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**Theme**

# Homogenization of piezoelectric structure by the energy method

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# Dedication

♢ To the one who brought me into the classroom when I was five years old,"My dear father",to the biggest supporter in my life,"My dear mother",to the one who made me proud "My dear brothers and sisters",to everyone who wished me success in my studies,"My friends and my large family ",to everyone who did not skimp on me with information,no matter how small,I stand today to offer all expressions of thanks and gratitude. ♢

**Thank you all**

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# NOTATIONS

▲ $div$:denotes the full divergence operator.

▲ $\nabla$:denotes the full gradient operator.

▲ $H_0^1(\Omega)$:denotes the closure of $\mathcal{D}(\Omega)$.

▲ $H^{-1}(\Omega)$:denotes the Banach space defined by $H^{-1}(\Omega) = (H_0^{1'}(\Omega))$.

▲ $H_\sharp^1(Y) = H_{per}^1(Y)$:denotes the subspace of functions in $H_{loc}^1(\mathbb{R}^n)$,which are Y-periodic.

▲ $< . >= \mathcal{M}_Y(.)$: denotes the mean operator which is defined by $< . >= \dfrac{1}{Y}\int_Y dy$,where $|Y|$ is the measure of $Y$.

▲ $\mathcal{M}(\alpha, \beta, \Omega)$:denotes the set of all possibly non-symmetric ▲

$$\mathcal{M}(\alpha, \beta, \Omega) = A(x) \in L^\infty(\Omega, M^{n\times n})\ such\ that|\xi|^2 \leq A(x)\xi.\xi \leq \beta|\xi|^2\ for\ any\xi \in \mathbb{R}^n.$$

▲ $\dfrac{\partial}{\partial x_j} = \partial_j$:denotes the partial drive.

▲ $n_j$:the outer normal.

▲ $\delta_{ik}$:represents the symbol of Kronecker.

▲ $\rightharpoonup$:denotes the weak convergence.

▲ $\longrightarrow$: denotes the strong convergence.

▲ $\rightharpoonup *$ :denotes the weak convergence star(*)

# Introduction

Homogenization method is a mathematical theory of **averaging**,which allows the calculation of composite **effective properties**,knowing the the topology of the composite unit cell and the replacement of the composite medium by an "**equivalent**" homogeneous medium to solve the global problem.Among it's advantage in relation to other methods that it needs only the information about the unit cell and this last can have any complex shape. Note that the homogenization method used to study differential operators with rapidly oscillating coefficients,boundary value problems with rapidly changing boundary conditions and equations in perforated domains. Several homogenization methods were developed in the 1970s,and homogenization became a subject in Mathematics. The methods introduce include**Young Measures,Method of Asymptotic Expansion,G-convergence ,$\Gamma$-convergence ,H-convergence and Energy method**.However in this dissertation we are interested with the both Asymptotic Expansion and energy methods ,such that our aim is shed light on the homogenization of a three dimensional piezoelectric heterogeneous structure by this two methods,see[1].This dissertation contained chapters organized as following:

**Chapter 1: The Energy method** In this chapter we provide a detailed exposition of the energy method of the Tartar ,such that in order to get the hang of this method we consider the model problem of diffusion in a periodic medium .

**Chapter 2: Homogenization Of Piezoelectric Structure By Asymptotic Expansion Method** In this chapter we derive the homogenised problem of a three-dimensional piezoelectric heterogeneous structure by the Asymptotic Expansion. **Chapter 3 :The Limit Problem Of The Piezoelectric Structure** In this chapter we establish the convergence of three-dimensional piezoelectric problem to the homogenized one by using the Energy method.

Chapter 1

# THE ENERGY METHOD

## 1.1 Setting of a Model Problem

A very elegant and efficient method for homogenizing partial differential has been devised by Tartar,which has later been called the energy method although it has nothing to do with any kind of energy.To expose the energy method in its full generality may hide the Kay ideas of the method in a lot of technicalities.Therefore,for clarity ,we prefer to present the energy method on a model problem of periodic homogenization.We consider a model problem of diffusion in a periodic medium ,a usual example in all textbooks on homogenization,but of course,the energy method converse many other problems with slight changes.

We consider now the problem

$$(\mathcal{P}_\varepsilon) \begin{cases} -div(A^\varepsilon \nabla u^\varepsilon) = f \ in \ \Omega, \\ u^\varepsilon = 0 \ on \ \partial\Omega, \end{cases} \tag{1.1}$$

where :$f \in H^{-1}(\Omega)$ and $A^{\varepsilon}(x)$ satisfies:

$$\begin{cases} A^{\varepsilon} \in \mathcal{M}(\alpha, \beta, \Omega), \\ A^{\varepsilon}(x) = (a_{ij}^{\varepsilon})_{1 \leq i,j \leq n}, \\ (A^{\varepsilon}(x)\lambda, \lambda) \geq \alpha |\lambda|^2, \\ |A^{\varepsilon}(x)\lambda| \leq \beta |\lambda|. \end{cases} \tag{1.2}$$

### 1.1.1 The main result of convergence

In this subsection we present the convergence result of the classical problem of homogenization and we give a rigorous proof of this result following a general method due to tartar (see Mural.F,Tartar.L,H-convergence.In R.V.Kohn,editor,Séminaire d'Analyse Fonctionnelle et Numérique de l'Université d'Alger 1977,and Cioranescu.D,Murat.F,Un terme étrange venu d'ailleurs(i),(ii).Brizis and Lions,editors,Nonlinear partial differential equations and their applications. Researech notes in mathematics,volume 60,70 of college de france Seminar,pages 98-138,154-178,London,1982.Pitman.). This method relies on the construction a class of oscillating test functions.

#### 1.1.1.1 The main convergence result

**Theorem 1.1.1** *Let* $f \in H^{-1}(\Omega)$ *and* $u^\varepsilon$ *be the solution of(1.1) . Then*

$$\begin{cases} u^\varepsilon \rightharpoonup u^0 \text{ weackly in } H_0^1(\Omega), \\ A^\varepsilon \nabla u^\varepsilon \rightharpoonup A^* \nabla u^0 \text{ in } (L^2(\Omega))^n, \end{cases} \tag{1.3}$$

*where* $u^0(x)$*is the unique solution in* $H_0^1(\Omega)$ *of the homogenized problem*

$$\begin{cases} -div(A^* \nabla u^0) = f \ in \Omega, \\ u^0 = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.4}$$

*In (1.4)the homogenized diffusion tensor* $A^* = (a^*_{ij})_{1 \le i,j \le n}$ *is constant,elliptic and given by :*

$$A^*_{ij} = \int_Y a_{ik}(y)(\delta_{kj} + \nabla_{y_k} \chi^j(y)) \mathrm{d}y,$$

where $\chi^j(y)$ are defined ,as the unique solution in $H^1_\sharp(Y)/\mathbb{R}$ of the so-called cell problems

$$\begin{cases} -div_y(A(\vec{e}_j + \nabla_y \chi^j(y))) = 0 \in Y, \\ y \mapsto \chi^j(y) Y - periodic, \end{cases} \tag{1.5}$$

with $(\vec{e}_j)_{1 \le j \le n}$ the canonical basis of $\mathbb{R}^n$

**Theorem 1.1.2** *Let* $1 \leq p \leq \infty$ *and* $f$ *be a Y-periodic function in* $L^p(Y)$*.Set*

$$f_\varepsilon(x) = f\left(\frac{x}{\varepsilon}\right) \ a.e \ on \ \mathbb{R}^n.$$

*Then,if* $p < \infty$*,as* $\varepsilon \longrightarrow 0$

$$f_\varepsilon \rightharpoonup \mathcal{M}_Y(f) = \frac{1}{|Y|}\int_Y f(y)\mathrm{d}y \ weackly \ in \ L^p(\Omega)$$

*,for any bounded open subset* $\Omega$ $of$ $\mathbb{R}^n$*.*

*If* $p = \infty$*,one has*

$$f_\varepsilon \rightharpoonup \mathcal{M}_Y(f) = \frac{1}{|Y|}\int_Y f(y)\mathrm{d}y \ \text{ weackly* in } \ L^\infty(\Omega)$$

**Proof :**

see( [2] page 34 ) ■

#### 1.1.1.2 Proof of the main convergence

we proved the Theorem 1.1.1

**Proof :**

This proof will be divided in to 3steps:

**Step 1:** **Existence and uniqueness**

The variational formulation of $(1.1)$ is :

$$\int_\Omega A^\varepsilon(x)\nabla u^\varepsilon \nabla\vartheta \mathrm{d}x \ = \int_\Omega f\vartheta \mathrm{d}x \ ,\forall \in H_0^1(\Omega). \tag{1.6}$$

**Proposition 1.1.3** *The problem (1.1) admit an unique solution* $u^\varepsilon$*.*

**proof**:we verified the Lax-Milgram Theorem.

- The space $H_0^1(\Omega)$ is a Hilbert space.

- **The continuity**

  By using Cauchi-Scharz,we get :

$$|\int_{\Omega} f\vartheta dx| \leq c\|f\|_{L^2(\Omega)}\|\vartheta\|_{H_0^1(\Omega)},$$

  and

$$|\int_{\Omega} A^{\varepsilon}(x)\nabla u^{\varepsilon}\nabla\vartheta dx| \leq \beta\|u^{\varepsilon}\|_{H_0^1(\Omega)}\|\vartheta\|_{H_0^1},$$

  then the linear and bilinear forms are continued

- **The ellipticity**

  By using Poincaré inequality,we get :

$$\int_{\Omega} A^{\varepsilon}(x)\nabla\vartheta\nabla\vartheta dx \geq \frac{\alpha}{1+c_p}\|\vartheta\|_{H_0^1(\Omega)},$$

  then the bilinear form is elliptic.

According the Lax-Milgram Theorem $(1.1)$ admit an unique solution $u^{\varepsilon} \in H_0^1(\Omega)$.

**Step 2:**<u>**A priori estimation**</u>

Let be $u^{\varepsilon}$ the solution of (1.1),we know that there exists a subsequence such that :

$$\begin{cases} u^{\varepsilon} \rightharpoonup u^0 \text{ in } H_0^1(\Omega), \\ u^{\varepsilon} \longrightarrow u^0 \text{ in } L^2(\Omega), \end{cases} \tag{1.7}$$

**Remark 1.1.4** *$u^{\varepsilon} \rightharpoonup u^0$ in $H_0^1(\Omega)$, then: $\nabla u^{\varepsilon} \rightharpoonup \nabla u^0$ in $H_0^1(\Omega)$.*

Introduce now

$$\xi^{\varepsilon} = (\xi_1^{\varepsilon}, ..., \xi_n^{\varepsilon}) = (\sum_{j=1}^{n} a_{1j}^{\varepsilon}\partial_j u^{\varepsilon}, ..., a_{nj}^{\varepsilon}\partial_j u^{\varepsilon}) = A^{\varepsilon}(x)\nabla u^{\varepsilon}(x) \tag{1.8}$$

From (1.6), it is easily seen that $\xi^{\varepsilon}$, satisfies :

$$\int_{\Omega} \xi^{\varepsilon}\nabla\vartheta \mathrm{d}x = \int_{\Omega} f\vartheta \mathrm{d}x \ \forall\vartheta \in H_0^1(\Omega) \tag{1.9}$$

It is self -evident that using the ellipticity of the matrix $A^\varepsilon$ and

$$\|\xi^\varepsilon\|_{(L^2)^n} \leq c \tag{1.10}$$

Hence ,we can extract a subsequence still denoted by $\xi^*$ such that

$$\xi^\varepsilon \rightharpoonup \xi^* \text{ in } (L^2(\Omega))^n \tag{1.11}$$

Passing to limit in (1.9) :

$$\begin{aligned}\lim_{\varepsilon\longrightarrow 0}\int_\Omega \xi^\varepsilon \nabla\vartheta \mathrm{d}x = \int_\Omega f\vartheta \mathrm{d}x &\Longrightarrow \int_\Omega \xi^* \nabla\vartheta \mathrm{d}x = \int_\Omega f\vartheta \mathrm{d}x \\ &\Longrightarrow \int_\Omega -div\xi^*\vartheta \mathrm{d}x = \int_\Omega f\vartheta \mathrm{d}x \\ &\Longrightarrow -div\xi^* = f\end{aligned} \tag{1.12}$$

i.e:

$$\int_\Omega \xi^* \nabla\vartheta \mathrm{d}x = \int_\Omega f\vartheta \mathrm{d}x. \tag{1.13}$$

**Step 3:<u>The limit problem (the homogenized problem)</u>**

Showing now :

$$\xi^* = A^*\nabla u^0. \tag{1.14}$$

Set

$$\omega_\varepsilon^j = \varepsilon\omega^j(\frac{x}{\varepsilon}) = e_j.x - \varepsilon\hat{\chi}^j \; j = 1, ..., n, \tag{1.15}$$

it is obvious that:

$$\begin{cases}\omega_\lambda^j \rightharpoonup \lambda.x \; in \; H_0^1(\Omega), \\ \omega_\varepsilon^j \longrightarrow \lambda.x \; in \; L^2(\Omega), \\ \nabla\omega_\varepsilon^j \rightharpoonup \lambda \; in \; (L^2(\Omega))^2,\end{cases} \tag{1.16}$$

we consider

$$\begin{cases}-div(A_\varepsilon^t(\nabla_y\hat{\chi}^j + \vec{e}_j)) = 0 \; inY \\ y \longmapsto \chi^j(y) \; Y - periodic\end{cases} \tag{1.17}$$

Set now :

$$\eta_\lambda^\varepsilon = (\sum_{j=1}^n a_{j1}^\varepsilon \partial_j \omega_j^\varepsilon, ..., \sum_{j=1}^n a_{jn}^\varepsilon \partial_j \omega_j^\varepsilon) = (A^\varepsilon)^t \nabla\omega_\varepsilon^j. \tag{1.18}$$

Observe that

$$\eta_\varepsilon^j = ((A^\varepsilon)^t \nabla \hat{\chi}^j + A^t \vec{e}_j)(\frac{x}{\varepsilon}).$$

Since ${}^tA^\varepsilon(x)$ and ${}^tA^\varepsilon \nabla \hat{\chi}^j \frac{x}{\varepsilon}$ are $Y-periodic$. Hence,we can applying (1.1.2)

$$\eta_\varepsilon^j \rightharpoonup \mathcal{M}_Y({}^tA^\varepsilon \nabla \omega_\varepsilon^j) = ({}^tA^*)\vec{e}_j \text{weackly in } (L^2(\Omega))^n \tag{1.19}$$

We can show easily that $\eta_\lambda^\varepsilon$ verifies :

$$\int_\Omega \eta_\varepsilon^j \nabla \vartheta = 0, \forall \vartheta \in H_0^1(\Omega). \tag{1.20}$$

Let $\varphi \in \mathcal{D}(\Omega)$ and choose $\varphi\omega_\varepsilon^j$ as test function in(1.9) and $\varphi u^\varepsilon$ as test function in (1.20), we have respectively

$$\begin{cases} \int_\Omega \xi^\varepsilon \nabla(\varphi\omega_\varepsilon^j)\mathrm{d}x = \int_\Omega f\varphi\omega_\varepsilon^j \mathrm{d}x \\ \int_\Omega \eta_\lambda^\varepsilon \nabla(\varphi u^\varepsilon)\mathrm{d}x = 0 \end{cases} \tag{1.21}$$

$\Longrightarrow$

$$\begin{cases} \int_\Omega \xi^\varepsilon \nabla\varphi\omega_\varepsilon^j \mathrm{d}x + \int_\xi^\varepsilon \nabla\omega_\varepsilon^j \varphi \mathrm{d}x = \int_\Omega f\varphi\omega_\varepsilon^j \mathrm{d}x \\ \int_\Omega \eta_\varepsilon^j \nabla\varphi u^\varepsilon \mathrm{d}x + \int_\Omega \eta_\lambda^\varepsilon \nabla u^\varepsilon \varphi \mathrm{d}x = 0 \end{cases} \tag{1.22}$$

we have

$$\xi^\varepsilon \nabla \omega_\varepsilon^j = A^\varepsilon \nabla u^\varepsilon \nabla \omega_\varepsilon^j =^t A^\varepsilon \nabla \omega_\varepsilon^j \nabla u^\varepsilon = \eta_\varepsilon^j \nabla u^\varepsilon \tag{1.23}$$

Now :

$$\int_\Omega \xi^\varepsilon \nabla\varphi\omega_\varepsilon^j + \int_\Omega \eta_\varepsilon^j \nabla u^\varepsilon \varphi = \int_\Omega f\varphi\omega_\varepsilon^j \tag{1.24}$$

$$\int_\Omega \eta_\varepsilon^j \nabla\varphi u^\varepsilon + \int_\Omega \eta_\lambda^\varepsilon \nabla u^\varepsilon \varphi = 0 \tag{1.25}$$

for(1.24)-(1.25) we will get :

$$\int_\Omega \xi^\varepsilon \nabla\varphi\omega_\varepsilon^j \mathrm{d}x - \int_\Omega \eta_\varepsilon^j \nabla\varphi u^\varepsilon \mathrm{d}x = \int_\Omega f\varphi\omega_\varepsilon^j \mathrm{d}x \tag{1.26}$$

**Remark 1.1.5** *The objective of this step is eliminated the functions which converge a weak convergence* passing to limit:

$$\int_\Omega \xi^* \vec{e}_j x \nabla\varphi \mathrm{d}x - \int_\Omega^t A^* \vec{e}_j \nabla\varphi u^0 \mathrm{d}x = \int_\Omega f\varphi\lambda.x\mathrm{d}x$$
$$\Longrightarrow \int_\Omega \xi^* \nabla(\vec{e}_j.x\varphi)\mathrm{d}x - \int_\Omega \xi^*\vec{e}_j\varphi\mathrm{d}x - \int_\Omega^t A^*\vec{e}_j\nabla\varphi u^0\mathrm{d}x = \int_\Omega f\vec{e}_j.x\varphi\mathrm{d}x \quad (1.27)$$

now,we choose $\vartheta = \vec{e}_j.x\varphi$, and replace in(1.24), then:

$$\int_\Omega \xi^*\nabla(\vec{e}_j.x\varphi)\mathrm{d}x = \int_\Omega \xi^*\vec{e}_j\mathrm{d}x + \int_\Omega \xi^*\lambda.x\nabla\mathrm{d}x = \int_\Omega f\lambda.x\varphi\mathrm{d}x \quad (1.28)$$

we replace by this equation in(1.13),then we get:

$$\int_\Omega \xi^*\varphi\vec{e}_j = -\int_\Omega (A^\varepsilon)^t\nabla\varphi u^0\mathrm{d}x \quad (1.29)$$

we use integrate by part:

$$\int_\Omega \vec{e}_j\varphi\mathrm{d}x = \int_\Omega (A^\varepsilon)^t\vec{e}_j\nabla u^0\varphi\mathrm{d}x \Longrightarrow ((A^\varepsilon)^t\nabla u^0 - \xi^*)\vec{e}_j\varphi = 0, \forall \in \mathcal{D}(\Omega) \quad (1.30)$$
$$\Longrightarrow^t A^*\nabla u^0 = \xi^*$$

■

Chapter 2

# HOMOGENIZATION OF PIEZOELECTRIC Structure BY ASYMPTOTIC EXPANSION METHOD

The homogenization of piezoelectric materials is a challenging task due to their intricate electro-mechanical coupling effects .In this chapter ,we explore the use of the asymptotic method to analyse and understand the effective behaviour of piezoelectric composites . We begin by introducing the piezoelectric problem .

## 2.1 Setting of the problem

### 2.1.1 Notations and geometry:

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain occupied by piezoelectric material with Lipschitz boundary $\Gamma = \partial\Omega$, points of $\Omega$ are denoted by

$$x = (x_1, x_2, x_3).$$

We consider two decompositions of the boundary $\Gamma$,

$$\begin{cases} \Gamma^m = \Gamma_0^m \cup \Gamma_1^m \ with \ \Gamma_0^m \cap \Gamma_1^m = \phi, \ and \ meas(\Gamma_0^m) > 0, \\ \Gamma^e = \Gamma_0^e \cup \Gamma_1^e \ with \ \Gamma_0^e \cap \Gamma_1^e = \phi, \ and \ meas(\Gamma_0^e) > 0. \end{cases} \tag{2.1}$$

Let $Y = [0, y_1] \times [0, y_2] \times [0, y_3]$, denotes the basic periodic points of $Y$are denoted by $(y_1, y_2, y_3) = \left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\right)$, where $\varepsilon$ denotes the size of the periodic. We consider the three-dimensional piezoelectric model

$$(\mathcal{P}_\varepsilon) : \begin{cases} -\partial_j[\sigma_{ij}^\varepsilon] = f_i \ in \ \Omega, \\ \partial_i[D_i^\varepsilon] = r \ in \ \Omega, \\ \sigma_{ij}^\varepsilon n_j = g_i \ on \ \Gamma_1^m, \\ D_i n_i = 0 \text{ on } \Gamma_1^e, \\ u^\varepsilon = 0 \text{ on } \Gamma_0^m, \\ \varphi^\varepsilon = 0 \text{ on } \Gamma_0^e, \end{cases} \tag{2.2}$$

where:

$$\begin{cases} \sigma_{ij}^\varepsilon = C_{ijkl}^\varepsilon(x) e_{kl}(u^\varepsilon) + P_{kij}^\varepsilon(x)\partial_k \varphi^\varepsilon, \\ D_i^\varepsilon = P_{ikl}^\varepsilon(x) e_{kl}(u^\varepsilon) - d_{ik}^\varepsilon \partial_k \varphi^\varepsilon. \end{cases} \tag{2.3}$$

Note that unknown of the piezoelectric structure model (2.2) is the pair $(u^\varepsilon, \varphi^\varepsilon)$ where:

| **Notation** | **Designation** |
|---|---|
| $f \in (L^2(\Omega))^3$ | is the density of the mechanical volume force, |
| $g \in (L^2(\Gamma_1^m))^3$ | is the density of mechanical surface traction, |
| $r \in (L^2(\Omega))^3$ | is the density of the electric volume charge, |
| $u^\varepsilon : \Omega \longrightarrow \mathbb{R}^3$ | denotes the displacement vector field , |
| $\varphi^\varepsilon : \Omega \longrightarrow \mathbb{R}$ | is the electric potential, a scalar field, |
| $\sigma_{ij}^\varepsilon : \Omega \longrightarrow \mathbb{R}^9$ | is the stress tensor, |
| $D_i^\varepsilon : \Omega \longrightarrow \mathbb{R}^3$ | is the electric displacement vector , |
| $e_{kl}(u^\varepsilon)$ | is the linear strain tensor, |
| $C_{ijkl}^\varepsilon(x) = C_{ijkl}(\frac{x}{\varepsilon})$ | is the elastic fourth-order tensor field , |
| $P_{ijk}^\varepsilon(x) = P_{ijk}(\frac{x}{\varepsilon})$ | is the piezoelectric third-order tensor field , |
| $d_{ij}^\varepsilon = d_{ij}(\frac{x}{\varepsilon})$ | is the dielectric second-order tensor field. |

Table 2.1: Notations and designations of the piezoelectric problem

We assume that the elastic tensor $C_{ijkl}$ is symmetric,positive defined,it verifies:

$$\begin{cases} C_{ijkl} = C_{klij} = C_{jikl} = C_{ijlk}, \\ C_{ijkl} \in L^{\infty}(\Omega), \\ \exists c > 0 : C_{ijkl}(x) X_{ij} X_{kl} \geq c X_{ij} X_{kl},\ \forall x \in \Omega, \text{for every symmetric } 3 \times 3 \text{ real matrix } X_{ij}, \\ Y - periodic, \end{cases} \tag{2.4}$$

the piezoelectric third order tensor $P_{ijk}$ is symmetric ,it verifies

$$\begin{cases} P_{ijk} = P_{ikj}, \\ P_{ijk} \in L^{\infty}(\Omega), \\ Y - periodic, \end{cases} \tag{2.5}$$

the dielectric tensor $d_{ij}$ is symmetric,positive defined,it verifies

$$\begin{cases} d_{ij} = d_{ji}, \\ d_{ij} \in L^{\infty}(\Omega), \\ \exists c > 0 : d_{ij} X_i X_j \geq c, \forall x \in \Omega, \text{ for any vector } X_i \in \mathbb{R}^n, \end{cases} \tag{2.6}$$

and the linear strain tensor $e_{ij}(u^{\varepsilon})$ defined by :

$$e_{ij}(u^{\varepsilon}) = \frac{1}{2}(\partial_i u_j^{\varepsilon} + \partial_j u_i^{\varepsilon}). \tag{2.7}$$

## 2.2 The variational formulation of $(\mathcal{P}_{\varepsilon})$

Let us define the two following spaces :

$$\mathcal{V}(\Omega) = \vartheta \in (H^1(\Omega))^3, \vartheta = 0 \text{ on } \Gamma_0^m \tag{2.8}$$

and

$$\Psi(\Omega) = \psi \in H^1(\Omega), \psi = 0 \text{ on } \Gamma_0^e \tag{2.9}$$

equipped with the two norms (equivalent to the usual norm usual ($H^1$),such that :

$$\|\vartheta\|_{\mathcal{V}} = \left( \sum_{i,j=1}^{3} \int_{\Omega} \left( \frac{\partial \vartheta_i}{\partial x_j} \right)^2 \right)^{\frac{1}{2}},$$

$$\|\psi\|_{\Psi} = \left( \sum_{i=1}^{3} \int_{\Omega} \left( \frac{\partial \psi}{\partial x_i} \right)^2 \right)^{\frac{1}{2}},$$

$$\|\vartheta, \psi\|_{\mathcal{V}\times\Psi} = \|\vartheta\|_{\mathcal{V}} + \|\psi\|_{\Psi}.$$

Multiplying the first equation of (2.2) by a test function $\vartheta \in \mathcal{V}$,and the second one by $\psi \in \Psi$,and summing the two obtained equations we get the following variational problem:

$$a((u^\varepsilon, \varphi^\varepsilon); (\vartheta, \psi)) = L(\vartheta, \psi),$$

where

$$\begin{cases} a((u^\varepsilon, \varphi^\varepsilon); (\vartheta, \psi)) = \int_\Omega C^\varepsilon_{ijkl}(x) e_{kl}(u^\varepsilon) e_{ij}(\vartheta) + P^\varepsilon_{kij}(x) \dfrac{\partial \varphi^\varepsilon}{\partial x_k} e_{ij}(\vartheta) dx \\ + \int_\Omega -P^\varepsilon_{jkl}(x) e_{kl}(u^\varepsilon) \dfrac{\partial \psi}{\partial x_j} + d^\varepsilon_{jk}(x) \dfrac{\partial \varphi^\varepsilon}{\partial x_k} \dfrac{\partial \psi}{\partial x_j} dx \\ L(\vartheta, \psi) = \int_\Omega f_i \vartheta_i dx + \int_{\Gamma_1^m} g_i \vartheta_i d\Gamma^m + \int_\Omega r \psi dx. \end{cases} \tag{2.10}$$

## 2.3 Existence and uniqueness

- The spaces $\mathcal{V}(\Omega)$ and $\Psi(\Omega)$ are Hilbert spaces

- **The continuity**

  By using **Cauchi-Scharz inequality** with**Trace theorem**,we get :

$$|L(\vartheta, \psi)| \leq c(\|f\|_{L^2(\Omega)} \|\vartheta\|_{\mathcal{V}(\Omega)} + \|g\|_{L^2(\Gamma_1^m)} \|\vartheta\|_{\mathcal{V}(\Omega)} + \|r\|_{L^2(\Omega)} \|\psi\|_{\Psi(\Omega)}),$$

  and

$$|a((u^\varepsilon, \varphi^\varepsilon); (\vartheta, \psi)| \leq c_p(\|u^\varepsilon\|_{\mathcal{V}(\Omega)} \|\vartheta\|_{\mathcal{V}(\Omega)} + \|\varphi^\varepsilon\|_{\Psi(\Omega)} \|\vartheta\|_{\mathcal{V}(\Omega)} + \|u^\varepsilon\|_{\mathcal{V}(\Omega)} \|\psi\|_{\Psi(\Omega)} + \|\varphi^\varepsilon\|_{\Psi(\Omega)} \|\psi\|_{\Psi(\Omega)}).$$

  Now the linear and bilinear forms are continue

- **The ellipticity**

  We set $u^\varepsilon = \vartheta$ and $\varphi^\varepsilon = \psi$,then by using the ellipticity of tensors $C_{ijkl}, d_{ij}$ with **Poincaré** and **Korn** inequalities,we get :

$$a((\vartheta,\psi);(\vartheta,\psi) \geq c_1\|\vartheta\|^2_{\mathcal{V}(\Omega)} + c_2\|\psi\|^2_{\Psi(\Omega)},$$

  now the bilinear form is elliptic.

According the Lax-Milgram theorem the problem$(2.2)$ admits an unique solution $(u^\varepsilon, \varphi^\varepsilon)$.

## 2.4 Homogenization of piezoelectric structure by asymptotic method

We postulate the following ersatz for the the solution $(u^\varepsilon(x), \varphi^\varepsilon(x))$

$$u^\varepsilon(x) = u^0(x,y) + \varepsilon u^1(x,y) + \varepsilon^2 u^2(x,y) + o(\varepsilon^k u^k(x,y)) \tag{2.11}$$

and

$$\varphi^\varepsilon(x) = \varphi^0(x,y) + \varepsilon\varphi^1(x,y) + \varepsilon^2\varphi^2(x,y) + o(\varepsilon^k\varphi^k(x,y)) \tag{2.12}$$

where each functions $u^\varepsilon(x), \varphi^\varepsilon(x)$ are Y-periodic with respect to $y = \dfrac{x}{\varepsilon}$

Suppose that a function $\Psi^\varepsilon = \Psi(x,y)$ depends on both the slow and the fast coordinates.We make use of the chain rule of differentiation we obtain the following relation:

$$\frac{\partial\Psi(x,y)}{\partial x} = \frac{\partial\Psi(x,y)}{\partial x} + \frac{1}{\varepsilon}\frac{\partial\Psi(x,y)}{\partial y} \tag{2.13}$$

we have

$$\begin{aligned} e_{ij}(\vartheta) &= \frac{1}{2}\left(\frac{\partial\vartheta_i}{\partial x_j} + \frac{\partial\vartheta_j}{\partial x_i}\right) \\ &= \frac{1}{2}\left[\left(\frac{\partial}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_i}\right)(\vartheta_j) + \left(\frac{\partial}{\partial_j} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_j}\right)(\vartheta_i)\right] \\ &= eij_x(\vartheta) + \frac{1}{\varepsilon}e_{ij_y}(\vartheta) \end{aligned} \tag{2.14}$$

Plunging the asymptotic expansions (2.11) and (2.12), taking into account (2.13) and (2.14), and identifying different powers of $\varepsilon$ yields the following problems:

**The problem of order** $\varepsilon^{-2}$

$$(\mathcal{P}_1)\begin{cases}-\dfrac{\partial}{\partial y_j}\left[C^{\varepsilon}_{ijkl}e_{kl_y}(u^0)+P^{\varepsilon}_{kij}(x)\dfrac{\partial\varphi^0}{\partial y_k}\right]=0\\ \dfrac{\partial}{\partial y_i}\left[P^{\varepsilon}_{ikl}(x)e_{kl_y}(u^0)-d^{\varepsilon}_{ik}(x)\dfrac{\partial\varphi^0}{\partial y_k}\right]=0\end{cases}\tag{2.15}$$

**The problem of order** $\varepsilon^{-1}$

$$(\mathcal{P}_2)\begin{cases}-\dfrac{\partial}{\partial x_j}\left[C^{\varepsilon}_{ijkl}(x)e_{kl_y}(u^0)+P^{\varepsilon}_{kij}(x)\dfrac{\partial\varphi^0}{\partial y_k}\right]\\ -\dfrac{\partial}{\partial y_j}\left[C^{\varepsilon}_{ijkl}(x)\left[e_{kl_x}(u^0)+e_{kl_y}(u^1)\right]+P^{\varepsilon}_{kij}(x)\left[\dfrac{\partial\varphi^0}{\partial x_k}+\dfrac{\partial\varphi^1}{\partial y_k}\right]\right]=0\\ \dfrac{\partial}{\partial x_i}\left[P^{\varepsilon}_{ikl}(x)e_{kl_y}(u^0)-d^{\varepsilon}_{ik}\dfrac{\partial\varphi^0}{\partial y_k}\right]+\dfrac{\partial}{\partial y_i}\left[P^{\varepsilon}_{ikl}(x)[e_{kl_x}(u^0)+e_{kl_y}(u^1)]-d^{\varepsilon}_{ik}(x)\left[\dfrac{\partial\varphi^0}{\partial x_k}+\dfrac{\partial\varphi^1}{\partial y_k}\right]\right]=0\end{cases}\tag{2.16}$$

# The problem of order $\varepsilon^0$

$$(\mathcal{P}_3)\begin{cases}-\dfrac{\partial}{\partial x_j}[C^{\varepsilon}_{ijkl}(x)[e_{kl_x}(u^0)+e_{kl_y}(u^1)]+P^{\varepsilon}_{kij}(x)[\dfrac{\partial\varphi^0}{\partial x_k}+\dfrac{\partial\varphi^1}{\partial y_k}]]\\ -\dfrac{\partial}{\partial y_j}[C^{\varepsilon}_{ijkl}(x)[e_{kl_x}(u^1)+e_{kl_y}(u^2)]+P^{\varepsilon}_{kij}(x)[\dfrac{\partial\varphi^1}{\partial x_k}+\dfrac{\partial\varphi^2}{\partial y_k}]]=f_i\ in\ \Omega\\ \dfrac{\partial}{\partial x_i}[P^{\varepsilon}_{ikl}(x)[e_{kl_x}(u^0)+e_{kl_y}(u^1)]-d^{\varepsilon}_{ik}[\dfrac{\partial\varphi^0}{\partial x_k}+\dfrac{\partial\varphi^1}{\partial y_k}]]\\ +\dfrac{\partial}{\partial y_i}[P^{\varepsilon}_{ikl}(x)[e_{kl_x}(u^1)+e_{kl_y}(u^2)]-d^{\varepsilon}_{ik}(x)[\dfrac{\partial\varphi^1}{\partial x_k}+\dfrac{\partial\varphi^2}{\partial y_k}]]=r\ in\ \Omega.\end{cases}\tag{2.17}$$

The problems $(\mathcal{P}_1),(\mathcal{P}_2)and(\mathcal{P}_3)$ can be written in the following forms:

**The first system:**

$$(\mathcal{P}_1):\begin{cases} Find\ (u^0,\varphi^0)\in(\mathcal{V}(\Omega)\times\Psi(\Omega))\ such\ that\\ -\dfrac{\partial}{\partial y_j}[C^{\varepsilon}_{ijkl}(x)e_{kl_y}(u^0)+P^{\varepsilon}_{kij}(x)\dfrac{\partial\varphi^0}{\partial y_k}]=0\\ \dfrac{\partial}{\partial y_i}[P^{\varepsilon}_{ikl}(x)e_{kl_y}(u^0)-d^{\varepsilon}_{ik}(x)\dfrac{\partial\varphi^0}{\partial y_k}]=0\\ u^0\ and\ \varphi^0\ are\ Y-periodic,\end{cases} \tag{2.18}$$

**The second system:**

$$(\mathcal{P}_2):\begin{cases} Find\ (u^1,\varphi^1)\in(\mathcal{V}(\Omega)\times\Psi(\Omega))\ such\ that\\ \dfrac{\partial}{\partial x_j}[C^{\varepsilon}_{ijkl}(x)e_{kl_y}(u^0)+P^{\varepsilon}_{kij}(x)\dfrac{\partial\varphi^0}{\partial y_k}]\\ -\dfrac{\partial}{\partial y_j}[C^{\varepsilon}_{ijkl}(x)[e_{kl_x}(u^0)+e_{kl_y}(u^1)]+P^{\varepsilon}_{ikl}(x)[\dfrac{\partial\varphi^0}{\partial x_k}+\dfrac{\partial\varphi^1}{\partial y_k}]]=0\\ \dfrac{\partial}{\partial x_i}[P^{\varepsilon}_{ikl}(x)e_{kl_y}(u^0)-d^{\varepsilon}_{ik}(x)\dfrac{\partial\varphi^0}{\partial y_k}]+\dfrac{\partial}{\partial y_i}[P^{\varepsilon}_{ikl}(x)[e_{kl_x}(u^0)+e_{kl_y}(u^1)]-d^{\varepsilon}_{ik}(x)[\dfrac{\partial\varphi^0}{\partial x_k}+\dfrac{\partial\varphi^1}{\partial y_k}]])=0\\ u^1\ and\ \varphi^1\ are\ Y-periodic,\end{cases} \tag{2.19}$$

**The third system**

$$(\mathcal{P}_3):\begin{cases} Find(u^2,\varphi^2)\in\mathcal{V}(\Omega)\times\Psi(\Omega)\ such\ that:\\ \Big(-\dfrac{\partial}{\partial x_j}[C^{\varepsilon}_{ijkl}(x)[e_{kl_x}(u^1)+e_{kl_y}(u^2)]+P^{\varepsilon}_{kij}(x)[\dfrac{\partial\varphi^0}{\partial x_k}+\dfrac{\partial\varphi^1}{\partial y_k}]]\\ -\dfrac{\partial}{\partial y_j}[C^{\varepsilon}_{ijkl}(x)[e_{kl_x}(u^1)+e_{kl_y}(u^2)]+P^{\varepsilon}_{kij}(x)[\dfrac{\partial\varphi^1}{\partial x_k}+\dfrac{\partial\varphi^2}{\partial y_k}]]\Big)=f_i\ \text{in}\ \Omega\\ (\dfrac{\partial}{\partial x_i}[P^{\varepsilon}_{ikl}(x)[e_{kl_x}(u^0)+e_{kl_y}(u^1)]-d^{\varepsilon}_{ik}(x)[\dfrac{\partial\varphi^0}{\partial x_k}+\dfrac{\partial\varphi^1}{\partial y_k}]]\\ +\dfrac{\partial}{\partial y_i}[P^{\varepsilon}_{ikl}(x)[e_{kl_x}(u^1)+e_{kl_y}(u^2)]-d^{\varepsilon}_{ik}[\dfrac{\partial\varphi^1}{\partial x_k}+\dfrac{\partial\varphi^2}{\partial y_k}]]=r\ in\ \Omega\\ u^2\ and\ \varphi^2 areY-periodic\end{cases} \tag{2.20}$$

### 2.4.1 Determination of homogenized problem

**Lemma 2.4.1 (Fredholm)** *We confider the problem*

$$(\mathcal{P}) : \begin{cases} A\phi = f \\ \phi \text{ is } Y - periodic \end{cases} \tag{2.21}$$

*where*

$$A = \frac{-\partial}{\partial y_i}[a_{ij}(y)\frac{\partial}{\partial y_j}]$$

*we say that* $(\mathcal{P})$ *admits a unique solution if:*

$$\int_Y f(y)\mathrm{d}y = 0 \tag{2.22}$$

**Study the problem** $(\mathcal{P}_1)$: According to Fredhom's , the problem $(\mathcal{P}_1)$ admits an unique solution has a additive constant.

1- We multiply the first equation of $(\mathcal{P}_1)$ by $u^0(x,y) \in \mathcal{V}(\Omega)$ and integrate in $Y$:

$$\int_Y -\frac{\partial}{\partial y_j}(C_{ijkl}(y))e_{kl_y}(u^0) + P_{kij}(y)\frac{\partial \varphi^0}{\partial y_k})u^0\mathrm{d}y \;= 0$$

applying **Green's formula**,then:

$$\int_Y [C_{ijkl}(y)e_{kl_y}(u^0) + P_{kij}(y)\frac{\partial \varphi^0}{\partial y_k}]e_{ij}(u^0)\mathrm{d}y = 0 \tag{2.23}$$

2- We multiply the second equation of $(\mathcal{P}_1)$ by $\varphi^0 \in \Psi(\Omega)$ and integrate in Y:

$$\int_Y \frac{\partial}{\partial y_i}(P_{ikl}(y)e_{kl_y}(u^0) - d_{ik}(y)\frac{\partial \varphi^0}{\partial y_k})\varphi^0$$

applying **Green's formula**,then:

$$\int_Y [-P_{ikl}(y)e_{kl_y}(u^0) + d_{ik}(y)\frac{\partial \varphi^0}{\partial y_k}]\frac{\partial \varphi^0}{\partial y_i}\mathrm{d}y = 0 \tag{2.24}$$

3- for summing the two equations (2.23) and (2.24), we get :

$$0=\int_Y C_{ijkl}(y)(e_{kl_y}(u^0))^2+d_{ik}(y)(\frac{\partial\varphi^0}{\partial y_k})^2\mathrm{d}y \geq \alpha\|e_{kl_y}(u^0)\|^2_{L^2}+\beta\|\frac{\varphi^0}{\partial y_k}\|^2_{L^2} \tag{2.25}$$

$\Longrightarrow$

$$\|e_{kl_y}(u^0)\|_{L^2}=0 \Longrightarrow \boxed{u^0(x,y)=u_0(x)} \tag{2.26}$$

and

$$\left\|\frac{\partial\varphi^0}{\partial y_k}\right\|_{L^2}=0 \Longrightarrow \boxed{\varphi^0(x,y)=\varphi_0(x)} \tag{2.27}$$

#### 2.4.1.1 Solving the second system:

As $u_0(x)$ and $\varphi_0(x)$ independent for $y$,then we can write the $(\mathcal{P}_2)$ a following

$$\begin{cases}\frac{\partial}{\partial y_j}[C_{ijkl}(y)e_{kl_x}(u^0)+P_{kij}(y)\frac{\partial\varphi^0}{\partial x_k}]=0\\ \frac{\partial}{\partial y_i}[P^{\varepsilon}_{ikl}(x)e_{kl_y}(u^1)\\ -\frac{\partial}{\partial y_i}[P_{ikl}(y)e_{kl_x}(u^0)-d_{ik}(y)\frac{\partial\varphi^0}{\partial x_k}]=0.\end{cases} \tag{2.28}$$

As $C_{ijkl}(y)e_{kl_x}(u_0)$ , $P_{ikl}(y)\frac{\partial\varphi_0}{\partial x_k}$,$P_{ikl}(y)e_{kl_x}(u_0)$ and $d_{ik}(y)\frac{\partial\varphi_0}{\partial x_k}$ are $Y-periodic$, now

$$\int_Y \frac{\partial}{\partial y_j}[C^{\varepsilon}_{ijkl}(x)e_{kl}(u_0)+P^{\varepsilon}_{ikj}(x)\frac{\partial\varphi_0}{\partial x_k}]=0$$

and

$$\int_Y \frac{\partial}{\partial y_i}[P^{\varepsilon}_{ikl}(x)e_{kl}(u_0)-d^{\varepsilon}_{ik}\frac{\partial\varphi_0}{\partial x_k}]=0,$$

according Fredholm the problem $\mathcal{P}_2$ admits a unique solution $(u^1,\varphi^1)$ where:

$$u^1(x,y)=\chi^{mn}_k(y)e_{mn_x}(u_0)+\Phi^m_k(y)\frac{\partial\varphi_0}{\partial x_m} \tag{2.29}$$

and

$$\varphi^1(x,y)=\varPsi^{mn}(y)e_{mn_x}(u_0)+\mathcal{R}^m(y)\frac{\partial\varphi_0}{\partial x_m} \tag{2.30}$$

Where :$\chi^{mn}(y)$, $\Phi^m(y)$, $\Psi^{mn}(y)$ and $\mathcal{R}^m(y)$ are $Y-periodic$ the solution of :

$$\begin{cases} -\dfrac{\partial}{\partial y_j}[C_{ijkl}(y)(e_{kl_y}(\chi^{mn})e_{mnx}(u_0)+\dfrac{\partial\varphi_0}{\partial x_m}\dfrac{\partial\varphi_0}{\partial x_m})+ \\ P_{kij}(y)(e_{mn_x}(u_0)\dfrac{\partial\Psi^{mn}}{\partial y_k}+\dfrac{\partial\varphi_0}{\partial x_m}\dfrac{\partial\mathcal{R}^m}{\partial y_k})]=\dfrac{\partial}{\partial y_j}[C_{ijmn}(y)e_{mn_x}(u_0(x))+P_{mij}(y)\dfrac{\partial\varphi_0}{\partial x_m}] \\ -\dfrac{\partial}{\partial y_i}[P^{\varepsilon}_{jkl}(x)(e_{mn_x}(u_0)e_{kl_y}(\chi^{mn})+\dfrac{\partial\varphi_0}{\partial x_m}e_{kl_y}(\Phi^m)) \\ -d_{jk}(y)(e_{mn_x}(u_0)\dfrac{\partial\Psi^{mn}}{\partial y_k}+\dfrac{\partial\varphi_0}{\partial x_m}\dfrac{\partial\mathcal{R}^m}{\partial y_k})]=\dfrac{\partial}{\partial y_i}[P_{jmn}(y)e_{mn_x}(u_0)-d_{jm}(y)\dfrac{\partial\varphi_0}{\partial x_m}], \end{cases} \tag{2.31}$$

#### 2.4.1.2 Solving the third system:

As $C_{ijkl}(y)e_{kl_x}(u^1)$, $P_{kij}\dfrac{\partial\varphi^1}{\partial x_k}$, $P_{ikl}e_{kl_x}(u_1)$ and $d_{ik}\dfrac{\partial\varphi^1}{\partial x_k}$ are $Y-periodic$, then we can write the problem $(\mathcal{P}_3)$ a following:

$$\begin{cases} -\dfrac{\partial}{\partial y_j}[C_{ijkl}(y)e_{kl_y}(u^2)+P_{kij}(y)\dfrac{\partial\varphi^2}{\partial y_k}]=\dfrac{\partial}{\partial x_j}[C_{ijkl}(y)[e_{kl_x}(u_0)+e_{kl_y}(u^1)]+P_{kij}(y)[\dfrac{\partial\varphi_0}{\partial x_k}+\dfrac{\partial\varphi^1}{\partial y_k}]]+f_i \\ -\dfrac{\partial}{\partial y_i}[P_{ikl}(y)e_{kl_y}(u^2)-d_{ik}(y)\dfrac{\partial\varphi^2}{\partial y_k}]=r-\dfrac{\partial}{\partial x_i}[P_{ikl}(y)[e_{kl_x}(u_0)+e_{kl}(u^1)]-d_{ik}(y)[\dfrac{\partial\varphi_0}{\partial x_k}+\dfrac{\partial\varphi^1}{\partial y_k}]] \end{cases} \tag{2.32}$$

now,the problem $(\mathcal{P}_3)$ admits a unique solution

$$\begin{cases} \int_Y -\dfrac{\partial}{\partial x_j}[C_{ijkl}(y)[e_{kl_x}(u_0)+e_{kl_y}(u^1)+P_{kij}(y)[\dfrac{\partial\varphi_0}{\partial x_k}+\dfrac{\partial\varphi^1}{\partial y_k}]]\mathrm{d}y \;=\int_Y f_i\mathrm{d}y \\ \int_Y \dfrac{\partial}{\partial x_i}[P_{ikl}(y)[e_{kl_x}(u_0)+e_{kl_y}(u^1)]-d_{ik}(y)[\dfrac{\partial\varphi_0}{\partial x_k}+\dfrac{\partial\varphi^1}{\partial y_k}]]dy \;=\int_Y r dy\ , \end{cases} \tag{2.33}$$

$\Longrightarrow$

$$\begin{cases} -\dfrac{\partial}{\partial x_j}(\int_Y C_{ijkl}(y)[e_{kl_x}(u_0)+e_{kl_y}(u^1)]+P_{kij}(y)[\dfrac{\partial\varphi_0}{\partial x_k}+\dfrac{\partial\varphi^1}{\partial y_k}]\mathrm{d}y\ )=|Y|f_i \\ \dfrac{\partial}{\partial x_i}(\int_Y P_{ikl}(y)[e_{kl_x}(u_0)+e_{kl_y}(u^1)]-d_{ik}(y)[\dfrac{\partial\varphi_0}{\partial x_k}+\dfrac{\partial\varphi^1}{\partial y_k}]\mathrm{d}y\ )=|Y|r \end{cases} \tag{2.34}$$

$\Longleftrightarrow$

$$\begin{cases} -\dfrac{1}{|Y|}(\dfrac{\partial^2 u_0(x)}{\partial x_k\partial x_j}\int_Y C_{ijkl}(y)+C_{ijmn}e_{mn_y}(\chi_i^{kl})dy \\ +\dfrac{\partial^2\varphi_0(x)}{\partial x_j\partial x_k}\int_Y P_{mij}(y)\dfrac{\mathcal{R}^k}{\partial y_m}+P_{kij}(y)+C_{ijmn}\dfrac{\partial\theta_k^m}{\partial y_n}dy)=f_i \\ \dfrac{1}{|Y|}(\dfrac{\partial^2 u_0(x)}{\partial x_m\partial x_j}\int_Y P_{jmn}(y)+P_{jkl}e_{kl_y}(\chi_k^{mn})-d_{jk}\dfrac{\partial\Psi^m}{\partial y_k}dy \\ +\dfrac{\partial^2\varphi_0(x)}{\partial x_j\partial x_i}\int_Y d_{jk}(x)-P_{jkl}(y)(\theta^m)+d_{jm}(y)dy)=r \end{cases} \tag{2.35}$$

Now the homogenized problem is :

$$(\mathcal{P}_h) : \begin{cases} -\dfrac{\partial}{\partial x_j}(C^H_{ijkl}\dfrac{\partial u_k}{\partial x_l} + P^H_{kij}\dfrac{\partial \varphi}{\partial x_k}) = f_i \in \Omega \\ \dfrac{\partial}{\partial x_j}(P^H_{jmn}\dfrac{\partial u_m}{\partial x_n} - d^H_{jm}\dfrac{\partial \varphi}{\partial x_m}) = r \in \Omega \end{cases} \tag{2.36}$$

where:

$$\begin{cases} C^H_{ijkl} = \dfrac{1}{|Y|}\displaystyle\int_Y \left( C_{ijmn} e_{mn_y}(\chi^{kl}) + P_{mij}\dfrac{\partial \Psi^{kl}}{\partial y_m} + C_{ijkl} \right) \mathrm{d}y \\ P^H_{kij} = \dfrac{1}{|Y|}\displaystyle\int_Y \left( P_{mij}(y)\dfrac{\mathcal{R}^k}{\partial y_m} + P_{kij} + C_{ijmn}\dfrac{\partial \Phi^k_m}{\partial y_n} \right) \mathrm{d}y \\ P^H_{jmn} = \dfrac{1}{|Y|}\displaystyle\int_Y \left( P_{jkl} e_{kl_y}(\chi^{mn}) + P_{jmn}(y) - d_{jk}\dfrac{\partial \Psi^{mn}}{\partial y_k} \right) \mathrm{d}y \\ d^H_{jm} = \dfrac{1}{|Y|}\displaystyle\int_Y \left( -d_{jk}(y)\dfrac{\partial \mathcal{R}^m}{\partial y_k} + P_{jkl} e_{kl_y}(\Phi^m) + d_{jm}(y) \right) \mathrm{d}y \end{cases} \tag{2.37}$$

Chapter 3

# THE LIMIT PROBLEM OF THE PIEZOELECTRIC STRICTURE

The results obtained this chapter are token from[1].

## 3.1 Theorem

**Theorem 3.1.1** *Let* $(u^\varepsilon, \varphi^\varepsilon)$ *be the unique solution of* (2.2)*, then*

$$\begin{cases} u^\varepsilon \rightharpoonup u^0 \text{ in } H^1(\Omega), \\ \varphi^\varepsilon \rightharpoonup \varphi^0 \text{ in } H^1, \\ \sigma^\varepsilon_{ij} \rightharpoonup \sigma^*_{ij} = C^H_{ijkl}\dfrac{\partial u^0_k}{\partial x_i} + P^H_{kij}\dfrac{\partial \varphi^0}{\partial x_k} \text{ in } L^2(\Omega), \\ D^\varepsilon_i \rightharpoonup D^*_i \text{ in } L^2(\Omega). \end{cases} \tag{3.1}$$

*Where* $(u^0, \varphi^0)$ *are the unique solution in* $H^1(\Omega)^2$ *of the homogenized problem*

$$\begin{cases} C^H_{ijkl}\dfrac{\partial^2 u^0_k}{\partial x_j \partial x_l} + P^H_{kij}\dfrac{\partial^2 \varphi^0}{\partial x_j \partial x_k} = f \ in \ \Omega, \\ P^H_{ikl}\dfrac{\partial^2 u^0_k}{\partial x_i \partial x_l} - d^H_{ij}\dfrac{\partial^2 \varphi^0}{\partial x_i \partial x_j} = r \ in \ \Omega, \\ < \sigma^*_{ij} > n_j = g_i \ on \ \Gamma^m_1, \\ < D^*_i > n_i = 0 \ on \ \Gamma^e_1, \\ u^0 = 0 \ on \ \Gamma^m_0, \\ \varphi^0 = 0 \ on \ \Gamma^e_0, \end{cases} \tag{3.2}$$

where the homogenized coefficients $C^h_{ijkl}, P^h_{kij}, P^h_{ikl}, d^h_{ij}$,

$$C^H_{ijkl} = \frac{1}{|Y|}\int_Y (C_{ijmn}(y)e_{mn_y}(\chi^{kl}) + C_{ijkl}(y) + P_{mij}(y)\frac{\partial \Psi^{kl}}{\partial y_m})\mathrm{d}y$$

$$P^H_{kij} = \frac{1}{|Y}\int_Y (C_{ijmn}(y)e_{mn_y}(\Phi^k) + P_{mij}(y)\frac{\partial(\mathcal{R}^k + y_k)}{\partial y_m})\mathrm{d}y,$$

$$P^H_{ikl} = \frac{1}{|Y|}\int_Y (P_{imn}(y)e_{mny}(\chi^{kl}) + P_{ikl}(y) - d_{im}(y)\frac{\partial \Psi^{kl}}{\partial y_m})\mathrm{d}y,$$

$$d^H_{ij} = \frac{1}{|Y|}\int_Y (P_{jmn}(y)e_{mn_y}(\Phi^i - d_{jm}\frac{\partial(\mathcal{R}^i + y_i)}{\partial y_m}).$$

## 3.2 The proof of theorem

**Proof :**

The proof will be divided into 4 steps.

**Step 1:<u>The variational formulation</u>**

See[chapter 2.section 3]

**Step 2:<u>A priori estimates</u>**

**Lemma 3.2.1** *The solution of (2.2) are bounded.*

**Proof**: We take $u^\varepsilon = \vartheta$ and $\varphi^\varepsilon = \psi$ in(2.10) we get:

$$\int_\Omega C^\varepsilon_{ijkl}(x)e_{kl}(u^\varepsilon)e_{ij}(u^\varepsilon) + P^\varepsilon_{kij}(x)e_{kl}(u^\varepsilon)\frac{\partial \varphi^\varepsilon}{\partial x_l}dx - \int_\Omega P^\varepsilon_{ikl}(x)e_{kl}(u^\varepsilon)\frac{\partial \varphi^\varepsilon}{\partial x_i} - d^\varepsilon_{ik}\frac{\partial \varphi^\varepsilon}{\partial x_k}\frac{\partial \varphi^\varepsilon}{\partial x_i}dx = \int_\Omega f u^\varepsilon(x)dx + \int_\Omega r\varphi^\varepsilon(x)dx + \int_{\Gamma^m_1} g u^\varepsilon d\Gamma^m,$$

we use the symmetry of $P_{kij}$,then we get:

$$\int_{\Omega} C^{\varepsilon}_{ijkl}(x)e_{ij}(u^{\varepsilon})e_{kl}(u^{\varepsilon})dx + \int_{\Omega} d^{\varepsilon}_{ik}\frac{\partial\varphi^{\varepsilon}}{\partial x_i}\frac{\partial\varphi^{\varepsilon}}{\partial x_k}dx = \int_{\Omega} fu^{\varepsilon}dx + \int_{\Gamma^m_1} gu^{\varepsilon}d\Gamma^m + \int_{\Omega} r\varphi^{\varepsilon}dx. \quad (3.3)$$

On the hand, taking advantage of ellipticity of $C_{ijkl}$ and $d_{ij}$ we obtain

$$c_1\|\frac{\partial\varphi^{\varepsilon}}{\partial x_k}\|^2_{L^2(\Omega)} + \|e_{ij}(u^{\varepsilon})\|^2_{L^2(\Omega)} \leq \int_{\Omega} C^{\varepsilon}_{ijkl}(x)e_{kl}(u^{\varepsilon})e_{ij}(u^{\varepsilon}) + d^{\varepsilon}_{ik}\frac{\partial\varphi^{\varepsilon}}{\partial x_k}\frac{\partial\varphi^{\varepsilon}}{\partial x_i}dx. \quad (3.4)$$

By applying **Korn's inequality, Poincarr's inequality, Cauchi-Scharz inequality** and **trace theorem**, we get :

$$c\left(\|u^{\varepsilon} + \varphi^{\varepsilon}\|^2_{\mathcal{V}}\right) \leqslant c(\|(f\|_{L^2} + \|r\|_{L^2})(\|u^{\varepsilon}\|_{\mathcal{V}}) + \|r\|_{L^2}\|\varphi^{\varepsilon}\|_{\mathcal{V}}) = c_p(\|(u^{\varepsilon},\varphi^{\varepsilon})\|_{\mathcal{V}\times\Psi}), \quad (3.5)$$

which leads to

$$\|u^{\varepsilon}\|_{\mathcal{V}} + \|\varphi^{\varepsilon}\|_{\Psi} \leqslant c, \quad (3.6)$$

which means that

$$\begin{cases} \|u^{\varepsilon}\|_{\mathcal{V}} \leqslant C, \\ \|\varphi^{\varepsilon}\|_{\Psi} \leqslant C. \end{cases} \quad (3.7)$$

So we can extract a subsequences still denoted by $u^0, \varphi^0$ , such tat:

$$\begin{cases} u^{\varepsilon} \rightharpoonup u^0 \ in \ H^1(\Omega), \\ u^{\varepsilon} \longrightarrow u^0 \ in \ L^2(\Omega), \\ \varphi^{\varepsilon} \rightharpoonup \varphi^0 \ in \ H^1(\Omega), \\ \varphi^{\varepsilon} \longrightarrow \varphi^0 \ in \ L^2(\Omega). \end{cases} \quad (3.8)$$

**Remark 3.2.2** *$u^{\varepsilon}$ and $\varphi^{\varepsilon}$ are bounded than : $\frac{\partial u^{\varepsilon}}{\partial x_j}$ and $\frac{\partial\varphi^{\varepsilon}}{\partial x_j}$ are bounded.*

Hence we conclude

$$\begin{cases} u^\varepsilon \rightharpoonup u^0 \text{ in } H^1(\Omega), \\ u^\varepsilon \longrightarrow u^0 \text{ in } L^2(\Omega), \\ \dfrac{\partial u^\varepsilon}{\partial x_j} \rightharpoonup \dfrac{\partial u^0}{\partial x_j} \text{ in } L^2(\Omega), \\ \varphi^\varepsilon \rightharpoonup \varphi^0 \text{ in } H^1(\Omega), \\ \varphi^\varepsilon \longrightarrow \varphi^0 \text{ in } L^2(\Omega), \\ \dfrac{\partial \varphi^\varepsilon}{\partial x_j} \rightharpoonup \dfrac{\partial \varphi^0}{\partial x_j} \text{ in } L^2(\Omega). \end{cases} \tag{3.9}$$

Set

$$\wp^\varepsilon_{ij} = C^\varepsilon_{ijkl}(x)e_{kl}(u^\varepsilon) + P^\varepsilon_{kij}(x)\frac{\partial \varphi^\varepsilon}{\partial x_k}$$

$$\Lambda^\varepsilon_i = P^\varepsilon_{ikl}(x)e_{kl}(u^\varepsilon) - d_{ik}\frac{\partial \varphi^\varepsilon}{\partial x_k}$$

we have

$$\|\wp^\varepsilon_{ij}\|^2_{L^2(\Omega)} = \int_\Omega |\wp^\varepsilon_{ij}|^2 \mathrm{d}x = \int_\Omega |C^\varepsilon_{ijkl}(x)e_{kl}(u^\varepsilon) + P^\varepsilon_{kij}\frac{\partial \varphi^\varepsilon}{\partial x_j}\mathrm{d}x|^2 \tag{3.10}$$

by:

$$(a+b)^2 \leq 2a^2 + 2b^2$$

$\Longrightarrow$

$$\begin{aligned} \|\wp^\varepsilon_{ij}\|^2_{L^2(\Omega)} &\leq 2\int_\Omega |C^\varepsilon_{ijkl}e_{kl}(u^\varepsilon)|^2 + |P^\varepsilon_{kij}\partial_k\varphi^\varepsilon|^2 \mathrm{d}x \\ &\preccurlyeq \|e_{kl}(u^\varepsilon)\|^2_{L^2(\Omega)} + \|\partial_k\varphi^\varepsilon\|^2_{L^2(\Omega)} \\ &\preccurlyeq \|\nabla u^\varepsilon\|^2_{L^2} + \|\partial_k\varphi^\varepsilon\|^2_{L^2(\Omega)} \\ &= c\|u^\varepsilon\|^2_{\mathcal{V}} + \|\varphi^{\varepsilon\|^2}_\Psi \\ &\leq C_p. \end{aligned}$$

Hence,we deduce that we can extract a subsequence still by $\wp^\varepsilon_{ij}$ such that : $\wp^\varepsilon \rightharpoonup \wp^*_{ij} \in L^2(\Omega)$.

With same way we find

$$\Lambda^\varepsilon_i \rightharpoonup \Lambda^*_i \tag{3.11}$$

so, we conclude

$$\wp_{ij}^{\varepsilon} \rightharpoonup \wp_{ij}^{*} \text{ in } L^2(\Omega) \tag{3.12}$$

$$\Lambda_i^{\varepsilon} \rightharpoonup \Lambda_i^{*} \text{ in } L^2(\Omega) \tag{3.13}$$

It is worth that $\wp_{ij}^{*}$ satisfies

$$-\partial_j(\wp_{ij}^{*}) = f_i \tag{3.14}$$

Inned, taking $\psi = 0$ in (2.10) brings us to

$$\int_{\Omega}[C_{ijkl}^{\varepsilon}(x)e_{kl}(u^{\varepsilon}) + P_{kij}^{\varepsilon}\frac{\partial\varphi^{\varepsilon}}{\partial x_k}]e_{ij}(\vartheta)\lceil x = \int_{\Omega} f_i\vartheta_i \mathrm{d}x + \int_{\Gamma_1^m} g_i\vartheta_i \mathrm{d}s \tag{3.15}$$

$$\iff \int_{\Omega}\sum_{ij}^{\varepsilon} e_{ij}(\vartheta)\mathrm{d}x = \int_{\Omega} f_i\vartheta_i \mathrm{d}x + \int_{\Gamma_1^m} g_i\vartheta_i \mathrm{d}s, \forall \in \mathcal{V} \tag{3.16}$$

passing to limit(taking $\vartheta \in \mathcal{D}(\Omega)$)

$$\lim_{\varepsilon \longrightarrow 0}\int_{\Omega}\wp_{ij}^{\varepsilon}e_{ij}(\vartheta) = \int_{\Omega} f_i\vartheta_i \tag{3.17}$$

$$\Longrightarrow \int_{\Omega}\wp^{*}e_{ij}(\vartheta) = \int_{\Omega} -\partial_j[\wp_{ij}^{*}]\vartheta \mathrm{d}x = \int_{\Omega} f_i\vartheta_i \mathrm{d}x \tag{3.18}$$

$$\Longrightarrow \int_{\Omega}(-\partial_j[\wp_{ij}^{*}] - f_i)\vartheta \mathrm{d}x = 0\forall \in \mathcal{D}(\Omega) \tag{3.19}$$

$$\Longrightarrow -\partial_j\wp_{ij}^{*} = f_i \tag{3.20}$$

**Remark 3.2.3** *we use the same way with $\Lambda_{ij}^{*}$ for taking $\vartheta = 0, \forall\psi \in \mathcal{D}(\Omega)$*

so

$$\begin{cases} -\partial_j[\wp_{ij}^{*}] = f_i \\ \partial_i[\Lambda_{ij}^{\varepsilon}] = r \end{cases} \tag{3.21}$$

**Step 3:**<u>**The introduction of the oscillating test functions**</u>

$$\begin{cases} \rho_i^{\varepsilon,mn}(x) = \varepsilon\chi_i^{mn}(\frac{x}{\varepsilon}) + \delta_{im}x_n, \\ \theta^{\varepsilon,mn}(x) = \varepsilon\Psi^{mn}(\frac{x}{\varepsilon}), \\ \pi_i^{\varepsilon,m}(x) = \varepsilon\Phi_i^{m}(\frac{x}{\varepsilon}), \\ \mathcal{I}^{\varepsilon,m}(x) = \varepsilon\mathcal{R}^{m} + x_m \end{cases} \tag{3.22}$$

where:$(\chi_i^{mn}(y), \Psi^{mn}(y)) \in H^1_\sharp \times H^1_\sharp$is the solution of:

$$(P_{\chi^{mn},\Psi^{mn}}) : \begin{cases} -\dfrac{\partial}{\partial y_j}[C_{ijkl}(y)e_{kl_y}(\chi^{mn}(y) + \tau^{kl}_{mn}) + P_{kij}(y)\dfrac{\partial \Psi^{mn}}{\partial y_k}] = 0 \\ -\dfrac{\partial}{\partial y_i}[P_{jkl}(y)(e_{kl_y}(\chi^{mn}(y) + \tau^{kl}_{mn})) - d_{jk}(y)\dfrac{\partial \Psi^{mn}}{\partial y_k}] = 0 \\ \int_Y \chi^{mn} = 0, \int_Y \Psi^{mn} = 0 \\ \chi^{mn}, \Psi^{mn} Y - periodic, \end{cases} \tag{3.23}$$

and :$(\Phi^m, \mathcal{R}^m) \in H^1_\sharp \times H^1_\sharp$ is the solution of:

$$(P_{\Phi^m,\mathcal{R}^m}) : \begin{cases} -\dfrac{\partial}{\partial y_j}[C_{ijkl}(y)e_{kl_y}(\Phi^m(y)) + P_{kij}(y)(\delta_{km} + \dfrac{\partial \mathcal{R}^m}{\partial y_k} = 0)] \\ -\dfrac{\partial}{\partial y_i}[P_{jkl}(y)e_{kl_y}(\Phi^m(y)) - d_{jk}(y)(\delta_{km} + \dfrac{\partial \mathcal{R}^m}{\partial y_k})] = 0 \\ \int_Y \Phi^m = 0, \int_Y \mathcal{R}^m = 0 \\ \Phi^m, \mathcal{R}^m \; Y - periodic, \end{cases} \tag{3.24}$$

with

$$\tau^{kl}_{mn} = \frac{1}{2}[\delta_{km}\delta_{ln} + \delta_{kn}\delta_{lm}], 1 \leq k, l, m, n \leq 3,$$

is the unit tensor of the fourth-order. Such that $(\rho_i^{\varepsilon,mn}(x), \theta^{\varepsilon,mn}(x))$ solution of

$$(P^\varepsilon_{\rho,\theta}) : \begin{cases} -\dfrac{\partial}{\partial x_j}[C^\varepsilon_{ijkl}(x)e_{kl}(\rho_i^{\varepsilon,mn}) + P^\varepsilon_{kij}(x)\dfrac{\partial \theta^{\varepsilon,mn}}{\partial x_k}] = 0 \\ \dfrac{\partial}{\partial x_j}[P^\varepsilon_{jkl}(x)e_{kl}(\rho_i^{\varepsilon,mn}) - d^\varepsilon_{jk}\dfrac{\partial \theta^{\varepsilon,mn}}{\partial x_k}] = 0, \end{cases} \tag{3.25}$$

hence the variational formulation of $(\mathcal{P}^\varepsilon_{\rho,\theta})$ is :

$$\int_\Omega C^\varepsilon_{ijkl}(x)e_{kl}(\rho_i^{\varepsilon,mn})e_{ij}(\vartheta) + P^\varepsilon_{kij}(x)[\frac{\partial \theta^{\varepsilon,mn}}{\partial x_k}e_{ij}(\vartheta) - e_{kl}(\rho_i^{\varepsilon,mn})\frac{\partial \psi}{\partial x_j}] + d^\varepsilon_{jk}\frac{\partial \theta^{\varepsilon,mn}}{\partial x_j}\frac{\partial \psi}{\partial x_j} = 0 \tag{3.26}$$

and:

$$(\mathcal{P}^\varepsilon_{\pi,\mathcal{I}}) : \begin{cases} -\dfrac{\partial}{\partial x_j}[C^\varepsilon_{ijkl}(x)e_{kl}(\pi_i^{\varepsilon,m}) + P^\varepsilon_{kij}\dfrac{\partial \mathcal{I}^{\varepsilon,m}}{\partial x_k}] = 0 \\ \dfrac{\partial}{\partial x_j}[P^\varepsilon_{ikl}(x)e_{kl}(\pi_i^{\varepsilon,m}) - d^\varepsilon_{ik}\dfrac{\partial \mathcal{I}^{\varepsilon,m}}{\partial x_k}] = 0, \end{cases} . \tag{3.27}$$

then, the variational of $(\mathcal{P}^\varepsilon_{\pi,\mathcal{I}})$ is :

$$\int_\Omega C^\varepsilon_{ijkl}(x)e_{kl}(\pi_i^{\varepsilon,m})e_{ij}(\vartheta) + P^\varepsilon_{kij}[\frac{\partial \mathcal{I}^{\varepsilon,m}}{\partial x_k}e_{ij}(\vartheta) - e_{kl}(\pi_i^{\varepsilon,m})\frac{\partial \psi}{\partial x_j}] + d^\varepsilon_{jk}\frac{\partial \mathcal{I}^{\varepsilon,m}}{\partial x_k}\frac{\partial \psi}{\partial x_j} = 0 \tag{3.28}$$

**Lemma 3.2.4** *We have the following convergence,* $\varepsilon \longrightarrow 0$ *:*

$\rho_i^{\varepsilon,mn}(x)\underrightarrow{L^2(\Omega)}\delta_{im}x_n,$
$\theta^{\varepsilon,mn}(x)\underrightarrow{L^2(\Omega)}0,$
$\mathcal{I}^{\varepsilon,m}(x)\underrightarrow{L^2(\Omega)}x_m,$
$\pi_i^{\varepsilon,m}(x)\underrightarrow{L^2(\Omega)}0.$

Set :

$$\begin{cases} \Im^{1,\varepsilon}_{ijmn} = C^{\varepsilon}_{ijkl}e_{kl}(\rho_i^{\varepsilon,mn}) + P_{kij}\dfrac{\partial\theta^{\varepsilon,mn}}{\partial x_k} \\ S^{1,\varepsilon}_{jmn} = P^{\varepsilon}_{jkl}e_{kl}(\rho_i^{\varepsilon,mn}) - d^{\varepsilon}_{jk}(x)\dfrac{\partial\theta^{\varepsilon,mn}}{\partial x_k} \end{cases} \tag{3.29}$$

then,we can write the problem as: $(\mathcal{P}^{\varepsilon}_{\rho,\theta})$

$$(\mathcal{P}^{\varepsilon}_{\rho,\theta}) : \begin{cases} -\dfrac{\partial\Im^{1,\varepsilon}_{ijmn}}{\partial x_j} = 0, \\ \dfrac{\partial S^{1,\varepsilon}_{jmn}}{\partial x_j}, = 0 \\ \Im^{1,\varepsilon}_{ijmn} \text{ and } S^{1,\varepsilon}_{jmn} \text{ are } Y-periodic \end{cases} \tag{3.30}$$

And set :

$$\begin{cases} \Im^{2,\varepsilon}_{ijm} = C^{\varepsilon}_{ijkl}(x)e_{kl}(\pi_i^{\varepsilon,m}) + P^{\varepsilon}_{kij}\dfrac{\partial\mathcal{I}^{\varepsilon,m}}{\partial x_k} \\ S^{2,\varepsilon}_{jm} = P^{\varepsilon}_{ikl}(x)e_{kl}(\pi_i^{\varepsilon,m}) - d^{\varepsilon}_{ik}\dfrac{\partial\mathcal{I}^{\varepsilon,m}}{\partial x_k} \end{cases} \tag{3.31}$$

then we can write the problem $(\mathcal{P}^{\varepsilon}_{\pi,\mathcal{I}})$ as:

$$(\mathcal{P}_{\pi,\mathcal{I}}) : \begin{cases} -\dfrac{\partial\Im^{2,\varepsilon}_{ijm}}{\partial x_j} = 0, \\ \dfrac{\partial S^{2,\varepsilon}_{jm}}{\partial x_j} = 0, \\ \Im^{2,\varepsilon}_{ijm} \; and \; S^{2}_{jm} \text{ are } Y-periodic. \end{cases} \tag{3.32}$$

Where $\Im^{1,\varepsilon}_{ijmn}, \Im^{2,\varepsilon}_{ijm}, S^{1,\varepsilon}_{jmn}$ and $S^{2,\varepsilon}_{jm}$ are periodic functions ,then we can applying(**??**),then:

$$\Im^{1,\varepsilon}_{ijmn} \rightharpoonup <\Im^{1,\varepsilon}_{ijmn}>= \frac{1}{|Y|}\int_Y C_{ijkl}(y)e_{kl_y}(\chi_i^{mn}) + C_{ijmn} + P_{kij}(y)\frac{\partial\Psi^{mn}}{\partial y_k}\mathrm{d}y \tag{3.33}$$

$$\Im^{2,\varepsilon}_{ijmn} \rightharpoonup <\Im^{2,\varepsilon}_{ijmn}>= \frac{1}{Y}\int_Y C_{ijkl}(y)e_{kl_y}(\Phi^m(y)) + P_{mij}(y) + P_{kij}(y)\frac{\partial\mathcal{R}^m}{\partial y_k}\mathrm{d}y \tag{3.34}$$

$$S^{1,\varepsilon}_{jmn} \rightharpoonup < S^{1,\varepsilon}_{jmn} >= \frac{1}{|Y|}\int_Y P_{jkl}(y)e_{kl_y}(\chi_i^{mn}) + P_{jmn}(y) - d_{kj}(y)\frac{\partial\Psi^{mn}}{y_k}\mathrm{d}y \tag{3.35}$$

$$S^{2,\varepsilon}_{jmn} \rightharpoonup < S^{2,\varepsilon}_{jmn} >= \frac{1}{|Y|}\int_Y P_{ikl}(y)e_{kl_y}(\Phi^n) - d_{ik}(y)\frac{\partial(\mathcal{R}(y)+y_n)}{\partial y_n}\mathrm{d}y \tag{3.36}$$

**Step 4:<u>The homogenized coefficients</u>**

We can write equation (2.10)

$$\int_\Omega C^\varepsilon_{ijkl}(x)e_{kl}(u^\varepsilon)e_{ij}(\vartheta) + P^\varepsilon_{kij}(x)[\frac{\partial\varphi^\varepsilon}{\partial x_k}e_{ij}(\vartheta) - e_{kl}(u^\varepsilon)\frac{\partial\psi}{\partial x_k}] + d^\varepsilon_{jk}(x)\frac{\partial\varphi^\varepsilon}{\partial x_k}\frac{\partial\psi}{\partial x_i} = \int_\Omega f_i\vartheta_i\mathrm{d}x + \int_\Omega r\psi\mathrm{d}x + \int_{\Gamma_1^m} g_i\vartheta_i\; d\Gamma^m, \tag{3.37}$$

taking in (3.37)$\vartheta_i(x) = -\omega(x)\rho_i^{\varepsilon,mn}(x),\ where\ \omega \in \mathcal{D}(\Omega)$ and $\psi(x) = \omega(x)\theta^{\varepsilon,mn}(x)$,then

$$e_{ij}(\vartheta) = -\omega(x)e_{ij}(\rho_i^{\varepsilon,mn}) - \frac{\partial\omega}{\partial x_j}\rho_i^{\varepsilon,mn},$$

and

$$\frac{\partial\psi}{\partial x_k} = \omega(x)\frac{\partial\theta^{\varepsilon,mn}}{\partial x_k} + \frac{\partial\omega}{\partial x_k}\theta^{\varepsilon,mn}.$$

We obtain

$$-\int_\Omega C^\varepsilon_{ijkl}(x)e_{kl}(u^\varepsilon)[\omega(x)e_{ij}(\rho_i^{\varepsilon,mn}) + \frac{\partial\omega}{\partial x_j}\rho_i^{\varepsilon,mn}]dx - \int_\Omega P^\varepsilon_{kij}(x)\frac{\partial\varphi^\varepsilon}{\partial x_k}[\omega(x)e_{ij}(\rho_i^{\varepsilon,mn}) + \frac{\partial\omega}{\partial x_j}\rho_i^{\varepsilon,mn}]dx - \int_\Omega P^\varepsilon_{kij}e_{ij}(u^\varepsilon)[\omega(x)\frac{\partial\theta^{\varepsilon,mn}}{\partial x_k}]dx + \int_\Omega d^\varepsilon_{jk}\frac{\partial\varphi^\varepsilon}{\partial x_k}[\frac{\partial\theta^{\varepsilon,mn}}{\partial x_k}\omega(x) + \frac{\partial\omega}{\partial x_k}\theta^{\varepsilon,mn}] = -\int_\Omega f_i\rho_i^{\varepsilon,mn}\omega(x)dx + \int_\Omega r\theta^{\varepsilon,mn}\omega(x)dx, \forall(x) \in \mathcal{D}(\Omega), \tag{3.38}$$

taking in (3.26)

$$\begin{cases}\vartheta_i(x) = -\omega(x)u_i^\varepsilon(x) \Longrightarrow e_{ij}(\vartheta) = -\omega(x)e_{ij}(u^\varepsilon) - \dfrac{\partial\omega}{\partial x_j}u_i^\varepsilon(x)\\ \psi(x) = \omega(x)\varphi^\varepsilon(x) \Longrightarrow \varphi^\varepsilon\dfrac{\partial\omega}{\partial x_k} + \omega(x)\dfrac{\partial\varphi^k}{\partial x_k}.\end{cases}$$

We obtain:

$$-\int_\Omega C^\varepsilon_{ijkl}(x)e_{kl}(\rho_i^{\varepsilon,mn})[\omega(x)e_{ij}(u^\varepsilon)+\frac{\partial\omega}{\partial x_j}u_i^\varepsilon(x)]dx-\int_\Omega P^\varepsilon_{kij}\frac{\partial\theta^{\varepsilon,mn}}{\partial x_k}[\omega(x)e_{ij}(u^\varepsilon)$$
$$+\frac{\partial\omega}{\partial x_j}u_i(x)]-\int_\Omega P^\varepsilon_{kij}e_{kl}(\rho_i^{\varepsilon,mn})[\omega(x)\frac{\partial\varphi^\varepsilon}{\partial x_k}+\frac{\partial\omega}{\partial x_k}\varphi^\varepsilon]dx$$
$$+\int_\Omega d^\varepsilon_{jk}\frac{\partial\theta^{\varepsilon,mn}}{\partial x_k}[\omega(x)\frac{\partial\varphi^\varepsilon}{\partial x_k}+\varphi^\varepsilon\frac{\partial\omega}{\partial x_k}]dx\ =0, \quad (3.39)$$

by subtraction between( (3.38),3.39),we get :

$$-\int_\Omega C^\varepsilon_{ijkl}(x)e_{kl}(u^\varepsilon)[\omega(x)e_{ij}(\rho_i^{\varepsilon,mn})+\rho^{\varepsilon,mn}\frac{\partial\omega}{\partial x_j}]dx+\int_\Omega C^\varepsilon_{ijkl}(x)e_{kl}(\rho_i^{\varepsilon,mn})[\omega(x)e_{ij}(u^\varepsilon)$$
$$+\frac{\partial\omega}{\partial x_j}u_i^\varepsilon(x)]dx-\int_\Omega P^\varepsilon_{kij}(x)\frac{\partial\varphi^\varepsilon}{\partial x_j}[\omega(x)e_{ij}(\rho_i^{\varepsilon,mn})+\rho_i^{\varepsilon,mn}\frac{\partial\omega}{\partial x_j}]dx$$
$$+\int_\Omega P^\varepsilon_{kij}\frac{\partial\theta^{\varepsilon,mn}}{\partial x_k}[\omega(x)e_{ij}(u^\varepsilon)+\frac{\partial\omega}{\partial x_j}u_i^\varepsilon(x)]dx-\int_\Omega P^\varepsilon_{kij}(x)e_{kl}(u^\varepsilon)[\frac{\partial\omega}{\partial x_j}\theta^{\varepsilon,mn}+\omega(x)\frac{\partial\theta^{\varepsilon,mn}}{\partial x_j}]d$$
$$+\int_\Omega P^\varepsilon_{kij}(x)e_{kl}(\rho^{\varepsilon,mn})[\omega(x)\frac{\partial\varphi^\varepsilon}{\partial x_k}+\varphi^\varepsilon\frac{\partial\omega}{\partial x_k}]dx+\int_\Omega d^\varepsilon_{jk}(x)\frac{\partial\varphi^\varepsilon}{\partial x_k}[\frac{\partial\omega}{\partial x_j}\theta^{\varepsilon,mn}+\omega(x)\frac{\partial\theta^{\varepsilon,mn}}{\partial x_j}]dx$$
$$-\int_\Omega d^\varepsilon_{jk}\frac{\partial\theta^{\varepsilon,mn}}{\partial x_k}[\omega(x)\frac{\partial\varphi^\varepsilon}{\partial x_k}+\varphi^\varepsilon\frac{\partial\omega}{\partial x_k}]$$
$$=\int_\Omega -f_i\rho_i^{\varepsilon,mn}\omega(x)dx+\int_\Omega r\theta^{\varepsilon,mn}\omega(x)dx \quad (3.40)$$

$\Longrightarrow$

$$-\int_\Omega[C^\varepsilon_{ijkl}(x)e_{kl}(u^\varepsilon)+P^\varepsilon_{kij}\frac{\partial\varphi^\varepsilon}{\partial x_k}]\rho_i^{\varepsilon,mn}\frac{\partial\omega}{\partial x_j}$$
$$+\int_\Omega[C^\varepsilon_{ijkl}(x)e_{kl}(\rho_i^{\varepsilon,mn})+P^\varepsilon_{kij}(x)\frac{\partial\theta^{\varepsilon,mn}}{\partial x_j}]\frac{\partial\omega}{\partial x_j}u_i(x)$$
$$+\int_\Omega[-P^\varepsilon_{kij}(x)e_{kl}(u^\varepsilon)+d^\varepsilon_{jk}\frac{\partial\varphi^\varepsilon}{\partial x_k}]\frac{\partial\omega}{\partial x_j}\theta^{\varepsilon,mn}(x)dx$$
$$+\int_\Omega[P^\varepsilon_{kij}e_{kl}(\rho_i^{\varepsilon,mn})-d^\varepsilon_{jk}\frac{\partial\theta^{\varepsilon,mn}}{\partial x_k}]\frac{\partial\omega}{\partial x_k}\varphi^\varepsilon dx$$
$$=\int_\Omega -f_i\rho_i^{\varepsilon,mn}\omega(x)dx+\int_\Omega r\theta^{\varepsilon,mn}\omega(x)dx, \quad (3.41)$$

$\Longleftrightarrow$

$$-\int_\Omega \wp^\varepsilon_{ijkl}\frac{\partial\omega}{\partial x_j}\rho_i^{\varepsilon,mn}(x)dx+\int_\Omega\Im^{1,\varepsilon}_{ijmn}\frac{\partial\omega}{\partial x_j}u_i^\varepsilon(x)dx-\int_\Omega\Lambda^\varepsilon_j\frac{\partial\omega}{\partial x_j}\theta^{\varepsilon,mn(x)}dx$$
$$-\int_\Omega S^{1,\varepsilon}_{jmn}\frac{\partial\omega}{\partial x_j}\varphi^\varepsilon dx=-\int_\Omega f_i\rho_i^{\varepsilon,mn}(x)\omega(x)dx+\int_r\theta_{\varepsilon,mn}(x)\omega(x), \quad (3.42)$$

by passing to the limit we obtain :

$$\int_\Omega -\wp^*_{ij}(\delta_{im}x_n)\frac{\partial\omega}{\partial x_j} + \int_\Omega < \Im^1_{ijmn} > u^0_i\frac{\partial\omega}{\partial x_j}dx - \int < S^1_{jmn} > \varphi^0\frac{\partial\omega}{\partial x_j}dx$$
$$= -\int_\Omega f_i(\delta_{im}x_n)\omega(x)dx, \quad (3.43)$$

we know that :

$$\int_\Omega f\frac{\partial\varphi}{\partial x_j}dx = -\int_\Omega \frac{\partial f}{\partial x_j}\varphi, \forall\varphi \in \mathcal{D}(\Omega),$$

now,we obtain:

$$\int_\Omega \frac{\partial\wp^*_{ij}}{\partial x_j}\partial x_j(\delta_{im}x_n)\omega dx - < \Im^1_{ijmn} > \int_\Omega \frac{\partial u^0}{\partial x_j}\omega + < S^1_{jmn} > \int_\Omega \frac{\partial\varphi^0}{\partial x_j}\omega$$
$$= -\int_\Omega f_i(\delta_{im}x_n)\omega dx, \quad (3.44)$$

$$\Longrightarrow \int_\Omega \wp^*_{mn}\omega dx - < \Im^1_{ijmn} > \int_\Omega \frac{\partial u^0_i}{\partial x_j}\omega dx - < S^1_{jmn} > \int_\Omega \frac{\partial\varphi^0}{\partial x_j}\omega dx = 0 \quad (3.45)$$
$$\Longrightarrow \int_\Omega (\wp^*_{mn} - < \Im_{ijmn} > \frac{\partial u^0_i}{\partial x_j} - < S^1_{jmn} > \frac{\partial\varphi^0}{\partial x_j})\omega dx = 0, \forall \in (\mathcal{D}(\Omega))$$
$$\Longrightarrow \wp^*_{mn} =< \Im^1_{ijmn} > \frac{\partial u^0_i}{\partial x_j} + < S^1_{jmn} > \frac{\partial\varphi^0}{\partial x_j}$$

$\Longrightarrow$

$$\wp^*_{mn} = \frac{1}{|Y|}\int_Y (C_{ijkl}(y)e_{kl_y}(\chi^{mn}_i) + C_{ijmn}(y) + P_{kij}(y)\frac{\partial\Psi^{mn}}{\partial y_k})dy\frac{\partial u^0_i}{\partial x_j}$$
$$+ \frac{1}{|Y|}\int_Y (P_{jkl}(y)e_{kl_y}(\chi^{mn}_i) + P_{jmn} - d_{jk}\frac{\partial\Psi^{mn}}{\partial y_k})dy\frac{\partial\varphi^0}{\partial y_k}$$

With the same way , taking in (3.37)

$$\begin{cases} \vartheta(x) = -\omega(x)\pi^{\varepsilon,m}_i(x) \\ \psi(x) = \omega(x)\mathcal{I}^{\varepsilon,m}(x), \end{cases}$$

now we obtain:

$$\int_\Omega -C^\varepsilon_{ijkl}(x)e_{kl}(u^\varepsilon)[\omega e_{ij}(\pi^{\varepsilon,m}) + \frac{\partial\omega}{\partial x_j}\pi^{\varepsilon,m}] - P^\varepsilon_{kij}\frac{\partial\varphi^\varepsilon}{\partial x_k}[\omega e_{ij}(\pi_i^{\varepsilon,m}) + \frac{\partial\omega}{\partial x_i}\pi_i^{\varepsilon,m}]$$
$$- P^\varepsilon_{kij}(x)[\omega\frac{\partial\mathcal{I}^{\varepsilon,m}}{\partial x_k} + \frac{\partial\omega}{\partial x_k}\mathcal{I}] + d^\varepsilon_{jk}(x)\frac{\partial\varphi^\varepsilon}{\partial x_k}[\omega\frac{\partial\mathcal{I}^{\varepsilon,m}}{\partial x_j} + \frac{\partial\omega}{\partial x_j}\mathcal{I}^{\varepsilon,m}]dx$$
$$= -\int_\Omega f_i\omega(x)\pi_i^{\varepsilon,m}dx + \int_\Omega r\omega\mathcal{I}^{\varepsilon,m}dx \quad (3.46)$$

and taking in (3.28)

$$\begin{cases}\vartheta(x) = -\omega(x)u_i^\varepsilon(x)\\ \psi(x) = \omega(x)\varphi^\varepsilon(x),\end{cases}$$

we obtain :

$$\int_\Omega -C^\varepsilon_{ijkl}(x)e_{kl}(\pi_i^{\varepsilon,m})[\omega e_{ij}(u^\varepsilon) + \frac{\partial\omega}{\partial x_j}u_i^\varepsilon] - P^\varepsilon_{kij}(x)\frac{\partial\mathcal{I}^{\varepsilon,m}}{\partial x_k}[\omega e_{ij}(u^\varepsilon) + \frac{\partial\omega}{\partial x_j}u_i^\varepsilon]$$
$$- P^\varepsilon_{kij}e_{ij}(\pi_i^{\varepsilon,m})[\omega\frac{\partial\varphi^\varepsilon}{\partial x_k} + \frac{\partial\omega}{\partial x_k}\varphi^\varepsilon] + d^\varepsilon_{jk}\frac{\partial\mathcal{I}^{\varepsilon,m}}{\partial x_k}[\omega\frac{\partial\varphi^\varepsilon}{\partial x_k} + \frac{\partial\omega}{\partial x_j}\varphi^\varepsilon]dx = 0, \quad (3.47)$$

subtracting (3.46) and(3.47) we get :

$$-\int_\Omega (C^\varepsilon_{ijkl}(x)e_{kl}(u^\varepsilon) + P^\varepsilon_{kij}(x)\frac{\partial\varphi^\varepsilon}{\partial x_k})\pi_i^{\varepsilon,m}\frac{\partial\omega}{\partial x_j}dx$$
$$-\int_\Omega (P^\varepsilon_{jkl}(x)e_{kl}(u^\varepsilon) - d^\varepsilon_{jk}(x)\frac{\partial\varphi^\varepsilon}{\partial x_k})\mathcal{I}^{\varepsilon,m}\frac{\partial\omega}{\partial x_j}dx$$
$$+\int_\Omega (C^\varepsilon_{ijkl}(x)e_{kl}(\pi_i^{\varepsilon,m}) + P^\varepsilon_{kij}(x)\frac{\partial\mathcal{I}^{\varepsilon,m}}{\partial x_k})u_i^\varepsilon\frac{\partial\omega}{\partial x_j}dx$$
$$+\int_\Omega (P^\varepsilon_{jkl}(x)e_{kl}(\pi_i^{\varepsilon,m}) - d^\varepsilon_{jk}(x)\frac{\partial\mathcal{I}^{\varepsilon,m}}{\partial x_k})\varphi^\varepsilon\frac{\partial\omega}{\partial x_j}dx$$
$$= -\int_\Omega f_i\omega\pi_i^{\varepsilon,m}(x)dx + \int_\Omega r\omega\mathcal{I}^{\varepsilon,m}dx, \quad (3.48)$$

$\Longleftrightarrow$

$$-\int_\Omega \wp^\varepsilon_{ij}\pi_i^{\varepsilon,m}\frac{\partial\omega}{\partial x_j}dx - \int_\Omega \Lambda^\varepsilon_j\mathcal{I}^{\varepsilon,m}\frac{\partial\omega}{\partial x_j}dx + \int_\Omega \Im^{2,\varepsilon}_{ijm}u_i^\varepsilon\frac{\partial\omega}{\partial x_j}dx + \int_\Omega S^{2,\varepsilon}_{jm}\varphi^\varepsilon\frac{\partial\omega}{\partial x_j}$$
$$= -\int_\Omega f_i\pi_i^{\varepsilon,m}\omega dx + \int_\Omega r\mathcal{I}^{\varepsilon,m}dx. \quad (3.49)$$

By passing to the limit we get :

$$-\Lambda_m^* x_m \frac{\partial \omega}{\partial x_j} dx + < \Im_{ijm}^2 > \int_\Omega u_i^0 \frac{\partial \omega}{\partial x_j} dx + < S_{jm}^2 > \varphi^0 \frac{\partial \omega}{\partial x_j} dx = \int_\Omega r x_m \omega(x) dx, \tag{3.50}$$

$\Longleftrightarrow$

$$\int_\Omega (\frac{\partial \Lambda_j^* x_m}{\partial x_j}) \omega - < \Im_{ijm}^{2,\varepsilon} > \int_\Omega \frac{\partial u_i^0}{\partial x_j} \omega dx - < S^{2,\varepsilon_{jm}} > \int_\Omega \frac{\partial \varphi^0}{\partial x_j} \omega dx = \int_\Omega r x_m \omega(x) dx \tag{3.51}$$

$\Longleftrightarrow$

$$\Lambda_j^* = < \Im_{ijm}^2 > \frac{\partial u_i^0}{\partial x_j} + < S_{jm}^2 > \frac{\partial \varphi^0}{\partial x_j} \tag{3.52}$$

■

# Conclusion

In conclusion,this desertion has explored homogenization of piezoelectric materials using two different methods:the asymptotic method and the energy method .

Overall,this desertion has contribution to the understanding of homogenization techniques for piezoelectric materials.By analysing the energy method and its application to periodic functions,as well as utilizing the asymptotic method and the energy method for homogenization,we have obtained valuable insights into the effective properties and behaviour of piezoelectric composites. The knowledge gained form this desertion can inform the design and analysis of piezoelectric devices and contribute to advancement in the field of solid mechanics.

As further desertion is conducted in this area,it is crucial to continue exploring and refining homogenization methods for piezoelectric materials. The combination of multiple techniques and development of advanced numerical models can lead to more accurate prediction of the macroscopic behaviour of these materials. Additionally,investigating the effects of micro structural variations,non-linear behaviour,and dynamic loading conditions would provide a more comprehensive understanding of piezoelectric materials.

In conclusion,the study of homogenization techniques for piezoelectric materials has proven to be a fascinating and challenging field. By applying the asymptotic method and the energy method,we can unravel the complex behaviour of theses materials and derive their effective prosperities. This desertion opens up new possibilities for the design and optimization of piezoelectric devices and contributes to the advancement of materials science and engineering

# Abstract

In this dissertation,we will study the convergence between the piezoelectric problem $(\mathcal{P}_\varepsilon)$and the homogenized problem $(\mathcal{P}_h)$ by energy method,such that by choosing an appropriate oscillating test functions ,we can get easily The limit problem.

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**Keywords:** Energy method, homogeneous problem,convergence,test functions, oscillating.

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# Résumé

Dans cette recherche,nous étudierons la convergence entre le probl eme piézoélectrique et le probleme homogénéisé par la méthode ńergétique, telle qu'en choisissant des fonctions test oscillantes appropriée, on obtienne facilement le probleme limite.

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**Mots clés :** Méthode d'énergie, le probleme homogene, la convergence,oscilantes, fonctions de test

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# الملخص

في هذه المذكرة، سوف ندرس التقارب بين مسألة الكهرضغطية $(P_\varepsilon)$ والمسألة المتجانسة $(P_h)$ باعتماد طريقة الطاقة، مثل ذلك باختيار دوال اختبارية متذبذبة مناسب، يمكننا الحصول بسهولة على مشكلة الحد.

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**كلمات مفتاح:** طريقة الطاقة، مشكلة متجانسة، التقارب، التذبذب، وظائف الاختبار

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