KASDI MERBAH UNIVERSITY OUARGLA

Faculty of mathematics and materials science

N° d'ordre :
N° de série :

**DEPARTMENT OF mathematics
MATHEMATICS**

Master

Specialty: Mathematics

Option: Modeling and AN

by: Haddache Soraya

Theme

Numerical Analysis of Some Variational Inequalities

version : June 14, 2014

Before the jury:

Mr.Ghezal Abderrazek	M.A. Kasdi Merbah University - Ouargla	President
Mr. Merabet Smail	M.C. Kasdi Merbah University - Ouargla	examiner
Mr.Bensayah Abdallah	M.A. Kasdi Merbah University - Ouargla	protractor

Dedication

Idedicate my work to my family and aspecial feeling of gratitude to my parents,whose are the source of my success.

Idedicate my work to may sisters meriem and sonia and may brothers walid and wahib and ghelese who have supported me throughout the process.

Iwill always appreciate all they have done.
Idedicate this work and give special thanks to students in my class .

In the end Idedicate this memory to my colleagues and all those who are dear to me.

Acknowledgement

At the begining I want to thank God that guide us to complete this work.

I would like to thank Professor **Mr Abdallah Bebsayah**, who carry within us the suffering of this work .

also Iwant to thank the teachers of the department of mathematics and Saide and all the help given by us.

Thank special gratitude to the members of the council Have you wish to Improve and Evaluate this work.

Iwant to thank all colleagues who accompanied me during the years of study Adia, Hadia, Hanan, Soumia, Rabab, saida, roumaysa, Noura.Thank you all help us in one way or

another.Thank you for all.

Preliminaries and Notations

V: real Hilbert space with scalar product $(V, \| \ \|_V)$. We are also given K closed non empty subsets of V with K $\subset V$.

$a(,)$: V $\times V \to \mathbb{R}$ bilinear continuous and V. Elliptic from on $V \times V$.

continue : $\exists \ c \ > 0 \ \forall u, v \in V \ \ |a(u,v)| \leq c\|u\|_V \|v\|_V$.

ceorcive :$\exists \ \alpha > 0 \ \forall u, v \in V \ \ |a(u,v)| \geq \alpha\|u\|_V^2$.

$V^{'}$: the dual space of V.

$L : V \longrightarrow \mathbb{R}$ continuous, linear functional.

In general we do not assume a(.) to be symmetric, since in some applications non-symmetric bilinear forms may occur naturally.

$j(.) : V \longrightarrow \bar{\mathbb{R}} = \mathbb{R} \bigcup \{\infty\}$is convex, lower semi-continuous and proper .

Contents

Introduction

Numerous problems in Mechanics, Physics and Control Theory lead to the study of systems of partial differential inequalities, the solution of which leans heavily on the techniques of so-called variational inequalities(see [18]). In the last fifty years, variational inequalities have become a useful tool in etide nonlinear problems in physics and mechanics. The theory of variational inequalities were made from the results of unilateral problems obtained by Signorini (see[1]) and Fichera(see [17]). The mathematical theory obtained by Stampacchia (see [8]), Lions and Stampacchia (see [15]) and then developed by: Brézis (see [12]), (see [13]), Stampacchia(see [11]), Lions(see [16]), Mosco (see[22]), Kinderlehrer (see [5]) and Stampacchia (see [23]), and the approximation of variational inequalities are reminded, the contributions Mosco (see[20]), Lions and Trémoliéres or Glowinski ([21]). The unilateral contact of elastic bodies with or without friction often encountered in modélisation. In 1964, that G.Fichera (see [9]). a pu résoudre ce probléme en utilisant quelques propriétés des inéquations variationnelles elliptique. The mathematical study of problems contact began in 1972, with the work of Duvaut and Lions, or there are results of existence and uniqueness of several problems contact, but in the linear case. In this memory we present in the first chapter. Useful mathematical preliminries. In second chapter we study the uniqueness results for EVI of first kind and second kind. Next in the third chapter investigate an abstract internal approximation of EVI first kind and second one. As au example we use the Finite Element Method on a specefic, simplified obstacle problem. In the end we conclude our work by a conclusion involiving the main result and some perspectives.

Chapter 1

Mathematical preliminaries

1.1 Some functional spaces

We recall below some definitions,(see [3]) and theorems of classical functional analysis that will be used in later later chapters, here all the functions considered are real-valued real, let x $\in \mathbb{R}^n$ Ω over in $\mathbb{R}$, $K \subset \Omega$, m positive integer, α is an integer multiple,$|\alpha| = \sum_{i=1}^{n} \alpha$ then we define the differential operator:

$$D^\alpha = D_1^{\alpha_1}......D_n^{\alpha i} = \frac{d^{|\alpha|}}{d_{x_1}^{\alpha_1}....d_{x_n}^{\alpha_n}}$$

we denote by $C(\Omega)$,the space of continuous real functions on Ω,they say it is relatively K , compact in Ω,if the adhesion of K,is a compact (closed and bounded) included in Ω was noted by$K \subset\subset \Omega$ also be denoted by:

$$C^m = \{v \in C(\Omega) : \ D^\alpha \ v \in \ C(\Omega) \ for \ |\alpha| \ \leqslant \ m\}$$

called support of function v defined on Ω all closed

$$suppv = \{x \in \Omega, v(x) \neq 0\}$$

we say that the function v is compactly supported in $\Omega, if notes : ssupv \subset\subset \Omega$

$$C^m = \{v \in C^m(\Omega) : \ v \ is \ a \ support \ compact \ in \ \Omega\}$$

$$C^\infty(\Omega) = \bigcap_{m=0}^{\infty} C^m(\Omega)$$

we will denote by $D(\Omega)$ called the space of test function, space C_0^∞ indefinitely differentiable functions with compact support in $(\Omega)$with the topology of inductive limit as in the theory of distributions of L. Schwarz we notes $D^{'}(\Omega)$ the dual space of D (Ω), therefore the space of continuous linear forms on $D(\Omega)$, $D'(\Omega)$ is called the space of distribution (or generalized function) on (Ω), and is provided with the dual topology strong $(f_i \longrightarrow f$ in $D^{'}$ if $< f, \varphi >$ $\forall \varphi \in D(\Omega)$) ou $< ., . >$ is the product of duality between $D^{'}(\Omega)$ et $D(\Omega)$, for given by :

$$L^p(\Omega) = \{v \ mesurables \ on \ \Omega \ , \|v\|_p = (\int_\Omega |v|^p dx)^{\frac{1}{p}} \ < \ \infty\}$$

we recall that $(L^p(\Omega)), \|\|_p)$ a Banach space is separable, and for $1 < p < \infty$ $reflexive.$

for p=2, space is a Hilbert space with the scalar product:

$$< u, v >= \int_\Omega u(x)v(x)dx$$

we will identify the space L^2 to its dual, for $p = \infty$ we denote by:

$$L^\infty(\Omega) = \{v \ as \ measured \ on \ \Omega; \ such \ as \ \|v\|_\infty = supess_{x\in\Omega}|v(x)| = inf\{\{C, |v(x)| \leqslant Ca, e \ x \in \Omega\}$$

reminder that ($L^\infty(\Omega), \|\|)_\infty$), there is a space of banach, for all $1 < p < \infty$ one inequality of holder :

$$\int_\Omega u(x) \ v(x)dx \leqslant \|u\|_p \|v\|_p$$

Theorem 1 *The space $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$ $\forall$ $1 < p < \infty$. we say that $X \hookrightarrow Y$, for $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ norms space, means $X \subset Y$ with continuous injection, that is to say there exists a constant C such that*

$$\|u\|_Y \leq C\|u\|_X \ \ \forall u \in X.$$

1.2 Sobolev spaces

$1 \leq p \leq \infty$, *we have* $D(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow D'(\Omega)$ *We will define the Sobolev space(see [14])*

$$W^{m,p}(\Omega) = \{v, D^\alpha v \in L^p(\Omega), \ \ for \ \ |\alpha| \leq m\},$$

with the norm

$$\|v\|_{W^{m,p}} = \left(\Sigma_{|\alpha|\leq m} \|D^{\alpha}v\|_p^p\right)^{1\backslash p} \quad if\ p \in [1,\infty)$$
$$\|u\|_{W^{m,p}} = \max_{|\alpha|\leq m} \|D^{\alpha}v\|_{\infty},$$

is a Banach space . We denote by $W_0^{m,p}(\Omega)$ *adherence of* C_0^{∞} *in the space* $W^{m,p}(\Omega)$*; For all* $p \in [1,\infty)$ *we have*

$$W_0^{m,p}(\Omega) \hookrightarrow W^{m,p}(\Omega)^p \hookrightarrow L^p(\Omega).$$

In the case $p = 2$ *we use the notation*

$$H^m(\Omega) = W^{m,p}(\Omega).$$

equipped with the scalar product

$$\langle u, v\rangle_{2,m} = \Sigma_{|\alpha|\leq m} \langle D^{\alpha}u, D^{\alpha}v\rangle .$$

The space $H^m(\Omega)$ *is a Hilbert space. We also posed* $H_0^m(\Omega) = W_0^{m,p}(\Omega)$ *the negative Sobolev spaces are dual spaces of spaces* $W_0^{m,p}(\Omega)$

$$W_0^{-m,p'}(\Omega) = (W_0^{m,p}(\Omega))',$$

with the norm

$$\|u\|_{W_0^{-m,p'}(\Omega)} = \sup_{u\in W_0^{m,p}(\Omega)} \frac{\langle u, v\rangle}{\|u\|_{W_0^{m,p}(\Omega)}}$$

The space $W_0^{-m,p'}(\Omega)$ *is Banach(separable and reflexive ,if* $1 < p < \infty$*). Since* $D(\Omega)$ *is dense in* $H_0^1(\Omega)$*, then we have* $H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$.

Theorem 2 *Suppose that* Ω *satisfies the property of the cone and* $1 \leq p < \infty$*. Then*

1. $C(\bar{\Omega}) \hookrightarrow W_0^{m,p}(\Omega)$ *with the dense injection.*
2. *if* $mp \geq n$ *then* $W_0^{m,p}(\Omega) \hookrightarrow C^k(\bar{\Omega})$ *whatever integer* k *with* $\dfrac{mp-n}{p} - 1 \leq k \leq \dfrac{mp-n}{p}$.

1.3 Coercivity

Definition 3 *a:*$H \times H \longrightarrow \mathbb{R}$ *is called coercive if there exist a constant* $c > 0$*, such that* $a(x,x) \geqslant c\|x\|^2$ *for all* x *in* H
(see [6])

1.4 Strong convergence

Definition 4 *A sequence* $\{x\} \subset X$ *in a normed space converges strongly to* $x \in Xif$ $\|x_n - x\| \longrightarrow 0$ *as* $n \longrightarrow \infty$.
Notationally
$\lim_{n\longrightarrow\infty} x_n = x$ *or* $x_n \longrightarrow x$ *Notice that this is just convergence in the norm of X, that is, it is vergence as we normally think of it. The terminology "strong" is useful to avoid confusion with the following other type of convergence.*

1.5 Weak convergence

Definition 5 *A sequence* $\{x\} \subset X$ *in a normed space converges weakly to* $x \in X$ *if for every* $f \in x_0$ *, we have that* $|f(x_n) - f(x)| \longrightarrow 0$ *as* $n \longrightarrow \infty$*. That is, the sequence* $\{f(x_n\} \subset F$ *converges to* $f(x) \in F$.
Notationally,
$x_n \longrightarrow x$*, Strong and weak convergence are the same on nite-dimensional normed spaces, which is why the distinction is not made in calculus. But they are not the same in general, (see [7])*

1.6 Convex

Definition 6 *(Convexity of function) Let f be function f:* $I \longrightarrow \mathbb{R}$ *is called convex if :*
$\forall x_1, x_2 \in X \ \forall t \in [0.1]$.

$$f(tx_1 + (1-t)x_2) \leqslant tf(x_1) + (1-t)f(x_2).$$

Definition 7 *(Convexity of set) all said C is convex if :*
$\forall x_1, x_2 \in X \ \forall t \in [0.1]$, $\quad tx_1 + (1-t)x_2 \in C$

1.7 Stampacchia theorem

Theorem 8 *(Stampacchia) Let H be a Hilbert space and let a(.,.) be a continuous and coercive bilinear form on H . Let K be a closed and convex subset of H. Then given $f \in H$ there exists a unique $u \in K$ such that*
a(u, v -u) $\geqslant$ (f, v - u), for all $v \in K$.
Proof. *(see [14])* ■

1.8 Riesz representation theorem

Theorem 9 *Let H is a Hilbert space. For all $F \in H'$ (dual to H), there is a unique $v \in H$, such that : f(v)= $< u, v >$ $\forall u \in H$, more we have: $\|F\|_{H'} = \|v\|_h$.(see [19])*

1.9 Schauder theorem

Theorem 10 *E is a Banach space and $k \subset E$ convex and compact*
then any continuous mapping, $f : K \longrightarrow K$, has a fixed point.
i.e.,$\exists$ x $\in K$ such that f(x)=x.
Proof. *(see [4])* ■

1.10 Contracting and strictly contracting

Definition 11 *Let X space complet let $\varphi : X \longrightarrow X$ be*
contracting *if :*

$$\|\varphi(u) - \varphi(v)\|_X \leqslant C\|u - v\|_X, \forall u, v \in X \text{ and } 0 \leqslant C \leqslant 1$$

Strictly contracting *if :*

$$\|\varphi(u) - \varphi(v)\|_X \leqslant C\|u - v\|_X, \forall u, v \in X \text{ and } 0 \leqslant C < 1$$

Theorem 12 *Let X space complet let, $\varphi : X \longrightarrow X$ be strictly contracting , then φ has a unique fixed point x $(that is \varphi(x) = x)$*

Chapter 2

Variational Inequalities

2.1 EVI of first kind

any inequality of the form

$$(P1)\begin{cases} a(u, v-u) \geq L(v-u)\ \forall\ v\ \in K \\ u \in K \end{cases} \tag{2.1}$$

called a variational inequality of first kind, where $K \subset V$ *and* $a(,) : V \times V \to \mathbb{R}$ *bilinear.*

2.1.1 Existence And Uniqueness Results For EVI of First Kind

Theorem 13 *if a (.,.) bilinear form continuous coercive, on* $V \times V$ *and* <.,.>*, defines a continuous linear form, on* $V \longrightarrow \mathbb{R}$ *and K closed convex in V.*

then :
the problem(P1) has one and only one solution (by stampacchia)

Proof. Uniqueness

Let u_1 *and* u_2 *be solution of* (P_1) *, we have then*

$$a(u_1, v-u_1) \geq L(v-u_1) \forall v \in K, u_1 \in K \tag{2.2}$$

$$a(u_2, v-u_2) \geq L(v-u_2) \forall v \in K, u_2 \in K \tag{2.3}$$

putting u_1 *for v in (2.2) , and* u_2 *for v in (2.3) , we get :*

$$a(u_1, u_2 - u_1) \geq L(u_2 - u_1) \tag{2.4}$$

$$a(u_2, u_1 - u_2) \geq L(u_1 - u_2) \tag{2.5}$$

by the add (2.4) and (2.5) we get:

$$a(u_2 - u_1, u_1 - u_2) \geq 0$$

we using the coercivity of a (,):

$$-a(u_2 - u_1, u_2 - u_1) \geq 0 \Rightarrow \alpha \|u_2 - u_1\|_V^2 \leq 0$$

wich proves $u_1 = u_2$ *since* $\alpha > 0$

Proof. Existence

We will reduce the problem (P1), to fixed point problem.
By the Riesz representation there exist: (Au, v) = a(u, v) $\forall u, v \in V$ *and L(v) = (l, v)* $\forall v \in V$.

$$(w, v - w) \geq \rho(l, \phi) + (u, \phi) - \rho a(u, \phi) \ \forall \ v \ \in K$$

admits a fixed point u=Tx $\longrightarrow$ *u solution for (P1) existence for all* $u \in K, \rho > 0 is$

$$(P1*) \begin{cases} find \ u \in K \\ (w, v - w) \geq (F_{\rho,u}, v - w) \ \forall \ v \ \in K \end{cases} \tag{2.6}$$

(P1)admits uniquensse solution* $w = P_k F_{\rho,u}$*(according to the projection theorem w=* P_k $F_{\rho,u}$ *there is unique in K [10])*
$T : u \longmapsto w$

$$(F_{\rho,u}, \phi) = \rho(l, \phi) + (u, \phi) - \rho a(u, \phi)$$

to prove that T_ρ admits a fixed point, it suffices to prove that it is strictly contracting .

$$\|T_{\rho,u_1} - T_{\rho,u_2}\|_V \leqslant \ C \ \|u_1 - u_2\|_V \ for \ C \ < \ 1$$

$$\|w_1 - w_2\|_V \leqslant C\|u_1 - u_2\|_V$$

for C < 1

$$(w_1, v - w_1) \geqslant (F_{\rho,u_1}, v - w_1)$$

for $v = w_2$
$(w_2, v - w_2) \geqslant (F_{\rho,u_2}, v - w_1) for \ v = w_1$
$(w_1, w_2 - w_1) - (w_2, w_2 - w_1) \geqslant (F_{\rho,u_1}, w_2 - w_1) - (F_{\rho,u_2}, w_2 - w_1)$

$$(w_1 - w_2, w_2 - w_2) \geqslant (F_{\rho,u_1} - F_{\rho,u_2}, w_2 - w_1)$$

$$\|w_2 - w_1\|_V^2 \leqslant (F_{\rho,u_2} - F_{\rho,u_1}, w_2 - w_1)$$

$$\|w_2 - w_1\|_V^2 \leq \|F_{\rho,u_2} - F_{\rho,u_1}\|_V \|w_2 - w_1\|_V$$

$$\|w_2 - w_1\|_V \leq \|F_{\rho,u_1} - F_{\rho,u_2}\|_V$$

$$(F_{\rho,u_2} - F_{\rho,u_1}, \phi) = \rho(l, \phi) + (u_2, \phi) - \rho a(u_2, \phi) - \rho(l, \phi) - (u_1, \phi) + \rho a(u_1, \phi)$$

$$(F_{\rho,u_2} - F_{\rho,u_1}, \phi) = (u_2 - u_1, \phi) - \rho a(u_2 - u_1, \phi)$$

$$(F_{\rho,u_2} - F_{\rho,u_1}, \phi) = (u_2 - u_1) - \ \rho \ A(u_2 - u_1, \phi)$$

$$|(F_{\rho,u_2} - F_{\rho,u_1}, \phi), \phi)| \leq \|I - \rho A\|_V \|u_2 - u_1\|_V \|\phi\|_V \Longrightarrow \|F_{\rho,u_2} - F_{\rho,u_1}\|_V \leq \|I - \rho A\|_V \ \|u_2 - u_1\|_V$$

if $\|I - \rho A\|_V < 1$

$$\|(I - \rho A)v\|_V^2 = \|(I - \rho A)v, (I - \rho A)v)\|_V$$

$$\|(I - \rho A)v\|_V^2 = \|v\|_V^2 - 2\rho(Av, v) + \rho^2\|Av\|_V^2$$

$$\|(I - \rho A)v\|_V^2 \leqslant \|v\|^2 - 2\rho\alpha\|v\|^2 + \rho^2\|A\|^2\|v\|_V^2 \leq (1 - 2\rho\alpha + \rho^2\|A\|_V^2)\|v\|_V^2$$

$$(1 - 2 \ \rho \ \alpha \ + \ \rho^2\|A\|_V^2) \ < \ 1 \ \Longrightarrow \ 0 < \ \rho \ \leqslant \ \frac{2\alpha}{\|A\|_V^2}$$

for $\rho \in]\ 0,\ \frac{2\alpha}{\|A\|_V^2}]$
T_ρ *strictly contracting admis a fixed point* $T_\rho u = u\ ,\ w = u$

$$(u, v - u) \geqslant (F_{\rho,u}, v - u)\ \forall u \in K$$

$$(u, v - u) \geqslant \rho(l, v - u) + (u, v - u) - \rho a(u, v - u) \Longrightarrow\ a(u, v - u) \geqslant L(v - u)$$

u is the solution of problem (P1). ■

2.2 EVI of Second Kind

Any inequality of the form

$$(P2) \begin{cases} a(u, v - u) + j(v) - j(u) \geq L(v - u)\ \forall\ v\ \in K \\ u \in K \end{cases} \tag{2.8}$$

called a variational inequality of Second Kind, where $j(.) : V \longrightarrow \mathbb{R}$.

2.2.1 Existence And Uniqueness Results For EVI of Second Kind

Theorem 14 *If a (.,.) continuous coercive, on* $V \times\ V$ *and, that K the set is* $\neq \phi$ *and K closed convex in V, and j(.) is semi-continuous convex function.*

then:
the problem (P2) has one and only one solution

Proof. Uniqueness

Let u_1 *and* u_2 *be tow solution of (P2),then we have*

$$a(u_1, v - u_1) + j(v) - j(u_1) \geqslant L(v - u_1)\ \forall u_1 \in V$$

$$a(u_2, v - u_2) + j(v) - j(u_2) \geqslant L(v, u_1 - u_2)\ \forall u_1 \in V$$

we putting u_1 *in v and* u_2 *in v by the adde we take :*

$$a(u, u_2 - u_1) + a(u_2, u_1 - u_2 \geqslant 0$$

$\Longrightarrow -a(u_1 - u_2, u_1 - u_2 \geqslant 0$

a($u_1 - u_2, u_1 - u_2 \leqslant 0$ *by the coercivity*

$$\alpha \parallel u_1 - u_2 \parallel_V^2 \leqslant a(u_1 - u_2, u_1 - u_2) \leqslant 0$$

$$\|u_1 - u_2\|_V = 0 \Longrightarrow u_1 = u_2$$

Proof. Existence
For $u \in V$ *and* $\rho > 0$ *we associate a problem of typ (P2) defined as below:*
(w,v-w)+ $\rho\ j(v)\ -\ \rho\ j(w)\ \geq\ \rho\ L(v-w)\rho -\ a(u, v-w)\ +\ (u, v-w)$

$$(P2*) \begin{cases} find\ w\ \in K \\ a(w, v-w) + \rho j(v) - \rho j(w) \geq (u, v-w) + \rho L(v-w) - \rho a(u, v-w) \end{cases} \quad (2.9)$$

$f_\rho : u \longrightarrow w = f_\rho(u)$
if f_ρ *admits a fixed point* $u = Tx$ *and,* $f(u) = u\ ,\ \rho > 0\ .$

to prove that f_ρ *admits a fixed point, it suffices to prove that it is strictly contracting*
$0 \leq c < 1$

$$\|f_\rho(u_1) - f_\rho(u_2\|_V \leqslant c\|u_1 - u_2\|_V\ \forall u_1, u_2 \in V$$

let $w_1 = f_\rho(u_1)$ *and* $w_2 = f_\rho(u_2)$ *such that:*

$$(w_1, v - w_1) + j(v) - j(w_1) \geqslant (v - w_1) - \rho a(u_1, v - w_1) + (u_1, v - w_1)$$

$$for\ w_2 = v :\ (w_1, w_2 - w_1) + j(w_2) - j(w_1) \geqslant (w_2 - w_1) - \rho a(u_1, w_2 - w_1) + (u_1, w_2 - w_1) \quad (2.10)$$

$$forw_1 = v : (w_2, w_1 - w_2) + j(w_1) - j(w_2) \geqslant (w_1 - w_2) - \rho a(u_2, w_1 - w_2) + (u_2, w_1 - w_2) \quad (2.11)$$

adding 2.10 and 2.11 we have then :

$$(w_1, w_2 - w_1) + (w_2, w_1 - w_2) \geqslant -\rho a(u_1, w_2 - w_1) - \rho a(u_2, w_2 - w_1) + (u_2, w_2 - w_1) + (u_2, w_1 - w_2)$$

$$(w_2 - w_1, w_1 - w_2) \geqslant \rho a(u_1 - u_2, w_1 - w_2) - (u_1 - u_2, w_1 - w_2)$$

$$-(w_1 - w_2, w_2 - w_1) \geqslant \rho a(u_1 - u_2, w_1 - w_2) - (u_1 - u_2, w_1 - w_2)$$

$$\|w_1 - w_2\|_V^2 \leqslant (u_1 - u_2, w_1 - w_2) - \rho a(u_1 - u_2, w_1 - w_2) \tag{2.12}$$

we have

$$(u_1 - u_2, w_1 - w_2) \leqslant \|u_1 - u_2\|_V \|w_1 - w_2\|_V$$

$$a(u, v) = A(u, v)$$

$$a(u_1 - u_2, w_1 - w_2) = (A(u_1 - u_2), w_1 - w_2)$$

$$a(u_1 - u_2, w_1 - w_2) = ((u_1 - u_2) - \rho A(u_1 - u_2), w_1 - w_2)$$

$$a(u_1 - u_2, w_1 - w_2) = ((I - \rho A)(u_1 - u_2), w_1 - w_2) \tag{2.13}$$

$$|(I - \rho A)(u_1 - u_2), w_1 - w_2)| \leqslant \|(I - \rho A\| \|u_1 - u_2\| \|w_1 - w_2\|$$

with Compensation (2.13) in (2.12) we get:

$$\|w_1 - w_2\|_V \leqslant \|I - \rho A\|_V \|u_1 - u_2\|_V$$

if

$$\|I - \rho A\|_V \leqslant 1$$

$$\|(I - \rho A)v\|_V^2 = (I - \rho A)v, (I - \rho A)v)$$

$$\|(I - \rho A)v\|_V^2 = \|v\|_V^2 - 2\rho(Av, v) + \rho^2 \|A\|_V^2 \|v\|_V^2$$

$$\|(I - \rho A)v\|_V^2 \leqslant (1 - 2\alpha\rho + \rho^2 \|A\|_V^2) \|v\|_V^2$$

$$(1 - 2\rho\alpha + \rho^2 \|A\|_V^2) \Longleftarrow\!\!\!\Longrightarrow \; 0 < \rho \leqslant \frac{2\alpha}{\|A\|_V^2}$$

f_ρ *strictly contracting* $\exists u \in V,\ f_\rho(u) = u$ *for* $\rho \in]\ 0,\ \frac{2\alpha}{\|A\|_V^2}]$

$f_\rho(u) = w$ *admis a fixed point* $w = u$

$$(w, v - w) + (v) - (w) \geqslant \rho L(v - w) - \rho a(u, v - w) + (u, v - w)$$

$$(u, v - u) + (v) - (u) \geqslant \rho L(v - u) - \rho a(u, v - u) + (u, v - u)$$

$$a(u, v - u) + j(v) - j(u) \geq L(v - u)$$

$\Longrightarrow$ *u is the solution of problem (P2).*

Chapter 3

Numerical approximation of Variational Inequalities

3.1 Internal approximation of EVI of first kind

3.1.1 The approximation of V and K

We are given a parameter h converging to 0 and a family $(V_h)_h$ *of closed subspaces of V. (In practice* V_h *are finite dimensional and the parameter h varies over a sequence). We are also given a family* $(K_h)_h$ *of closed, convex, non-empty subsets of V with* $K_h \subset V_h \ \forall h$ *(in general we do not assume* $K_h \subset K$*) such that* $(K_h)_h$ *satisfies the following two conditions : (see [19])*

- *(i) If* $(v_h)_h$ *is such that* $V_h \in K_h \ \forall h$ *and* $(v_h)_h$ *is bounded in V then the weak cluster points of* $(v_h)_h$ *belong to K.*
- *(ii) Assume there exist* $X \subset V, \bar{X} = K$ *and* $r_h : X_h \longrightarrow V$ *such that* $\lim_{h \longrightarrow 0} r_h v = v$ *strongly in*V, $\forall v \in V$

Remarques

1. *If* $K_h \subset K \quad \forall h$ *then (i) is trivially satisfied because K is weakly closed*
2. *A useful variant of condition (ii), for* r_h *is (ii) Assume . there exists a subset* $X \subset V$ *such that* $\bar{X} = K$ *and* $r_h : X \longrightarrow V_h$ *with the property that for each* $v \in X$*, there exists* $h_0 = h_0(v)$ *with* $r_h v \in K_h$ *for all* $\lim_{h \longrightarrow 0} r_h v = v$ *strongly in V.*

3.1.2 Approximation of (P1)

V real Hilbert space with scalar product not $(V_h, \|\|_h)$ *,we are also given* $(K_h)_h$ *is closed, convex ,non emply subsets of V with* $K_h \subset V_h$ *.*
$a(,) : V \times V \longrightarrow \mathbb{R}$ *bilinaire, and L :*$V \longrightarrow \mathbb{R}$ *continue.*
continue

$$\exists c > 0, \forall u_h, v_h \in V_h \ |a(u_h, v_h)| \leqslant c\|u_h\|_V \|v_h\|_V$$

coercive

$$\exists c > 0, \forall u_h, v_h \in V_h \ |a(u_h, v_h)| \geqslant \alpha\|v_h\|_V^2$$

$$(P_h1) \begin{cases} a(u_h, v_h - u_h) \geq L(v_h - u_h) \ \forall \ v_h \ \in K_h \\ u_h \in K_h \end{cases} \tag{3.1}$$

The problem (P_h1) *has one and only one solution (by the theorem 13).*

3.2 Convergence results

Theorem 15 *With the above assumptions on K and* $(K_h)_h$*we have* $\lim_{h \longrightarrow 0} \|u_h - u\|_V = 0$ *with* u_h*thesolutionof* (P_h1) *and u the solution of (P1).*

(1)Estimation for u_h

We will now show that there exist constants C_1 *and* C_2 *independent of h such that*

$$\|u_h\|_V^2 \leqslant C_1\|u_h\|_V + C_2, \forall h \tag{3.2}$$

Since u_h *is the solution of* (P_h1) *we have*

$$a(u_h, v_h - u_h \geqslant L(v_h - u_h), \forall v_h \in K_h \tag{3.3}$$

$$a(u_h, u_h) \leqslant a(u_h, v_h) - L(v_h - u_h)$$

by coercivity we get :

$$\alpha\|u_h\|_V^2 \leqslant \|A\|_V \|u_h\|_V \|v_h\|_V + \|L\|_V(\|v_h\|_V + \|u_h\|_V), \forall v_h \in K_h \tag{3.4}$$

Let $v_0 \in X$ *and* $v_h = r_h v_0 \in K_h$*. By condition (ii) on* K_h*we have* $r_h v_0 \longrightarrow v_0$ *strongly in V and hence* $\|v_h\|_V$ *is uniformly bounded by a constant m. Hence (3.4) can be written as*

$$\|u_h\|_V^2 \leqslant \frac{1}{\alpha}((m\|A\|_V + \|L\|_V)\|u_h\|_V + \|L\|_V m) = C_1\|u_h\|_V + C_2$$

where $C_1 = \frac{1}{\alpha}(m\|A\|_V + \|L\|_V)$ *and* $C_2 = \frac{m}{\alpha}\|L\|_V$ *implies*

$$\|u_h\|_V \leqslant C \tag{3.5}$$

Weak convergence of $(u_h)_h$

Relation (3.5) gives u_h *is uniformly bounded. Hence there exists a subsequence say* u_{h_i} *such that* u_{h_i} *converges to* u^* *weakly in V. By condition (i) on* $(K_h)_h$ *we have* $u^* \in K$*. We will prove that* u^* *is a solution for* (P_1)*, We have:*

$$a(u_{h_i}, u_{h_i}) \leqslant a(u_{h_i}, v_{h_i}) - L(v_{h_i}, u_{h_i}) \ \forall v_{h_i} \in K_{h_i} \tag{3.6}$$

let $v \in X$ *and* $v_{h_i} = r_{h_i} v$ *then (3.6) becomes*

$$a(u_{h_i}, u_{h_i}) \leqslant a(u_{h_i}, r_{h_i} v) - L(r_{h_i} v, u_{h_i}) \ \forall r_{h_i} v \in K_{h_i} \tag{3.7}$$

Since $r_{h_i} v$ *converges strongly to v and* u_{h_i} *to* u^* *weakly as* $h_i \longrightarrow 0$ *taking the limit in (3.7) we get*

$$\lim_{h \longrightarrow 0} inf \ a(u_{h_i}, u_{h_i}) \leqslant a(u^*, v) - L(v, u^*) \ \forall v \in X \tag{3.8}$$

also we have

$$0 \leqslant a(u_{h_i} - u^*, u_{h_i} - u^*) \leqslant a(u_{h_i}, u_{h_i}) - a(u_{h_i}, u^*) - a(u^*, u_{h_i}) + a(u^*, u^*)$$

$$a(u_{h_i}, u^*) + a(u^*, u_{h_i}) - a(u^*, u^*) \leqslant a(u_{h_i}, u_{h_i})$$

by taking limit we obtain

$$a(u^*, u^*) \leqslant \lim_{h_i \longrightarrow 0} \inf a(u_{h_i}, u_{h_i}) \tag{3.9}$$

from (3.8) and (3.9) we get

$$a(u^*, u^*) \leqslant \lim_{h_i \longrightarrow 0} \inf a(u_{h_i}, u_{h_i}) \leqslant a(u^*, v) - L(v, u^*) \quad \forall \ v \ \in X$$

therefore we have

$$\begin{cases} a(u^*, v - u^*) \geq L(v - u^*) \ \forall \ v \ \in X \\ u^* \in K \end{cases} \tag{3.10}$$

Since X is dense in K and a(.,.), L are continuous, we get from (3.10)

$$\begin{cases} a(u^*, v - u^*) \geq L(v - u^*) \ \forall \ v \ \in K \\ u^* \in K \end{cases} \tag{3.11}$$

Hence u^* is a solution of (P1). By Theorem (Lions − stampacchia) the solution for (P1) is unique and hence $u^* = u$ is the unique solution,Hence u is the only cluster point of $(u_h)_h$ in the weak topology of V. Hence the whole $\{u_h\}_h$ converges to u weakly.

Strong convergence

we have by coercivity of a(.,.)

$$0 \leqslant \alpha \|u_h - u\|_v^2 \leqslant a(u_h - u, u_h - u) = a(u_h, u_h) - a(u_h, u) + a(u, u_h) + a(u, u) \tag{3.12}$$

wher u_h is the solution of $(P_h 1)$ and u is the solution of (P1). Since u_h is the solution of $(P_h 1)$ and $r_h v \in K_h$ for any $v \in X$, we get by $(P_h 1)$

$$a(u_h, u_h) \leqslant a(u_h, r_h v) - L(r_h v, u_h) \ \forall v \in X \tag{3.13}$$

Since $\lim_{h \longrightarrow 0} u_h = u$ weakly in V and $\lim_{h \longrightarrow 0} r_h v = v$ strongly in V (by condition (ii)) we obtain, from (3.12), (3.13) and after taking the lim, that $\forall \ v \in X$ we have:

$$0 \leqslant \lim inf \ \alpha \|u_h - u\|_V^2 \leqslant \alpha \lim sup \ \|u_h - u\|_V^2 \leqslant a(u, v - u) - L(v - u) \tag{3.14}$$

By density and continuity, (3.14) also holds $\forall \ v \in K$, then taking $v = u$ in (3.14) we obtain that

$$\lim_{h \longrightarrow 0} \|u_h - u\|_V^2 = 0 \tag{3.15}$$

3.3 Internal Approximation of EVI of Second Kind

3.3.1 Approximation of V

Given a real parameter h converging to 0 and a family $(V_h)_h$ of closed subspaces of V (in practice we will take V_h to be finite dimensional and h to vary over a sequence), we assume that $(V_h)_h$ satisfies.

- *(i) there exists $U \subset V$ such that $\bar{U} = V$ and for each h, a map $r_h : U \longrightarrow V_h$ such that $\lim_{h \longrightarrow 0} r_h \ v = v$ strongly in V, $\forall v \in U$.*

3.3.2 Approximation of j(.)

We approximate the functional j(·) by $(j_h)_h$ *where for each h,* j_h *satisfies*

$$\begin{cases} j_h : V_h \longrightarrow \bar{R} \\ j_h \ is\ convex,\ l, s, c\ and\ uniformly\ proper\ in\ h. \end{cases} \tag{3.16}$$

The family $(j_h)_h$ *is said to be uniformly proper in h if there exist* $\lambda \in V^*$ *and* $\mu \in R$ *such that*

$$j(v_h) \geqslant \lambda(v_h) + \mu \quad \forall v_h \in V_h, \forall h \tag{3.17}$$

Furthermore we assume that $(j_h)_h$ *satisfies*

(ii) if $v_h \longrightarrow v$ *weakly in V then*

$$\lim_{h \longrightarrow 0} inf\, j_h(v_h) \geqslant j(v)$$

(iii) $\lim_{h \longrightarrow 0} j_h(r_h v) = j(v) \quad \forall v \in U$

Remarques

1. *In all the applications we know, if j(·) is a continuous functional then it is always possible to construct continuous* j_h *satisfying(ii) and (iii)*

2. *In some cases we are fortunate enough to have* $j_h(v_h) = j(v_h) \forall v_h \forall h$ *and then (ii) and (iii) are trivially satisfied.*

3.3.3 Approximation of (P2)

We approximate $(P2)$ *by*

$$(P_h 2) \begin{cases} a(u_h, v_h - u_h) + j_h(v_h) - j_h(u_h) \geq L(v_h - u_h)\ \forall\ v_h\ \in V_h \\ u_h \in V_h \end{cases} \tag{3.18}$$

the problem $(P_h 2)$ *has one only solution (by theorem 14).*

3.3.4 Convergence results

Theorem 16 *Under the above assumptions on* $(V_h)_h$ *and* $(j_h)_h$ *we have*

$$\begin{cases} \lim\limits_{h\longrightarrow 0} \|u_h - u\|_V = 0 \\ \lim\limits_{h\longrightarrow 0} j_h(u_h) = j(u) \end{cases} \tag{3.19}$$

Ase in the proof of Theorem 15, we divide the proof into three parts.

(1) Estimation for u_h

We will show that there exist positive constants C1 and C2 independent of h such that

$$\|u_h\|_V^2 \leqslant C1\|u_h\|_V + C2 \tag{3.20}$$

Since u_h *is the solution of* (P_h2) *we have By using relation (3.17) we get*

$$\alpha\|u_h\|_V^2 + <\lambda, u_h> + \mu \leqslant \|A\|\|u_h\|\|v_h\|_V + |j_h(v_h)| + \|L\|_V(\|v_h\|_V + \|u_h\|_V)$$

$$\alpha\|u_h\|_V^2 \leqslant \|\lambda\|_V\|u_h\|_V + |\mu| + \|A\|_V\|u_h\|_V\|v_h\|_V + |j_h(v_h)| + \|L\|_V(\|v_h\|_V + \|u_h\|_V) \tag{3.21}$$

Let $v_0 \in U$ *and* $v_h = r_h v_0$*. By using condition (i) and (iii) there exists a constant* m*, independent of h such that* $\|v_h\|_v \leqslant m$ *, and* $|j_h(v_h| \leqslant m$*. Therefore (3.21) can be written as*

$$\|u_h\|_V^2 \leqslant \frac{1}{\alpha}(\|\lambda\|_V + \|A\|_V m + \|L\|_V)\|u_h\|_V + \frac{m}{\alpha}(1 + \|L\|_V) + \frac{|\mu|}{\alpha}$$

$$\|u\|_V^2 \leqslant C1\|u_h\|_V + C2$$

$$C1 = \frac{1}{\alpha}(\|\lambda\|_V + \|A\|_V m + \|L\|_V)$$

and

$$C2 = \frac{m}{\alpha}(1 + \|L\|_V) + \frac{|\mu|}{\alpha}$$

and (3.20) implies

$$\|u_h\|_V \leqslant C \quad \forall h \tag{3.22}$$

where C is aconstant

(2) weak convergence of $(u_h)_h$

Relation (3.22) gives that u_h *is uniformly bounded. Therefore there exists a subsequence*

$(u_{h_i})_{h_i}$ such that $u_{h_i} \longrightarrow u_h$ weakly in V. Since u_h is the solution of $(P_h 1)$ and $r_h v \in V_h \forall h$ and $\forall v \in$ U we get:

$$a(u_{h_i}, u_{h_i}) + j_{h_i}(u_{h_i}) \leqslant a(u_{h_i}, r_{h_i} v) + j_{h_i}(r_{h_i} v) - L(r_{h_i} v - u_{h_i}) \tag{3.23}$$

By condition (iii) and weak convergence of $\{u_{h_i}\}$ we get :

$$\lim_{h \longrightarrow 0} inf[a(u_{h_i}, u_{h_i}) + j_{h_i}(u_{h_i})] \leqslant a(u^*, v) + j(v) - L(v - u^*) \quad \forall v \in U \tag{3.24}$$

As in (3.9) and using condition (ii), we get :

$$a(u^*, u^*) + j(u^*) \leqslant \lim_{h \longrightarrow 0} inf[a(u_{h_i}, u_{h_i}) + j_{h_i}(u_{h_i})] \tag{3.25}$$

From (3.24), (3.25) and using the density of U we have

$$\begin{cases} a(u^*, v\_u^*) + j(v) - j(u^*) \geq L(v - u^*) \ \forall \ v \ \in V \\ u* \in V \end{cases} \tag{3.26}$$

This implies u^* is a solution of (P2). Hence $u^* = u$ is the unique solution 3.26 of (P2) and this shows that (u_h) converges to u weakly.

***(3) Strong convergence of** $(u_h)_h$*

We have by V -ellipticity of a(.,.) and by $(p_2 h)$

$$\alpha \|u_h - u\|_V^2 + j_h(u_h) \leqslant a(u_h - u, u_h - u) + j_h(u_h)$$

$$= a(u_h, u_h) - a(u, u_h) - a(u_h, u) + a(u, u) + j_h(u_h) \leqslant$$

$$a(u_h, r_h v) + j_h(r_h v) - L(r_h v - u_h) - a(u, u_h) - a(u_h, u) + a(u, u) \ \forall v \in U \tag{3.27}$$

The right hand side of inequality (3.27) tends to a(u, v - u) + j(v) - L(v - u) as $h \longrightarrow 0$ $\forall v \in$ U. Therefore we have

$$\lim inf_{h \longrightarrow 0} j_h(u_h) \leqslant \lim inf_{h \longrightarrow 0}[\alpha \|u_h - u\|_V^2 + j_h(u_h)] \leqslant$$

$$\leqslant \lim sup_{h \longrightarrow 0}[\alpha \|u_h - u\|_V^2 + j_h(u_h)] \leqslant$$

$$\leqslant a(u, v - u) + j(v) - L(v - u) \ \forall v \in U \tag{3.28}$$

By density of U, (3.28) holds $\forall v \in V$. Replacing V by u in (3.28) and using condition (ii) we obtain

$$j(u) \leqslant \lim_{h \longrightarrow 0} inf\ j_h(u_h) \leqslant \lim_{h \longrightarrow 0} sup\ [\alpha \|u_h - u\|_V^2 + j_h(u_h)] \leqslant j(u) \tag{3.29}$$

this implies that

$$\lim_{h \longrightarrow 0} sup\ [\alpha \|u_h - u\|_V^2 + j_h(u_h)] - j(u) = 0$$

$$\lim_{h \longrightarrow 0} sup\ [\alpha \|u_h - u\|_V^2] + [\lim_{h \longrightarrow 0} sup\ j_h(u_h) - j(u)] = 0$$

$$\begin{cases} \lim\limits_{h \longrightarrow 0} sup\ j_h(u_h) = j(u) \\ \lim\limits_{h \longrightarrow 0} inf\ \ j_h(u_h) = j(u) \end{cases} \tag{3.30}$$

by (3.29) and (3.30)implies we have :

$$\lim_{h \longrightarrow 0} sup\ j_h(u_h) = j(u)$$

and

$$\lim_{h \longrightarrow 0} sup\ \|u_h - u\|_V = 0$$

this proves the theorem.

3.4 The continuous problem

The physical interpretation of this problem is the following: let an elastic membrane occupy a region in the x_1, x_2 plane; this membrane is fixed along the boundary Γ of . When there is no obstacle, from the theory of elasticity the vertical displacement u, obtained by applying a vertical force F

C.P

$$\begin{cases} -\triangle\ u = f\ \ in\ \ \ \Omega \\ u|_\Gamma = 0 \\ u(x) \geqslant \psi(x) \forall x \in \Omega \end{cases} \tag{3.31}$$

$a(u,v) = \int_\Omega \nabla u \nabla v$
$L(v) = \int_\Omega fv \quad f \in L^2(\Omega)$
$f \in V^* = H^{-1}(\Omega)$ *and* $v \in V$
$V = H_0^1(\Omega)=\{v \in H^1(\Omega) : v|\Gamma = trace\ de\ v\ sure\ \Gamma = 0\}$
let $\psi \in H^1(\Omega) \cap C^0(\bar{\Omega})$ *and* $\psi|_\Gamma \leqslant 0$.
$K = \{v \in V,\ v \geqslant \psi\ \ ae\ on\ \Omega\}$.

$$(V.P)\begin{cases} find\ u \in K \\ a(u, v-u) \geqslant L(v-u) \forall v \in K \end{cases} \tag{3.32}$$

Remarques

is given by the Dirichlet problem :

$$\begin{cases} -\triangle u = f\ in\ \ \Omega \\ u|_\Gamma = 0 \end{cases} \tag{3.33}$$

Where f= $\frac{F}{t}$, *t being the tension. when there is an obstacle, we have a free boundary problem and the displacement u satisfies the variational inequality (3.32) with being the height of the obstacle. Similar EVI also occur.*

3.5 Finite Element Approximations of (3.32)

Henceforth we shall assume that is a polygonal domain of $\mathbb{R}^2$. *Consider a " classical" triangulation of* φ_h , φ_h*is a finite set of triangles T such that*
$T \subset \bar{\Omega}\ \forall T \in \varphi_h \quad , \cup_{\varphi_h} T = \bar{\Omega}$
$T_1^0 \bigcap T_2^0 = \phi, \quad \forall\ T_1, T_2 \in \varphi_h$ *and* $T_1 \neq T_2$
oreover $\forall T_1, T_2 \in \varphi_h$ *and* T_1 , T_2, *exactly one of the following conditions must hold*

1. $T_1^0 \bigcap T_2^0 = \phi$.

2. T_1 *and* T_2 *have only one common vertex.*

3. T_1 and T_2 have only a whole common edge.

As usual h will be the length of the largest edge of the triangles in the triangulation. From now on we restrict ourselves to piecewise linear and piecewise quadratic finite element approximations.

3.5.1 Approximation of V and K

P_k:space of polynomials in x_1 and x_2 of degree less than or equal to k.

$\sum_h = \{P \in \bar{\Omega}$:P *is a vertex of* $T \in \varphi_h\}$

$\sum_h^0 = \{P \in \sum_h : P \notin \Gamma\}$

$\sum_h^{'} = \{P \in \bar{\Omega} :$ *P is the mid point of an edge of* $T \in \varphi_h\}$

$\sum_h^{'0} = \{P \in \sum_h^{'} : P \notin \Gamma\}$

$\sum_h^1 = \sum_h$ *and* $\sum_h^2 = \sum_h \bigcup \sum_h^{'}$

We have tringle arbitrary illustrates some further notations associated with an arbitrary triangle T. we have $m_{iT} \in \sum_h^{'}$, $M_{iT} \in \sum_h$. The centroid of the triangle T is denoted by G_T . The space $v = H_0^1(\Omega)$is approximated by the family of subspaces $(V_h^k)_h$ with k = 1 or 2 where

$$V_h^k = \{v_h \in C^0(\bar{\Omega}) : v_h|_\Gamma = 0 \textit{ and } v_h|_T \in P_k \ \forall\ T \in \varphi_h\}\ k = 1, 2$$

It is clear that V_h^k are finite dimensional , It is then quite natural to approximate K by

$$K_h^k = \{v \in V_h^k : v_h(p) \geqslant \psi(p)\ \forall P \in\ \Sigma_h^k\ \}\ k = 1, 2$$

3.5.2 The approximate problems

For $k = 1, 2$ the approximate problems are defined by

$$(P_{1h}^k)\begin{cases} a(u_h^k, v_h - u_h^k) \geq L(v_h - u_h^k)\ \forall\ v_h\ \in K_h^k \\ u_h^k \in K_h^k \end{cases} \tag{3.34}$$

(P_{1h}^k) *has a unique solution for $k = 1$ and 2.*

3.6 convergence result

In other to simplify the convergence proof we shall assume in this section that let $\psi \in H^1(\Omega) \cap C^0(\bar{\Omega})$and $\psi \leqslant 0$ in a neighbourhood of Γ.

Trapezoidal Rule and Simpson's Integral formula

we have triangle arbitrary , prove the following identities for any triangle T.

$$\int_T w dx = \frac{meas.(T)}{3} \sum_{i=1}^{3} w(M_{iT})\ \forall w \in P_1. \tag{3.35}$$

$$\int_T w dx = \frac{meas.(T)}{3} \sum_{i=1}^{3} w(m_{iT})\ \forall w \in P_2. \tag{3.36}$$

Theorem 17 *Suppose that the angles of the triangles of φ_h are uniformly bounded below by $\theta_0 > 0$ as $h \longrightarrow 0$; then for $k = 1, 2$*

$$\lim_{h \longrightarrow 0} \|u_h^k - u\|_{H_0^1(\Omega)} = 0 \tag{3.37}$$

where u_h^k and u are respectively the solutions of $P_{1h}^k$and(3.32) .
In this proof we shall use the following density result to be proved later:

$\overline{D(\Omega) \bigcap K} = K$

To prove (3.37) we shall use Theorem 15 ; To do this we have to verify that the following two properties hold (for $k = 1, 2$):

- *(i) $If(v_h)_h$ is such that $v_h \in K_h^k$ $\forall h$ and converges weakly to v as $h \longrightarrow 0$; then $v \in K$.*

- *(ii) There exists X, $\bar{X} = K$ and $r_h^k : X \longrightarrow K_h^k$ such that $\lim_{h\longrightarrow 0} r_h^k v = v$ strongly in V $\forall v \in X$*

Verification of (i)

Using the notations of triangle arbitrary and considering $\phi \in D(\Omega)$with $\phi \geqslant 0$, we define ϕ_h by

ϕ_h $=\sum_{T\in\varphi_h} \phi(G_t)X_T$ where X_T is the characteristic function of T and G_T is the centroid of T. It is easy to see from the uniform continuity of ϕ that

$$\lim_{h\longrightarrow 0} \phi_h = \phi \textit{ strongly in } L^\infty(\Omega) \tag{3.38}$$

Then we approximate $\psi$by ψ_h such that

$$\begin{cases} \psi_h \in C^0(\bar{\Omega}), \psi_h|_T \in P_k \ \forall\ T \ \in \varphi_h \\ \psi_h(p) = \psi(p)\ \forall P \in \Sigma_h^k \end{cases} \tag{3.39}$$

This function ψ_h satisfies

$$\lim_{h\longrightarrow 0} \psi_h = \psi \textit{ strongly in } L^\infty(\Omega) \tag{3.40}$$

Let us consider a sequence $(v_h)_h, v_h \in K_h^k$ $\forall h$ such that

$$\lim_{h\longrightarrow 0} v_h = v \textit{ weakly in } V.$$

then $\lim_{h\longrightarrow 0} v_h$ $= v$ strongly in $L^2(\Omega)$ which, using (3.38) and (3.40), implies that

$$\lim_{h\longrightarrow 0} \int_\Omega (v_h - \psi_h)\phi_h dx = \int_\Omega (v - \psi)\phi dx \tag{3.41}$$

actually since $\phi_h \longrightarrow \phi$ strongly in $L^\infty(\Omega)$ the weak convergence of v_h in $L^2(\Omega)$ is enough to prove (3.41). We have

$$\int_\Omega (v_h - \psi_h)\phi_h dx = \sum_{T\in\varphi} \phi(G_t) \int_T (v_h - \psi_h) dx \tag{3.42}$$

From (3.35). (3.36) and the definition of ψ_h we obtain for all $T \in \varphi_h$.

$$\int_\Omega (v_h - \psi_h) dx = \frac{meas.(T)}{3} \sum_{i=1}^{3} [v_h(M_{iT}) - \psi_h(M_{iT})] \ if k = 1 \tag{3.43}$$

$$\int_\Omega (v_h - \psi_h)dx = \frac{meas.(T)}{3} \sum_{i=1}^{3} [v_h(m_{iT}) - \psi_h(m_{iT})] \ if k = 2 \tag{3.44}$$

(see [19])

Using the fact that $\phi_h \geqslant 0$, the definition of $K_h^k$and the relations (3.43), (3.44) it follows from (3.42) that

$$\int_\Omega (v_h - \psi_h)\phi_h dx \geqslant 0 \ \forall \phi \in D(\Omega), \ \ \phi \geqslant 0. \tag{3.45}$$

so that as $h \longrightarrow 0$

$$\int_\Omega (v - \psi)\phi dx \geqslant 0 \ \forall \phi \in D(\Omega), \ \ \phi \geqslant 0. \tag{3.46}$$

which in turn implies $v \geqslant \psi$ a.e. in Hence (i) is verified.

Verification of (ii).

From $\overline{D(\Omega) \bigcap K} = K$ it is natural to take $X = D \bigcap K$.we define r_h^k:$H_0^1(\Omega) \bigcap C^0(\bar{\Omega}) \longrightarrow V_h^k$ as the " linear " interpolation operator when k = 1 and " quadratic" interpolation operator when k = 1, i.e.

$$\begin{cases} r_h^k v \in V_h^k \ \ \forall v \in H_0^1(\Omega) \bigcap C^0(\bar{\Omega}) \\ (r_h^k v)(p) = v(p) \ \forall P \in \Sigma_h^k \ for k = 1, 2. \end{cases} \tag{3.47}$$

On the one hand it is known that under the assumptions made on φ_h in statement of Theorem 17 , we have :

$$\|r_h^k v - v\|_v \leqslant Ch^k \|v\|_{H^{k+1}(\Omega)} \forall v \in D(\Omega) \ \ k = 1, 2. \tag{3.48}$$

with C independent of h and v . This implies that

$$\lim_{h \longrightarrow 0} \|r_h^k v - v\|_v = 0 \ ,\forall v \in X \ \ k = 1, 2. \tag{3.49}$$

on the other hand it is obvious that

$$r_h^k v \in K_h^k \ \ \forall v \in K \bigcap C^0(\bar{\Omega}) \tag{3.50}$$

so that $r_h^k \in V_h^k$ *and*

$$K_h^k = \{v \in V_h^k : v_h(p) \geqslant \psi(p)\ \forall P \in\ \Sigma_h^k\ \}$$

$v \in X = K \bigcap D$

$$v(p) \geqslant \psi(p)$$

$$(r_h^k v)(p) \geqslant \psi(p)$$

if $v \in X \quad r_h^k v \in K_h^k$ *for K=1,2 In conclusion with the above X and* r_h^k *, (ii) is satisfied. Hence we have proved the Theorem 17.*

Corollary 18 *(see [19]) If v+ and v- denote the positive and the negative parts of v for v* $\in H^1(\Omega)$ *(respectively* $H_0^1(\Omega)$*) then the map* $v \longrightarrow \{v+, v-\}$ *is continuous from* $H^1(\Omega) \longrightarrow H^1(\Omega) \times H^1(\Omega)$*(respectively* $H^1(\Omega) \longrightarrow H^1(\Omega) \times H^1(\Omega)$ *.also* $v \longrightarrow |v|$ *is continuous*

Lemma 1 *Under the assumptions we have* $\overline{D(\Omega) \bigcap K} = K$

Proof

Let us prove the Lemma in two steps.

Step 1

Let us show that

$$\wp = \{v \in K \bigcap C^0\bar{(}\Omega) : v \text{ compact support in } \Omega\} \text{ is dense in } K. \tag{3.51}$$

let $v \in K$ *,* $K \subset H_0^1(\Omega)$*implies that exists a sequence* $\{\phi_h\}_h$ *in* $D(\Omega)$ *such that*

$$\lim_{h \longrightarrow \infty} \phi_n = v \text{ strongly in } V.$$

define v_h *by*

$$v_n = max(\psi, \phi) \tag{3.52}$$

so that

$$v_n = \frac{1}{2}[(\psi + \phi_h + |\psi - \phi_n|)]$$

Since $v \in K$, from Corollary 18 ,and relations (3.52) follows that

$$\lim_{n \longrightarrow 0} v_n = \frac{1}{2}[(\psi + v + |\psi - v|)] = max(\psi, v) = v \tag{3.53}$$

strongly in V

A From $\overline{D(\Omega) \bigcap K} = K$ and (3.52) it follows that each v_n has a compact support in Ω

$$v_n \in K \bigcap C^0(\bar{\Omega}) \tag{3.54}$$

From (3.53) - (3.54) we obtain (3.51)

Step2

Let us show that

$$D(\Omega) \bigcap \wp \text{ is dense } \wp \tag{3.55}$$

This proves from Step 1, that $D(\Omega) \bigcap K$ is dense in K. Let ρ_n be a sequence of mollifiers .

$$\begin{cases} \rho_h \in D(\mathbb{R}^2), \rho_n \geqslant 0 \\ \displaystyle\int_{R^2} \rho_n(y) = 1 \\ \displaystyle\bigcap_{n=1} supp\rho_n = \{0\}, \{supp\rho_n\} \text{ is a decreasing sequence} \end{cases} \tag{3.56}$$

Let $v \in \wp$. Let $\tilde{v}$ extension of v defined by

v(x) if $x \in \Omega$

and 0 $if x \notin \Omega$ then $\tilde{v} \in H^1(\mathbb{R}^2)$,let $\tilde{v}_n = \tilde{v} * \rho_n i.e$

$$\tilde{v}(x) = \int_{R^2} \rho_n(x - y)\tilde{v}(y)dy \tag{3.57}$$

then

$$\begin{cases} \tilde{v_n} \in D(\mathbb{R}^2) \\ supp\ \tilde{v_n} \subset supp\ v + supp\ \rho_n^{'} \\ \displaystyle\lim_{n \longrightarrow \infty} \tilde{v_n} = \tilde{v} \text{ strongly in } H^1(\mathbb{R}^2) \end{cases} \tag{3.58}$$

Hence from (3.59) and (each v_n has a compact support in Ω) we have

$supp(\tilde{v_n}) \subset \Omega$ *for n large enough* (3.59)
We also have (since supp $(\tilde{v})$*is bounded)*

$$\lim \tilde{v_h} = \tilde{v} \ strongly \ in \ L^{\infty}(\mathbb{R}^2) \tag{3.60}$$

Define $v_n = \tilde{v}|_{\Omega}$ *,then (3.59)–(3.60) imply*

$$\begin{cases} v_n \in D(\Omega) \\ \lim\limits_{n \longrightarrow \infty} v_n = v \ strongly \ in \ H_0^1(\Omega) \bigcap C^0(\bar{\Omega}) \end{cases} \tag{3.61}$$

$v \in \ \wp$ *and* $\psi \leqslant 0$ *in a neighbourhood of* Γ *imply that there exists a* $\sigma > 0$ *such that*

$$v = 0, \ \psi \leqslant 0 \ on \ \Omega_{\sigma} \tag{3.62}$$

where $\Omega_{\sigma} = \{x \in \Omega : d(x, \Gamma) < \sigma\}$ *From (3.60) and (3.62) it follows that* $\forall \xi > 0$ *there exists an* $n_0 = n_0(\xi)$ *such that*

$$\begin{cases} v(x) - \xi \leqslant v_n(x) \leqslant v(x) + \xi \forall x \in \Omega - \Omega_{\frac{\sigma}{2}} \\ v_n(x) = v(x) = 0 \ for x \in \Omega_{\frac{\sigma}{2}} \end{cases} \tag{3.63}$$

Since $\bar{\Omega} - \Omega_{\frac{\sigma}{2}}$ *is a compact subset of* $\bar{\Omega}$ *there exists a functions* θ *such that*

$$\begin{cases} \theta \in D(\Omega) \ \theta \ \geqslant 0 \ in \ \Omega \\ \theta(x) = 1 \ \forall x \in \bar{\Omega} - \Omega_{\frac{\delta}{2}} \end{cases} \tag{3.64}$$

finally define $W_n^{\xi} = v_n + \xi\theta$
Then from (3.61), (3.63) and (3.3.2) we have

$$w_n^{\xi} \in D(\Omega)$$

$$\lim_{\xi \longrightarrow 0} = v \ strongly \ in H_0^1(\Omega)$$

$$\lim_{n \longrightarrow \infty} = v \ strongly \ in H_0^1(\Omega)$$

$$\lim_{n \geqslant n_0(\xi)} = v \ strongly \ in H_0^1(\Omega)$$

with $w_n^{\xi} \geqslant v(x) \geqslant \psi(x) \forall x \in \Omega$ *so step2 is proved.*

Conclusion

The result of this work we have estabished the existence and uniqueness results For EVI of First Kind and second knd and the Internal approximation of EVI of first kind and secend one. the main result we use Finite Element Approximations on an obstacle problem. As perpectives are the use of FEM for:

- dynamical signorini problem without friction

Bibliography

[1] A.Signorini, Sopra alcune questioni d elastostatica, Atti Societa Italiana per il Progresso della Scienze, 1933.

[2] An introduction to variational inequalities and their application,guido stampacchia.

[3] A. Capatina, Inéquations variationnelles et problèmes de contact avec frottement. 10=2011; Bucuresti, ISSN 0250 3638.

[4] Clémence MINAZZO-Kelsey RIDER, Théorémes du Point Fixe et Applications aux Equations Différentielles.

[5] D.Kinderlehrer, G.Stampacchia, An introduction to Variational Inequalities an their application, Academic Press, New York, 1980.

[6] Estimates near the boundary for solution of elliptic partial differential equations satisfying general boundary conditions.

[7] Fengbo Hang, Functional Analysis, Spring 2009 (I). Comm. Pure Applied Math., 12, (1959), pp. 623-727.

[8] G.Stampacchia,Formes bilinéaires coercives sur les ensembles convexes, C.R.Acad. Sci. paris, 258,1964.

[9] G.Fichera, problem elastostatici con vincoli unilaterali:ilproblema di signorini con ambigue condizioni al contorno, Mem. Accad.Naz.Lince Ser,91-140(1964)

[10] Georges Skandalis.Topologie et analyse 3ème année.Objectif Agregation pour un paragraphe consacré aux applications du théorème de projection sur un convexe fermé.

[11] G. Stampacchia, variational inequalities, Theory and application of monotone operators,Proceeding of a NATO Advanced Study Institute, Venice, Italy, 1968.

[12] H.Brézis,problémes unilatéraux,J.Math.Pures Appl, 51,1,1-168,1972.

[13] H.Brézis, équations etinéquations non linéires dans les espaces vectoriels en dualité, Ann. Inst. Fourier, Grenoble,18,1,115-175,1968.

[14] H. Brézis, Analyse fonctionnelle : théorie et applications, Dunod, 1999.

[15] J.L.Lions, G.Stampacchia, Variational inequalites, Comm.Pure Appl.Math,XX,493-519,1967.

[16] J.L.Lions,quelques methodes de résolution des problémes aux lmites non linéaires,Dunod, paris,1969.

[17] Q. Fichera, Problemi elastostatici con vincoli, unilaterali, il problema di Signorini conambigue condizioni al contoro, Mem.Aem.Accad.Naz.dei Lincei, 91-140,1964.

[18] R.Glowinski,J.Lions,R.Tremolieres,Numerocal analysis of variational inequalities, Amsterdam : North-Holland, 1981.

[19] R. Glowinski Lectures on Numerical Methods For Non-Linear Variational Problems, Berlin Heidelberg New York.

[20] R.Glowinski,J,L.Lions,R.Trémoliéres, Numerical analysis of variational inequalities, Amsterdam: NorthHlland, 1981.

[21] R.Glowinski, Numerical methods for nonlinear variational problems,Berlin Heidel-berg New York,Spriger,1984.

[22] *U.Mosco, Implicit variational problems and quasi-variational, Lect.Notes in Math,543,83-156,1975.*

[23] *U.Mosco, An introduction to approximate solution of variational inequalities, Constructive aspects of function analysis, Erice 1971,497-684, Crempnese,1973.*