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Homogenization of a periodic plate with variable thickness by the unfolding method

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Dedication

This study is wholeheartedly dedicated to the Almighty God for the guidance, strength, power of mind, protection and skills, and for giving us a healthy life. To my beloved parents who have been my source of inspiration and gave me the strength when I thought of giving up, who continually provide their moral, spiritual, emotional, and financial support. And lastly, to my sisters , brothers , relatives, mentors, friends , and classmates who shared their words of advice and encouragement to finish this study.

Thanks

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Notations and Conventions

Notations

- $\partial_i = \frac{\partial}{\partial x_i}$: partial differentiation with respect to x_i.
- $C^m(\Omega)$: Space of m-times continuously differentiable functions on Ω , for $m \in \mathbb{R}$.
- $L^p(\Omega)$: Space of the integrable functions on Ω with respect to the Lebesgue mesure dx , for $p \in [1, +\infty[$.
- $L^\infty(\Omega)$: Space of bounded functions on Ω.
- $H^m(\Omega)$: Sobolev Space of order m , for $m \in \mathbb{N}$.
- $H^1_0(\Omega)$: Space of functions in $H^1(\Omega)$ vanishing on the boundary .
- $\|.\|_V$: The norm in the space V .
- $|.|_V$: Semi-norm (which may be a norm) .
- $(n_i) : \partial\Omega \to \mathbb{R}^3$: Unit outer normal vector along the boundary $\partial\Omega$ of Ω .
- π^δ : Bijection from on $\overline{\Omega}$ on to $\overline{\Omega}^{\delta\varepsilon}$.
- $e^{\delta\varepsilon}(u^{\delta\varepsilon}) = (e^{\delta\varepsilon}_{ij}(u^{\delta\varepsilon}))$: Linearized strain tensor .
- $\sigma^{\delta\varepsilon} = (\sigma^{\delta\varepsilon}_{ij})$: Stress tensor.
- $\to$: Strong convergence.

- $\rightharpoonup$: Weak convergence.

(i) Latin indices and exponents: $i, j, ...$, take their values in the set $\{1, 2, 3\}$, unless otherwise indicated, as when they are used for indexing sequences.

(ii) Greek indices and exponents: $\alpha, \beta, ..., except \varepsilon$, take their values in the set $\{1, 2\}$.

(iii) The repeated index summation convention is systematically used in conjunction with rules (i) and (ii).

Conventions

- Plate mid-surface limit displacements:

  For every $\mathcal{V}$ belonging to $\mathbb{D}_M$, $\mathcal{V} = (\mathcal{V}_m, \mathcal{V}_3) = (\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3)$ define the symmetric matrix $E_M(\mathcal{V})$ as

$$E_M(\mathcal{V}) = \begin{pmatrix} E_{11}(\mathcal{V}) & E_{12}(\mathcal{V}) & 0 \\ E_{11}(\mathcal{V}) & E_{12}(\mathcal{V}) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

  Where

$$E_{\alpha\beta}(\mathcal{V}) = e_{\alpha\beta}(\mathcal{V}_m) - \delta z_3 h \partial^2 \mathcal{V}_3 \partial x_\alpha \partial x_\beta$$

- The warping displacements:

  Set

$$\mathcal{W} = \{\psi^0 \in H^1(-1,1)^3 \mid \int_{-1}^{1} \psi^0(y_3) dy_3 = 0\}$$

  For every $v^0 \in L^2(w, \mathcal{W})$, define the symmetric matrix $E_w(v^0)$ by

$$E_w(v^0) = \begin{pmatrix} 0 & 0 & \frac{1}{2}\frac{\partial v_1^0}{y_3} \\ 0 & 0 & \frac{1}{2}\frac{\partial v_2^0}{y_3} \\ \frac{1}{2}\frac{\partial v_1^0}{y_3} & \frac{1}{2}\frac{\partial v_1^0}{y_3} & \frac{\partial v_3^0}{\partial y_3} \end{pmatrix}$$

- The microscopic displacements or the limit periodic cell displacements:

  Set

$$\mathcal{H}_{per}(Y) = \left\{\phi \in H^1(Y) \mid \phi_{|y_1=0} = \phi_{|y_1=1},\ \phi_{|y_2=0} = \phi_{|y_2=1}\right\}$$

$$D = \{\widehat{\psi} \in \mathcal{H}_{per}(Y) \mid \int_{Y'} \widehat{\psi}(y'y_3)dy' = 0 \ \text{ for a.e. } y_3 \in (-1,1)\}$$

  where $Y = Y' \times (-1,1)$. Introduce the symetric tensor

$$\mathcal{E}_y(\widehat{\psi}) = \begin{pmatrix} e_{11,y'}(\widehat{\psi}) & e_{12,y'}(\widehat{\psi}) & \frac{1}{2}\left(\frac{\partial\widehat{\Psi}_3}{y_1} + \frac{\partial\widehat{\psi}_1}{\partial y_3}\right) \\ e_{12,y'}(\widehat{\psi}) & e_{22,y'}(\widehat{\psi}) & \frac{1}{2}\left(\frac{\partial\widehat{\Psi}_3}{y_2} + \frac{\partial\widehat{\psi}_2}{\partial y_3}\right) \\ \frac{1}{2}\left(\frac{\partial\widehat{\Psi}_3}{y_2} + \frac{\partial\widehat{\psi}_2}{\partial y_3}\right) & \frac{1}{2}\left(\frac{\partial\widehat{\Psi}_3}{y_2} + \frac{\partial\widehat{\psi}_2}{\partial y_3}\right) & \frac{\partial\widehat{\Psi}_3}{\partial y_3} \end{pmatrix}$$

Introduction

The theory of thin structures like plates, shallow shells, shells and junctions between them, originated in the need on the part of engineers for tractable models to analyze and predict the response of thin structures to various kinds of loading. The basic idea is to exploit the **thinness** of the structure to represent the mechanics of the actual thin three-dimensional body under consideration by a more tractable two-dimensional theory associated with an interior surface. In this way, the relatively complex three-dimensional continuum mechanics of the thin body is replaced by a far more tractable two-dimensional theory. To ensure that the resulting model is predictive, it is necessary to compensate for this 'dimension reduction' by assigning additional kinematical and dynamical descriptors to the surface whose deformations are modeled by the simpler two-dimensional theory. An efficient method for obtaining dimension reduction in thin 3D structures is the asymptotic method developed by Ciarlet and Destuynder.

For thin elastic plates, there are several models used to describe their behavior, each with different assumptions and levels of complexity. Some of the most common models are:

Kirchhoff-Love model, Reissner-Mindlin model, Donnell model, Vlasov model.

Each of these models has its own advantages and limitations, and the choice of model depends on the specific application and the level of accuracy required.

In our study, we are interested in the asymptotic behavior of a thin elastic plate of variable thickness with a heterogeneous periodic structure, within the framework of classical linear elasticity. In this case, the thickness depends on the microscopic and macroscopic scales. Then the effective behavior of the plate is influenced by the variations in thickness across the structure which can lead to non-uniform stress and strain distributions throughout the plate.

For example, in a plate with a thicker region, the stiffness and strength of that region will be greater than those of a thinner region. This can lead to a concentration of stress and strain in the thinner region, which may result in localized deformation or failure. Moreover, the variations in thickness can also affect the natural frequencies and modes of vibration of the plate. This can be particularly important in applications where vibration control is critical, such as in aerospace or automotive engineering.

Periodic plates with variable thickness have a wide range of potential applications in various fields, including engineering, materials science, and physics. Here are a few examples:

1. Aerospace engineering: In the design of lightweight and high-strength structures for aerospace applications, such as aircraft wings and fuselages. By varying the thickness of the plate in a periodic manner, it is possible to achieve a favorable balance between stiffness, strength, and weight, which is crucial for optimizing the performance of aerospace structures.

2. Metamaterials: To design metamaterials with unique mechanical properties, such as negative Poisson's ratio, which have potential applications in areas such as vibration damping, energy absorption, and acoustic insulation.

3. Biomechanics: The design of artificial bone implants. By varying the thickness of the plate in a periodic manner, it is possible to mimic the natural structure of bone, which has a complex hierarchical structure with variations in thickness and density.

4. Energy harvesting: To design energy harvesting devices, such as piezoelectric plates. By varying the thickness of the plate in a periodic manner, it is possible to optimize the energy harvesting properties of the plate and increase its efficiency.

Homogenization is a collection of methods to approximate a heterogeneous problem by homogeneous one.

Classicaly, the theory of homogenization studies the behavior of a model (typically, a PDE or an energy functional) with heterogeneous coefficients that periodically oscillate on a small scale, say ε.

There are different methods of homogenization:

- Two scale asymptotic expansions method for periodic media.
- **H** or **G convergence** method for general media.
- Stochastic homogenization.
- Variational methods (Γ - convergence).
- Two-scale convergence method.
- Unfoding method.

The homogenization of periodic structures has been a topic of interest in the field of mechanics, as it allows us to study the effective behavior of materials composed of a large number of small substructures. One challenge in homogenizing of periodic plates with variable thickness is that the thickness variations can cause stress concentrations, which can significantly affect the overall behavior of the structure.

The periodic unfolding method is a powerful technique for studying the homogenization of periodic structures. The main idea is the introduction of an operator $\mathcal{T}_\varepsilon$, which maps a function ϕ_ε defined on a finely structured periodic domain $\Omega_\varepsilon \subset \mathbb{R}^n$ to a function $\mathcal{T}_\varepsilon(\phi_\varepsilon)$ defined on a fixed domain $\Omega \times Y$ even for varying ε, where

$Y =]0,1[^n$ is the periodicity cell. Thus, we may use standard convergence results from functional analysis. For more details we refer to [3].

In this Master thesis, we investigate the homogenization of periodic plates with variable thickness using the unfolding method. We consider a periodic plate with a periodicity in two dimensions, and we assume that the thickness of the plate depends on the local and global variables. Our goal is to derive the effective behavior of the plate, which can be described by an equivalent homogeneous plate.

We begin by introducing the basic concepts of the unfolding method and its application to the homogenization of periodic structures. We then derive the equations governing the behavior of the periodic plate with variable thickness using the new "decomposition for the plate displacements" proposed by Griso (see [3]), and at the end we explain how to apply the unfolding method to homogenize the heterogeneous plate where plate thickness and period size are of the same order of magnitude.

Chapter 1

The Unfolding Method :

This chapter will tackle the new methods that gained the scientists' interest for study the asymptotic behavior and the homogenization of structures formed by large numbers of rods, plates or shells, which is **the Unfolding Method** . It starts with the cell $Y \subset \mathbb{R}^n$ which is defined from the set of macroscopic periods attached to the considered problem. Next, we will define three operators intimately connected to the εY-tiling of the domain, are define for measurable functions and functions in a Lebesgue space.

In the first section will tackle the first operator which is **Unfolding Operator** $\mathcal{T}_\varepsilon$. By the following definition, the operator $\mathcal{T}_\varepsilon$ associates to any function $u \in L^p(\Omega)$, a function $\mathcal{T}_\varepsilon(u) \in L^p(\Omega \times Y)$. An immediate property of $\mathcal{T}_\varepsilon$ is that it enables to transform any integral over Ω in an integral over $\Omega \times Y$. It is given as follows,

$$\int_\Omega u(x)dx \sim \frac{1}{|Y|}\int_{\Omega\times Y} \mathcal{T}_\varepsilon u(x,y)dxdy, \quad \forall u \in L^1(\Omega). \tag{1.1}$$

1.1 The Unfolding operator $\mathcal{T}_\varepsilon$

Let Ω an open set of $\mathbb{R}^n$, $n \geqslant 2$, and let $Y =]0, b_1[\times....\times]0, b_n[$, where $(b_1, ..., b_n) \in \mathbb{R}^n$, be a reference cell. Let $x \in \mathbb{R}^n$, we denote by $[z]_Y$ the integer part of z with respect to cell Y. This is the unique integer combination $(k_1, ..., k_n)$ such that $\sum_{j=1}^N k_j b_j$ belongs to Y. For all $z \in \mathbb{R}^n$, we define $\{z\}_Y = z - [z]_Y \in Y$, it is the fractional part of z with respect to Y.

Then for each $x \in \mathbb{R}^n$, one has:

$$x = \varepsilon\left(\left[\frac{x}{\varepsilon}\right]_Y + \left\{\frac{x}{\varepsilon}\right\}_Y\right) \quad \text{a.e. for } x \in \mathbb{R}^n.$$

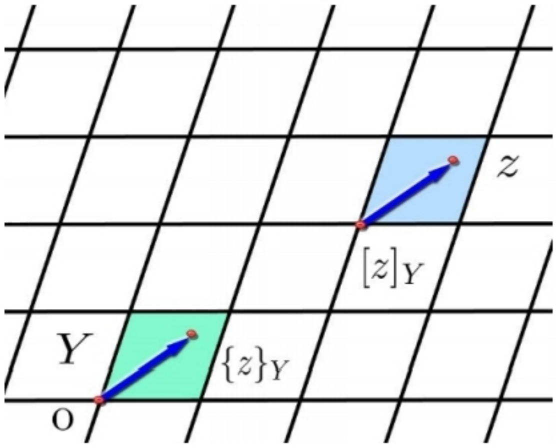

Figure 1.1: Definition of $[z]_Y$ and $\{z\}_Y$

For Ω a domain in $\mathbb{R}^n$, consider a covering using the notations:

$$\begin{cases} \Xi_\varepsilon = \{\xi \in \mathbb{Z}^n,\ \varepsilon(\xi + Y) \subset \Omega\}, \\ \widehat{\Omega}_\varepsilon = \text{ interior} \left\{ \bigcup\limits_{\xi \in \Xi_\varepsilon} \varepsilon(\xi + \overline{Y}) \right\}, \\ \Lambda_\varepsilon = \Omega \backslash \widehat{\Omega}_\varepsilon. \end{cases} \tag{1.2}$$

The following figure will show these sets:

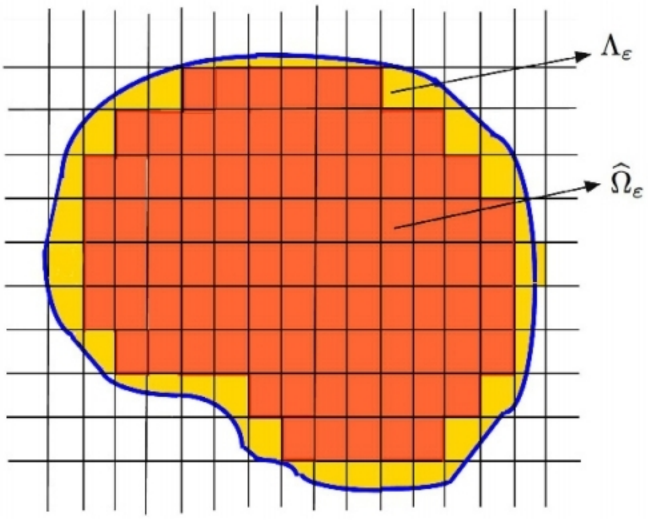

Figure 1.2: The domains $\widehat{\Omega}_\varepsilon$ and Λ_ε

The set $\widehat{\Omega}_\varepsilon$ is the largest subset of cells εY contained in Ω , while Λ_ε is the subset of Ω containing the cells εY which intersect the boundary of Ω (See Figure 1.2). Such that if $\Omega = \mathbb{R}^N$ and $\partial\Omega$ is bounded , the subset Λ_ε is a null set and we can write:

$$|\Lambda_\varepsilon| = measure(\Lambda_\varepsilon) \to 0.$$

Definition 1.1 (The Unfolding operator $\mathcal{T}_\varepsilon$).

For ϕ Lebesgue−measurable on $\widehat{\Omega}_\varepsilon$, the unfolding operator $\mathcal{T}_\varepsilon$ is defined as follows :

$$\mathcal{T}_\varepsilon\phi(x,y) = \begin{cases} \phi\left(\varepsilon\left[\dfrac{x}{\varepsilon}\right]_Y + \varepsilon Y\right) & for \quad a.e. \quad (x,y) \in \widehat{\Omega}_\varepsilon \times Y \\ 0 \qquad for \qquad a.e. & (x,y) \in \Lambda_\varepsilon \times Y, \end{cases} \tag{1.3}$$

For this definition it is obvious that $\mathcal{T}_\varepsilon$ is a linear operator (See[1]) . Moreover it verifies a number of properties:

$$\begin{cases} (i) \quad \mathcal{T}_\varepsilon(uv) = \mathcal{T}_\varepsilon(u)\mathcal{T}_\varepsilon(v), \quad \textit{for } u \textit{ and } v \textit{ Lebesgue-measurable} \\ (ii) \quad \mathcal{T}_\varepsilon(\phi)\left(x, \left\{\frac{x}{\varepsilon}\right\}_Y\right) = \phi(x), \\ (iii) \quad \mathcal{T}_\varepsilon(\phi)\,(x,y) = \phi(y), \quad \textit{for all } \phi \in L^p(Y) \textit{ Y-periodic on } \mathbb{R}^n \end{cases}$$

Proposition 1.1.

*For every $\phi \in L^1(\widehat{\Omega}_\varepsilon), \psi \in L^1(\Omega)$, the operator $\mathcal{T}_\varepsilon$ is **linear** and **continuous** from $L^p(\widehat{\Omega}_\varepsilon)$ to $L^p(\Omega \times Y)$ for all $p \in [1, +\infty[$, and satisfy the following properties:*

$$\begin{cases} (i) \quad \frac{1}{|Y|}\int_{\Omega\times Y}\mathcal{T}_\varepsilon(\phi)(x,y)dxdy = \int_{\widehat{\Omega}_\varepsilon}\phi(x)dx, \\ (ii) \quad \left|\int_\Omega \psi dx - \frac{1}{|Y|}\int_{\Omega\times Y}\mathcal{T}_\varepsilon(\psi)dxdy\right| \leqslant \int_{\Lambda_\varepsilon}|\psi|dx, \\ (iii) \quad \|\mathcal{T}_\varepsilon(u)\|_{L^p(\Omega\times Y)} \leqslant |Y|^{\frac{1}{p}}\|u\|_{L^p(\Omega)}, \\ (iv) \quad \nabla_y(\mathcal{T}_\varepsilon(u)) = \varepsilon\mathcal{T}_\varepsilon(\nabla_x u), \quad \textit{for all } u \in W^{1,p}(\Omega). \end{cases}$$

Proof. See ([3], p11). ∎

Definition 1.2 (The mean value operator $\mathcal{M}_Y$)**.**

For $p \in [1, +\infty]$, the mean value operator $\mathcal{M}_Y : L^p(\Omega \times Y) \longrightarrow L^p(\Omega)$, *is defined for φ in $L^p(\Omega \times Y)$, as follows :*

$$\mathcal{M}_Y(\varphi)(x) = \frac{1}{|Y|} \int_Y \varphi(x, y) dy \quad \textit{for a.e.} \quad x \in \Omega, \tag{1.4}$$

which satisfy the following estimate:

$$\|\mathcal{M}_Y(\varphi)\|_{L^p(\Omega)} \leqslant |Y|^{-\frac{1}{p}} \|\varphi\|_{L^p(\Omega \times Y)}.$$

Proposition 1.2.

For f measurable on Y, extended by $Y-$periodicity to the whole of $\mathbb{R}^N$, define the sequence $\{f_\varepsilon\}_\varepsilon$ by

$$f_\varepsilon(x) = f\left(\frac{x}{\varepsilon}\right) \quad \textit{for a.e } x \in \mathbb{R}^N.$$

Then

$$\mathcal{T}_\varepsilon\left(f_{\varepsilon|\Omega}\right)(x, y) = \begin{cases} f(y) & \textit{for a.e.} \quad (x, y) \in \widehat{\Omega}_\varepsilon \times Y, \\ 0 & \textit{for a.e.} \quad (x, y) \in_\varepsilon \times Y. \end{cases}$$

If f belongs to $L^p(Y)$, $p \in [1, +\infty[$, and if Ω is bounded ,

$$\mathcal{T}_\varepsilon\left(f_{\varepsilon|\Omega}\right) \longrightarrow f \ \textit{strongly in} \ L^p(\Omega \times Y). \tag{1.5}$$

Then

$$f_{\varepsilon|\Omega} \rightharpoonup \mathcal{M}_Y(f) \quad \textit{weakly in} \ L^p_{loc}(\Omega). \tag{1.6}$$

Furthermore, this convergence is strong if and only if f is constant.

Proof. of the convergence (1.5) and (1.6) , See ([3], p.9, p.14) ∎

1.1.1 Unfolding operator and Limits convention:

Proposition 1.3 ([3])**.**

Let $p \in [1, +\infty]$. *Suppose that a sequence* $\{u_\varepsilon\}_\varepsilon$ *is bounded in* $L^p(\Omega)$ *and* $v \in L^q(\Omega)$, *where* $\frac{1}{p} + \frac{1}{q} = 1$.

(i) *If* $\lim_{\varepsilon \longrightarrow 0} \int_{\Lambda_\varepsilon} u_\varepsilon v dx = 0$, *then*

$$\int_\Omega u_\varepsilon(x)v(x)dx - \frac{1}{|Y|}\int_{\Omega\times Y} \mathcal{T}_\varepsilon(u_\varepsilon)(x,y)\mathcal{T}_\varepsilon(v)(x,y)dxdy \longrightarrow 0. \tag{1.7}$$

(ii) $\mathcal{T}_\varepsilon(u) \longrightarrow u$ *strongly in* $L^p(\Omega)$.

(iii) *If* $\{u_\varepsilon\}$ *strongly convergence to* u *in* $L^p(\Omega)$, *then*

$$\mathcal{T}_\varepsilon(u_\varepsilon) \longrightarrow u \quad \textit{strongly in } L^p(\Omega \times Y),$$

and the same for the weak convergence of this sequence.

(iv) *If there exists a function* $\widehat{u} \in L^p(\Omega \times Y)$, *such that :*

$$\mathcal{T}_\varepsilon(u_\varepsilon) \rightharpoonup \widehat{u} \quad \textit{weakly in } L^p(\Omega \times Y),$$

then

$$u_\varepsilon \rightharpoonup \mathcal{M}_Y(\widehat{u}) \quad \textit{weakly in } L^p(\Omega).$$

1.2 The local average operator

Definition 1.3 (The local average operator)**.**

Let $p \in [1, +\infty]$ *and* $\varphi \in L^p(\Omega \times Y)$, *we defined the local average operator*

$\mathcal{M}_\varepsilon : L^p(\Omega) \to L^p(\Omega)$, *as follows:*

$$\mathcal{M}_\varepsilon(\varphi)(x) = \begin{cases} \frac{1}{|Y|} \int_Y \varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon z\right) dz & \text{if } x \in \widehat{\Omega}_\varepsilon, \\ 0 & \text{if } x \in \Lambda_\varepsilon, \end{cases}$$

where $\widehat{\Omega}_\varepsilon$ *and* Λ_ε *are defined in (1.2).*

And we can show the connection between $\mathcal{M}_\varepsilon$ *and the operator* $\mathcal{T}_\varepsilon$*, as follows:*

$$\mathcal{M}_\varepsilon(\varphi) = \frac{1}{|Y|}\mathcal{T}_\varepsilon(\varphi)(.,y) = \mathcal{M}_Y \circ \mathcal{T}_\varepsilon(\varphi).$$

It is easily seen that: $\mathcal{M}_\varepsilon \circ \mathcal{M}_\varepsilon = \mathcal{M}_\varepsilon$ *and* $\mathcal{T}_\varepsilon \circ \mathcal{M}_\varepsilon = \mathcal{M}_\varepsilon$.

and satisfy:

$$\|\mathcal{M}_\varepsilon(\varphi)\|_{L^p(\Omega)} \leqslant \|\varphi\|_{L^p(\Omega)}. \tag{1.8}$$

Proof. See ([3], p.18) ∎

1.3 The averaging operator

The adjoint of $\mathcal{T}_\varepsilon$ is the averaging operator $\mathcal{A}_\varepsilon$ defined as follows:

Definition 1.4 (The Averaging Operator)**.**

For $p \in [1, +\infty]$ *the averaging operator* $\mathcal{A}_\varepsilon : L^p(\Omega \times Y) \to L^p(\Omega)$ *is defined as follows:*

$$\mathcal{A}_\varepsilon(\varphi)(x) = \begin{cases} \frac{1}{|Y|} \int_Y \varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon z, \left\{\frac{x}{\varepsilon}\right\}_Y\right) dz & \text{for a.e. } x \in \widehat{\Omega}_\varepsilon, \\ 0 & \text{for a.e. } x \in \Lambda_\varepsilon. \end{cases}$$

From this definition we can remark:

$$\mathcal{A}_\varepsilon\left(\mathcal{T}_\varepsilon(\varphi)\right)(x) = \begin{cases} \varphi(x) & \text{for a.e. } x \in \widehat{\Omega}_\varepsilon, \\ 0 & \text{for a.e. } x \in \Lambda_\varepsilon. \end{cases}$$

Then

$$\mathcal{T}_\varepsilon\left(\mathcal{A}_\varepsilon(\varphi)\right)(x,y) = \begin{cases} \frac{1}{|Y|}\int_Y \varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon z, y\right) dz & \text{for a.e. } (x,y) \in \widehat{\Omega}_\varepsilon \times Y, \\ 0 & \text{for a.e. } (x,y) \in \Lambda_\varepsilon \times Y. \end{cases}$$

Let us recall some convergence properties of this operator.

Proposition 1.4.

Let $p \in [1, +\infty[$ and $\varphi \in L^p(\Omega \times Y)$, one has

- *Let $\{\varphi_\varepsilon\}_\varepsilon$ be a sequence such that $\varphi_\varepsilon \rightharpoonup \varphi$ weakly in $L^p(\Omega \times Y)$, then*

$$\begin{cases} (i)\ \mathcal{A}_\varepsilon(\varphi) \rightharpoonup \mathcal{M}_Y(\varphi) = \frac{1}{|Y|}\int_Y \varphi(x,y)dy & \text{weakly in } L^p(\Omega). \\ (ii)\ \mathcal{T}_\varepsilon \circ \mathcal{A}_\varepsilon(\varphi_\varepsilon) \rightharpoonup \varphi & \text{weakly in } L^p(\Omega \times Y). \\ (iii)\ \mathcal{M}_\varepsilon \circ \mathcal{A}_\varepsilon(\varphi_\varepsilon) \rightharpoonup \mathcal{M}_Y(\varphi) & \text{weakly in } L^p(\Omega). \end{cases}$$

- *If $\varphi \in L^p(\Omega)$ and does not depend upon y, then*

$$\mathcal{A}_\varepsilon(\varphi) = \mathcal{M}_\varepsilon(\varphi),$$

and therefore

$$\mathcal{A}_\varepsilon(\varphi) \to \varphi \quad \text{strongly in } L^p(\Omega).$$

Moreover, let $\{\varphi_\varepsilon\}$ be a sequence of $L^p(\Omega)$ and $\widehat{\varphi} \in L^p(\Omega \times Y)$, one has

$$\mathcal{T}_\varepsilon(\varphi_\varepsilon) \to \widehat{\varphi} \ \text{ strongly in } L^p(\Omega \times Y) \iff \varphi_\varepsilon - \mathcal{A}_\varepsilon(\widehat{\varphi}) \to 0 \ \text{ strongly in } L^p(\Omega).$$

Proof. See([3], p.21). ∎

1.4 Unfolding and gradients

In this section will tackle the properties of the unfolding operator to the space $W^{1,p}(\Omega)$.

Firstly, we recall the definitions of **Sobolev spaces** on a domain Ω in $\mathbb{R}^N$ where $p \in [1, +\infty]$.

- $W^{1,p}(\Omega) = \left\{\psi \in L^p(\Omega) : \nabla\psi \in L^p(\Omega)^N \text{ in the sense of distributions in } \Omega\right\}$,
- $W_0^{1,p}(\Omega) =$ the closure of the space C_c^∞in $W^{1,p}(\Omega)$,
- $W_{per}^{1,p}(Y) = \left\{\psi \in W_{loc}^{1,p}(\mathbb{R}^N) \mid \psi \text{ is } \mathbf{Y}\text{-periodic }\right\}$,
- $W_{per,0}^{1,p}(Y) = \left\{\psi \in W_{per}^{1,p}(Y) \mid \mathcal{M}_Y(\psi) = 0\right\}$.

The second space plays a central role in the unfolding method for gradients, such that for $p = 2$ these spaces will be denoted H_{per}^1 and $H_{per,0}^1$.

The next propositions states the relationship between $\mathcal{T}_\varepsilon$ and gradients.

Proposition 1.5.

Suppose $p \in [1, +\infty]$, the operator $\mathcal{T}_\varepsilon$ maps $W^{1,p}(\Omega)$ into $L^2(\Omega; W^{1,p}(Y))$ and for all $u \in W^{1,p}(\Omega)$,

$$\nabla_y(\mathcal{T}_\varepsilon(u)) = \varepsilon\mathcal{T}_\varepsilon(\nabla u) \quad a.e. \ in \ \Omega \times Y.$$

Proof. See ([3], p25) ■

Theorem 1.1. *Suppose $p \in [1, +\infty]$, let $\{u_\varepsilon\}_\varepsilon$ be sequence in $W^{1,p}(\Omega)$ and $u \in L^p(\Omega)$ such that :*

$$u_\varepsilon \rightharpoonup u \quad weakly \ in \ W^{1,p}(\Omega),$$

then

$$\mathcal{T}_\varepsilon(u_\varepsilon) \rightharpoonup u \quad weakly \ in \ L^p(\Omega; W^{1,p}(\Omega)).$$

Furthermore, if

$$u_\varepsilon \to u \quad strongly \ in \ L^p(\Omega),$$

then

$$\mathcal{T}_\varepsilon(u_\varepsilon) \to u \quad strongly \ in \ L^p(\Omega).$$

Particular case :

If $\{u_\varepsilon\}_\varepsilon$ satisfy : $\|u_\varepsilon\|_{L^p(\Omega)} + \varepsilon\|\nabla u_\varepsilon\|_{L^p(\Omega)} \leqslant C$, *then there exists a subsequence $\widehat{u}$*

in $L^p(\Omega; W^{1,p}_{per,0}(Y))$*, such that*

$$\begin{aligned}
&\varepsilon \nabla u_\varepsilon \rightharpoonup 0 \quad \textit{weakly in } L^p(\Omega)^N, \\
&\mathcal{T}_\varepsilon(u_\varepsilon) \rightharpoonup u + \widehat{u} \quad \textit{weakly in } L^p(\Omega; W^{1,p}(Y)), \\
&\mathcal{T}_\varepsilon(\varepsilon \nabla u_\varepsilon) = \nabla_y(\mathcal{T}_\varepsilon(u_\varepsilon)) \rightharpoonup \nabla_y \widehat{u} \quad \textit{weakly in } L^p(\Omega \times Y)^N.
\end{aligned}$$

Proof. See ([3], p. 26, 27) ∎

Proposition 1.6.

Suppose $p \in [1, +\infty]$*, let* $\{u_\varepsilon\}_\varepsilon$ *be a sequence in the space* $W^{1,p}(\Omega)$ *such that,*

$$u_\varepsilon \to u \quad \textit{strongly in } W^{1,p}(\Omega).$$

Then

(i) $\mathcal{T}_\varepsilon(\nabla u_\varepsilon) \to \nabla u$ *strongly in* $L^p(\Omega \times Y)$.

(ii) $\frac{1}{\varepsilon}(\mathcal{T}_\varepsilon(u_\varepsilon) - \mathcal{M}_\varepsilon(u_\varepsilon)) \to \sum_{j=1}^N y_j^c \frac{\partial u}{\partial x_j}$ *strongly in* $L^p(\Omega; W^{1,p}(\Omega))$,

where

$$y^c = y - \mathcal{M}_Y(y) \ .$$

Proof. See ([3], p.29) ∎

Remark 1.1. *For the case* $Y = (0,1)^N$ *we have*

$$y^c = (y_1 - \frac{1}{2}, ..., y_N - \frac{1}{2}) \quad \textit{and} \quad \sum_{j=1}^N y_j^c \frac{\partial u}{\partial x_j} = y^c . \nabla u.$$

Proposition 1.7.

Suppose $p \in [1, +\infty]$*, then for every* ψ *in* $W^{1,p}(\Omega)$*,*

$$\|\psi - \mathcal{M}_\varepsilon(\psi)\|_{L^p(\widehat{\Omega}_\varepsilon)} \leqslant C\varepsilon \|\nabla \psi\|_{L^p(\Omega)},$$

$$\|\psi - \mathcal{T}_\varepsilon(\psi)\|_{L^p(\widehat{\Omega}_\varepsilon \times Y)} \leqslant C\varepsilon \|\nabla \psi\|_{L^p(\Omega)},$$

with the constant C depending only on Y. Moreover,

$$\frac{1}{\varepsilon}(\psi \boldsymbol{1}_{\widehat{\Omega}_\varepsilon} - \mathcal{M}_\varepsilon(\phi)) \rightharpoonup 0 \quad \text{weakly in } L^p(\Omega). \tag{1.9}$$

Theorem 1.2.

Suppose $p \in [1, +\infty]$, let $\{u_\varepsilon\}_\varepsilon$ be a sequence in $W^{1,p}(\Omega)$, such that

$$u_\varepsilon \rightharpoonup u \quad \text{weakly in } W^{1,p}(\Omega).$$

Then for subsequence, there exists some $\widehat{u}$ in $L^p(\Omega; W^{1,p}_{per,0}(Y))$ such that

(i) $\frac{1}{\varepsilon}(\mathcal{T}_\varepsilon(u_\varepsilon) - \mathcal{M}_\varepsilon(u_\varepsilon)) \rightharpoonup y^c.\nabla u + \widehat{u}$ *weakly in* $L^p(\Omega; W^{1,p}(Y))$.

(ii) $\mathcal{T}_\varepsilon(\nabla u_\varepsilon) \rightharpoonup \nabla u + \nabla_y \widehat{u}$ *weakly in* $L^p(\Omega \times Y)^N$.

Moreover

$$\|\widehat{u}\|_{L^p(\Omega; W^{1,p}_{per}(Y))} \leqslant C \limsup_{\varepsilon \to 0} \|u_\varepsilon\|_{W^{1,p}(\Omega)},$$

$$\|\nabla u + \nabla_y \widehat{u}\|_{L^p(\Omega \times Y)} \leqslant |Y|^{\frac{1}{p}} \liminf_{\varepsilon \to 0} \|\nabla u_\varepsilon\|_{L^p(\Omega)},$$

where the constant C only depends on the Poincaré-Wirtinger constant of Y.

Proof. See ([3], p.30). ■

Corollary 1.1.

Under the assumptions of theorem 1.2, one has

(i) $\frac{1}{\varepsilon}(u_\varepsilon \boldsymbol{1}_{\widehat{\Omega}_\varepsilon} - \mathcal{M}_\varepsilon(u_\varepsilon)) \rightharpoonup 0$ *weakly in* $L^p(\Omega)$,

as well as

(ii) $\frac{1}{\varepsilon}(\mathcal{T}_\varepsilon(u_\varepsilon) - \boldsymbol{1}_{\widehat{\Omega}_\varepsilon}) \rightharpoonup y^c.\nabla u + \widehat{u}$ *weakly in* $L^p(\Omega; W^{1,p}(Y))$.

These result is a complement to convergence (1.9).

Proof. See ([3], p.32) ■

Chapter 2

Decomposition of the plate displacements

2.1 The tree-dimensional problem of linearly plate:

Let ω be a bounded open subset of $\mathbb{R}^2$ with a Lipschitz-continuous boundary γ, and let γ_0 and γ_1 be two subsets of γ such that:

$$meas(\gamma_0) > 0, \quad meas(\gamma_1) > 0 \quad \text{with} \quad \gamma_1 = \gamma - \gamma_0.$$

Let us define the sets:

$$\Omega^{\delta\varepsilon} = \omega \times] - \delta h^\varepsilon, \delta h^\varepsilon[,$$

$$\begin{cases} \Gamma_0^{\delta\varepsilon} = \gamma_0 \times] - \delta h^\varepsilon, \delta h^\varepsilon[, \\ \Gamma_1^{\delta\varepsilon} = \gamma_1 \times] - \delta h^\varepsilon, \delta h^\varepsilon[, \end{cases} \qquad \begin{cases} \Gamma_+^{\delta\varepsilon} = \omega \times \{+\delta h^\varepsilon\}, \\ \Gamma_-^{\delta\varepsilon} = \omega \times \{-\delta h^\varepsilon\}, \end{cases}$$

where δ is a parameter $(0 < \delta \leq 1)$, and h^ε is a function defined in ω.

We consider a three-dimensional linear heterogenous elastic thin plate with variable thickness. This plate is occupying the set $\overline{\Omega}^{\delta\varepsilon}$, which is the reference configuration, with a middle surface $\overline{\omega}$ such that:

$$\overline{\Omega}^{\delta\varepsilon} = \overline{\omega} \times [-\delta h^\varepsilon, \delta h^\delta].$$

We denote the boundary of the set $\Omega^{\delta\varepsilon}$ by $\Gamma^{\delta\varepsilon}$, which is defined as follows:

$$\Gamma^{\delta\varepsilon} = \Gamma_L^{\delta\varepsilon} \cup \Gamma_-^{\delta\varepsilon} \cup \Gamma_+^{\delta\varepsilon},$$

where $\Gamma_L^{\delta\varepsilon} = \gamma \times] - \delta h^\varepsilon, \delta h^\varepsilon[$ is the laterale face of the plate. This face be divided into two portions, while the plate is clamped on the first portion $\Gamma_0^{\delta\varepsilon}$, and the remaining portion $\Gamma_1^{\delta\varepsilon}$ of this laterale face is free of all action, and $\Gamma_\pm^{\delta\varepsilon} = \omega \times \{\pm\delta h^\varepsilon\}$ the upper and lower faces.

The plate is subjected to applied body forces, of density $(f_i^\delta) : \Omega^{\delta\varepsilon} \longrightarrow \mathbb{R}^3$ per unit volume in its interior $\Omega^{\delta\varepsilon}$ and to applied surface forces acting on the upper and lower faces of density $(g_i^\delta) : \Gamma_+^{\delta\varepsilon} \cup \Gamma_-^{\delta\varepsilon} \longrightarrow \mathbb{R}^3$ per unit area, where: $\begin{cases} f_i^\delta \in L^2(\Omega^{\delta\varepsilon}), \\ g_i^\delta \in L^2(\Gamma_+^{\delta\varepsilon} \cup \Gamma_-^{\delta\varepsilon}). \end{cases}$

Let $(x_1, x_2) = (x_1^\delta, x_2^\delta)$ and $x^\delta = (x_1, x_2, x_3^\delta)$ denote the generic points in the sets $\overline{\omega}$ and $\overline{\Omega}^{\delta\varepsilon}$, respectively, and let $n^\delta = (n_i^\delta)$ denotes the unit outer normal vector along $\Gamma^{\delta\varepsilon}$ (the boundary of the set $\Omega^{\delta\varepsilon}$).

Remark 2.1.

Since δ is a dimensionless parameter, the thickness of the plate should be more appropriately written as $2\delta h^\varepsilon$, where h^ε represents the variation in thickness, which depends on microscopic and macroscopic variables. There are two variable parameters,

$$\begin{cases} \delta \text{ denotes the order of the variable thickness of the a plate,} \\ \varepsilon \text{ denotes the order of magnitude of the period of a plate.} \end{cases}$$

We suppose that the thickness of the plate satisfy:

$$\begin{cases} h^\varepsilon \in W^{2,\infty}(\omega) \ , \ h^\varepsilon(x_1, x_2) \geqslant h_0 > 0. \\ h^\varepsilon(x_1, x_2) = h(x_1, x_2; \frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}). \end{cases}$$

We consider the classical linear elasticity problem posed on a periodic structure $\Omega^{\delta\varepsilon}$:

$$(P(\Omega^{\delta\varepsilon})) \begin{cases} -\partial_j^\delta \sigma_{ij}^{\delta\varepsilon} = f_i^\delta & \text{in } \Omega^{\delta\varepsilon} \\ u_i^{\delta\varepsilon} = 0 & \text{on } \Gamma_0^{\delta\varepsilon} \\ \sigma_{ij}^{\delta\varepsilon} \eta_j^\delta = \begin{cases} g_i^\delta & \text{on } \Gamma_-^{\delta\varepsilon} \cup \Gamma_+^{\delta\varepsilon}, \\ 0 & \text{on } \Gamma_1^{\delta\varepsilon}. \end{cases} \end{cases}$$

The unknown of this problem is the displacement vector field $u^{\delta\varepsilon} = (u_i^{\delta\varepsilon}) : \overline{\Omega}^{\delta\varepsilon} \longrightarrow \mathbb{R}^3$, where, $\sigma^{\delta\varepsilon} = (\sigma_{ij}^{\delta\varepsilon})$ is called **the stress tensor** which is related to $u^{\delta\varepsilon}$, such that:

$$\sigma_{ij}^{\delta\varepsilon} = a_{ijkl}^{\delta\varepsilon} e_{kl}^{\delta\varepsilon}(u^{\delta\varepsilon})$$

where

$$e_{kl}^{\delta\varepsilon}(u^{\delta\varepsilon}) = \frac{1}{2}\left(\partial_k u_l^{\delta\varepsilon} + \partial_l u_k^{\delta\varepsilon}\right).$$

The tensor $e^{\delta\varepsilon}(u^{\delta\varepsilon}) = (e_{kl}^{\delta\varepsilon}(u^{\delta\varepsilon}))$ is **the linearized strain tensor**, and the constants $a_{ijkl}^{\delta\varepsilon}$ denote the components of **the three-dimensional elasticity tensor** in cartesian coordinates.

Moreover, these heterogeneous elastic coefficients satisfy the following conditions:

- $a_{ijkl}^{\delta\varepsilon} \in L^{\infty}(\Omega^{\delta\varepsilon})$ such that :

$$a_{ijkl}^{\delta\varepsilon}(x_1, x_2, x_3) = a_{ijkl}\left(\tfrac{x_1}{\varepsilon}, \tfrac{x_2}{\varepsilon}, \tfrac{x_3}{\delta}\right)$$

- **Symetry :**

$$a_{ijkl}^{\delta\varepsilon} = a_{jikl}^{\delta\varepsilon} = a_{ijlk}^{\delta\varepsilon} = a_{klij}^{\delta\varepsilon}$$

- **Ellipticity :**

$$a_{ijkl}^{\delta\varepsilon}\tau_{kl}\tau_{ij} \geq c\tau_{ij}\tau_{ij}, \quad \forall \tau_{ij} = \tau_{ji}.$$

2.1.1 The variational formulation

After all the reminders in the first appendix, we can affirm that the displacement field $u^{\delta\varepsilon}$ satisfies the following weak formulation :

$$(P_v(\Omega^{\delta\varepsilon}))\begin{cases} \text{Find } u^{\delta\varepsilon} \in V^{\delta\varepsilon} = \{v^{\delta\varepsilon} \in \left[H^1(\Omega^{\delta\varepsilon})\right]^3 : v^{\delta\varepsilon} = 0 \text{ on } \Gamma_0^{\delta\varepsilon}\} \text{ such that} \\ B^{\delta}(u^{\delta\varepsilon}, u^{\delta\varepsilon}) = L^{\delta}(v) \ \ \forall v \in V^{\delta\varepsilon}. \end{cases},$$

where

$$\begin{cases} B^{\delta}(u^{\delta\varepsilon}, v) = \displaystyle\int_{\Omega^{\delta\varepsilon}} a_{ijkl}^{\delta\varepsilon} e_{kl}^{\delta\varepsilon}(u^{\delta\varepsilon}) e_{kl}^{\delta\varepsilon}(v) dx^{\delta} \\ L^{\delta}(v) = \displaystyle\int_{\Omega^{\delta\varepsilon}} f^{\delta} v dx^{\delta} \end{cases}$$

2.1.2 The existence and unicity of the solution of problem

To proof the existence and unicity of $u^{\delta\varepsilon}$ the unknown of the problem $P(\Omega^{\delta\varepsilon})$, we assume that $f^{\delta} \in \left[L^2(\Omega^{\delta\varepsilon})\right]^3$ and $a_{ijkl}^{\delta\varepsilon}(x)$ satisfy the hypotheses above, so we have $\alpha \leqslant a_{ijkl}^{\delta\varepsilon}(x) \leqslant M$, with C, M, α are positive real numbers.

1. **The continuity** :

$$|L^\delta(v)| = |\int_{\Omega^{\delta\varepsilon}} f^\delta v dx^\delta|$$
$$\leqslant \|f^\delta\|_{L^2(\Omega^{\delta\varepsilon})}\|v\|_{L^2(\Omega^{\delta\varepsilon})}$$
$$\leqslant C\|v\|_{H^1(\Omega^{\delta\varepsilon})}.$$

Using the Cauchy Schwarz's inequality, so $L^\delta(.)$ is continuous, with $C = \|f^\delta\|_{L^2(\Omega^{\delta\varepsilon})}$.

On the other hand, using the Cauchy Schwarz's inequality, we get

$$|B^\delta(u^{\delta\varepsilon}, v)| = |\int_{\Omega^{\delta\varepsilon}} a^{\delta\varepsilon}_{ijkl}(x)e^{\delta\varepsilon}_{kl}(u^{\delta\varepsilon})e^{\delta\varepsilon}_{ij}(v)dx^\delta|$$
$$\leqslant M\|e(u^{\delta\varepsilon})\|_{L^2(\Omega^{\delta\varepsilon})}\|e(v)\|_{L^2(\Omega^{\delta\varepsilon})}$$
$$\leqslant M\|u^{\delta\varepsilon}\|_{H^1(\Omega^{\delta\varepsilon})}\|v\|_{H^1(\Omega^{\delta\varepsilon})}.$$

So the bilinear form $B^{\delta\varepsilon}(.,.)$ is continuous .

2. **$V^{\delta\varepsilon}$-elliptic** :

Using the ellipticity of the elastic coefficients, we get

$$B^\delta(v, v) = \int_{\Omega^{\delta\varepsilon}} a^{\delta\varepsilon}_{ijkl}(x)e^{\delta\varepsilon}_{kl}(v)e^{\delta\varepsilon}_{ij}(v)dx^\delta \geqslant M \left\|e^{\delta\varepsilon}(v)\right\|^2_{L^2(\Omega^{\delta\varepsilon})},$$ where M is positive constant.

Using the Korn's inequality with a boundary condition we find:

$$B^\delta(v, v) \geqslant \gamma\|v\|^2_{H^1(\Omega^{\delta\varepsilon})},$$

with $\gamma = M.C_k^{-2}$, Then the bilinear form $B^{\delta\varepsilon}$ is $V^{\delta\varepsilon}$-elliptic. Consequently, via Lax-Milgram Theorem the variational problem $P(\Omega^{\delta\varepsilon})$ has one and only one solution $u^{\delta\varepsilon}$.

2.2 A Decomposition for the plate displacements :

In order to simplify the notations, we omit the parameter ε (as it only concerns homogenization) in the rest of this chapter. In this section, we use the new decomposition of the displacement with some properties such that every displacement u^δ in $\left[H^1(\Omega^\delta)\right]^3$ is uniquely decomposed as the sum :

$$u^\delta = u_e^\delta + u_r^\delta \tag{2.1}$$

of an elementary displacement u_e^δ and a residual displacement u_r^δ .

Definition 2.1 (elementary displacements[3]).

Elementary displacements are elements u_e^δ *of* $\left[H^1(\Omega^\delta)\right]^3$ *satisfying for a.e.* $x = (x', x_3) \in \Omega^\delta$*, where* $x' \in \omega$*:*

$$\begin{cases} u_{e,1}^\delta(x) = \mathcal{U}_1^\delta(x') + x_3\mathcal{R}_1^\delta(x') \\ u_{e,2}^\delta(x) = \mathcal{U}_2^\delta(x') + x_3\mathcal{R}_2^\delta(x') \\ u_{e,3}^\delta(x) = \mathcal{U}_3^\delta(x'). \end{cases},$$

where

$$\mathcal{U}^\delta = \left(\mathcal{U}_1^\delta, \mathcal{U}_2^\delta, \mathcal{U}_3^\delta\right) \in \left[H^1(\omega)\right]^3 \quad \textit{and} \quad \mathcal{R}^\delta = \left(\mathcal{R}_1^\delta, \mathcal{R}_2^\delta\right) \in \left[H^1(\omega)\right]^2.$$

Elementary displacements are a generalization of the notion of Kirchhoff-Love and Reissner-Mindlin displacements. The first part $\mathcal{U}^\delta(x')$ *of* u_e^δ *is mid-surface displacement at the point* $x' \in \omega$*, while* $x_3\mathcal{R}^\delta(x')$ *represents the small linearized rotation of the fiber* $\{x'\} \times (-\delta, \delta)$.

Definition 2.2 (Residual displacements[3]).

The residual displacements are elements of $u_r^\delta = (u_{r,1}^\delta, u_{r,2}^\delta, u_{r,3}^\delta)$*, which satisfy the condition:*

$$\int_{-\delta}^{\delta} u_r^\delta(x', x_3)dx_3 = \int_{-\delta}^{\delta} x_3 u_{r,1}^\delta(x', x_3)dx_3 = \int_{-\delta}^{\delta} x_3 u_{r,2}^\delta(x', x_3)dx_3 = 0 \quad \textit{a.e. } x' \in \omega. \tag{2.2}$$

Due to the properties of the residual part, the components $\mathcal{U}^\delta$ and $\mathcal{R}^\delta$ of u_e^δ are given for a.e. $x' \in \omega$, by

$$\begin{aligned} \mathcal{U}^\delta(x') &= \frac{1}{2\delta}\int_{-\delta}^{\delta} u^\delta(x', x_3)dx_3 \\ \mathcal{R}_\alpha^\delta(x') &= \frac{3}{2\delta^3}\int_{-\delta}^{\delta} x_3 u_\alpha^\delta(x', x_3)dx_3, \ \alpha = 1, 2. \end{aligned}$$

Proof. you can see this proof in the second appendix. ■

The sum

$$\mathcal{U}_m^\delta = \mathcal{U}_1^\delta e_1 + \mathcal{U}_2^\delta e_2$$

is the membrane displacement, while $\mathcal{U}_3^\delta$ represents the bending of the mid-surface. The residual part u_r^δ is the warping; it stands for the deformation of the fibers $\{x'\} \times (-\delta, \delta)$. With the above notations, the explicit expressions of the components of the strain tensor of u^δ are

- $e_{11}(u^\delta) = \frac{\partial \mathcal{U}_1^\delta}{\partial x_1} + x_3 \frac{\partial \mathcal{R}_1^\delta}{\partial x_1} + \frac{\partial u_{r,1}^\delta}{\partial x_1}$,
- $e_{22}(u^\delta) = \frac{\partial \mathcal{U}_2^\delta}{\partial x_2} + x_3 \frac{\partial \mathcal{R}_2^\delta}{\partial x_2} + \frac{\partial u_{r,2}^\delta}{\partial x_2}$,
- $e_{12}(u^\delta) = \frac{1}{2}\left(\left[\frac{\partial \mathcal{U}_1^\delta}{\partial x_2} + \frac{\partial \mathcal{U}_2^\delta}{\partial x_1}\right] + x_3 \left[\frac{\partial \mathcal{R}_1^\delta}{\partial x_2} + \frac{\partial \mathcal{R}_2^\delta}{\partial x_1}\right] + \left[\frac{\partial u_{r,1}^\delta}{\partial x_2} + \frac{\partial u_{r,2}^\delta}{\partial x_1}\right]\right)$,
- $e_{13}(u^\delta) = \frac{1}{2}\left(\left[\frac{\partial \mathcal{U}_3^\delta}{\partial x_1} + \mathcal{R}_1\right] + \left[\frac{\partial u_{r,3}^\delta}{\partial x_1} + \frac{\partial u_{r,1}^\delta}{\partial x_3}\right]\right)$,
- $e_{23}(u^\delta) = \frac{1}{2}\left(\left[\frac{\partial \mathcal{U}_3^\delta}{\partial x_2} + \mathcal{R}_2\right] + \left[\frac{\partial u_{r,3}^\delta}{\partial x_2} + \frac{\partial u_{r,2}^\delta}{\partial x_3}\right]\right)$,
- $e_{33}(u^\delta) = \frac{\partial u_{r,3}^\delta}{\partial x_3}$.

Proof. See in the second appendix. ■

Theorem 2.1 ([3]).

Let u^δ be a displacement in $\left[H^1(\Omega^\delta)\right]^3$ and $\left(\mathcal{U}^\delta, \mathcal{R}^\delta, u_r^\delta\right)$ be its decomposition given by,

$$\begin{aligned} \delta \left\| e_{\alpha\beta}(\mathcal{R}^\delta) \right\|_{L^2(\omega)} + \left\| e_{\alpha\beta}(\mathcal{U}^\delta) \right\|_{L^2(\omega)} &\leqslant \frac{C}{\delta^{\frac{1}{2}}} \left\| e(u^\delta) \right\|_{L^2(\Omega^\delta)}, \\ \left\| \frac{\partial \mathcal{U}_3^\delta}{\partial x_\alpha} + \mathcal{R}_\alpha^\delta \right\|_{L^2(\omega)} &\leqslant \frac{C}{\delta^{\frac{1}{2}}} \left\| e(u^\delta) \right\|_{L^2(\Omega^\delta)}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} \left\| u_r^\delta \right\|_{L^2(\Omega^\delta)} &\leqslant C\delta \left\| e(u^\delta) \right\|_{L^2(\Omega^\delta)}, \\ \left\| \nabla u_r^\delta \right\|_{L^2(\Omega^\delta)} &\leqslant C \left\| e(u^\delta) \right\|_{L^2(\Omega^\delta)}. \end{aligned}$$

The constant does not depend on δ.

Proof. One has :

$$\begin{aligned}
\left\|e(u^\delta)\right\|^2_{L^2(\Omega^\delta)} &= \sum_{i,j=1}^{3} \left\|e_{ij}(u^\delta)\right\|^2_{L^2(\Omega^\delta)} \\
&= \sum_{\alpha,\beta=1}^{2} \left\|e_{\alpha\beta}(u^\delta)\right\|^2_{L^2(\Omega^\delta)} + 2\sum_{\alpha=1}^{2} \left\|e_{\alpha 3}(u^\delta)\right\|^2_{L^2(\Omega^\delta)} + \left\|e_{33}(u^\delta)\right\|^2_{L^2(\Omega^\delta)} \\
&\geqslant \sum_{\alpha,\beta=1}^{2} \left\|e_{\alpha\beta}(u^\delta)\right\|^2_{L^2(\Omega^\delta)}.
\end{aligned}$$

Step 1: Proof of the first inequality (2.3),

$$\begin{aligned}
\left\|e_{\alpha\beta}(u^\delta)\right\|^2_{L^2(\Omega^\delta)} &= \int_{\Omega^\delta} |e_{\alpha\beta}(u^\delta)|^2 dx \\
&= \int_{\Omega^\delta} e^2_{\alpha\beta}(\mathcal{U}^\delta)dx + \int_{\Omega^\delta} \left\{x_3 e_{\alpha\beta}(\mathcal{R}^\delta)\right\}^2 dx + \int_{\Omega^\delta} e^2_{\alpha\beta}(u^\delta_r)dx \\
&= \int_\omega e^2_{\alpha\beta}(\mathcal{U}^\delta)\left(\int_{-\delta h}^{\delta h} dx_3\right)dx' + \int_\omega e^2_{\alpha\beta}(\mathcal{R}^\delta)\left(\int_{-\delta h}^{\delta h} x_3^2 dx_3\right)dx' + \int_{\Omega^\delta} e^2_{\alpha\beta}(u^\delta_r)dx \\
&= 2\delta\int_\omega h\ e^2_{\alpha\beta}(\mathcal{U}^\delta)dx' + \frac{2\delta^3}{3}\int_\omega h^3 e^2_{\alpha\beta}(\mathcal{R}^\delta)dx' + \int_{\Omega^\delta} e^2_{\alpha\beta}(u^\delta_r)dx \\
&\geqslant 2\delta h_0 \left\|e_{\alpha\beta}(\mathcal{U}^\delta)\right\|^2_{L^2(\omega)} + \frac{2}{3}h_0^3\delta^3 \left\|e_{\alpha\beta}(\mathcal{R}^\delta)\right\|^2_{L^2(\omega)} + \left\|e_{\alpha\beta}(u^\delta_r)\right\|^2_{L^2(\Omega^\delta)} \\
&\geqslant 2\delta h_0 \left\|e_{\alpha\beta}(\mathcal{U}^\delta)\right\|^2_{L^2(\omega)} + \frac{2}{3}h_0^3\delta^3 \left\|e_{\alpha\beta}(\mathcal{R}^\delta)\right\|^2_{L^2(\omega)}
\end{aligned}$$

and by $a^2+b^2 \geqslant \frac{1}{2}(a+b)^2$, we get

$$\begin{aligned}
\left\|e_{\alpha\beta}(u^\delta)\right\|^2_{L^2(\Omega^\delta)} &\geqslant \frac{1}{2}\left\{\sqrt{2\delta h_0}\|e_{\alpha\beta}(\mathcal{U}^\delta)\|_{L^2(\omega)} + \frac{\sqrt{2h_0\delta^3}}{\sqrt{3}}(\delta h_0)\|e_{\alpha\beta}(\mathcal{R}^\delta)\|_{L^2(\omega)}\right\}^2 \\
& \quad \delta h_0\left\{\|e_{\alpha\beta}(\mathcal{U}^\delta)\|_{L^2(\omega)} + \frac{\delta h_0}{\sqrt{3}}\|e_{\alpha\beta}(\mathcal{R}^\delta)\|_{L^2(\omega)}\right\}^2
\end{aligned}$$

and after that we get

$$\left\|e_{\alpha\beta}(u^\delta)\right\|_{L^2(\Omega^\delta)} \geqslant \sqrt{\delta h_0}\left\{\|e_{\alpha\beta}(\mathcal{U}^\delta)\|_{L^2(\omega)} + \frac{\delta h_0}{\sqrt{3}}\|e_{\alpha\beta}(\mathcal{R}^\delta)\|_{L^2(\omega)}\right\},$$

which gives:

$$\delta\left\|e_{\alpha\beta}(\mathcal{R}^\delta)\right\|_{L^2(\omega)} + \left\|e_{\alpha\beta}(\mathcal{U}^\delta)\right\|_{L^2(\omega)} \leqslant \frac{C}{\sqrt{\delta}}\left\|e(u^\delta)\right\|_{L^2(\Omega^\delta)}.$$

with $C = min\left\{\sqrt{h_0}, \frac{h_0}{\sqrt{3}}\right\}$.

Step 2: Proof of the second inequality of (2.3):

$$\begin{aligned}
\left\|e_{\alpha 3}(u^\delta)\right\|^2_{L^2(\Omega^\delta)} &= \int_{\Omega^\delta} |e_{\alpha 3}(u^\delta)|^2 dx \\
&= \int_{\Omega^\delta} e^2_{\alpha 3}(u^\delta_e)dx + \int_{\Omega^\delta} e^2_{\alpha 3}(u^\delta_r)dx + 2\int_{\Omega^\delta} e_{\alpha 3}(u^\delta_e)e_{\alpha 3}(u^\delta_r)dx \\
&= \int_{\Omega^\delta} \left\{\frac{1}{2}\left(\frac{\partial u^\delta_{e,\alpha}}{\partial x_3} + \frac{\partial u^\delta_{e,3}}{\partial x_\alpha}\right)\right\}^2 dx + \int_{\Omega^\delta} e^2_{\alpha 3}(u^\delta_r)dx \\
&= \int_{\Omega^\delta} \left\{\frac{1}{2}\left(\mathcal{R}^\delta_\alpha(x') + \frac{\partial \mathcal{U}^\delta_3}{\partial x_\alpha}\right)\right\}^2 dx + \int_{\Omega^\delta} e^2_{\alpha 3}(u^\delta_r)dx \\
&= \frac{1}{4}\int_\omega \left\{\mathcal{R}^\delta_\alpha(x') + \frac{\partial \mathcal{U}^\delta_3}{\partial x_\alpha}\right\}^2 \left(\int_{-\delta h}^{\delta h} dx_3\right) dx' + \int_\Omega e^2_{\alpha 3}(u^\delta_r)dx \\
&= \frac{\delta}{2}\int_\omega h\left\{\mathcal{R}^\delta_\alpha + \frac{\partial \mathcal{U}^\delta_3}{\partial x_\alpha}\right\}^2 dx' + \int_{\Omega^\delta} e^2_{\alpha 3}(u^\delta_r)dx \\
&\geqslant \frac{h_0}{2}\delta\left\|\mathcal{R}^\delta_\alpha + \frac{\partial \mathcal{U}^\delta_3}{\partial x_\alpha}\right\|^2_{L^2(\omega)}.
\end{aligned}$$

Then we get

$$\left\|e_{\alpha 3}(u^\delta)\right\|_{L^2(\Omega^\delta)} \geqslant \sqrt{\frac{h_0\delta}{2}}\left\|\mathcal{R}^\delta_\alpha(x') + \frac{\partial \mathcal{U}^\delta_3}{\partial x_\alpha}\right\|^2_{L^2(\omega)},$$

which gives

$$\left\|\mathcal{R}^\delta_\alpha + \frac{\partial \mathcal{U}^\delta_3}{\partial x_\delta}\right\|_{L^2(\omega)} \leqslant \frac{C}{\sqrt{\delta}}\left\|e(u^\delta)\right\|_{L^2(\Omega^\delta)},$$

with $C = \frac{2}{\sqrt{h_0}}$. ∎

Proposition 2.1.

$$\|\mathcal{U}_\alpha\|_{H^1(\omega)} + \delta\left(\|\mathcal{U}_3\|_{H^1(\omega)} + \|\mathcal{R}_\alpha\|_{H^1(\omega)}\right) \leqslant \frac{C}{\delta^{\frac{1}{2}}}\|e(u(\delta))\|_{L^2(\Omega^\delta)} \tag{2.4}$$

Proof.

We use the displacement $\tilde{U}_e = U_{e,\alpha} + \delta U_{e,3}$ belongs to $[H^1(\Omega)]^3$, one has

$$\begin{cases} U_{e,1}(z', z_3) = \mathcal{U}_1(z') + z_3 \delta \mathcal{R}_1(z') \\ U_{e,2}(z', z_3) = \mathcal{U}_2(z') + z_3 \delta \mathcal{R}_2(z') \\ U_{e,3}(z', z_3) = \mathcal{U}_3(z') \end{cases}$$

which give

$$C \|\tilde{U}_e\|_{H^1(\Omega)} \leqslant \|e^z(\tilde{U}_e)\|_{L^2(\Omega)} \leqslant \frac{C}{\delta^{\frac{1}{2}}} \|e(u(\delta))\|_{L^2(\Omega^\delta)}$$

and via the definition of $\tilde{U}_e$, One has

$$\begin{aligned} \|\tilde{U}_e\|^2_{H^1(\Omega)} &= \|\tilde{U}_e\|^2_{L^2(\Omega)} + \|\nabla \tilde{U}_e\|^2_{L^2(\Omega)} \\ &= \|U_{e,\alpha}\|^2_{L^2(\Omega)} + \|\delta U_{e,3}\|^2_{L^2(\Omega)} \\ &= \|\mathcal{U}_\alpha(z') + z_3 \delta \mathcal{R}_\alpha(z')\|^2_{L^2(\Omega)} + \|\delta \mathcal{U}_3(z')\|^2_{L^2(\Omega)} \\ &= 2 \|\mathcal{U}_\alpha(z')\|^2_{L^2(\omega)} + \frac{2}{3}\delta^2 \|\mathcal{R}_\alpha(z')\|^2_{L^2(\omega)} + 2\delta^2 \|\mathcal{U}_3(z')\|^2_{L^2(\omega)} \\ &\leqslant C \|e^z(\tilde{U}_e)\|^2_{L^2(\Omega)}. \end{aligned}$$

from the previous results we deduce that

$$\sqrt{2} \|\mathcal{U}_\alpha\|_{L^2(\omega)} + \frac{\sqrt{2}}{\sqrt{3}} \delta \|\mathcal{R}_\alpha\|_{L^2(\omega)} + \sqrt{2}\delta \|\mathcal{U}_3\|_{L^2(\omega)} \leqslant \frac{1}{\delta^{\frac{1}{2}}} \|e(u(\delta))\|_{L^2(\Omega^\delta)} ,$$

Then

$$\|\mathcal{U}_\alpha\|_{L^2(\omega)} + \delta \left(\|\mathcal{R}_\alpha\|_{L^2(\omega)} + \|\mathcal{U}_3\|_{L^2(\omega)} \right) \leqslant \frac{C}{\delta^{\frac{1}{2}}} \|e(u(\delta))\|_{L^2(\Omega^\delta)}.$$

The constant does not depend on δ. ■

2.3 Transformation into a problem posed over a domain independent of δh :

2.3.1 The fixed domain

Firstly, recall that $\Omega^\delta = \omega\times]-\delta h, \delta h[$, with thickness variable $h_0 > 0$, such that $0 < \delta \leqslant 1$. Since the displacement fields u^δ is defined on the set $\overline{\Omega}^\delta = \overline{\omega} \times [-\delta h, \delta h]$ which depends on δ and h.

In this part we will transform the problem $P(\Omega^{\delta\varepsilon})$ into a problem on a fixed domain that does not depend on δ. Let us define the fixed domain:

$$\Omega = \omega\times]-1, 1[\ ,$$

$$\begin{cases} \Gamma_0 = \gamma_0 \times [-1, 1], \\ \Gamma_1 = \gamma_1 \times [-1, 1], \end{cases} \qquad \begin{cases} \Gamma_+ = \omega \times \{1\}\,, \\ \Gamma_- = \omega \times \{-1\}\,. \end{cases}$$

Let $z = (z_1, z_2, z_3)$ denote a current point of $\overline{\Omega}$, where $z_\alpha = x_\alpha$ and $z_3 = \frac{x_3}{\delta h}$. With each point $x^\delta \in \overline{\Omega}^\delta$, we associate the point $z \in \overline{\Omega}$ through the correspondance

$$z = (z_1, z_2, z_3) \in \overline{\Omega} \longrightarrow x^\delta = (x_1, x_2, x_3^\delta) \in \overline{\Omega}^\delta,\ \text{with } x_3^\delta = \delta h z_3.$$

Such that for any function $\psi^\delta : \overline{\Omega}^\delta \to \mathbb{R}$ we associate the corresponding function $\psi(\delta)$: $\overline{\Omega} \to \mathbb{R}$. We have

$$\begin{cases} \partial_\alpha^\delta \psi^\delta &= \partial_\alpha^z \psi(\delta) - \frac{1}{h} z_3 \partial_\alpha^z h\ \partial_3^z \psi(\delta), \\ \partial_3^\delta \psi^\delta &= \frac{1}{\delta h} \partial_3^z \psi(\delta). \end{cases} \tag{2.5}$$

And for any ψ^δ integrable over Ω^δ and $\Gamma_-^\delta \cup \Gamma_+^\delta$, we have

$$\int_{\Omega^\delta} \psi^\delta(x^\delta) dx^\delta = \delta \int_\Omega h\ \psi(\delta)(z) dz$$

$$\int_{\Gamma_-^\delta \cup \Gamma_+^\delta} \psi^\delta(x^\delta) d\Gamma^\delta = \int_{\Gamma_- \cup \Gamma_+} h^* \psi(\delta)(z) d\Gamma,$$

where $h^* = \{1 + \delta^2[(\partial_1^z h)^2 + (\partial_2^z h)^2]\}^{\frac{1}{2}}$.

2.3.2 Assumptions on the Data and decomposition of the Unknowns.

In order to fined the displacement field $u(\delta) = (u_i(\delta)) : \overline{\Omega} \to \mathbb{R}^3$ which does not depend on δ, we use the new decomposition that was defined in section 2.2 as follows :

$$u^\delta = u_e^\delta + u_r^\delta$$

Under the hypothesis $g = (g_1, g_2, g_3) = 0,\ f = (f_1, f_2, f_3) \in [L^2(\omega)]^3$ and $t = (t_1, t_2) \in [L^2(\omega)]^2$, we make the assumptions that the applied body forces f^δ are of the form:

$$f^\delta(x) = (\delta f_\alpha(x') + x_3 t_\alpha(x'))\, e_\alpha + \delta^2 f_3(x')e_3 \quad \text{for a.e. } x \in \Omega^\delta$$

Now, by using the new displacement and the assumptions on the data and the relations(2.5), we can reformulation the variational problem $P(\Omega^\delta)$ to problem on a fixed domain, which is denoted $P(\Omega)$ in the following equivalent form:

$$\begin{cases} u(\delta) \in V(\Omega) = \{v = (v_i) \in H^1(\Omega); v = 0 \text{ on } \Gamma_0\}, \\ \delta \int_\Omega h\ \sigma_{ij}(\delta) H_{ij}(v) dz = L(v) \quad \text{for all } v \in V, \end{cases}$$

where

$$\sigma_{ij}(\delta) = a_{ijkl}^\delta(z) H_{kl}(u(\delta)) \quad , \quad L(v) = \delta \int_\Omega f(\delta) v\ dz,$$

and the expressions of the strain tensor H are defined as follows:

1. For the test-functions:

- $H_{\alpha\beta}^\delta(v) = e_{\alpha\beta}^z(v) - \frac{1}{2h} z_3 \left[\partial_\alpha^z h\ \partial_3^z v_\beta + \partial_\beta^z h\ \partial_3^z v_\alpha\right]$,
- $H_{\alpha 3}^\delta(v) = \frac{1}{2}\left[\frac{1}{\delta h}\partial_3^z v_\alpha + \partial_\alpha^z v_3 - \frac{1}{h} z_3 \partial_\alpha^z h\ \partial_{3^z} v_3\right]$,
- $H_{33}^\delta(v) = \frac{1}{\delta h}\partial_3^z v_3$.

2. For the Unknown:

- $H^{\delta}_{\alpha\beta}(u(\delta)) = e^{z}_{\alpha\beta}(u(\delta)) - \frac{z_3}{2h}\left\{\left[\mathcal{R}_{\beta}(\delta)\partial^{z}_{\alpha}h \ + \mathcal{R}_{\alpha}(\delta)\partial^{z}_{\beta}h\right] + \left[\partial^{z}_{\alpha}h\ \partial^{z}_{3}u_{r,\beta}(\delta) + \partial^{z}_{\beta}h\ \partial^{z}_{3}u_{r,\alpha}(\delta)\right]\right\}.$
- $H^{\delta}_{\alpha 3}(u(\delta)) = \frac{1}{2}\left[\partial^{z}_{\alpha}\mathcal{U}_3(\delta) + \frac{1}{\delta h}\mathcal{R}_{\alpha}(\delta)\right] + H^{\delta}_{\alpha 3}(u_r(\delta))$
- $H^{\delta}_{33}(u(\delta)) = \frac{1}{\delta h}\partial^{z}_{3}u_{r,3}(\delta).$

Proof.

By the relations (2.5) we find:

$$\begin{aligned}
H^{\delta}_{\alpha\beta}(u(\delta)) &= e^{z}_{\alpha\beta}(u(\delta)) - \frac{z_3}{2h}\left[\partial^{z}_{\alpha}h\ \partial^{z}_{3}u_{\beta}(\delta) + \partial^{z}_{\beta}h\ \partial^{z}_{3}u_{\alpha}(\delta)\right],\\
H^{\delta}_{\alpha 3}(u(\delta)) &= \frac{1}{2}\left[\frac{1}{\delta h}\partial^{z}_{3}u_{\alpha}(\delta) + \partial^{z}_{\alpha}u_3(\delta) - \frac{1}{h}z_3\partial^{z}_{\alpha}h\ \partial^{z}_{3}u_3(\delta)\right],\\
H^{\delta}_{33}(u(\delta)) &= \frac{1}{\delta h}\partial^{z}_{3}u_3(\delta).
\end{aligned}$$

and for the decomposition of $u(\delta)$, we have

$$\begin{aligned}
H_{ij}(u(\delta)) &= H_{ij}\left\{\mathcal{U}(\delta) + z_3\mathcal{R}(\delta) + u_r(\delta)\right\} &(2.6)\\
&= H^{\delta}_{ij}(\mathcal{U}(\delta)) + H^{\delta}_{ij}(z_3\mathcal{R}(\delta)) + H^{\delta}_{ij}(u_r(\delta)) &(2.7)
\end{aligned}$$

Now, we can proof these expressions for $1 \leqslant i, j \leqslant 3$,

Step 1:. . . . .

$$\begin{aligned}
H^{\delta}_{\alpha\beta}(\mathcal{U}(\delta)) &= e^{z}_{\alpha\beta}(\mathcal{U}(\delta)) + \frac{x_3}{2h}\left[\partial^{z}_{\alpha}h\ \partial^{z}_{3}\mathcal{U}_{\beta}(\delta) + \partial^{z}_{\beta}h\ \partial^{z}_{3}\mathcal{U}_{\alpha}(\delta)\right],\\
H^{\delta}_{\alpha\beta}(z_3\mathcal{R}(\delta)) &= e^{z}_{\alpha\beta}(z_3\mathcal{R}(\delta)) + \frac{z_3}{2h}\left[\partial^{z}_{\alpha}h\ \partial^{z}_{3}(z_3\mathcal{R}(\delta)) + \partial^{z}_{\beta}h\ \partial^{z}_{3}(z_3\mathcal{R}_{\alpha}(\delta))\right]\\
&= z_3e^{z}_{\alpha\beta}(\mathcal{R}(\delta)) + \frac{z_3}{2h}\left[\mathcal{R}_{\beta}(\delta)\ \partial^{z}_{\alpha}h + \mathcal{R}_{\alpha}(\delta)\ \partial^{z}_{\beta}h\right],\\
H^{\delta}_{\alpha\beta}(u_r(\delta)) &= e^{z}_{\alpha\beta}(u_r(\delta)) + \frac{z_3}{2h}\left[\partial^{z}_{\alpha}h\ \partial^{z}_{3}u_{r,\beta}(\delta) + \partial^{z}_{\beta}h\ \partial^{z}_{3}u_{r,\alpha}(\delta)\right].
\end{aligned}$$

Then

$$H_{\alpha\beta}(u(\delta)) = e^{z}_{\alpha\beta}(u(\delta)) + \frac{z_3}{2h}\left\{\left[\mathcal{R}_{\beta}(\delta)\ \partial^{z}_{\alpha}h + \mathcal{R}_{\alpha}(\delta)\ \partial^{z}_{\beta}h\right] + \left[\partial^{z}_{\alpha}h\ \partial^{z}_{3}u_{r,\beta}(\delta) + \partial^{z}_{\beta}h\ \partial^{z}_{3}u_{r,\alpha}(\delta)\right]\right\},$$

Step 2:. . . . .

$$\begin{aligned}
H^\delta_{\alpha 3}(\mathcal{U}(\delta)) &= \frac{1}{2}\left[\partial^z_\alpha \mathcal{U}_3(\delta) - \frac{1}{h} z_3 \partial^z_\alpha h\ \partial^z_3 \mathcal{U}_3(\delta) + \frac{1}{\delta h}\partial^z_3 \mathcal{U}_\alpha(\delta)\right] \\
&= \frac{1}{2}\partial^z_\alpha \mathcal{U}_3(\delta), \\
H^\delta_{\alpha 3}(z_3\mathcal{R}(\delta)) &= \frac{1}{2}\left[\partial^z_\alpha (z_3\mathcal{R}(\delta))_3 - \frac{1}{h} z_3 \partial^z_\alpha h\ \partial^z_3 (z_3\mathcal{R}(\delta))_3 + \frac{1}{\delta h}\partial^z_3 (z_3\mathcal{R}_\alpha(\delta))\right] \\
&= \frac{1}{2\delta h}\mathcal{R}_\alpha(\delta), \\
H^\delta_{\alpha 3}(u_r(\delta)) &= \frac{1}{2}\left[\partial^z_\alpha u_{r,3}(\delta)) - \frac{1}{h} z_3 \partial^z_\alpha h\ \partial^z_3 u_{r,3}(\delta) + \partial^z_3 u_{r,\alpha}(\delta)\right].
\end{aligned}$$

Then

$$H^\delta_{\alpha\beta}(u(\delta)) = \frac{1}{2}\left[\partial^z_\alpha \mathcal{U}_3(\delta) + \frac{1}{\delta h}\mathcal{R}_\alpha(\delta)\right] + H^\delta_{\alpha 3}(u_r(\delta)),$$

Step 3: . . . .

Because $\mathcal{U}(\delta)$ and $\mathcal{R}(\delta)$ do not depend of z_3, the terms $H^\delta_{33}(\mathcal{U}(\delta))$ and $H^\delta_{33}(x_3\mathcal{R}(\delta))$ are null, then

$$H^\delta_{33}(u(\delta)) = H^\delta_{33}(u_r(\delta)) = \frac{1}{\delta h}\partial^z_3 u_{r,3}(\delta)$$

■

Proposition 2.2.

On a domain fixed, the strain tensor satisfy the following inequality:

$$\|H^\delta_{\alpha\beta}(\mathcal{R}(\delta))\|_{L^2(\omega)} + \|H^\delta_{\alpha\beta}(\mathcal{U}(\delta))\|_{L^2(\omega)} \leqslant C\|H^\delta(u(\delta))\|_{L^2(\Omega)}, \tag{2.8}$$

$$\|h\partial^z_\alpha \mathcal{U}_3(\delta) + \mathcal{R}_\alpha(\delta)\|_{L^2(\omega)} \leqslant C\|H^\delta(u(\delta))\|_{L^2(\Omega)}. \tag{2.9}$$

The constant does not depend on δ.

Proof.

$$
\begin{aligned}
\left\|H^{\delta}(u(\delta))\right\|_{L^2(\Omega)}^2 &= \sum_{i,j=1}^{3} \left\|H_{ij}^{\delta}(u(\delta))\right\|_{L^2(\Omega)}^2 \\
&= \sum_{\alpha,\beta=1}^{2} \left\|H_{\alpha\beta}^{\delta}(u(\delta))\right\|_{L^2(\Omega)}^2 + 2\sum_{\alpha=1}^{2} \left\|H_{\alpha 3}^{\delta}(u(\delta))\right\|_{L^2(\Omega)}^2 + \left\|H_{33}^{\delta}(u(\delta))\right\|_{L^2(\Omega)}^2 \\
&\geqslant \sum_{\alpha,\beta=1}^{2} \left\|H_{\alpha\beta}^{\delta}(u(\delta))\right\|_{L^2(\Omega)}^2 .
\end{aligned}
$$

Step 1: Proof of (2.8),

$$
\begin{aligned}
\left\|H_{\alpha\beta}^{\delta}(u(\delta))\right\|_{L^2(\Omega)}^2 &= \int_{\Omega} |H_{\alpha\beta}^{\delta}(u(\delta))|^2 dz \\
&= \int_{\Omega} \left\{H_{\alpha\beta}^{\delta}(\mathcal{U}(\delta))\right\}^2 dz + \int_{\Omega} \left\{z_3 H_{\alpha\beta}^{\delta}(\mathcal{R}(\delta))\right\}^2 dz + \int_{\Omega} \left\{H_{\alpha\beta}^{\delta}(u_r(\delta))\right\}^2 dz \\
&= \int_{\omega} \left\{H_{\alpha\beta}^{\delta}(\mathcal{U}(\delta))\right\}^2 \left(\int_{-1}^{1} dz_3\right) dz' + \int_{\omega} \left\{H_{\alpha\beta}^{\delta}(\mathcal{R}(\delta))\right\}^2 \left(\int_{-1}^{1} z_3^2 dz_3\right) dz' \\
&+ \left\|H_{\alpha\beta}^{\delta}(u_r(\delta))\right\|_{L^2(\Omega)}^2 \\
&= 2\int_{\omega} \left\{H_{\alpha\beta}^{\delta}(\mathcal{U}(\delta))\right\}^2 dz' + \frac{2}{3}\int_{\omega} \left\{H_{\alpha\beta}^{\delta}(\mathcal{R}(\delta))\right\}^2 dz' + \left\|H_{\alpha\beta}^{\delta}(u_r(\delta))\right\|_{L^2(\Omega)}^2 \\
&= 2\left\|H_{\alpha\beta}^{\delta}(\mathcal{U}(\delta))\right\|_{L^2(\omega)}^2 + \frac{2}{3}\left\|H_{\alpha\beta}^{\delta}(\mathcal{R}(\delta))\right\|_{L^2(\omega)}^2 + \left\|H_{\alpha\beta}^{\delta}(u_r(\delta))\right\|_{L^2(\Omega)}^2 \\
&\geqslant 2\left\|H_{\alpha\beta}(\mathcal{U}(\delta))\right\|_{L^2(\omega)}^2 + \frac{2}{3}\left\|H_{\alpha\beta}(\mathcal{R}(\delta))\right\|_{L^2(\omega)}^2 .
\end{aligned}
$$

Using the inequality

$$a^2 + b^2 \geqslant \tfrac{1}{2}(a+b)^2$$

we get

$$
\begin{aligned}
\|H_{\alpha\beta}^{\delta}(u(\delta))\|_{L^2(\Omega)}^2 &\geqslant \frac{1}{2}\left\{\sqrt{2}\|H_{\alpha\beta}^{\delta}(\mathcal{U}(\delta))\|_{L^2(\omega)} + \frac{\sqrt{2}}{\sqrt{3}}\|H_{\alpha\beta}^{\delta}(\mathcal{R}(\delta))\|_{L^2(\omega)}\right\}^2 \\
&\geqslant \left\{\|H_{\alpha\beta}^{\delta}(\mathcal{U}(\delta))\|_{L^2(\omega)} + \frac{1}{\sqrt{3}}\|H_{\alpha\beta}^{\delta}(\mathcal{R}(\delta))\|_{L^2(\omega)}\right\}^2 ,
\end{aligned}
$$

then we have

$$\left\|H_{\alpha\beta}^{\delta}(u(\delta))\right\|_{L^2(\Omega)} \geqslant \left\{\left\|H_{\alpha\beta}^{\delta}(\mathcal{U}(\delta))\right\|_{L^2(\omega)} + \frac{1}{\sqrt{3}}\left\|H_{\alpha\beta}(\mathcal{R}(\delta))\right\|_{L^2(\omega)}\right\},$$

which gives:

$$\|H^\delta_{\alpha\beta}(\mathcal{R}(\delta))\|_{L^2(\omega)} + \|H^\delta_{\alpha\beta}(\mathcal{U}(\delta))\|_{L^2(\omega)} \leqslant \|H^\delta(u(\delta))\|_{L^2(\Omega)}.$$

Step 2: Proof of (2.9):

$$\begin{aligned}
\left\|H^\delta_{\alpha 3}(u(\delta))\right\|^2_{L^2(\Omega)} &= \int_\Omega |H^\delta_{\alpha 3}(u(\delta))|^2 dz \\
&= \int_\Omega (H^\delta_{\alpha 3}(u_e(\delta)))^2 dz + \int_\Omega (H^\delta_{\alpha 3}(u_r(\delta)))^2 dz \\
&= \int_\Omega (H^\delta_{\alpha 3}(\mathcal{U}(\delta)))^2 dz + \int_\Omega (H^\delta_{\alpha 3}(z_3\mathcal{R}(\delta)))^2 dz + \int_\Omega (H^\delta_{\alpha 3}(u_r(\delta)))^2 dz \\
&= \int_\Omega \left\{\frac{1}{2}\left(\partial^z_\alpha \mathcal{U}_3(\delta)\right)\right\}^2 dz + \int_\Omega \left\{\frac{1}{2h}\left(\mathcal{R}_\alpha(\delta)\right)\right\}^2 dz + \int_\Omega (H^\delta_{\alpha 3}(u_r(\delta)))^\delta dz. \\
&\geqslant \frac{1}{8}\int_\omega \left\{\frac{1}{h}\mathcal{R}_\alpha(\delta) + \partial^z_\alpha \mathcal{U}_3(\delta)\right\}^2 dz' + \int_\Omega (H^\delta_{\alpha 3}(u_r(\delta)))^2 dz \\
&\geqslant \frac{1}{4\|h\|^2_{L^\infty(\omega)}}\int_\omega \left\{\mathcal{R}_\alpha(\delta) + h\partial^z_\alpha \mathcal{U}_3(\delta)\right\}^2 dz' + \int_\Omega (H^\delta_{\alpha 3}(u_r(\delta)))^2 dz \\
&\geqslant C^* \left\|\mathcal{R}_\alpha(\delta) + h\partial^z_\alpha \mathcal{U}_3(\delta)\right\|^2_{L^2(\omega)} + \left\|H^\delta_{\alpha 3}(u_r(\delta))\right\|^2_{L^2(\Omega)}
\end{aligned}$$

Then one has

$$\left\|H^\delta_{\alpha 3}(u(\delta))\right\|_{L^2(\Omega)} \geqslant C^* \left\|\mathcal{R}_\alpha(\delta) + h\partial^z_\alpha \mathcal{U}_3(\delta)\right\|_{L^2(\omega)}.$$

Which gives

$$\left\|\mathcal{R}_\alpha(\delta) + h\partial^\delta_\alpha \mathcal{U}_3(\delta)\right\|_{L^2(\omega)} \leqslant C\left\|H^\delta(u(\delta))\right\|_{L^2(\Omega)} \tag{2.10}$$

■

Lemma 2.1.

The maps : $v \in V \to \|H(v)\|_{0,\Omega}$

is a norm equivalent to $\|.\|_V$ *, such that* $V = H^1(\Omega)$.

Chapter 3

Homogenization of The Plate

In this chapter, we gives the asymptotic behavior of the rescaled strain tensor in fixed domain Ω Then in the subset of Ω. Next, we presents the unfoldes and rescaled limit elasticity problem. finally, gives the homogenized limit.

From the last chapter we deduce that:

$$\|u(\delta)\|_{H^1(\Omega)} \leqslant C^* \|H^\delta_{kl}(u(\delta))\|_{L^2(\Omega)} \leqslant C.$$

So $H_{kl}(u(\delta))$ is bounded and There exist an subsequence, such that :

- $H^\delta_{kl}(u^\varepsilon(\delta)) \rightarrow H^*_{kl}(u(\delta))$ strongly in $L^2(\Omega)$,
- $H^\delta_{kl}(u^\varepsilon(\delta)) \rightharpoonup H^*_{kl}(u(\delta))$ weakly in $H^1(\Omega)$.

3.1 Unfolding the rescaled plate

From now on, we will use the usual unfolding operator in ω, as well as in Ω. The subset of ω included in the ε-cells intersecting its boundary $\partial\omega$ is $\tilde{\Lambda}_\varepsilon$. At some point we may identify $\Omega \times Y'$ with $\omega \times Y$, where

$$Y' = (0,1)^2, \qquad Y = Y' \times (-1,1).$$

For $x' \in \mathbb{R}^2$, one has

$$x' = [x'] + \{x'\}, \qquad [x'] \in \mathbb{Z}^2, \quad \{x'\} \in Y'.$$

3.2 Asymptotic behavior of the tensor

We are now in position to give the limits of the rescaled and unfolded strain tensor, for that in the following proposition we show that

$$\frac{1}{\delta}(H^\delta(u^\varepsilon(\delta))) \rightharpoonup E_M(\mathcal{U}) + E_w(u^0) \text{ weakly in } \left[L^2(\Omega)\right]^9$$

Hence

$$\frac{1}{\delta}\mathcal{T}_\varepsilon(H^\delta(u^\varepsilon(\delta))) \rightharpoonup E_M(\mathcal{U}) + E_w(u^0) \quad \text{weakly in } \left[L^2(\Omega\times Y)\right]^9$$

where

$$u^0 = u_r + y_3 z_\alpha e_\alpha,$$

under the same hypotheses with the notation of (proposition 11.12, See [3]) and by the same way, we obtain these convergence (See [3], p 373). The difference is in the symetric tensor $E_M(\mathcal{U})$ such that

$$\mathcal{U} = (\mathcal{U}_m, \mathcal{U}_3), \qquad E_{\alpha\beta}(\mathcal{U}) = e^z_{\alpha\beta}(\mathcal{U}_m) - \delta z_3 h\frac{\partial^2\mathcal{U}_3}{\partial z_\alpha \partial z_\beta} + \delta z_3 \partial^z_\alpha h \frac{\partial \mathcal{U}_3}{\partial z_\beta} \tag{3.1}$$

Furthermore, there exist

$$\widehat{U}_\alpha, \widehat{R}_\alpha \in L^2(\omega; H^1_{per,0}(Y')) \quad \text{and} \quad \widehat{Z}_\alpha \in L^2(\omega\times Y'),$$

with $M_{Y'}(\widehat{Z}_\alpha) = 0$ a.e. in ω, such that

$$\begin{aligned}
&\frac{1}{\delta}\mathcal{T}_\varepsilon(\nabla \mathcal{U}^\varepsilon_\alpha(\delta)) \rightharpoonup \nabla\mathcal{U}_\alpha + \nabla_{y'}\widehat{U}_\alpha \quad \text{weakly in } \left[L^2(\omega\times Y')\right]^2,\\
&\mathcal{T}_\varepsilon(\nabla \mathcal{R}^\varepsilon(\delta)) \rightharpoonup -D^2\mathcal{U}_3 + \nabla_{y'}\widehat{R} \quad \text{weakly in } \left[L^2(\omega\times Y')\right]^{2\times 2},\\
&\frac{1}{\delta}\mathcal{T}_\varepsilon(h^\varepsilon \nabla \mathcal{U}^\varepsilon_3(\delta) + \mathcal{R}^\varepsilon(\delta)) \rightharpoonup Z + \widehat{Z} \quad \text{weakly in } \left[L^2(\omega\times Y')\right]^2.
\end{aligned}$$

Proposition 3.1 below completes the results of Proposition 11.12 (See [3], p372).

Proposition 3.1. *Under the hypotheses and with the notations of Proposition 11.12,*

$$\frac{1}{\delta}\mathcal{T}_\varepsilon(H(u^\varepsilon(\delta))) \rightharpoonup E_M(\mathcal{U}) + E_w(u^0) + \mathcal{E}_y(\widehat{u}) \quad \textit{weakly in } \left[L^2(\omega\times Y)\right]^9, \tag{3.2}$$

Such that

$$\widehat{u}(.,y) = \widehat{U}(.,y') + y_3\widehat{R}(.,y') + \widehat{u}_r(.,y), \qquad \widehat{u} \in L^2(\omega; \mathcal{D}). \tag{3.3}$$

Proof. of Proposition 3.1 it is enough to the limits of the following two terms :

$$\frac{1}{\delta}\mathcal{T}_\varepsilon(H_{11}(u^\varepsilon(\delta))) \qquad \frac{1}{\delta}\mathcal{T}_\varepsilon(H_{13}(u^\varepsilon(\delta))),$$

Since the limit of all the terms of the forms $\frac{1}{\delta}\mathcal{T}_\varepsilon(H_{\alpha\beta}(u^\varepsilon(\delta)))$ are obtained in the same way as the former , and the limit of $\frac{1}{\delta}\mathcal{T}_\varepsilon(H_{23}(u^\varepsilon(\delta)))$ is obtained in the same way as the latter. The limit of $\frac{1}{\delta}\mathcal{T}_\varepsilon(H_{33}(u^\varepsilon(\delta)))$ was already obtained in proposition 11.12 (See [3])

$$
\begin{aligned}
\frac{1}{\delta}\mathcal{T}_\varepsilon(H_{11}(u^\varepsilon(\delta))) &= \frac{1}{\delta}\mathcal{T}_\varepsilon\left(\frac{\partial \mathcal{U}_1^\varepsilon(\delta)}{\partial z_1}\right) + \mathcal{T}_\varepsilon(z_3 h^\varepsilon)\mathcal{T}_\varepsilon\left(\frac{\partial \mathcal{R}_1^\varepsilon(\delta)}{\partial z_1}\right) + \frac{1}{\delta}\mathcal{T}_\varepsilon\left(\frac{\partial u_{r,1}^\varepsilon(\delta)}{\partial z_1}\right)\\
&\quad - \mathcal{T}_\varepsilon(z_3\partial_1^\varepsilon h^\varepsilon)\mathcal{T}_\varepsilon(\mathcal{R}_1^\varepsilon(\delta)) + \mathcal{T}_\varepsilon(z_3\partial_1^z h^\varepsilon)\mathcal{T}_\varepsilon(\partial_3^z u_{r,1}^\varepsilon(\delta))\\
&\rightharpoonup \frac{\partial \mathcal{U}_1(\delta)}{\partial z_1} + \frac{\partial \widehat{U}_1(\delta)}{\partial y_1} + z_3 h\left(\frac{\partial \mathcal{R}_1}{\partial z_1} + \frac{\partial \widehat{R}_1}{\partial y_1}\right) + \frac{\partial \widehat{u}_{r,1}(\delta)}{\partial y_1} + z_3\partial_1^z h\frac{\mathcal{U}_3(\delta)}{\partial z_1} - z_3\partial_1^z h\frac{\partial u_{r,1}(\delta)}{\partial y_3}.
\end{aligned}
$$

and with the help of the field $\widehat{u}$ (See (3.3)),

$$
\begin{aligned}
\frac{1}{\delta}\mathcal{T}_\varepsilon(H_{13}(u^\varepsilon(\delta))) &= \frac{1}{2}\left\{\frac{1}{\delta}\mathcal{T}_\varepsilon\left(\frac{\partial \mathcal{U}_3^\varepsilon(\delta)}{\partial z_1} + \frac{1}{\delta h^\varepsilon}\mathcal{R}_1^\varepsilon(\delta)\right) + \frac{1}{\delta}\mathcal{T}_\varepsilon\left(\partial_1^z u_{r,3}^\varepsilon(\delta)\right) - \mathcal{T}_\varepsilon(z_3\partial_1^\varepsilon h^\varepsilon)\mathcal{T}_\varepsilon(\partial_3^z u_{r,3}^\varepsilon(\delta))\right.\\
&\quad \left. + \frac{1}{\delta^2}\mathcal{T}_\varepsilon\left(\frac{1}{h^\varepsilon}\partial_3^z u_{r,1}^\varepsilon(\delta)\right)\right\}\\
&\rightharpoonup \frac{1}{2}\left\{Z \ + \frac{1}{h}\widehat{Z} + \frac{\partial \widehat{u}_{r,3}(\delta)}{\partial y_1} + z_3\partial_1^z h\frac{\partial u_{r,3}(\delta)}{\partial y_3} + \frac{1}{h}\left(\frac{\partial \widehat{u}_{r,1}(\delta)}{\partial y_3} + \frac{\partial \widehat{u}_{r,1}}{\partial y_3}\right)\right\}.
\end{aligned}
$$

Where

$$
\widehat{Z}_1 = \frac{\partial \widehat{U}_3}{\partial y_1} + \widehat{R}_1 \quad and \quad \frac{\partial u_1^0}{\partial y_3} = Z_1 + \frac{\partial u_{r,1}}{\partial y_3}
$$

So we get the convergence (3.2). ■

Corollary 3.1. *For the rescaled and unfolded stress tensor, one has the convergence*

$$
\frac{1}{\delta}\mathcal{T}_\varepsilon(\sigma^\varepsilon(\delta)) \rightharpoonup \Sigma \quad weakly\ in\ L^2(\omega\times Y),
$$

i.e, for $1 \leqslant i,j,k,l \leqslant 3$,

$$
\frac{1}{\delta}\mathcal{T}_\varepsilon(\sigma_{ij}^\varepsilon(u^\varepsilon(\delta))) \rightharpoonup \Sigma_{ij} = a_{ijkl}E_{kl,M}(\mathcal{U}) + a_{ijkl}E_{kl,w}(u^0) + a_{ijkl}\mathcal{E}_{kl,y}(\widehat{u}).
$$

3.3 The Unfolding Limit Problems

Theorem 3.1. *Let $u^\varepsilon(\delta)$ be the solution of the elasticity problem $P(\Omega^\delta)$. The following convergence holds:*

$$\frac{1}{\delta}\mathcal{T}_\varepsilon(H^\delta(u^\varepsilon(\delta))) \to E_M(\mathcal{U}) + E_w(u^0) + \mathcal{E}_y(\widehat{u}) \quad \textit{strongly in } \left[L^2(\omega \times Y)\right]^9, \tag{3.4}$$

and $u = (\mathcal{U}, u^0, \widehat{u})$ belonging to V^δ is the solution of the rescaled and unfolded problem :

$$\begin{cases} \dfrac{1}{2}\displaystyle\int_{\omega\times Y} a_{ijkl}(E_{kl,M}(\mathcal{U}) + E_{kl,w}(u^0) + \mathcal{E}_{kl,y}(\widehat{u})) \\ \qquad\qquad \times (E_{ij,M}(\mathcal{V}) + E_{ij,w}(v^0) + \mathcal{E}_{ij,y}(\widehat{v}))dx'dy \\ \qquad\qquad = \displaystyle\int_\omega \left(f.\mathcal{V} - \frac{1}{3}g_\alpha\frac{\partial\mathcal{V}}{\partial x_\alpha}\right)dx', \quad \forall v = (\mathcal{V}, v^0, \widehat{v}) \in V^\delta \end{cases} \tag{3.5}$$

Proof. Due to hypotheses of previous section and lemma 11.18 (See [3], p384), the lax-Milgram theorem applies to Problem 3.5 which, has a unique solution. This uniqueness implies it is enough to prove convergence (3.4) for a subsequence, as is done now. ■

Introduce the set

$$V = \{v = (\mathcal{V}, v^0, \widehat{v}) \in V_M \times L^2(w, \mathcal{W}) \times L^2(w, \mathcal{D})\},$$

To every V^δ, we associate the symmetric tensor.

$$E_M(\mathcal{V}) + E_w(v^0) + \mathcal{E}_y(\widehat{v})$$

and the norm

$$\|v\| = \|E_M(\mathcal{V}) + E_w(v^0) + \mathcal{E}_y(\widehat{v})\|.$$

With $\mathcal{V} = (\mathcal{V}_m, \mathcal{V}_3) \in V_M$ and $(v^0, \widehat{v})$ in $V^0 \times V'_{per}$, where

$V^0 = \{\Psi \in C^1(\overline{\omega} \times [-1,1])^3 \mid \Psi(., z_3) = 0 \;\text{ on } \partial\omega \forall z_3 \in [-1,1]\}.$

$V_{per} = \{\Psi \in C^1(\overline{\omega} \times \overline{Y})^3 \mid \Psi Y'\text{-periodic and } \Psi(., y) = 0 \text{ on } \partial\omega\ \forall y \in \overline{Y}\}.$

Consider the following test displacement:

$$w^\varepsilon(\delta) = v(\delta) + v^\varepsilon(\delta),$$

$v(\delta)(z) = \delta\mathcal{V}_\alpha(z') - z_3\frac{\partial\mathcal{V}_3}{\partial z_\alpha}(z') + \delta^2 v^0_\alpha(z', z_3)\Big]\, e_\alpha + [\mathcal{V}_3(z') + \delta^2 v^0_3(z', z_3)]\, e_3,$

$v^\varepsilon(\delta)(x) = \delta\widehat{v}(z', \{\frac{z'}{\delta}\}, z_3).$

A straightforward computation gives

$$\frac{1}{\delta}(H^\delta(v^\varepsilon(\delta))) \to E_M(\mathcal{V} + E_w(v^0) \quad \text{strongly in } \left[L^2(\Omega)\right]^9.$$

Hence,
$$\frac{1}{\delta}\mathcal{T}_\varepsilon(H^\delta(v^\varepsilon(\delta))) \to E_M(\mathcal{V} + E_w(v^0) \quad \text{strongly in } \left[L^2(\omega \times Y)\right]^9.$$

Also the function $\widehat{v}$ is defined in [3],

It is easily seen that

$$\frac{1}{\delta}\mathcal{T}_\varepsilon(H^\delta(u^\varepsilon(\delta))) \to \mathcal{E}_M(\widehat{v}) \quad \text{strongly in } \left[L^2(\omega \times Y)\right]^9.$$

then we get,

$$\frac{1}{\delta}\mathcal{T}_\varepsilon(H^\delta(u^\varepsilon(\delta))) \to E_M(\mathcal{V}) + E_w(v^0) + \mathcal{E}_y(\widehat{v}) \quad \text{strongly in } \left[L^2(\omega \times Y)\right]^9.$$

Taking $v^\varepsilon(\delta)$ as test displacement in 2.1, unfolding the equality with $\mathcal{T}_\varepsilon$, dividing by $2\delta^3$, and passing to the limit, give 3.5 with v as the test function. The density of the product space $V^0 \times V_{per}$ in $L^2(\omega; \mathcal{W}) \times L^2(\omega; \mathcal{D})$ give 3.5 for every $v \in V$.

3.4 Homogenization

With the choice $\mathcal{V} = 0$, Problem 3.5 becomes

$$\frac{1}{2}\int_{\omega\times Y} a_{ijkl}(E_{kl,M}(\mathcal{U}) + E_{kl,w}(u^0) + \mathcal{E}_{kl,y}(\widehat{u}))(E_w(v^0) + \mathcal{E}_{ij,y}(\widehat{v}))dx'dy = 0.$$

Define the space

$$\mathcal{WD} = \left\{ \psi^0 + \widehat{\psi} \mid \psi^0 \in \mathcal{W}, \widehat{\psi} \in \mathcal{D} \right\}.$$

For every function $\tilde{\psi}$ in $\mathcal{W}$,associate the symmetric tensor

$$\mathbb{E}_y(\tilde{\psi}) = E_w(\psi^0) + \mathcal{E}_y(\widehat{\psi}).$$

Due to the properties of the functions in $\mathcal{D}$, one has

$$\begin{aligned}\|\mathbb{E}_y(\tilde{\psi})\|^2_{L^2(Y)} &= \|E_w(\psi^0)\|^2_{L^2(Y)} + \|\mathcal{E}_y(\widehat{\psi})\|^2_{L^2(Y)} \\ &= \frac{1}{4}\|\frac{\partial \psi^0_\alpha}{\partial y_3}\|^2_{L^2(-1,1)} + \|\frac{\partial \psi^0_3}{\partial y_3}\|^2_{L^2(-1,1)} + \|\mathcal{E}_y(\widehat{\psi})\|^2_{L^2(Y)}.\end{aligned}$$

Set

$$M^{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad M^{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad M^{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

One introduces the correctors:

$$\widetilde{\mathcal{X}_m^{\alpha\beta}} \in \mathcal{WD}, \quad \widetilde{\mathcal{X}_b^{\alpha\beta}} \in \mathcal{WD}, \quad \widetilde{\mathcal{X}_c^{\alpha\beta}} \in \mathcal{WD}, \quad (\alpha,\beta) \in \{(1,1),(1,2),(2,2)\},$$

defined respectively by

$$\begin{aligned}&\int_Y a_{ijkl}(y)(\mathbf{M}^{\alpha\beta}_{kl} + \mathbb{E}_{kl,y}(\widetilde{\mathcal{X}_m^{\alpha\beta}})(y))\mathbb{E}_{ij,y}(\tilde{\psi})(y)dy = 0, \\ &\int_Y a_{ijkl}(y)(y_3\mathbf{M}^{\alpha\beta}_{kl} + \mathbb{E}_{kl,y}(\widetilde{\mathcal{X}_b^{\alpha\beta}})(y))\mathbb{E}_{ij,y}(\tilde{\psi})(y)dy = 0, \\ &\int_Y a_{ijkl}(y)(y_3\mathbf{M}^{\alpha\beta}_{kl} + \mathbb{E}_{kl,y}(\widetilde{\mathcal{X}_c^{\alpha\beta}})(y))\mathbb{E}_{ij,y}(\tilde{\psi})(y)dy = 0 \\ &\qquad \forall \tilde{\psi} \in \mathcal{WD}.\end{aligned} \tag{3.6}$$

As a consequence ,$\tilde{u}$ can be written in the form

$$\begin{aligned}\tilde{u}(.,y) &= u^0(.,y_3) + \widehat{u}(.,y) \\ &= e^z_{\alpha\beta}(\mathcal{U}_m)\widetilde{\mathcal{X}_m^{\alpha\beta}}(y) + \frac{\partial^2 \mathcal{U}_3}{\partial z_\alpha \partial z_\beta}\widetilde{\mathcal{X}_b^{\alpha\beta}}(y) + \partial^z_\alpha h \frac{\partial \mathcal{U}_3}{\partial z_\beta}\widetilde{\mathcal{X}_c^{\alpha\beta}}(y) \quad \text{for a.e. } y \in Y.\end{aligned} \tag{3.7}$$

3.4.1 The limit problems in the mid surface

Theorem 3.2. *The limit displacement*

$\mathcal{U} = (\mathcal{U}_m, \mathcal{U}_3)$ *belongs to* $V_M = \left[H^1_{\gamma_0}(\omega)\right]^2 \times H^2_{\gamma_0}(\omega)$.

It is the solution of the homogenized problem :

$$\begin{cases} \displaystyle\int_\omega a^{hom}_{\alpha\beta\alpha'\beta'} \left\{ e^z_{\alpha\beta}(\mathcal{U}_m) e^z_{\alpha\beta}(\mathcal{V}_m) + b^{hom}_{\alpha\beta\alpha'\alpha'} \left(e^z_{\alpha\beta}(\mathcal{U}_m) \frac{\partial^2 \mathcal{V}_3}{\partial z_\alpha \partial z_\beta} + e^z_{\alpha\beta}(\mathcal{V}_m) \frac{\partial^2 \mathcal{U}_3}{\partial z_\alpha \partial z_\beta} \right) \right. \\ \quad + c^{hom}_{\alpha\beta\alpha'\beta'} \left(\left[e^z_{\alpha\beta}(\mathcal{U}_m) + \frac{\partial^2 \mathcal{U}_3}{\partial z_\alpha \partial z_\beta} \right] \partial^z_\alpha h \frac{\partial \mathcal{V}_3}{\partial z_\beta} + \left[e^z_{\alpha\beta}(\mathcal{V}_m) + \frac{\partial^2 \mathcal{V}_3}{\partial z_\alpha \partial z_\beta} \right] \partial^z_1 h \frac{\partial \mathcal{U}_3}{\partial z_\beta} \right) \\ \quad \left. + d^{hom}_{\alpha\beta\alpha'\beta'} (\partial^z_\alpha h)^2 \frac{\partial \mathcal{U}_3}{\partial z_\alpha} \frac{\partial \mathcal{V}_3}{\partial z_\beta} + k^{hom}_{\alpha\beta\alpha'\beta'} \frac{\partial^2 \mathcal{U}_3}{\partial z_\alpha \partial z_\beta} \frac{\partial^2 \mathcal{V}_3}{\partial z_\alpha \partial z_\beta} \right] dx' \\ \quad = \displaystyle\int_\omega f.\mathcal{V} dx' - \frac{1}{3} \int_\omega g_\alpha \frac{\partial \mathcal{V}_3}{\partial x_\alpha} dx', \quad \forall \mathcal{V} = (\mathcal{V}_m, \mathcal{V}_3) \in V_M, \end{cases} \tag{3.8}$$

Where:

$$\begin{aligned} a^{hom}_{\alpha\beta\alpha'\beta'} &= \frac{1}{2} \int_Y a_{ijkl}(y) [\boldsymbol{M}^{\alpha\beta}_{kl} + \mathbb{H}_{kl,y}(\widetilde{\mathcal{X}^{\alpha\beta}_m})] \boldsymbol{M}^{\alpha'\beta'}_{ij} dy, \\ b^{hom}_{\alpha\beta\alpha'\beta'} &= \frac{1}{2} \int_Y a_{ijkl}(y) [z_3 \boldsymbol{M}^{\alpha\beta}_{kl} + \mathbb{H}_{kl,y}(\widetilde{\mathcal{X}^{\alpha\beta}_b})] \boldsymbol{M}^{\alpha'\beta'}_{ij} dy, \\ c^{hom}_{\alpha\beta\alpha'\beta'} &= \frac{1}{2} \int_Y a_{ijkl}(y) [z_3 \boldsymbol{M}^{\alpha\beta}_{kl} + \mathbb{H}_{kl,y}(\widetilde{\mathcal{X}^{\alpha\beta}_c})] z_3 \boldsymbol{M}^{\alpha'\beta'}_{ij} dy, \\ d^{hom}_{\alpha\beta\alpha'\beta'} &= \frac{1}{2} \int_Y a_{ijkl}(y) [z_3 \boldsymbol{M}^{\alpha\beta}_{kl} + \mathbb{H}_{kl,y}(\widetilde{\mathcal{X}^{\alpha\beta}_c})] z_3 \boldsymbol{M}^{\alpha'\beta'}_{ij} dy, \\ k^{hom}_{\alpha\beta\alpha'\beta'} &= \frac{1}{2} \int_Y a_{ijkl}(y) [z_3 \boldsymbol{M}^{\alpha\beta}_{kl} + \mathbb{H}_{kl,y}(\widetilde{\mathcal{X}^{\alpha\beta}_b})] z_3 \boldsymbol{M}^{\alpha'\beta'}_{ij} dy, \end{aligned} \tag{3.9}$$

Proof. In problem 3.5, choose as test displacement $\mathcal{V} = (\mathcal{V}_m, \mathcal{V}_3)$ in V_M and $v^0 = 0, \widehat{v} = 0$.

Replacing $\tilde{u}$ by is expression (3.7), yields

$$\begin{aligned} \int_{\omega \times Y} a_{ijkl}(y) \left[e^z_{\alpha\beta}(\mathcal{U}_m) \right. & \left. (\mathbf{M}^{\alpha\beta}_{kl} + \mathbb{E}_{kl,y}(\widetilde{\mathcal{X}^{\alpha\beta}_m})) + \frac{\partial^2 \mathcal{U}_3}{\partial z_\alpha \partial z_\beta} (z_3 \mathbf{M}^{\alpha\beta}_{kl} + \mathbb{E}_{kl,y}(\widetilde{\mathcal{X}^{\alpha\beta}_b})) \right] + \partial^z_\alpha h \frac{\partial \mathcal{U}_3}{\partial z_\beta} \\ & \times \mathbf{M}^{\alpha'\beta'}_{ij} \left[e^z_{\alpha'\beta'}(\mathcal{V}_m) + z_3 \frac{\partial^2 \mathcal{V}_3}{\partial z_{\alpha'} \partial z_{\beta'}} + z_3 \partial_\alpha h \frac{\partial \mathcal{V}_3}{\partial z_{\beta'}} \right] dx' dy \\ & = 2 \int_\omega f.\mathcal{V} dx' - \frac{2}{3} \int_\omega g_\alpha \frac{\partial \mathcal{V}_3}{\partial x_\alpha} dx'. \end{aligned} \tag{3.10}$$

We can obtained the homogenized coefficients of problem 3.9 by a simple computation. (see [3] ,p 390).

Taking into account the variational problems 3.6 satisfied by the corectors, it is easily seen that the homogenized coefficients are also given by the following expressions:

$$\begin{aligned}
a^{hom}_{\alpha\beta\alpha'\beta'} &= \frac{1}{2}\int_Y a_{ijkl}(y)[\mathbf{M}^{\alpha\beta}_{kl} + \mathbb{H}_{kl,y}(\widetilde{\mathcal{X}^{\alpha\beta}_b})][z_3\mathbf{M}^{\alpha'\beta'}_{ij} + \mathbb{H}_{ij,y}(\widetilde{\mathcal{X}_m})]dy,\\
b^{hom}_{\alpha\beta\alpha'\beta'} &= \frac{1}{2}\int_Y a_{ijkl}(y)[\mathbf{M}^{\alpha\beta}_{kl} + \mathbb{H}_{kl,y}(\widetilde{\mathcal{X}^{\alpha\beta}_b})][z_3\mathbf{M}^{\alpha'\beta'}_{ij} + \mathbb{H}_{ij,y}(\widetilde{\mathcal{X}_b})]dy\\
c^{hom}_{\alpha\beta\alpha'\beta'} &= \frac{1}{2}\int_Y a_{ijkl}(y)[z_3\mathbf{M}^{\alpha\beta}_{kl} + \mathbb{H}_{kl,y}(\widetilde{\mathcal{X}^{\alpha\beta}_b})][z_3\mathbf{M}^{\alpha'\beta'}_{ij} + \mathbb{H}_{ij,y}(\widetilde{\mathcal{X}_c})]dy,\\
d^{hom}_{\alpha\beta\alpha'\beta'} &= \frac{1}{2}\int_Y a_{ijkl}(y)[z_3\mathbf{M}^{\alpha\beta}_{kl} + \mathbb{H}_{kl,y}(\widetilde{\mathcal{X}^{\alpha\beta}_b})][z_3\mathbf{M}^{\alpha'\beta'}_{ij} + \mathbb{H}_{ij,y}(\widetilde{\mathcal{X}_c})]dy,\\
k^{hom}_{\alpha\beta\alpha'\beta'} &= \frac{1}{2}\int_Y a_{ijkl}(y)[y_3\mathbf{M}^{\alpha\beta}_{kl} + \mathbb{H}_{kl,y}(\widetilde{\mathcal{X}^{\alpha\beta}_b})][z_3\mathbf{M}^{\alpha'\beta'}_{ij} + \mathbb{H}_{ij,y}(\widetilde{\mathcal{X}_b})]dy,
\end{aligned} \tag{3.11}$$

■

Remark 3.1. *This part in our study was similar to [3] , but the difference lies in our use of the tensor H, which added another term in the expression of the strees tensor (See 3.1), and the same test function in [3].*

Appendix I

In this appendix, we will present some basics preliminaries including spaces definitions and theories, we need to get weak formulation of studied problem.

Preliminaries

Let Ω be a domain in $\mathbb{R}^3$, $H^m(\Omega)$ and $H_0^m(\Omega)$ denote the usual Sobolev spaces, in particular for $m = 1$,

$$H^1(\Omega) = \left\{ v \in L^2(\Omega) \ ; \ \partial_i v \in L^2(\Omega) \ 1 \leqslant i \leqslant 3 \right\},$$

$$H_0^1(\Omega) = \left\{ v \in H^1(\Omega) \ ; \ v = 0 \text{ on } \Gamma \right\},$$

and for $m = 0$,

$$H^0(\Omega) = L^2(\Omega).$$

And there norms are defined as follows:

$$\|v\|_{L^2(\Omega)}^2 = \sum_{i=1}^{3} \|v_i\|_{L^2(\Omega)}^2 \quad \text{for all } v = (v_i) \in L^2(\Omega),$$

$$\|v\|_{H^1(\Omega)}^2 = \sum_{i=1}^{3} \|v_i\|_{H^1(\Omega)}^2 \quad \text{for all } v = (v_i) \in H^1(\Omega),$$

Theorem 3.3 (***Young's Inequality***[1]).

Let a and b be two positive real numbers. For $p, q \in]1, +\infty[$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}.$$

Theorem 3.4 (***Hôlder Inequality***).

Suppose $p, q \in]1, +\infty[$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let Ω be a domain in $\mathbb{R}^N$ and $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then $fg \in L^1(\Omega)$ and

$$\|fg\|_{L^1(\Omega)} \leqslant \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$

Remark 3.2.

Cauchy Schwarz's Inequality is particular case of Holder Inequality for $p = 2$ and $q = 2$.

Proof. See([4], p706). ∎

Theorem 3.5 (***(Poincaré-Wirtinger's inequality)***[1]**)**.

Let Ω be a connected open set of class C^1 and $1 \leqslant p \leqslant \infty$. Then there exists a constant C such that

$$\|u - \overline{u}\|_{L^p(\Omega)} \leqslant C \, \|\nabla u\|_{L^p(\Omega)} \quad \forall u \in W^{1,p}(\Omega), \quad \textit{where } \overline{u} = \frac{1}{meas(\Omega)} \int_\Omega u.$$

Theorem 3.6 (***Poincaré's Inequality***)**.**

Let Ω be a bounded open set in $\mathbb{R}^N$. Then there exists a constant C_p (depending on Ω and $p \in [1, +\infty[$), such that for all $v \in W_0^{1,p}(\Omega)$,

$$\|v\|_{L^p(\Omega)} \leqslant C_p \|\nabla v\|_{L^p(\Omega)}.$$

Proof. See ([1], p.220) ∎

Theorem 3.7 (***Korn's Inequality With a Boundary Condition***)**.**

Let Ω be a domain in $\mathbb{R}^3$ and Γ_0 be a mesurable subset of the boundary Γ such that

$\boxed{meas(\Gamma_0) > 0}$. *Given a vector field* $v = (v_i)_{i=1}^3 \in H^1(\Omega)$. *Not that*

$$e_{i,j}(v) = \frac{1}{2}(\partial_j v_i + \partial_i v_j) \in L^2(\Omega),$$

$$\|e(v)\|^2_{L^2(\Omega)} = \sum_{i,j=1}^{3} \|e_{i,j}(v)\|^2_{L^2(\Omega)}$$

Therefore, there exists a constant C_k *such that*

$$\|v\|_{H^1(\Omega)} \leqslant C_k \|e(v)\|_{L^2(\Omega)}, \quad \forall v \in H^1(\Omega) \textit{vanishing on } \Gamma_0.$$

Proof. See ([2], p.11). ■

Theorem 3.8 (***Trace Theorem***).

Let Ω *be a bounded open set in* $\mathbb{R}^n$. *We define the trace map,*

$$\gamma_0 : C^1(\overline{\Omega}) \longrightarrow L^2(\partial\Omega)$$
$$v \longmapsto \gamma_0(v) = v_{|\partial\Omega}$$

This map is linear and continuous on $L^2(\partial\Omega)$, *then there exists a positive constant* C_r *such that*

$$\|\gamma_0(v)\|_{L^2(\partial\Omega)} \leqslant C_r \|v\|_{H^1(\Omega)} \quad \textit{for all } v \in H^1(\Omega).$$

Proof. See ([4], p.272). ■

Theorem 3.9 (***Green's Formulation***).

Let Ω *be a bounded open in* $\mathbb{R}^3$ *and* $d\Gamma$ *be a sufficiently smooth boundary* Γ, *Let* $v, w \in C^1(\Omega)$, *one has the Green's Integration by parts formula:*

$$\int_\Omega v \ \partial_i u dx = -\int_\Omega u \ \partial_i v dx + \int_\Gamma uv \ n_i d\Gamma$$

where ***n****=(****n****$_i$)* ***is the outer normal on*** Γ***.***

Proof. See ([4], p.712). ■

Proposition 3.2.

Let E be a Hilbert space and $(x_n)_{n\geqslant 0}$ be a sequence converges weakly to x^* in E. Then $(x_n)_{n\geqslant 0}$ is bounded and satisfy

$$\|x^*\|_E \leqslant \lim_{n\to\infty} inf\|x_n\|_E$$

and this sequence converges strongly to x^* in E if

$$\lim_{n\to\infty} \|x_n\|_E = \|x\|_E$$

Proof. See ([6], p.124). ■

Theorem 3.10.

Let E be a Reflexive Banach space and let (x_n) be a bounded sequence in E. Then There exists a subsequence (x_{n_k}) converges weakly to x^*

Proof. See ([5], p.496). ■

Theorem 3.11 (***Lax-Milgram Theorem*** [1])**.**

Let $V^{\delta\varepsilon}$ be a Hilbert space and suppose that:

(i) The symetric bilinear form $B^\delta : V^{\delta\varepsilon} \times V^{\delta\varepsilon} \to \mathbb{R}$ is continuous,

$$\exists M < +\infty, \quad \forall (u^{\delta\varepsilon}, v) \in V^{\delta\varepsilon} \times V^{\delta\varepsilon}, \qquad |B^\delta(u^{\delta\varepsilon}, v)| \leqslant M\|u^{\delta\varepsilon}\|_{V^{\delta\varepsilon}}\|v\|_{V^{\delta\varepsilon}};$$

(ii) The bilinear form B^δ is $V^{\delta\varepsilon}$-elliptic,

$$\exists \gamma > 0, \quad \forall v \in V^{\delta\varepsilon}, \qquad B^\delta(v,v) \geqslant \gamma\ \|v\|^2_{V^{\delta\varepsilon}};$$

(iii) The linear form $L^\delta : V^{\delta\varepsilon} \to \mathbb{R}$ is continuous,

$$\exists C < +\infty, \quad \forall v \in V^{\delta\varepsilon}, \qquad |L^\delta(v)| \leqslant C\|v\|_{V^{\delta\varepsilon}}.$$

Then the variational problem $P(\Omega^{\delta\varepsilon})$ has one and only one solution.

Appendix II

In the second appendix, we will mention some of the proofs tackled in the second chapter.

Proof. One has after (2.1):

$$\begin{aligned}
\int_{-\delta}^{\delta} u_1^\delta dx_3 &= \int_{-\delta}^{\delta} u_{e,1}^\delta dx_3 + \int_{-\delta}^{\delta} u_{r,1}^\delta dx_3 \\
&= \int_{-\delta}^{\delta} \left\{\mathcal{U}_1^\delta(x') + x_3\mathcal{R}_1^\delta(x')\right\} dx_3 + \int_{-\delta}^{\delta} u_{r,1}^\delta dx_3 \\
&= \int_{-\delta}^{\delta} \mathcal{U}_1^\delta(x') dx_3 + \int_{-\delta}^{\delta} x_3\mathcal{R}_1^\delta(x') dx_3 \\
&= \mathcal{U}_1^\delta(x')\left(\int_{-\delta}^{\delta} dx_3\right) + \mathcal{R}_1^\delta(x')\left(\int_{-\delta}^{\delta} x_3 dx_3\right) \\
&= 2\delta\mathcal{U}_1^\delta(x') + \frac{2\delta^3}{3}\mathcal{R}_1^\delta(x')
\end{aligned}$$

$$\begin{aligned}
\int_{-\delta}^{\delta} u_2^\delta dx_3 &= \int_{-\delta}^{\delta} u_{e,2}^\delta dx_3 + \int_{-\delta}^{\delta} u_{r,2}^\delta dx_3 \\
&= \int_{-\delta}^{\delta} \left\{\mathcal{U}_2^\delta(x') + x_3\mathcal{R}_2^\delta(x')\right\} dx_3 + \int_{-\delta}^{\delta} u_{r,2}^\delta dx_3 \\
&= \int_{-\delta}^{\delta} \mathcal{U}_2^\delta(x') dx_3 + \int_{-\delta}^{\delta} x_3\mathcal{R}_2^\delta(x') dx_3 \\
&= \mathcal{U}_2^\delta(x')\left(\int_{-\delta}^{\delta} dx_3\right) + \mathcal{R}_2^\delta(x')\left(\int_{-\delta}^{\delta} x_3 dx_3\right) \\
&= 2\delta\mathcal{U}_2^\delta(x') + \frac{2\delta^3}{3}\mathcal{R}_2^\delta(x').
\end{aligned}$$

■

The strain tensor $(e_{ij})_{i,j\leqslant 3}$ defined for $u^\delta \in [H^1(\Omega^\delta)]^3$ as follows :

$$e_{i,j}(u^\delta) = \frac{1}{2}\left(\frac{\partial u_j^\delta}{\partial x_i} + \frac{\partial u_i^\delta}{\partial x_j}\right),$$

and by the decomposition (2.1), one has:

$$e_{11}(u^\delta) = \frac{\partial u_1^\delta}{\partial x_1} = \frac{\partial \mathcal{U}_1^\delta}{\partial x_1} + x_3 \frac{\partial \mathcal{R}_1^\delta}{\partial x_1} + \frac{\partial u_{r,1}^\delta}{\partial x_1}.$$

$$e_{22}(u^\delta) = \frac{\partial u_2^\delta}{\partial x_2} = \frac{\partial \mathcal{U}_2^\delta}{\partial x_2} + x_3 \frac{\partial \mathcal{R}_2^\delta}{\partial x_2} + \frac{\partial u_{r,2}^\delta}{\partial x_2}.$$

$$\begin{aligned} e_{12}(u^\delta) &= \frac{1}{2}\left(\frac{\partial u_1^\delta}{\partial x_2} + \frac{\partial u_2^\delta}{\partial x_1}\right) \\ &= \frac{1}{2}\left(\left[\frac{\partial \mathcal{U}_1^\delta}{\partial x_2} + x_3 \frac{\partial \mathcal{R}_1^\delta}{\partial x_2} + \frac{\partial u_{r,1}^\delta}{\partial x_2}\right] + \left[\frac{\partial \mathcal{U}_2^\delta}{\partial x_1} + x_3 \frac{\partial \mathcal{R}_2^\delta}{\partial x_1} + \frac{\partial u_{r,2}^\delta}{\partial x_1}\right]\right) \\ &= \frac{1}{2}\left(\left[\frac{\partial \mathcal{U}_1^\delta}{\partial x_2} + \frac{\partial \mathcal{U}_2^\delta}{\partial x_1}\right] + x_3\left[\frac{\partial \mathcal{R}_1^\delta}{\partial x_2} + \frac{\partial \mathcal{R}_2^\delta}{\partial x_1}\right] + \left[\frac{\partial u_{r,2}^\delta}{\partial x_1} + \frac{\partial u_{r,1}^\delta}{\partial x_2}\right]\right). \end{aligned}$$

$$\begin{aligned} e_{13}(u^\delta) &= \frac{1}{2}\left(\frac{\partial u_1^\delta}{\partial x_3} + \frac{\partial u_3^\delta}{\partial x_1}\right) \\ &= \frac{1}{2}\left(\left[\mathcal{R}_1^\delta + \frac{\partial u_{r,1}^\delta}{\partial x_3}\right] + \frac{\partial \mathcal{U}_3^\delta}{\partial x_1} + \frac{\partial u_{r,3}^\delta}{\partial x_1}\right) \\ &= \frac{1}{2}\left(\left[\frac{\partial \mathcal{U}_3^\delta}{\partial x_1} + \mathcal{R}_1^\delta\right] + \left[\frac{\partial u_{r,1}^\delta}{\partial x_1} + \frac{\partial u_{r,3}^\delta}{\partial x_1}\right]\right). \end{aligned}$$

$$\begin{aligned} e_{23}(u^\delta) &= \frac{1}{2}\left(\frac{\partial u_2^\delta}{\partial x_3} + \frac{\partial u_3^\delta}{\partial x_2}\right) \\ &= \frac{1}{2}\left(\left[\mathcal{R}_2^\delta + \frac{\partial u_{r,2}^\delta}{\partial x_3}\right] + \frac{\partial \mathcal{U}_3^\delta}{\partial x_2} + \frac{\partial u_{r,3}^\delta}{\partial x_2}\right) \\ &= \frac{1}{2}\left(\left[\frac{\partial \mathcal{U}_3^\delta}{\partial x_2} + \mathcal{R}_2^\delta\right] + \left[\frac{\partial u_{r,2}^\delta}{\partial x_2} + \frac{\partial u_{r,3}^\delta}{\partial x_2}\right]\right). \end{aligned}$$

$$e_{33}(u^\delta) = \frac{\partial u_3^\delta}{\partial x_3} = \frac{\partial u_{r,3}^\delta}{\partial x_3}.$$

Conclusion

The work presented in this Master's thesis concerns the homogenization, using **the periodic unfolding method**, of heterogeneous elastic plates with a periodic structure and variable thickness. We consider the case where the orders of magnitude of thickness and period size are identical. Using the displacement decomposition method proposed by Griso, and applying the unfolding techniques, we obtain the homogenized two-dimensional plate model.

Perspectives

This work can be extended to cases:

- The orders of magnitude of thickness and period size are different: $(\lim\limits_{\delta\longrightarrow 0}(\lim\limits_{\varepsilon\longrightarrow 0} P^{\delta\varepsilon}), \lim\limits_{\varepsilon\longrightarrow 0}(\lim\limits_{\delta\longrightarrow 0} P^{\delta\varepsilon}))$.
- Linear elastic shallow shell
- Linear elastic thin shell
- Micro-fissured plates and shells

Abstract

In our work, we study the homogenization of a periodic heterogeneous elastic thin plate with variable thickness in the case when the order of magnitude of the period and the thickness are the same. Starting from the equations governing the equilibrium of a three dimensional linear elastic heterogeneous body formed the thin periodic plate.then we use the decomposition of the displacements cite above in the fixed domain, we get the limit (homogenization) problem by using the periodic unfolding method.

.**<u>Key words</u>**: Asymptotic analysis, homogenization, linear elasticity, thin plate, unfolding method.

Résumé

Dans notre travail, nous étudions l'homogénéisation d'une plaque mince élastique hétérogène périodique d'épaisseur variable dans le cas où l'ordre de grandeur de la période et les épaisseurs sont les mémes. Partant des équations régissant l'équilibrium d'un corps hétérogène élastique linéaire tridimensionnel formé le mince périodique plate puis on utilise la décomposition des déplacements cités plus haut dans le domaine fixe, on obtient le problème limite (homogénisation) en utilisant la méthode d'éclatement.

<u>Mots clés</u>: Analyse asymptotique, homogénéisation, élasticité linéaire, plaque mince, méthode d'éclatement.

ملخص

في عملنا هذا، ندرس تجانس صفيحة رقيقة مرنة غير متجانسة دورية ذات سمك متغير في الحالة التي يكون فيها ترتيب حجم الدور والسمك متماثلين، بدءا من المعادلات التي تحكم توازن جسم غير متجانس خطي مرن ثلاثي الابعاد شكلت اللوحة الدورية الرقيقة، ثم استخدمنا تحلل الازاحة المذكورة أعلاه في المجال الثابت، نحصل على مسألة التجانس باستخدام طريقة الفك الدوري.

<u>الكلمات المفتاحية</u>: التحليل المقارب، التجانس، المرونة الخطية، الصفيحة الرقيقة، طريقة الفك الدوري.

Bibliography

[1] H. Brezis, *Functional Analysis,* Sobolev and Partial Differential Equations, Universitext series, (2011).

[2] P. G. Ciarlet, Mathematical Elasticity, *Vol 02 : Theory of Plates,* North Holland, (1997).

[3] D. Cioranescu, A. Damlamian and G. Griso, *The periodic Unfolding Method,* Theory and Applications to Partial Differential Equations, Springer (2018).

[4] L. C. Evans , *Partial Differential Equations,* American Mathematical Society, 2nd ed., (2010).

[5] V. Kadets, *A Course in Functional Analysis and Measure Theory,* Springer, (2018).

[6] K. Yosida, *Functional Analysis,* Springer-Verlag, 6th ed., (1980).