)}}{T_{2n-(2p-1)}})x_{i-p} + \frac{T_{2n-(2p-2)}}{T_{2n-(2p-1)}}\right)}{T_{2n-(2p-2)}\left(y_{i-(p+1)}x_{i-p} + (\frac{T_{2n-(2p-1)}}{T_{2n-(2p-2)}} + \frac{T_{2n-(2p-2)}}{T_{2n-(2p-2)}})x_{i-p} + \frac{T_{2n-(2p-3)}}{T_{2n-(2p-2)}}\right)}, \\
&= \left(\frac{y_{i-(p+1)}x_{i-p} + (\alpha^2-\alpha)x_{i-p} + \alpha}{y_{i-(p+1)}x_{i-p} + (\frac{1}{\alpha}+1)x_{i-p} + \alpha}\right)\left(\lim_{n\longrightarrow+\infty} \frac{T_{2n-(2p-1)}}{T_{2n-(2p-2)}}\right), \\
&= \lim_{n\longrightarrow+\infty} \frac{T_{2n-(2p-1)}}{T_{2n-(2p-2)}}, \\
&= \frac{1}{\alpha}, \\
&= a.
\end{aligned}$$

Similarly, we get

$$\lim_{n\longrightarrow+\infty} y_{2pn-i} = a, \quad \lim_{n\longrightarrow+\infty} y_{2pn-(i-1)} = a.$$

So,

$$\lim_{n\rightarrow+\infty} (x_n, y_n) = \chi.$$

As a result, the equilibrium point $\chi$ is globally asymptotically stable. ■

**Example 3.2** *For confirming the results of this section, we consider the following numerical example.*

**(i)** *In figure 3.3, we assume that* $x_{-2} = 5.24$, $x_{-1} = 2.54, x_0 = 14, y_{-2} = -7.12, y_{-1} = 0.12, y_0 = 8.16.$

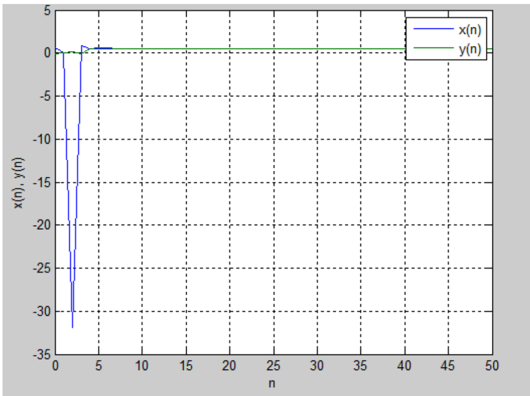


Figure 3.3: The sequences $(x_n)_{n\geq 0}$ (blue) and $(y_n)_{n\geq 0}$ (green) of solution of the system (3.6) for $p = 2$.

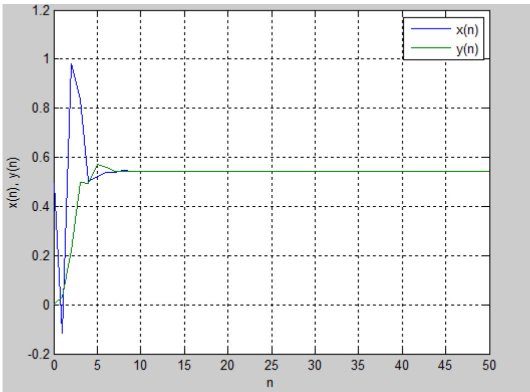


Figure 3.4: The sequences $(x_n)_{n\geq 0}$ (blue) and $(y_n)_{n\geq 0}$ (green) of solution of the system (3.6) for $p = 3$.

**(ii)** *In figure 3.4, we assume that* $x_{-3} = -9.5, x_{-2} = 42.24, x_{-1} = -2.54, x_0 = 14.2, y_{-3} = 17, y_{-2} = 0.12, y_{-1} = 1.2, y_0 = 6.16.$

*We can see from the preceding figures that the sequences* $\{x_n, y_n\}_{n\geq -p}$ *of systems (3.6) converge to the following equilibrium point.*

$$\chi := (a, a, \ldots, a, a, a, \ldots, a) \in I^{p+1} \times J^{p+1}.$$

## 3.3 Periodicity of solutions

In this section, we look at the periodicity of the difference equations systems in heiger-order shown below

$$\begin{cases} x_{n+1} = \frac{y_n(x_{n-(p-1)}+y_{n-p})}{y_{n-p}+x_{n-(p-1)}-y_n}, \\ \\ y_{n+1} = \frac{x_{n-(p-2)}(x_{n-(p-2)}+y_{n-(p-1)})}{2x_{n-(p-2)}+y_{n-(p-1)}}, \end{cases} \quad n \in \mathbb{N}_0, \tag{3.10}$$

where the initial conditions of the negative index terms : $x_{-(p-1)}, x_{-(p-2)}, \ldots, x_{-2}$, $x_{-1}, x_0, y_{-p}, y_{-(p-1)}, y_{-(p-2)}, \ldots, y_{-2}, y_{-1}, y_0$ are nonzero real numbers. Which $n = 0, 1, 2, ...$ and $p \geq 2$, such that
$\frac{y_{-(p-1)}}{x_{-(p-2)}}, \frac{y_{-(p-2)}}{x_{-(p-3)}}, \ldots, \frac{y_{-2}}{x_{-1}}, \frac{y_{-1}}{x_0} \notin \left\{-\frac{F_{2n+3}}{F_{2n+2}}, n = 0, 1, 2, \ldots\right\}$,
and

$$\frac{x_{-(p-1)} + y_{-p}}{y_0} \notin \{1\} \cup \left\{\frac{F_{2n}}{F_{2n+2}}, n = 0, 1, 2, \ldots\right\}.$$

From (3.10) and when $n = 0$, we get

$$\begin{aligned} x_1 &= \frac{y_0(x_{-(p-1)} + y_{-p})}{y_{-p} + x_{-(p-1)} - y_0}, \\ &= \frac{\frac{y_0(x_{-(p-1)}+y_{-p})}{y_0}}{\frac{y_{-p}+x_{-(p-1)}-y_0}{y_0}}, \\ &= \frac{x_{-(p-1)} + y_{-p}}{\left(\frac{x_{-(p-1)}+y_{-p}}{y_0}\right) - 1}. \end{aligned}$$

As a result, for $x_1$ to exist, it is necessary that $\frac{x_{-(p-1)}+y_{-p}}{y_0} \neq 1$.

Similarly, when we have

$$\begin{aligned} y_{pn} &= \left(\frac{y_0(x_{-(p-1)}+y_{-p})}{y_{-p}+x_{-(p-1)}-y_0}\right)\left[\frac{(x_{-(p-1)}+y_{-p})F_{2n+1}-y_0F_{2n-1}}{(x_{-(p-1)}+y_{-p})F_{2n+2}-y_0F_{2n}}\right], \\ &= \left(\frac{y_0(x_{-(p-1)}+y_{-p})}{y_{-p}+x_{-(p-1)}-y_0}\right)\left(\frac{\frac{(x_{-(p-1)}+y_{-p})}{y_0}F_{2n-1}-\frac{y_0}{y_0}F_{2n-1}}{\frac{(x_{-(p-1)}+y_{-p})}{y_0}F_{2n+2}-\frac{y_0}{y_0}F_{2n}}\right). \end{aligned}$$

It is therefore necessary that $\frac{(x_{-(p-1)}+y_{-p})}{y_0} \neq \frac{F_{2n}}{F_{2n+2}}$ for the existence of $y_{pn}$.

**Lemma 3.3** *Let* $(x_n)_{n\geq-(p-1)}$, $(y_n)_{n\geq-p}$ *be the solutions of system* (3.10). *Then, for* $n = 0, 1, \ldots$ *we have*

$$x_{n+p} = x_n,$$

*that is* $(x_n)_{n\geq-(p-1)}$ *is eventually periodic with period* $p$ $(p \geq 2)$.

**Proof.** [39] We have

$$y_{n+(p-1)} = y_{n+(p-2)+1} = \frac{x_n(x_n+y_{n-1})}{2x_n+y_{n-1}}.$$

So

$$\begin{aligned} x_{n+p} = x_{n+(p-1)+1} &= \frac{y_{n+(p-1)}(x_{n+(p-1)}+y_{n+p-2-(p-1)})}{y_{n+(p-1)-p}+x_{n+(p-1)+(p-1)}-y_{n+(p-1)}}, \\ &= \frac{\frac{x_n(x_n+y_{n-1})}{2x_n+y_{n-1}}(x_n+y_{n-1})}{(y_{n-1}+x_n)-x_n\left(\frac{x_n+y_{n-1}}{2x_n+y_{n-1}}\right)}, \\ &= \frac{x_n(x_n+y_{n-1})^2}{(x_n+y_{n-1})(2x_n+y_{n-1}-x_n)}, \\ &= x_n. \end{aligned}$$

Then

$$x_{n+p} = x_n.$$

■

**Theorem 3.5** *[39] Let $(x_n)_{n\geq-(p-1)}, (y_n)_{n\geq-p}$ be the solutions of system (3.10). Then, for $n = 0, 1, \ldots,$ we have*

$$\begin{aligned}
x_{pn} &= x_0, \\
x_{pn+1} &= \frac{y_0(x_{-p+1} + y_{-p})}{y_{-p} + x_{-p+1} - y_0}, \\
x_{pn+2} &= x_{-p+2} = x_2, \\
x_{pn+3} &= x_{-p+3} = x_3, \\
&\ldots \\
&\ldots \\
&\ldots \\
x_{pn+(p-2)} &= x_{-2}, \\
x_{pn+(p-1)} &= x_{-1}.
\end{aligned}$$

**Proof.** According to Lemma 3.3 one has the formula $x_{k+p} = x_k$ for any integer $k$, such that $0 \leq k \leq p-1$. Thus one has

$$x_{k+2p} = x_{(k+p)+p} = x_{k+p} = x_k.$$

So, from system (3.10), we have

$$x_1 = \frac{y_0(x_{-(p-1)} + y_{-p})}{y_{-p} + x_{-(p-1)} - y_0},$$

and

$$y_1 = \frac{x_{-(p-2)}(x_{-(p-2)} + y_{-(p-1)})}{2x_{-(p-2)} + y_{-(p-1)}}.$$

Now, we have

$$\begin{aligned} x_2 &= \frac{y_1\big(x_{-(p-2)}+y_{-(p-1)}\big)}{y_{-(p-1)}+x_{-(p-2)}-y_1}, \\ &= \frac{\frac{x_{-(p-2)}(x_{-(p-2)}+y_{-(p-1)})}{2x_{-(p-2)}+y_{-(p-1)}}\big(x_{-(p-2)}+y_{-(p-1)}\big)}{x_{-(p-2)}+y_{-(p-1)}-\frac{x_{-(p-2)}(x_{-(p-2)}+y_{-(p-1)})}{2x_{-(p-2)}+y_{-(p-1)}}}, \\ &= \frac{x_{-(p-2)}\big(x_{-(p-2)}+y_{-(p-1)}\big)^2}{\big(2x_{-(p-2)}+y_{-(p-1)}\big)\big(x_{-(p-2)}+y_{-(p-1)}\big)-x_{-(p-2)}\big(x_{-(p-2)}+y_{-(p-1)}\big)}, \\ &= x_{-(p-2)}. \end{aligned}$$

Similarly, from system (3.10), we obtained

$$y_2=\frac{x_{-(p-3)}\big(x_{-(p-3)}+y_{-(p-2)}\big)}{2x_{-(p-3)}+y_{-(p-2)}}.$$

So,

$$\begin{aligned} x_3 &= \frac{y_2\big(x_{-(p-3)}+y_{-(p-2)}\big)}{y_{-(p-2)}+x_{-(p-3)}-y_2}, \\ &= \frac{\frac{x_{-(p-3)}(x_{-(p-3)}+y_{-(p-2)})}{2x_{-(p-3)}+y_{-(p-2)}}\big(x_{-(p-3)}+y_{-(p-2)}\big)}{x_{-(p-3)}+y_{-(p-2)}-\frac{x_{-(p-3)}(x_{-(p-3)}+y_{-(p-2)})}{2x_{-(p-3)}+y_{-(p-2)}}}, \\ &= \frac{x_{-(p-3)}\big(x_{-(p-3)}+y_{-(p-2)}\big)^2}{\big(2x_{-(p-3)}+y_{-(p-2)}\big)\big(x_{-(p-3)}+y_{-(p-2)}\big)-x_{-(p-3)}\big(x_{-(p-3)}+y_{-(p-2)}\big)}, \\ &= \frac{x_{-(p-3)}\big(x_{-(p-3)}+y_{-(p-2)}\big)^2}{\big(x_{-(p-3)}+y_{-(p-2)}\big)\big(2x_{-(p-3)}-x_{-(p-3)}+y_{-(p-2)}\big)}, \\ &= x_{-(p-3)}. \end{aligned}$$

We complete the proof of the following relationships by using the same method

$$\begin{aligned} x_4 &= x_{-(p-4)}, \\ x_5 &= x_{-(p-5)}, \\ &\dots \\ &\dots \\ x_{p-2} &= x_{-2}, \\ x_{p-1} &= x_{-1}. \end{aligned}$$

From Lemma 3.3 we got the sequence $\{x_n\}_{n\geq-(q-1)}$ is repeat. Then,

$$\begin{aligned} x_{pn} &= x_0, \\ x_{pn+1} &= x_1 = \frac{y_0(x_{-(p-1)} + y_{-p})}{y_{-p} + x_{-(p-1)} - y_0}, \\ x_{pn+2} &= x_2 = x_{-(p-2)}, \\ x_{pn+3} &= x_3 = x_{-(p-3)}, \\ &\dots \\ &\dots \\ x_{pn+(p-2)} &= x_{p-2} = x_{-(p-(p-2))} = x_{-2}, \\ x_{pn+(p-1)} &= x_{p-1} = x_{-(p-(p-1))} = x_{-1}, \\ x_{pn+p} &= x_0 = x_{pn}. \end{aligned}$$

By induction one sees that $x_{k+np} = x_k$ for any integer $k$, such that $0 \leq k \leq p-1$. ■

**Theorem 3.6** *Let $(x_n)_{n\geq-(p-1)}, (y_n)_{n\geq-p}$ be the solutions of system (3.10). Then* *one has*

$$y_{pn+k} = x_{k+1}\left(\frac{x_{k+1}F_{2n+2} + y_{-(p-k)}F_{2n+1}}{x_{k+1}F_{2n+3} + y_{-(p-k)}F_{2n+2}}\right), \quad for \quad 0 \leq k \leq p-1.$$

**Proof.** [39] for $n = 0$ and from system (3.10) one has

$$y_k = x_{k+1}\left(\frac{x_{k+1} + y_{-(p-k)}}{2x_{k+1} + y_{-(p-k)}}\right),$$

and we see that

$$y_k = x_{k+1}\left(\frac{x_{k+1}F_2 + y_{-(p-k)}F_1}{x_{k+1}F_3 + y_{-(p-k)}F_2}\right), \quad for \quad 0 \leq k \leq p-1.$$

Suppose that the property is true up to rank $n$, that is

$$y_{pn+k} = x_{k+1}\left(\frac{x_{k+1}F_{2n+2} + y_{-(p-k)}F_{2n+1}}{x_{k+1}F_{2n+3} + y_{-(p-k)}F_{2n+2}}\right), \quad for \quad 0 \leq k \leq p-1.$$

Let us show that it is true at rank $n+1$ .i.e.

$$y_{p(n+1)+k} = x_{k+1}\left(\frac{x_{k+1}F_{2n+4} + y_{-(p-k)}F_{2n+3}}{x_{k+1}F_{2n+5} + y_{-(p-k)}F_{2n+4}}\right), \quad for \quad 0 \leq k \leq p-1.$$

By system (3.10) one has

$$y_{p(n+1)+k} = y_{(p(n+1)+k-1)+1} = x_{k+1}\left(\frac{x_{k+1} + y_{pn+k}}{2x_{k+1} + y_{pn+k}}\right),$$

$$\begin{aligned}
x_{k+1} + y_{pn+k} &= x_{k+1}\left(1 + \frac{x_{k+1}F_{2n+2} + y_{-(p-k)}F_{2n+1}}{x_{k+1}F_{2n+3} + y_{-(p-k)}F_{2n+2}}\right), \\
&= x_{k+1}\left(\frac{x_{k+1}(F_{2n+2} + F_{2n+3}) + y_{-(p-k)}(F_{2n+1} + F_{2n+2})}{x_{k+1}F_{2n+3} + y_{-(p-k)}F_{2n+2}}\right), \\
&= x_{k+1}\left(\frac{x_{k+1}F_{2n+4} + y_{-(p-k)}F_{2n+3}}{x_{k+1}F_{2n+3} + y_{-(p-k)}F_{2n+2}}\right).
\end{aligned}$$

So

$$\begin{aligned} 2x_{k+1}+y_{pn+k} &= x_{k+1}\left(2+\frac{x_{k+1}F_{2n+2}+y_{-(p-k)}F_{2n+1}}{x_{k+1}F_{2n+3}+y_{-(p-k)}F_{2n+2}}\right), \\ &= x_{k+1}\left(\frac{x_{k+1}(F_{2n+2}+2F_{2n+3})+y_{-(p-k)}(F_{2n+1}+2F_{2n+2})}{x_{k+1}F_{2n+3}+y_{-(p-k)}F_{2n+2}}\right), \\ &= x_{k+1}\left(\frac{x_{k+1}F_{2n+5}+y_{-(p-k)}F_{2n+4}}{x_{k+1}F_{2n+3}+y_{-(p-k)}F_{2n+2}}\right). \end{aligned}$$

It follows that

$$\begin{aligned} & y_{p(n+1)+k} \\ &= x_{k+1}\left(\frac{x_{k+1}+y_{pn+k}}{2x_{k+1}+y_{pn+k}}\right), \\ &= x_{k+1}\times\left(\frac{x_{k+1}F_{2n+4}+y_{-(p-k)}F_{2n+3}}{x_{k+1}F_{2n+3}+y_{-(p-k)}F_{2n+2}}\right)\left(\frac{x_{k+1}F_{2n+3}+y_{-(p-k)}F_{2n+2}}{x_{k+1}F_{2n+5}+y_{-(p-k)}F_{2n+4}}\right), \\ &= x_{k+1}\left(\frac{x_{k+1}F_{2n+4}+y_{-(p-k)}F_{2n+3}}{x_{k+1}F_{2n+5}+y_{-(p-k)}F_{2n+4}}\right). \end{aligned}$$

■

**Theorem 3.7** *[39] Let $(x_n)_{n\geq-(p-1)}, (y_n)_{n\geq-p}$ be the solutions of system (3.10). For $y_0=-\alpha x_1$ one has*

$$y_{pn+k}=y_k, \quad for \quad 0\leq k\leq p-1.$$

**Proof.** By induction

For $n=0$, we get

$$y_{p(0)+k}=y_k, \quad for \quad 0\leq k\leq p-1,$$

the result holds for $n = 0$.

Now, we Suppose that $n > 0$ and that our assumption holds for $n$. That is,

$$y_{pn+k} = y_k, \quad for \quad 0 \leq k \leq p-1.$$

From system (3.10) we have

$$\begin{aligned} y_{p(n+1)+k} &= y_{(pn+p+k-1)+1} \\ &= \frac{x_{(pn+p+k-1)-(p-2)}\big(x_{(pn+p+k-1)-(p-2)} + y_{(pn+p+k-1)-(p-1)}\big)}{2x_{(pn+p+k-1)-(p-2)} + y_{(pn+p+k-1)-(p-1)}}, \end{aligned}$$

which we prove already that the sequence $(x_n)_{n \geq -(p-1)}$ is p-periodic.

So

$$y_{p(n+1)+k} = \frac{x_{k+1}\big(x_{k+1} + y_{pn+k}\big)}{2x_{k+1} + y_{pn+k}} = \frac{x_{k+1}\big(x_{k+1} + y_k\big)}{2x_{k+1} + y_k} = y_{k+p} = y_k,$$

and for $p = 0$, we have

$$\begin{aligned} y_k &= \frac{x_{k+1}\big(x_{k+1} + y_k\big)}{2x_{k+1} + y_k}, \\ &\Rightarrow \; y_k(2x_{k+1} + y_k) = x_{k+1}(x_{k+1} + y_k), \\ &\Rightarrow \; y_k^2 + x_{k+1}y_k - {x_{k+1}}^2 = 0, \end{aligned}$$

which

$$\Delta = (x_{k+1})^2 - 4(-x_{k+1})^2 = 5(x_{k+1})^2,$$

this equation having two roots $y_{k_1}$ and $y_{k_2}$, such that

$$y_{k_1} = \frac{-x_{k+1} + x_{k+1}\sqrt{5}}{2} = -\left(\frac{1-\sqrt{5}}{2}\right) x_{k+1} = -\beta x_{k+1},$$

and

$$y_{k_2} = \frac{-x_{k+1} - x_{k+1}\sqrt{5}}{2} = -\left(\frac{1+\sqrt{5}}{2}\right) x_{k+1} = -\alpha x_{k+1}.$$

Then, for $y_0 = -\alpha x_1$, (and we can also put $y_0 = -\beta x_1$), one has

$$y_{p(n+1)+k} = y_k, \quad for \quad 0 \leq k \leq p-1.$$

Since the periodicity of the sequence $(x_n)_{n \geq -(p-1)}$ has been established through Theorem 3.5. The periodicity of the sequence $(y_n)_{n \geq -p}$ should also be the same. It is seen that these sequences are dependent on each other. ■

# Chapter 4

# Applications of differential equations to Fibonacci and Tribonacci polynomials

## 4.1 Introduction

Differential equations have been used since the time of Newton in understanding the physical, engineering and biological sciences. In addition to its contribution to the study of mathematical analysis and its uses extended to economic and social sciences.
Differential equations have developed and become increasingly important in all fields of science and their applications.
In the following, we will look at how to write a differential equation with polynomial coefficients and how to solve it.

**Theorem 4.1** *[30] Consider the characteristic equation*

$$r^k - C_1 r^{k-1} - C_2 r^{k-2} - \ldots - C_k = 0,$$

*and the following recurrent relation*

$$a_n = C_1 a_{n-1} + C_2 a_{n-2} + \ldots + C_k a_{n-k}. \tag{4.1}$$

*Assume that* $r_1, r_2, \ldots, r_m$ *satisfy the equation (4.1).*

*Let* $\alpha_1, \alpha_2, \ldots, \alpha_m$ *be any constants. Then,*

$$a_n = \alpha_1 {r_1}^n + \alpha_2 {r_2}^n + \ldots + \alpha_m {r_m}^n. \tag{4.2}$$

**Existence and uniqueness of differential equation solutions**

**Theorem 4.2** *[30] Let the differential equation*

$$y' = f(t, y), \tag{4.3}$$

*and the initial condition*

$$y(t_0) = y_0, \tag{4.4}$$

*which* $f(t, y)$ *is a function defined in a closed and bounded rectangular domain*

$$R : |t - t_0| \leq a, \quad |y - y_0| \leq b, \tag{4.5}$$

*such that* $a$ *and* $b$ *are both positive constants. Therefore*

**(i)** *If M is a real number, where* $M = \sup\{M_1, M_2, M_3, \ldots\}$ $(M_1, M_2, M_3, \ldots \in \mathbb{R})$*, such that*

$$|f(t, y)| \leq M.$$

*Then, the function* $f(t, y)$ *is continuous, so bounded.*

**(ii)** *Because the function* $f(t, y)$ *has a partial derivative with respect to* $y$*, it can be defined as bounded. That is*

$$\frac{\partial f(t,y)}{\partial y} \leq K,$$

*with* $K$ *is positive number.*

*If conditions* **(i)** *and* **(ii)** *are satisfied. Hence, the equation (4.3) has only one solution* $y$ *in the interval*

$$R :|t - t_0|\leq h, \tag{4.6}$$

*with* $h = \min(a, M)$*. That satisfies the initial condition (4.4).*

## 4.2 Applications of differential equations to Fibonacci polynomials

Firstly, we consider the convergence radius $R(x)$ from the whole series of the real variable $t$.

We define the characteristic equation of (FRR) as follows

$$r^n = xr^{n-1} + r^{n-2}$$

where $r$ verifed the characteristic equation of (FRR).

Dividing both sides by $r^{n-2}$, we obtained

$$r^2 = xr + 1 \Leftrightarrow r^2 - xr - 1 = 0.$$

### 4.2.1 Proof of R(x) > 0

**Proposition 4.1** *The radius of convergence* $R(x)$ *is positive.*

**Proof.** Let $x$ be a real positive number, and we have

$$\begin{cases} f_n(x) = x f_{n-1}(x) + f_{n-2}(x), \quad n \geq 3, \\ \\ f_1(x) = 1, \quad f_2(x) = x. \end{cases} \tag{4.7}$$

So,

$$|f_n(x)| = |x f_{n-1}(x) + f_{n-2}(x)| \leq |x||f_{n-1}(x)| + |f_{n-2}(x)|. \tag{4.8}$$

Hence,

$$|f_n(x)| \leq |x||f_{n-1}(x)| + |f_{n-2}(x)|.$$

Then, consider the recurrent sequence

$$u_n(x) = |x| u_{n-1}(x) + u_{n-2}(x); \quad \forall n \in \mathbb{N},$$

such that $u_{n-1}(x) = |f_{n-1}(x)|$ and $u_{n-2}(x) = |f_{n-2}(x)|$.

The benomial $T(y) = y^2 - |x|y - 1$, has two real roots $\alpha$ and $\beta$ which

$$\alpha = \frac{|x| + \sqrt{x^2+4}}{2}; \quad \beta = \frac{|x| - \sqrt{x^2+4}}{2}.$$

(where $x$ is an real number).

According to theorem (4.1), there is two constants $A$ and $B$ such that for all $n \geq 3$, we have

$$\begin{aligned} f_n = A\alpha^n + B\beta^n \Rightarrow |f_n| &= |A\alpha^n + B\beta^n|, \\ &\leq |A\alpha^n| + |B\beta^n|, \\ &\leq |A||\alpha^n| + |B||\alpha^n|, \\ &= (|A| + |B|)\alpha^n. \end{aligned}$$

(Because $|\beta| < \alpha$).

The radius of convergence is given by

$$\begin{aligned}\frac{1}{R(x)} &= \lim_{n\mapsto+\infty}\frac{|f_{n+1}|}{|f_n|},\\ &\leq \lim_{n\mapsto+\infty}\frac{(|A|+|B|)\alpha^{n+1}}{(|A|+|B|)\alpha^n},\\ &= \alpha.\end{aligned}$$

Then

$$R(x) \geq \frac{1}{\alpha} = \frac{2}{|x|+\sqrt{x^2+4}} > 0. \tag{4.9}$$

■

**Existence and uniqueness of differential equation solutions**

**Proposition 4.2** *For each $|t| \leq 1$, consider the differential equation below*

$$\begin{cases} f(t,y) = y' = \frac{x+2t}{1-xt-t^2}y + \frac{1}{1-xt-t^2}, \\ \\ y(0) = 0. \end{cases} \tag{4.10}$$

*(where $x$ is an real number).*

*Then, the equation (4.10) admits one solution.*

**Proof.** From Theorem 4.2, we create a rectangle with the center at $(0,0)$.

Let $a = b = 1$ such that

$$R : |t| \leq 1, \quad |y| \leq 1.$$

So,

$$\begin{aligned} |f(t,y)| &= |\frac{x+2t}{1-xt-t^2}y + \frac{1}{1-xt-t^2}|, \\ &\leq \left(|\frac{x}{1-xt-t^2}|+|\frac{2t}{1-xt-t^2}|\right)|y|+|\frac{1}{1-xt-t^2}|, \\ &= |\frac{1}{1-xt-t^2}|((|x|+|2t|)|y|+1)\,, \end{aligned}$$

and by taking the values $a = 1$, and $b = 1$, we get

$$|f(t,y)|\leq \frac{3+|x|}{|x|} = M.$$

Where $x$ is an real number.

Then, by Theorem (4.2), the equation $y^{'} = f(x,y)$ has only one solution for $|t|\leq \frac{|x|}{3+|x|}$ that satisfies the initial condition $y(0) = 0$. ■

### 4.2.2 Differential equation verifed by $g : t \mapsto h(x,t) \in ]-R(x), R(x)[$

**Theorem 4.3** *[8] Let $x$ is an real number and $g$ be a function defined by $g : t \mapsto h(x,t)$ is in class $C^\infty$ on $]-R(x), R(x)[$. Then $g$ satisfied the following differential equation*

$$\begin{cases} g'(t)(1-xt-t^2) - (x+2t)g(t) = 1, \\ \\ g(0) = 0. \end{cases} \tag{4.11}$$

**Proof.** Let $g$ a generating function defined on $]-R(x), R(x)[$ by

$$g(t) = \sum_{n=0}^{+\infty} f_n(x)t^n \Rightarrow g'(t) = \sum_{n=0}^{+\infty} n f_n(x)t^{n-1}.$$

For $n \geq 3$, we express $f_n(x)$ on function of $f_{n-2}(x)$ and $f_{n-1}(x)$

$$\begin{aligned}
g'(t) &= \sum_{n=0}^{+\infty} n(xf_{n-1}(x) + f_{n-2}(x))t^{n-1}, \\
&= \sum_{n=0}^{+\infty} nxf_{n-1}(x)t^{n-1} + \sum_{n=0}^{+\infty} nf_{n-2}(x)t^{n-1}, \\
&= x\left(0 \times f_{-1}(x)t^{-1} + 1 \times f_0(x)t^0 + 2 \times f_1(x)t + \sum_{n=3}^{+\infty} nf_{n-1}(x)t^{n-1}\right) \\
&+ \sum_{n=0}^{+\infty} nf_{n-2}(x)t^{n-1}, \\
&= 2xt + x\sum_{n=3}^{+\infty} nf_{n-1}(x)t^{n-1} + \sum_{n=0}^{+\infty} nf_{n-2}(x)t^{n-1}.
\end{aligned}$$

So

$$g'(t) = 2xt + xA(t) + B(t). \tag{4.12}$$

Now, we find $A(t)$ and $B(t)$

$$\begin{aligned}
A(t) &= \sum_{n\geq 3} nf_{n-1}(x)t^{n-1}, \\
&= \sum_{n\geq 2} (n+1)f_n(x)t^n, \\
&= \sum_{n\geq 2} nf_n(x)t^n + \sum_{n\geq 2} f_n(x)t^n, \\
&= t\left(\sum_{n\geq 2} nf_n(x)t^{n-1}\right) + \sum_{n\geq 2} f_n(x)t^n, \\
&= t\left(\sum_{n=0}^{+\infty} nf_n(x)t^{n-1} - 1\right) + \left(\sum_{n=0}^{+\infty} f_n(x)t^n - t\right), \\
&= t(g'(t) - 1) + g(t) - t = tg'(t) + g(t) - 2t.
\end{aligned}$$

So

$$A(t) = tg'(t) + g(t) - 2t, \tag{4.13}$$

and

$$\begin{aligned} B(t) &= \sum_{n=0}^{+\infty} n f_{n-2}(x) t^{n-1}, \\ &= 1 + \sum_{n\geq 3} n f_{n-2}(x) t^{n-1}, \\ &= 1 + \sum_{n\geq 1} (n+2) f_n(x) t^{n+1}, \\ &= 1 + \sum_{n=0}^{+\infty} (n+2) f_n(x) t^{n+1}. \end{aligned}$$

So

$$B(t) = 1 + (t^2 g(t))', \tag{4.14}$$

because

$$\begin{aligned} \left(t^2 g(t)\right)' &= \left(t^2 \sum_{n\geq 0} f_n(x) t^n\right)', \\ &= \left(\sum_{n\geq 0} f_n(x) t^{n+2}\right)', \\ &= \sum_{n\geq 0} (n+2) f_n(x) t^{n+1}, \end{aligned}$$

as a result of compensation (4.13) and (4.14) in the equation (4.12), we get

$$\begin{aligned} g'(t) &= 2xt + x(tg'(t) + g(t) - 2t) + \left(\left(t^2 g(t)\right)' + 1\right), \\ &= 2xt + 1 + xtg'(t) + xg(t) - 2xt + 2tg(t) + t^2 g'(t)) \\ &\Rightarrow g'(t)(1 - xt - t^2) - (x + 2t) g(t) = 1. \end{aligned}$$

■

### 4.2.3 Integration of the differential equation

**Theorem 4.4** *[8] If $r \leq 1$, then the equation (4.11) accepts only one solution in the interval $]-r, r[$ given as*

$$t \mapsto \frac{t}{1 - xt - t^2}.$$

*(where $x$ is an real number).*

**Proof.** If $r \leq 1$, we will integrate the equation (4.11) on $]-r, r[$ , (where $r > 0$).

Then, the solution to a homogeneous differential equation is given as follows

$$\begin{aligned} \frac{g'(t)}{g(t)} &= \frac{x + 2t}{1 - xt - t^2} \\ &\Rightarrow \ln(|g(t)|) = -\ln(|1 - xt - t^2|) + k. \end{aligned}$$

Then,

$$g(t) = \frac{e^k}{1 - xt - t^2} = \frac{C}{1 - xt - t^2}$$

where $k$ and $C$ are optional constants ($C = e^k$).

By using the method of designation the optional constants, we have

$$\begin{aligned} g(t) &= \frac{C(t)}{1 - xt - t^2}, \\ \Rightarrow \quad g'(t) &= \frac{C'(t)(1 - xt - t^2) - C(t)(-x - 2t)}{(1 - xt - t^2)^2}, \end{aligned}$$

the results obtained by replacing into the equation (4.11). So, we get

$$\frac{C'(t)}{1 - xt - t^2} = \frac{1}{1 - xt - t^2} \Rightarrow C(t) = t,$$

So,

$$g_0(t) = \frac{t}{1 - xt - t^2}.$$

Then, the solution to the differential equation (4.11) is

$$g(t) = g_0(t) + \frac{C}{1 - xt - t^2} = \frac{t + C}{1 - xt - t^2},$$

if this solution meets the original condition $g(0) = 0$. Thus

$$\begin{aligned} g(t) &= \frac{t + C}{1 - xt - t^2} \\ &\Rightarrow g(0) = C = 0 \\ &\Rightarrow C = 0. \end{aligned}$$

Hence, the solution to the equation (4.11) is given as follows

$$g(t) = \sum_{0}^{+\infty} f_n(x) t^n = \frac{t}{1 - xt - t^2}.$$

■

## 4.3 Applications of differential equations to Tribonacci polynomials

To begin, we look at the convergence radius $R$ from the whole series of the real variable $x$.

The characteristic equation of (TRR) is defined as follows

$$r^n = r^{n-1} + r^{n-2} + r^{n-3},$$

where $r$ verifed the characteristic equation of (TRR).

Dividing both sides by $r^{n-3}$. Then, we obtained

$$r^3 = r^2 + r + 1 \Leftrightarrow r^3 - r^2 - r - 1 = 0.$$

### 4.3.1 Proof of R > 0

we have

$$\begin{cases} t_n(1) = x^2 t_{n-1}(1) + x t_{n-2}(1) + t_{n-3}(1), \quad n \geq 3 \\ \\ t_0(1) = 0, \quad t_1(1) = 1, \quad t_2(1) = 1. \end{cases} \tag{4.15}$$

$$|t_n(1)| = |t_{n-1}(1) + t_{n-2}(1) + t_{n-3}(1)| \leq |t_{n-1}(1)| + |t_{n-2}(1)| + |t_{n-3}(1)|. \tag{4.16}$$

Then, consider the recurrent sequence

$$u_n(x) = u_{n-1}(x) + u_{n-2}(x) + u_{n-3}(x); \quad \forall n \in \mathbb{N},$$

such that $u_{n-1}(x) = |t_{n-1}(x)|$, $u_{n-2}(x) = |t_{n-2}(x)|$ and $u_{n-3}(x) = |t_{n-3}(x)|$.

The cubic equation $x^3 - x^2 - x - 1 = 0$ has three roots : $\alpha$ ,$\beta$ and $\gamma$ ($\alpha$ is a real root, $\beta$ and $\gamma$ are conjugated compound roots), which

$$\begin{aligned} \alpha = \quad & \frac{1+a+b}{3} & = \frac{1+\sqrt[3]{19+3\sqrt{33}}+\sqrt[3]{19-3\sqrt{33}}}{3}, \\ \beta = \quad & \frac{1+\omega a+\omega^2 b}{3} & = \frac{1+\omega\sqrt[3]{19+3\sqrt{33}}+\omega^2\sqrt[3]{19-3\sqrt{33}}}{3}, \\ \gamma = \quad & \frac{1+\omega^2 a+\omega b}{3} & = \frac{1+\omega^2\sqrt[3]{19+3\sqrt{33}}+\omega\sqrt[3]{19-3\sqrt{33}}}{3}, \end{aligned}$$

where

$$a = \sqrt[3]{19+3\sqrt{33}}\,, \quad b = \sqrt[3]{19-3\sqrt{33}},$$

and $\omega = \frac{-1+i\sqrt{3}}{2} = exp(\frac{2\pi i}{3})$ is a primitive cube root of unity.
Furthermore, the roots $\alpha, \beta$, and $\gamma$ satisfied the following identities

$$\alpha + \beta + \gamma = 1,$$
$$\alpha\beta + \alpha\gamma + \beta\gamma = -1,$$
$$\alpha\beta\gamma = 1.$$

According to Theorem (4.1), there is three constants $A$, $B$ and $C$ such that for all $n \geq 3$ we have

$$t_n = A\alpha^n + B\beta^n + C\gamma^n.$$

So,

$$\begin{aligned} |t_n| &= |A\alpha^n + B\beta^n + C\gamma^n|, \\ &\leq |A\alpha^n| + |B\beta^n| + |C\gamma^n|, \\ &= |A||\alpha^n| + |B||\alpha^n| + |C||\alpha^n|, \\ &= (|A| + |B| + |C|)\alpha^n. \end{aligned}$$

(Because $|\beta| < \alpha$ and $|\gamma| < \alpha$).
The radius of convergence defined by

$$\begin{aligned} \frac{1}{R} &= \lim_{n \mapsto +\infty} \frac{|t_{n+1}|}{|t_n|}, \\ &\leq \lim_{n \mapsto +\infty} \frac{(|A| + |B| + |C|)\alpha^{n+1}}{(|A| + |B| + |C|)\alpha^n} = \alpha. \end{aligned}$$

So,

$$R \geq \frac{1}{\alpha} = \frac{3}{1+a+b}. \tag{4.17}$$

### Existence and uniqueness of solutions to the differential equation

**Proposition 4.3** *For each $|x| \leq 1$, consider the differential equation below*

$$\begin{cases} f(t,y) = y^{'} = \frac{1+2x+3x^2}{1-x-x^2-x^3}y + \frac{1}{1-x-x^2-x^3}, \\ y(0) = 0. \end{cases} \tag{4.18}$$

*Then, the equation (4.18) accepts one solution.*

**Proof.** From theorem (4.2), we create a rectangle with the center at $(0,0)$. Let $a = b = 1$ such that

$$R : |x| \leq 1, \quad |y| \leq 1,$$

we have

$$\begin{aligned} |f(x,y)| &= \left| \frac{1+2x+3x^2}{1-x-x^2-x^3}y + \frac{1}{1-x-x^2-x^3} \right|, \\ &\leq \left( \left| \frac{1}{1-x-x^2-x^3} \right| + \left| \frac{2x}{1-x-x^2-x^3} \right| + \left| \frac{3x^2}{1-x-x^2-x^3} \right| \right), \\ &+ \ |y| + \left| \frac{1}{1-x-x^2-x^3} \right|, \\ &= \left| \frac{1}{1-x-x^2-x^3} \right| \left( (1+|2x|+|3x^2|)|y|+1 \right), \end{aligned}$$

and by taking the values $a = 1, b = 1$. Then, we obtained

$$|f(t,y)| \leq \frac{7}{2} = M.$$

So, by Theorem (4.2), the equation $y^{'} = f(x,y)$ has only one solution for $|x| \leq \frac{2}{7}$, that satisfies the condition $y(0) = 0$. ■

### 4.3.2 Differential equation verifed by $g : x \mapsto g(x) \in ]-R, R[$

**Theorem 4.5** *[8] Let $g$ be a function defined in class $C^{\infty}$ on $]-R, R[$, Then $g$ satisfied the following differential equation*

$$\begin{cases} g'(x)(1-x-x^2-x^3)-(1+2x+3x^2)g(x)=1, \\ g(0)=0. \end{cases} \tag{4.19}$$

**Proof.** Let $g$ a generating function defined on $]-R, R[$ by

$$g(x)=\sum_{n=0}^{+\infty} t_n(1)x^n \Rightarrow g'(x)=\sum_{n=0}^{+\infty} nt_n(1)x^{n-1}.$$

For $n \geq 3$, we express $t_n(1)$ on function of $t_{n-3}(1), t_{n-2}(1)$ and $t_{n-1}(1)$, that is

$$\begin{aligned} g'(x) &= \sum_{n=0}^{+\infty} n(t_{n-1}(1)+t_{n-2}(1)+t_{n-3}(1))x^{n-1}, \\ &= \sum_{n=0}^{+\infty} nt_{n-1}(1)x^{n-1}+\sum_{n=0}^{+\infty} nt_{n-2}(1)x^{n-1}+\sum_{n=0}^{+\infty} nt_{n-3}(1)x^{n-1}, \\ &= \left(2t_1(1)x+\sum_{n=3}^{+\infty} nt_{n-1}(1)x^{n-1}\right)+\sum_{n=3}^{+\infty} nt_{n-2}(1)x^{n-1}, \\ &+ \left(1+\sum_{n=3}^{+\infty} nt_{n-3}(1)x^{n-1}\right), \\ &= 2x+1+\sum_{n=3}^{+\infty} nt_{n-1}(1)x^{n-1}+\sum_{n=3}^{+\infty} nt_{n-2}(1)x^{n-1}+\sum_{n=3}^{+\infty} nt_{n-3}(1)x^{n-1}. \end{aligned}$$

So,

$$g'(x)=2x+1+A(x)+B(x)+C(x). \tag{4.20}$$

Now, we're looking for $A(x)$, B(x), and $C(x)$, that is

$$\begin{aligned} A(x) &= \sum_{n\geq 3} nt_{n-1}(1)x^{n-1}, \\ &= \sum_{n\geq 2}(n+1)t_n(1)x^n, \\ &= \sum_{n\geq 0}(n+1)t_n(1)x^n - 2x, \\ &= (xg(x))^{'} - 2x. \end{aligned}$$

So,

$$A(x) = (xg(x))^{'} - 2x. \tag{4.21}$$

Similarly, we get

$$\begin{aligned} B(x) &= \sum_{n=3}^{+\infty} nt_{n-2}(1)x^{n-1}, \\ &= \sum_{n\geq 2}(n+1)t_{n-1}(1)x^n, \\ &= \sum_{n\geq 1}(n+2)t_n(1)x^{n+1}, \\ &= \sum_{n=0}^{+\infty}(n+2)t_n(1)x^{n+1}. \end{aligned}$$

So,

$$B(t) = (x^2g(x))^{'}, \tag{4.22}$$

and

$$\begin{aligned} C(x) &= \sum_{n=3}^{+\infty} nt_{n-3}(1)x^{n-1} \\ &= \sum_{n\geq 0}(n+3)t_n(1)x^{n+2}. \end{aligned}$$

So

$$C(x) = (x^3 g(x))^{'}, \tag{4.23}$$

by compensation (4.21), (4.22) and (4.23) in the equation (4.20), we get

$$\begin{aligned}
&g'(t) \\
&= (2x+1) + (xg(x))^{'} - 2x + (x^2 g(x))^{'} + (x^3 g(x))^{'} \\
&= 2xt + 1 + xg'(x) + g(x) - 2x + 2xg(x) + x^2 g'(x) + 3x^2 g(x) + x^3 g'(x) \\
&\Rightarrow (1 - x - x^2 - x^3) g'(x) - (1 + 2x + 3x^2) g(x) = 1.
\end{aligned}$$

■

### 4.3.3 Integration of the differential equation

**Theorem 4.6** *[8] If $r \leq 1$, then the equation (4.19) accepts only one solution in the interval $]-r, r[$ given by*

$$x \mapsto \frac{x}{1 - x - x^2 - x^3}.$$

**Proof.**

If $r \leq 1$, we integrate the equation (4.19) on $]-r, r[$ , (with $r > 0$),

Then, the solution to a homogeneous differential equation is given as follows

$$\begin{aligned}
\frac{g^{'}(x)}{g(x)} &= \frac{1 + 2x + 3x^2}{1 - x - x^2 - x^3}, \\
\ln(|g(x)|) &= -\ln(|1 - x - x^2 - x^3|) + k.
\end{aligned}$$

Then,

$$g(x) = \frac{e^k}{1 - x - x^2 - x^3} = \frac{C}{1 - x - x^2 - x^3},$$

where $k$ and $C$ are an optional constants $(C = e^k)$.

By using the method of designation the optional constants, we have

$$g(x) = \frac{C(x)}{1 - x - x^2 - x^3},$$

then

$$g'(x) = \frac{C'(x)(1 - x - x^2 - x^3) - C(x)(-1 - 2x - 3x^2)}{(1 - x - x^2 - x^3)^2},$$

and by substitute into the equation (4.19), we get

$$\frac{C'(x)}{1 - x - x^2 - x^3} = \frac{1}{1 - x - x^2 - x^3} \Rightarrow C(x) = x.$$

So,

$$g_0(x) = \frac{x}{1 - x - x^2 - x^3}$$

Then, the solution to the differential equation (4.19) is given as follows

$$g(x) = g_0(x) + \frac{C}{1 - x - x^2 - x^3} = \frac{x + C}{1 - x - x^2 - x^3},$$

where this solution satisfies the initial condition $g(0) = 0$.

So,

$$\begin{aligned} g(x) &= \frac{x + C}{1 - x - x^2 - x^3}, \\ &\Rightarrow g(0) = C = 0, \\ &\Rightarrow C = 0. \end{aligned}$$

Hence, the solution to the differential equation (4.19) is

$$g(x) = \sum_{0}^{+\infty} t_n(1) x^n = \frac{x}{1 - x - x^2 - x^3}.$$

■

# Conclusion

In this thesis, we deal certain recurrent linear sequences, (Fibonacci and Tribonacci sequences). We then analysed its applications. On the subject of systems of difference equations and differential equations. Beginning with a study of the solutions to various systems of higher-order difference equations, these solutions are provided in their form related to the Fibonacci and Tribonacci sequences. The stability and periodicity of these solutions were then investigated. We also discovered first-order linear differential equations based on the Fibonacci and Tribonacci polynomials.

The applications of linear recurrent sequences and polynomial are still being addressed, and here we raise the difficulty of finding the Fibonacci and Tribonacci polynomial interpolations.

# Bibliography


[1] R. P. Agarwal, D. O'Regan, Ordinary and partial differential equations: with special functions, Fourier series, and boundary value problems, Springer Science., Business Media, 2008.

[2] R.P. Agarwal and V. Lakshmikantham, Uniqueness and nonuniqueness criteria for ordinary differential equations, World Scientific, **6**, 1993.

[3] S. Badidja and A.A. Mokhtar and O. Ozer, Representation of integers by k-Generalized Fibonacci Sequences and applications in cryptographe, World Scientific Publishing Company, **14**(9), (2021), 150-157.

[4] S. Badidja, A. Boudaoud, A unique representation of positive integers as a sum of distinct Tribonacci numbers, J.Math. Statistics, **13**(1), (2017), 57-61.

[5] S. Badidja, Generalization and some characteristics of Tribonacci sequences, International Journal of Scientific Engineering and Research,

**7**, (2016).

[6] D. H. Bailey, J. M. Borwein, R. E. Crandall and C. Pomerance, n the binary expansions of algebraic numbers, Journal of theorie the nombres of Bordeaux, **16**(3), (2004), 487-518.

[7] J. Banasiak, Mathematical modelling in one dimension: an introduction via difference and differential equations, Cambridge University Press, 2013.

[8] J. P. Barani, Note de cours, Sur la série génératrice des polynômes de Laguerre, 2002.

[9] P. Barry, On integer-sequence-based constructions of generalized Pascal triangles, J. Integer Sequences, chool of Computer Science, University of Waterloo, Canada, **9**, (2006).

[10] L. Brand, A sequence defined by a difference equation. Am. Math. Mon. **62**(7), (1955), 489-492.

[11] C. Cinar , On the positive solutions of the difference equation system $x_{n+1} = \frac{1}{y_n}, y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}}$, Appl. Math. Comput., **158**(2), (2004), 303-305.

[12] J. Deng, L. Ma, Existence and uniqueness of solutions of initial value problems for nonlinear fractional differential equations, Applied Mathematics Letters, **23**(6), (2010), 676-680.

[13] E. M. Elabbasy, H. El-Metwally and E. M. Elsayed, On the solutions of a class of difference equations systems, Demonstr. Math., **41**(1), (2008), 109-122.

[14] E. M. Elsayed, On the solutions of a rational system of difference equations, Fasc. Math., **45**, (2010), 25-36.

[15] E. M. Elsayed, M. Mansour and M. M. El-Dessoky, Solutions of fractional systems of difference equations, Ars Comb., **110**, (2013), 469-479.

[16] E. M. Elsayed, On the solutions of higher order rational system of recursive sequences, Math.Balk., **21**(3-4), (2008), 287-296.

[17] E. M. Elsayed, Solution for systems of difference equations of rational form of order two, J. Comput. Appl. Math., **33**(3), (2014).

[18] K. Eriksson, D. Estep, P. Hansbo, and C. Johnson, Introduction to Computational Methods for Differential Equations, Clarendon Press, Oxford, 1995.

[19] M. Feinberg, Fibonacci-Tribonacci, The Fibonacci Quarterly, **1**(1), (1963), 70-74.

[20] D. Franco, J. Nieto and D.O'regan, Existence of solutions for first order ordinary differential equations with nonlinear boundary conditions, Elsevier. pp., **153**(3), (2004), 793-802.

[21] Y. Halim, A System of Difference Equations with Solutions Associated to Fibonacci Numbers, International Journal of Difference Equations, ISSN 0973-6069., **11**(1), (2016), 65-77.

[22] Y. Halim, Global character of systems of rational difference equations, Electron. J. Math. Analysis Appl, **3**(1), (2015), 204-214.

[23] Y. Halim, N. Touafek and E.M. Elsayed, Closed form solution of some systems of rational difference equations in terms of Fibonacci numbers. Mathematical Analysis **21**(6), (2014), 473-486.

[24] Y. Halim, J. Fergy and T. Rabago, On some solvable systems of difference equations with solutions associated to Fibonacci numbers, Electronic Journal of Mathematical Analysis and Applications,**5**(1), (2017), 166-178.

[25] Y. Halim, N. Touafek and E.M. Elsayed, Closed form solutions of some system of rational difference equations in terms of Fibonacci numbers, Mathematical Analysis **21**(6), (2014), 473-486.

[26] A. Hamdi, S. Badidja, Some identities of generalized Tribonacci and Jacobsthal polynomials, Notes on Number Theory and Discrete Math, **21**, (2021), 137-147.

[27] V.E. Hoggatt, M. Bicknell, Generalized Fibonacci polynomials and Zeckendorf's theorem, The Fibonacci Quarterly, **11**(4), (1973), 399-419.

[28] A.F. Horadam, Extension of a synthesis for a class of polynomial sequences, The Fibonacci Quarterly, vol. **(34)**, no. 1, pp., (1996), 68-73.

[29] T. F. Ibrahim, Closed form solution of a symmetric competitive system of rational difference equations, Stud. Math. Sci., **5**(1), (2012), 49-57.

[30] E.L. Ince, Ordinary differential equations, Courier Corporation, 1956.

[31] I. Okumuş, Y. Soykan, On the solutions of systems of difference equations via Tribonacci numbers, arXiv preprint arXiv:1906.09987, 2019.

[32] P. Kabal, R.P. Ramachandran, The computation of line spectral frequencies using Chebyshev polynomials, IEEE transactions on acoustics, speech and signal processing., **34**(6), (1986), 1419-1426.

[33] V. L. Kocic, G. Ladas, Global behavior of nonlinear difference equations of higher order with applications, Kluwer Academic Publishers, Dordrecht, **256** (1993).

[34] T. Koshy, Fibonacci and Lucas Numbers with Applications, Wiley-Inter science Publication., **2**, 2019.

[35] M. R. S. Kulenovic, G.Ladas, Dynamics of second ordre rational difference equations with open problems and conjectures, Chapman and Hall/Crc, Boca Raton, London, New York, Washington, D.C, 2001.

[36] M.S. Solary, Numerical experiments for finding roots of the polynomials in Chebyshev basis, applications and Applied Mathematics: An International Journal (AAM), **12**(2), (2017), 12.

[37] S. Steviċ, Representation of solutions of bilinear difference equations in terms of generalized Fibonacci sequences, Electron. J. Qual. Theory Differ. Equ., No. (67),(2014), 1-15.

[38] S. Steviċ, On some solvable systems of difference equations, Appl. Math. Comput., **218**(9), (2012), 5010-5018.

[39] I. Talha, S. Badidja, Periodicity of the solutions of general system of rational difference equations related to fibonacci numbers, Asian-European .J.Math (2021).

[40] Y. Tasyurdu, On the Sums of Tribonacci and Tribonacci-Lucas Numbers, Applied Mathematical Sciences, **13**(24), (2019), 1201-1208.

[41] D. T. Tollu, Y. Yazlik and N. Taskara, On the solutions of two special types of Riccati difference equation via Fibonacci numbers, Adv. Difference Equ., Article ID, (1),(2013), 1-7.

[42] N. Touafek and E. M. Elsayed, On the periodicity of some systems of nonlinear difference equations, Bull. Math. Soc. Sci. Math. Roum., Nouv. Sr., **55**(103), (2012), 217-224.

[43] N. Touafek and E. M. Elsayed, On the solutions of systems of rational difference equations, Math. Comput. Modelling, **55**(7-8), (2012), 1987-1997.

[44] N. Touafek and Y. Halim, On max type difference equations : expressions of solutions, Int. J. Nonlinear Sci., **11**,(2011), 396-402.

[45] I. Yalçinkaya, On the global asymptotic behavior of a system of two nonlinear difference equations, Ars Comb., **95**, Article ID, (2008), 151-159.

[46] I. Yalçinkaya, On the global asymptotic stability of a second-order system of difference equations, Discrete Dyn. Nat. Soc., Article ID 860152, (2008).

[47] Y. Yazlik, D.T. Tollu, and N. Taskara, On the solutions of difference equations systems with Padovan numbers, Appl.Math, **4**(12A), (2013), 15-20.

[48] N. Yilmaz, N. Taskara, Incomplete Tribonacci-Lucas Numbers and Polynomials, Springer, **25**(3), (2015), 741-753.