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Before the jury:

Mr.Chacha A.Djamal	Prof.	University of Kasdi Merbah-Ouargla	Chairman
Mr.Merabet Ismail	Prof.	University of Kasdi Merbah-Ouargla	Examiner
Mr.Ghazal Abdrrazek	M.C.A	University of Kasdi Merbah-Ouargla	Examiner
Mr.Bensayah Abdallah	M.C.A	University of Kasdi Merbah-Ouargla	Supervisor

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Dedication

I am forever grateful to my late father, "**Ahmed**", for being an unwavering pillar of support, encouragement, and inspiration.
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Contents

Notation

X, Y	real Banach spaces.
E	metric space.
$\|.\|_X$	the norm of X.
X^*	topological dual space of X.
$\langle .,. \rangle_X$	duality product of X and X^*.
$\mathcal{P}(X)$	the set of all subsets of X.
$\mathcal{P}_{cl}(X)$	the set of all closed subsets of X.
$\rightarrow$	the strong convergence.
$\rightharpoonup$	the weak convergence.
$\mathcal{L}(X,Y)$	the space of continuous linear operators from X to Y.
∇	the gradient.
$\Re(z)$	the real part of z.
$\|.\|$	the euclidean norm of $\mathbb{R}^n$.
$C(E,F)$	the space of continuous functions from E to F.
$CB(X)$	the collection of all nonempty bounded and closed subsets of X.
$L^p(\Omega)$	the space of measurable functions on Ω such that: $\int_\Omega \|u\|dx < \infty$, $1 \leq p < \infty$.
$L^\infty(\Omega)$	the space of essentially bounded functions on Ω.

Fractional derivatives will lead to a paradox, from which one day useful consequences will be drawn.

-Gottfried Leibniz

Introduction

The seeds of fractional derivatives were planted over 300 years ago, although it only contained some pure mathematical manipulations with little to no use, but around 30 years ago this paradigm started to shift more towards the applied mathematics allowing the subject of fractional differential equations and inclusions as well to emerge as an important area of investigation. In fact, many systems in physics and engineering and some biological phenomena can be modeled more accurately by fractional derivatives or fractional integrals than traditional integer order derivatives or integrals,Applications include bio-mechanics, behaviors of viscoelastic materials, control, electrochemical processes, dielectric polarization, colored noise and chaos (See:[1],[2],[3],[4] and[5]).

Traditionally, differential equations describe the relationship between a function and its derivatives by using integer-order derivatives. However, in many real-world applications, phenomena such as anomalous diffusion, viscoelasticity, and fractional calculus arise, indicating the need to consider fractional derivatives, which extends the concept of differentiation and integration to non-integer orders. Unlike integer-order derivatives, fractional derivatives capture the memory of past states, making them suitable for modeling systems with long-range dependencies and complex dynamics.

The notion of inclusions involves a family of differential equations or inequalities, encompassing a range of possible solutions. This allows for a more flexible modeling of systems where the dynamics are uncertain or exhibit multiple possible trajectories.

In this thesis we attempt to study some of these fractional inclusion problems.

Precisely, the first chapter is devoted to recalling some notions and properties of set valued operators, exploring the fractional calculus, and citing theorems and principles that are used in the following chapters.

The second chapter, is intended to highlighting the connection between the variational inequalities and inclusions. Presenting types of inclusion problems solvable by the existence theorems for on the pseudo-monotone perturbation of the maximal monotone set valued operator, as well as another type which follows another approach like a hybrid integro-differential inclusion model.

And in the third chapter, we study the existence and uniqueness of a generalized Caputo fractional inclusion of the thermostat model using the endpoint concept and finishing with numerical application by the ANN method.

Chapter 1

Preliminaries

In this section we are going to present some important definitions as well as useful theorems related to our field of research.

1.1 Single valued functions

Here we consider the single valued function $A : X \to X^*$.

Definition 1.1 *[6, page 11]If* $(X, \mathcal{M})$ *and* $(Y, \mathcal{N})$ *are measurable spaces, a mapping* $f : X \to Y$ *is called* $(\mathcal{M}, \mathcal{N})$***-measurable*** *or just* ***measurable*** *if* $f^{-1}(E) \in \mathcal{M}, \forall E \in \mathcal{N}$.

Definition 1.2 *[7, page 12]*

- *A map* $f : X \to Y$ *is said to be* ***continuous*** *at* $x_0 \in X$ *if for a given* $\epsilon > 0$ *there exists* $\delta > 0$,$\delta = \delta(\epsilon, x_0)$,*such that:*

  $||f(x) - f(x_0)||_Y < \epsilon$ *whenever* $||x - x_0||_X < \delta$

  f *is continuous on* X *if it is continuous on every point in* X.

- *A map* $f : X \to Y$ *is said to be* ***uniformly continuous*** *at* $x_0 \in X$ *if for a given* $\epsilon > 0$ *there exists* $\delta > 0$ *independent of* x_0,*such that:*

  $||f(x) - f(x_0)||_Y < \epsilon$ *whenever* $||x - x_0||_X < \delta$

Definition 1.3 *[7, page 109]A family of functions A mapping a metric space E into the real or complex numbers is said to be* ***equi-continuous*** *if:* $\forall x_0 \in E$ *and* $\epsilon > 0, \exists \delta > 0$, *such that:*

$$d(x, x_0) < \epsilon \ \ \textit{implies} \ \ |f(x) - f(x_0)| < \delta, \quad \forall f \in A$$

Note that δ may depends on x_0 or ϵ but not on f.

Definition 1.4 *[8, page 8]Let the function $\varphi : X \rightarrow]-\infty, +\infty]$:*

$\rhd$ *φ is said to be **lower semi-continous** if for all $\lambda \in \mathbb{R}$*

the set $\{x \in X : \varphi(x) \leq \lambda\}$ is closed.

or, if for every sequence $(x_n)_{n\in\mathbb{N}}$ in Xsuch that $x_n \to x$, we have

$$\liminf_{n\longrightarrow\infty} \varphi(x_n) \geq \varphi(x).$$

$\rhd$ *φ is said to be **upper semi-continous** if for all $\lambda \in \mathbb{R}$*

the set $\{x \in X : \varphi(x) \geq \lambda\}$ is closed.

or, if for every sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $x_n \to x$, we hav

$$\limsup_{n\longrightarrow\infty} \varphi(x_n) \leq \varphi(x).$$

Definition 1.5 *[9, page 500,501]*

$\rhd$ *The operator A is said to be **monotone** if and only if*

$$\langle Au - Av, u - v\rangle_X \geq 0 \quad \textit{for all } u, v \in X.$$

$\rhd$ *The operator A is said to be **strictly monotone** if and only if*

$$\langle Au - Av, u - v\rangle_X > 0 \quad \textit{for all } u, v \in X \textit{ with } u \neq v.$$

$\rhd$ *The operator A is said to be **coercive** if and only if*

$$\lim_{\|u\|\longrightarrow\infty} \frac{\langle Au, u\rangle_X}{\|u\|_X} = +\infty.$$

Definition 1.6 *[10, page 546] The operator A is said to be* ***maximal monotone*** *if it is monotone and if that: $\langle Au - w, u - v\rangle_X \geq 0$ for all $u \in X$ implies $w = Av$.*

Definition 1.7 *[9, page 515] The operator $A : M \subset X \to X^*$ is said to be* ***pseudo-monotone*** *if and only if for all $u \in M$ and for every sequence (u_n) in M:*

$$u_n \rightharpoonup u \quad \text{when } n \to \infty \quad \text{and} \quad \limsup_{n \longrightarrow \infty} \langle Au_n, u_n - u\rangle_X \leq 0,$$

implies that:

$$\liminf_{n \longrightarrow \infty} \langle Au_n, u_n - w\rangle_X \geq \langle Au, u - w\rangle_X \qquad \text{for all} \quad w \in X.$$

Definition 1.8 *[9, page 554] The operator A is said to be* ***hemi-continuous*** *if and only if the real function*

$$t \mapsto \langle A(u + tv), w\rangle_X$$

is continuous on [0,1] for all $u, v, w \in X$.

Definition 1.9 *[9, page 554] A is said to be* ***demi-continuous*** *if and only if*

$$u_n \to u \quad \text{when} \quad n \to \infty$$

implies $Au_n \rightharpoonup Au$ when $n \to \infty$.

Definition 1.10 *[7, page 53] An operator $T \in \mathcal{L}(X, Y)$ is said to be* ***compact*** *if T maps bounded sets of a Banach space X into relatively compact sets of a Banach space Y.*
An equivalent definition is that T is linear and for any bounded sequence $\{x_k\}$ in X,$\{Tx_k\}$ has a convergent sub-sequence in Y.
(this definition can be extended to set-valued functions).

Definition 1.11 *[11, page 3] A mapping $p : I \times \mathbb{R} \to \mathbb{R}$ is said to be* ***Carathéodory*** *if:*

- *$t \to p(t, u)$is measurable for each $u \in \mathbb{R}$.*
- *$u \to p(t, u)$is continuous a.e. for each $t \in I$.*

1.2 Set-valued functions

Unlike ordinary functions, set-valued functions can return multiple values for a single input. These functions are widely used in many branches of mathematics, such as optimization, game theory, and topology.

Definition 1.12 *[9, page 850] Let $T : X \to \mathcal{P}(Y)$ be* ***a set-valued*** *function that is T associates for each element $u \in X$ a subset $Tu \subset Y$.(it also can be referred to as multi-function or multi-valued map)*

- *The set $D(T) = \{u \in X : Tu \neq \phi\}$ is called* ***the domain*** *of T.*
- *The set $R(T) = \bigcup_{u \in X} Tu$ is called* ***the range*** *of T.*
- *The set $G(T) = \{(u, v) \in X \times Y : u \in D(T), v \in Tu\}$ is called* ***the graph*** *of T.*

Definition 1.13 *[9, page 851]**The inverse** of a set-valued function $T^{-1} : Y \to \mathcal{P}(X)$ is defined by:*

$$T^{-1}(v) = \{u \in X : v \in Tu\}$$

such that $D(T^{-1}) = R(T)$

and: $(u, v) \in G(T)$ if and only if $(v, u) \in G(T^{-1})$

Definition 1.14 *[9, page 851] Let $M \subseteq X$.*

For the given multi-functions

$$A, B : M \to \mathcal{P}(Y)$$

and for $\alpha, \beta \in \mathbb{R}$, we define the linear composition

$$\alpha A + \beta B : M \to \mathcal{P}(Y)$$

By:

$$(\alpha A + \beta B)(u) = \begin{cases} \alpha Au + \beta Bu & \text{if } u \in D(A) \cap D(B), \\ \phi & \text{else.} \end{cases}$$

and we have: $D(\alpha A + \beta B) = D(A) \cap D(B)$.

Definition 1.15 *[12, page 4] For a set $B$of real numbers, let $\{I_k\}_{k=1}^{\infty}$ be an open cover for B. We define the set function **Lebesgue outer measure**, sometimes called an exterior measure, as:*

$$m^*(B) = \inf\{\sum_{k=1}^{\infty} l(I_k) \ \ | \ \ B \subseteq \bigcup_{k=1}^{\infty} I_k\}$$

With:

$$l(I) = \begin{cases} b-a, if I =]a,b[\\ \\ \infty, if I \text{ is unbounded.} \end{cases}$$

Definition 1.16 *[12, page 7] A subset M of $\mathbb{R}$ is said to be **Lebesgue measurable** if for given $\epsilon > 0$, there exists an open set G such that:*

$$M \subset G \quad \text{and} \quad m^*(G \backslash M) < \epsilon$$

Definition 1.17 *[13, page 7] A multi-valued map $T : I \to \mathcal{P}_{cl}(\mathbb{R})$ is said to be **measurable** if:*

For every open $U \subset \mathbb{R}$,the set $\{t \in I, T(t) \subset U\}$is Lebesgue measurable in I.

Definition 1.18 *[14, page 36] If the images of the set valued map T are closed, we say that T is **closed-valued**.*

Definition 1.19 *[15, page 160] A multi-function $F : \Omega \to \mathcal{P}(\mathbb{X})$ is called **integrably bounded** if there is a $\rho \in L^1(\Omega, \mathbb{X})$ such that:*

$$||x|| \leq \rho(\omega)$$

for any x and ω with: $x \in F(\omega)$.

Definition 1.20 *[9, page 851] The operator B:$M \to \mathcal{P}(Y)$ is called an **extension** of the operator A:$M \to \mathcal{P}(Y)$ if and only if $G(A) \subseteq G(B)$.*

Definition 1.21 *[14, page 38]A set valued map $T : X \to \mathcal{P}(Y)$ is called **upper semi continuous** at $x \in D(T)$ if and only if for any neighborhood V of $T(x)$, $\exists \eta > 0$ such* *that:*

$$\forall x' \in B_X(x,\eta), \quad T(x') \subset V.$$

T is said to be upper semi continuous if it is upper semi continuous in any point of $D(T)$.

Definition 1.22 *[16, page 47] We say that a multi-function operator $A : M \subseteq X \to \mathcal{P}(Y)$ is* ***bounded*** *if its graph $G(A)$ is bounded in $X \times Y$,*
that is, let $(x_n, y_n) \in M \times Y$ such that $y_n \in Ax_n$ for all $n \in \mathbb{N}$, and $x_n \to x$ in X and $y_n \to y$ in Y implies $y \in Ax$.

Definition 1.23 *[9, page 851] Let $A : M \to \mathcal{P}(X^*)$ be a set valued function, where M is a sub set of the Banach space X.*

(a) *A subset S of $M \times X^*$ is called monotone if and only if*

$$\langle u^* - v^*, u - v\rangle_X \geqslant 0 \qquad \text{for all} \quad (u,u^*),(v,v^*) \in S$$

(b) *A subset S of $M \times X^*$ is called maximal monotone if and only if it is monotone and that it has no proper monotonic extension in $M \times X^*$.*

(c) *The multi-function A is called monotone if and only if the graph $G(A)$ is a monotone set in $M \times X^*$, that is:*

$$\langle u^* - v^*, u - v\rangle_X \geqslant 0 \qquad \text{for all} \quad (u,u^*),(v,v^*) \in G(A)$$

(d) *The multi-function A is called maximal monotone if and only if the graph $G(A)$ is a maximal monotone set in $M \times X^*$, that is:*
A monotone and $(u,u^*) \in M \times X^*$ and

$$\langle u^* - v^*, u - v\rangle_X \geqslant 0 \qquad \text{for all} \quad (v,v^*) \in G(A)$$

implies that $(u,u^*) \in G(A)$.

Definition 1.24 *[10, page 546] Let X be a real reflexive Banach space.*

*The multi-function operator $T:X \to \mathcal{P}(X^*)$ is **generalized pseudo-monotone** if for every sequence $\{u_n\} \subset X$ and $\{u_n^*\} \subset X^*$ such that $u_n \rightharpoonup u$ in $X, u_n^* \in Tu_n$ for $n \geqslant 1$,*

$u_n^* \rightharpoonup u^*$ and $\limsup\limits_{n\longrightarrow\infty}\langle u_n^*, u_n - u\rangle_X \leqslant 0$,

we have $u^* \in Tu$

and $\lim\limits_{n\longrightarrow\infty} \langle u_n^*, u_n\rangle_X = \langle u^*, u\rangle_X$.

For the following, let (X, d) be a metric space.

Definition 1.25 *[17, page 132] Let $F : X \to \mathcal{P}(X)$ be a set valued function.*

*An element $x \in X$ is said to be **a fixed point** of F if: $x \in Fx$. An element $x \in X$ is said to be **an endpoint** (or stationary point) of F if: $Fx = \{x\}$.*

Definition 1.26 *[17, page 132]We say that a set-valued map $F : X \to \mathcal{P}(X)$ has **the approximate endpoint property** if:*

$$\inf_{x\in X} \sup_{y\in Fx} d(x, y) = 0$$

Definition 1.27 *[18, page 10]For each $u \in X$, we define the set of selection of T by*

$$S_T = \{\nu \in L^1([0, 1], \mathbb{R}) : \nu(t) \in T(t, u(t))\}$$

Definition 1.28 *[19, page 21] A countable family $\{f_n\}_{n=1}^{\infty} \subset S_F$ is said to be **casting representation** of F if:*

$$\overline{\cup_{n=1}^{\infty} f_n(t)} = F(t)$$

for μ-a.e. $t \in I$.

1.3 Some useful lemmas and theorems

Theorem 1.1 ***Arzela-Ascoli Theorem****[7, page 109-110]*

Let E be a compact metric space,and $C(E)$ the Banach space of real or complex valued *continuous functions with respect to the norm:*

$$||f|| = \sup_{x \in E} |f(x)|.$$

If $A = \{f_n\}$ *is a sequence in* $C(E)$ *such that:*

- $f_n(x)$ *is uniformly bounded (i.e.* $\sup_{n \geq 1} \sup_{x \in E} |f_n(x)| < \infty$*).*
- $\{f_n\}$ *is equi-continuous.*

Then $\bar{A}$ *is compact.*

Theorem 1.2 ***Lebesgue dominated convergence****[7, page 30]*

If $\{u_n\}$ *is a sequence of measurable functions on* E *such that:* $u_n(\omega) \to u(\omega)$ *as* $n \to \infty$ *a.e. on* E*, and* $|u_n(\omega)| \leq \nu(\omega)$*a.e. on* E*, where* ν *is an integrable function on* E*. Then:*

$$\int_E u \quad d\mu = \lim_{n \to \infty} \int_E u_n \quad d\mu$$

Theorem 1.3 ***the Schauder-Tychonoff fixed point theorem****[20, page 161]*

Let E *be a separated LCTVS, and* K *a non empty convex subset of a* E*, and let* T *be a continuous mapping of* K *into itself. Then* T *has a fixed point in* K*.*

Lemma 1.1 *[21, page 236] Let* $\psi : [0, \infty) \to [0, \infty)$ *be an upper semi continuous function such that:*

- $\liminf_{t \to \infty}(t - \psi(t)) > 0, \forall t > 0.$
- $\psi(t) < t, \forall t > 0.$

With (X, d) *a complete metric space and* $T : X \to CB(X)$ *a multi function such that:*

$$H_d(T(x), T(y)) \leq \psi(d(x, y)), \quad \forall x, y \in X,$$

where $H_d(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\}$*.*

Then T *has a unique endpoint if and only if* T *has approximated endpoint property.*

Lemma 1.2 *[18, page 3]Let E be a separable metric space and let $G : [a, b] \to \mathcal{P}(E)$ be a measurable set-valued map with closed values. Then G has a measurable selection.*

Theorem 1.4 *[19, page 22]*

Let X be a separable Banach space, For a multi function $F : I \to K(X)$ the following conditions are equivalent:

(1) F is measurable .

(2) for every countable dense subset $\{x_n\}_{n=1}^{\infty}$ of X the functions $\{g_n\}_{n=1}^{\infty}$, $g_n : I \to \mathbb{R}$ are measurable.

(3) F has a casting representation.

(4) F is strongly measurable.

(5) is measurable as a single-valued map from I into a metric space $(K(X), h)$.

(6) F has the Lusin property: for every $\delta > 0$ there exists a closed subset $I_\delta \subset I$ such that $\mu(I \backslash I_\delta) \leq \delta$ and the restriction of F on I_δ is continuous.

Theorem 1.5 *[19, page 29]*

Let X, X_0 (not necessarily separable) Banach spaces, and let $F : I \times X_0 \to K(X)$ be a multi-function, such that:

(1) for every $x \in X_0$ the multi-function $F(., x) : I \to K(X)$ has a strongly measurable selection.

(2) for μ -a.e. $t \in I$, the multi-map $F(t, .) : X_0 \to K(X)$ is upper semi continuous.

Then for every strongly measurable function $q : I \to X_0$ there exists a strongly measurable selection $\varphi : I \to X$ of the multi-function $\mathcal{F} : I \to K(X)$, such that:

$$\mathcal{F}(t) = F(t, q(t))$$

1.4 Fractional calculus

Have you ever wondered what lies between the function and its derivative?

The main idea of fractional calculus is to extend the concept of integer-order differentiation and integration to non-integer orders. This involves defining fractional derivatives and integrals which we are going to introduce in this section.

1.4.1 Gamma function

Euler's Gamma function is certainly one of the fundamental applications of fractional calculus, as it extends the notion of factorial to non-integer and even complex values of n.

Definition 1.29 *[22, page 1] The Gamma function is defined by the integral:*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

Which converges in the right half of the complex plane $\Re(z) > 0$.

Properties [22, page 2,4]

Based on the information provided, we can draw several crucial properties of the Gamma function, some of which are outlined below.:

1. $\Gamma(z+1) = z\Gamma(z)$, for all $z \in \mathbb{Z}$.
2. $\Gamma(n) = (n-1)!$, for all $n \in \mathbb{N}$.
3. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.
4. $\Gamma(1) = 1$.

Example 1.1 *We are welling to express the fractional derivative of the function $f(x) = x^n$ by the Gamma function:*

Since the classic integer derivative of order p of $f(x) = x^n$ is given by:

$$f^{(p)}(x) = \frac{n!}{(n-p)!}x^{n-p}$$

Then for $\alpha \in \mathbb{R}$ and by using the proprieties of Gamma function, we arrive at

$$f^{(\alpha)}(x) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}x^{n-\alpha}$$

1.4.2 Riemann-Liouville fractional integrals

The Riemann-Liouville integral is a generalization of the Riemann integral that allows for the integration of functions with singularities or other irregularities.

Definition 1.30 *[23, page 69] Let $[a, b]$ be a finite interval. The Riemann-Liouville integrals of f with respect to x over the interval $[a, b]$ are defined as:*

$$(I_{a+}^{\alpha}f)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \qquad (x > a, \Re(\alpha) > 0)$$

$$(I_{b-}^{\alpha}f)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \qquad (x < b, \Re(\alpha) > 0)$$

Where $\Gamma(\alpha)$ is the Gamma function, these integrals are called the left-sided and the right-sided fractional integrals.
For $n = 0$, we set:

$$I_{a+}^{\alpha}f(x) = f(x)$$

Lemma 1.3 *[23, page 73] If $\Re(\alpha) > 0$ and $\Re(\beta) > 0$, then the equations*

$$(I_{a+}^{\alpha}I_{a+}^{\beta}f)(x) = I_{a+}^{\alpha+\beta}f(x)$$

is satisfied almost on every point $x \in [a, b]$ for $f \in L_p(a,b)$,if $\alpha + \beta > 1$ then the equation holds at any point of $[a, b]$.

1.4.3 Riemann–Liouville derivative

Definition 1.31 *[18, page 2] Let $n-1<\alpha<n$. The Riemann–Liouville derivative of a continous function $f:[0,+\infty[\to\mathbb{R}$ of the order α is given by:*

$$\mathcal{D}^{\alpha}_{0+}f(t)=\frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^{n}\int_{0}^{t}(t-s)^{n-\alpha-1}f(s)ds$$

where $n=[\alpha]+1$ and $[\alpha]$ denotes the greatest integer number less than α.

The following lemma presents some important properties of the Riemann–Liouville derivative.

Lemma 1.4 *[18, page 2] Let $u\in L^{1}([0,1],\mathbb{R})$ and $\beta>\alpha>0$. Then,*

- $\mathcal{D}^{\alpha}_{0+}I^{\beta}_{0+}u(t)=I^{\beta-\alpha}_{0+}u(t)$.
- $\mathcal{D}^{\alpha}_{0+}I^{\alpha}_{0+}u(t)=u(t)$.

1.4.4 Caputo derivative

The Caputo derivative is a type of fractional derivative that generalizes the classical derivative to non-integer orders and even the Riemann-Liouville derivative. It is named after Michel Caputo, who introduced it in 1967 as a way to solve fractional differential equations.

Definition 1.32 *[21, page 235] The Caputo derivative of the f of the order α, denoted by ${}^{C}\mathcal{D}^{\alpha}_{a+}$, is defined as:*

$${}^{C}\mathcal{D}^{\alpha}_{a+}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}(t-\tau)^{n-\alpha-1}\frac{d^{n}}{d\tau^{n}}f(\tau),d\tau$$

where n is the smallest integer greater than or equals to α, Γ is the Gamma function, and f is in $C([a,b],\mathbb{R})$.

Lemma 1.5 *[23, page 95] Let $\Re(\alpha) > 0$ and let $y(t) \in L_\infty([a,b])$ or $y(t) \in C([a,b])$. If $\Re(\alpha) \notin \mathbb{N}$ or $\alpha \in \mathbb{N}$, then*

$$({}^C\mathcal{D}^\alpha_{a+} I^\alpha_{a+} y)(t) = y(t)$$

Lemma 1.6 *[22, page 81]Let $\Re(\alpha) > 0$, $m \in \mathbb{N}$ and let $y(t) \in C([a,b])$.*

$${}^C\mathcal{D}^\alpha_{a+}({}^C\mathcal{D}^m_{a+} y(t)) =^C \mathcal{D}^{\alpha+m}_{a+} y(t)$$

Remak 1.1 *Using the definitions of both the Caputo derivative and Riemann-Liouville integral, we can easily prove that:*

$$({}^C\mathcal{D}^\alpha_{a+} I^n_{a+} y)(t) = I^{n-\alpha}_{a+} y^{(n)}(t)$$

Remak 1.2 *If $\alpha \notin \mathbb{N}_0$ $(\Re(\alpha) > 0)$ and $y(x)$ is a continuous function. Caputo derivative and Riemann-Liouville derivative are connected with each other by the following relations:*

$$({}^C\mathcal{D}^\alpha_{a+} y)(x) = (\mathcal{D}^\alpha_{a+} y)(x) - \sum_{n-1}^{k=0} \frac{y^{(k)}(a)}{\Gamma(k-\alpha+1)}(x-a)^{k-\alpha} \quad (n = [\Re(\alpha)] + 1)$$

$$({}^C\mathcal{D}^\alpha_{b-} y)(x) = (\mathcal{D}^\alpha_{b-} y)(x) - \sum_{n-1}^{k=0} \frac{y^{(k)}(b)}{\Gamma(k-\alpha+1)}(b-x)^{k-\alpha} \quad (n = [\Re(\alpha)] + 1)$$

Chapter 2

Some inclusion problems

The existence and uniqueness of solutions are key problems in studying set-valued inclusions. Generally, solutions to set-valued inclusions are not unique and may not exist at all.

One should also know that there's a certain type of inclusions which is differential inclusions this type is used to model systems that are subject to uncertain or unpredictable influences such as systems of noise or disturbances, or systems with multiple possible modes of operation.

Definition 2.1 *[14, page 383] A differential inclusion is simply defined as:*

$$x'(t) \in F(t, x(t))$$

Where F is set-valued map from $\mathbb{R} \times X$ to finite dimensional victor space X.

In this chapter, we distinguish two types of inclusion problems: one in which we can apply the theorems of perturbed and non-perturbed inclusion on, and another type in which we can't, so instead these inclusions are analyzed individually using the fixed point theorems and so on.

2.1 The main theorem on the pseudo-monotone perturbation of the maximal monotone set valued operator

Maximal monotone mappings are set-valued mappings with a well-defined inverse, composition closeness, and the ability to generate other set-valued mappings. They are important in studying set-valued mappings. Pseudo-monotone mappings, though weaker, are also useful as their graphs are subsets of maximal monotone operators' graphs.
This part includes some important theorems on the existence of the solution related to maximal monotone and pseudo-monotone multi functions (the theorems of perturbed and non-perturbed inclusion).

Theorem 2.1 (Browder (1968)) *[9, page 867]*
The objective of this theorem is to solve the following perturbed inclusion:

$$b \in Au + Bu, \qquad u \in C \tag{2.1}$$

where $A : C \subseteq X \to \mathcal{P}(X^*)$ is maximal monotone and $B : C \to X^*$ is pseudo-monotone. Explicitly, the inclusion (2.1) means the following: For a given $b \in X^*$, find $u \in C$ such that

$$b = v + w, \qquad \textit{where} \quad v \in Au \quad \textit{and} \quad w \in Bu.$$

Suppose that:

($\mathcal{H}1$) C is a convex, closed and non empty set in a reflexive Banach space X.

($\mathcal{H}2$) The multi-function operator $A : C \to \mathcal{P}(X^*)$ is maximal monotone.

($\mathcal{H}3$) The multi-function operator $B : C \to X^*$ is pseudo-monotone, bounded and demi-continuous.

($\mathcal{H}4$) If the set C is non bounded, then the operator B is A-coercive with respect to the fixed element $b \in X^*$, that is, there exists a point $u_0 \in C \cap D(A)$ and a number

$r > 0$ such that:

$$\langle Bu, u - u_0 \rangle > \langle b, u - u_0 \rangle \qquad \text{for all} \quad u \in C \quad \text{with} \quad \|u\| > r.$$

Let $b \in X^*$ and suppose that ($\mathcal{H}1$), ($\mathcal{H}2$), ($\mathcal{H}3$) and ($\mathcal{H}4$) are satisfied, then the original problem (2.1) admits at least one solution. If A and B are single valued, then (2.1) is equivalent to the equation

$$b = Au + Bu, \qquad u \in C.$$

This theorem represents a fundamental result in the theory of monotone operators .

Proof: (See[9], page 868) ■

Now we will introduce the non perturbed inclusion:

$$b \in Au, \qquad u \in C \tag{2.2}$$

Corollary 2.1 *([9], page 868) we suppose that:*

(i) C est un ensemble non vide convexe et fermé dans un espace de Banach réflexif réel X.

(ii) The multi-function operator $A : C \to \mathcal{P}(X^*)$ is maximal monotone.

(iii) If the set C is non bounded, them the operator A is coercive with respect to the fixed element $b \in X^*$, that is, there exists $u_0 \in D(A)$ and r>0 such that:

$$\langle u^*, u - u_0 \rangle > \langle b, u - u_0 \rangle \qquad \text{for all} \quad (u, u^*) \in G(A) \quad \text{with} \quad \|u\| > r$$

Then, the inclusion (2.2) admits at least one solution.

Proof: (See[9], page 868) ■

2.2 Variational inequalities and inclusions

Variational inequalities and inclusions are related concepts, with inclusions being a generalization of variational inequalities. Both concepts are important in optimization and analysis.

The term "variational" in variational inequality comes from the fact that the problem can be formulated as a variational problem which is a type of mathematical optimization problem that involves finding the function that minimizes or maximizes a certain functional.

Definition 2.2 *[24, page 283]Given a Banach space X, a subset K of X, and a functional $F : K \to X^*$, the variational inequality problem is the problem of seeking the variable x belonging to K that satisfies:*

$$\langle F(x), y - x\rangle_X \leq 0 \qquad \text{for all } y \in K$$

Definition 2.3 *[9, page 856] Let $f : X \to [-\infty, \infty]$ be a function in the real Banach space X.*

▷ *The function u^* in X^* is called the **sub-gradient** of f in the point u if and only if $f \neq \pm\infty$ and the following inequality is satisfied:*

$$f(v) \geqslant f(u) + \langle u^*, v - u\rangle_X \qquad \text{for all} \quad v \in X. \tag{2.3}$$

▷ *The set of sub-gradients of f in the point u is called **sub-differential** $\partial f(u)$ in a point u.*

2.2.1 Illustrative examples

First example

Considering the following variational inequality

$$\langle b - Au, v - u\rangle_X + \varphi(u) \leq \varphi(v) \qquad \text{for all } v \in M \tag{2.4}$$

for $u \in M$. Alongside this, the multi-function inclusion :

$$b \in Au + \partial\varphi(u), \qquad u \in M \tag{2.5}$$

According to these assumptions:

($\mathcal{H}_1$) X is a real and reflexive Banach space.

($\mathcal{H}_2$) M is a non-empty, closed, and convex subset of X.

($\mathcal{H}_3$) $\varphi : M \to]-\infty, +\infty]$ is convex, lower semi-continuous and $\varphi \not\equiv +\infty$.

In what follows, we assume that the extension of φ in X is defined by $\varphi(v) = +\infty$(by definition) for $v \in X - M$. Then $\varphi : X \to]-\infty, +\infty]$ is convex and lower semi-continuous.

($\mathcal{H}_4$) $A : M \subseteq X \to X^*$ is pseudo-monotone, demi-continuous, and bounded.

For now, these assumptions are satisfied when $A : M \subseteq X \to X^*$ is monotone, hemi-continuous, and bounded.

($\mathcal{H}_5$) **Coercivity.** If M is non bounded, then there exists $u_0 \in M$,$v_0 \in X^*$ such that $v_0 \in \partial\varphi(u_0)$, that is, $\varphi(u_0) < +\infty$ and

$$\varphi(u_0) + \langle v_0, u - u_0\rangle_X \leq \varphi(v) \qquad \text{for all } v \in M$$

also

$$\frac{\langle Au, v - u_0\rangle_X}{\|u\|} \to +\infty \qquad \text{when } \|u\| \to +\infty, \quad u \in M.$$

$(\mathcal{H}_6)$ b is a fixed element in X^*.

Theorem 2.2 *[25, page 551] With the hypotheses $(\mathcal{H}_1)$-$(\mathcal{H}_6)$, the following two assertions are true:*

(a) ***Equivalence.*** *(2.4) and (2.5) are equivalents.*

(b) ***Existence.*** *(2.4) admits at least one solution.*

Proof: (see [25], page 551) ■

Second example

We are going to extend the inequality (2.4) by replacing the map φ by a map of two variables:

$$\langle b - Au, v - u\rangle_X + \varphi(u,u) \leq \varphi(u,v) \qquad \text{for all } v \in M \tag{2.6}$$

where:

- X is a real and reflexive Banach space.
- M is a non-empty, closed and convex subset.
- $A : X \to X^*$ is an operator.

Theorem 2.3 *We assume the following assumptions:*

$$A \text{ is a monotone, hemi-continuous and bounded operator.} \tag{2.7}$$

$$\varphi \text{ is inferiorly weakly semi-continuous in } M \times M. \tag{2.8}$$

$$\begin{gathered} \forall v \in X, \varphi(.,v) : M \to]-\infty, +\infty] \text{ is superiorly weakly} \\ \text{semi-continuous in } M. \end{gathered} \tag{2.9}$$

$$\begin{gathered} \forall u \in M, \varphi(u,.) : M \to]-\infty, +\infty] \text{ is convex and inferiorly} \\ \text{semi-continuous in } M \text{ and } \varphi(u,.) \not\equiv +\infty. \end{gathered} \tag{2.10}$$

Then, the quasi-variational inequality (2.6) admits at least one solution in M if one of the two conditions are satisfied:

$$M \text{ is bounded.} \tag{2.11}$$

$$\exists v_0 \in M \text{ such that } \lim_{\substack{\|v\| \longrightarrow +\infty \\ v \in M}} \frac{\langle Av, v - v_0 \rangle_X + \varphi(v,v) - \varphi(v,v_0)}{\|v\|} = +\infty \tag{2.12}$$

Sketch of the proof:

- For a fixed $u \in M$, We consider the following auxiliary problem:

$$\begin{cases} \text{find } w \in M \text{ such that} \\ \langle b - Aw, v - w \rangle_X + \varphi(u,w) \leq \varphi(u,v) \quad \forall v \in M \end{cases} \tag{2.13}$$

  Which is equivalent to the inclusion

$$b \in Aw + \partial \varphi_u(w) \quad \forall w \in M \tag{2.14}$$

  such that $\forall u \in M, \varphi_u(.) = \varphi(u,.)$.

- We will define an operator $S : M \to 2^M$ such that

$$S(u) = \{w \in M : w \text{ is a solution of}(2.13)\}$$

- We show that S has a fixed point.

2.3 Model of a hybrid integro-differential inclusion

Now we shall investigate the solutions of an inclusion in which we can't use the previous theorems on. Let's consider the following non-local problem of the Chandrasekhar hybrid second-order functional integro-differential inclusion:

$$-\frac{d^2}{dt^2}\left(\frac{x(t)}{g(t,x(t))}\right) \in \int_0^1 \frac{t}{t+s} \Phi\left(s, \int_0^1 \frac{s}{s+\tau} \psi(\tau, x(\tau)) d\tau\right) ds, \qquad t \in [0,1] \tag{2.15}$$

with the non-local hybrid boundary value conditions:

$$\begin{cases} \mathcal{D}\left(\frac{x(t)}{g(t,x(t))}\right)\Big|_{t=0} = 0, \\ \lambda^{c}\mathcal{D}^{\varrho}\left(\dfrac{x(t)}{g(t,x(t))}\right)\Big|_{t=\sigma} + \left(\dfrac{x(t)}{g(t,x(t))}\right)\Big|_{t=\eta} = 0, \qquad \varrho \in (0,1]\,, \sigma \ \in (0,1]\,, \eta \in (0,1]\,, \end{cases} \tag{2.16}$$

where $\mathcal{D} = \frac{d}{dt}$, λ is a positive real parameter, ${}^{c}\mathcal{D}^{\varrho}$ is the Caputo derivative of order ϱ, $\Phi : I \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multi-valued map, $\psi : I \times \mathbb{R} \to \mathbb{R}$ and $g : I \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$.

2.3.1 The existence of the solution

Considering the problem (2.15)-(2.16) together with the following assumptions:

$(\mathcal{H}_1)$ Let $\Phi : I \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a nonempty,closed, and convex subset for all $(t, u) \in I \times \mathbb{R}$ such that:

- $\Phi(t, .)$ is upper semicontinous in $u \in \mathbb{R}, \forall t \in I$.
- $\Phi(., u)$ is measurable in $t \in I, \forall u \in \mathbb{R}$.
- there exist two integrable functions $m, k_1 : I \to I$ such that:

$$|\Phi(t,u)| = \sup\{|\phi| : \phi \in \Phi(t,u)\} \leq m(t) + k_1|u|, \quad t \in I$$

  and:

$$\int_0^1 |m(\tau)|d\tau = m^*, \qquad \int_0^1 |k_1(\tau)|d\tau = k_1^*$$

$(\mathcal{H}_2)$ $\psi \in C(I \times \mathbb{R}, \mathbb{R})$ and there exists a continuous function $k_1 : I \times I \to \mathbb{R}$ and a continuous non decreasing map $\chi : [0, \infty) \to [0, \infty)$, such that:

$$|\psi(t,u)| \leq k_2(t)\chi(\|u\|), \qquad \forall t \in I, \forall \tau \in \mathbb{R}$$

and:

$$\int_0^1 |k_2(\tau)|d\tau = k_2^*.$$

$(\mathcal{H}_3)$ $g \in C(I \times \mathbb{R}, \mathbb{R}\backslash\{0\})$ and there is a positive constant ω, such that:

$$|g(t, \mu_1) - g(t, \mu_2)| \leq \omega|\mu_1(t) - \mu_2(t)|, \qquad \forall t \in I, \forall \mu_1, \mu_2 \in \mathbb{R}.$$

$(\mathcal{H}_4)$ There is a positive root r of the equation:

$$(m^* + k_1^* k_2^* \chi(r))(r\omega + G)\Lambda = r$$

where

$$G = \sup_{t \in I} |g(t, 0)|, \quad \Lambda = \lambda + 2$$

Remak 2.1 *From $(\mathcal{H}_1)$ the set of selections of Φ is non empty.*

Proof: *Φ is a measurable set valued function, by using the theorem (1.4) we conclude that it has a casting representation.*

As a result there exists a sequence f_m measurable and $f_m(t) \in \Phi(t, u), \forall m \geq 1$. So f is strongly measurable selection.

From $(\mathcal{H}_1)$, the multimap is upper semi continuous.

By the theorem (1.5), there exists $\varphi : I \to \mathbb{R}$ such that: $\varphi(t) \in \Phi(t, u(t))$.

The only thing left is to prove that this selection φ is in fact in $L^1(I, \mathbb{R})$ which is a direct result of third condition in $(\mathcal{H}_1)$, so $\int_0^1 |\varphi(t)|dt \leq m + k_1 \leq \infty$.

Which proves that S_ϕ is not empty.

■

Thus there exists a function $\phi(t, u)$ which is measurable in $t \in I, \forall u \in \mathbb{R}$, and continuous in $u \in \mathbb{R}, \forall t \in I$, in other words $\phi(t, u)$ is a Carathédory function, such that:

$$|\phi(t, u)| \leq m(t) + k_1|u|, \quad t \in I$$

And it satisfies the non-local problem of the Chandrasekhar hybrid second-order functional integro-differential equation:

$$-\frac{d^2}{dt^2}\left(\frac{x(t)}{g(t,x(t))}\right) = \int_0^1 \frac{t}{t+s}\phi\left(s, \int_0^1 \frac{s}{s+\tau}\psi(\tau,x(\tau))d\tau\right) ds, \qquad t \in [0,1] \tag{2.17}$$

with the condition (2.16). It is clear that if the problem (2.17)-(2.16) has a solution, then the main problem (2.15)-(2.16) has a solution as well.

Remak 2.2 *By using the assumption* $(\mathcal{H}_3)$, *and having* $\mu_1 = \mu, \mu_2 = 0$ *one can get:*

$$|g(t,\mu)| - |g(t,0)| \leq |g(t,\mu) - g(t,0)| \leq \omega|\mu(t) - 0|$$

Then

$$|g(t,\mu)| \leq \omega|\mu(t)| + |g(t,0)|$$

$$|g(t,\mu)| \leq \omega|\mu(t)| + \sup_{t\in I}|g(t,0)|$$

Finally

$$|g(t,\mu)| \leq \omega|\mu(t)| + G.$$

Now, we present a key lemma for the existence of the solution $x(t) \in C(I)$:

Lemma 2.1 *[26, page 5]*

A function $x \in C(I)$ *is a solution for the hybrid differential equation:*

$$\frac{d^2}{dt^2}\left(\frac{x(t)}{g(t,x(t))}\right) + \varphi(t,x(t)) = 0, \qquad t \in I. \tag{2.18}$$

with the nonlocal hybrid condition (2.16), if and only if $x \in C(I)$ *is a solution for the* *integral equation:*

$$x(t) = g(t, x(t))[-\int_0^t (t-s)\varphi(s, x(s))ds \\ + \lambda \int_0^\sigma \frac{(\sigma - s)^{1-\varrho}}{\Gamma(2-\varrho)} \varphi(s, x(s))ds + \int_0^\eta (\eta - s)\varphi(s, x(s))ds]. \tag{2.19}$$

Proof:

▷ Let x be the solution for the hybrid fractional equation (2.18), then by integration:

$$\int_0^t \frac{d^2}{ds^2}(\frac{x(s)}{g(s, x(s))})ds = -\int_0^t \varphi(s, x(s))ds$$
$$\frac{d}{ds}(\frac{x(s)}{g(s, x(s))})\Big|_0^t = \frac{d}{dt}(\frac{x(t)}{g(t, x(t))}) - \frac{d}{dt}(\frac{x(t)}{g(t, x(t))})\Big|_{t=0} = -\int_0^t \varphi(s, x(s))ds.$$

Using the condition(2.16), we get

$$\frac{d}{dt}(\frac{x(t)}{g(t, x(t))}) = -\int_0^t \varphi(s, x(s))ds. \tag{2.20}$$

And by integration for the second time

$$(\frac{x(t)}{g(t, x(t))}) = -\int_0^t (t-s)\varphi(s, x(s))ds + c_0. \tag{2.21}$$

The Riemann Liouville integration of φ is given by:

$$I^2\varphi(t, x(t)) = \frac{1}{\Gamma(2)}\int_0^t (t-s)\varphi(s, x(s))ds = \int_0^t (t-s)\varphi(s, x(s))ds. \tag{2.22}$$

Then

$$(\frac{x(t)}{g(t, x(t))}) = c_0 - I^2\varphi(t, x(t)). \tag{2.23}$$

When $t = \eta$

$$(\frac{x(t)}{g(t, x(t))})|_{t=\eta} = c_0 - I^2\varphi(t, x(t))\Big|_{t=\eta}. \tag{2.24}$$

Now, according to the remark 1.1 we obtain

$$\lambda^c\mathcal{D}^\varrho(\frac{x(t)}{g(t,x(t))})\Big|_{t=\sigma} = \lambda I^{1-\varrho}\frac{d}{dt}(\frac{x(t)}{g(t,x(t))})\Big|_{t=\sigma}. \tag{2.25}$$

In the other hand we already have from (2.20)

$$\frac{d}{dt}(\frac{x(t)}{g(t,x(t))}) = -I\varphi(t,x(t)).$$

Then by using the proprieties of RLI, we get

$$\lambda I^{1-\varrho}\frac{d}{dt}(\frac{x(t)}{g(t,x(t))}) = \lambda I^{1-\varrho}(-I\varphi(t,x(t))) = -\lambda I^{2-\varrho}\varphi(t,x(t)). \tag{2.26}$$

Using the latter in (2.25), we obtain

$$\lambda^c\mathcal{D}^\varrho(\frac{x(t)}{g(t,x(t))})\Big|_{t=\sigma} = -\lambda I^{2-\varrho}\varphi(t,x(t))|_{t=\sigma}. \tag{2.27}$$

Substituting (2.24) and (2.27) in the condition(2.16)

$$-\lambda I^{2-\varrho}\varphi(t,x(t))|_{t=\sigma} + c_0 - I^2\varphi(t,x(t))|_{t=\eta} = 0. \tag{2.28}$$

So

$$c_0 = \lambda I^{2-\varrho}\varphi(t,x(t))|_{t=\sigma} + I^2\varphi(t,x(t))|_{t=\eta}. \tag{2.29}$$

From the equation (2.23), it results

$$c_0 = (\frac{x(t)}{g(t,x(t))}) + I^2\varphi(t,x(t)). \tag{2.30}$$

Together with (2.29) we can have

$$x(t) = g(t,x(t))[\lambda I^{2-\varrho}\varphi(t,x(t))|_{t=\sigma} + I^2\varphi(t,x(t))|_{t=\eta} - I^2\varphi(t,x(t))]. \tag{2.31}$$

And finally since

$$\begin{aligned} I^2\varphi(t,x(t)) &= \int_0^t (t-s)\varphi(s,x(s))ds. \\ I^2\varphi(t,x(t))|_{t=\eta} &= \int_0^\eta (\eta-s)\varphi(s,x(s))ds. \\ I^{2-\varrho}\varphi(t,x(t))|_{t=\sigma} &= \frac{1}{\Gamma(2-\varrho)}\int_0^t (t-s)^{1-\varrho}\varphi(s,x(s))ds|_{t=\sigma} = \int_0^\sigma \frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\varphi(s,x(s))ds. \end{aligned}$$

Hence, we obtain the desired result

$$x(t) = g(t,x(t))[-\int_0^t (t-s)\varphi(s,x(s))ds + \lambda\int_0^\sigma \frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\varphi(s,x(s))ds + \int_0^\eta (\eta-s)\varphi(s,x(s))ds].$$

▷ Conversely, from (2.19) one can get

$$(\frac{x(t)}{g(t,x(t))}) = -I^2\varphi(t,x(t)) + \lambda I^{2-\varrho}\varphi(\sigma,x(\sigma)) + I^2\varphi(\eta,x(\eta)). \tag{2.32}$$

By derivation

$$\frac{d}{dt}(\frac{x(t)}{g(t,x(t))}) = -I\varphi(t,x(t)). \tag{2.33}$$

Then

$$\frac{d^2}{dt^2}(\frac{x(t)}{g(t,x(t))}) = -\varphi(t,x(t)).$$

Which leads to (2.18).

Also we have from (2.33)

$$\frac{d}{dt}(\frac{x(t)}{g(t,x(t))})\Big|_{t=0} = -\int_0^t \varphi(s,x(s))ds\Big|_{t=0} = 0. \tag{2.34}$$

From (2.32), with $t=\eta$, one can gain

$$(\frac{x(t)}{g(t,x(t))})\Big|_{t=\eta} = -I^2\varphi(\eta,x(\eta)) + \lambda I^{2-\varrho}\varphi(\sigma,x(\sigma)) + I^2\varphi(\eta,x(\eta)) = \lambda I^{2-\varrho}\varphi(\sigma,x(\sigma)). \tag{2.35}$$

Now, operating by $\lambda^c\mathcal{D}^\varrho$ in the equation (2.32) with $t=\sigma$, along with the propriety:

$$\lambda^c\mathcal{D}^\varrho(\frac{x(t)}{g(t,x(t))})\Big|_{t=\sigma} = -\lambda^c\mathcal{D}^\varrho I^2\varphi(t,x(t))\Big|_{t=\sigma} = -\lambda I^{2-\varrho}\varphi(\sigma,x(\sigma)). \tag{2.36}$$

And we finally conclude the second bounday condition by adding (2.36) to (2.35). ■

Corollary 2.2 *If the solution $x \in C(I)$ of the non-local problem (2.17)-(2.16) exists, then it is given by the integral equation:*

$$\begin{aligned} x(t) = g(t,x(t))[&-\int_0^t (t-s)\int_0^1 \frac{s}{s+\tau}\phi\left(\tau, \int_0^1 \frac{\tau}{\tau+s}\psi(\varsigma,x(\varsigma))d\varsigma\right) d\tau ds \\ &+ \lambda\int_0^\sigma \frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\int_0^1 \frac{s}{s+\tau}\phi\left(\tau, \int_0^1 \frac{\tau}{\tau+s}\psi(\varsigma,x(\varsigma))d\varsigma\right) d\tau ds \\ &+ \int_0^\eta (\eta-s)\int_0^1 \frac{s}{s+\tau}\phi\left(\tau, \int_0^1 \frac{\tau}{\tau+s}\psi(\varsigma,x(\varsigma))d\varsigma\right) d\tau ds]. \end{aligned} \tag{2.37}$$

Proof: Immediately from Lemma(2.1) , with :

$$\varphi(t,x(t)) = \int_0^1 \frac{t}{t+s}\phi\left(s, \int_0^1 \frac{s}{s+\tau}\psi(\tau,x(\tau))d\tau\right) ds, \qquad t\in[0,1]$$

■

Theorem 2.4 *Assuming that the assumptions $(\mathcal{H}_1)-(\mathcal{H}_4)$ are satisfied, if: $(m+k_1k_2\chi(r))\Lambda \le 1$. Then, there exists at least one solution for the problem (2.17)-(2.16).*

Proof: We define the operator $\mathcal{A}$ as follows:

$$\begin{aligned} (\mathcal{A}x)(t) = g(t,x(t))[&-\int_0^t (t-s)\int_0^1 \frac{s}{s+\tau}\phi\left(\tau, \int_0^1 \frac{\tau}{\tau+s}\psi(\varsigma,x(\varsigma))d\varsigma\right) d\tau ds \\ &+ \lambda\int_0^\sigma \frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\int_0^1 \frac{s}{s+\tau}\phi\left(\tau, \int_0^1 \frac{\tau}{\tau+s}\psi(\varsigma,x(\varsigma))d\varsigma\right) d\tau ds \\ &+ \int_0^\eta (\eta-s)\int_0^1 \frac{s}{s+\tau}\phi\left(\tau, \int_0^1 \frac{\tau}{\tau+s}\psi(\varsigma,x(\varsigma))d\varsigma\right) d\tau ds]. \end{aligned} \tag{2.38}$$

Considering the ball $\mathcal{V}_r = \{x \in C(I) : \|x\| = \|x\|_{C(I)} \leq r\}$.

Clearly $\mathcal{V}_r$ is a closed, convex and bounded subset of the Banach space C(I).

Let $x \in \mathcal{V}_r$ and $t \in I$, then:

$$\begin{aligned}\big|(\mathcal{A}x)(t)\big| = \big|g(t,x(t))\big|\Big| &- \int_0^t (t-s)\int_0^1 \frac{s}{s+\tau}\phi\left(\tau, \int_0^1 \frac{\tau}{\tau+s}\psi(\varsigma, x(\varsigma))d\varsigma\right) d\tau ds \\ &+ \lambda\int_0^\sigma \frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\int_0^1 \frac{s}{s+\tau}\phi\left(\tau, \int_0^1 \frac{\tau}{\tau+s}\psi(\varsigma, x(\varsigma))d\varsigma\right) d\tau ds \\ &+ \int_0^\eta (\eta-s)\int_0^1 \frac{s}{s+\tau}\phi\left(\tau, \int_0^1 \frac{\tau}{\tau+s}\psi(\varsigma, x(\varsigma))d\varsigma\right) d\tau ds\Big|.\end{aligned}$$

$$\begin{aligned}\big|(\mathcal{A}x)(t)\big| \leq \big|g(t,x(t))\big|\Big| &- \int_0^t (t-s)\int_0^1 \frac{s}{s+\tau}\Big|\phi\left(\tau, \int_0^1 \frac{\tau}{\tau+\varsigma}\psi(\varsigma, x(\varsigma))d\varsigma\right)\Big| d\tau ds \\ &+ \lambda\int_0^\sigma \frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\int_0^1 \frac{s}{s+\tau}\Big|\phi\left(\tau, \int_0^1 \frac{\tau}{\tau+\varsigma}\psi(\varsigma, x(\varsigma))d\varsigma\right)\Big| d\tau ds \\ &+ \int_0^\eta (\eta-s)\int_0^1 \frac{s}{s+\tau}\Big|\phi\left(\tau, \int_0^1 \frac{\tau}{\tau+\varsigma}\psi(\varsigma, x(\varsigma))d\varsigma\right)\Big| d\tau ds\Big|.\end{aligned}$$

$$\begin{aligned}\big|(\mathcal{A}x)(t)\big| \leq \big|g(t,x(t))\big|\Big| &- \int_0^t (t-s)\int_0^1 \frac{s}{s+\tau}\Big[m(\tau) + k_1(\tau)\int_0^1 \frac{\tau}{\tau+\varsigma}\big|\psi(\varsigma, x(\varsigma))\big|d\varsigma\Big] d\tau ds \\ &+ \lambda\int_0^\sigma \frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\int_0^1 \frac{s}{s+\tau}\Big[m(\tau) + k_1(\tau)\int_0^1 \frac{\tau}{\tau+\varsigma}\big|\psi(\varsigma, x(\varsigma))\big|d\varsigma\Big] d\tau ds \\ &+ \int_0^\eta (\eta-s)\int_0^1 \frac{s}{s+\tau}\Big[m(\tau) + k_1(\tau)\int_0^1 \frac{\tau}{\tau+\varsigma}\big|\psi(\varsigma, x(\varsigma))\big|d\varsigma\Big] d\tau ds\Big|.\end{aligned}$$

By $(\mathcal{H}_2)$:

$$\begin{aligned}\big|(\mathcal{A}x)(t)\big| \leq \big[\omega|x(t)| + G\big]\Big| &- \int_0^t (t-s)\int_0^1 \frac{s}{s+\tau}\Big[|m(\tau)| + |k_1(\tau)|\int_0^1 \frac{\tau}{\tau+\varsigma}\big|k_2(\varsigma)\big|\chi(\|x\|)d\varsigma\Big] d\tau ds \\ + \lambda\int_0^\sigma \frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}&\int_0^1 \frac{s}{s+\tau}\Big[|m(\tau)| + |k_1(\tau)|\int_0^1 \frac{\tau}{\tau+\varsigma}\big|k_2(\varsigma)\big|\chi(\|x\|)d\varsigma\Big] d\tau ds \\ + \int_0^\eta (\eta-s)&\int_0^1 \frac{s}{s+\tau}\Big[|m(\tau)| + |k_1(\tau)|\int_0^1 \frac{\tau}{\tau+\varsigma}\big|k_2(\varsigma)\big|\chi(\|x\|)d\varsigma\Big] d\tau ds\Big|.\end{aligned}$$

Using the supremum:

$$\sup_{s\in I}(t-s)=t\leq 1 \qquad \sup_{\tau\in I}\frac{s}{s+\tau}=1 \qquad \sup_{\varsigma\in I}\frac{\tau}{\tau+\varsigma}=1$$

we get:

$$\begin{aligned}\big|(\mathcal{A}x)(t)\big| \leq \big[\omega|x(t)|+G\big]\Big(&\int_0^t\int_0^1\Big[|m(\tau)|+|k_1(\tau)|\int_0^1\big|k_2(\varsigma)\big|\chi(\|x\|)d\varsigma\Big]d\tau ds\\ &+\lambda\int_0^\sigma\frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\int_0^1\Big[|m(\tau)|+|k_1(\tau)|\int_0^1\big|k_2(\varsigma)\big|\chi(\|x\|)d\varsigma\Big]d\tau ds\\ &+\int_0^\eta\int_0^1\Big[|m(\tau)|+|k_1(\tau)|\int_0^1\big|k_2(\varsigma)\big|\chi(\|x\|)d\varsigma\Big]d\tau ds\Big).\end{aligned}$$

Calculating the integrals:

$$\begin{aligned}\big|(\mathcal{A}x)(t)\big| \leq \big[\omega|x(t)|+G\big]\Big(&\int_0^t\int_0^1\Big[|m(\tau)|+|k_1(\tau)|k_2\chi(\|x\|)\Big]d\tau ds\\ &+\lambda\int_0^\sigma\frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\int_0^1\Big[|m(\tau)|+|k_1(\tau)|k_2\chi(\|x\|)\Big]d\tau ds\\ &+\int_0^\eta(\eta-s)\int_0^1\Big[|m(\tau)|+|k_1(\tau)|k_2\chi(\|x\|)\Big]d\tau ds\Big).\\ &\leq \big[\omega|x(t)|+G\big]\Big(m^*+k_1^*k_2^*\chi(\|x\|)+\lambda\int_0^\sigma\frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}(m^*+k_1^*k_2^*\chi(\|x\|))ds\\ &+m^*+k_1^*k_2^*\chi(\|x\|)\Big).\end{aligned}$$

Calculating the remaining integral:

$$\int_0^\sigma\frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}ds=\frac{1}{\Gamma(2-\varrho)}\Big[-\frac{(\sigma-s)^{2-\varrho}}{2-\varrho}\Big]_0^\sigma=\frac{1}{\Gamma(2-\varrho)}\times\frac{(\sigma-s)^{2-\varrho}}{(2-\varrho)}$$

Using the properties of the Γ function and the fact that $\sigma\in I$ we get

$$\int_0^\sigma\frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}ds=\frac{(\sigma)^{2-\varrho}}{\Gamma(3-\varrho)}\leq\frac{1}{\Gamma(3-\varrho)}$$

Now, taking the supremum over $t \in I$ we obtain

$$\begin{aligned}
\big|\big|(\mathcal{A}x)(t)\big|\big| &\leq \big[r\omega + G\big]\left(m^* + k_1^*k_2^*\chi(r) + \lambda\frac{\lambda}{\Gamma(3-\varrho)}(m^* + k_1^*k_2^*\chi(r))ds + m^* + k_1^*k_2^*\chi(r))\right). \\
&\leq (m^* + k_1^*k_2^*\chi(r))(r\omega + G)\Lambda = r.
\end{aligned}$$

then:$\big|\big|(\mathcal{A}x)(t)\big|\big| = r$.

Hence $(\mathcal{A}x)(t) : \mathcal{V}_r \to \mathcal{V}_r$, and the class $\{\mathcal{A}x\}$ is uniformly bounded on $\mathcal{V}_r$. ■

Now, for the continuity of $\mathcal{A}$, let $\{x_n\}$ be a converging sequence towards a certain point $x \in \mathcal{V}_r$.

By using the continuity of the function $g(t, x(t))$ and the Lebesgue Dominated Convergence Theorem together with the assumptions $(\mathcal{H}_1), (\mathcal{H}_2)$ on ϕ and ψ (since m, k_1 and k_2 are Lebesgue integrable functions) we gain

$$\begin{aligned}
\lim_{n\to\infty}(\mathcal{A}x_n)(t) &= \lim_{n\to\infty} g(t, x_n(t))[-\lim_{n\to\infty}\int_0^t (t-s)\int_0^1 \frac{s}{s+\tau}\phi\left(\tau, \int_0^1 \frac{\tau}{\tau+s}\psi(\varsigma, x_n(\varsigma))d\varsigma\right)d\tau ds \\
&+ \lim_{n\to\infty}\lambda\int_0^\sigma \frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\int_0^1 \frac{s}{s+\tau}\phi\left(\tau, \int_0^1 \frac{\tau}{\tau+s}\psi(\varsigma, x_n(\varsigma))d\varsigma\right)d\tau ds \\
&+ \lim_{n\to\infty}\int_0^\eta (\eta-s)\int_0^1 \frac{s}{s+\tau}\phi\left(\tau, \int_0^1 \frac{\tau}{\tau+s}\psi(\varsigma, x_n(\varsigma))d\varsigma\right)d\tau ds]. \\
&= g(t, x(t))[-\int_0^t (t-s)\int_0^1 \frac{s}{s+\tau}\lim_{n\to\infty}\phi\left(\tau, \int_0^1 \frac{\tau}{\tau+s}\psi(\varsigma, x_n(\varsigma))d\varsigma\right)d\tau ds \\
&+ \lambda\int_0^\sigma \frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\int_0^1 \frac{s}{s+\tau}\lim_{n\to\infty}\phi\left(\tau, \int_0^1 \frac{\tau}{\tau+s}\psi(\varsigma, x_n(\varsigma))d\varsigma\right)d\tau ds \\
&+ \int_0^\eta (\eta-s)\int_0^1 \frac{s}{s+\tau}\lim_{n\to\infty}\phi\left(\tau, \int_0^1 \frac{\tau}{\tau+s}\psi(\varsigma, x_n(\varsigma))d\varsigma\right)d\tau ds]. \\
&= g(t, x(t))[-\int_0^t (t-s)\int_0^1 \frac{s}{s+\tau}\phi\left(\tau, \int_0^1 \frac{\tau}{\tau+s}\lim_{n\to\infty}\psi(\varsigma, x_n(\varsigma))d\varsigma\right)d\tau ds \\
&+ \lambda\int_0^\sigma \frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\int_0^1 \frac{s}{s+\tau}\phi\left(\tau, \int_0^1 \frac{\tau}{\tau+s}\lim_{n\to\infty}\psi(\varsigma, x_n(\varsigma))d\varsigma\right)d\tau ds \\
&+ \int_0^\eta (\eta-s)\int_0^1 \frac{s}{s+\tau}\phi\left(\tau, \int_0^1 \frac{\tau}{\tau+s}\lim_{n\to\infty}\psi(\varsigma, x_n(\varsigma))d\varsigma\right)d\tau ds].
\end{aligned}$$

$$
\begin{aligned}
&= g(t,x(t))[-\int_0^t (t-s)\int_0^1 \frac{s}{s+\tau}\phi\left(\tau,\int_0^1 \frac{\tau}{\tau+s}\psi(\varsigma,x(\varsigma))d\varsigma\right)d\tau ds \\
&+\lambda\int_0^\sigma \frac{(\sigma-s)^{1-\varrho}}{\Gamma(2-\varrho)}\int_0^1 \frac{s}{s+\tau}\phi\left(\tau,\int_0^1 \frac{\tau}{\tau+s}\psi(\varsigma,x(\varsigma))d\varsigma\right)d\tau ds \\
&+\int_0^\eta (\eta-s)\int_0^1 \frac{s}{s+\tau}\phi\left(\tau,\int_0^1 \frac{\tau}{\tau+s}\psi(\varsigma,x(\varsigma))d\varsigma\right)d\tau ds]. \\
&= (\mathcal{A}x)(t).
\end{aligned}
$$

As a result:

$$\lim_{n\to\infty}(\mathcal{A}x_n)(t) = (\mathcal{A}x)(t).$$

Hence, $\mathcal{A}$ is continuous.

We define the following set:

$$\theta_g(\delta) = \sup\{\left|g(t_2,x) - g(t_1,x)\right| : t_1,t_2 \in I, t_1 < t_2, |t_1 - t_2| < \delta, |x| < \epsilon\}$$

Therefore, based on the uniform continuity of the function $\phi : I \times \mathcal{V}_r \to \mathbb{R}$ using the assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_3)$, we can conclude that $\theta_g(\delta) \to 0$ as $\delta \to 0$ independent of $x \in \mathcal{V}_r$.

Let $t_1, t_2 \in I, |t_2 - t_1| < \delta$. Then

$$
\begin{aligned}
\left|(\mathcal{A}x)(t_2) - (\mathcal{A}x)(t_1)\right| = &\left|g(t_2,x(t_2))\int_0^{t_2}(t_2-s)\int_0^1 \frac{s}{s+\tau}\phi\left(\tau,\int_0^1 \frac{\tau}{\tau+s}\psi(\varsigma,x(\varsigma))d\varsigma\right)d\tau ds\right. \\
&\left.- g(t_1,x(t_1))\int_0^{t_1}(t_1-s)\int_0^1 \frac{s}{s+\tau}\phi\left(\tau,\int_0^1 \frac{\tau}{\tau+s}\psi(\varsigma,x(\varsigma))d\varsigma\right)d\tau ds\right|
\end{aligned}
$$

Then adding the following values:

$$
\begin{aligned}
&\left|(\mathcal{A}x)(t_2) - (\mathcal{A}x)(t_1)\right| \\
&= \left|g(t_2,x(t_2))\int_0^{t_2}(t_2-s)\int_0^1 \frac{s}{s+\tau}\phi\left(\tau,\int_0^1 \frac{\tau}{\tau+s}\psi(\varsigma,x(\varsigma))d\varsigma\right)d\tau ds\right. \\
&- g(t_2,x(t_2))\int_0^{t_2}(t_1-s)\int_0^1 \frac{s}{s+\tau}\phi\left(\tau,\int_0^1 \frac{\tau}{\tau+s}\psi(\varsigma,x(\varsigma))d\varsigma\right)d\tau ds
\end{aligned}
$$

$$
\begin{aligned}
&+ g(t_2, x(t_2)) \int_0^{t_2} (t_1 - s) \int_0^1 \frac{s}{s+\tau} \phi\left(\tau, \int_0^1 \frac{\tau}{\tau+s} \psi(\varsigma, x(\varsigma)) d\varsigma\right) d\tau ds \\
&- g(t_2, x(t_2)) \int_0^{t_1} (t_1 - s) \int_0^1 \frac{s}{s+\tau} \phi\left(\tau, \int_0^1 \frac{\tau}{\tau+s} \psi(\varsigma, x(\varsigma)) d\varsigma\right) d\tau ds \\
&+ g(t_2, x(t_2)) \int_0^{t_1} (t_1 - s) \int_0^1 \frac{s}{s+\tau} \phi\left(\tau, \int_0^1 \frac{\tau}{\tau+s} \psi(\varsigma, x(\varsigma)) d\varsigma\right) d\tau ds \\
&- g(t_1, x(t_1)) \int_0^{t_1} (t_1 - s) \int_0^1 \frac{s}{s+\tau} \phi\left(\tau, \int_0^1 \frac{\tau}{\tau+s} \psi(\varsigma, x(\varsigma)) d\varsigma\right) d\tau ds\Big| \\
&\leq \big|g(t_2, x(t_2))\big| \times \int_0^{t_2} \big((t_2 - s) - (t_1 - s)\big) \int_0^1 \frac{s}{s+\tau} \Big|\phi\left(\tau, \int_0^1 \frac{\tau}{\tau+s} \psi(\varsigma, x(\varsigma)) d\varsigma\right)\Big| d\tau ds \\
&+ \big|g(t_2, x(t_2))\big| \int_{t_1}^{t_2} (t_1 - s) \int_0^1 \frac{s}{s+\tau} \Big|\phi\left(\tau, \int_0^1 \frac{\tau}{\tau+s} \psi(\varsigma, x(\varsigma)) d\varsigma\right)\Big| d\tau ds \\
&+ \big|g(t_2, x(t_2)) - g(t_1, x(t_1))\big| \times \int_0^{t_1} (t_1 - s) \int_0^1 \frac{s}{s+\tau} \Big|\phi\left(\tau, \int_0^1 \frac{\tau}{\tau+s} \psi(\varsigma, x(\varsigma)) d\varsigma\right)\Big| d\tau ds
\end{aligned}
$$

By using the assumptions, we get

$$
\begin{aligned}
&\big|(\mathcal{A}x)(t_2) - (\mathcal{A}x)(t_1)\big| \\
&\leq \big[|x(t_2)\omega + G|\big] \times \int_0^{t_2} \big((t_2 - s) - (t_1 - s)\big) \int_0^1 \frac{s}{s+\tau} \left[m(\tau) + k_1(\tau) \int_0^1 \frac{\tau}{\tau+\varsigma} \big|\psi(\varsigma, x(\varsigma))\big| d\varsigma\right] d\tau ds \\
&+ \big[|x(t_2)\omega + G|\big] \int_{t_1}^{t_2} (t_1 - s) \int_0^1 \frac{s}{s+\tau} \left[m(\tau) + k_1(\tau) \int_0^1 \frac{\tau}{\tau+\varsigma} \big|\psi(\varsigma, x(\varsigma))\big| d\varsigma\right] d\tau ds \\
&+ \theta_g(\delta) \times \int_0^{t_1} (t_1 - s) \int_0^1 \frac{s}{s+\tau} \left[m(\tau) + k_1(\tau) \int_0^1 \frac{\tau}{\tau+\varsigma} \big|\psi(\varsigma, x(\varsigma))\big| d\varsigma\right] d\tau ds \\
&\leq \big[||x(t_2)||\omega + G\big] \times \int_0^{t_2} \big((t_2 - s) - (t_1 - s)\big) \int_0^1 \left[|m(\tau)| + |k_1(\tau)| \int_0^1 \big|k_2(\varsigma)\big| \chi(\|x\|) d\varsigma\right] d\tau ds \\
&+ \big[||x(t_2)||\omega + G\big] \int_{t_1}^{t_2} (t_1 - s) \int_0^1 \left[|m(\tau)| + |k_1(\tau)| \int_0^1 \big|k_2(\varsigma)\big| \chi(\|x\|) d\varsigma\right] d\tau ds \\
&+ \theta_g(\delta) \times \int_0^{t_1} (t_1 - s) \int_0^1 \left[|m(\tau)| + |k_1(\tau)| \int_0^1 \big|k_2(\varsigma)\big| \chi(\|x\|) d\varsigma\right] d\tau ds \\
&\leq \big[||x(t_2)||\omega + G\big] \times \int_0^{t_2} \big((t_2 - s) - (t_1 - s)\big) \int_0^1 \big[|m(\tau)| + |k_1(\tau)| k_2 \chi(\|x\|)\big] d\tau ds \\
&+ \big[||x(t_2)||\omega + G\big] \int_{t_1}^{t_2} (t_1 - s) \int_0^1 \big[|m(\tau)| + |k_1(\tau)| k_2 \chi(\|x\|)\big] d\tau ds \\
&+ \theta_g(\delta) \times \int_0^{t_1} (t_1 - s) \int_0^1 \big[|m(\tau)| + |k_1(\tau)| k_2 \chi(\|x\|)\big] d\tau ds \\
&\leq \big[r\omega + G\big] \big[m^* + k_1^* k_2^* \chi(r)\big] \left[\int_0^{t_2} \big((t_2 - s) - (t_1 - s) ds + \int_{t_1}^{t_2} (t_1 - s) ds\right] \\
&+ \theta_g(\delta) \big[m + k_1 k_2 \chi(r)\big] \int_0^{t_1} (t_1 - s) ds.
\end{aligned}
$$

Calculating the integrals, it results

$$\begin{aligned}\big|(\mathcal{A}x)(t_2)-(\mathcal{A}x)(t_1)\big| &\le \big[r\omega+G\big]\big[m^*+k_1^*k_2^*\chi(r)\big]\Big|\delta t_2+\big(\frac{-(t_1-s)^2}{2}\big)\Big|_{t_1}^{t_2}\Big| \\ &+\theta_g(\delta)\big[m^*+k_1^*k_2^*\chi(r)\big]\Big|\big(\frac{-(t_1-s)^2}{2}\big)\Big|_0^{t_1}\Big|. \\ &\le \big[r\omega+G\big]\big[m^*+k_1^*k_2^*\chi(r)\big](\delta+\frac{\delta^2}{2})+\theta_g(\delta)\big[m^*+k_1^*k_2^*\chi(r)\big]\times\frac{1}{2}.\end{aligned}$$

Therefore, since it only depends on δ, the class $\{\mathcal{A}x\}$ is equi-continuous.

By relying on the Arzela-Ascoli Theorem, we conclude that the operator $\mathcal{A}$ is compact.

Since the operator $\mathcal{A}:\mathcal{V}_r\to\mathcal{V}_r$ is continuous, then it satisfies the hypotheses of Schauder Fixed Point Theorem, hence $\mathcal{A}$ admits at least one fixed point $x\in\mathcal{V}_r$.

As a result of the corollary(2.2), the problem (2.15)-(2.16) has a solution in $C(I)$.

2.3.2 Example

By this example we proceed to investigate the existence of the solution for the following Chandrasekhar hybrid second order integrodifferential inclusion

$$-\frac{d^2}{dt^2}\left(\frac{x(t)}{\frac{t|x(t)|^2}{1+|x(t)|^2}+4}\right)\in\left[\int_0^1\frac{t}{t+s}\left(\frac{s}{100}+\frac{1}{10}\int_0^1\frac{s}{s+\tau}\frac{\tau cos^2(2\pi\tau)cos(x(\tau))}{200}d\tau\right)ds,0\right],t\in[0,1] \tag{2.39}$$

with the boundary conditions

$$\begin{cases}\mathcal{D}\left(\frac{x(t)}{\frac{t|x(t)|^2}{1+|x(t)|^2}+4}\right)\Big|_{t=0}=0,\\ \frac{7}{3}\times{}^c\mathcal{D}^{\frac{4}{3}}\left(\frac{x(t)}{\frac{t|x(t)|^2}{1+|x(t)|^2}+4}\right)\Big|_{t=1}+\left(\frac{x(t)}{\frac{t|x(t)|^2}{1+|x(t)|^2}+4}\right)\Big|_{t=0.76}=0,,\end{cases} \tag{2.40}$$

Consider the continuous map $g:[0,1]\times\mathbb{R}\to\mathbb{R}\backslash\{0\}$ such that

$$g(t,x(t))=\frac{x(t)}{\frac{t|x(t)|^2}{1+|x(t)|^2}+4}$$

And the set valued map $\Phi : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ by

$$\Phi\Big(t, \int_0^1 \frac{t}{t+\tau}\psi(\tau, x(\tau))d\tau\Big) = \left[\frac{t}{100} + \frac{1}{10}\int_0^1 \frac{t}{t+\tau}\frac{\tau cos^2(2\pi\tau)cos(x(\tau))}{200}ds, 0\right]$$

Then, for $\phi \in \Phi(t, x(t))$ we have:

$$\phi(t, x(t)) = \frac{t}{100} + \frac{1}{10}x(t), \quad \psi(t, x(t)) = \frac{tcos^2(2\pi t)cos(x(t))}{200}$$

We have

$$g(t, x(t)) = \frac{t|x(t)|^2}{1+|x(t)|^2} + 4 = \frac{|x(t)|^2}{1+|x(t)|^2} + 4 = |x(t)| \times \frac{|x(t)|}{1+|x(t)|^2} + 4 = |x(t)| + 4$$

Which leads to $\omega = 1$ and $G = 4$. Obviously $m(t) = \frac{t}{100}$ and $k_1(t) = \frac{1}{10}$ because:

$$|\phi(t, x(t))| = |\frac{t}{100} + \frac{1}{10}x(t)| \leq \frac{t}{100} + \frac{1}{10}|x(t)|$$

We also can get $k_2(t) = \frac{1}{200}$ and $\chi(||x||) = 1$ because:

$$|\psi(t, x(t))| = |\frac{tcos^2(2\pi t)cos(x(t))}{200}| \leq \frac{1}{200}$$

In this case we obtain $\Lambda = 0.0357 \leq 1$. By using the theorem 2.4, then the fractional hybrid inclusion(2.39) with the boundary conditions(2.40) has at least one solution.

Chapter 3

Analytical and numerical study of some fractional differential inclusions

3.1 Analytical study

3.1.1 Main problem

Inspired by the work of Infante and Webb in 2006 [27], and that of Nieto and Pimentel in 2013 [28] concerning the thermostat model, we investigate the existence aspects of solutions for the following generalized Caputo fractional inclusion:

$$\begin{cases} {}^{c}\mathcal{D}_{0^+}^{\mu} u(\mathbf{t}) \in \mathcal{F}\big(\mathbf{t}, u(\mathbf{t}), {}^{c}\mathcal{D}_{0^+}^{\sigma} u(\mathbf{t}), \big), \qquad \mathbf{t} \in \mathbb{O} = [0,1], \\ {}^{c}\mathcal{D}_{0^+}^{1} u(0) = \beta_1 \sum_{k=1}^{m} u(\xi_k), \qquad \beta_2^{c}\mathcal{D}_{0^+}^{\mu-1} u(1) + u(\xi) = \beta_3 \sum_{k=1}^{m} u(\xi_k), \end{cases} \tag{3.1}$$

in which $\mu \in (1,2]$, $\sigma \in (0,1)$, $0 < \xi_1 < \xi_2 < \cdots < \xi_m < \xi < 1$, $m \in \mathbb{N}$, $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$ and ${}^{c}\mathcal{D}_{0^+}^{1} = \frac{\mathrm{d}}{\mathrm{d}\mathbf{t}}$, ${}^{c}\mathcal{D}_{0^+}^{\varrho}$ displays the Caputo fractional derivation of order $\varrho \in \{\mu, \sigma, 1, \mu-1\}$ and $\mathcal{F} : \mathbb{O} \times \mathbb{R}^2 \longrightarrow \mathcal{P}(\mathbb{R})$ is a compact set-valued function.

It is known that $\mathbb{X} = \{u : u, {}^{c}\mathcal{D}_{0^+}^{\sigma} u \in C(\mathbb{O}, \mathbb{R})\}$ together with the norm $||u||_{\mathbb{X}} = \sup_{t\in\mathbb{O}} |u(t)| + \sup_{t\in\mathbb{O}} |{}^{c}\mathcal{D}_{0^+}^{\sigma} u(t)|$ is a Banach space (see [29, page 65-66]).

3.1.2 Existence result

Lemma 3.1 *[28, page 3]*

$$I^{\mu c}\mathcal{D}^{\mu}_{0+}u(t) = u(t) + a_0 + a_1 t + a_2 t^2 + \cdots + a_{n-1}t^{n-1}$$

Where $a_i \in \mathbb{R}, i = 0, 1, \ldots, n-1 (n-1 < \mu < n)$.

We provide the general solution of

$$^{c}\mathcal{D}^{\mu}_{0+}u(\mathbf{t}) = h(t),$$

which is given by:

$$u(t) = \frac{1}{\Gamma(\mu)}\int_0^t (t-\tau)^{\mu-1}h(\tau)d\tau + c_0 + c_1 t. \tag{3.2}$$

Proof: Let h, u be two functions in $C[0,1]$, such that:

$$^{c}\mathcal{D}^{\mu}_{0+}u(\mathbf{t}) = h(t).$$

Since $1 < \mu \leq 2$, then $n = 2$.

Using Lemma 3.1, we obtain:

$$u(t) + a_0 + a_1 t = I^{\mu}h(t)$$

$$u(t) = I^{\mu}h(t) + c_0 + c_1 t,$$

with: $c_0 = -a_0$, $c_1 = -a_1$, for some constants $c_0, c_1 \in \mathbb{R}$. As a result:

$$u(t) = \frac{1}{\Gamma(\mu)}\int_0^t (t-\tau)^{\mu-1}h(\tau)d\tau + c_0 + c_1 t.$$

■

Now we calculate the first and second derivatives:

$$u'(t) = \frac{1}{\Gamma(\mu-1)} \int_0^t (t-\tau)^{\mu-2} h(\tau) d\tau + c_1, \tag{3.3}$$
$$u''(t) = \frac{1}{\Gamma(\mu-2)} \int_0^t (t-\tau)^{\mu-3} h(\tau) d\tau. \tag{3.4}$$

For the first boundary condition:

$${}^c\mathcal{D}^1_{0+} u(0) = u'(0) = \beta_1 \sum_{k=1}^{m} u(\xi_k). \tag{3.5}$$

By substituting $t = 0$ in (3.3) we get:

$$u'(0) = c_1.$$

Then

$$c_1 = \beta_1 \sum_{k=1}^{m} u(\xi_k). \tag{3.6}$$

We compute the value of $u(\xi_k)$, by using (3.2):

$$u(\xi_k) = \frac{1}{\Gamma(\mu)} \int_0^{\xi_k} (\xi_k - \tau)^{\mu-1} h(\tau) d\tau + c_0 + c_1 \xi_k.$$

As a result

$$c_1 = \beta_1 \sum_{k=1}^{m} \frac{1}{\Gamma(\mu)} \int_0^{\xi_k} (\xi_k - \tau)^{\mu-1} h(\tau) d\tau + \beta_1 \sum_{k=1}^{m} c_0 + \beta_1 \sum_{k=1}^{m} c_1 \xi_k$$
$$c_1 = \beta_1 \sum_{k=1}^{m} \frac{1}{\Gamma(\mu)} \int_0^{\xi_k} (\xi_k - \tau)^{\mu-1} h(\tau) d\tau + \beta_1 m c_0 + \beta_1 \sum_{k=1}^{m} c_1 \xi_k$$
$$(1 - \beta_1 \sum_{k=1}^{m} \xi_k) c_1 = \beta_1 \sum_{k=1}^{m} \frac{1}{\Gamma(\mu)} \int_0^{\xi_k} (\xi_k - \tau)^{\mu-1} h(\tau) d\tau + \beta_1 m c_0$$
$$c_1 = \frac{\beta_1}{(1 - \beta_1 \sum_{k=1}^{m} \xi_k)} \sum_{k=1}^{m} \frac{1}{\Gamma(\mu)} \int_0^{\xi_k} (\xi_k - \tau)^{\mu-1} h(\tau) d\tau + m \frac{\beta_1}{(1 - \beta_1 \sum_{k=1}^{m} \xi_k)} c_0. \tag{3.7}$$

From the last equation, we can conclude the value of $\sum_{k=1}^{m} u(\xi_k)$ which depends only on

c_0:

$$\sum_{k=1}^{m} u(\xi_k) = \frac{1}{(1-\beta_1 \sum_{k=1}^{m} \xi_k)} \sum_{k=1}^{m} \frac{1}{\Gamma(\mu)} \int_0^{\xi_k} (\xi_k - \tau)^{\mu-1} h(\tau) d\tau + m \frac{1}{(1-\beta_1 \sum_{k=1}^{m} \xi_k)} c_0. \tag{3.8}$$

For the second condition:

$$\beta_2 {}^c\mathcal{D}_{0^+}^{\mu-1} u(1) + u(\xi) = \beta_3 \sum_{k=1}^{m} u(\xi_k).$$

Firstly, we calculate ${}^c\mathcal{D}_{0^+}^{\mu-1} u(t)$. Since we have $1 < \mu \leq 2$ then $0 < \mu - 1 \leq 1$ so $n = 1$,

$$\begin{aligned}
{}^c\mathcal{D}_{0^+}^{\mu-1} u(t) &= \frac{1}{\Gamma(1-\mu+1)} \int_0^t (t-\tau)^{1-\mu+1-1} u'(\tau) d\tau \\
&= \frac{1}{\Gamma(2-\mu)} \int_0^t (t-\tau)^{1-\mu} u'(\tau) d\tau \\
&= \frac{1}{\Gamma(2-\mu)} \int_0^t (t-\tau)^{1-\mu} [\int_0^\tau \frac{(\tau-s)^{\mu-2} h(s)}{\Gamma(\mu-1)} ds + c_1] d\tau \\
&= \frac{1}{\Gamma(2-\mu)} \int_0^t (t-\tau)^{1-\mu} \int_0^\tau \frac{(\tau-s)^{\mu-2} h(s)}{\Gamma(\mu-1)} ds d\tau + \frac{c_1}{\Gamma(2-\mu)} \int_0^t (t-\tau)^{1-\mu} d\tau,
\end{aligned} \tag{3.9}$$

Which leads to

$${}^c\mathcal{D}_{0^+}^{\mu-1} u(1) = \frac{1}{\Gamma(2-\mu)} \int_0^1 (1-\tau)^{1-\mu} \int_0^\tau \frac{(\tau-s)^{\mu-2} h(s)}{\Gamma(\mu-1)} ds d\tau + \frac{c_1}{\Gamma(2-\mu)} \int_0^1 (1-\tau)^{1-\mu} d\tau.$$

Integral calculation:

$$\int_0^1 (1-\tau)^{1-\mu} d\tau = -\frac{(1-\tau)^{2-\mu}}{2-\mu} \Big|_0^1 = \frac{1}{2-\mu}$$

$${}^c\mathcal{D}_{0^+}^{\mu-1} u(1) = \frac{1}{\Gamma(2-\mu)} \int_0^1 (1-\tau)^{1-\mu} \int_0^\tau \frac{(\tau-s)^{\mu-2} h(s)}{\Gamma(\mu-1)} ds d\tau + \frac{c_1}{\Gamma(3-\mu)}.$$

Based on the value of c_1:

$$\beta_2 {}^c\mathcal{D}_{0^+}^{\mu-1}u(1) = \frac{\beta_2}{\Gamma(2-\mu)}\int_0^1 (1-\tau)^{1-\mu}\int_0^\tau \frac{(\tau-s)^{\mu-2}h(s)}{\Gamma(\mu-1)}dsd\tau$$
$$+\frac{\beta_1\beta_2}{(1-\beta_1\sum_{k=1}^m \xi_k)\Gamma(3-\mu)}\sum_{k=1}^m \frac{1}{\Gamma(\mu)}\int_0^{\xi_k}(\xi_k-\tau)^{\mu-1}h(\tau)d\tau$$
$$+m\frac{\beta_1\beta_2}{(1-\beta_1\sum_{k=1}^m \xi_k)\Gamma(3-\mu)}c_0. \tag{3.10}$$

We already have:

$$u(\xi) = \frac{1}{\Gamma(\mu)}\int_0^\xi (\xi-\tau)^{\mu-1}h(\tau)d\tau + c_0 + c_1\xi.$$

By using the value of c_1 from (3.7), we can have:

$$u(\xi) = \frac{1}{\Gamma(\mu)}\int_0^\xi (\xi-\tau)^{\mu-1}h(\tau)d\tau + c_0 + \frac{\beta_1\xi}{(1-\beta_1\sum_{k=1}^m \xi_k)}\sum_{k=1}^m \frac{1}{\Gamma(\mu)}\int_0^{\xi_k}(\xi_k-\tau)^{\mu-1}h(\tau)d\tau$$
$$+m\frac{\beta_1\xi}{(1-\beta_1\sum_{k=1}^m \xi_k)}c_0. \tag{3.11}$$

Substituting the given values from (3.8), (3.10) and (3.11) into the second condition, yields

$$\frac{\beta_2}{\Gamma(2-\mu)}\int_0^1 (1-\tau)^{1-\mu}\int_0^\tau \frac{(\tau-s)^{\mu-2}h(s)}{\Gamma(\mu-1)}dsd\tau$$
$$+\frac{\beta_1\beta_2}{(1-\beta_1\sum_{k=1}^m \xi_k)\Gamma(3-\mu)}\sum_{k=1}^m \frac{1}{\Gamma(\mu)}\int_0^{\xi_k}(\xi_k-\tau)^{\mu-1}h(\tau)d\tau + m\frac{\beta_1\beta_2}{(1-\beta_1\sum_{k=1}^m \xi_k)\Gamma(3-\mu)}c_0$$
$$+\frac{1}{\Gamma(\mu)}\int_0^\xi (\xi-\tau)^{\mu-1}h(\tau)d\tau + c_0 + \frac{\beta_1\xi}{(1-\beta_1\sum_{k=1}^m \xi_k)}\sum_{k=1}^m \frac{1}{\Gamma(\mu)}\int_0^{\xi_k}(\xi_k-\tau)^{\mu-1}h(\tau)d\tau$$
$$+m\frac{\beta_1\xi}{(1-\beta_1\sum_{k=1}^m \xi_k)}c_0 = \frac{\beta_3}{(1-\beta_1\sum_{k=1}^m \xi_k)}\sum_{k=1}^m \frac{1}{\Gamma(\mu)}\int_0^{\xi_k}(\xi_k-\tau)^{\mu-1}h(\tau)d\tau + m\frac{\beta_3}{(1-\beta_1\sum_{k=1}^m \xi_k)}c_0.$$

So

$$\frac{\Gamma(3-\mu)\big[m\beta_3-1+\beta_1(\sum_{k=1}^m\xi_k-m\xi)\big]-m\beta_1\beta_2}{\Gamma(3-\mu)(1-\beta_1\sum_{k=1}^m\xi_k)}c_0=\frac{\beta_2}{\Gamma(2-\mu)}\int_0^1(1-\tau)^{1-\mu}\int_0^\tau\frac{(\tau-s)^{\mu-2}h(s)}{\Gamma(\mu-1)}dsd\tau$$
$$+\frac{1}{\Gamma(\mu)}\int_0^\xi(\xi-\tau)^{\mu-1}h(\tau)d\tau+\frac{\beta_1\beta_2+(\beta_1\xi-\beta_3)\Gamma(3-\mu)}{\Gamma(3-\mu)(1-\beta_1\sum_{k=1}^m\xi_k)}\sum_{k=1}^m\frac{1}{\Gamma(\mu)}\int_0^{\xi_k}(\xi_k-\tau)^{\mu-1}h(\tau)d\tau.$$

This means:

$$c_0=\frac{\Gamma(3-\mu)(1-\beta_1\sum_{k=1}^m\xi_k)}{\Gamma(3-\mu)\big[m\beta_3-1+\beta_1(\sum_{k=1}^m\xi_k-m\xi)\big]-m\beta_1\beta_2}\frac{\beta_2}{\Gamma(2-\mu)}\int_0^1(1-\tau)^{1-\mu}\int_0^\tau\frac{(\tau-s)^{\mu-2}h(s)}{\Gamma(\mu-1)}dsd\tau$$
$$+\frac{\Gamma(3-\mu)(1-\beta_1\sum_{k=1}^m\xi_k)}{\Gamma(3-\mu)\big[m\beta_3-1+\beta_1(\sum_{k=1}^m\xi_k-m\xi)\big]-m\beta_1\beta_2}\frac{1}{\Gamma(\mu)}\int_0^\xi(\xi-\tau)^{\mu-1}h(\tau)d\tau$$
$$+\frac{\beta_1\beta_2+(\beta_1\xi-\beta_3)\Gamma(3-\mu)}{\Gamma(3-\mu)\big[m\beta_3-1+\beta_1(\sum_{k=1}^m\xi_k-m\xi)\big]-m\beta_1\beta_2}\sum_{k=1}^m\frac{1}{\Gamma(\mu)}\int_0^{\xi_k}(\xi_k-\tau)^{\mu-1}h(\tau)d\tau.$$

The next step is to substitute the value of c_0 in (3.7), we have

$$c_1=\frac{\beta_1}{(1-\beta_1\sum_{k=1}^m\xi_k)}\sum_{k=1}^m\frac{1}{\Gamma(\mu)}\int_0^{\xi_k}(\xi_k-\tau)^{\mu-1}h(\tau)d\tau$$
$$+\frac{m\beta_1\Gamma(3-\mu)}{\Gamma(3-\mu)\big[m\beta_3-1+\beta_1(\sum_{k=1}^m\xi_k-m\xi)\big]-m\beta_1\beta_2}\frac{\beta_2}{\Gamma(2-\mu)}\int_0^1(1-\tau)^{1-\mu}\int_0^\tau\frac{(\tau-s)^{\mu-2}h(s)}{\Gamma(\mu-1)}dsd\tau$$
$$+\frac{m\beta_1\Gamma(3-\mu)}{\Gamma(3-\mu)\big[m\beta_3-1+\beta_1(\sum_{k=1}^m\xi_k-m\xi)\big]-m\beta_1\beta_2}\frac{1}{\Gamma(\mu)}\int_0^\xi(\xi-\tau)^{\mu-1}h(\tau)d\tau$$
$$+\frac{m\beta_1}{(1-\beta_1\sum_{k=1}^m\xi_k)}\frac{\beta_1\beta_2+(\beta_1\xi-\beta_3)\Gamma(3-\mu)}{\Gamma(3-\mu)\big[m\beta_3-1+\beta_1(\sum_{k=1}^m\xi_k-m\xi)\big]-m\beta_1\beta_2}\sum_{k=1}^m\frac{1}{\Gamma(\mu)}\int_0^{\xi_k}(\xi_k-\tau)^{\mu-1}h(\tau)d\tau.$$

Finally

$$c_1=\frac{m\beta_1\Gamma(3-\mu)}{\Gamma(3-\mu)\big[m\beta_3-1+\beta_1(\sum_{k=1}^m\xi_k-m\xi)\big]-m\beta_1\beta_2}\frac{\beta_2}{\Gamma(2-\mu)}\int_0^1(1-\tau)^{1-\mu}\int_0^\tau\frac{(\tau-s)^{\mu-2}h(s)}{\Gamma(\mu-1)}dsd\tau$$
$$+\frac{m\beta_1\Gamma(3-\mu)}{\Gamma(3-\mu)\big[m\beta_3-1+\beta_1(\sum_{k=1}^m\xi_k-m\xi)\big]-m\beta_1\beta_2}\frac{1}{\Gamma(\mu)}\int_0^\xi(\xi-\tau)^{\mu-1}h(\tau)d\tau$$
$$+\frac{\beta_1\big[\Gamma(3-\mu)\big[m\beta_3-1+\beta_1(\sum_{k=1}^m\xi_k-m\xi)\big]-m\beta_1\beta_2\big]+m\beta_1\big[\beta_1\beta_2+(\beta_1\xi-\beta_3)\Gamma(3-\mu)\big]}{(1-\beta_1\sum_{k=1}^m\xi_k)\big[\Gamma(3-\mu)\big[m\beta_3-1+\beta_1(\sum_{k=1}^m\xi_k-m\xi)\big]-m\beta_1\beta_2\big]}$$
$$\times\sum_{k=1}^m\frac{1}{\Gamma(\mu)}\int_0^{\xi_k}(\xi_k-\tau)^{\mu-1}h(\tau)d\tau.$$

We get

$$\begin{aligned}
c_1 =& \frac{m\beta_1\Gamma(3-\mu)}{\Gamma(3-\mu)\big[m\beta_3 - 1 + \beta_1(\sum_{k=1}^{m}\xi_k - m\xi)\big] - m\beta_1\beta_2}\frac{\beta_2}{\Gamma(2-\mu)}\int_0^1 (1-\tau)^{1-\mu}\int_0^\tau \frac{(\tau-s)^{\mu-2}h(s)}{\Gamma(\mu-1)}dsd\tau \\
&+ \frac{m\beta_1\Gamma(3-\mu)}{\Gamma(3-\mu)\big[m\beta_3 - 1 + \beta_1(\sum_{k=1}^{m}\xi_k - m\xi)\big] - m\beta_1\beta_2}\frac{1}{\Gamma(\mu)}\int_0^\xi (\xi-\tau)^{\mu-1}h(\tau)d\tau \\
&- \frac{\beta_1\Gamma(3-\mu)}{\Gamma(3-\mu)\big[m\beta_3 - 1 + \beta_1(\sum_{k=1}^{m}\xi_k - m\xi)\big] - m\beta_1\beta_2}\sum_{k=1}^{m}\frac{1}{\Gamma(\mu)}\int_0^{\xi_k} (\xi_k-\tau)^{\mu-1}h(\tau)d\tau.
\end{aligned}$$

We can note that

$$\Delta = \Gamma(3-\mu)\big[m\beta_3 - 1 + \beta_1(\sum_{k=1}^{m}\xi_k - m\xi)\big] - m\beta_1\beta_2.$$

Now we find $c_1 t + c_0$:

$$\begin{aligned}
&\frac{\beta_2(2-\mu)(m\beta_1 t + 1 - \beta_1\sum_{k=1}^{m}\xi_k)}{\Delta}\int_0^1 (1-\tau)^{1-\mu}\int_0^\tau \frac{(\tau-s)^{\mu-2}h(s)}{\Gamma(\mu-1)}dsd\tau \\
&+ \frac{\Gamma(3-\mu)(m\beta_1 t + 1 - \beta_1\sum_{k=1}^{m}\xi_k)}{\Delta\Gamma(\mu)}\int_0^\xi (\xi-\tau)^{\mu-1}h(\tau)d\tau \\
&+ \frac{\Gamma(3-\mu)\big(\beta_1(\xi-t)-\beta_3\big)+\beta_1\beta_2}{\Delta\Gamma(\mu)}\sum_{k=1}^{m}\int_0^{\xi_k}(\xi_k-\tau)^{\mu-1}h(\tau)d\tau.
\end{aligned}$$

Then, the solution is

$$\begin{aligned}
u(t) =& \frac{1}{\Gamma(\mu)}\int_0^t (t-\tau)^{\mu-1}h(\tau)d\tau \\
&+ \frac{\beta_2(2-\mu)(m\beta_1 t + 1 - \beta_1\sum_{k=1}^{m}\xi_k)}{\Delta}\int_0^1 (1-\tau)^{1-\mu}\int_0^\tau \frac{(\tau-s)^{\mu-2}h(s)}{\Gamma(\mu-1)}dsd\tau \\
&+ \frac{\Gamma(3-\mu)(m\beta_1 t + 1 - \beta_1\sum_{k=1}^{m}\xi_k)}{\Delta\Gamma(\mu)}\int_0^\xi (\xi-\tau)^{\mu-1}h(\tau)d\tau \\
&+ \frac{\Gamma(3-\mu)\big(\beta_1(\xi-t)-\beta_3\big)+\beta_1\beta_2}{\Delta\Gamma(\mu)}\sum_{k=1}^{m}\int_0^{\xi_k}(\xi_k-\tau)^{\mu-1}h(\tau)d\tau. \qquad (3.12)
\end{aligned}$$

We can write

$$\begin{aligned}
u(t) = & \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} h(\tau) d\tau \\
& + \frac{\beta_2(2-\mu)A(t)}{\Delta} \int_0^1 (1-\tau)^{1-\mu} \int_0^\tau \frac{(\tau-s)^{\mu-2}h(s)}{\Gamma(\mu-1)} ds d\tau \\
& + \frac{\Gamma(3-\mu)A(t)}{\Delta\Gamma(\mu)} \int_0^\xi (\xi-\tau)^{\mu-1} h(\tau) d\tau \\
& + \frac{\Gamma(3-\mu)B(t) + \beta_1\beta_2}{\Delta\Gamma(\mu)} \sum_{k=1}^m \int_0^{\xi_k} (\xi_k-\tau)^{\mu-1} h(\tau) d\tau, \qquad (3.13)
\end{aligned}$$

with

$$A(t) = m\beta_1 t + 1 - \beta_1 \sum_{k=1}^m \xi_k \qquad (3.14)$$

$$B(t) = \beta_1(\xi - t) - \beta_3, \qquad (3.15)$$

and $\Delta \neq 0$.

Remak 3.1 *Since $t \in [0,1]$,$\beta_1, \beta_2, \beta_3 \in \mathbb{R}$ and $\xi_i < 1, \forall i \in \{0, \dots, m\}$, notice that*

$$\begin{aligned}
|A(t)| &= |m\beta_1 t + 1 - \beta_1 \sum_{k=1}^m \xi_k| \\
&\leq |m\beta_1| + 1 + |\beta_1 \sum_{k=1}^m 1| \\
&\leq 2|m\beta_1| + 1 = A_0
\end{aligned}$$

$$\begin{aligned}
|B(t)| &= |\beta_1(\xi - t) - \beta_3| \\
&\leq |\beta_1| + |\beta_3| = B_0.
\end{aligned}$$

We also have $\sigma \in (0,1)$, so

$$
\begin{aligned}
{}^c\mathcal{D}^{\sigma}_{0^+}A(t) &= \int_0^t \frac{(t-\tau)^{1-\sigma-1}}{\Gamma(1-\sigma)}A'(\tau)d\tau \\
&= \int_0^t \frac{(t-\tau)^{-\sigma}}{\Gamma(1-\sigma)}m\beta_1 d\tau.
\end{aligned}
$$

By calculating the integral we obtain

$$
{}^c\mathcal{D}^{\sigma}_{0^+}A(t) = \frac{m\beta_1 t^{1-\sigma}}{\Gamma(2-\sigma)}.
$$

In the same way we have

$$
{}^c\mathcal{D}^{\sigma}_{0^+}B(t) = \frac{-\beta_1 t^{1-\sigma}}{\Gamma(2-\sigma)}.
$$

As a result

$$
\begin{aligned}
|{}^c\mathcal{D}^{\sigma}_{0^+}A(t)| &\leq \frac{m|\beta_1|}{\Gamma(2-\sigma)} = A_1 \\
|{}^c\mathcal{D}^{\sigma}_{0^+}B(t)| &\leq \frac{|\beta_1|}{\Gamma(2-\sigma)} = B_1.
\end{aligned}
$$

Remak 3.2 *The function $u \in \mathbb{X}$ is a solution for the boundary problem (3.1) if it satisfies the boundary conditions and if there exists a function (selection) $\nu \in S_{\mathcal{F},u}$, such that:*

$$
\begin{aligned}
u(t) &= \frac{1}{\Gamma(\mu)}\int_0^t (t-\tau)^{\mu-1}\nu(\tau)d\tau \\
&+ \frac{\beta_2(2-\mu)A(t)}{\Delta}\int_0^1 (1-\tau)^{1-\mu}\int_0^{\tau}\frac{(\tau-s)^{\mu-2}\nu(s)}{\Gamma(\mu-1)}dsd\tau \\
&+ \frac{\Gamma(3-\mu)A(t)}{\Delta\Gamma(\mu)}\int_0^{\xi}(\xi-\tau)^{\mu-1}\nu(\tau)d\tau \\
&+ \frac{\Gamma(3-\mu)B(t)+\beta_1\beta_2}{\Delta\Gamma(\mu)}\sum_{k=1}^{m}\int_0^{\xi_k}(\xi_k-\tau)^{\mu-1}\nu(\tau)d\tau.
\end{aligned}
\tag{3.16}
$$

Where $A(t), B(t)$ are defined in (3.14) and (3.15).

Let the set valued operator $N : \mathbb{X} \to \mathcal{P}(\mathbb{X})$ be defined as follows:

$$N(u) = \begin{cases} f \in \mathbb{X} : \\ f(t) = \begin{cases} \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} \nu(\tau) d\tau \\ +\frac{\beta_2(2-\mu)A(t)}{\Delta} \int_0^1 (1-\tau)^{1-\mu} \int_0^\tau \frac{(\tau-s)^{\mu-2}\nu(s)}{\Gamma(\mu-1)} ds d\tau \\ +\frac{\Gamma(3-\mu)A(t)}{\Delta\Gamma(\mu)} \int_0^\xi (\xi-\tau)^{\mu-1} \nu(\tau) d\tau \\ +\frac{\Gamma(3-\mu)B(t)+\beta_1\beta_2}{\Delta\Gamma(\mu)} \sum_{k=1}^m \int_0^{\xi_k} (\xi_k-\tau)^{\mu-1} \nu(\tau) d\tau, \end{cases} \end{cases}$$

such that $\nu \in S_{\mathcal{F},u}$.

Theorem 3.1 *Let $\mathcal{F} : \mathbb{O} \times \mathbb{R}^2 \longrightarrow \mathcal{P}(\mathbb{R})$ be a set-valued map such that:*

($\mathcal{A}_1$) $\mathcal{F}$ is an integrably bounded set-valued function with closed values, and $\mathcal{F}(., u, w) : \mathbb{O} \longrightarrow \mathcal{P}(\mathbb{R})$ is measurable for all $u, w \in \mathbb{R}$.

($\mathcal{A}_2$) There exists $\psi : [0, \infty[\to [0, \infty[$ a non-decreasing upper semi continuous mapping such that:

1. $\liminf_{t\to\infty}(t - \psi(t)) > 0, \forall t > 0.$
2. $\psi(t) < t, \forall t > 0.$

($\mathcal{A}_3$) There exists a function $\gamma \in C(\mathbb{O}, [0, \infty[)$ which satisfies

$$H_d(\mathcal{F}(t, u_1, u_2), \mathcal{F}(t, u_1', u_2')) \leq \frac{1}{\Lambda_1 + \Lambda_2} \gamma(t) \psi\Big(\sum_{i=1}^{2} |u_i - u_i'|\Big),$$

for all $t \in \mathbb{O}$ and $u_i, u_i' \in \mathbb{R}$, where

$$\Lambda_1 = ||\gamma|| \left(\frac{1}{\Gamma(\mu+1)} + \frac{|\beta_2| A_0}{\Gamma(\mu)|\Delta|} + \frac{\Gamma(3-\mu)A_0}{\Gamma(\mu+1)|\Delta|} + \frac{(\Gamma(3-\mu)B_0 + |\beta_1||\beta_2|)m}{\Gamma(\mu+1)|\Delta|} \right),$$

$$\Lambda_2 = ||\gamma|| \left(\frac{1}{\Gamma(\mu)\Gamma(2-\sigma)} + \frac{|\beta_2| A_1(2-\mu)}{\Gamma(\mu)|\Delta|} + \frac{\Gamma(3-\mu)A_1}{\Gamma(\mu+1)|\Delta|} + \frac{\Gamma(3-\mu)B_1 m}{\Gamma(\mu+1)|\Delta|} \right).$$

($\mathcal{A}_4$) The set-valued map N has the approximate endpoint property.

If $(\mathcal{A}_1)$-$(\mathcal{A}_4)$ are satisfied. Then the boundary value inclusion (3.1) has a solution.

Proof:

Firstly, we prove that $S_{\mathcal{F},u} \neq \phi$ for all $u \in \mathbb{X}$:

Since $\mathcal{F}(.,u,w)$ is measurable for all $u, w \in \mathbb{R}$ with closed values, then by relaying on the lemma 1.2, we deduce that: $\exists \nu(t) \in \mathcal{F}\big(\mathbf{t}, u(\mathbf{t}), {}^{c}\mathcal{D}^{\sigma}_{0+}u(\mathbf{t})\big)$ for all $t \in [0,1]$.

Moreover, $\mathcal{F}$ is integrably bounded, which means: for all $\nu \in \mathcal{F}\big(\mathbf{t}, u(\mathbf{t}), {}^{c}\mathcal{D}^{\sigma}_{0+}u(\mathbf{t})\big)$, there exists a function $g \in L^1(\mathbb{O}, \mathbb{R})$ satisfies:

$$|\nu(t)| \leq g(t)$$

So:

$$|\nu(t)| \leq |g(t)|$$

Then one can get:

$$\int_0^1 |\nu(s)|ds \leq \int_0^1 |g(s)|ds \leq \infty$$

This proves that $\nu \in L^1(\mathbb{O}, \mathbb{R})$, which ends the proof.

Secondly, we prove that $N(u)$ is closed.

Assume that $u \in \mathbb{X}$ and $\{Z_n\}_{n\geq 1}$ be a sequence in $N(u)$ which tends towards Z. For every $n \in \mathbb{N}$, we choose $\nu_n \in S_{\mathcal{F},u}$ such that

$$\begin{aligned}
Z_n(t) &= \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1}\nu_n(\tau)d\tau \\
&+ \frac{\beta_2(2-\mu)A(t)}{\Delta} \int_0^1 (1-\tau)^{1-\mu} \int_0^\tau \frac{(\tau-s)^{\mu-2}\nu_n(s)}{\Gamma(\mu-1)} ds d\tau \\
&+ \frac{\Gamma(3-\mu)A(t)}{\Delta\Gamma(\mu)} \int_0^\xi (\xi-\tau)^{\mu-1}\nu_n(\tau)d\tau \\
&+ \frac{\Gamma(3-\mu)B(t) + \beta_1\beta_2}{\Delta\Gamma(\mu)} \sum_{k=1}^m \int_0^{\xi_k} (\xi_k-\tau)^{\mu-1}\nu_n(\tau)d\tau,
\end{aligned}$$

for all $t \in \mathbb{O}$.

By compactness of $\mathcal{F}$, the sequence $\{\nu_n\}_{n\geq 1}$ has a subsequence converging to a certain $\nu \in L^1(\mathbb{O}, \mathbb{R})$, obviously $\nu \in S_{\mathcal{F},u}, \forall t \in \mathbb{O}$. Denoting this subsequence again by $\{\nu_n\}_{n\geq 1}$, one can obtain

$$\begin{aligned}
\lim_{n\to\infty} Z_n(t) &= \lim_{n\to\infty} \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1}\nu_n(\tau)d\tau \\
&+ \lim_{n\to\infty} \frac{\beta_2(2-\mu)A(t)}{\Delta} \int_0^1 (1-\tau)^{1-\mu} \int_0^\tau \frac{(\tau-s)^{\mu-2}\nu_n(s)}{\Gamma(\mu-1)} dsd\tau \\
&+ \lim_{n\to\infty} \frac{\Gamma(3-\mu)A(t)}{\Delta\Gamma(\mu)} \int_0^\xi (\xi-\tau)^{\mu-1}\nu_n(\tau)d\tau \\
&+ \lim_{n\to\infty} \frac{\Gamma(3-\mu)B(t)+\beta_1\beta_2}{\Delta\Gamma(\mu)} \sum_{k=1}^m \int_0^{\xi_k} (\xi_k-\tau)^{\mu-1}\nu_n(\tau)d\tau.
\end{aligned}$$

Then

$$\begin{aligned}
\lim_{n\to\infty} Z_n(t) &= \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} \lim_{n\to\infty} \nu_n(\tau)d\tau \\
&+ \frac{\beta_2(2-\mu)A(t)}{\Delta} \int_0^1 (1-\tau)^{1-\mu} \int_0^\tau \frac{(\tau-s)^{\mu-2}}{\Gamma(\mu-1)} \lim_{n\to\infty} \nu_n(s)dsd\tau \\
&+ \frac{\Gamma(3-\mu)A(t)}{\Delta\Gamma(\mu)} \int_0^\xi (\xi-\tau)^{\mu-1} \lim_{n\to\infty} \nu_n(\tau)d\tau \\
&+ \frac{\Gamma(3-\mu)B(t)+\beta_1\beta_2}{\Delta\Gamma(\mu)} \sum_{k=1}^m \int_0^{\xi_k} (\xi_k-\tau)^{\mu-1} \lim_{n\to\infty} \nu_n(\tau)d\tau
\end{aligned}$$

$$\begin{aligned}
Z(t) &= \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1}\nu(\tau)d\tau \\
&+ \frac{\beta_2(2-\mu)A(t)}{\Delta} \int_0^1 (1-\tau)^{1-\mu} \int_0^\tau \frac{(\tau-s)^{\mu-2}}{\Gamma(\mu-1)}\nu(s)dsd\tau \\
&+ \frac{\Gamma(3-\mu)A(t)}{\Delta\Gamma(\mu)} \int_0^\xi (\xi-\tau)^{\mu-1}\nu(\tau)d\tau \\
&+ \frac{\Gamma(3-\mu)B(t)+\beta_1\beta_2}{\Delta\Gamma(\mu)} \sum_{k=1}^m \int_0^{\xi_k} (\xi_k-\tau)^{\mu-1}\nu(\tau)d\tau.
\end{aligned}$$

This shows that $Z \in N(u)$, as a result N is closed-valued.

Furthermore, $N(u)$ is a bounded set for all $u \in \mathbb{X}$ due to the following result:

Let $f \in N(u)$ and $g \in L^1(\mathbb{O}, \mathbb{R})$, we have:

$$
\begin{aligned}
|f(t)| = \Big| & \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} \nu(\tau) d\tau \\
& + \frac{\beta_2(2-\mu)A(t)}{\Delta} \int_0^1 (1-\tau)^{1-\mu} \int_0^\tau \frac{(\tau-s)^{\mu-2}\nu(s)}{\Gamma(\mu-1)} ds d\tau \\
& + \frac{\Gamma(3-\mu)A(t)}{\Delta\Gamma(\mu)} \int_0^\xi (\xi-\tau)^{\mu-1} \nu(\tau) d\tau \\
& + \frac{\Gamma(3-\mu)B(t) + \beta_1\beta_2}{\Delta\Gamma(\mu)} \sum_{k=1}^m \int_0^{\xi_k} (\xi_k-\tau)^{\mu-1} \nu(\tau) d\tau \Big| \\
& \leq \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} |\nu(\tau)| d\tau \\
& + \frac{|\beta_2|(2-\mu)A_0}{|\Delta|} \int_0^1 (1-\tau)^{1-\mu} \int_0^\tau \frac{(\tau-s)^{\mu-2}|\nu(s)|}{\Gamma(\mu-1)} ds d\tau \\
& + \frac{\Gamma(3-\mu)A_0}{|\Delta|\Gamma(\mu)} \int_0^\xi (\xi-\tau)^{\mu-1} |\nu(\tau)| d\tau \\
& + \frac{\Gamma(3-\mu)B_0 + |\beta_1||\beta_2|}{|\Delta|\Gamma(\mu)} \sum_{k=1}^m \int_0^{\xi_k} (\xi_k-\tau)^{\mu-1} |\nu(\tau)| d\tau \\
& \leq \frac{1}{\Gamma(\mu)} \int_0^1 |\nu(\tau)| d\tau \\
& + \frac{|\beta_2|(2-\mu)A_0}{|\Delta|\Gamma(\mu-1)} \int_0^1 \int_0^1 |\nu(s)| ds d\tau \\
& + \frac{\Gamma(3-\mu)A_0}{|\Delta|\Gamma(\mu)} \int_0^1 |\nu(\tau)| d\tau \\
& + \frac{\Gamma(3-\mu)B_0 + |\beta_1||\beta_2|}{|\Delta|\Gamma(\mu)} m \int_0^1 |\nu(\tau)| d\tau
\end{aligned}
$$

From $(\mathcal{A}_1)$ we know that $\mathcal{F}$ is integrably bounded, so we obtain

$$
\int_0^1 |\nu(\tau)| d\tau \leq \int_0^1 |g(\tau)| d\tau = ||g||_{L^1(\mathbb{O},\mathbb{R})}
$$

Then, there exists $C > 0$ independent of u such that: $|f(t)| \leq C$ for all $t \in [0, 1]$. Hence $N(u)$ is bounded.

Finally, we show that

$$
H_d(N(u), N(w)) \leq \psi(||u - w||)
$$

Let $u, w \in \mathbb{X}$ and $y_1 \in N(u), y_2 \in N(w)$, choosing $\nu_1 \in S_{\mathcal{F},u}$ such that

$$\begin{aligned} y_1(t) &= \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} \nu_1(\tau) d\tau \\ &+ \frac{\beta_2(2-\mu)A(t)}{\Delta} \int_0^1 (1-\tau)^{1-\mu} \int_0^\tau \frac{(\tau-s)^{\mu-2}}{\Gamma(\mu-1)} \nu_1(s) ds d\tau \\ &+ \frac{\Gamma(3-\mu)A(t)}{\Delta\Gamma(\mu)} \int_0^\xi (\xi-\tau)^{\mu-1} \nu_1(\tau) d\tau \\ &+ \frac{\Gamma(3-\mu)B(t) + \beta_1\beta_2}{\Delta\Gamma(\mu)} \sum_{k=1}^m \int_0^{\xi_k} (\xi_k-\tau)^{\mu-1} \nu_1(\tau) d\tau, \end{aligned}$$

for all $t \in \mathbb{O}$.

Since we assumed in $(\mathcal{A}_3)$ that there exists a function $\gamma \in C(\mathbb{O}, [0, \infty[)$ which satisfies

$$H_d(\mathcal{F}(t, u(t), {}^c\mathcal{D}^\sigma_{0^+} u(t)), \mathcal{F}(t, w(t), {}^c\mathcal{D}^\sigma_{0^+} w(t))) \leq \frac{1}{\Lambda_1 + \Lambda_2} \gamma(t) \psi\Big(|u(t) - w(t)| + |{}^c\mathcal{D}^\sigma_{0^+} u(t) - {}^c\mathcal{D}^\sigma_{0^+} w(t)|\Big).$$

Then $\exists \nu_2 \in S_{\mathcal{F},w}$ satisfies:

$$|\nu_1(t) - \nu_2(t)| \leq \frac{1}{\Lambda_1 + \Lambda_2} \gamma(t) \psi\Big(|u(t) - w(t)| + |{}^c\mathcal{D}^\sigma_{0^+} u(t) - {}^c\mathcal{D}^\sigma_{0^+} w(t)|\Big),$$

for all $t \in \mathbb{O}$.

Thus we define $y_2(t)$ as follows:

$$\begin{aligned} y_2(t) &= \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} \nu_2(\tau) d\tau \\ &+ \frac{\beta_2(2-\mu)A(t)}{\Delta} \int_0^1 (1-\tau)^{1-\mu} \int_0^\tau \frac{(\tau-s)^{\mu-2}}{\Gamma(\mu-1)} \nu_2(s) ds d\tau \\ &+ \frac{\Gamma(3-\mu)A(t)}{\Delta\Gamma(\mu)} \int_0^\xi (\xi-\tau)^{\mu-1} \nu_2(\tau) d\tau \\ &+ \frac{\Gamma(3-\mu)B(t) + \beta_1\beta_2}{\Delta\Gamma(\mu)} \sum_{k=1}^m \int_0^{\xi_k} (\xi_k-\tau)^{\mu-1} \nu_2(\tau) d\tau, \end{aligned}$$

Let $\sup_{t\in\mathbb{O}}|\gamma(t)| = ||\gamma||$, thus one can obtain

$$\begin{aligned}|y_1(t)-y_2(t)| \leq & \frac{1}{\Gamma(\mu)}\int_0^t (t-\tau)^{\mu-1}|\nu_1(\tau)-\nu_2(\tau)|d\tau \\ & + \frac{|\beta_2|(2-\mu)|A(t)|}{|\Delta|}\int_0^1 (1-\tau)^{1-\mu}\int_0^\tau \frac{(\tau-s)^{\mu-2}}{\Gamma(\mu-1)}|\nu_1(s)-\nu_2(s)|dsd\tau \\ & + \frac{\Gamma(3-\mu)|A(t)|}{|\Delta|\Gamma(\mu)}\int_0^\xi (\xi-\tau)^{\mu-1}|\nu_1(\tau)-\nu_2(\tau)|d\tau \\ & + \frac{\Gamma(3-\mu)\big|B(t)| + |\beta_1||\beta_2|}{|\Delta|\Gamma(\mu)}\sum_{k=1}^{m}\int_0^{\xi_k}(\xi_k-\tau)^{\mu-1}|\nu_1(\tau)-\nu_2(\tau)|d\tau.\end{aligned}$$

So

$$\begin{aligned}|y_1(t)-y_2(t)| \leq & \frac{1}{\Lambda_1+\Lambda_2}||\gamma||\psi(||u-w||)\Big[\frac{1}{\Gamma(\mu)}\int_0^t (t-\tau)^{\mu-1}d\tau \\ & + \frac{|\beta_2|(2-\mu)|A(t)|}{|\Delta|}\int_0^1 (1-\tau)^{1-\mu}\int_0^\tau \frac{(\tau-s)^{\mu-2}}{\Gamma(\mu-1)}dsd\tau \\ & + \frac{\Gamma(3-\mu)|A(t)|}{|\Delta|\Gamma(\mu)}\int_0^\xi (\xi-\tau)^{\mu-1}d\tau \\ & + \frac{\Gamma(3-\mu)\big|B(t)| + |\beta_1||\beta_2|}{|\Delta|\Gamma(\mu)}\sum_{k=1}^{m}\int_0^{\xi_k}(\xi_k-\tau)^{\mu-1}d\tau\Big].\end{aligned}$$

By calculating the integrals and using the results of Remark 3.1, we get

$$\begin{aligned}|y_1(t)-y_2(t)| \leq & \frac{1}{\Lambda_1+\Lambda_2}||\gamma||\psi(||u-w||)\Big\{\frac{1}{\Gamma(\mu+1)} + \frac{|\beta_2|A_0}{\Gamma(\mu)|\Delta|} \\ & + \frac{\Gamma(3-\mu)A_0}{\Gamma(\mu+1)|\Delta|} + \frac{(\Gamma(3-\mu)B_0 + |\beta_1||\beta_2|)m}{\Gamma(\mu+1)|\Delta|}\Big\} \\ |y_1(t)-y_2(t)| \leq & \frac{\Lambda_1}{\Lambda_1+\Lambda_2}\psi(||u-w||). \end{aligned} \tag{3.17}$$

On the other hand $\sigma \in (0,1)$ which allows us to have

$${}^c\mathcal{D}^\sigma_{0^+}y_1(t) = \int_0^t \frac{(t-\omega)^{-\sigma}}{\Gamma(1-\sigma)}y_1'(\omega)d\omega,$$

with

$$\begin{aligned} y_1'(\omega) &= \frac{(\mu-1)}{\Gamma(\mu)}\int_0^\omega (\omega-\tau)^{\mu-2}\nu_2(\tau)d\tau \\ &+ \frac{\beta_2(2-\mu)A'(\omega)}{\Delta}\int_0^1 (1-\tau)^{1-\mu}\int_0^\tau \frac{(\tau-s)^{\mu-2}}{\Gamma(\mu-1)}\nu_2(s)dsd\tau \\ &+ \frac{\Gamma(3-\mu)A'(\omega)}{\Delta\Gamma(\mu)}\int_0^\xi (\xi-\tau)^{\mu-1}\nu_2(\tau)d\tau \\ &+ \frac{\Gamma(3-\mu)B'(\omega)}{\Delta\Gamma(\mu)}\sum_{k=1}^m \int_0^{\xi_k}(\xi_k-\tau)^{\mu-1}\nu_2(\tau)d\tau, \end{aligned}$$

$$\begin{aligned} y_1'(\omega) &= \frac{(\mu-1)}{\Gamma(\mu)}\int_0^\omega (\omega-\tau)^{\mu-2}\nu_1(\tau)d\tau \\ &+ \frac{m\beta_1\beta_2(2-\mu)}{\Delta}\int_0^1 (1-\tau)^{1-\mu}\int_0^\tau \frac{(\tau-s)^{\mu-2}}{\Gamma(\mu-1)}\nu_1(s)dsd\tau \\ &+ \frac{m\beta_1\Gamma(3-\mu)}{\Delta\Gamma(\mu)}\int_0^\xi (\xi-\tau)^{\mu-1}\nu_1(\tau)d\tau \\ &- \frac{\beta_1\Gamma(3-\mu)}{\Delta\Gamma(\mu)}\sum_{k=1}^m \int_0^{\xi_k}(\xi_k-\tau)^{\mu-1}\nu_1(\tau)d\tau. \end{aligned}$$

Thus

$$\begin{aligned} {}^c\mathcal{D}_{0^+}^\sigma y_1(t) &= \int_0^t \frac{(t-\omega)^{-\sigma}}{\Gamma(1-\sigma)}\Big[\frac{(\mu-1)}{\Gamma(\mu)}\int_0^\omega (\omega-\tau)^{\mu-2}\nu_1(\tau)d\tau \\ &+ \frac{m\beta_1\beta_2(2-\mu)}{\Delta}\int_0^1 (1-\tau)^{1-\mu}\int_0^\tau \frac{(\tau-s)^{\mu-2}}{\Gamma(\mu-1)}\nu_1(s)dsd\tau \\ &+ \frac{m\beta_1\Gamma(3-\mu)}{\Delta\Gamma(\mu)}\int_0^\xi (\xi-\tau)^{\mu-1}\nu_1(\tau)d\tau \\ &- \frac{\beta_1\Gamma(3-\mu)}{\Delta\Gamma(\mu)}\sum_{k=1}^m \int_0^{\xi_k}(\xi_k-\tau)^{\mu-1}\nu_1(\tau)d\tau\Big]d\omega. \end{aligned}$$

So we have

$$
\begin{aligned}
|{}^c\mathcal{D}^{\sigma}_{0+}y_1(t) - {}^c\mathcal{D}^{\sigma}_{0+}y_2(t)| \leq & \int_0^t \frac{(t-\omega)^{-\sigma}}{\Gamma(1-\sigma)}\Big[\frac{(\mu-1)}{\Gamma(\mu)}\int_0^{\omega}(\omega-\tau)^{\mu-2}|\nu_1(\tau)-\nu_2(\tau)|d\tau \\
& + \frac{m|\beta_1||\beta_2|(2-\mu)}{|\Delta|}\int_0^1(1-\tau)^{1-\mu}\int_0^{\tau}\frac{(\tau-s)^{\mu-2}}{\Gamma(\mu-1)}|\nu_1(s)-\nu_2(s)|dsd\tau \\
& + \frac{m|\beta_1|\Gamma(3-\mu)}{|\Delta|\Gamma(\mu)}\int_0^{\xi}(\xi-\tau)^{\mu-1}|\nu_1(\tau)-\nu_2(\tau)|d\tau \\
& + \frac{|\beta_1|\Gamma(3-\mu)}{|\Delta|\Gamma(\mu)}\sum_{k=1}^{m}\int_0^{\xi_k}(\xi_k-\tau)^{\mu-1}|\nu_1(\tau)-\nu_2(\tau)|d\tau\Big]d\omega,
\end{aligned}
$$

$$
\begin{aligned}
|{}^c\mathcal{D}^{\sigma}_{0+}y_1(t) - {}^c\mathcal{D}^{\sigma}_{0+}y_2(t)| \leq & \frac{1}{\Lambda_1+\Lambda_2}||\gamma||\psi(||u-w||)\Big[\int_0^t \frac{(t-\omega)^{-\sigma}}{\Gamma(1-\sigma)}\Big[\frac{(\mu-1)}{\Gamma(\mu)}\int_0^{\omega}(\omega-\tau)^{\mu-2}d\tau \\
& + \frac{m|\beta_1||\beta_2|(2-\mu)}{|\Delta|}\int_0^1(1-\tau)^{1-\mu}\int_0^{\tau}\frac{(\tau-s)^{\mu-2}}{\Gamma(\mu-1)}dsd\tau \\
& + \frac{m|\beta_1|\Gamma(3-\mu)}{|\Delta|\Gamma(\mu)}\int_0^{\xi}(\xi-\tau)^{\mu-1}d\tau \\
& + \frac{|\beta_1|\Gamma(3-\mu)}{|\Delta|\Gamma(\mu)}\sum_{k=1}^{m}\int_0^{\xi_k}(\xi_k-\tau)^{\mu-1}d\tau\Big]d\omega\Big].
\end{aligned}
$$

By calculating the integrals and using the results of remark 3.1, we get

$$
\begin{aligned}
|{}^c\mathcal{D}^{\sigma}_{0+}y_1(t) - {}^c\mathcal{D}^{\sigma}_{0+}y_2(t)| &\leq \frac{1}{\Lambda_1+\Lambda_2}||\gamma||\psi(||u-w||)\Big\{\frac{1}{\Gamma(\mu)\Gamma(2-\sigma)} + \frac{|\beta_2|A_1(2-\mu)}{\Gamma(\mu)|\Delta|} \\
&\quad + \frac{\Gamma(3-\mu)A_1}{\Gamma(\mu+1)|\Delta|} + \frac{\Gamma(3-\mu)B_1 m}{\Gamma(\mu+1)|\Delta|}\Big\} \\
|{}^c\mathcal{D}^{\sigma}_{0+}y_1(t) - {}^c\mathcal{D}^{\sigma}_{0+}y_2(t)| &\leq \frac{\Lambda_2}{\Lambda_1+\Lambda_2}\psi(||u-w||). \qquad (3.18)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
||y_1(t)-y_2(t)|| &= \sup_{t\in\mathbb{O}}|y_1(t)-y_2(t)| + \sup_{t\in\mathbb{O}}|{}^c\mathcal{D}^{\sigma}_{0+}y_1(t) - {}^c\mathcal{D}^{\sigma}_{0+}y_2(t)| \\
&\leq \frac{1}{\Lambda_1+\Lambda_2}\psi(||u-w||)(\Lambda_1+\Lambda_2) = \psi(||u-w||).
\end{aligned}
$$

Hence $H_d(N(u), N(w)) \leq \psi(||u-w||)$ for all $u, w \in \mathbb{X}$. Using lemma 1.1. Since the set valued function N has approximate endpoint property by $(\mathcal{A}_4)$, then there exists $u^* \in \mathbb{X}$

such that $N(u^*) = \{u^*\}$, which leads to the existence of at least one solution for the inclusion problem (3.1), and that completes the proof. ∎

Remak 3.3 *Notice that 0 is always a solution if $\mathcal{F}(t, x(t), y(t))$ contains 0.*

3.1.3 Uniqueness approach

Theorem 3.2 *Let the hypotheses of theorem 3.1 be satisfied, and assume that:*

- *$\mathcal{F}(t, ., .)$ is Lipschitzean, that is $\exists k > 0$ such that $\forall x, y \in \mathbb{R}$*

$$\mathcal{F}(t, x, .) \subset \mathcal{F}(t, y, .) + B(0, k|x - y|)$$

  where $B(0, k|x - y|) = [-k|x - y|, +k|x - y|]$.

- *The following condition is satisfied: $(\Lambda_1 + \Lambda_2)k < 1$*

Then the problem main inclusion (3.1) has a unique solution.

Proof: Let u_1, u_2 be two solutions of the main inclusion(3.1). This leads to $\{u_1\} = N(u_1)$ and $\{u_2\} = N(u_2)$ with:

$$\begin{aligned}
u_1(t) &= \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} \nu_1(\tau) d\tau \\
&+ \frac{\beta_2(2-\mu)A(t)}{\Delta} \int_0^1 (1-\tau)^{1-\mu} \int_0^\tau \frac{(\tau - s)^{\mu-2}\nu_1(s)}{\Gamma(\mu-1)} ds d\tau \\
&+ \frac{\Gamma(3-\mu)A(t)}{\Delta\Gamma(\mu)} \int_0^\xi (\xi - \tau)^{\mu-1} \nu_1(\tau) d\tau \\
&+ \frac{\Gamma(3-\mu)B(t) + \beta_1\beta_2}{\Delta\Gamma(\mu)} \sum_{k=1}^m \int_0^{\xi_k} (\xi_k - \tau)^{\mu-1} \nu_1(\tau) d\tau
\end{aligned}$$

$$
\begin{aligned}
u_2(t) = & \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} \nu_2(\tau) d\tau \\
& + \frac{\beta_2(2-\mu)A(t)}{\Delta} \int_0^1 (1-\tau)^{1-\mu} \int_0^\tau \frac{(\tau-s)^{\mu-2}\nu_2(s)}{\Gamma(\mu-1)} ds d\tau \\
& + \frac{\Gamma(3-\mu)A(t)}{\Delta\Gamma(\mu)} \int_0^\xi (\xi-\tau)^{\mu-1} \nu_2(\tau) d\tau \\
& + \frac{\Gamma(3-\mu)B(t) + \beta_1\beta_2}{\Delta\Gamma(\mu)} \sum_{k=1}^m \int_0^{\xi_k} (\xi_k-\tau)^{\mu-1} \nu_2(\tau) d\tau,
\end{aligned}
$$

$$
\begin{aligned}
|u_1(t) - u_2(t)| = & \Big| \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} (\nu_1(\tau) - \nu_2(\tau)) d\tau \\
& + \frac{\beta_2(2-\mu)A(t)}{\Delta} \int_0^1 (1-\tau)^{1-\mu} \int_0^\tau \frac{(\tau-s)^{\mu-2}(\nu_1(s) - \nu_2(s))}{\Gamma(\mu-1)} ds d\tau \\
& + \frac{\Gamma(3-\mu)A(t)}{\Delta\Gamma(\mu)} \int_0^\xi (\xi-\tau)^{\mu-1} (\nu_1(\tau) - \nu_2(\tau)) d\tau \\
& + \frac{\Gamma(3-\mu)B(t) + \beta_1\beta_2}{\Delta\Gamma(\mu)} \sum_{k=1}^m \int_0^{\xi_k} (\xi_k-\tau)^{\mu-1} (\nu_1(\tau) - \nu_2(\tau)) d\tau \Big|, \\
\leq & \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} |\nu_1(\tau) - \nu_2(\tau)| d\tau \\
& + \frac{|\beta_2|(2-\mu)|A(t)|}{|\Delta|} \int_0^1 (1-\tau)^{1-\mu} \int_0^\tau \frac{(\tau-s)^{\mu-2}|\nu_1(s) - \nu_2(s)|}{\Gamma(\mu-1)} ds d\tau \\
& + \frac{\Gamma(3-\mu)|A(t)|}{|\Delta|\Gamma(\mu)} \int_0^\xi (\xi-\tau)^{\mu-1} |\nu_1(\tau) - \nu_2(\tau)| d\tau \\
& + \frac{\Gamma(3-\mu)|B(t)| + |\beta_1||\beta_2|}{|\Delta|\Gamma(\mu)} \sum_{k=1}^m \int_0^{\xi_k} (\xi_k-\tau)^{\mu-1} |\nu_1(\tau) - \nu_2(\tau)| d\tau
\end{aligned}
$$

Relaying on the first hypothesis we get

$$
|\nu_1(t) - \nu_2(t)| \leq k||u_1 - u_2||_{\mathbb{X}}
$$

It results that

$$\begin{aligned}|u_1(t)-u_2(t)| \leq & \frac{1}{\Gamma(\mu)}\int_0^t (t-\tau)^{\mu-1}k||u_1-u_2||_{\mathbb{X}}(\tau)d\tau \\ & + \frac{|\beta_2|(2-\mu)|A(t)|}{|\Delta|}\int_0^1 (1-\tau)^{1-\mu}\int_0^\tau \frac{(\tau-s)^{\mu-2}k||u_1-u_2||_{\mathbb{X}}}{\Gamma(\mu-1)}dsd\tau \\ & + \frac{\Gamma(3-\mu)|A(t)|}{|\Delta|\Gamma(\mu)}\int_0^\xi (\xi-\tau)^{\mu-1}k||u_1-u_2||_{\mathbb{X}}d\tau \\ & + \frac{\Gamma(3-\mu)|B(t)|+|\beta_1||\beta_2|}{|\Delta|\Gamma(\mu)}\sum_{k=1}^m \int_0^{\xi_k}(\xi_k-\tau)^{\mu-1}k||u_1-u_2||_{\mathbb{X}}d\tau\end{aligned}$$

$$\begin{aligned}|u_1(t)-u_2(t)| \leq & \Big\{\frac{1}{\Gamma(\mu+1)}+\frac{|\beta_2|A_0}{\Gamma(\mu)|\Delta|} \\ & + \frac{\Gamma(3-\mu)A_0}{\Gamma(\mu+1)|\Delta|}+\frac{(\Gamma(3-\mu)B_0+|\beta_1||\beta_2|)m}{\Gamma(\mu+1)|\Delta|}\Big\}k||u_1-u_2||_{\mathbb{X}} \\ |u_1(t)-u_2(t)| \leq & k\Lambda_1||u_1-u_2||_{\mathbb{X}}\end{aligned}$$

On the other hand, in the same way we have

$$\begin{aligned}|{}^c\mathcal{D}^\sigma_{0^+}u_1(t)-{}^c\mathcal{D}^\sigma_{0^+}u_2(t)| \leq & \Big\{\frac{1}{\Gamma(\mu)\Gamma(2-\sigma)}+\frac{|\beta_2|A_1(2-\mu)}{\Gamma(\mu)|\Delta|} \\ & + \frac{\Gamma(3-\mu)A_1}{\Gamma(\mu+1)|\Delta|}+\frac{\Gamma(3-\mu)B_1 m}{\Gamma(\mu+1)|\Delta|}\Big\}k||u_1-u_2||_{\mathbb{X}} \\ |{}^c\mathcal{D}^\sigma_{0^+}u_1(t)-{}^c\mathcal{D}^\sigma_{0^+}u_2(t)| \leq & k\Lambda_2||u_1-u_2||_{\mathbb{X}}\end{aligned}$$

Then

$$||u_1-u_2||_{\mathbb{X}} = \sup_{t\in[0,1]}|u_1(t)-u_2(t)|+\sup_{t\in[0,1]}|{}^c\mathcal{D}^\sigma_{0^+}u_1(t)-{}^c\mathcal{D}^\sigma_{0^+}u_2(t)| \leq k(\Lambda_1+\Lambda_2)||u_1-u_2||_{\mathbb{X}}$$

Using the second condition of the theorem 3.2

$$\big(1-k(\Lambda_1+\Lambda_2)\big)||u_1-u_2||_{\mathbb{X}} \leq 0.$$

$$||u_1-u_2||_{\mathbb{X}} \leq 0$$

This means that

$$u_1(t) = u_2(t) \quad \forall t \in [0,1]$$

$\blacksquare$

3.1.4 First numerical example

Let's consider the following fractional differential inclusion:

$$^{c}\mathcal{D}_{0^+}^{3/2}u(\mathbf{t}) \in \Big[0, \frac{4|u(t)|}{\Gamma(1/2)}\Big], \quad \mathbf{t} \in \mathbb{O} = [0,1], \tag{3.19}$$

With the boundary conditions:

$$\begin{cases} u'(0) = 0, \\ \Gamma(1/2)^{c}\mathcal{D}_{0^+}^{1/2}u(1) + u(0.8) = 1.3375 \sum_{i=1}^{6} u(\xi_i) \end{cases} \tag{3.20}$$

with $\xi_1 = 0.2, \xi_2 = 0.3, \xi_3 = 0.4, \xi_4 = 0.5, \xi_5 = 0.6, \xi_6 = 0.7$

It's clear that

$\mu = 3/2, m = 6, \beta_1 = 0, \beta_2 = \Gamma(1/2), \beta_3 = 1.3375$

Now we define: $\mathcal{F} : \mathbb{O} \times \mathbb{R}^2 \longrightarrow \mathcal{P}(\mathbb{R})$ by:

$$\mathcal{F}(t, x(t), y(t)) = \Big[0, \frac{4|x(t)|}{\Gamma(1/2)}\Big], \quad \mathbf{t} \in \mathbb{O} = [0,1],$$

In this case we choose $\gamma : [0,1] \to [0, \infty[$ such that $\gamma(t) = \frac{8}{\Gamma(1/2)}$ and $\psi(t) = \frac{t}{2}$

Obviously we have

- $\liminf_{t\to\infty}(t - \frac{t}{2}) = \liminf_{t\to\infty}(\frac{t}{2}) > 0.$
- $\psi(t) = \frac{t}{2} < t \;\; \forall t > 0.$

We aim to prove

$$H_d(\mathcal{F}(t, x_1, y_1), \mathcal{F}(t, x_2, y_2)) \le \frac{1}{\Lambda_1 + \Lambda_2}\gamma(t)\psi(|x_1(t) - x_2(t)|)$$

So we consider

$$\mathcal{F}^*(t, x^*(t), y^*(t)) = \Big[0, \frac{0.04|x^*(t)|}{\Gamma(1/2)}\Big], \qquad \mathbf{t} \in \mathbb{O} = [0, 1],$$

As a result

$$\begin{aligned} H_d(\mathcal{F}(t, x_1(t), y_1(t)), \mathcal{F}(t, x_2(t), y_2(t))) &\leq \frac{0.04}{\Gamma(1/2)}|x_1^*(t) - x_2^*(t)| \\ &\leq \frac{0.08}{\Gamma(1/2)} \frac{|x_1^*(t) - x_2^*(t)|}{2} \\ &\leq \gamma^*(t)\psi(|x_1^*(t) - x_2^*(t)|) \end{aligned}$$

Now we calculate some important values using $\gamma^*(t) = \frac{0.08}{\Gamma(1/2)}$ we get

$$\Lambda_1^* = 0.0597493012793817, \quad \Lambda_2^* = 0.056416336022008656$$

$$\Lambda_1^* + \Lambda_2^* = 0.11616563730139036$$

Then, for any $x^*, y^* \in \mathbb{R}$, we have

$$H_d(\mathcal{F}^*(t, x_1^*, y_1^*), \mathcal{F}(t, x_2^*, y_2^*)) \leq \frac{\gamma^*(t)}{\Lambda_1^* + \Lambda_2^*}\psi(|x_1^*(t) - x_2^*(t)|)$$

Taking $x_1(t) = 100x_1^*(t), \gamma(t) = 100\gamma^*(t)$ gives us

$$H_d(\mathcal{F}(t, x_1(t), y_1(t)), \mathcal{F}(t, x_2(t), y_2(t))) \leq \frac{1}{\Lambda_1 + \Lambda_2}\gamma(t)\psi(|x_1(t) - x_2(t)|)$$

Let $\mathbb{X} = \{u : u, {}^c\mathcal{D}_{0^+}^{3/2} u \in C(\mathbb{O}, \mathbb{R})\}$, we define the operator $N : (X) \to \mathcal{P}(\mathbb{R})$ such that

$$\begin{aligned} N(u) = \{f \in \mathbb{X} : \exists \nu \in S_{\mathcal{F},u} \\ f(t) &= \frac{1}{\Gamma(3/2)} \int_0^t (t-\tau)^{1/2} \nu(\tau) d\tau \\ &+ \frac{1}{2\Delta} \int_0^1 (1-\tau)^{-1/2} \int_0^\tau (\tau - s)^{-1/2} \nu(s) ds d\tau \\ &+ \frac{1}{\Delta} \int_0^{0.8} (0.8-\tau)^{1/2} \nu(\tau) d\tau \\ &- \frac{1.3375}{\Delta} \sum_{i=1}^{6} \int_0^{\xi_i} (\xi_i - \tau)^{1/2} \nu(\tau) d\tau \} \end{aligned}$$

Since $\sup_{u \in N(0)} ||u||_{\mathbb{X}} = 0$, it results $\inf_{u \in \mathbb{X}} \sup_{w \in N(u)} ||u - w||_{\mathbb{X}} = 0$.

Thus, N has the approximate endpoint property, therefore the conditions of the theorem 3.1 are satisfied which implies that the fractional differential inclusion problem (3.19) has a solution.

3.1.5 Second numerical example

Let's consider the following fractional differential inclusion:

$$ {}^c\mathcal{D}_{0^+}^{3/2} u(\mathbf{t}) \in \{\frac{4|u(t)|}{\Gamma(1/2)}\}, \qquad \mathbf{t} \in \mathbb{O} = [0, 1], \tag{3.21}$$

With the boundary conditions:

$$\begin{cases} u'(0) = 0, \\ \Gamma(1/2) {}^c\mathcal{D}_{0^+}^{1/2} u(1) + u(0.8) = 1.3375 \sum_{i=1}^{6} u(\xi_i) \end{cases} \tag{3.22}$$

with $\xi_1 = 0.2, \xi_2 = 0.3, \xi_3 = 0.4, \xi_4 = 0.5, \xi_5 = 0.6, \xi_6 = 0.7$

It's clear that

$\mu = 3/2, m = 6, \beta_1 = 0, \beta_2 = \Gamma(1/2), \beta_3 = 1.3375$

In a similar way as the first example, we can verify the existence theorem. As a result there exist a solution for the problem (3.21).

For the uniqueness, and since $\mathcal{F}(t, x(t), y(t))) = \{\frac{4|u(t)|}{\Gamma(1/2)}\}$, then the inequality

$$|\nu_1(t) - \nu_2(t)| \leq k||u - v||_{\mathbb{X}}$$

holds for every $k > 0$.

Choosing $k = 0.04$ and with $\Lambda_1 = 11.94986025587634, \Lambda_2 = 11.28326720440173$ we get:

$$(\Lambda_1 + \Lambda_2)k = 0.9293250984111228 < 1$$

Then the problem (3.21) admits only one solution.

3.1.6 Third numerical example

Let's consider the following fractional differential inclusion:

$$^c\mathcal{D}_{0^+}^{3/2} u(\mathbf{t}) \in \{\frac{8|u(t)|}{\Gamma(1/2)}\}, \qquad \mathbf{t} \in \mathbb{O} = [0, 1], \tag{3.23}$$

With the boundary conditions:

$$\begin{cases} u'(0) = -12.5 \sum_{i=1}^{6} u(\xi_i), \\ \Gamma(1/2)^c\mathcal{D}_{0^+}^{1/2} u(1) + u(0.8) = 1.3375 \sum_{i=1}^{6} u(\xi_i) \end{cases} \tag{3.24}$$

with $\xi_1 = 0.2, \xi_2 = 0.3, \xi_3 = 0.4, \xi_4 = 0.5, \xi_5 = 0.6, \xi_6 = 0.7$

It's clear that

$\mu = 3/2, m = 6, \beta_1 = -12.5, \beta_2 = \Gamma(1/2), \beta_3 = 1.3375$

Now we define: $\mathcal{F} : \mathbb{O} \times \mathbb{R}^2 \longrightarrow \mathcal{P}(\mathbb{R})$ by:

$$\mathcal{F}(t, x(t), y(t)) = \{\frac{8|x(t)|}{\Gamma(1/2)}\}, \qquad \mathbf{t} \in \mathbb{O} = [0, 1],$$

In this case we choose $\gamma : [0, 1] \to [0, \infty[$ such that $\gamma(t) = \frac{16}{\Gamma(1/2)}$ and $\psi(t) = \frac{t}{2}$

Obviously we have

- $\liminf_{t\to\infty}(t - \frac{t}{2}) = \liminf_{t\to\infty}(\frac{t}{2}) > 0.$

- $\psi(t) = \frac{t}{2} < t \quad \forall t > 0.$

We aim to prove

$$H_d(\mathcal{F}(t, x_1, y_1), \mathcal{F}(t, x_2, y_2)) \leq \frac{1}{\Lambda_1 + \Lambda_2}\gamma(t)\psi(|x_1(t) - x_2(t)|)$$

So we consider

$$\mathcal{F}^*(t, x^*(t), y^*(t)) = \{\frac{0.08|u(t)|}{\Gamma(1/2)}\}, \qquad \mathbf{t} \in \mathbb{O} = [0, 1],$$

As a result

$$\begin{aligned} H_d(\mathcal{F}(t, x_1(t), y_1(t)), \mathcal{F}(t, x_2(t), y_2(t))) &\leq \frac{0.08}{\Gamma(1/2)}|x_1^*(t) - x_2^*(t)| \\ &\leq \frac{0.16}{\Gamma(1/2)}\frac{|x_1^*(t) - x_2^*(t)|}{2} \\ &\leq \gamma^*(t)\psi(|x_1^*(t) - x_2^*(t)|) \end{aligned}$$

Now we calculate some important values using $\gamma^*(t) = \frac{0.16}{\Gamma(1/2)}$ we get

$$\Lambda_1^* = 0.3524008759077773, \quad \Lambda_2^* = 0.22057069743272514$$

$$\Lambda_1^* + \Lambda_2^* = 0.5729715733405024$$

Then, for any $x^*, y^* \in \mathbb{R}$, we have

$$H_d(\mathcal{F}^*(t, x_1^*, y_1^*), \mathcal{F}(t, x_2^*, y_2^*)) \leq \frac{\gamma^*(t)}{\Lambda_1^* + \Lambda_2^*}\psi(|x_1^*(t) - x_2^*(t)|)$$

Taking $x_1(t) = 100x_1^*(t), \gamma(t) = 100\gamma^*(t)$ gives us

$$H_d(\mathcal{F}(t, x_1(t), y_1(t)), \mathcal{F}(t, x_2(t), y_2(t))) \leq \frac{1}{\Lambda_1 + \Lambda_2}\gamma(t)\psi(|x_1(t) - x_2(t)|)$$

Let $\mathbb{X} = \{u : u, {}^c\mathcal{D}^{3/2}_{0^+}u \in C(\mathbb{O}, \mathbb{R})\}$, we define the operator $N : (X) \to \mathcal{P}(\mathbb{R})$ such that

$$\begin{aligned} N(u) = \{f \in \mathbb{X} : \exists \nu \in S_{\mathcal{F},u} \\ f(t) &= \frac{1}{\Gamma(3/2)} \int_0^t (t-\tau)^{1/2}\nu(\tau)d\tau \\ &+ \frac{A(t)}{2\Delta} \int_0^1 (1-\tau)^{-1/2} \int_0^\tau (\tau - s)^{-1/2}\nu(s)dsd\tau \\ &+ \frac{A(t)}{\Delta} \int_0^{0.8} (0.8-\tau)^{1/2}\nu(\tau)d\tau \\ &+ \frac{\Gamma(3/2)B(t) - 12.5\Gamma(1/2)}{\Delta\Gamma(3/2)} \sum_{i=1}^{6} \int_0^{\xi_i} (\xi_i - \tau)^{1/2}\nu(\tau)d\tau\} \end{aligned}$$

with $A(t) = -75t + 3.7$ and $B(t) = -12.5(0.8-t) - \Gamma(1/2)$.

Since $N(u)$ contains only one element because the set of selections contains only one element. Thus, N has the approximate endpoint property, therefore the conditions of the theorem 3.1 are satisfied which implies that the fractional differential inclusion problem (3.23) has a solution.

For the uniqueness, and since $\mathcal{F}(t, x(t), y(t))) = \{\frac{8|u(t)|}{\Gamma(1/2)}\}$, then the inequality

$$|\nu_1(t) - \nu_2(t)| \leq k||u - v||_{\mathbb{X}}$$

holds for every $k > 0$.

Choosing $k = 0.01$ and with $\Lambda_1 = 35.240087590777726, \Lambda_2 = 22.057069743272514$ we get:

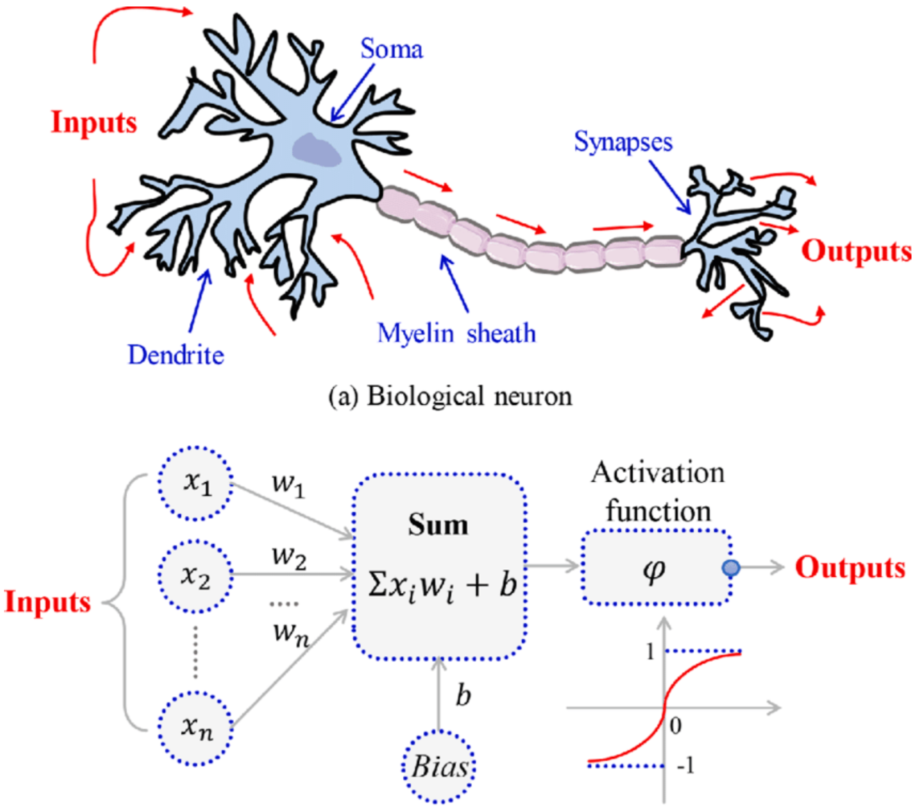
$$(\Lambda_1 + \Lambda_2)k = 0.5729715733405024 < 1$$

Then the problem (3.23) admits only one solution.

3.2 Numerical application

3.2.1 What is the ANN?

ANN stands for Artificial Neural Network. It is a computational model inspired by the structure and functioning of biological neural networks, such as the human brain (see the underneath figure[1]). ANNs are widely used in the field of machine learning and are designed to mimic the way neurons in the brain process information.



(a) Biological neuron

(b) Artificial neuron

Figure 3.1: Comparison between biological neuron and artificial neuron

ANNs are capable of learning and recognizing complex patterns and relationships in data, making them valuable for tasks such as classification, regression, pattern recognition, and decision making.

[1]This figure was taking from the article [30] by Xianlin Wang, Yuqing Liu, and Haohui Xin.

3.2.2 Approximating the solution using artificial neural networks

The initial endeavor to simulate a biological neuron was accomplished by McCulloch and Pitts[31] in 1943, where they combined a step function and an affine combination of signals. In 1958, Rosenblatt[32] introduced a multilayer perceptron as a basic model for a neuron network. This model consists of an input layer, hidden layers, and an output layer, as depicted in Figure 1. By utilizing this structure, a function A is constructed:

$$A : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$$

$$x \longmapsto x_L$$

where x_L is the result of the following scheme:

- Through input layer: $x_0 = x$.
- Through hidden layer: $x_l = \rho(A_l(x_{l-1})), \quad \text{for } l = 1, \dots, L-1$.
- Through output layer: $x_L = A_L(x_{L-1})$.

where $\rho : \mathbb{R} \to \mathbb{R}$ is an activation function, which acts as follows: $\rho(x_1, \dots, x_{N_l}) = (\rho(x_1), \dots, \rho(x_{N_l}))$, and $A_l = (x) = W_l x + b_l$, with $(W_l)_{l=1}^{L} \in \mathbb{R}^{N_l \times N_{l-1}}$ denotes the weights, and $b_l \in \mathbb{R}^{N_l}$ denotes the biases. The structure of such a neural network is defined by $N_0, N_1, \dots, N_L$ and the activation function ρ. We denote the set of such functions by $\mathcal{NN}$. The density of this set is important in the approximation.

Many theorems of density were obtained such as Cybenko[33] in 1989, Kurt Hornik[34] in 1991 and more, following that multiple outcomes are obtained regarding the quantity of neurons in a hidden layer to demonstrate the density of artificial neural networks within diverse function spaces.

Note that $\mathcal{NN}^{\rho}_{n,m,k}$ is the class of functions $\mathbb{R}^n \to \mathbb{R}^m$ characterized by feedforward neural networks with n neurons in the input layer, m neurons in the output layer, and k neurons in the hidden layers. In this problem we use the following theorem of density:

Theorem 3.3 *[35, page 2] Let $\rho : \mathbb{R} \to \mathbb{R}$ be any nonaffine continuous function which is continuously differentiable at at least one point, with nonzero derivative at that point. Let $K \in \mathbb{R}^n$ be compact. Then $\mathcal{NN}^{\rho}_{n,m,n+m+2}$ is dense in $\mathcal{C}(K, \mathbb{R}^m)$ with respect to the uniform norm.*

3.2.3 The used algorithm

In the following $u(t)$ represents the solution while $y(t)$ represents the selection.

- We are looking for a certain $u_\theta(t)$ such that:

$$\begin{cases} {}^c\mathcal{D}^{\mu}_{0+} u_\theta(\mathbf{t}) \in \mathcal{F}\big(\mathbf{t}, u_\theta(\mathbf{t}), {}^c\mathcal{D}^{\sigma}_{0+} u_\theta(\mathbf{t})\big), \qquad \mathbf{t} \in \mathbb{O} = [0,1], \\ u'_\theta(0) - \beta_1 \displaystyle\sum_{k=1}^{m} u_\theta(\xi_k) = 0, \ldots (I_0) \\ \beta_2^c \mathcal{D}^{\mu-1}_{0+} u_\theta(1) + u_\theta(\xi) - \beta_3 \sum_{k=1}^{m} u_\theta(\xi_k) = 0, \ldots (I_1) \end{cases}$$

- To make sure that ${}^c\mathcal{D}^{\mu}_{0+} u_\theta(\mathbf{t})$ is in the set of selections we verify if the projection of this element on the set of selections is itself, that is:

$$P_{\mathcal{S}_\mathcal{F}} {}^c\mathcal{D}^{\mu}_{0+} u_\theta(\mathbf{t}) - {}^c\mathcal{D}^{\mu}_{0+} u_\theta(\mathbf{t}) = 0, \ldots (I(t))$$

  with $P_{[a,b]}x = \max(a, \min(x, b))$

- We chose the lost function formula according to our objective, in general we have:

$$Loss_\theta = \min |I(t)| + |I_0| + |I_1|$$

- For θ we have the following formula:

$$\theta_{n+1} = \theta_n - l \times d_n$$

where

- l is the learning rate.
- d_n is given by $d_n = \nabla_\theta Loss(\theta_n)$

- The number of layers and nodes were taken according to the theorem 3.3, we can see it in the following figure:

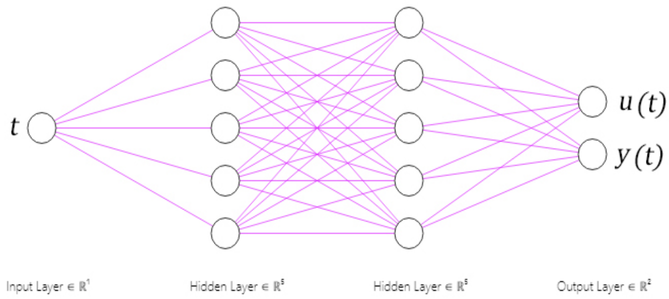


Figure 3.2: Neurons structure

- The activation function used was **Sigmoid**, while the optimizer chosen was **Adam**.

3.2.4 The obtained results

Applying the ANN method on the example (3.19) revealed the following results with the number of epochs is 150 :

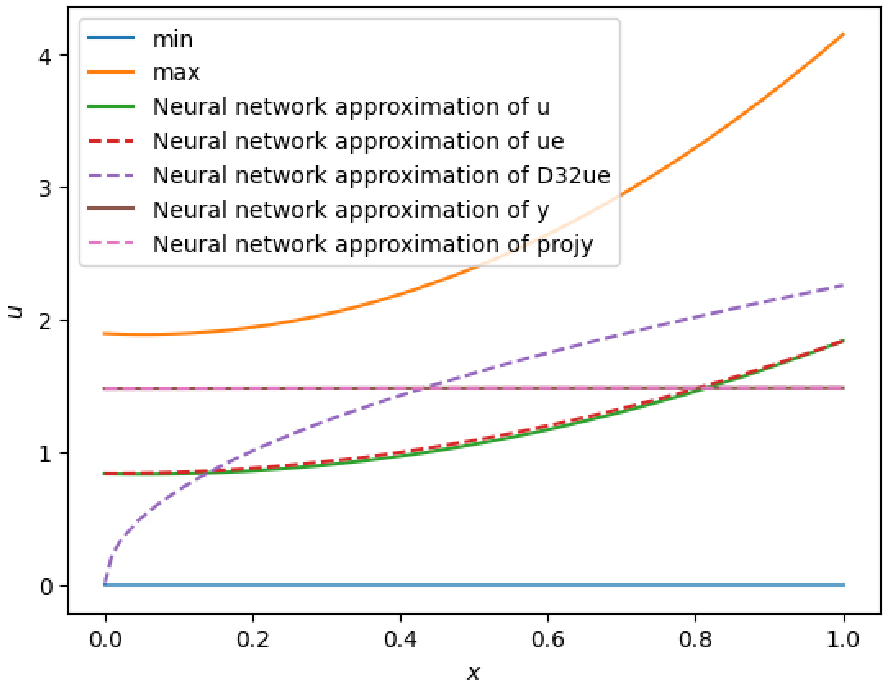


Figure 3.3: ANN results

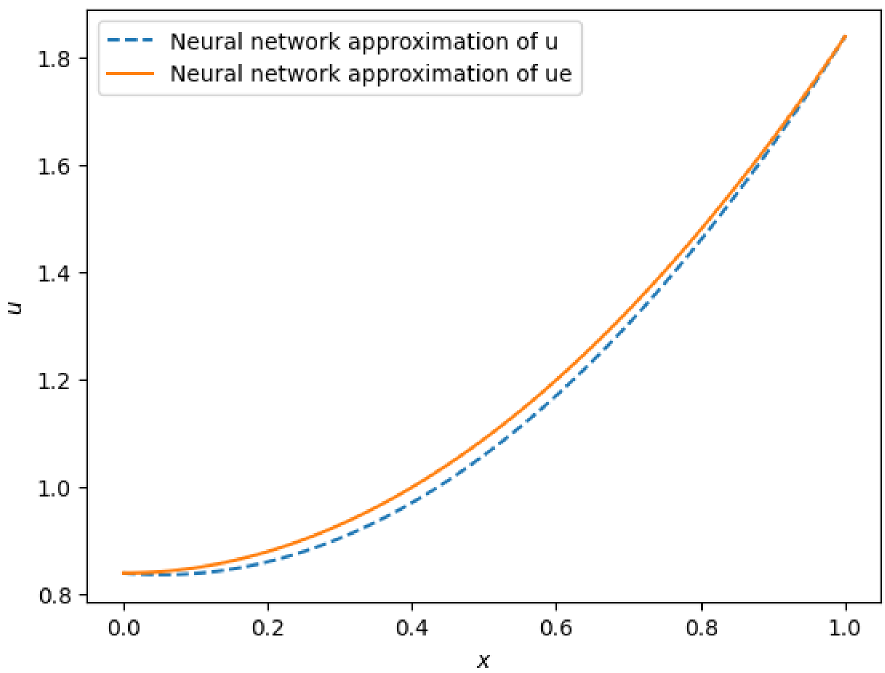


Figure 3.4: Comparison between the exact and the approximate solutions

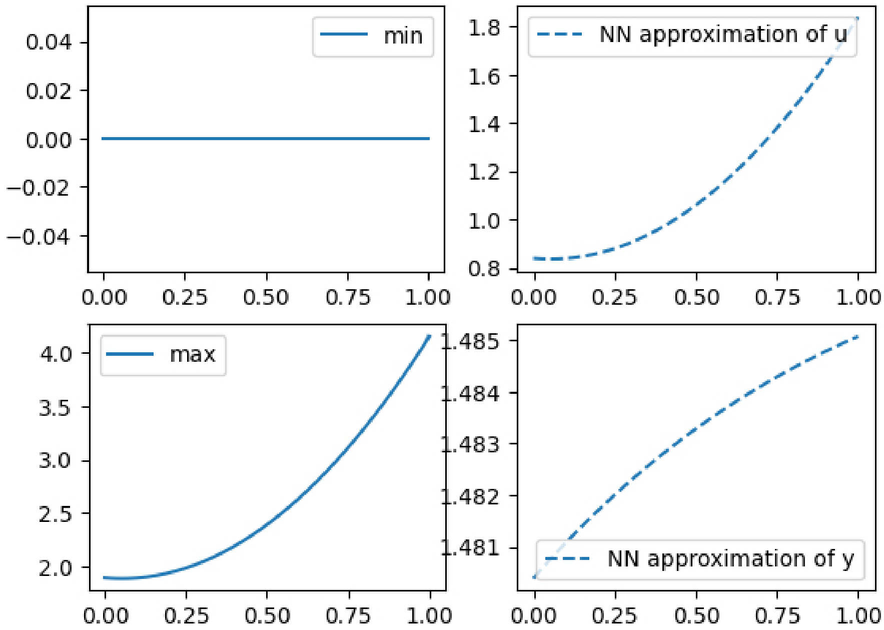


Figure 3.5: The approximate solution graphs

And on the example (3.21) with the number of epochs is 350 :

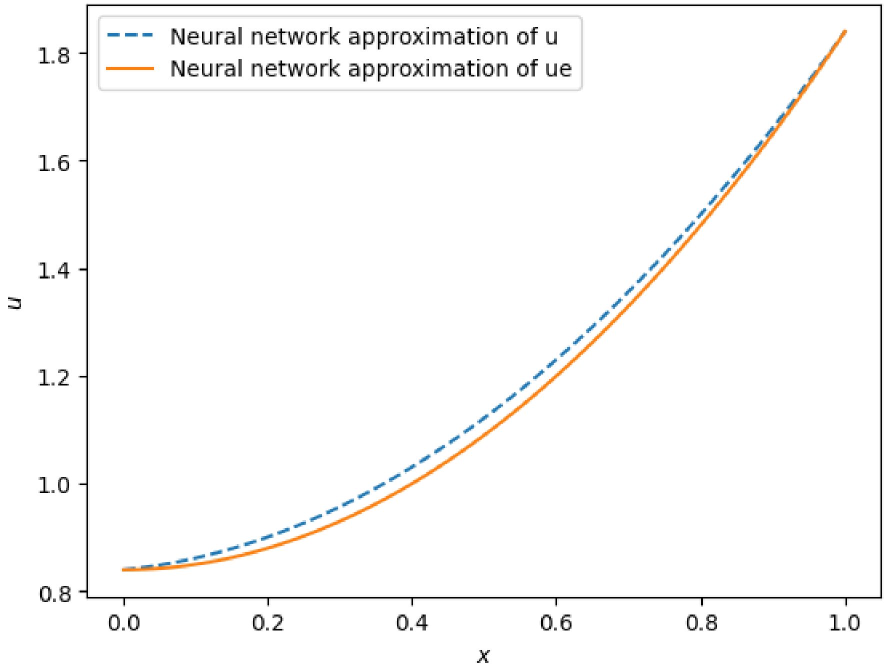


Figure 3.6: ANN results

And on the example (3.23) with the number of epochs is 400 :

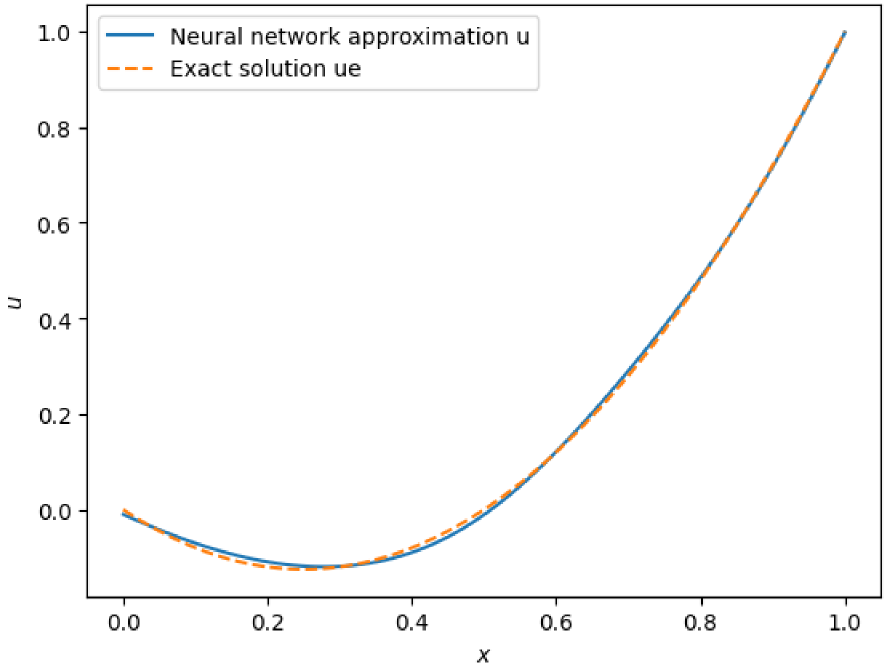


Figure 3.7: ANN results

Observation:

From observing the figures above, we can see that the approximated function of u by the ANN is indeed close to the exact solution u_e in the three examples.
However, this is not the case for the selection y and the Caputo derivative ${}^{c}\mathcal{D}_{0+}^{3/2}u_e(\mathbf{t})$, even though both of them are in the range of $\mathcal{F}$. We interpret this by the possibility that the equivalence between the inclusion problem (3.1) and the integral equation (3.16) does not hold on the numerical level, so it can be seen as an implication only.

Conclusion

In conclusion, this master thesis has explored the fascinating field of fractional differential inclusions and their applications. Through a comprehensive study of fractional derivatives, inclusions, and their existence theorems, we have studied the existence of perturbed and unperturbed inclusions. We also investigated the existence and uniqueness of the solution for a thermostat inclusion model with numerical application.

Upon completing this study, several questions arise:

What would be the difference if we chose the fixed point approach? will it provide a better results?

Is there a way to extend the results to some Hilbert spaces?

In comparison to the integer derivative, where exactly can we spot the difference?

In the end, I wish that this work can provide even a small spotlight in this field.

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ملخص:
في هذا العمل، قمنا بدراسة وجود ووحدانية الحلول لبعض الإنتماءات التفاضلية الكسرية، باستعمال مفهوم نقطة النهاية، ثم طبقنا ال ANN على النتائج المتحصلة.
الكلمات المفتاحية: الانتماءات التفاضلية الكسرية، دوال المجموعات، الحساب الكسري.

Résumé :
Dans ce travail, nous avons étudié l'existence et l'unicité des solutions pour certaines inclusions différentielles fractionnaires, en utilisant le concept de point final, puis nous avons appliqué l'ANN sur les résultats obtenus.
mots clés: Inclusions différentielles fractionnaires, fonctions multivoques, calcul fractionnaire.

Summary:
In this work, we investigated the existence and uniqueness of the solutions for some fractional differential inclusions, using the concept of endpoint, then we applied the ANN on the obtained results.
Key words: Fractional differential inclusions, set-valued functions, fractional calculus.