kasdi Merbah University of Ouargla
Faculty of Mathematics and Material Science
Department of Mathematics

Mathematics
Specialisation : Modelling and Numerical Analysis

N° d'ordre
N° de sÃľrie

A Thesis Submitted to Obtain the Degree of
MASTER

Asymptotic Modelling of Viscoelastic von Karman Plate Model

By :
Bouras RABAB IBTISSAM

Presented Publicly On : 25/06/2023

Examination Committee :

Med El Hadi MEZABIA	MCA	Kasdi Merbah University of Ouargla	President
Abderrezak GHEZAL	MCA	Kasdi Merbah University of Ouargla	Examiner
Djamal Ahmed CHACHA	Pr	Kasdi Merbah University of Ouargla	Supervisor

Academic Year
2022 - 2023

Dedication

This work is dedicated to my parents you are the sunshine that brightens my days.

The pillars of strength that hold me up and the source of endless love and joy in my life.

This dedication is a tribute to the incredible bond we share, filled with cherished memories and countless moments of laughter. Your unwavering support and encouragement have shaped me into the person I am today, and I am eternally grateful for your presence in my life. With all my heart, I dedicate this to you, my beloved family, for being my rock and my inspiration.

Acknowledgement

First of all, I would like to thank Allah for all his abundant blessings. Then, I would like to thank my family for the needed support they provided during my thesis journey.

I would like to express my sincere gratitude to my supervisor Professor Djamal Ahmed CHACHA for suggesting this interesting topic, guiding me through the process of writing my thesis, and equipping me with the necessary research skills.

I am also grateful to Med El Hadi MEZABIA**,** Dr. Abderrezak GHEZAL, and for honouring us with their presence in the examination committee, and accepting to evaluate my thesis, without forgetting all of my teachers and lecturers who have helped me during my years in college.

Finally,I would like to thank those who helped me answer my questions during the writing and preparation process. Dr.Ibtissam TALHA, Dr. Bochra AZZAOUI, Tarek ASSILA, Maroua GNAWA .

Contents

Introduction **1**

1 Viscoelasticity: Introduction and Equations **3**

1.1 Introduction 4

1.2 Governing Equations 6

1.2.1 Examples of Materials Exhibiting Viscoelastic Behavior 6

1.2.2 Properties of viscoelastic materials 6

1.2.3 Stress-Strain Constitutive law 6

1.2.4 Motivations for studying viscoelasticity 8

2 Modeling of Viscoelastic von Karman Plate **10**

2.1 Geometry of the plate 11

2.2 Problem of the Plate 14

2.3 The mixed variational problem 16

2.3.1 Preliminaries 16

2.3.2 Weak Formulations 17

3 Asymptotic Analysis **20**

3.1 The Scaled Three-Dimensional Problem 21

3.1.1 The Fixed Domain 21

3.1.2 Scaling of the Data and the Unknowns 22
3.1.3 The Scaled Weak Formulation 23
3.2 The limit three dimentional problem 29
3.2.1 Solution of the limit problem 30

Appendix 1 **36**

Appendix 2 **37**

Appendix 3 **38**

Notations

- ➤ ε : A small positive parameter characterizing the half-thickness.
- ➤ $x^\varepsilon = \pi^\varepsilon x = (x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon) = (x_1, x_2, \varepsilon x_3)$: an arbitrary point of Ω^ε.
- ➤ $\partial_i = \frac{\partial}{\partial x_i}$: Partial differentiation with respect to x_i.
- ➤ $\partial_i^\varepsilon = \frac{\partial}{\partial x_i^\varepsilon}$: Partial differentiation with respect to x_i^ε.
- ➤ Ω^ε : Reference configuration of a thin plate.
- ➤ ω Middle surface of the plate.
- ➤ $\Gamma_\pm^\varepsilon$: The upper and lower faces respectively of Ω^ε.
- ➤ Γ_0^ε : Portin of the lateral face where a plate is clamped.
- ➤ a_{ijkl}^ε : Relaxation matrix.

Conventions

➤ $A = (a_{ijkl})$

1/The symmetry:

$a_{ijkl} = a_{jikl} = a_{klij} = a_{klji}$

2/Ellipticity:

$\exists c > 0 \ \ \forall \varepsilon_{ij} = \varepsilon_{ji} \ \ (\varepsilon = (\varepsilon_{ij}))$

$a_{ijkl} = \varepsilon_{kl}\varepsilon_{ij} \geq c\varepsilon_{ij}\varepsilon_{ij}$

3/$a_{ijkl} \in \mathbf{L}^{\infty}(\Omega)$

➤ Latin indices $\{i, j, k, l\}$ over the set $\{1, 2, 3\}$.

➤ $[L^2(\Omega)]^3$ or $[L^2(\Omega)]^9$.

➤ $\mathbb{H}^1(\Omega)$ and $\mathbb{H}^2(\Omega)$ represent $[H^1(\Omega)]^3$ and $[H^2(\Omega)]^3$, respectively .

➤ $\sum_{ij}^{\varepsilon} = \sigma_{ij}^{\varepsilon} + \sigma_{kj}^{\varepsilon}\partial_k^{\varepsilon}u_i^{\varepsilon}$

➤ $\sigma_{ij}^{\varepsilon}(x^{\varepsilon}, t) = \displaystyle\int_0^t a_{ijkl}^{\varepsilon}(t-s)\frac{\partial E_{kl}^{\varepsilon}(u^{\varepsilon}(x^{\varepsilon}, s))}{\partial s}ds = a_{ijkl}^{\varepsilon} * \frac{\partial E_{kl}^{\varepsilon}}{\partial s}$

➤ $a_{ijkl}^{\varepsilon}(\theta) = \lambda^{\varepsilon}(\theta)\delta_{ij}\delta_{kl} + \mu^{\varepsilon}(\theta)\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$ (the relaxation modulus)

➤ $E_{kl}^{\varepsilon} = \frac{1}{2}\left(\partial_l^{\varepsilon}u_k^{\varepsilon} + \partial_k^{\varepsilon}u_l^{\varepsilon} + \partial_k^{\varepsilon}u_m^{\varepsilon}\partial_l^{\varepsilon}u_m^{\varepsilon}\right)$ (the components of strain tensor)

Introduction

In 1910 **Theodore von Karman** introduced a system of "*two fourth order elliptic quasilinear partial differential equations*" which can be used to describe the large deflections and stresses produced in a thin elastic plate subjected to compressive forces along its edge. The most interesting phenomenon associated with this non linear situation is the appearance of "**buckling**", i.e. the plate may deflect out of its plane when these forces reach a certain magnitude. Mathematically this circumstance is expressed by the multiplicity of solutions of the boundary value problem associated with von Karman's equations.

The formulation of von Kármán equations is as follows: We consider a thin elastic body occuping a domain $\Omega \subset \mathbb{R}^2$, that is flat in its undeformed state subjected to a compressive force (of magnitude λ) acting on the boundary $\partial\Omega$ of Ω. Then the stresses produced in Ω, as measured by the **Airy stress** function, $f(x, y) + \lambda F_0(x, y)$ and the displacement $u(x, y)$ of the plate are defined by the following quasilinear elliptic system:

$$\begin{cases} \Delta^2 f = -\frac{1}{2}\left[u, u\right], \ \text{ in } \ \Omega \\ \Delta^2 u = \lambda\left[F_0, u\right] + \left[f, u\right], \ \text{ in } \ \Omega \\ u = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0 \ \text{ on } \ \partial\Omega, \\ f = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \ \text{ on } \ \partial\Omega, \end{cases}$$

where Δ^2 denotes the biharmonic operator and

$$[v, w] = \frac{\partial^2 v}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 v}{\partial y^2} - 2\frac{\partial^2 v}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y}.$$

Here $F_0(x,y)$ is the function obtained by solving an associated inhomogeneous linear problem, and is a measure of the stress produced in the undeflected plate if it were prevented from deflecting.

In the past decades many models of physics and mechanics have been derived and justified by the use of the asymptotic expansion method. Ciarlet-Rabier and Ciarlet-Paumier were successfully to justify the von Karman and Marguerre von-Karman equations in framework of nonlinear elasticity by the asymptotic methode [2]. Nevertheless, elasticity models cannot describe important mechanical phenomena such as hardening, memory or relaxation of the materials involved. For more details about the von Karman and Marguerre von Karman equations see [3].

Thus the objective of our work is the justification of the von Karman equations in the viscoelastic case using the mixed asymptotic expansions method.

The constitutive law of viscoelastic materials is of long memory type and can be expressed in the following form:

$$\sigma_{ij}(x,t) = a_{ijkl}(x,t)E_{kl}(x,0) + \int_0^t a_{ijkl}(x,t-\theta)\frac{\partial E(x,\theta)}{\partial \theta}\mathrm{d}\theta,$$

where $a_{ijkl}(x,t)$ are the relaxation functions, describe the mechanical properties of the material.

The thesis is divided into three chapters, each has a main focus and purpose.

The first chapter provides a broad definition of viscoelasticity as well as its fundamental equations. We also explain the study's aim.

In the second chapter, We introduce the geometry of the von karman plate with mentioning the types of forces applied on this plate, then we address the problem of the plate and give The elastodynamic viscoelastic 3D plate model von Karman. We then turn our problem into a formulation variationall.

The third chapter is the core of our thesis. We proceed our asymptotic analysis by posing the problem satisfied by the scaled solution on a fixed interval using appropriate specific scalings.

Chapter 1

Viscoelasticity: Introduction and Equations

The first chapter provides a broad definition of viscoelasticity as well as its fundamental equations. We also explain the study aim.

1.1 Introduction

A viscoelasticity is the property of materials that exhibit both viscous (dashpot-like) and elastic (spring like) characteristics when undergoing deformation. It combines aspects of both fluid mechanics (viscosity) and solid mechanics (elasticity) to describe the unique response of these materials to applied forces or deformations. In simple terms, viscoelastic materials possess the ability to deform under stress and also exhibit a time-dependent behavior. The viscoelastic behavior of materials arises from the internal molecular structure and interactions within the material. These interactions can involve various mechanisms, such as the sliding and reordering of molecular chains, the breaking and reformation of bonds, or the diffusion of molecules. These mechanisms contribute to the material's ability to store and dissipate energy under applied loads. The viscoelastic response of a material is typically characterized by several important parameters, including elastic modulus, viscous modulus, and time-dependent functions. The elastic modulus represents the material's resistance to deformation, while the viscous modulus characterizes its resistance to flow. The time-dependent functions, such as relaxation modulus and creep compliance, describe the material's behavior over time under constant stress or strain. Viscoelasticity finds applications in various fields, including engineering, biomechanics, polymer science, and materials engineering. Understanding the viscoelastic behavior of materials is crucial for designing and predicting the performance of structures and products subjected to dynamic or time-varying loads. Overall, the study of viscoelasticity provides insights into the complex mechanical properties of materials, helping researchers and engineers develop better materials and designs for a wide range of applications. in this work we study the viscoelasticity modeling of a von Karman plate involves considering the material behavior of the plate as both elastic and viscous. A von Karman plate refers to a thin plate with a moderate amount of curvature. The viscoelastic modeling takes into account the time-dependent deformation and stress relaxation of the material.

To describe the viscoelastic behavior of a von Karman plate, a constitutive equation is used that combines the elastic and viscous responses. One commonly used model for viscoelasticity is the Maxwell model.

In the context of a von Karman plate, the strain and stress tensors would be formulated to describe the plate's deformation and internal forces, respectively. The strain rate tensor captures the rate at which the strain changes over time.

By incorporating the Maxwell model into the governing equations for a von Karman plate, such as the Kirchhoff-Love equations, it is possible to simulate the time-dependent behavior of the plate under various loading conditions. These equations involve terms related to the plate's curvature, moments, and distributed loads.

It's worth noting that the viscoelastic behavior of a von Karman plate can be more complex than the simplified Maxwell model. Depending on the specific material properties and desired accuracy, other viscoelastic models such as the Kelvin-Voigt model or the generalized Maxwell model may be employed. These models can capture additional viscoelastic phenomena such as creep, stress relaxation, and frequency-dependent behavior.

The viscoelastic modeling of von Karman plates is a topic of ongoing research, and various numerical methods, such as finite element analysis, are commonly used to solve the governing equations and simulate the plate's behavior.

1.2 Governing Equations

1.2.1 Examples of Materials Exhibiting Viscoelastic Behavior

Synthetic polymers, wood, soil, human biological tissue (bone , tendons , ligaments , muscles and articular cartilage), plastics , and metals at elevated temperatures .

1.2.2 Properties of viscoelastic materials

Viscoelastic materials are those for which the relationship between stress and strain depends on time, and they possess the following three important properties:

◊ **Stress relaxation:** If the strain is held constant , the stress **decreases** with time.

◊ **Creep:** If the stress is held constant , the strain **increases** with time.

◊ **Hysteresis:** If cyclic loading is applied , a phase lag occurs , leading to a dissipation of mechanical energy.

1.2.3 Stress-Strain Constitutive law

The **stress-strain** relationship for a viscoelastic material is not unique but is a function of the time or the rate at which the stresses (σ) and strains (e) are developed in the material:

$$\sigma = \sigma(e, \dot{e}, ..., t), \text{ where } \dot{e} = \frac{de}{dt}.$$

There are several types of constitutive laws to describe the behavior of viscoelastic materials: Maxwell model , Kelvin-Voigt model, Burgers model,...,etc. The choice of model depends on the specific properties of the material being studied and the type of deformation being applied.

The strain behavior over time of a viscoelastic material is a function of the creep function and the stress , while the stress behavior over time is a function of the stress relaxation function and the strain . **Boltzmann** (1844-1906) first generalized these observations by saying that for a simple bar subject to a stress $\sigma(t)$, that the increment in stress over a small time interval dt would be:

$$\frac{d\sigma}{dt} = \frac{d\sigma}{d\tau}d\tau.$$

This assumes that the stress is continuous and differentiable in time . Given that the stress is related to the strain via the creep function , **Boltzmann** postulated that an increament of strain de , which depends on the complete stress history up to time t , would be related to the increment of stress $d\sigma$ at the specific time increment from τ to t through the creep function J at the time $t-\tau$ as:

$$de(t) = J(t-\tau)\frac{d\sigma(t)}{d\tau}d\tau.$$

The complete **strain** at a time t would then be obtained by integrating the strain increments from time 0 to time t, over all the increments dt:

$$e(t) = \int_0^t J(t-\tau)\frac{d\sigma(t)}{d\tau}d\tau. \tag{1.1}$$

We can also make the same argument for an increment of **stress** $d\sigma$ through time t depending on the increment of strain de at time t and the stress relaxation over the time $t-\tau$ as:

$$\sigma(t) = \int_0^t G(t-\tau)\frac{de(t)}{d\tau}d\tau. \tag{1.2}$$

The above constitutive relationships (1.1) and (1.2) can be generalized to three dimensions in tensorial form for a strain history from time $t=-\infty$ to t as:

$$\sigma_{ij}(x,t) = \int_{-\infty}^t G_{ijkl}(x,t-\tau)\frac{\partial e_{kl}(x,t)}{\partial\tau}d\tau \tag{1.3}$$

where G_{ijkl} is a tensorial stress relaxation function satisfying:

$$G_{ijkl} = G_{jikl} = G_{ijlk}.$$

We can write a similar relationship for **strain** in terms of a prescribed **stress history** and a tensorial **creep** function J_{ijkl} as:

$$e_{ij}(x,t) = \int_{-\infty}^{t} J_{ijkl}(x, t-\tau)\frac{\partial \sigma_{kl}(x,t)}{\partial \tau} d\tau \tag{1.4}$$

where the functions J_{ijkl} satisfy:

$$J_{ijkl} = J_{jikl} = J_{ijlk}.$$

If we know that the loading starts at $t = 0$ and the stress and strain are zero at this point , we may write the above constitutive relationships as:

$$\sigma_{ij}(x,t) = G_{ijkl}(x,t)e_{kl}(x,0) + \int_{0}^{t} G_{ijkl}(x, t-\tau)\frac{\partial e_{kl}(x,t)}{\partial \tau} d\tau$$

and

$$e_{ij}(x,t) = J_{ijkl}(x,t)\sigma_{kl}(x,0) + \int_{0}^{t} J_{ijkl}(x, t-\tau)\frac{\partial \sigma_{kl}(x,t)}{\partial \tau} d\tau$$

For a viscoelastic body , some of the strain energy is stored in the body as a potential energy and some of it is dissipated as heat. This dissipation is also known as hysteresis . In comparison, elastic materials do not exhibit energy dissipation or hysteresis . Indeed, the fact that all energy due to deformation is stored is a characteristic of elastic materials.

1.2.4 Motivations for studying viscoelasticity

Understanding and studying the behavior of viscoelasticity is important and of interest in a wide range of fields, including materials science, engineering, physics, and biology.
First, materials used for structural applications of practical interest may exhibit viscoelastic behavior which has a profound influence on the performance of that material . Materials used in engineering applications may exhibit viscoelastic behavior as an unintentional side effect . In applications, one may deliberately make use of the viscoelasticity of certain materials in the design process , to achieve a particular goal.

Second , the mathematics underlying viscoelasticity theory is of interest within the applied mathematics community.

Third , viscoelasticity is of interest in some branches of materials science , metallurgy , and solid-state physics since it is causally linked to a variety of microphysical processes and can be used as an experimental probe of those processes.

Fourth , the causal links between viscoelasticity and microstructure are exploited in the use of viscoelastic tests as an inspection tool.

Chapter 2

Modeling of Viscoelastic von Karman Plate

In the second chapter, We introduce the geometry of the von karman plate with mentioning the types of forces applied on this plate, then we address the problem of the plate and give The elastodynamic viscoelastic 3D plate model von Karman. We then turn our problem into a formulation variationnel.

2.1 Geometry of the plate

In this model , the plate is assumed to be thin , which means its thickness is small compared to its lateral dimensions . The geometry of the plate is typically described in terms of its mid-surface and the displacement field. The mid-surface is a two-dimensional surface that represents the undeformed shape of the plate.

To define the geometry of a viscoelastic von Karman plate , we need to consider the following parameters :

Mid-surface shape : The mid-surface of the plate is typically defined by a mathematical function that describes its shape in its undeformed state . Common choices include a flat plate , a cylindrical plate , or a spherical plate. The specific equation describing the mid-surface shape depends on the problem at hand.

Displacement field : The displacement field describes the deformation of the plate from its undeformed state . It is typically represented by a vector field that specifies the displacement of each point on the mid-surface in the plate's thickness direction. The displacement field is usually decomposed into in-plane displacements (in the plane of the mid-surface) and out-of-plane displacements (perpendicular to the mid-surface).

Plate dimensions : The dimensions of the plate include its length, width, and thickness. The length and width define the lateral dimensions of the plate, while the thickness represents the distance between the undeformed mid-surface and the deformed surface .

Boundary conditions : The boundary conditions specify how the plate is supported and constrained at its edges . Common boundary conditions include clamped edges (zero displacement and zero slope) , simply supported edges (zero displacement but non-zero slope) , and free edges (no constraint).

These parameters collectively define the geometry of a viscoelastic von Karman plate model. Once the geometry is established , the governing equations of motion and boundary conditions can be applied to solve for the plate's deformation and response under external load .

Let ω be a domain in the plane spanned by the vectors e_α. We denote by ν_α and τ_α the unit outer normal vector and unit tangent vector along the boundary γ of ω , related by $\tau_1 = -\nu_2, \tau_2 = \nu_1$. Given $\varepsilon > 0$, let

$$\begin{cases} \Omega = \omega \times]-\varepsilon, +\varepsilon[\\ \Gamma_0 = \gamma \times [-\varepsilon, +\varepsilon] \\ \Gamma_\pm = \omega \times \{\pm\varepsilon\} \end{cases} \tag{2.1}$$

where ω is a bounded domain of R^2 with a LIPSCHITZ-boundary, and ε is a small real parameter $(0 < \varepsilon \leq 1)$.

So that the boundary $\partial\Omega^\varepsilon$ of the set Ω^ε is partitioned into the lateral face $\gamma \times [-\varepsilon, \varepsilon]$ and the upper and lower faces Γ_+^ε and Γ_-^ε. Finally , we let (n_i^ε): $\partial\Omega^\varepsilon \to \mathbb{R}^3$, denotes the unit outer normal vector along $\partial\Omega^\varepsilon$; hence $(n_i^\varepsilon) = (\nu_1, \nu_1, 0)$ along the lateral face

$$\gamma \times [-\varepsilon, \varepsilon].$$

We assume that , for each $\varepsilon > 0$, the set Ω^ε is the reference configuration of a nonlinearly elastic plate , subjected to three kinds of applied forces:

➤ applied body forces acting in Ω^ε , of density (f_i^ε): $\Omega^\varepsilon \to \mathbb{R}^3$,

➤ applied surface forces acting on the upper and lower faces ,
of density (g_i^ε): $\Gamma_+^\varepsilon \bigcup \Gamma_-^\varepsilon \to \mathbb{R}^3$,

➤ applied surface forces parallel to the plane spanned by the vectors e_α acting on the lateral face $\gamma \times [-\varepsilon, \varepsilon]$, whose only the resultant density $(h_1^\varepsilon, h_2^\varepsilon, 0)$: $\gamma \to \mathbb{R}^3$ a per unit length, obtained by integration across the thickness, is known along the boundary Ω of the middle surface of the plate.

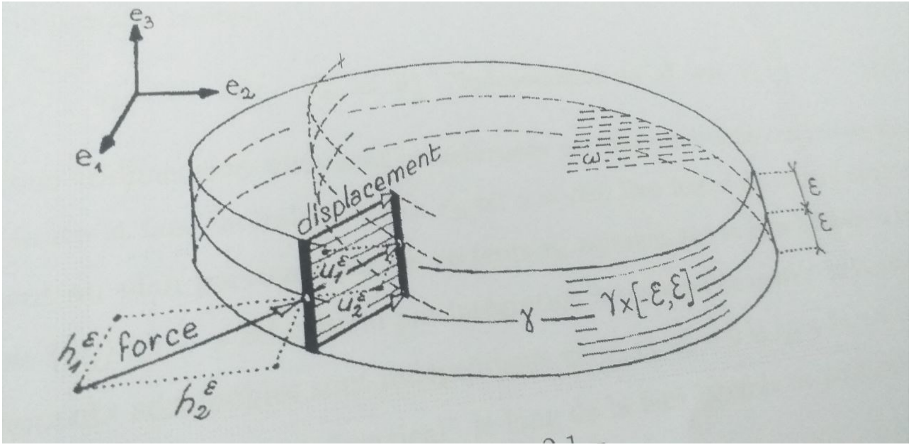

Figure 2.1: A von Kàrmàn plate

The three-dimensional equations are characterized by specific boundary conditions on the whole lateral face $\gamma \times [-\varepsilon, \varepsilon]$, where $\gamma = \partial\omega$. Applied surface forces parallel to the plane spanned by the vectors are acting on the lateral face through their resultant $(h^{\varepsilon}_{\alpha})= \gamma \to \mathbf{R^2}$ obtained by integration across the thickness of the plate . The admissible displacements u^{ε}_{α} are independent of x^{ε}_3 and $u^{\varepsilon}_3 = 0$ along $\gamma \times [-\varepsilon, \varepsilon]$ in other words, any "vertical" segment along the lateral face can only undergo "horizontal" translations.
Finally, all applied forces are "vertical",$i.e.$, $f^{\varepsilon}_{\alpha} = 0$ and $g^{\varepsilon}_{\alpha} = 0$.
The coefficients are defined as follows:
Let $a^{\varepsilon}_{ijkl}(t-s)$ be bounded functions . these functions satisfy the following conditions:

$$\begin{cases} a^{\varepsilon}_{ijkl}(y) = a^{\varepsilon}_{jikl}(y) = a^{\varepsilon}_{klij}(y) \;\; a.e \;\; y \in \Omega^{\varepsilon} \\ \exists m > 0 \;\text{ such that }\; \forall \tau = (\tau^{\varepsilon}_{ij}), \tau^{\varepsilon}_{ij} = \tau^{\varepsilon}_{ji} \\ m\; \tau^{\varepsilon}_{ij}\tau^{\varepsilon}_{ij} \leq a_{ijkl}(y)\tau^{\varepsilon}_{ij}\tau_{kl} \;\; a.e \;\; y \in \Omega^{\varepsilon} \\ \exists M > 0 \;\text{ such that }\; M = sup \;\; a^{\varepsilon}_{ijkl}(y), y \in \Omega, \;\; (k,h = 1,2,3). \end{cases} \tag{2.2}$$

2.2 Problem of the Plate

The elastodynamic viscoelastic 3D plate model von Karman is a mathematical model used to describe the behavior of viscoelastic plates subjected to dynamic loading . It combines the principles of elastodynamics , which describes the motion of elastic materials under dynamic loads, with the viscoelastic behavior of the material. The model is based on the von Karman plate theory, it considers the plate to be thin and assumes that the displacements are small compared to the plate dimensions. The viscoelastic behavior is incorporated into the model by introducing a time-dependent constitutive equation that describes the stress-strain relationship of the material. This constitutive equation includes both elastic and viscous components , allowing the material to exhibit time-dependent and rate-dependent behavior. We use in the following the conventions and notations : Greek indices , belong to the set $\{1,2\}$ while Latin indices belong to the set $\{1,2,3\}$. δ_{ij} is the Kronecker symbols , the symbols of differentiation $\partial_i^\varepsilon = \dfrac{\partial}{\partial x_i^\varepsilon}, \partial_i = \dfrac{\partial}{\partial x_i}$.

For each $\varepsilon > 0$, the plate is subjected to thereee kinds of applied forces:

- Body forces acting on its interior, of density
  $(f_i^\varepsilon) : \Omega^\varepsilon \times]0,\infty[\longrightarrow \mathbb{R}^3$.

- Surface forces acting on its upper and lower faces, of density
  $(g_i^\varepsilon) : \left(\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon\right) \times]0,\infty[\longrightarrow \mathbb{R}^3$.

- Horizontal forces of von Karman type acting on its lateral boundary , of density $(h_1^\varepsilon, h_2^\varepsilon, 0) : \partial\omega \times]0,\infty[\longrightarrow \mathbb{R}^2$, such that
  $h_\alpha^\varepsilon = \frac{1}{2\varepsilon} \displaystyle\int_{-\varepsilon}^{\varepsilon} \sum_{\alpha\beta}^{\varepsilon} \nu_\beta dx_3^\varepsilon$.

The ELASTODYNAMIC VISCOELASTIC 3D plate model with von Karman conditions is formulated as follows:

$$
(\mathrm{P}^\varepsilon)\begin{cases}
\begin{array}{l}
\text{Find } u^\varepsilon(x^\varepsilon,t) : \Omega^\varepsilon\times]0,\infty[\longrightarrow \mathbb{R}^3 \text{ solution of} \\
\qquad \rho^\varepsilon \dfrac{\partial^2 u_i^\varepsilon}{\partial t^2} - \partial_j^\varepsilon \sum_{ij}^\varepsilon = f_i^\varepsilon \ \text{ in } \Omega^\varepsilon\times]0,\infty[
\end{array} \\
\begin{cases}
u_1^\varepsilon, u_2^\varepsilon \text{ independent of } x_3^\varepsilon \text{ and } u_3^\varepsilon = 0 \text{ on } \Gamma_0^\varepsilon\times]0,\infty[\\
\frac{1}{2\varepsilon}\displaystyle\int_{-\varepsilon}^{\varepsilon} \Sigma_{\alpha\beta}^\varepsilon \nu_\beta dx_3^\varepsilon = h_\alpha^\varepsilon \ \text{ on } \ \partial\omega\times]0,\infty[
\end{cases} \\
\Sigma_{ij}^\varepsilon n_j^\varepsilon = g_i^\varepsilon \text{ on } \ \left(\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon\right)\times]0,\infty[\\
u^\varepsilon(x^\varepsilon,0) = \dfrac{\partial u^\varepsilon(x^\varepsilon,0)}{\partial t} = 0 \ \text{ in } \ \Omega^\varepsilon
\end{cases} \tag{2.3}
$$

where

$$
\begin{cases}
\rho^\varepsilon : \text{the mass density} \\
\sum_{ij}^\varepsilon = \sigma_{ij}^\varepsilon + \sigma_{kj}^\varepsilon \partial_k^\varepsilon u_i^\varepsilon \\
\sigma_{ij}^\varepsilon(x^\varepsilon,t) : \text{the components of stress tensor} \\
\sigma_{ij}^\varepsilon(x^\varepsilon,t) = \displaystyle\int_0^t a_{ijkl}^\varepsilon(t-s)\frac{\partial E_{kl}^\varepsilon(u^\varepsilon(x^\varepsilon,s))}{\partial s} ds = a_{ijkl}^\varepsilon * \frac{\partial E_{kl}^\varepsilon}{\partial s} \\
E_{kl}^\varepsilon : \text{the components of strain tensor,} \\
E_{kl}^\varepsilon = \frac{1}{2}\left(\partial_l^\varepsilon u_k^\varepsilon + \partial_k^\varepsilon u_l^\varepsilon + \partial_k^\varepsilon u_m^\varepsilon \partial_l^\varepsilon u_m^\varepsilon\right)
\end{cases}
$$

2.3 The mixed variational problem

The aim of this section is to determine an appropriate weak formulation to our problem . To archieve this objective , we will give some preliminaries and define some spaces needed to carry on our study.

2.3.1 Preliminaries

Now, We define the spaces

$$\mathbf{V}(\Omega^\varepsilon) : \left\{ v^\varepsilon = (v_i^\varepsilon) : \ v_i^\varepsilon \in W^{1,4}(\Omega^\varepsilon), \ \ v_\alpha^\varepsilon|_{\Gamma_0^\varepsilon} \text{ independent of } x_3^\varepsilon \text{, and } \ v_3^\varepsilon|_{\Gamma_0^\varepsilon} = 0 \right\}, \tag{2.4}$$

$$\mathbf{H}(\Omega^\varepsilon) = \left\{ \tau = (\tau_{ij}) : \tau_{ij}^\varepsilon \in L^2(\Omega^\varepsilon), \ \tau_{ji}^\varepsilon = \tau_{ij}^\varepsilon \right\}. \tag{2.5}$$

Theorem 2.1 (Green's Integration by Parts Formula) *Let Ω be a bounded open domain in $\mathbb{R}^3$ with a sufficiently smooth boundary Γ and $\mathbf{n}$ is the outward normal. Then for all $u, v \in \mathcal{C}^1(\overline{\Omega})$*

$$\int_\Omega \partial_i u(x) v(x) dx = -\int_\Omega u(x) \partial_i v(x) dx + \int_\Gamma u(x) v(x) n_i d\Gamma.$$

Proof. See ([7]). ■

2.3.2 Weak Formulations

To avoid complexity , we will derive the weak formulation of each equation in the boundary value problem (2.3.2) , then obtain the weak formulation of the whole problem . We suppose the data of the problem satisfy the regularity conditions stated in the previous subsection. Multiplying equation $(2.2)_1$ by a test-function $\mathbf{v}^\varepsilon \in \mathbb{V}^\varepsilon(\Omega^\varepsilon)$ and integrating on Ω^ε gives

$$\int_{\Omega^\varepsilon} \left(\rho^\varepsilon \frac{\partial^2 u_i^\varepsilon}{\partial t^2}(x^\varepsilon,t) v_i^\varepsilon dx^\varepsilon\right) - \int_{\Omega^\varepsilon} \partial_j^\varepsilon \Sigma_{ij}^\varepsilon v_i^\varepsilon dx^\varepsilon = \int_{\Omega^\varepsilon} f_i^\varepsilon v_i^\varepsilon dx^\varepsilon . (2.6)$$

By applying GREEN's formula on the integral equation (2.6) and using the appropriate boundary conditions we find, for all $\mathbf{v}^\varepsilon \in \mathbb{V}^\varepsilon(\Omega^\varepsilon)$ and $t \in]0,T[$

$$\int_{\Omega^\varepsilon} \rho^\varepsilon \frac{\partial^2 u_i^\varepsilon}{\partial t^2}(x^\varepsilon,t) v_i^\varepsilon dx^\varepsilon - \int_{\partial\Omega^\varepsilon} \Sigma_{ij}^\varepsilon n_j v_i^\varepsilon dx^\varepsilon + \int_{\Omega^\varepsilon} \Sigma_{ij}^\varepsilon \partial_j^\varepsilon v_i^\varepsilon dx^\varepsilon = \int_{\Omega^\varepsilon} f_i^\varepsilon v_i^\varepsilon dx^\varepsilon \tag{2.7}$$

$$\begin{cases} \int_{\Omega^\varepsilon} \Sigma_{ij}^\varepsilon n_j v_i^\varepsilon dx^\varepsilon = \int_{\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon} g_i^\varepsilon v_i^\varepsilon d\Gamma + \int_{\Gamma_0^\varepsilon} \Sigma_{\alpha\beta}^\varepsilon \eta_\beta v_\alpha^\varepsilon d\Gamma \\ \\ = \int_{\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon} g_i^\varepsilon v_i^\varepsilon d\Gamma + \int_\gamma \left(\Sigma_{\alpha\beta}^\varepsilon n_\beta dx_3^\varepsilon\right) v_\alpha^\varepsilon dx_1 dx_2 \\ \\ = \int_{\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon} g_i^\varepsilon v_i^\varepsilon d\Gamma + \frac{1}{2}\int_{-\varepsilon}^{\varepsilon} h_\alpha^\varepsilon v_\alpha^\varepsilon d\gamma \end{cases}$$

We can rewrite this equation in the form :

$$A^\varepsilon\left(u^\varepsilon(x,t); v^\varepsilon\right) = L^\varepsilon\left(v^\varepsilon\right). \tag{2.8}$$

Where A^ε and L^ε are given by

$$A^\varepsilon(u^\varepsilon(x,t); v^\varepsilon) = \frac{d^2}{dt^2}\left\{\rho^\varepsilon \int_{\Omega^\varepsilon} u_i^\varepsilon v_i^\varepsilon dx^\varepsilon\right\} + \int_{\Omega^\varepsilon} \Sigma_{i,j}^3 \frac{\partial v_i^\varepsilon}{\partial x_j^\varepsilon} dx^\varepsilon \tag{2.9}$$

$$L^\varepsilon(v^\varepsilon) = \int_{\Omega^\varepsilon} f_i^\varepsilon v_i^\varepsilon dx^\varepsilon + \int_{\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon} g_i^\varepsilon v_i^\varepsilon d\Gamma^\varepsilon + \int_{\partial\omega}\left(\int_{-\varepsilon}^{\varepsilon} v_\alpha^\varepsilon dx_3^\varepsilon\right) h_\alpha^\varepsilon d\gamma \tag{2.10}$$

$$E_{ij}^\varepsilon = b_{ijkl}^\varepsilon * \frac{\partial \sigma_{kl}^\varepsilon}{\partial \theta} = \int_0^t b_{ijkl}^\varepsilon(t.\theta) \frac{\partial \sigma_{kl}^\varepsilon}{\partial \theta} d\theta$$

where

$\mathrm{b}^\varepsilon = (b_{ijkl}^\varepsilon)$ is the inverse of $a^\varepsilon = (a_{ijkl}^\varepsilon)$.(2.11)

Multiplying equation (2.3.2) by a test-function $\tau_{ij}^{\varepsilon} \in H\left(\Omega^{\varepsilon}\right)$ and integrating over Ω^{ε} gives :

$$\int_{\Omega^{\varepsilon}} E_{ij}^{\varepsilon} \tau_{ij}^{\varepsilon} dx^{\varepsilon} = \int_{\Omega^{\varepsilon}} \left[\int_0^t b_{ijkl}^{\varepsilon}(t-s) \left(\frac{\partial \sigma_{kl}^{\varepsilon}}{\partial s}(x^{\varepsilon}, s) \right) ds \right] \tau_{ij}^{\varepsilon} dx^{\varepsilon} \tag{2.12}$$

$$\int_{\Omega^{\varepsilon}} E_{ij}^{\varepsilon} \tau_{ij}^{\varepsilon} dx^{\varepsilon} - \int_{\Omega^{\varepsilon}} \left[\int_0^t b_{ijkl}^{\varepsilon}(t-s) \left(\frac{\partial \sigma_{kl}^{\varepsilon}(x^{\varepsilon}, s)}{\partial s} \right) ds \right] \tau_{ij}^{\varepsilon} dx^{\varepsilon} = 0, \tag{2.13}$$

$$\forall \tau^{\varepsilon} = (\tau_{ij}^{\varepsilon}) \in \mathbf{H}(\Omega^{\varepsilon}) \ \ , u^{\varepsilon}(x^{\varepsilon}, 0) = \frac{\partial u^{\varepsilon}(x^{\varepsilon}, 0)}{\partial t} = 0 \ \text{ in } \ \Omega^{\varepsilon}. \tag{2.14}$$

We formulate the problem (P^ε) as a time-dependent variational mixed problem

$$
\left\{
\begin{array}{l}
\text{Find } (u^\varepsilon(.,t), \sigma^\varepsilon(.,t)) \in \mathbf{V}(\Omega^\varepsilon) \times \mathbf{H}(\Omega^\varepsilon) \text{ such that } \forall t \geq 0 : \\[2ex]
\dfrac{d^2}{dt^2}\left\{\rho^\varepsilon \displaystyle\int_{\Omega^\varepsilon} u_i^\varepsilon v_i^\varepsilon dx^\varepsilon\right\} + \displaystyle\int_{\Omega^\varepsilon} \sum_{i,j}^{3} \dfrac{\partial v_i^\varepsilon}{\partial x_j^\varepsilon} dx^\varepsilon = \displaystyle\int_{\Omega^\varepsilon} f_i^\varepsilon v_i^\varepsilon dx^\varepsilon + \displaystyle\int_{\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon} g_i^\varepsilon v_i^\varepsilon d\Gamma^\varepsilon \\[2ex]
\qquad + \displaystyle\int_{\partial\omega} \left(\int_{-\varepsilon}^{\varepsilon} v_\alpha^\varepsilon dx_3^\varepsilon\right) h_\alpha^\varepsilon d\gamma \ \ , \forall v^\varepsilon \in \mathbf{V}(\Omega^\varepsilon) \\[2ex]
\displaystyle\int_{\Omega^\varepsilon} E_{ij}^\varepsilon \tau_{ij}^\varepsilon dx^\varepsilon - \int_{\Omega^\varepsilon} \left[\int_0^t b_{ijkl}^\varepsilon(t-s) \left(\dfrac{\partial \sigma_{kl}^\varepsilon(x^\varepsilon, s)}{\partial s}\right) ds\right] \tau_{ij}^\varepsilon dx^\varepsilon = 0, \\[2ex]
\forall \tau = (\tau_{ij}) \in \mathbf{H}(\Omega^\varepsilon) \ \ , u^\varepsilon(x^\varepsilon, 0) = \dfrac{\partial u^\varepsilon(x^\varepsilon, 0)}{\partial t} = 0 \quad \text{in } \Omega^\varepsilon.
\end{array}
\right.
$$

■

Chapter 3

Asymptotic Analysis

In this chapter, we proceed our asymptotic analysis by posing the problem satisfied by the scaled solution on a fixed interval using appropriate specific scalings.

3.1 The Scaled Three-Dimensional Problem

In this section, we use the method called "asymptotic analysis" developed by CIARLET & DESTUYNDER [5], it involves three steps: first we fix our domain by getting rid of the thickness parameter ε, then we give appropriate scaling assumptions on the data, and scale the unknown $u^\varepsilon(x^\varepsilon, t)$ by defining a new unknown $u(\varepsilon)(x,t)$ on the fixed domain found in the first step, finally we transform our weak problem (2.3.2) into a "scaled" weak problem posed on the fixed domain.

3.1.1 The Fixed Domain

Recall that $\Omega^\varepsilon = \omega\times]-\varepsilon, \varepsilon[$, and $0 < \varepsilon \leq 1$. Since the unknowns $(u^\varepsilon, \sigma^\varepsilon)$ are defined over the set $\Omega^\varepsilon \times [0,T]$ which depends on ε, we want to transform our problem into a problem that is established on a fixed domain that does not depend on ε. For that, we let

$$\Omega = \omega\times]-1,1[,$$
$$\Gamma_\pm = \omega \times \{\pm 1\},$$
$$\Gamma_0 = \partial\omega \times \,]-1,1[\,.$$

Let $x = (x_1, x_2, x_3)$ denotes a current point in Ω . For each point $x \in \Omega$, we associate the point $x^\varepsilon \in \Omega^\varepsilon$ with the bijective mapping

$$\pi^\varepsilon : \Omega \to \Omega^\varepsilon$$
$$x \mapsto x^\varepsilon,$$

where $x^\varepsilon = (x_1, x_2, \varepsilon x_3)$. Note that

$$\partial^\varepsilon_\alpha = \partial_\alpha, \qquad \partial^\varepsilon_3 = \frac{1}{\varepsilon}\partial_3. \tag{3.1}$$

With the space $\mathbb{V}^\varepsilon$, $\mathbb{H}(\Omega^\varepsilon)$, we associate the spaces

$$\begin{cases} \mathbb{V} = \{v \in W^{1,4}(\Omega), v_\alpha|_{\Gamma_0} \text{ is independent of } x_3, \text{ and } v_3|_{\Gamma_0} = 0\,\}, \\ \mathbb{H}(\Omega) = \{\tau = (\tau_{ij}) : \tau_{ij} \in L^2(\Omega),\ \tau_{ji} = \tau_{ij}\}\,. \end{cases}$$

3.1.2 Scaling of the Data and the Unknowns

Before finding the scaled weak formulation, which will be denoted (3.2) , we first need to decide how will the data, the unknowns , and the test-functions , in the appropriate spaces, be mapped over the set Ω . Also , we should control the way the material coefficients depend on ε.

First we start with the data . We scale the applied forces as in [2]

$$
\begin{aligned}
f_\alpha^\varepsilon\left(x^\varepsilon, t\right)=0 \quad, f_3^\varepsilon\left(x^\varepsilon, t\right)=\varepsilon^3 f_3(x, t), \quad x \in \Omega, \\
g_\alpha^\varepsilon\left(x^\varepsilon, t\right)=0 \quad, g_3^\varepsilon\left(x^\varepsilon, t\right)=\varepsilon^4 g_3(x, t), \quad x \in \Gamma_{ \pm}. \\
h_\alpha^\varepsilon\left(x^\varepsilon, t\right)=\varepsilon^2 h_\alpha(x, t).
\end{aligned}
$$

For the unknowns , the scaling of the mechanical displacement and the stress:

$$
\begin{aligned}
u_\alpha^\varepsilon\left(x^\varepsilon, t\right)=\varepsilon^2 u_\alpha(\varepsilon)(x, t), \quad u_3^\varepsilon\left(x^\varepsilon, t\right)=\varepsilon u_3(\varepsilon)(x, t), \quad x \in \Omega, \\
\sigma_{\alpha \beta}^\varepsilon\left(x^\varepsilon, t\right)=\varepsilon^2 \sigma_{\alpha \beta}(\varepsilon)(x, t), \quad \sigma_{\alpha 3}^\varepsilon\left(x^\varepsilon, t\right)=\varepsilon^3 \sigma_{\alpha 3}(\varepsilon)(x, t), \\
\sigma_{33}^\varepsilon\left(x^\varepsilon, t\right)=\varepsilon^4 \sigma_{33}(\varepsilon)(x, t),
\end{aligned}
$$

The test-functions are scaled in a similar way but independent of t

$$
\begin{aligned}
v_\alpha^\varepsilon\left(x^\varepsilon\right)=\varepsilon^2 v_\alpha(x), \quad v_3^\varepsilon\left(x^\varepsilon\right)=\varepsilon v_3(x), \quad x \in \Omega, \\
\tau_{\alpha \beta}^\varepsilon(x^\varepsilon)=\varepsilon^2 \tau_{\alpha \beta}(x), \; \tau_{\alpha 3}^\varepsilon(x^\varepsilon)=\varepsilon^3 \tau_{\alpha 3}(x) \; \tau_{33}^\varepsilon(x^\varepsilon)=\varepsilon^4 \tau_{33}(x) \quad x \in \Omega.
\end{aligned}
$$

From now on, we will be writing $\mathbf{u}(\varepsilon), \sigma(\varepsilon)$, instead of $\mathbf{u}(\varepsilon)(x, t), \sigma(\varepsilon)(x, t)$ respectively . Similarly , we write $u(\varepsilon)$ instead of $u(\varepsilon)(t)$. Therefore , the test-functions $V^\varepsilon=\left(v_i^\varepsilon\right)$ and the unknown $u(\varepsilon)(t)=(\mathbf{u}(\varepsilon), \sigma(\varepsilon))$ belong to the space $\mathbf{V}(\Omega) \times \mathbb{H}(\Omega)$. We suppose that the material coefficients remain the same, meaning that they are already independent of ε.

We assume that the mass density ρ^{ε} takes the following scalings:

$$\rho^{\varepsilon} = \varepsilon^{2}\rho.$$

3.1.3 The Scaled Weak Formulation

After fixing the domain and scaling the data and the unknowns, we are now in a position to pose the weak formulation on the fixed domain Ω. For any test-function $v \in \mathbb{V}(\Omega)$.

The following theorem gives the scaled weak formulation (3.2) equivalent to the weak formulation

The scaled weak formulation satisfies the following variational problem:

$$\begin{cases} \text{Find } \ u(\varepsilon)(x), \sigma(\varepsilon)) \in \mathbb{V}(\Omega) \times \mathbb{H}(\Omega), \forall t \in \,]0,T\,[\ \text{ such that:} \\ \\ A(\varepsilon)\left(u(\varepsilon)(x); v\right) = L(\varepsilon)\left(v\right), \forall v \in \mathbb{V}(\Omega), \forall t \in]0,T[, \\ \\ \mathbf{u}(\varepsilon)(x,0) = 0, \ \dot{\mathbf{u}}(\varepsilon)(x,0) = 0, \quad \text{in } \ \Omega, \end{cases} \tag{3.2}$$

where

$$\begin{aligned} A(\varepsilon)\left(u(\varepsilon); v\right) = \frac{d^2}{dt^2}[\rho \int_{\Omega} u_3(x,t) v_3 dx] + \int_{\Omega} \sigma_{ij}(x,t)[\partial_j v_i \\ + \partial_i u_3(\varepsilon)\partial_j v_3] dx + \varepsilon^2 \left\{ \frac{d^2}{dt^2}[\rho \int_{\Omega} u_\alpha(\varepsilon) v_\alpha dx] \right. \\ \left. + \int_{\Omega} \sigma_{ij}(x,t)[\partial_i u_\alpha(\varepsilon)\partial_j v_\alpha(\varepsilon)] dx \right\} \end{aligned}$$

$$\begin{aligned} L(\varepsilon)\left(\mathcal{V}\right) = \int_{\Omega} f_3 v_3 dx + \int_{\Gamma_+ \cup \Gamma_-} g_3 v_3 d\Gamma \\ + \int_{\partial\omega} (\int_{-1}^{1} \nu_\alpha dx_3) h_\alpha d\gamma \ \ \forall v \in \mathbb{V}(\Omega) \end{aligned}$$

Proof. Proving this theorem goes through scaling our weak formulation (2.3.2) using the scaling assumptions we mentioned above. To do this, we have to scale all the terms of A^ε and L^ε. The calculations are long but simple, so we will work only on some of the terms to build the idea. We will scale the terms $(\mathbf{f}^\varepsilon, \mathbf{v}^\varepsilon)$ and $(\mathbf{g}^\varepsilon, \mathbf{v}^\varepsilon)_{\Gamma^\varepsilon_\pm}$ and $(\mathbf{h}^\varepsilon_\alpha, \mathbf{v}^\varepsilon)$ in L^ε and the terms $(\rho^\varepsilon \mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon)$ and $\Sigma^\varepsilon_{ij}(\sigma^\varepsilon, \mathbf{v}^\varepsilon)$ in A^ε.

• Scaling the terms of L^ε:

➤ The scalings on the data f^ε_i and the functions v^ε_i, give

$$(\mathbf{f}^\varepsilon, \mathbf{v}^\varepsilon) = \int\limits_{\Omega^\varepsilon} f^\varepsilon_i v^\varepsilon_i dx^\varepsilon = \int\limits_{\Omega^\varepsilon} \left(f^\varepsilon_\alpha v^\varepsilon_\alpha + f^\varepsilon_3 v^\varepsilon_3\right) dx^\varepsilon$$

➤ The scalings on the data g^ε_i and the functions v^ε_i, give:

$$(\mathbf{g}^\varepsilon, \mathbf{v}^\varepsilon)_{\Gamma^\varepsilon_+ \cup \Gamma^\varepsilon_-} = \int\limits_{\Gamma^\varepsilon_+ \cup \Gamma^\varepsilon_-} g^\varepsilon_i v^\varepsilon_i d\Gamma^\varepsilon =$$

$$\int\limits_{\Gamma^\varepsilon_+ \cup \Gamma^\varepsilon_-} \left(g^\varepsilon_\alpha v^\varepsilon_\alpha + g^\varepsilon_3 v^\varepsilon_3\right) d\Gamma^\varepsilon$$

➤ The scalings on the data h^ε_α and the functions v^ε_α, give:

$$\int_\gamma h^\varepsilon_\alpha \left(\int_{-\varepsilon}^{\varepsilon} v_\alpha(x^\varepsilon_1, x^\varepsilon_2, x^\varepsilon_3) dx^\varepsilon_3 \right) d\gamma = \varepsilon^5 \int_\gamma h_\alpha \left(\int_{-1}^{1} v_\alpha(x_1, x_2, x_3) dx_3 \right) d\gamma$$

• Scaling the terms of A^ε:

➤ scalings on ρ^ε, the mechanical displacement u_i^ε, and the functions v_i^ε :

$$\int_{\Omega^\varepsilon} \rho^\varepsilon \frac{\partial^2 u_i^\varepsilon}{\partial t^2} v_i^\varepsilon dx^\varepsilon = \int_{\Omega^\varepsilon} \Big(\rho^\varepsilon \frac{\partial^2 u_\alpha^\varepsilon}{\partial t^2} v_\alpha^\varepsilon + \int_{\Omega^\varepsilon} \rho^\varepsilon \frac{\partial^2 u_3^\varepsilon}{\partial t^2} v_3^\varepsilon) dx^\varepsilon$$
$$= \varepsilon^7 \int_\Omega \rho \frac{\partial^2 u_\alpha(\varepsilon)}{\partial t^2} v_\alpha(x) dx + \varepsilon^5 \int_\Omega \rho \frac{\partial^2 u_3(\varepsilon)}{\partial t^2} v_3(x) dx$$

➤ scalings on Σ_{ij}^ε the functions v_i^ε :

$$\int_{\Omega^\varepsilon} \Sigma_{ij}^\varepsilon \frac{\partial v_i^\varepsilon}{\partial x_j^\varepsilon} dx^\varepsilon = \int_{\Omega^\varepsilon} \sigma_{ij}^\varepsilon \frac{\partial v_i^\varepsilon}{\partial x_j^\varepsilon} dx^\varepsilon + \int_{\Omega^\varepsilon} \sigma_{kj}^\varepsilon \partial_k^\varepsilon u_i^\varepsilon \frac{\partial v_i^\varepsilon}{\partial x_j^\varepsilon} dx^\varepsilon$$

we have :

$$\star \int_{\Omega^\varepsilon} \sigma_{ij}^\varepsilon \frac{\partial v_i^\varepsilon}{\partial x_j^\varepsilon} dx^\varepsilon = \varepsilon \int_\Omega \Big[\varepsilon^2 \sigma_{\alpha\beta}(\varepsilon) \varepsilon^2 \frac{\partial v_\alpha}{\partial x_\beta} + \varepsilon^3 \sigma_{\alpha 3}(\varepsilon) \varepsilon \frac{\partial v_\alpha}{\partial x_3} + \varepsilon^3 \sigma_{3\beta}(\varepsilon) \varepsilon \frac{\partial v_3}{\partial x_\beta}$$
$$+ \varepsilon^4 \sigma_{33}(\varepsilon) \frac{\varepsilon}{\varepsilon} \frac{\partial v_3}{\partial x_3} \Big] dx = \varepsilon^5 \int_\Omega \sigma_{ij}(\varepsilon) \frac{\partial v_i}{\partial x_j} dx$$

$$\star \int_{\Omega^\varepsilon} \sigma_{kj}^\varepsilon \partial_k^\varepsilon u_i^\varepsilon \frac{\partial v_i^\varepsilon}{\partial x_j^\varepsilon} dx^\varepsilon = \int_{\Omega^\varepsilon} \sigma_{\alpha j}^\varepsilon \partial_\alpha^\varepsilon u_i^\varepsilon \partial_j^\varepsilon v_i^\varepsilon + \int_{\Omega^\varepsilon} \sigma_{3j}^\varepsilon \partial_3^\varepsilon u_i^\varepsilon \partial_j^\varepsilon v_i^\varepsilon$$
$$= \varepsilon^7 [\int_\Omega \sigma_{\alpha\gamma}(\varepsilon) \partial_\alpha u_\beta(\varepsilon) \partial_\gamma v_\beta + \sigma_{\alpha 3}(\varepsilon) \partial_\alpha u_\beta(\varepsilon) \partial_3 v_\beta$$
$$+ \sigma_{3\gamma}(\varepsilon) \partial_3 u_\alpha(\varepsilon) \partial_\gamma v_\alpha + \sigma_{33}(\varepsilon) \partial_3 u_\alpha(\varepsilon) \partial_3 v_\alpha] dx$$
$$+ \varepsilon^5 [\int_\Omega \sigma_{\alpha\gamma}(\varepsilon) \partial_\alpha u_3(\varepsilon) \partial_\gamma v_3 + \sigma_{\alpha 3}(\varepsilon) \partial_\alpha u_3(\varepsilon) \partial_3 v_3$$
$$+ \sigma_{3\gamma}(\varepsilon) \partial_3 u_3(\varepsilon) \partial_\gamma v_3 + \sigma_{33}(\varepsilon) \partial_3 u_3(\varepsilon) \partial_3 v_3] dx$$

[1]. ■

[1]Voir APPENDIX 1

• Scaling the terms of $\mathbf{E}^{\varepsilon}_{ij}\left(\mathbf{u}^{\varepsilon}(x^{\varepsilon},t)\right)$

$$\begin{aligned}\mathbf{E}^{\varepsilon}_{ij}(\mathbf{u}^{\varepsilon}(x^{\varepsilon},t)) &= \int_0^t b^{\varepsilon}_{ijkl}(x^{\varepsilon},t-\theta)\frac{\partial\sigma^{\varepsilon}}{\partial\theta}(x^{\varepsilon},\theta)d\theta = b^{\varepsilon}_{ijkl} * \frac{\partial\sigma^{\varepsilon}_{kl}}{\partial\theta}\\ &= e^{\varepsilon}_{ij}(u^{\varepsilon}) + \frac{1}{2}\partial^{\varepsilon}_i u^{\varepsilon}_m \partial^{\varepsilon}_j u^{\varepsilon}_m\end{aligned}$$

$$\int_{\Omega^{\varepsilon}} E^{\varepsilon}_{ij}\tau^{\varepsilon}_{ij}dx^{\varepsilon} - \int_{\Omega^{\varepsilon}}\Big(b^{\varepsilon}_{ijkl} * \frac{\partial\sigma^{\varepsilon}_{kl}}{\partial\theta}\Big)\tau^{\varepsilon}_{ij}dx^{\varepsilon} = 0$$

$$\begin{cases}\mathbf{E}^{\varepsilon}_{\alpha\beta} \rightsquigarrow \varepsilon^2\left[e_{\alpha\beta}(u(\varepsilon)) + \frac{1}{2}\partial_{\alpha}u_3\partial_{\beta}u_3\right] + \varepsilon^4\frac{1}{2}\partial_{\alpha}u_{\gamma}\partial_{\beta}u_{\gamma}\\ \qquad = \varepsilon^2\gamma_{\alpha\beta} + \varepsilon^4 I_{\alpha\beta}\\ \mathbf{E}^{\varepsilon}_{\alpha3} \rightsquigarrow \varepsilon\left[e_{\alpha3}(u(\varepsilon)) + \frac{1}{2}\partial_{\alpha}u_3\partial_3u_3\right] + \varepsilon^3\frac{1}{2}\partial_{\alpha}u_{\gamma}\partial_3u_{\gamma}\\ \qquad = \varepsilon\gamma_{\alpha3} + \varepsilon^3 I_{\alpha3}\\ \mathbf{E}^{\varepsilon}_{33} \rightsquigarrow \varepsilon\left[e_{33}(u(\varepsilon)) + \frac{1}{2}\partial_{\alpha}u_3(\partial_3u_3)^2\right] + \varepsilon^2\frac{1}{2}\partial_3u_{\gamma}\partial_3u_{\gamma}\\ \qquad = \gamma_{33} + \varepsilon^2 I_{33}\end{cases} \tag{3.3}$$

we define:

$$\begin{cases} A = \int_{\Omega^{\varepsilon}} E^{\varepsilon}_{ij}\tau^{\varepsilon}_{ij}dx^{\varepsilon}\\ B = \int_{\Omega^{\varepsilon}}\left(b^{\varepsilon}_{ijkl} * \frac{\partial\sigma^{\varepsilon}_{kl}}{\partial\theta}\right)\tau^{\varepsilon}_{ij}dx^{\varepsilon}\end{cases} \tag{3.4}$$

where :

$$\mathbf{A} = \varepsilon^3 \int_\Omega (\varepsilon^2\gamma_{\alpha\beta} + \varepsilon^4 I_{\alpha\beta})\tau_{\alpha\beta}dx + 2\varepsilon^4 \int_\Omega (\varepsilon\gamma_{\alpha 3} + \varepsilon^3 I_{\alpha 3})\tau_{\alpha 3}dx$$

$$+\varepsilon^5 \int_\Omega (\gamma_{33} + \varepsilon^2 I_{33})\,\tau_{33}dx = \varepsilon^5 \int_\Omega \gamma_{ij}\tau_{ij}dx + \varepsilon^7 \int_\Omega I_{ij}\tau_{ij}dx$$

$$\mathbf{B} \ = \varepsilon^3 \int_\Omega \left[\varepsilon^2 \left(b_{\alpha\beta\gamma\delta} * \frac{\partial\sigma_{\gamma\delta}}{\partial\theta}\right) + 2\varepsilon^3 \left(b_{\alpha\beta\gamma 3} * \frac{\partial\sigma_{\gamma 3}}{\partial\theta}\right) + \varepsilon^4 \left(b_{\alpha\beta 33} * \frac{\partial\sigma_{33}}{\partial\theta}\right)\right]\tau_{\alpha\beta}dx \tag{3.5}$$

$$+2\varepsilon^4 \int_\Omega \left[\varepsilon^2 \left(b_{\alpha 3\gamma\delta} * \frac{\partial\sigma_{\gamma\delta}}{\partial\theta}\right) + 2\varepsilon^3 \left(b_{\alpha 3\gamma 3} * \frac{\partial\sigma_{\gamma 3}}{\partial\theta}\right) + \varepsilon^4 \left(b_{\alpha 333} * \frac{\partial\sigma_{33}}{\partial\theta}\right)\right]\tau_{\alpha 3}dx$$

$$+\varepsilon^5 \int_\Omega \left[\varepsilon^2 \left(b_{33\gamma\delta} * \frac{\partial\sigma_{\gamma\delta}}{\partial\theta}\right) + 2\varepsilon^3 \left(b_{\alpha 3\gamma 3} * \frac{\partial\sigma_{\gamma 3}}{\partial\theta}\right) + \varepsilon^4 \left(b_{3333} * \frac{\partial\sigma_{33}}{\partial\theta}\right)\right]\tau_{33}dx$$

$$A - B = 0 \Leftrightarrow$$

$$\left\{\begin{array}{c}
-\varepsilon^9 \int_\Omega \left(b_{3333} * \frac{\partial\sigma_{33}}{\partial\theta}\right)\tau_{33}dx - 2\varepsilon^8 \int_\Omega \left[\left(b_{\alpha 333} * \frac{\partial\sigma_{33}}{\partial\theta}\right)\tau_{\alpha 3}\right. \\
\left. + \left(b_{33\gamma 3} * \frac{\partial\sigma_{\gamma 3}}{\partial\theta}\right)\tau_{33}\right]dx + \varepsilon^7 \int_\Omega \left[I_{ij}\tau_{ij} - \left(b_{\alpha\beta 33} * \frac{\partial\sigma_{33}}{\partial\theta}\right)\tau_{\alpha\beta}\right. \\
\left. -4\left(b_{\alpha 3\gamma 3} * \frac{\partial\sigma_{\gamma 3}}{\partial\theta}\right)\tau_{\alpha 3} - \left(b_{33\gamma\delta} * \frac{\partial\sigma_{\gamma\delta}}{\partial\theta}\right)\tau_{33}\right] dx \\
-2\varepsilon^6 \int_\Omega \left[\left(b_{\alpha\beta\gamma 3} * \frac{\partial\sigma_{\gamma 3}}{\partial\theta}\right)\tau_{\alpha\beta} + \left(b_{\alpha 3\gamma\delta} * \frac{\partial\sigma_{\gamma\delta}}{\partial\theta}\right)\tau_{\alpha 3}\right\}dx \\
+\varepsilon^5 \int_\Omega \left\{\gamma_{ij}\tau_{ij} - \left(b_{\alpha\beta\gamma\delta} * \frac{\partial\sigma_{\gamma\delta}}{\partial\theta}\right)\tau_{\alpha\beta}\right] = 0
\end{array}\right. \tag{3.6}$$

We rewritten in the following form:

$$
\begin{cases}
\text{Find}(u(\varepsilon),\sigma(\varepsilon)) \in \mathbb{V}(\Omega)\times\Sigma(\Omega) \ \ \forall t\in[0,T], \ \ \text{such that} \\
\dfrac{d^2}{dt^2}\left[\rho\int_\Omega u_3(x,t)v_3dx\right] + \int_\Omega \sigma_{ij}(x,t)[\partial_j v_i + \partial_i u_3 \partial_j v_3]dx \\
\ +\varepsilon^2\left\{\dfrac{d^2}{dt^2}\left[\rho\int_\Omega u_\alpha(\varepsilon)v_\alpha dx\right] + \int_\Omega \sigma_{ij}(x,t)\partial_i u_\alpha \partial_j v_\alpha dx\right\} \\
= \int_\Omega f_3 v_3 dx + \int_{\Gamma_+\cup\Gamma_-} g_3 v_3 d\Gamma + \int_{\partial\omega}(v_\alpha dx_3)h_\alpha d\gamma \ \ \forall v\in V \\
\\
\qquad \int_\Omega \left[\gamma_{ij}\tau_{ij} - (b_{\alpha\beta\gamma\delta} * \dfrac{\partial\sigma_{\gamma\delta}}{\partial\theta})\tau_{\alpha\beta}\right]dx = 0, \quad \forall\tau\in H, \\
\qquad\qquad \mathbf{u}(\varepsilon)(x,0)=0, \ \dot{\mathbf{u}}(\varepsilon)(x,0)=0, \quad \text{in } \ \Omega.
\end{cases} \tag{3.7}
$$

3.2 The limit three dimentional problem

The form of the problem (3.6) makes it amenable to the formal asymptotic expansion method . We assume a priori that the solution $((u(\varepsilon),(\sigma(\varepsilon))$ of the problem can be sought in the form of the following expansion :

$$((u(\varepsilon),(\sigma(\varepsilon)) = (u^0,\sigma^0)+\varepsilon(u^1,\sigma^1)+\varepsilon^2(u^2,\sigma^2)+...... \tag{3.8}$$

with

$$u^0=(u_i^0)\in\mathbb{V}(\Omega),\partial_3 u_3^0\in C^0(\Omega),u^p=(u_i^p)\in W^{1,4}(\Omega;R^3)\forall p\geqslant 1.$$

We assume that the relaxation tensor is independent of ε , such that

$$a^{\varepsilon}_{ijkl}(x^{\varepsilon},t)=a_{ijkl}(x,t),$$

where the functions $a_{ijkl}(x,t)$ satisfies the symmetry and coercivity . We substitute the formal asymptotic expansion (3.8) into the variational problem $(P(\varepsilon))$, we obtain the following limit three-dimensional problem. The leading term (u^0,σ^0) satisfies the following variational problem:

$$(P^0(\Omega))\begin{cases}\textbf{Find}(u^0,\sigma^0)\in\mathbb{V}(\Omega)\times\Sigma(\Omega)\forall\ t\in[0,T],\ \text{such that}\\[2ex] \dfrac{d^2}{dt^2}\left[\rho\int_\Omega u_3^0(x,t)v_3dx\right]+\int_\Omega\sigma_{ij}^0[\partial_j v_i+\partial_i u_3^0\partial_j v_3]dx=L(v)\ \ \forall v\in V\\[2ex] \int_\Omega\left[\gamma_{ij}^0\tau_{ij}-(b_{\alpha\beta\gamma\delta}*\dfrac{\partial\sigma_{\gamma\delta}}{\partial\theta})\tau_{\alpha\beta}\right]dx=0,\quad\forall\tau\in H,\\[2ex] \mathbf{u^0}(\varepsilon)(x,0)=0,\ \dot{\mathbf{u}}^{\mathbf{0}}(\varepsilon)(x,0)=0,\quad\text{in}\ \ \Omega\end{cases}\tag{3.9}$$

3.2.1 Solution of the limit problem

The solution (u^0, σ^0) :

♦ *Firstly we choose in (3.9) the second equation :*

➤ $\tau_{33} \neq 0, \quad \tau_{\alpha 3} = \tau_{\alpha\beta} = 0 \quad \forall \tau_{33} \in L^2(\Omega); \forall t \in]0, T[$

$$\int_\Omega \gamma^0_{33}\tau_{33}dx = 0 \Rightarrow \gamma^0_{33} = \partial_3 u^0_3(1 + \frac{1}{2}\partial_3 u^0_3) = 0$$

$$\partial_3 u^0_3 = 0 \text{ or } \partial_3 u^0_3 = (-2)$$

since we have assumed that $\partial_3 u^0_3 \in C^0(\Omega)$ and $u^0_3 = 0$ on $\gamma \times [-1, 1]$ the solution

$\partial_3 u^0_3 = (-2)$ is eliminated. Hence, we obtain

$$\partial_3 u^0_3 = 0 \quad \Leftrightarrow u^0_3 = \xi_3$$

➤ $\tau_{\alpha 3} \neq 0, \quad \tau_{33} = \tau_{\alpha\beta} = 0 \quad \forall \tau_{\alpha 3} \in L^2(\Omega) \forall \ t \in]0, T[$

$$\int_\Omega \gamma^0_{\alpha 3}\tau_{\alpha 3}dx = 0 \Rightarrow \gamma^0_{\alpha 3} = e_{\alpha 3}(u^0) + \frac{1}{2}\partial_\alpha u^0_3 \partial_3 u^0_3 = 0 \Rightarrow e_{\alpha 3}(u^0) = 0$$

$$\Rightarrow u^0_\alpha = \xi_\alpha - x_3\frac{\partial \xi_3}{\partial x_\alpha}$$

➤ $\tau_{\alpha\beta} \neq 0, \quad \tau_{33} = \tau_{\alpha 3} = 0 \ \forall \tau_{\alpha\beta} \in L^2(\Omega) \forall \ t \in]0, T[$

$$\int_\Omega [\gamma^0_{\alpha\beta} - (b_{\alpha\beta\gamma\delta} * \frac{\partial \sigma_{\gamma\delta}}{\partial \theta})]\tau_{\alpha\beta}dx = 0$$

$$\gamma^0_{\alpha\beta} = b_{\alpha\beta\gamma\delta} * \frac{\partial \sigma^0_{\gamma\delta}}{\partial \theta}$$

We obtaine:

$$L(\gamma^0_{\alpha\beta})(s) = sL(b_{\alpha\beta\gamma\delta})L(\sigma^0_{\gamma\delta}) \quad \text{where } L \text{ is the Laplace transfor}$$

[2].

[2] Voir APPENDIX 3

♦ *Secondly we choose in (3.9) the first equation:*

$$\forall v \in V_{kl}(\Omega) \begin{cases} v_\alpha = \eta_\alpha(x_1,x_2) - x_3\partial_\alpha\eta_3(x_1,x_2) \\ \\ v_3 = \eta_3(x_1,x_2), \qquad \eta_3 \in \mathbb{H}^2(\omega) \quad , \eta_\alpha \in \mathbb{H}^1(\omega) \end{cases}$$

we obtain:

$$\frac{d^2}{dt^2}\left[\rho\int_\Omega u_3^0(x,t)v_3dx\right] + \int_\Omega \sigma_{ij}^0(x,t)[\partial_j v_i + \partial_i u_3^0(x,t)\partial_j v_3]dx = L(\eta)$$

$$\frac{d^2}{dt^2}\left[\rho\int_\Omega \xi_3(x_1,x_2,t)\eta_3(x_1,x_2)dx\right] + \int_\Omega \sigma_{\alpha\beta}^0\Big\{\partial_\beta[\eta_\alpha - x_3\partial_\alpha\eta_3] + \partial_\alpha\xi_3\partial_\beta\eta_3\Big\}dx$$
$$+\int_\Omega \sigma_{\alpha 3}^0[-\partial_\alpha\eta_3]dx + \int_\Omega \sigma_{3\beta}^0[\partial_\beta\eta_3]dx + \int_\Omega \sigma_{33}^0[0]dx = L(\eta)$$

We defined

$$N_{\alpha\beta}^0 = \int_{-1}^1 \sigma_{\alpha\beta}^0 dx_3 \quad M_{\alpha\beta}^0 = \int_{-1}^1 x_3\sigma_{\alpha\beta}^0 dx_3 \tag{3.10}$$

then using Fubini's formula :

$$\int_\Omega F dx = \int_\omega \left\{\int_{-1}^1 F dx_3\right\}dw$$

we have

$$\frac{d^2}{dt^2}\left[\rho\int_\Omega \xi_3(x_1,x_2,t)\eta_3(x_1,x_2)dx\right] = 2\rho\int_\omega \frac{\partial^2\xi_3}{\partial t^2}\eta_3 dw$$

$$\int_\Omega \sigma_{\alpha\beta}^0\partial_\beta\eta_\alpha dx = \int_\omega N_{\alpha\beta}^0\partial_\beta\eta_\alpha$$

$$\int_\Omega x_3\sigma_{\alpha\beta}^0\partial_\alpha\eta_3 dx = \int_\omega M_{\alpha\beta}\partial_{\alpha\beta}\eta_3 dw$$

$$\int_\Omega \sigma_{\alpha\beta}^0\partial_\alpha\xi_3\partial_\beta\eta_3 dx = \int_\omega N_{\alpha\beta}^0\partial_\alpha\xi_3\partial_\beta\eta_3 dw$$

$$\int_{\Omega} f_3\eta_3 dx + \int_{\Gamma_+\cup\Gamma_-} g_3\eta_3 d\Gamma = \int_{\omega}\left\{\int_{-1}^{1} f_3 dx_3 + g_3(.,+1) + g_3(.,-1)\right\}\eta_3 d\omega$$

$$\int_{\gamma}\left(\int_{-1}^{1}\eta_\alpha dx_3\right)h_\alpha d\gamma = \int_{\partial\omega}\left(\int_{-1}^{1}\eta_\alpha dx_3\,)h_\alpha d\omega\right.$$

♦ *Finally , we have to find* $\sigma^0_{\alpha\beta}$.

we have:

$$\hat{\gamma}^0_{\alpha\beta}(s) = s\hat{b}_{\alpha\beta\gamma\delta}(s)\hat{\sigma}^0_{\gamma\delta}(s)$$

If $\hat{c}_{\alpha\beta\gamma\delta}$ is the inverse of $\hat{b}_{\alpha\beta\gamma\delta}$, we show that

$$\hat{c}_{\alpha\beta\gamma\delta}(s)\hat{\gamma}^0_{\gamma\delta}(s) = s\hat{\sigma}^0_{\alpha\beta}(s)$$

$$\hat{\sigma}^0_{\alpha\beta}(s) = \frac{1}{s}\hat{c}_{\alpha\beta\gamma\delta}(s)\hat{\gamma}^0_{\gamma\delta}(s)$$

Note that

$$f * g(t) = \int_0^1 f(t-\tau)g(\tau)d\tau$$

[3]. We obtain

$$\hat{c}_{\alpha\beta\gamma\delta}(s)\hat{\gamma}_{\gamma\delta}(s) = L(c_{\alpha\beta\gamma\delta} * \gamma^0_{\gamma\delta})$$

$$L(1) = \int_0^\infty e^{-ts}dt = -\frac{1}{s}e^{-ts}]_0^\infty = -\frac{1}{s}[0-1] = \frac{1}{s}$$

$$\hat{1} = \frac{1}{s} \Rightarrow L^{-1}(\hat{1}) = 1 = L^{-1}(\frac{1}{s})$$

[3] Voir APPENDIX 2

Then

$$\begin{aligned}\frac{1}{s}\hat{c}_{\alpha\beta\gamma\delta}(s)\hat{\gamma}^0_{\gamma\delta}(s) &= L(1).L(c_{\alpha\beta\gamma\delta} * \gamma^0_{\gamma\delta}) \\ &= L(1 * c_{\alpha\beta\gamma\delta} * \gamma^0_{\gamma\delta})\end{aligned}$$

$$\hat{\sigma}^0_{\alpha\beta}(s) = L(1 * c_{\alpha\beta\gamma\delta} * \gamma^0_{\gamma\delta})$$

$$\sigma^0_{\alpha\beta}(s) = L^{-1}(\hat{\sigma}^0_{\alpha\beta}) = 1 * c_{\alpha\beta\gamma\delta} * \gamma^0_{\gamma\delta}$$

where $\hat{f}$ is the Laplace transform of f

$$L(f)(s) \equiv \hat{f}(s) = F(s)$$

[4]. Finally

$$\sigma^0_{\alpha\beta}(s) = 1 * c_{\alpha\beta\gamma\delta} * \gamma^0_{\gamma\delta}$$

$$\sigma^0_{\alpha\beta}(s) = \int_0^t (c_{\alpha\beta\gamma\delta} * \gamma^0_{\gamma\delta})(\tau)d\tau \tag{3.11}$$

we make up (3.11) in (3.10), we find:

$$\begin{aligned}N^0_{\alpha\beta} &= \int_{-1}^{1} \sigma^0_{\alpha,\beta}dx_3 &= \int_{-1}^{1}\left[\int_0^t (c_{\alpha\beta\gamma\delta} * \gamma^0_{\gamma\delta})(\tau)d\tau\right]dx_3 \\ M^0_{\alpha\beta} &= \int_{-1}^{1} x_3\sigma^0_{\alpha,\beta}dx_3 &= \int_{-1}^{1} x_3\left[\int_0^t (c_{\alpha\beta\gamma\delta} * \gamma^0_{\gamma\delta})(\tau)d\tau\right]dx_3\end{aligned}$$

[4] Voir APPENDIX 3

The DYNAMIC VISCOELASTIC von Karman plate conditions is formulated as follows:

$$\left\{\begin{array}{l} 2\rho\int_\omega \ddot{\xi}_3(x_1,x_2,t)\eta_3(x_1,x_2)dx_1dx_2+\int_\omega(\int_{-1}^{1}\sigma^0_{\alpha\beta}dx_3)\partial_\beta\eta_\alpha \\ \\ -\int_\omega(\int_{-1}^{1}x_3\sigma^0_{\alpha\beta}dx_3)\partial_\alpha\eta_3+\int_\omega(\int_{-1}^{1}\sigma^0_{\alpha\beta}dx_3)\partial_\alpha\xi_3\partial_\beta\eta_3dx_1dx_2 \\ \\ =2\rho\int_\omega\ddot{\xi}_3\eta_3dx_1dx_2 \\ \\ +\int_\omega N^0_{\alpha\beta}\partial_\beta\eta_\alpha-\int_\omega M^0_{\alpha\beta}\partial_\alpha\eta_3+\int_\omega N^0_{\alpha\beta}\partial_\alpha\xi_3\partial_\beta\eta_3dx_1dx_2=L(\eta) \end{array}\right. \tag{3.12}$$

where:

$$\left\{\begin{array}{l} L(\eta)=\int_\omega\left\{\int_{-1}^{1}f_3dx_3+g_3(.,+1)+g_3(.,-1)\right\}\eta_3d\omega \\ \\ +\int_\omega\left(\int_{-1}^{1}\eta_\alpha dx_3\right)h_\alpha d\gamma \\ \\ N^0_{\alpha\beta}=\int_{-1}^{1}\sigma^0_{\alpha,\beta}dx_3 \quad =\int_{-1}^{1}\left[\int_0^t(c_{\alpha\beta\gamma\delta}*\gamma^0_{\gamma\delta})(\tau)d\tau\right]dx_3 \\ M^0_{\alpha\beta}=\int_{-1}^{1}x_3\sigma^0_{\alpha,\beta}dx_3=\int_{-1}^{1}x_3\left[\int_0^t(c_{\alpha\beta\gamma\delta}*\gamma^0_{\gamma\delta})(\tau)d\tau\right]dx_3 \end{array}\right. \tag{3.13}$$

Conclusion

The work presented in this Master's thesis concerns the asymptotic approximation of the three-dimensional equilibrium equations of a viscoelastic plate in the nonlinear framework, with von Karman-type boundary conditions. The viscoelastic behavior law considered is of the long memory type. Using the techniques of formal asymptotic analysis, we obtain the variational formulation of the two-dimensional limit problem of the viscoelastic von Karman n plate.

This work needs to be completed in order to obtain the two-dimensional viscoelastic von Karman n system of nonlinear equations whose unknowns are the bending displacement and the Airy function. Then to compare the model obtained with existing models.

Perspectives: Some problems can be considered as perspectives to this work: - Asymptotic analysis of viscoelastic (linear and nonlinear) shallow shells.

- Asymptotic analysis of viscoelastic Marguerre von Karman shallow shell.
- Asymptotic analysis of thermo-viscoelastic von Karman plates.
- Asymptotic analysis of the contact problem (with and without friction) of viscoelastic von Karman plates.

Appendix 1

$$\int_{\Omega^\varepsilon} \sigma^\epsilon_{kj}\partial^\epsilon_k u^\epsilon_i \frac{\partial v^\varepsilon_i}{\partial x^\varepsilon_j} dx^\varepsilon = \sigma^\varepsilon_{\alpha j}\partial^\varepsilon_\alpha u^\varepsilon_i \partial^\varepsilon_j v^\varepsilon_i + \sigma^\varepsilon_{3j}\partial^\varepsilon_3 u^\varepsilon_i \partial^\varepsilon_j v^\varepsilon_i$$

$$\begin{aligned}
\Rightarrow \sigma^\varepsilon_{\alpha j}\partial^\varepsilon_\alpha u^\varepsilon_i \partial^\varepsilon_j v^\varepsilon_i &= \sigma^\varepsilon_{\alpha j}\left[\partial^\epsilon_\alpha u^\varepsilon_\beta \partial^\epsilon_j v^\epsilon_\beta + \partial^\epsilon_\alpha u^\epsilon_3 \partial^\varepsilon_j v^\varepsilon_3\right] \\
&= \sigma^\varepsilon_{\alpha\gamma}\left[\partial^\epsilon_\alpha u^\varepsilon_\beta \partial^\varepsilon_\gamma v^\epsilon_\beta \partial^\varepsilon_\alpha u^\varepsilon_3 \partial^\varepsilon_\gamma v^\varepsilon_3\right] + \sigma^\varepsilon_{\alpha 3}\left[\partial^\varepsilon_\alpha u^\varepsilon_\beta \partial^\varepsilon_3 v^\varepsilon_\beta + \partial^\varepsilon_\alpha u^\varepsilon_3 \partial^\varepsilon_3 v^\varepsilon_3\right] \\
&= \varepsilon^2 \sigma_{\alpha\gamma}(\varepsilon)\left[\varepsilon^4 \partial_\alpha u_\beta(\varepsilon)\partial_\gamma v_\beta + \varepsilon^2 \partial_\alpha u_3(\varepsilon)\partial_\gamma v_3\right] \\
&+ \varepsilon^3 \sigma_{\alpha 3}(\varepsilon)\left[\varepsilon^3 \partial_\alpha u_\beta(\varepsilon)\partial_3 v_\beta + \varepsilon \partial_\alpha u_3(\varepsilon)\partial_3 v_3\right]
\end{aligned}$$

$$\begin{aligned}
\int_{\Omega^\varepsilon} \sigma^\varepsilon_{\alpha j}\partial^\epsilon_\alpha u^\epsilon_i \partial^\varepsilon_j v^\epsilon_i = \varepsilon^7 &\int_\Omega \sigma_{\alpha\gamma}(\varepsilon)\partial_\alpha u_\beta(\varepsilon)\partial_\gamma v_\beta + \sigma_{\alpha 3}(\varepsilon)\partial_\alpha u_\beta(\varepsilon)\partial_3 v_\beta \\
+ \varepsilon^5 &\int_\Omega \sigma_{\alpha\gamma}(\varepsilon)\partial_\alpha u_3(\varepsilon)\partial_\gamma v_3 + \sigma_{\alpha 3}(\varepsilon)\partial_\alpha u_3(\varepsilon)\partial_3 v_3
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \sigma^\varepsilon_{3j}\partial^\varepsilon_3 u^\varepsilon_i \partial^\varepsilon_k v^\varepsilon_i &= \sigma^\varepsilon_{3j}\left[\partial^\varepsilon_3 u^\varepsilon_\alpha \partial^\varepsilon_j v^\varepsilon_\alpha + \partial^\varepsilon_3 u^\varepsilon_3 \partial_j v^\varepsilon_3\right] \\
&= \sigma^\varepsilon_{3\gamma}\left[\partial^\varepsilon_3 u^\varepsilon_\alpha \partial_\gamma v^\varepsilon_\alpha + \partial^\varepsilon_3 u^\varepsilon_3 \partial_\gamma v^\varepsilon_3\right] + \sigma^\varepsilon_{33}\left[\partial^\varepsilon_3 u^\varepsilon_\alpha \partial^\varepsilon_3 v^\varepsilon_\alpha + \partial^\varepsilon_3 u^\varepsilon_3 \partial^{\varepsilon\varepsilon}_3 v^\varepsilon_3\right] \\
&= \varepsilon^3 \sigma_{3\gamma}(\varepsilon)\left[\varepsilon^3 \partial_3 u_\alpha(\varepsilon)\partial_\gamma v_\alpha + \varepsilon \partial_3 u_3(\varepsilon)\partial_\gamma v_3\right] \\
&+ \varepsilon^4 \sigma_{33}(\varepsilon)\left[\varepsilon^2 \partial_3 u_\alpha(\varepsilon)\partial_3 v_\alpha + \varepsilon \partial_3 u_3(\varepsilon)\partial_3 v_3\right]
\end{aligned}$$

$$\begin{aligned}
\int_{\Omega^\varepsilon} \sigma^\varepsilon_{3j}\partial^\varepsilon_3 u^\varepsilon_i \partial^\varepsilon_j v^\varepsilon_i = \varepsilon^7 &\int_\Omega \left[\sigma_{3\gamma}(\varepsilon)\partial_3 u_\alpha(\varepsilon)\partial_\gamma v_\alpha + \sigma_{33}(\varepsilon)\partial_3 u_3(\varepsilon)\partial_3 v_\alpha\right] \\
+ \varepsilon^5 &\left[\sigma_{3\gamma}(\varepsilon)\partial_3 u_3(\varepsilon)\partial_\gamma v_3 + \sigma_{33}(\varepsilon)\partial_3 u_3(\varepsilon)\partial_3 v_3\right]
\end{aligned}$$

Appendix 2

<u>Definition</u> 3.1 *Let ϕ and ψ be functions defined on $[0,+\infty[$, and let the Riemann integral*

$$v(t)=\int_0^t \phi(t-\tau)\psi(\tau)d\tau$$

exists for all t in $[0,+\infty[$.

Then the function v, so defined on $[0,+\infty[$, is the Riemann convolution of ϕ and ψ. We also write

$$v=\phi*\psi$$

to denote this function.

(Properties of the Riemann convolution):- Let ϕ, ψ and χ be in $C^0\left([0,+\infty[\right)$. Then

(a) $\phi*\psi\in C^0\left([0,+\infty[\right)$

(b) $\phi*\psi=\psi*\phi$

(c) $\phi*(\psi*\chi)=(\phi*\psi)*\chi=\phi*\psi*\chi$

(d) $\phi*(\psi+\chi)=\phi*\psi+\phi*\chi$

(e) $\phi*\psi=0 \implies \phi=0$ or $\psi=0.$

Appendix 3

<u>Definition</u> 3.2 *The Laplace transform of a function $f(t)$, $0 < t < \infty$, is defined by*

$$L(f)(s) = F(s) = \int_0^\infty f(t)e^{-ts}dt.$$

Here s is the transformation parameter.

The corresponding inverse Laplace transform, computed by means of complex variable techniques, is

$$L^{-1}(F)(t) = f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}ds.$$

If

(i) f is a piecewise continuous on $[0, \infty[$.

(ii) there are constants C and α such that $|f(t)| \leq Ce^{\alpha t}$, $0 < t < \infty$,

then $L(f)(s) = F(s)$ exists for all $s > \alpha$.

(i) L and L^{-1} are linear.

(ii) If $u = u(x,t)$ and $L(u)(x,s) = U(x,s)$, then

$$\begin{aligned} L(\frac{\partial u}{\partial t})(x,s) &= sU(x,s) - u(x,0) \\ L(\frac{\partial^2 u}{\partial t^2})(x,s) &= s^2U(x,s) - su(x,0) - \frac{\partial u}{\partial t}(x,0). \end{aligned}$$

(iii) For the same type of function u, differentiation with respect to x and the Laplace transformation commute:

$$L(\frac{\partial u}{\partial x})(x,s) = \frac{\partial L(u)}{\partial x}(x,s).$$

(iv) If $(f * g)(t) = \int_0^t f(s)g(t-s)ds = \int_0^t f(t-s)g(s)ds$, then

$$L(f * g) = L(f) \cdot L(g).$$

Bibliography

[1] D. A. Chacha, Asymptotic analysis of nonlinear elastic plates, 2002.

[2] P. G. Ciarlet , Mathematical elasticity VOLUME II : Theory of plates, 1997.

[3] P. G. Ciarlet, Paumier . A justification of the Marguerre-von Karman equations. Comput. Mech.**1** 177-202, 1986.

[4] P. G. Ciarlet, L. Gratie From the classical to the generalized von Karman and Marguerre-von Karman equations. J. Comput. Appl. Math. **190**, 470-486, 2006.

[5] P. G. Ciarlet and P. Destuynder, Justification of the two-dimensional linear plate model, Journal de mecanique, **18**, 315-344, 1979.

[6] E. H. Dill, "CONTINUUM MECHANICS Elasticity, Plasticity, Viscoelasticity", CRC Press 2007.

[7] L. C. Evans, Partial Differential Equations, American Mathematical Society, 2nd ed., **19**, 2022.

[8] R. S. Lakes, "Viscoelastic Solids", CRC Press, 2018.

[9] J. L. SCHIFF, The Laplace Transform: Theory and Applications . Springer-Verlag Telos, October 1999

الملخص

الهدف من هذا المشروع هو دراسة النمذجة المقاربة لصفيحة فون كارمان اللزجة المطاطية. بدءًا من معادلات المرونة اللاخطية ثلاثية الأبعاد الكلاسيكية ذات الذاكرة الطويلة وقانون السلوك وباستخدام تقنيات طرق التقارب ، نحصل على نموذج ثنائي الأبعاد من المرونة اللزجة فون كارمان .

الكلمات المفتاحية

تحليل مقارب، مرونة غير خطية، صفيحة رقيقة، مرونة لزوجة، ذاكرة طويلة

Abstract

The aim of this project is the study of the asymptotic modeling of a viscoelastic von Karman plate. Starting from equations of the classical 3D nonlinear elasticity with a long memory behaviour law, and using the technics of asymptotic methods, we obtain the 2D model of viscoelastic von Karman plate model**.**

Key words

Asymptotic analysis, nonlinear elasticity, thin plate, viscoelasticity, long memory.

Résumé

Le but de ce projet est l’etude de la modelisation asymptotique d’une plaque viscoelastique de von Karman. A partir des equations de l’e lasticitenon lineaire 3D classique avec une loi de comportement elonguememoire, et en utilisant les techniques des methodes asymptotiques, nous obtenons le module 2D du module de plaque viscoelastique de von Karman.

Les mots-clés

Analyse asymptotique, elasticitenon lineaire, plaque mince, viscoelasticite, memoire longue**.**