PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA

UNIVERSITY KASDI MERBAH OUARGLA

Faculty of Mathematics and Material Sciences

DEPARTMENT OF MATHEMATICS

N° d'ordre :
N° de série :

A Thesis Submitted to Obtain the Degree of

MASTER

Specialty : Mathematics

Option : Functional Analysis

By : Badidja Zohra

Theme

Mathematical Study of von Kármán Equations with Memory and Delay

Publicly supported the: 04/06/2024

In front of the jury composed of:

Abderrezak Ghezal	Pr	University Kasdi Merbah Ouargla	President
Sadok Otmani	MCB	University Kasdi Merbah Ouargla	Examiner
Ilyes Lacheheb	MCB	University Kasdi Merbah Ouargla	Examiner
Djamal Ahmed Chacha	Pr	University Kasdi Merbah Ouargla	Supervisor

Promotion : 2023/2024

Dedication

I dedicate this effort and my success to:

Dear dad, **"Lakhdar"**, Allah Save and take care of him.

Dear mother, **"Nassira"**, may Allah prolong her life.

I hope that one day I will be able to give them back a little of what they have done for me, may Allah grant them happiness and long life.

My sister **"Salima, Maroua, sedra"**, My brother is **"Emad, Said, Ibrahim, Salh eddin, Ahemad"**, my friends and colleagues.

My big family is all in his name.

And my brother is **"Abd elhak"** may Allah have mercy on him.

All my teachers and those who supported me in this work financially or morally. And to everyone who has forgotten my pen and has not forgotten my heart... I ask Allah to accept it from me and benefit those who are after me.

Acknowledgements

First of all, I thank **"Allah"** for the success and determination he has given me to accomplish this humble work.

would like to thank my family for the needed support they provided during my thesis journey, especially my **father** for being the source of my inspiration.

I would like to express my sincere gratitude to my supervisor **Professor Djamal Ahmed Chacha** for suggesting this interesting topic, guiding me through the process of writing my thesis, and equipping me with the necessary research skills.

Thanks are extended to everyone who contributed to my education, even by letter, throughout my academic career from primary to university.

I would like to express my thanks and gratitude to the members of the **"Discussion Committee"** for accepting this memorandum in order to discuss it and present its ideas, and I would like to get their satisfaction.

Abbreviations

- **ODE** : Ordinary differential equation.
- **PDE** : Partial differential equation.

Symbols

$\mathbb{N}$	The set of natural numbers.
$\mathbb{R}$	The set of real numbers.
$\mathbb{C}$	The set of complex numbers.
$Re(z)$	A real part of a complex number Z.
$\partial_i = \frac{\partial}{\partial x_i}$	Partial differentiation with respect to x_i
$\partial_\nu = \frac{\partial}{\partial \nu}$	The directional derivative along the outer normal ν
$L^p(\Omega)$	Space of p-th integrable functions on Ω with respect to the Lebesgue measure dx, for $p \in [1, +\infty[$.
$L^\infty(\Omega)$	Space of essentially bounded functions on Ω.
$H^m(\Omega)$	Sobolev space of order m, for $m \in \mathbb{N}$
$H_0^1(\Omega)$	Space of functions in H^1 vanishing on the boundary
$C^m(\Omega)$	Space of m-times contimously differentiable functions on ω, for $m \in \mathbb{N}$
H^*	Conjugate function of H.
$\Delta u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$	Laplacien of u
$\|.\|$	A norm associated with a scalar product.
$(.,.)$	Scalar product.
$\mathcal{D}(\Omega) = C_c^\infty(\Omega)$	Test functions space.

Contents

Introduction

The von Karman plate model is a nonlinear "large deflection" elastic plate model, analogue of the Kirchhoff model. However, it is assumed that the vertical deflection is small in comparison with the lateral dimensions of the plate. This hypothesis (and others) leads to a coupled pair of fourth-order, nonlinear partial differential equations for the vertical displacement u and the Airy stress function v. One of the equations is elliptic while the other is hyperbolic if **rotational inertia** is taken into account; otherwise, it is parabolic. The coupling takes place through quadratic nonlinearities in the second-order spatial derivatives of u and v. Transverse shear effects are not modeled. In the study of the energy decay of the von Karman system, we are concerned with analyzing the rate at which the energy dissipates over time and the influence of memory and time delay on the stability of the system. The objective of studying the energy decay of the von Karman system is twofold:

Stability Analysis: The energy decay analysis helps determine the stability and long-term behavior of the system. By examining the rate at which the system's energy dissipates, one can assess whether the system tends to a stable equilibrium state or exhibits unstable behavior. Understanding the stability properties of the von Karman system is crucial for predicting and controlling the plate's response under different loads and boundary conditions.

System Optimization and Design: Analyzing the energy decay provides insights into the efficiency and performance of thin plate structures. The rate at which the energy dissipates can inform the design and optimization of plate-like structures to minimize energy losses, enhance stability, and improve overall structural integrity. By understanding how the energy decays in the system, engineers can make informed decisions regarding material selection, structural configurations, and load-bearing capacities.

We study the energy decay of von Karman equations with, or without, rotational forces inertia, with memory and discrete time delay. In the first chapter discuses some of the previous concepts and knowledge of the energy method and types of time delay. In the second chapter we consider the case of von Karman equations with rotational inertia, infinite memory and hinged boundary conditions.This chapter is based on the paper [36]. In the third chapter we study the same problem as in the second chapter with constant discrete time delay. In the last chapter we study the von Karman equations without rotational inertia, with Dirichlet boundary conditions, finite memory and discret variable time delay.This chapter is based on the paper [11].

Chapter 1

Preliminaries

1.1 The Spaces L^p for $p \in [1, +\infty]$

We assume known the definition of (Lebesgue) measurable functions and of the space $L^1(\Omega)$ of summable functions on Ω, endowed with the norm defined by $\|f\|_1 = \int_\Omega |f(x)|dx$.

Definition 1.1.1 The space of functions on Ω with summable functions p-th powers is defined by

$L^p(\Omega, \mathbb{C}) = \{u \quad \text{measurable on} \quad \Omega, \text{with measurable on in } \mathbb{C} \mid |u|^P \in L^1\}$

Let $L^\infty(\Omega)$ be the space of measurable functions f such that

$\exists\ \alpha > 0, mesE_\alpha = mes\{x||f(x)| > \alpha\} = 0$

This is a normed space with norm $\|f\|_\infty = inf\{\alpha \mid mes(E_\alpha) = 0\}$

see([2])

Theorem 1.1 – Young's Inequality with a Parameter

Let a and b two non-negative real numbers. For all $\lambda > 0$

$$ab \leq \tfrac{1}{2\lambda}a^2 + \tfrac{\lambda}{2}b^2$$

Proof. Suppose that a and b are in $[0, \infty[$, and $\lambda > 0$. We know that $(a - \lambda b)^2 \geqslant 0$. As a result

$2\lambda ab \leqslant a^2 + \lambda^2 b^2 \Longrightarrow ab \leq \frac{\lambda}{2}b^2 + \frac{1}{2\lambda}a^2$ □

Lemma 1.1.1 – Modified Young's Inequality

Let $a, b \geqslant 0,\ \varepsilon > 0,\ p, q > 1,\ \frac{1}{p} + \frac{1}{q} = 1.$

Then

$$ab \leq \frac{\varepsilon a^p}{p} + \frac{\varepsilon^{1-q} b^q}{q}.$$

If $p = q = 2$, we result that

$$ab \leq \frac{\varepsilon a^2}{2} + \frac{b^2}{2\varepsilon}$$

1.2 Hölder Inequality and the Completeness of L^p

For $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$ with real numbers p and p' satisfying $1 < p < \infty$ and $1/p + 1/p' = 1$, we have the inequality

$\int_\Omega \mid f(x)g(x) \mid dx \leq [\int_\Omega |f(x)|^p dx]^{1/p} \cdot [\int_\Omega |g(x)|^{p'} dx]^{1/p'}.$

This inequality can be generalized by considering real numbers $p_j > 1$ such that the sum of thier inverses euquals 1:

$\forall f_j \in L^{p_j}, \int_\Omega |\Pi f_j(x)| dx \leq \Pi[(\int_\Omega |f_j|^{p_j})^{1/p_j}]$

see([2])

1.3 (Lebesgue's) Dominated Convergence Theorem

Let (f_n) be a sequence of measurable functions on measure space (S, Σ, μ).

Suppose that (f_n) convergences pointwise to a function f an is dominated by some Lebesgue integral function g, i.e. $|f_n(x)| \leq g(x)\ \forall n$ and $\forall x \in S$.

Then, f is Lebesgue integrable, and

$\lim_{n \longrightarrow \infty} \int_S f_n d\mu = \int_S f d\mu.$

1.4 Sobolev Spaces and Embedding Theorems

Definition 1.4.1 Let Ω be an open subset of $\mathbb{R}^n$. For $m \in \mathbb{N}$ and $1 \leq p \leq +\infty$, the Sobolev space denoted by $W^{m,p}(\Omega)$ consists of the functions in $L^p(\Omega)$ whose partial derivatives up to order m, in the sense of distributions, can be identified with function in $L^p(\Omega)$.

For these derivatives, we set $\alpha = (\alpha_1, ..., \alpha_N)$and $\mid \alpha \mid = \sum_1^N \alpha_i$. Moreover, we use the notation

$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial^{\alpha_1} x_1 .... \partial^{\alpha_n} x_n}$ the definition above can now be written as

$W^{m,p}(\Omega) = \{u \in L^p(\Omega) \; |\forall \alpha \in \mathbb{N}^n, |\alpha| \leq m \Longrightarrow D^\alpha u \in L^p(\Omega)\}$

see([2]).

Remark 1.4.1

- For $p = 2$, the notation $W^{m,2}(\Omega)$ is generally replaced by $H^m(\Omega)$.
- When $\Omega = \mathbb{R}^{\mathbb{N}}$, we can use the Fourier transform $\xi \longmapsto \widehat{u}(\xi)$ of a function u in $L^2(\mathbb{R}^{\mathbb{N}})$ to give the following equivalent definition :

  $W^{m,2}(\mathbb{R}^N) = H^m(\mathbb{R}^N)$=$\{u \in L^2(\mathbb{R}^N) | \xi \longmapsto (1 + |\xi|^2)^{\frac{m}{2}} \widehat{u}(\xi) \in L^2(\mathbb{R}^N)\}$.

Proposition 1.4.1. The space $W^{m,p}(\Omega)$ endowed with the norm defined by

$$\|u\|_{W^{m,p}(\Omega)} = \begin{cases} [\sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|^p_{L^p(\Omega)}]^{1/p} & \text{if} \quad 1 \leq p \leq +\infty \\ \max_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_{L^\infty(\Omega)} \text{ if } p = +\infty. \end{cases}$$

is a Banach space. For $p \in [1, +\infty[$, this space is uniformly convex and there fore a reflexive space. The space H^m endowed with the inner product.

Definition 1.4.2 Let Ω be an open subset of $\mathbb{R}^{\mathbb{N}}$, either bounded or not. We let $W_0^{m,p}(\Omega)$ denote the closure of the space $\mathcal{D}(\Omega)$ in $W^{m,p}(\Omega)$ for the norm $\|.\|_{m,p}$. In general, finding an intrinsic characterization of the functions in $W_0^{m,p}(\Omega)$ is not obvious and depends strongly on the structure of Ω. When $\Omega = \mathbb{R}^{\mathbb{N}}$, a method involving truncation and regularizzation allows us to show the following result.

Proposition 1.4.2. The space $\mathcal{D}(\mathbb{R}^{\mathbb{N}})$ is dense in $W^{m,p}(\mathbb{R}^{\mathbb{N}})$, so that

$W^{m,p}(\mathbb{R}^{\mathbb{N}}) = W_0^{m,p}(\mathbb{R}^{\mathbb{N}})$

1.5 Sobolev Embeddings for $W^{m,p}(\mathbb{R}^n)$

1.5.1 Definition of Functional Spaces

Given an integer $j \geqslant 0$, we define the family of spaces $C_b^j(\mathbb{R}^{\mathbb{N}})$ by setting

$$C_b^j(\mathbb{R}^n) = \{u \in C^j(\mathbb{R}^n) \mid \forall \alpha \in \mathbb{N}^n, \mid \alpha \mid \leq j, \exists K_\alpha, \|D^{(\alpha)}u\|_\infty \leq K_\alpha\}$$

For a positive real number λ, the subspace $C_b^{j,\lambda}(\mathbb{R}^n)$ consist of the functions in $C_b^j(\mathbb{R}^n)$ such that if $|\alpha| \leq j$, then

$\exists C_{\alpha,\lambda}, \forall x, y \in \mathbb{R}^n, \mid D^{(\alpha)}u(x) - D^{(\alpha)}u(y) \mid \leq C_{\alpha,\lambda} \mid x - y \mid^\lambda$

see([2])

1.5.2 Sobolev Embedding Theorem

Definition 1.5.1 Let X and Y be two normed spaces. We say that X is embedded in Y if there exists a contiunous injection i from X to Y, that is an injection i and a constant $C > 0$ such that

$$\forall x \in X \quad \|i(x)\|_Y \leq C\|x\|_X.$$

we denote the embedding by $X \hookrightarrow Y$.

We call the embedding compact if the operator i is compact, that is, if it maps a bounded subet of X to a relatively compact subset of Y. We denote the compact embedding by

$$X \hookrightarrow_c Y$$

Theorem 1.2

For $p \geqslant 1$ and $m \in \mathbb{N}$, we have :

- If $N > mp$ and $p > 1$, then for every q satisfying $p \leq q \leq Np/(N-mp)$, we have $W^{m,p}(\mathbb{R}^{\mathbb{N}}) \hookrightarrow L^q(\mathbb{R}^{\mathbb{N}})$. More precisely, under the given condition, there exits a constant C such that

  $\forall \varphi \in W^{m,p}(\mathbb{R}^{\mathbb{N}}),\ \|\varphi\|_q \leq C\|\varphi\|_{W^{m,p}(\mathbb{R}^{\mathbb{N}})}$

- If $p = 1$, we have $W^{N,1}(\mathbb{R}^{\mathbb{N}}) \hookrightarrow C_b(\mathbb{R}^{\mathbb{N}})$.

- If $N = mp$ and $p > 1$, then for every q satisfying $p \leq q < \infty$, we have $W^{m,p}(\mathbb{R}^{\mathbb{N}}) \hookrightarrow L^q(\mathbb{R}^{\mathbb{N}})$.

- If $mp > N, N/p \notin \mathbb{N}$, and j satisfies $(j-1)p < N < jp$, then

  $0 < \lambda \leq j - N/p \Longrightarrow W^{m,p}(\mathbb{R}^{\mathbb{N}}) \hookrightarrow C_b^{m-j,\lambda}(\mathbb{R}^{\mathbb{N}})$.

  If $mp > N \in \mathbb{N}$ and $m \geqslant j = N/p+1$, then $W^{m,p}(\mathbb{R}^{\mathbb{N}}) \hookrightarrow C_b^{m-N/p-1,\lambda}(\mathbb{R}^{\mathbb{N}})$

- If $p > N$, then we have

  $0 < \lambda \leq 1 - N/p \Longrightarrow W^{1,p}(\mathbb{R}^{\mathbb{N}}) \hookrightarrow C_b^{0,\lambda}(\mathbb{R}^{\mathbb{N}})$

see([2])

Theorem 1.3

Given a Lipschitz open set Ω, we have:

- (1) If $N > mp$, then $W^{m,p}(\Omega) \hookrightarrow L^q$ for every $q < Np/(N-mp)$.

- (2) If $N = mp$, then $W^{m,p}(\Omega) \hookrightarrow L^q$ for every $q < \infty$. If $p = 1$, then $W^{N,1} \hookrightarrow C_b(\Omega)$

- (3) If $mp > N$ with $N/p \notin \mathbb{N}$ and if j satisfies $(j-1)p < N < jp$, then we have

$$W^{m,p}(\Omega) \hookrightarrow C_b^{m-j,\lambda}(\Omega),\ \forall \lambda \leq j - N/p.$$

- If $N/p \in \mathbb{N}$ and $m \geqslant j = N/p+1$, then $W^{m,p}(\Omega) \hookrightarrow C_b^{m-(N/p)-1,\lambda}(\Omega)$ for every $\lambda < 1$

see([2]).

Theorem 1.4

Let Ω be a bounded Lipschitz open subset of $\mathbb{R}^n$, where $N > 1$. If $N > mp$, then the embedding $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for $q < Np/(N - mp)$

see([2]).

Corollary 1.5.1. For any bounded Lipschitz open subset Ω of $\mathbb{R}^{\mathbb{N}}$ we have

$$W^{1,1}(\Omega) \hookrightarrow_c L^1(\Omega)$$

1.5.3 Trace theorem

The trace operator can be defined for functions in the Sobolev space $W^{m,p}(\Omega)$, with $1 < p < \infty$ see the section below for possible extensions of the trace to other spaces. Let $\Omega \subset \mathbb{R}^n$ for $n \in \mathbb{N}$ be a bounded domain with lipschitz boundary. Then there exists a bounded linear trace operator

$$T_m : W^{m,p}(\Omega) \longrightarrow \Pi_{l=0}^{m-l} W^{m-l-1/p,p}(\partial\Omega).$$

The operator T_m extends the classical normal traces in the sense that

$T_m u = (u|_{\partial\Omega}, \partial_n u|_{\partial\Omega}, ..., \partial_n^{m-1} u|_{\partial\Omega})$ for all $u \in W^{m,p}(\Omega) \cap C^{m-1}(\bar{\Omega})$.

Finally, the space $W_0^{m,p}(\Omega)$, the completion of $C_c^\infty(\Omega)$ in the $W^{m,p}(\Omega)$ norm can be characterized as the kernel of T_m i.e.

$W_0^{m,p}(\Omega) = \{u \in W^{m,p}(\Omega) | T_m u = 0\}$

see([2]).

Theorem 1.5 – Generalized Green's Formula

Let Ω be a class C^1 open subset of $\mathbb{R}^n$. Let U be element of $W^{1,p}(\Omega)$ and let $\varphi \in \mathcal{D}(\mathbb{R}^n)$; then

$\int_\Omega \nabla U(x) \cdot \varphi(x)dx + \int_\Omega U(x)\text{div}\varphi(x)dx = \int_{\partial\Omega} T_0 U(s)\varphi(s) \cdot \vec{n}(s)d\sigma(s).$

In this formula, $d\sigma(s)$ is the superficial density on $\partial\Omega$, $\vec{n}$ is the outwar-pointing unit normal to $\partial\Omega$, the terms $\nabla U(x) \cdot \varphi(x)$ and $\varphi(s) \cdot \vec{n}(s)$ are inner products of vectors in $\mathbb{R}^n$, and the divergence of φ is defined to be $div\varphi(x) = \sum_{i=1}^n \partial_i(\varphi_i)(x)$ see([2]).

Lemma 1.5.1 – Gronwall Inequality

We assume that $u \in C([0.T], \mathbb{R}) \quad T \in]0, \infty[$, satisfies the differential inequality

$$u' \leqslant a(t)u + b(t) \text{ on } (0, T),$$

for some $a, b \in L^1(0, T)$. Then u satisfies the primitive estimate

$$u(t) \leqslant e^{A(t)}u(0) + \int_0^t b(s)e^{A(t)-A(s)}ds, \ \forall t \in [0, T]$$

where we have defined the primitive function $A(t) := \int_0^t a(s)ds$
see([3]).

1.6 Differential Equations with Delay: Brief Overview

Hereditary systems are characterized by the dependence of the state of the system on a period or a certain moment in the past. Such systems are often modeled by introducing functions with delayed arguments into partial differential equations (PDEs). Let $u = u(x, t)$ be the unknown function and ω be the function with time delay. Then, we have several different possibilities:

1. PDEs with constant delay (or, usually, PDEs with delay) contain a function ω of

the form $\omega = u(x, t - \tau)$, where $\tau > 0$ is the constant delay time;

2. PDEs with proportional delay contain a function ω of the form $\omega = u(x, qt)$, where q is the scaling parameter, $0 < q < 1$;

3. PDEs with variable delay contain a function ω of the form $\omega = u(x, t - \tau(t))$, where $\tau(t) > 0$ is the variable delay[1].

Remark 1.6.1

Delays of types (1), (2), or (3) can also occur in the argument x.

Remark 1.6.2

If there is no dependence on x, one can easily have similar items for delay ordinary differential equations (ODEs)

In more complex cases, the variable delay can depend either on the spatial coordinate, i.e., be spatially anisotropic, $\tau = \tau(x)$, or on both spatial and time arguments, $\tau = \tau(x, t)$, or even on both arguments and the desired solution, $\tau = \tau(x, t, u)$.

1.7 PDEs with Spatially Anisotropic Time Delay

By saying a PDE with spatially anisotropic time delay (briefly, a PDE with anisotropic time delay) we mean a functional-differential equation that, in addition to the desired function $u(x, t)$, also contains a delayed function of the form $u(x, t\tau(x))$, where $\tau(x)$ is a given positive function, and partial derivatives of u with respect to x and t. PDEs with an anisotropic time delay can model delay systems in anisotropic and inhomogeneous media, in which the signal propagation speed depends on the chosen direction or varies at different points of the medium. For example, in medicine, this can be differences in the rate of transmission of a nerve impulse in healthy and sick tissues of the human body.

1.8 An Example on the Effect of Time Delays

This example is concerned with the effect of time delays in boundary feedback stabilization schemas for wave equations. The question to be adddressed is whether such delay can destabilize a system which is uniformly asymptotically stable in the absence of delays. Consider the problem:

$$(p)\begin{cases} u_{tt} - u_{xx} + 2au_t + a^2u = 0, \quad 0 < x < 1, \quad t > 0 \text{ .....(1)} \\ u(0,t) = 0, \quad t > 0 \text{ ......(2)} \\ u_x(1,t) = -ku_t(1,t), \quad t > 0 \text{ ....(3)} \end{cases}$$

where $a \geqslant 0$ and $k \geqslant 0$. The question to treated is the effect on stability of (1)-(3) of a time delay in the right side of (3) when $k > 0$.
On multiplying (1) by u_t, we easily derive that

$$E'(t) = -2a\int_0^1 u_t^2 dx - ku_t^2(1,t) \leq 0.$$

The energy functional associated to problem (p) is defined as

$$E(t) = \frac{1}{2}\int_0^1 (u_t^2 + u_x^2 + a^2u^2)dx$$

The spectrum can be exhibited by setting $u = e^{\omega t}\phi(x)$. The eigen-values ω and eigen-functions ϕ are then determined by problem (p)

$$\phi''(x) - (a+\omega)^2\phi(x) = 0, \quad 0 < x < 1 \text{ ....(4)}$$

$$\phi(0) = 0, \quad \phi'(1) + k\omega\phi(1) = 0 \text{ .....(5)}$$

The eigenvalues are the solutions of

$$e^{2(\omega+a)} = \frac{(k-1)\omega - a}{(k+1)\omega + a} \text{ ....(6)}$$

If $a = 0$ and $k \neq 1$, (5) can be solved explicitly to yield

$Re(\omega) = \frac{1}{2}\log|\frac{k-1}{k+1}| < 0.$

When $a = 0$ and $k = 1$, all solutions of (1)-(3) can be shown to vanish identically for $t > 2$. When $a \neq 0$ the solution ϕ_n of (5) can be shown to lie in a half-plane $Re\phi \leqq \alpha < 0$ and to satisfy $|Im\omega_n| \longrightarrow +\infty$.

Now let $\varepsilon > 0$ and suppose the boundary condition (3) is replaced by

$$u_x(1,t) = -ku_t(1,t-\varepsilon), \quad t > \varepsilon \text{ .....(7)}$$

We have the following results (see [5])

Let $L = e^{-2a}$. The system (1), (2), (7) has the following stability properties:

- If $0 < k < \frac{(1-L)}{(1+L)}$, for each $\varepsilon > 0$ there exists $\beta(\varepsilon) > 0$ such that the spectrum of the system lies in $Re(\omega) \leqq -\beta$.
- If $k = (1-L)/(1+L)$, for each $\varepsilon > 0$ the spectrum lies in $Re(\omega) < 0$, but there is a countably dense set M in $(0,\infty)$ such that for each ε in M there is a sequence $\{\omega_n\}$ in the spectrum such that

  $\lim_{n\to\infty} Re(\omega_n) = 0$
- If $k > (1-L)/(1+L)$, there is dense open set D in $(0,\infty)$ such that for each ε in D the system admits exponentially unstable solution.

1.9 Energy Method

Consider a dynamical system defined on the set $\Omega \times \mathbb{R}^+$, where $u(x,t)$ is the solution.

Let E denote the energy of the system, which is of the form

$$E(t) = \int_{\Omega} f(u, u_x, u_t, \ldots) dx \geq 0, \quad \forall t \geq 0.$$

There are two common properties of E:

1. If $\frac{dE}{dt} = 0$, then E is called conserved and the system is conservative.
2. If $\frac{dE}{dt} \leq 0$, then E is called dissipated and the system is dissipative.

Exemple1: Consider the problem (wave problem)

$$\begin{cases} u_{tt} - u_{xx} = 0 \text{ in }]0, L[\times \mathbb{R}^+ \\ u(0,t) = u(L,t) = 0 \;, \quad t > 0 \\ u(x,0) = u_0(x), \quad u_x(x,0) = u_1(x), \quad \text{in }]0, L[\end{cases} .$$

The functional energy of this system defined by

$$E(t) = \frac{1}{2} \int_0^L \left[(u_x)^2 + (u_t)^2 \right] dx$$

is conserved.

Indeed, using the wave equation and the boundary conditions we get

$$\frac{dE}{dt} = \int_0^L \left[u_x u_{tx} + u_t u_{tt} \right] dx = \int_0^L \left[u_x u_{tx} + u_t u_{xx} \right] dx = \int_0^L \left[u_x u_{tx} - u_{tx} u_x \right] dx + \underbrace{\left[u_t u_x \right]_{x=0}^{x=L}}_{=0} = 0.$$

Exemple2: Consider the problem (diffusion problem)

$$\begin{cases} u_t - u_{xx} = 0 \text{ in }]0, L[\times \mathbb{R}^+ \\ u(0,t) = u(L,t) = 0 \ , \quad t > 0 \\ u(x,0) = u_0(x) \quad \text{in }]0, L[\end{cases} \cdot$$

The functional energy of this system defined by

$$E(t) = \frac{1}{2} \int_0^L (u_x)^2 \, dx$$

is dissipated.

Indeed, using the diffusion equation and the boundary conditions we get

$$\frac{dE}{dt} = \int_0^L u_x u_{tx} dx = -\int_0^L u_{xx} u_t dx + \underbrace{[u_t u_x]_{x=0}^{x=L}}_{=0} = -\int_0^L (u_{xx})^2 \, dx < 0.$$

The stability of a system generally refers to its ability to return to its initial state when an external disturbance ceases. The Lyapunov stability theorem defines the stability of a system in terms of energy, the biggest advantage of which is that the stability can be determined without the need to solve the motion equation of the system.

If the system is dissipative, then the energy is decreasing. In order to study the stability of the system, it is interesting to know the decay rate of this energy.

◊ System stability is said to be **strong** if

$$\lim_{t \longrightarrow \infty} E(t) = 0.$$

◊ System stability is said to be **exponential** (or uniform) if

$$\exists c_1, c_2 > 0 : \ E(t) \leq c_1 \exp(-c_2 t), \quad \forall t \geq 0.$$

◊ System stability is said to be **polynomial** if

$$\exists c_1, c_2 > 0 : \; E(t) \leq c_1 t^{-c_2}, \quad \forall t > 0.$$

1.10 Legendre Transformation

The Legendre tranformation is a very useful mathematical tool: it transforms functions on a vector space to functions on the dual space. Legendre transformations are related to projective duality and tangential coordinates in algebraic geometry and the construction of dual Banach spaces in analysis. They are often encountered in physics (for example, in the definition of thermodynamic quantities).

Definition 1.10.1 Let $y = f(x)$ be a convex function, $f''(x) > 0$.

The Legendre transformation of the function f is a new function g of a new variable p which is constructed in the following way (Figure 1.1). We draw the graph of f in the x, y plane. Let p be a given number. Consider the

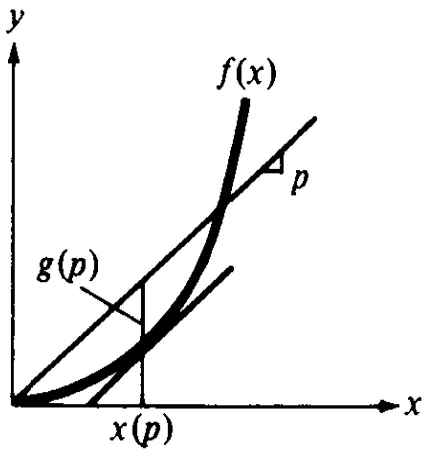

Figure 1.1: Legendre transformation

straight line $y = px$. We take the point $x = x(p)$ at which the curve is farthest from the straight line in the vertical direction: for each p the function $px - f(x) = F(p, x)$ has a maximun with respect to x at the point $x(p)$. Now we define $g(p) = F(p, x(p))$. The point $x(p)$ is defined by the extremal condition $\frac{\partial F}{\partial x} = 0$, i.e., $f'(x) = p$. Since

f is convex, the point $x(p)$ is unique, see([16]).

Exemples:

- Let $f(x) = x^2$. Then $F(p, x) = px - x^2$, $x(p) = \frac{1}{2}p$, $g(p) = \frac{1}{4}p^2$.
- Let $f(x) = \frac{x^\alpha}{\alpha}$. Then $g(p) = \frac{p^\beta}{\beta}$, where $(\frac{1}{\alpha}) + (\frac{1}{\beta}) = 1$

Definition 1.10.2 Two functions, f and g, which are the Legendre transforms of one another are called dual in the sense of Young.

By definition of the Legendre tranform, $F(x, p) = px - f(x)$ is less than or equal to $g(p)$ for any x and p. From this we have Young's inequality:

$$px \leq f(x) + g(p).$$

Example: If $f(x) = \frac{1}{2}x^2$, then $g(p) = \frac{1}{2}p^2$ and we obtain the well-known inequality

$$px \leq \frac{1}{2}x^2 + \frac{1}{2}p^2 \text{ for all } x \text{ and } p.$$

Definition 1.10.3 Let $f(x)$ be a real valued function defined on the interval $I = [a, b]$. f is said to be **convex** if for every $x_1, x_2 \in [a, b]$ and $\lambda \in [0, 1]$,

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2).$$

A function is said to be **strictly convex** if the inequality is strict for $x_1 \neq x_2$.

Theorem 1.6 – Jensen's Inequality

Let $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function. For any bounded open subset Ω of $\mathbb{R}^n$ and any nonnegative function $f \in L^1(\Omega)$, we have

$$\phi(\frac{1}{meas\Omega} \int_\Omega f(x)dx) \leqslant \frac{1}{meas\Omega} \int_\Omega \phi(f(x))dx$$

see[4].

Chapter 2

Energy Decay Rate for Nonlinear von Kármán Equations with Memory

2.1 Introduction

von Kármán equations describe vibrations of nonlinear plates with large deflection, having many applicaion in physics, control theory science, viscoelasticity, aerodynamic and so on. As useful equation comming from the real world, nonlinear von Kármán equations have attracted many researchers attention and we refer the reader to [9, 16] an the references therein for more information about the equation. In this chapter, we consider a nonlinear von Kármán equation with infinite memory, subject to the hinged boundary condition:

$$\begin{cases} u_{tt} + \Delta^2 u - \gamma\Delta u_{tt} - [u,v] - \int_{-\infty}^{t} g(t-s)\Delta^2 u(s)ds + f(u) = 0 & \text{in } \Omega \times [0,+\infty) \\ \Delta^2 v + [u,u] = 0 & \text{in } \Omega \times [0,+\infty) \\ u = \Delta u = 0, v = \frac{\partial v}{\partial \nu} = 0 & \text{in } \partial\Omega \times [0,+\infty) \\ u(x,-t) = u_0(x,t), u_t(x,-t) = \partial_t u_0(x,t) & \text{in } \Omega \times [0,+\infty) \end{cases} \tag{2.1}$$

where $u = u(x,t)$ represents the displacement of a plate, $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$, ν is the unit outward normal vector along the boundary, $\gamma > 0$, the memory kernel g is positive, $f(u)$ represents a source term, and $[u,v]$ is the

Von Kármán bracket defined by $[u, v] = u_{xx}v_{yy} + u_{yy}v_{xx} - 2u_{xy}v_{xy}$.

Hypotheses

i) The memory kernel function g satisfies:

$g : [0, \infty) \longrightarrow (0, \infty)$ is a non-increasing and locally absolutely continuous function satisfying

$$g'(t) \leq -\xi(t)g(t), \xi(t) > 0, \xi'(t) \leq 0, \forall t \geq 0,$$

and $1 - \int_0^\infty g(s)ds = l > 0$.

In addition, we have

$$meas(\Sigma) = 0, \text{where } \Sigma := \{s \geqslant 0 : g'(s) = 0\}, \tag{2.2}$$

and $\int_0^\infty \xi(s)ds = \infty$.

ii) The nonlinear function f satisfies:

$f \in C^1(\mathbb{R})$, $\mid f'(t) \mid \leq C_0(1+ \mid t \mid^p)$, $\forall t \in \mathbb{R}$ (with constants $C_0 > 0$ and $p \geqslant 0$), and

$$tf(t) \geqslant C_0F(t) \geqslant 0, \forall t \in \mathbb{R} \tag{2.3}$$

with a constant $C_0 > 0$, where $F(t) := \int_0^t f(s)ds$.

Definition 2.1.1 Let (.,.) and $\|.\|$ denote the inner product and the norm of the element in $L^2(\Omega)$ respectively. For a Hilbert space X, $L^2_g(\mathbb{R}^+$,X) is a X -valued function space equipped with the norm

$$\|u\|_{g,X} = \int_0^{+\infty} g(s)(u(s), u(s))_X ds$$

set $V = H^1_0 \cap H^2, V_1 = \{u \in H^3(\Omega) : u = \Delta u = 0 \ on \ \partial\Omega\}$

Then, we define the phase spaces

$\mathcal{H} = V \times H^1_0(\Omega) \times L^2_g(\mathbb{R}^+, V)$ with $\|(u, u_t, \eta)\|^2_{\mathcal{H}} := \|\Delta u\|^2 + \|\nabla v\|^2 + \|\eta\|^2_{g,V}$.

$\mathcal{H}' = V_1 \times V \times L^2_g(\mathbb{R}^+, V_1)$ with $\|(u, u_t, \eta)\|^2_{\mathcal{H}'} := \|\Delta u\|^2 + \|\nabla\Delta v\|^2 + \|\eta\|^2_g, V_1$.

Lemma 2.1.1

Let $u, \varphi, \psi \in H^2(\Omega)$. If at least one of them belongs to $H_0^2(\Omega)$, then

$$([u, \varphi], \psi) = ([u, \psi], \varphi). \tag{2.4}$$

If $\varphi = \varphi(u) := -(\Delta_D^2)^{-1}[u, u]$, then

$$\|[u, \varphi]\| \leq C_1 \|u\|_{H^2} \|\varphi\|_{W^{2,\infty}} \leq C_2 \|u\|_{H^2}^2, \text{with constant } C_1, C_2 > 0. \tag{2.5}$$

We refer the to [9] for proofs of the above lamma.

We follow Dafermos [6], we define, $\eta = \eta^t(x, s) = u(x, t) - u(x, t - s)$. Then (2.1) transforms into the system:

$$\begin{cases} u_{tt} + l\Delta^2 u_{tt} - \gamma \Delta u_{tt} - [u, v] - \int_0^\infty g(s) \Delta^2 \eta ds + f(u) = 0 & \text{in } \Omega \times [0, +\infty) \\ \Delta^2 v + [u, u] = 0, \eta_t + \eta_s = u_t & \text{in } \Omega \times [0, +\infty) \\ u = \Delta u = 0, v = \frac{\partial v}{\partial \nu} = 0, \eta = \Delta \eta = 0 & \text{in } \partial\Omega \times [0, +\infty) \\ u(x, 0) = u_0(x), u_t(x, 0) = u_t(x), \eta^0(x, s) = \eta_0(x, s) & \text{in } \Omega \times [0, +\infty) \end{cases} \tag{2.6}$$

here $\eta_0(x, s) := u_0(x) - u_0(x, s)$, and $l := 1 - \int_0^\infty g(s) ds$ (positive by hypothesis).

Proof. Let $\eta = \eta^t(x, s) = u(x, t) - u(x, t - s)$, and $z = t - s$:

$\eta_t = u_t(x, t) - u_t(x, z) = u_t(x, t) - \frac{\partial u}{\partial z} \times \frac{\partial z}{\partial t} = u_t(x, t) - \frac{\partial u}{\partial z}$

$\eta_s = u_s(x, t) - u_s(x, z) = -\frac{\partial u}{\partial z} \times \frac{\partial z}{\partial s} = \frac{\partial u(x,z)}{\partial z}$

then $\eta_s + \eta_t = u_t(x, t)$,

$\eta^0(x, s) = u(x, 0) - u(x, -s) = u_0(x, 0) - u_0(x, s) = u_0(x) - u_0(x, s) = \eta_0(x, s)$

and $\Delta^2 \eta^t(x, s) = \Delta^2 u(x, t) - \Delta^2 u(x, t - s)$, $z = t - s \Longrightarrow s = t - z$ and $ds = -dz$

$\int_{-\infty}^t g(t - s) \Delta^2 u(s) ds = \int_0^\infty g(z) \Delta^2 u(t - z) dz = \int_0^\infty g(s) \Delta^2 u(t - s) ds$

$\Delta^2 u(t - s) = \Delta^2 u(t) - \Delta^2 \eta^t(s)$, then $\int_0^\infty g(s)\Delta^2 u(t-s)ds = \int_0^\infty g(s)\Delta^2 u(t)ds - \int_0^\infty g(s)\Delta^2\eta^t(s)ds = (\int_0^\infty g(s)ds)\Delta^2 u(t) - \int_0^\infty g(s)\Delta^2\eta^t(s)ds.$ □

Now, we state a well posedness result about problem (2.1), which can be proved with the arguments in [14].

Proposition 2.1.1. (Well-Posedness) Let $T > 0$, for every $z_0 = (u_0, u_1, \eta_0) \in \mathcal{H}$, the problem (2.6) has a unique weak solution $z = (u, u_t, \eta) \in C([0,T];\mathcal{H})$. For each $t \in (0,T)$, the mapping $z_0 \longmapsto z(t)$ is continuous in $\mathcal{H}$. If furthermore $z_0 \in \mathcal{H}'$, then z has higher regularity

$$u \in L^\infty(0,T;V_1), u_t \in L^\infty(0,T;V),\ \eta \in L^\infty(0,T;L^2_g(\mathbb{R}^+,V_1)).$$

Corollary 2.1.1. Define the energy functional of the nonlinear von Kármán system by

$$E(t) = \tfrac{1}{2}\|u_t\|^2 + \tfrac{l}{2}\|\Delta u\|^2 + \tfrac{\gamma}{2}\|\nabla u_t\|^2 + \tfrac{1}{4}\|\Delta v\|^2 + \tfrac{1}{2}\int_0^\infty g(s)\|\Delta\eta\|^2 ds + \int_\Omega F(u)dx.$$

Noticing $v = -(\Delta_D^2)^{-1}[u,u]$, we have

$$\tfrac{d\|\Delta v\|^2}{dt} = -2(\tfrac{d[u,u]}{dt}, v) = -4([u,u_t],v) = -4([u,v],u_t)$$

in view of (2.6). Hence

$$\frac{dE(t)}{dt} = \frac{1}{2}\int_0^{+\infty} g'(s)\|\Delta\eta^t(s)\|^2 ds \le 0 \tag{2.7}$$

the energy is decreasing.

Proof. Let the system (2.6)

$$\begin{cases} u_{tt} + l\Delta^2 u_{tt} - \gamma\Delta u_{tt} - [u,v] - \int_0^\infty g(s)\Delta^2\eta ds + f(u) = 0 & \text{in } \Omega\times[0,+\infty) \\ \Delta^2 v + [u,u] = 0, \eta_t + \eta_s = u_t & \text{in } \Omega\times[0,+\infty) \\ u = \Delta u = 0, v = \frac{\partial v}{\partial\nu} = 0, \eta = \Delta\eta = 0 & \text{in } \partial\Omega\times[0,+\infty) \\ u(x,0) = u_0(x), u_t(x,0) = u_t(x), \eta^0(x,s) = \eta_0(x,s) & \text{in } \Omega\times[0,+\infty) \end{cases}$$

We multiply the first equation by u_t and integrate over Ω we get :

$\int_\Omega u_{tt}u_t dx + l\int_\Omega \Delta^2 u_{tt}u_t dx - \gamma\int_\Omega \Delta u_{tt}u_t dx - \int_\Omega [u,v]u_t dx + \int_\Omega(\int_0^\infty g(s)\Delta^2\eta ds)u_t ddx + \int_\Omega f(u)u_t dx = 0.$

- (1) Simplification of $\int_\Omega u_{tt}u_t dx$: we use $\frac{\partial}{\partial t}(u_t^2) = 2u_t u_{tt} \Rightarrow u_t u_{tt} = \frac{1}{2}\frac{\partial}{\partial t}u_t^2$

  then $\int_\Omega u_{tt}u_t dx = \frac{1}{2}\int_\Omega \frac{\partial}{\partial t}u_t^2 dx = \frac{\partial}{\partial t}\{\frac{1}{2}\|u_t\|^2\}$

- (2) Simplification of $\int_\Omega \Delta^2 u u_t dx$: using the Green's formula and the boundary conditions we get

  $\int_\Omega \Delta^2 u \cdot u_t dx = \int_\Omega \Delta(\Delta u)u_t dx = \int_\Omega \Delta u\Delta u_t dx + \int_{\partial\Omega}\partial_n(\Delta u)u_t d\sigma - \int_{\partial\Omega}\Delta u\partial_n u_t d\sigma = \int_\Omega \Delta u\Delta u_t dx.$

- (3) Simplification of $\int_\Omega \Delta u_{tt}u_t dx$ using the Green's formula and the boundary conditions we get

  $\int_\Omega \Delta u_{tt}u_t dx = \int_{\partial\Omega}\partial_n u_{tt}u_t d\sigma - \int_\Omega \nabla u_{tt}\nabla u_t dx = -\int_\Omega \nabla u_{tt}\nabla u_t dx$

  $= -1/2\frac{\partial}{\partial t}\int_\Omega(\nabla u_t)^2 dx = -1/2\frac{\partial}{\partial t}\|\nabla u_t\|^2.$

- (4) Simplification of $\int_\Omega [u,v]u_t dx$:

  $\int_\Omega [u,v]u_t dx = \int_\Omega [u,u_t]v dx$, using

  $\frac{d}{dt}[u,u] = [u,u_t] + [u_t,u] = 2[u,u_t]$

  $[u,u_t] = \frac{1}{2}\frac{d}{dt}[u,u]$

  then $\int_\Omega [u,v]u_t dx = \int_\Omega \frac{1}{2}\frac{d}{dt}[u,u]v dx,$

  using the equation $\Delta^2 v = -[u,u]$

  then

  $\int_\Omega [u,v]u_t dx = -\frac{1}{2}\int_\Omega \frac{d}{dt}(\Delta^2 v)v dx = -\frac{1}{2}\int_\Omega \Delta^2 v_t v dx =$

  $-\frac{1}{2}\int_{\partial\Omega}\partial_n\Delta v_t v d\sigma + \frac{1}{2}\int_{\partial\Omega}\Delta v_t\partial_n v d\sigma - \frac{1}{2}\int_\Omega \Delta v_t\Delta v dx = -\frac{1}{4}\frac{\partial}{\partial t}\int_\Omega(\Delta v)^2 dx = -\frac{1}{4}\frac{\partial}{\partial t}\|\Delta v\|^2.$

- (5) Simplification of $\int_\Omega \int_0^\infty g(s)\Delta^2\eta ds u_t dx$: using Fubini lemma and integration by parts and Gronwall inequality, we get
  $\int_\Omega \int_0^\infty g(s)\Delta^2\eta ds u_t dx = \int_\Omega \int_0^\infty g(s)\Delta^2\eta u_t ds dx = \int_0^\infty g(s)\int_\Omega \Delta^2\eta u_t dx ds$

  $\int_\Omega \Delta^2\eta u_t dx = \int_{\partial\Omega} \partial_n \Delta\eta u_t d\sigma - \int_{\partial\Omega} \Delta\eta \nabla u_t d\sigma + \int_\Omega \Delta\eta\Delta u_t dx,$

  given that $\Delta u_t = \Delta\eta_t + \Delta\eta_s$, we have

  $\int_0^\infty g(s)\int_\Omega \Delta^2\eta u_t dx ds = \int_0^\infty g(s)\int_\Omega \Delta\eta\Delta u_t dx ds = \int_0^\infty g(s)\int_\Omega \Delta\eta(\Delta\eta_t + \Delta\eta_s) dx ds$

  $= \int_0^\infty \int_\Omega \Delta\eta\Delta\eta_t dx ds + \int_0^\infty \int_\Omega \Delta\eta\Delta\eta_s dx ds = \frac{1}{2}\int_0^\infty g(s)\frac{\partial}{\partial t}\|\Delta\eta\|^2 ds + \frac{1}{2}\int_0^\infty g(s)\frac{\partial}{\partial s}\|\Delta\eta\|^2 ds$

  $= \frac{1}{2}\frac{\partial}{\partial t}\int_0^\infty g(s)\|\Delta\eta\|^2 ds + \frac{1}{2}[g(s)\|\Delta\eta\|^2]_0^\infty - \frac{1}{2}\int_0^\infty g'(s)\|\Delta\eta\|^2 ds$

  $= \frac{1}{2}\frac{\partial}{\partial t}\int_0^\infty g(s)\|\Delta\eta\|^2 ds - \frac{1}{2}\int_0^\infty g'(s)\|\Delta\eta\|^2 ds$

- (6) Simplification of $\int_\Omega f(u)u_t dx$:
  $\frac{dF}{du} = f(u)$,

  $\frac{dF}{dt} = \frac{dF}{du}\cdot\frac{du}{dt} = f(u)\cdot u_t$ then
  $\int_\Omega f(u)u_t dx = \int_\Omega \frac{dF}{dt} dx = \frac{d}{dt}\int_\Omega F(u) dx.$

  Then we have, by using (1)-(5) and (6) :
  $\int_\Omega u_{tt}u_t dx + l\int_\Omega \Delta^2 u_{tt}u_t dx - \gamma\int_\Omega \Delta u_{tt}u_t dx - \int_\Omega [u,v]u_t dx$
  $- \int_\Omega (\int_{-\infty}^0 g(s)\Delta^2\eta ds)u_t dx + \int_\Omega f(u)u_t dx$

  $= \frac{d}{dt}[\frac{1}{2}\|u_t\|^2] + \frac{d}{dt}[\frac{l}{2}\|\Delta u\|^2] + \frac{d}{dt}[\frac{\gamma}{2}\|\nabla u_t\|^2] +$

  $\frac{d}{dt}[\frac{1}{4}\|\Delta v\|^2] + \frac{1}{2}\frac{\partial}{\partial t}\int_0^\infty g(s)\|\Delta\eta\|^2 ds - \frac{1}{2}\int_0^\infty g'(s)\|\Delta\eta\|^2 ds + \frac{d}{dt}\int_\Omega F(u) dx = 0.$

Therefore

$\int_\Omega u_{tt}u_t dx + l\int_\Omega \Delta^2 u_{tt}u_t dx - \gamma\int_\Omega \Delta u_{tt}u_t dx$

$-\int_\Omega [u,v]u_t dx - \int_\Omega (\int_{-\infty}^0 g(s)\Delta^2\eta ds)u_t ddx + \int_\Omega f(u)u_t dx = \frac{d}{dt}[\frac{1}{2}\|u_t\|^2 + \frac{l}{2}\|\Delta u\|^2$

$+\frac{\gamma}{2}\|\nabla u_t\|^2\frac{1}{4}\|\Delta v\|^2 + \frac{1}{2}\int_0^\infty g(s)\|\Delta\eta\|^2 ds + \int_\Omega F(u)dx] = \frac{1}{2}\int_0^\infty g'(s)\|\Delta\eta\|^2 ds \leq 0$

$\Longrightarrow \frac{d}{dt}E(t) \leqslant 0.$

□

Now we present the energy decay rates. Write $h(t) = \int_t^\infty g(s)ds$.

Theorem 2.1

When the initial datum satifies $\int_0^\infty h(s)\|\Delta u_0(s)\|^2 ds < \infty$, the energy of (2.6) has the estimate

$$E(t) \leq \frac{C}{1+t},\ \forall t \geqslant 0$$

with a positive constant C.

To derive the energy decay rate, we introduce some auxiliary functions and give some estimates about them. Define

$$\varphi(t) = (u(t), u_t(t)) + \gamma(\nabla u(t), \nabla u_t(t)),$$
$$\psi(t) = -(\int_0^\infty g(s)\eta^t(s)ds, u_t(t)) - \gamma(\int_0^\infty g(s)\nabla\eta^t(s)ds, \nabla u_t(t)),$$
$$R(t) = \int_0^\infty h(s)\|\Delta u(t-s)\|^2 ds.$$

A simple calculation yields that for $t \geqslant 0$,

$R'(t) = \frac{d}{dt}\int_0^\infty h(s)\|\Delta u(t-s)\|^2 ds = \int_0^\infty h(s)\frac{d}{dt}(\|\Delta u(t-s)\|^2)ds =$

$-\int_0^\infty h(s)\frac{d}{ds}(\|\Delta u(t-s)\|^2)ds = -\int_0^\infty g(s)(\|\Delta u(t-s)\|^2)ds + h(0)\|\Delta u(t)\|^2$

$\leq \int_0^\infty g(s)[\|\Delta u(t)\|^2 - \frac{1}{2}\|\Delta u(t-s) - \Delta u(t)\|^2]ds + h(0)\|\Delta u(t)\|^2$

$\int_0^\infty g(s)[\|\Delta u(t)\|^2 - \frac{1}{2}\|\Delta u(t-s) - \Delta u(t)\|^2]ds = -\frac{1}{2}\int_0^\infty g(s)\|\Delta\eta^t(s)\|^2 ds + \int_0^\infty g(s)\|\Delta u(t)\|^2 ds$

$= -\frac{1}{2}\int_0^\infty g(s)\|\Delta\eta^t(s)\|^2 ds + h(0)\|\Delta u(t)\|^2.$

This gives

$$R'(t) \leq -\frac{1}{2}\int_0^\infty g(s)\|\Delta\eta^t(s)\|^2 ds + 2h(0)\|\Delta u(t)\|^2, \quad \forall t \geqslant 0. \tag{2.8}$$

Lemma 2.1.2

For any constant $\epsilon_3 > 0$, we have

$$\begin{aligned}\varphi'(t) &\leq \|u_t\|^2 + \gamma\|\nabla u_t\|^2 - l\|\Delta u\|^2 - \|\Delta v\|^2 - (u, f(u)) + \epsilon_3\|\Delta u\|^2 \\ &\quad + \frac{1}{4\epsilon_3}\int_0^\infty \frac{g(s)^2}{J_\delta(s)}ds \int_0^\infty J_\delta(s)\|\Delta\eta\|^2 ds, \ \forall t \geqslant 0.\end{aligned} \tag{2.9}$$

Here and in the sequel, $J_\delta(s) := \delta g(s) - g'(s)$, for $\delta > 0$.

Proof. By the definition of φ, and using the first equation in (2.6) and Young's Inequality With a Parameter shows

$\varphi'(t) = \frac{d}{dt}\varphi(t) = (u_t(t), u_t(t)) + (u(t), u_{tt}(t)) + \gamma(\nabla u_t(t), \nabla u_t(t)) + \gamma(\nabla u(t), \nabla u_{tt}(t))$

$= \|u_t(t)\|^2 + \gamma\|\nabla u_t\|^2 + (u(t), u_{tt}(t)) + \gamma(\nabla u(t), \nabla u_{tt}(t))$

$= \|u_t(t)\|^2 + \gamma\|\nabla u_t\|^2 + (u(t), -l\Delta^2 u + [u, v] - \int_0^\infty g(s)\Delta^2\eta ds - f(u))$

$+ (u, -\gamma\Delta u_{tt}) + \gamma(\nabla u(t), \nabla u_{tt}(t))$

$= \|u_t(t)\|^2 + \gamma\|\nabla u_t\|^2 + (u(t), -l\Delta^2 u + [u, v] - \int_0^\infty g(s)\Delta^2\eta ds - f(u))$.

From $([u, v], u) = ([u, u], v) = -(\Delta^2 v, v) = -\|\Delta v\|^2$, we have

$\varphi'(t) = \|u_t\|^2 + \gamma\|\nabla u_t\|^2 - l\|\Delta u\|^2 - \|\Delta^2 v\|^2 - (\int_0^\infty g(s)\Delta\eta ds, \Delta u(t)) - (u, f(u))$.

Using Hölder inequality:

$(\int_0^\infty g(s)\Delta\eta ds, \Delta u(t)) \leqslant \| \int_0^\infty g(s)\Delta\eta ds\| \|\Delta u\|$,

then

$\varphi'(t) \le \frac{1}{4\epsilon_3}\| \int_0^\infty \frac{g(s)}{\sqrt{J_\delta(s)}}\sqrt{J_\delta(s)}\Delta\eta ds\|^2 + \epsilon_3\|\Delta u\|^2 = \frac{1}{4\epsilon_3}\int_\Omega(\int_0^\infty \frac{g(s)}{\sqrt{J_\delta(s)}}\sqrt{J_\delta(s)}\Delta\eta ds)^2 dx + \epsilon_3\|\Delta u\|^2$

$\leqslant \frac{1}{4\epsilon_3}\int_\Omega(\int_0^\infty \frac{g(s)^2}{J_\delta(s)}ds)(\int_0^\infty J_\delta(s)\Delta^2\eta ds)dx + \epsilon_3\|\Delta u\|^2$

$\le \frac{1}{4\epsilon_3}\int_0^\infty \frac{g(s)^2}{J_\delta(s)}ds \int_0^\infty J_\delta(s)\|\Delta\eta\|^2 ds + \epsilon_3\|\Delta u\|^2$

and therefore

$\varphi'(t) \le \|u_t\|^2 + \gamma\|\nabla u_t\|^2 - l\|\Delta u\|^2 - \|\Delta v\|^2 - (u, f(u)) + \epsilon_3\|\Delta u\|^2$

$+ \frac{1}{4\epsilon_3}\int_0^\infty \frac{g(s)^2}{J_\delta(s)}ds \int_0^\infty J_\delta(s)\|\Delta\eta\|^2 ds, \ \forall t \geqslant 0.$

□

Lemma 2.1.3

For any positive constants $\epsilon_4, ..., \epsilon_8$, and $t \geqslant 0$, we have

$$\psi'(t) \leqslant (\frac{l}{4\epsilon_4} + \frac{c}{4\epsilon_5} + 1 + \frac{c}{4\epsilon_6}) \int_0^\infty \frac{g(s)^2}{J_\delta(s)} ds \int_0^\infty J_{\delta(s)} \|\Delta\eta\|^2 ds +$$

$$(\epsilon_4 l + c\epsilon_5 + c\epsilon_6)\|\Delta u\|^2 - (\int_0^\infty g(s)ds - \epsilon_7 g(0))\|u_t\|^2 -$$

$$(\frac{1}{4\epsilon_7} + \frac{cg(0)\gamma}{4\epsilon_8}) \int_0^\infty g'(s)\|\Delta\eta\|^2 ds - (\gamma(1-l) - \epsilon_8)\|\nabla u_t\|^2 \tag{2.10}$$

In the sequel, $c > 0$ denotes a generic constant.

Proof. Using the first equation in (2.6) again, we deduce

$$\psi'(t) = (\textstyle\int_0^\infty g(s)\eta_t^t(s)ds, u_t) - (\int_0^\infty g(s)\eta^t(s)ds, u_{tt}(t)) -$$

$$\gamma(\textstyle\int_0^\infty g(s)\nabla\eta_t^t(s)ds, \nabla u_t(t)) - \gamma(\int_0^\infty g(s)\nabla\eta^t(s)ds, \nabla u_{tt}(t)) =$$

$$-(\textstyle\int_0^\infty g(s)(u_t - \eta_s)ds, u_t(t)) - (\int_0^\infty g(s)\eta^t(s)ds, -l\Delta^2 u + \gamma\Delta u_{tt} + [u,v] - \int_0^\infty g(s)\Delta^2\eta ds -$$
$$f(u)) - \gamma(\textstyle\int_0^\infty g(s)(\nabla u_t - \nabla\eta_s)ds, \nabla u_t(t)) - \gamma(\int_0^\infty g(s)\nabla\eta^t(s)ds, \nabla u_{tt}(t)) =$$

$$-\textstyle\int_0^\infty g(s)(u_t - \eta_s, u_t)ds - \int_0^\infty g(s)(\eta^t(s), -l\Delta^2 u + [u,v] - \int_0^\infty g(\tau)\Delta^2\eta d\tau - f(u))ds$$

$$-\gamma\textstyle\int_0^\infty g(s)(\nabla u_t - \nabla\eta_s, \nabla u_t(t))ds$$

$$= \underbrace{l\int_0^\infty g(s)(\eta^t(s), \Delta^2 u)}_{I_1} - \underbrace{(\int_0^\infty g(s)\eta ds, [u,v])}_{I_2} + \underbrace{(\int_0^\infty g(s)\Delta\eta ds, \int_0^\infty g(s)\Delta\eta ds)}_{I_3}$$

$$+ \underbrace{(\int_0^\infty g(s)\eta ds, f(u))}_{I_4} - \gamma\textstyle\int_0^\infty g(s)\|\nabla u_t\|^2 ds - \int_0^\infty g(s)\|u_t\|^2 ds$$

$$+\underbrace{\int_0^\infty g(s)(\eta_s, u_t)ds}_{I_5} + \underbrace{\gamma\int_0^\infty g(s)(\nabla\eta_s^t, \nabla u_t)ds}_{I_6}.$$

Next, we estimate I_i,$1 \leq i \leq 6$, respectively. First,

$$I_1 = l(\int_0^\infty g(s)\Delta\eta^t(s)ds, \Delta u(t)) \leq l\epsilon_4\|\Delta u\|^2 + \frac{l}{4\epsilon_4}\int_\Omega(\int_0^\infty g(s)\Delta\eta^t ds)^2 dx$$

$$\leq l\epsilon_4\|\Delta u\|^2 + \frac{l}{4\epsilon_4}\int_\Omega(\int_0^\infty \frac{g(s)}{\sqrt{J_\delta}}\sqrt{J_\delta}\Delta\eta^t ds)^2 dx$$

$$\leq \epsilon_4 l\|\Delta u\|^2 + \frac{l}{4\epsilon_4}\int_0^\infty \frac{g(s)}{J_\delta(s)}ds\int_0^\infty J_\delta(s)\|\Delta\eta^t(s)\|^2 ds,$$

and

$$I_2 = -(\int_0^\infty g(s)\eta ds, [u, v])$$

$$\leqslant \|\int_0^\infty g(s)\eta ds\|\|[u, v]\| \leq c\|u\|\|\int_0^\infty g(s)\eta ds\| \quad (Cauchy - Schwarz)$$

$$\leq c\|\Delta u\|\|\int_0^\infty g(s)\eta ds\| \leq c\epsilon_5\|\Delta u\|^2 + \frac{c}{4\epsilon_5}\|\int_0^\infty g(s)\eta ds\|^2$$

$$\leq c\|\Delta u\|^2 + \frac{c}{4\epsilon_5}\int_0^\infty \frac{g(s)^2}{J_\delta(s)}ds\int_0^\infty J_\delta(s)\|\eta^t(s)\|^2 ds \leq c\epsilon_5\|\Delta u\|^2 + \frac{c}{4\epsilon_5}\int_0^\infty \frac{g(s)^2}{J_\delta(s)}ds\int_0^\infty J_\delta(s)\|\Delta\eta^t(s)\|^2 ds.$$

For I_3, we have

$$I_3 = \|\int_0^\infty g(s)\Delta\eta ds\|^2 = \int_\Omega(\int_0^\infty g(s)\Delta\eta ds)^2 dx$$

$$\leq \int_0^\infty \frac{g(s)^2}{J_\delta(s)}ds\int_0^\infty J_\delta(s)\|\Delta\eta^t(s)\|^2 ds.$$

Using the growth assumption of f, Sobolev embedding theorem, and boundedness of the energy, we know that $\|f(u)\|$ can be bounded by $\|\Delta u\|$; $\exists C > 0; \|f(u)\| \leq C\|\Delta u\|$.

Indeed $\|f(u)\| \leq c\|\Delta u\|$

$f(0) = 0,$

$f(u) - f(0) = \int_0^u f'(t)dt, \;\; |f(u)| \leq \int_0^u |f'(t)|dt$

$\leq c_0 \int_0^u (1 + |t|^p) dt = c_0[|u| + \frac{|u|^{p+1}}{p+1}] \leq c_0[|u| + |u|^{p+1}] \Longrightarrow \|f(u)\|^2 = \int_\Omega |f(u)|^2 dx \leq c_0 \int_\Omega (|u| + |u|^{p+1})^2 dx \Longrightarrow \|f(u)\|^2 \leq c_0 \int_\Omega |u|^2 [1 + |u|^p]^2 dx.$

We know that $H^2(\Omega) \hookrightarrow C^0(\bar{\Omega})$, then $u \in H_0^2(\Omega) \Longrightarrow u \in C^0(\bar{\Omega}) \Longrightarrow | u(x) |$

$\leq \sup_{x\in\Omega} | u(x) | \leq cte \Longrightarrow (1 + |u|^p) \leq 1 + (\sup_{x\in\Omega} |u|)^p \leq c'$.

$\|f(u)\|^2 \leq c\|u\|^2 \Longrightarrow \|f(u)\| \leq c\|\Delta u\|$.

Hence, $I_4 = (\int_0^\infty g(s)\eta ds, f(u)) \leq \| \int_0^\infty g(s)\eta ds\| \|f(u)\|$

then $I_4 \leq c\| \int_0^\infty g(s)\eta ds\| \|\Delta u\| \leq c\epsilon_6 \|\Delta u\|^2 + \frac{c}{4\epsilon_6} \| \int_0^\infty g(s)\eta ds\|^2 \leq$

$c\epsilon_6 \|\Delta u\|^2 + \frac{c}{4\epsilon_6} \int_0^\infty \frac{g(s)^2}{J_\delta(s)} ds \int_0^\infty J_\delta(s) \|\Delta\eta\|^2 ds.$

Finally, let us estimate I_5 and I_6.

$I_5 = \int_0^\infty g(s)(\eta_s, u_t) ds = \int_0^\infty g(s) \frac{d}{ds}(\eta, u_t) ds = -\int_0^\infty g'(s)(\eta, u_t) ds \leq -\int_0^\infty g'(s)\|\eta\| \|u_t\| ds \leq -\frac{c}{4\epsilon_7} \int_0^\infty g'(s) \|\Delta\eta\|^2 ds - \epsilon_7 \|u_t\|^2 g(s)|_0^\infty = -\frac{c}{4\epsilon_7} \int_0^\infty g'(s)\|\Delta\eta\|^2 ds + \epsilon_7 g(0) \|u_t\|^2$

$I_6 = \gamma \int_0^\infty g(s)(\nabla\eta_s^t(s), \nabla u_t(t))$ (integration by parts)

$I_6 = \gamma g(s)(\nabla\eta(s), \nabla u_t(t))|_0^\infty - \gamma \int_0^\infty g'(s)(\nabla\eta^t(s), \nabla u_t) ds \leq -\gamma \int_0^\infty g'(s)\|\nabla\eta^t\| \|\nabla u_t\| ds \leq \epsilon_8 \|\nabla u_t\|^2 + \frac{\gamma}{4\epsilon_8} (\int_0^\infty -g'(s)\|\nabla\eta^t\| ds)^2 \leq \epsilon_8 \|\nabla u_t\|^2 + \frac{\gamma}{4\epsilon_8} (\int_0^\infty \sqrt{-g'(s)}\sqrt{-g'(s)} \|\nabla\eta^t\| ds)^2 \leq \epsilon_8 \|\nabla u_t\|^2 - \frac{\gamma g(0)}{4\epsilon_8} \int_0^\infty -g'(s)\|\nabla\eta^t\|^2 ds.$

□

Proof. (Theorem 2.1). Put $L(t) = ME(t) + \epsilon_1\varphi(t) + \epsilon_2\psi(t) + \kappa R(t)$,

with positive constants $M, \epsilon_1, \epsilon_2, \kappa$ to be specifed later.

Combining (2.7)-(2.10) and (2.3) together, we obtain

$L'(t) = ME'(t) + \epsilon_1\varphi'(t) + \epsilon_2\psi(t) + \kappa R'(t) \leqslant \frac{M}{2}[(\int_0^\infty g'(s)\|\Delta\eta^t(s)\|^2 ds)] + \epsilon_1[(\|u_t\|^2 + \gamma\|\nabla u_t\|^2 - l\|\Delta u\|^2 - \|\Delta v\|^2 - (u, f(u)) + \epsilon_3\|\Delta u\|^2 + \frac{1}{4\epsilon_3} \int_0^\infty \frac{g(s)^2}{J_\delta(s)} ds \int_0^\infty J_\delta(s)\|\Delta\eta\|^2 ds)] +$

$\epsilon_2[(\frac{l}{4\epsilon_4} + \frac{c}{4\epsilon_5} + 1 + \frac{c}{4\epsilon_6}) \int_0^\infty \frac{g(s)^2}{J_\delta(s)} ds \int_0^\infty J_{\delta(s)} \|\Delta\eta\|^2 ds + (\epsilon_4 l + c\epsilon_5 + c\epsilon_6)\|\Delta u\|^2 - (\int_0^\infty g(s) ds - \epsilon_7 g(0))\|u_t\|^2 - (\frac{1}{4\epsilon_7} + \frac{cg(0)\gamma}{4\epsilon_8}) \int_0^\infty g'(s)\|\Delta\eta\|^2 ds - (\gamma(1-l) - \epsilon_8)\|\nabla u_t\|^2)] +$

$\kappa(-\frac{1}{2}\int_0^\infty g(s)\|\Delta\eta^t(s)\|^2 ds + h(0)\|\Delta u(t)\|^2)$

$\int_0^\infty g'(s)\|\Delta\eta^t(s)\|^2 = -\int_0^\infty J_\delta(s)\|\Delta\eta\|^2 ds + \int_0^\infty g(s)\|\Delta\eta^t(s)\|^2 ds$ such that $J_\delta(s) = \delta g(s) - g'(s)$

then $L'(t) \leq [(-\frac{M}{2} + \frac{\epsilon_2}{4\epsilon_7} + \frac{\epsilon_2 cg(0)\gamma}{4\epsilon_8}) + (\frac{\epsilon_1}{4\epsilon_3} + \frac{l\epsilon_2}{4\epsilon_4} + \frac{c\epsilon_2}{4\epsilon_5} + \epsilon_2 + \frac{c\epsilon_2}{4\epsilon_6})G_\delta] \int_0^\infty J_\delta(s)\|\Delta\eta\|^2 ds + [(\frac{M}{2} - \frac{\epsilon_2}{4\epsilon_7} - \frac{\epsilon_2 cg(0)\gamma}{4\epsilon_8})\delta - \frac{\kappa}{2}] \int_0^\infty g(s)\|\Delta\eta^t(s)\|^2 ds + (\epsilon_1 - \epsilon_2 \int_0^\infty g(s)ds + \epsilon_2\epsilon_7 g(0))\|u_t(t)\|^2 - \epsilon_1\|\Delta v(t)\|^2 + (-l\epsilon_1 + \epsilon_1\epsilon_3 + \epsilon_2\epsilon_4 l + c\epsilon_2\epsilon_5 + c\epsilon_2\epsilon_6 + 2h(0)\kappa)\|\Delta u(t)\|^2 - \epsilon_1 C_0 \int_\Omega F(u)dx + [\epsilon_1\gamma - (\gamma(1-l) - \epsilon_8)\epsilon_2]\|\nabla u_t(t)\|^2$

where $G_\delta := \int_0^\infty \frac{g(s)^2}{J_\delta(s)} ds$.

We will choose suitable positive constants $\epsilon_1, ..., \epsilon_8$ and κ, δ, M such that

$$\epsilon_1 - \epsilon_2 \int_0^\infty g(s)ds + \epsilon_7\epsilon_2 g(0) < 0 \tag{2.11}$$

$$-l\epsilon_1 + \epsilon_3\epsilon_1 + \epsilon_4\epsilon_2 l + c\epsilon_5\epsilon_2 + c\epsilon_2\epsilon_6 + 2h(0)\kappa < 0 \tag{2.12}$$

$$(\frac{M}{2} - \frac{\epsilon_2}{4\epsilon_7} - \frac{\epsilon_2 cg(0)\gamma}{4\epsilon_8})\delta - \frac{\kappa}{2} < 0 \tag{2.13}$$

$$(-\frac{M}{2} + \frac{\epsilon_2}{4\epsilon_7} + \frac{\epsilon_2 cg(0)\gamma}{4\epsilon_8}) + (\frac{\epsilon_1}{4\epsilon_3} + \frac{l\epsilon_2}{4\epsilon_4} + \frac{c\epsilon_2}{4\epsilon_5} + \epsilon_2 + \frac{c\epsilon_2}{4\epsilon_6})G_\delta) < 0 \tag{2.14}$$

$$\epsilon_1\gamma - (\gamma(1-l) - \epsilon_8)\epsilon_2 < 0 \tag{2.15}$$

To the end, we first take $\epsilon_2 > 0$ fixed. Second, take ϵ_1, ϵ_7 and ϵ_8 small enough such that (2.11) and (2.15) are satisfied. Then take $\epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \kappa$ small enough such that (2.12) holds. For (2.13) and (2.14) to be true, we take M, δ satisfying $\delta_1 < \frac{M}{2} < \delta_2$, where

$\delta_1 := \frac{\epsilon_2}{4\epsilon_7} + \frac{\epsilon_2 cg(0)\gamma}{4\epsilon_8} + C(\epsilon_1, ..., \epsilon_6)G_\delta$, $\delta_2 := \frac{\kappa}{2\delta} + \frac{\epsilon_2}{4\epsilon_7} + \frac{\epsilon_2 cg(0)\gamma}{4\epsilon_8}$

and the symbol $C(\epsilon_1, ..., \epsilon_6)$ is self-evident. The existence of such M and δ relies on the observation: the assumption (2.2) ensures

$\lim\limits_{\delta \longrightarrow 0} \frac{\delta g^2(s)}{\delta g(s) - g'(s)} = 0$, a.e. in $(0, \infty)$,

and so an application of Lebesgue's dominated convergence theorem gives

$\lim\limits_{\delta \longrightarrow 0} \delta G_\delta = \lim\limits_{\delta \longrightarrow 0} \int_0^\infty \frac{\delta g^2(s)}{\delta g(s) - g'(s)} ds = 0;$

thus, $\delta_1 < \delta_2$ when δ small enough.

Accordingly, we infer

$L'(t) \leqslant -C(\epsilon_1, ..., \epsilon_8, M, \delta, \kappa)E(t)$.

Note that $R(t) \geqslant 0$, and ψ, φ can be bounded by $E(t)$. We deduce that, $\forall t \geqslant 0$,

$\int_0^t C(\epsilon_1, ..., \epsilon_8, M, \delta, \kappa)E(s)ds \leqslant L(0) - L(t)$

$= L(0) - ME(t) - \epsilon_1\varphi(t) - \epsilon_2\psi(t) - \kappa R(t) \leq C(M, \epsilon_1, \epsilon_2)E(0)$.

Thus $\int_0^\infty E(t)dt \leqslant C'(\epsilon_1, ..., \epsilon_8, M, \delta, \kappa)E(0)$. Since

$\frac{d}{dt}((t+1)E(t)) = (t+1)E'(t) + E(t) \leqslant E(t)$,

it follows that $(t+1)E(t) - E(0) \leqslant \int_0^\infty E(s)ds$. Therefore, $E(t) \leqslant \frac{c}{t+1}$. □

Chapter 3

A General Stability Result for a von Kármán System with Infinite Memory and Constant Discrete Time Delay

In this chapter we consider the following von Kármán equations with memory and time delay, which is the extention of study carred out in chapter 2 the case with constant time delay.

$$(P)\begin{cases} u_{tt} + \Delta^2 u - \gamma\Delta u_{tt} - [u,v] - \int_{-\infty}^{t} g(t-s)\Delta^2 u(s)ds + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) + f(u) = 0 \\ \qquad \text{in } \Omega\times\mathbb{R}_+ \\ \Delta^2 v = -[u,u] \quad \text{in } \Omega\times\mathbb{R}_+ \\ u = \Delta u = 0, \quad v = \partial_\nu v = 0 \quad \text{on } \partial\Omega\times\mathbb{R}_+ \\ u(x,-t) = u_0(x,t) \quad \text{on } \Omega\times\mathbb{R}_+ \\ u_t(x,0) = u_1(x) \quad \text{in } \Omega \\ u_t(x,-t) = h(x,t) \quad \text{in } \Omega\times]0,\tau[\end{cases} \tag{3.1}$$

To treat the infinite memory (following [6]) and the discrete time delay (following [7], [8]), we introduce the following new auxiliary functions

$$\eta(x,t,s) \equiv \eta^t(x,s) = u(x,t) - u(x,t-s), \quad (x,t,s) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}_+$$

$$z(x,t,\rho) = u_t(x,t-\rho\tau), \quad (x,t,\rho) \in \Omega \times \mathbb{R}_+ \times]0,1[$$

then we have

$$\begin{cases} \eta_t^t(x,s) + \eta_s^t(x,s) = u_t(x,t) \ \text{ in } \ \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \\ \eta^t(x,0) = 0 \ \text{ in } \ \Omega \times \mathbb{R}_+ \\ \eta^0(x,s) = \eta_0(x,s) = u_0(x,0) - u(x,-s) = u_0(x) - u(x,-s) \ \text{ in } \ \Omega \times \mathbb{R}_+ \end{cases}$$

and

$$\begin{cases} \tau z_t(x,t,\rho) + z_\rho(x,t,\rho) = 0 \ \text{ in } \ \Omega \times \mathbb{R}_+ \times]0,1[\\ z(x,t,0) = u_t(x,t) \ \text{ in } \ \Omega \times \mathbb{R}_+ \\ z(x,0,\rho) = h(x,\rho\tau) \ \text{ in } \ \Omega \times]0,1[\end{cases}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded with a sufficiently smooth boundary $\partial\Omega$, μ_1 and μ_2 are real numbers with $\mu_1 > 0, \tau > 0$ respresents the constant time delay.

Then (P) transforms into the system

$$(P1) \begin{cases} u_{tt} + l\Delta^2 u - \gamma\Delta u_{tt} - [u,v] + \int_0^\infty g(s)\Delta^2\eta ds + \mu_1 u_t(x,t) + \mu_2 z(x,t,1) + f(u) = 0 \\ \text{in } \Omega \times \mathbb{R}_+ \\ \Delta^2 v = -[u,u] \quad \text{in } \Omega \times \mathbb{R}_+ \\ \eta_t^t(x,s) + \eta_s^t(x,s) = u_t(x,t) \ \text{ in } \ \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \\ \eta^t(x,0) = 0 \ \text{ in } \ \Omega \times \mathbb{R}_+, \ u = \Delta u = 0, v = \frac{\partial v}{\partial \nu} = 0, \eta = \Delta\eta = 0 \text{ in } \partial\Omega \times [0,+\infty) \\ \eta^0(x,s) = \eta_0(x,s) = u_0(x,0) - u(x,-s) = u_0(x) - u(x,-s) \ \text{ in } \ \Omega \times \mathbb{R}_+ \\ \tau z_t(x,t,\rho) + z_\rho(x,t,\rho) = 0 \ \text{ in } \ \Omega \times \mathbb{R}_+ \times]0,1[\\ z(x,t,0) = u_t(x,t) \ \text{ in } \ \Omega \times \mathbb{R}_+ \\ z(x,0,\rho) = h(x,\rho\tau) \ \text{ in } \ \Omega \times]0,1[\end{cases} \tag{3.2}$$

3.1 Preliminaries

We present some notations and state the main result. Let λ be the smallest positive constant such that

$$\lambda\|u\|^2 \leq \|\Delta u\|^2, \quad \forall u \in H_0^2(\Omega). \tag{3.1}$$

Define the energy functional of the nonlinear von Kármán equations with infinite memory and time delay

$$E(t) = \frac{1}{2}\|u_t\|^2+\frac{l}{2}\|\Delta u\|^2+\frac{\gamma}{2}\|\nabla u_t\|^2+\frac{1}{4}\|\Delta v\|^2+\frac{1}{2}\int_0^\infty g(s)\|\Delta\eta\|^2 ds+\int_\Omega F(u)dx+\frac{|\mu_2|\tau}{2}\int_0^1 \|z(x,t,\rho)\|^2 d\rho$$

Lemma 3.1.1

Under the condition $|\mu_2| \leq \mu_1$, we have $\dfrac{dE}{dt} \leq 0$, thus the energy functional $E(t)$ is decreasing.

Proof. We multiply the first equation of the problem (P1) by u_t and integrate over Ω we have :

$\int_\Omega u_{tt}u_t dx+l\int_\Omega \Delta^2 u_{tt}u_t dx-\gamma\int_\Omega \Delta u_{tt}u_t dx-\int_\Omega [u,v]u_t dx+\int_\Omega(\int_0^\infty g(s)\Delta^2\eta ds)u_t dx+\mu_1\int_\Omega u_t(x,t)u_t dx+$
$\mu_2\int_\Omega z(x,t,1)u_t dx + \int_\Omega f(u)u_t dx = 0.$

Then we have, by using (1)-(6) in chapter 2 :

$\int_\Omega u_{tt}u_t dx + l\int_\Omega \Delta^2 u_{tt}u_t dx - \gamma\int_\Omega \Delta u_{tt}u_t dx - \int_\Omega [u,v]u_t dx$
$+ \int_\Omega(\int_0^\infty g(s)\Delta^2\eta ds)u_t dx + \int_\Omega f(u)u_t dx$
$= \frac{d}{dt}[\frac{1}{2}\|u_t\|^2] + \frac{d}{dt}[\frac{l}{2}\|\Delta u\|^2] + \frac{d}{dt}[\frac{\gamma}{2}\|\nabla u_t\|^2] +$
$\frac{d}{dt}[\frac{1}{4}\|\Delta v\|^2] + \frac{1}{2}\frac{\partial}{\partial t}\int_0^\infty g(s)\|\Delta\eta\|^2 ds - \frac{1}{2}\int_0^\infty g'(s)\|\Delta\eta\|^2 ds + \frac{d}{dt}\int_\Omega F(u)dx$

$= -\mu_1 \int_\Omega u^2(x,t)dx - \mu_2 \int_\Omega z(x,t,1)u_t dx.$

Let us denote $L(t)$ the functional

$$L(t) = \frac{1}{2}\|u_t\|^2 + \frac{l}{2}\|\Delta u\|^2 + \frac{\gamma}{2}\|\nabla u_t\|^2 + \frac{1}{4}\|\Delta v\|^2 + \frac{1}{2}\int_0^\infty g(s)\|\Delta\eta\|^2 ds + \int_\Omega F(u)dx.$$

Then

$$\frac{dL(t)}{dt} = \frac{1}{2}\int_0^\infty g'(s)\|\Delta\eta\|^2 ds - \mu_1 \int_\Omega u^2(x,t)dx - \mu_2 \int_\Omega z(x,t,1)u_t dx.$$

Multiply by z and integrate over (0,1) the equation:

$\tau z_t(x,t,\rho) + z_\rho(x,t,\rho) = 0$, we get

$$\frac{\tau}{2}\int_0^1 \frac{\partial}{\partial t}\|z(x,t,\rho)\|^2 d\rho + \frac{1}{2}\int_0^1 \frac{\partial}{\partial \rho}\|z(x,t,\rho)\|^2 d\rho = 0,$$

then

$$\frac{\partial}{\partial t}(\frac{\tau}{2}\int_0^1 \|z(x,t,\rho)\|^2 d\rho) = \frac{1}{2}(\|u_t\|^2 - \|z(x,t,1)\|^2),$$

and

$$\frac{\tau|\mu_2|}{2}\frac{d}{dt}\int_0^1 \|z(x,t,\rho)\|^2 d\rho = \frac{|\mu_2|}{2}(\|u_t\|^2 - \|z(x,t,1)\|^2).$$

Using Cauchy-Schwarz and Young's inequalities, we get

$$-\mu_2 \int_\Omega z(x,t,1)u_t dx \le \frac{|\mu_2|}{2}\|z(x,t,1)\|^2 + \frac{|\mu_2|}{2}\|u_t\|^2.$$

We have

$$E'(t) = \frac{dL(t)}{dt} + \frac{\tau|\mu_2|}{2}\frac{d}{dt}\int_0^1 \|z(x,t,\rho)\|^2 d\rho = \frac{1}{2}\int_0^\infty g'(s)\|\Delta\eta\|^2 ds$$

$$-\mu_1 \int_\Omega u_t^2(x,t)dx - \mu_2 \int_\Omega z(x,t,1)u_t dx + \frac{|\mu_2|}{2}(\|u_t\|^2 - \|z(x,t,1)\|^2)$$

$$\leq -\mu_1\|u_t\|^2 + \frac{|\mu_2|}{2}\|z(x,t,1)\|^2 + \frac{|\mu_2|}{2}\|u_t\|^2 + \frac{|\mu_2|}{2}(\|u_t\|^2 - \|z(x,t,1)\|^2) \leq -(\mu_1 - |\mu_2|)\|u_t\|^2.$$

Then $E'(t) \leq 0$ if and only if $|\mu_2| \leq \mu_1$.

□

To derive the energy decay rate, we introduce some auxiliary functions and give some estimates about them. Define

$$\varphi(t) = (u(t), u_t(t)) + \gamma(\nabla u(t), \nabla u_t(t))$$

,

$$\psi(t) = -(\int_0^\infty g(s)\eta(s)ds, u_t(t)) - \gamma(\int_0^\infty g(s)\nabla\eta(s)ds, \nabla u_t(t))$$

,

$$R(t) = \int_0^\infty h(s)\|\Delta u(t-s)\|^2 ds$$

$$\Sigma(t) = \int_\Omega \int_0^1 e^{-2\tau\rho} z^2(x,t,\rho)dxd\rho.$$

A simple calculation yields that for $t \geqslant 0$,

$$\begin{aligned} R'(t) &= -\int_0^\infty h(s)\tfrac{d}{ds}(\|\Delta u(t-s)\|^2)ds \\ &= \int_0^\infty g(s)(\|\Delta u(t-s)\|^2)ds + h(0)\|\Delta u(t)\|^2 \\ &\leq \int_0^\infty g(s)[\|\Delta u(t)\|^2 - \tfrac{1}{2}\|\Delta u(t-s) - \Delta u(t)\|^2]ds + h(0)\|\Delta u(t)\|^2. \end{aligned}$$

This gives

$$R'(t) \leq -\frac{1}{2}\int_0^\infty g(s)\|\Delta\eta(s)\|^2 ds + 2h(0)\|\Delta u(t)\|^2, \quad \forall t \geqslant 0$$

Lemma 3.1.2

For any constant $\epsilon_3 > 0$, we have

$$\varphi'(t) \leq \|u_t\|^2 + \gamma\|\nabla u_t\|^2 - l\|\Delta u\|^2 - \|\Delta\phi\|^2 - (u, f(u)) + \epsilon_3\|\Delta u\|^2 +$$

$$\frac{1}{4\epsilon_3}\int_0^\infty \frac{g(s)^2}{J_\delta(s)} ds \int_0^\infty J_\delta(s)\|\Delta\eta\|^2 ds + \frac{1}{2k'\lambda}\|\Delta u\|^2 + \frac{k'|\mu_2|^2}{2}\|z(x,t,1)\|^2 +$$

$$\frac{1}{2k''\lambda}\|\Delta u|^2 + \frac{k''|\mu_1|^2}{2}\|u_t\|^2$$

Here and in the sequel, $J_\delta(s) := \delta g(s) - g'(s)$, for $\delta > 0$

Proof. By the definition of φ, and using the first equation in (3.2) and Young's Inequality with a Parameter and Cauchy Schwarz Inequality we have

$$\varphi'(t) = \frac{d}{dt}\varphi(t) = (u_t(t), u_t(t)) + (u(t), u_{tt}(t)) + \gamma(\nabla u_t(t), \nabla u_t(t)) + \gamma(\nabla u(t), \nabla u_{tt}(t))$$

$$= \|u_t(t)\|^2 + \gamma\|\nabla u_t\|^2 + (u(t), u_{tt}(t)) + \gamma(\nabla u(t), \nabla u_{tt}(t))$$

$$= \|u_t(t)\|^2 + \gamma\|\nabla u_t\|^2 + (u(t), -l\Delta^2 u + [u,v] - \int_0^\infty g(s)\Delta^2\eta ds - \mu_1 u_t(x,t) - \mu_2 z(x,t,1) - f(u))$$

$$+ (u, -\gamma\Delta u_{tt}) + \gamma(\nabla u(t), \nabla u_{tt}(t))$$
$$= \|u_t(t)\|^2 + \gamma\|\nabla u_t\|^2 + (u(t), -l\Delta^2 u + [u,v] - \int_0^\infty g(s)\Delta^2\eta ds - \mu_1 u_t(x,t) - \mu_2 z(x,t,1)$$

$- f(u))$.

From $([u,v],u) = ([u,u],v) = -(\Delta^2 v, v) = -\|\Delta v\|^2$

$\varphi'(t) = \|u_t\|^2 + \gamma\|\nabla u_t\|^2 - l\|\Delta u\|^2 - \|\Delta^2 v\|^2 - (\int_0^\infty g(s)\Delta\eta ds, \Delta u(t)) - \mu_1(u, u_t(x,t)) - \mu_2(u, z(x,t,1)) - (u, f(u))$

$(\int_0^\infty g(s)\Delta\eta ds, \Delta u(t)) \leqslant \| \int_0^\infty g(s)\Delta\eta ds\|\|\Delta u\| \quad (H\ddot{o}lder Inequality)$

$\leq \frac{1}{4\epsilon_3}\| \int_0^\infty g(s)\Delta\eta ds\|^2 + \epsilon_3\|\Delta u\|^2 = \frac{1}{4\epsilon_3}\int_\Omega (\int_0^\infty \frac{g(s)}{\sqrt{J_\delta(s)}}\sqrt{J_\delta(s)}\Delta\eta ds)^2 dx + \epsilon_3\|\Delta u\|^2$

$\leqslant \frac{1}{4\epsilon_3}\int_\Omega (\int_0^\infty \frac{g(s)^2}{J_\delta(s)} ds)(\int_0^\infty J_\delta(s)\Delta^2\eta ds) dx + \epsilon_3\|\Delta u\|^2$

$\leqslant \frac{1}{4\epsilon_3}\int_0^\infty (g(s))^2 ds \int_0^\infty \|\Delta\eta\|^2 ds + \epsilon_3\|\Delta u\|^2$

$\leq \frac{1}{4\epsilon_3}\int_0^\infty \frac{g(s)^2}{J_\delta(s)} ds \int_0^\infty J_\delta(s)\|\Delta\eta\|^2 ds + \epsilon_3\|\Delta u\|^2$

$-\mu_2(u, z(x,t,1)) \leq \|u\||\mu_2|\|z(x,t,1)\| \leq \frac{1}{2k'}\|u\|^2 + \frac{k'|\mu_2|^2}{2}\|z(x,t,1)\|^2 \leq \frac{1}{2k'\lambda}\|\Delta u\|^2 + \frac{k'|\mu_2|^2}{2}\|z(x,t,1)\|^2$

$-\mu_1(u, u_t) \leq \frac{1}{2k''\lambda}\|\Delta u\|^2 + \frac{k''|\mu_1|^2}{2}\|u_t\|^2 \quad k' and\ k''$positive constants

thus

$\varphi'(t) \leq \|u_t\|^2 + \gamma\|\nabla u_t\|^2 - l\|\Delta u\|^2 - \|\Delta v\|^2 - (u, f(u)) + \epsilon_3\|\Delta u\|^2$

$+ \frac{1}{4\epsilon_3}\int_0^\infty \frac{g(s)^2}{J_\delta(s)} ds \int_0^\infty J_\delta(s)\|\Delta\eta\|^2 ds + \frac{1}{2k'\lambda}\|\Delta u\|^2 + \frac{k'|\mu_2|^2}{2}\|z(x,t,1)\|^2 + \frac{1}{2k''\lambda}\|\Delta u\|^2 + \frac{k''|\mu_1|^2}{2}\|u_t\|^2$

□

Lemma 3.1.3

For any positive constants $\epsilon_4, ..., \epsilon_{10}$, and $t \geqslant 0$ we have

$$\psi'(t) \leqslant (\frac{l}{4\epsilon_4} + \frac{c}{4\epsilon_5} + 1 + \frac{c}{4\epsilon_6} + \frac{\lambda\mu_1}{4\epsilon_9} + \frac{\lambda\mu_2}{4\epsilon_{10}}) \int_0^\infty \frac{g(s)^2}{J_\delta(s)} ds \int_0^\infty J_{\delta(s)} \|\Delta\eta\|^2 ds +$$

$$(\epsilon_4 l + c\epsilon_5 + c\epsilon_6)\|\Delta u\|^2 - (\int_0^\infty g(s)ds - \epsilon_7 g(0) - \mu_1\epsilon_9)\|u_t\|^2 -$$

$$(\frac{1}{4\epsilon_7} + \frac{\lambda g(0)\gamma}{4\epsilon_8}) \int_0^\infty g'(s)\|\Delta\eta\|^2 ds - (\gamma(1-l) - \epsilon_8)\|\nabla u_t\|^2 + \epsilon_{10}\mu_2\|z(x,t,1)\|^2 \quad (3.2)$$

Proof. $\psi'(t) = (\int_0^\infty g(s)\eta_t^t(s)ds, u_t) - (\int_0^\infty g(s)\eta^t(s)ds, u_{tt}(t)) - \gamma(\int_0^\infty g(s)\nabla\eta_t^t(s)ds, \nabla u_t(t)) -$

$\gamma(\int_0^\infty g(s)\nabla\eta^t(s)ds, \nabla u_{tt}(t)) =$

$-(\int_0^\infty g(s)(u_t - \eta_s)ds, u_t(t)) - (\int_0^\infty g(s)\eta^t(s)ds, -l\Delta^2 u + \gamma\Delta u_{tt} + [u,v] - \int_0^\infty g(s)\Delta^2\eta ds -$
$\mu_1 u_t(x,t) - \mu_2 z(x,t,1) - f(u))$

$-\gamma(\int_0^\infty g(s)(\nabla u_t - \nabla\eta_s)ds, \nabla u_t(t)) - \gamma(\int_0^\infty g(s)\nabla\eta^t(s)ds, \nabla u_{tt}(t)) =$

$-\int_0^\infty g(s)(u_t - \eta_s, u_t)ds - \int_0^\infty g(s)(\eta^t(s), -l\Delta^2 u + [u,v] - \int_0^\infty g(\tau)\Delta^2\eta d\tau - \mu_1 u_t(x,t) -$
$\mu_2 z(x,t,1) - f(u))ds$

$-\gamma\int_0^\infty g(s)(\nabla u_t - \nabla\eta_s, \nabla u_t(t))ds$

$$= \underbrace{l\int_0^\infty g(s)(\eta^t(s), \Delta^2 u)}_{I_1} - \underbrace{(\int_0^\infty g(s)\eta ds, [u,v])}_{I_2} + \underbrace{(\int_0^\infty g(s)\Delta\eta ds, \int_0^\infty g(s)\Delta\eta ds)}_{I_3}$$

$$+\underbrace{(\int_0^\infty g(s)\eta ds, f(u))}_{I_4} - \gamma\int_0^\infty g(s)\|\nabla u_t\|^2 ds - \int_0^\infty g(s)\|u_t\|^2 ds$$

$$+\underbrace{\int_0^\infty g(s)(\eta_s, u_t)ds}_{I_5} + \gamma\underbrace{\int_0^\infty g(s)(\nabla\eta_s^t, \nabla u_t)ds}_{I_6} + \mu_1\underbrace{\int_0^\infty g(s)(\eta(s), u_t)ds}_{I_7} + \mu_2\underbrace{\int_0^\infty g(s)(\eta, z(x,t,1))ds}_{I_8}$$

Next, we estimate I_i, $1 \le i \le 8$, respectively. First,

$$I_1 = l(\int_0^\infty g(s)\Delta\eta^t(s)ds, \Delta u(t)) \le l\epsilon_4\|\Delta u\|^2 + \frac{1}{4\epsilon_4}\int_\Omega(\int_0^\infty g(s)\Delta\eta^t ds)^2 dx$$

$$\le l\epsilon_4\|\Delta u\|^2 + \frac{1}{4\epsilon_4}\int_\Omega(\int_0^\infty \frac{g(s)}{\sqrt{J_\delta}}\sqrt{J_\delta}\Delta\eta^t ds)^2 dx$$

$$\le \epsilon_4 l\|\Delta u\|^2 + \frac{l}{4\epsilon_4}\int_0^\infty \frac{g(s)}{J_\delta(s)}ds\int_0^\infty J_\delta(s)\|\Delta\eta^t(s)\|^2 ds,$$

and

$$I_2 = -(\int_0^\infty g(s)\eta ds, [u,v])$$

$$\leqslant \|\int_0^\infty g(s)\eta ds\|\|[u,v]\| \le c\|u\|\|\int_0^\infty g(s)\eta ds\| \quad \text{(Cauchy Schwarz)}$$

$$\le c\|\Delta u\|^2\|\int_0^\infty g(s)\eta ds\| \le c\epsilon_5\|\Delta u\|^2 + \frac{c}{4\epsilon_5}\|\int_0^\infty g(s)\eta ds\|$$

$$\le c\|\Delta u\|^2 + \frac{c}{4\epsilon_5}\int_0^\infty \frac{g(s)^2}{J_\delta(s)}ds\int_0^\infty J_\delta(s)\|\eta^t(s)\|^2 ds \le c\epsilon_5\|\Delta u\|^2 + \frac{c}{4\epsilon_5}\int_0^\infty \frac{g(s)^2}{J_\delta(s)}ds\int_0^\infty J_\delta(s)\|\Delta\eta^t(s)\|^2 ds.$$

For I_3, we have

$$I_3 = \|\int_0^\infty g(s)\Delta\eta ds\|^2 = \int_\Omega(\int_0^\infty g(s)\Delta\eta ds)^2 dx$$

$$\le \int_0^\infty \frac{g(s)^2}{J_\delta(s)}ds\int_0^\infty J_\delta(s)\|\Delta\eta^t(s)\|^2 ds.$$

Using the growth assumption of f, Sobolev embedding theorem, and boundedness of the energy, we know that $\|f(u)\|$ can be bounded by $\|\Delta u\|$. Hence,

$I_4 = (\int_0^\infty g(s)\eta ds, f(u)) =\le \|\int_0^\infty g(s)\eta ds\|\|f(u)\|$

Proof that $\|f(u)\| \leq c\|\Delta u\|$:

$f(0) = 0,$

$f(u) - f(0) = \int_0^u f'(t)dt \quad |f(u)| \leq \int_0^u |f'(t)|dt$

$\leq c_0 \int_0^u (1 + |t|^p)dt = c_0[|u| + \frac{|u|^{p+1}}{p+1}] \leq c_0[|u| + |u|^{p+1}] \Longrightarrow \|f(u)\|^2 = \int_\Omega |f(u)|^2 dx \leq$
$c_0 \int_\Omega (|u| + |u|^{p+1})^2 dx \Longrightarrow \|f(u)\|^2 \leq c_0 \int_\Omega |u|^2[1 + |u|^p]^2 dx.$

We know that $H^2(\Omega) \hookrightarrow C^0(\bar{\Omega})$ $u \in H_0^2(\Omega) \Longrightarrow u \in C^0(\bar{\Omega}) \Longrightarrow | u(x) |$

$\leq \sup_{x\in\Omega} | u(x) | \leq cte \Longrightarrow (1 + |u|^p) \leq 1 + (\sup_{x\in\Omega} |u(x)|)^p \leq c'$

$|f(u)|^2 \leq c * \|u\|^2 \Longrightarrow \|f(u)\| \leq c\|\Delta u\|$

then $I_4 \leq c\| \int_0^\infty g(s)\eta ds\|\|\Delta u\| \leq c\epsilon_6\|\Delta u\|^2 + \frac{c}{4\epsilon_6}\| \int_0^\infty g(s)\eta ds\|^2 \leq$
$c\epsilon_6\|\Delta u\|^2 + \frac{c}{4\epsilon_6} \int_0^\infty \frac{g(s)^2}{J_\delta(s)} ds \int_0^\infty J_\delta(s)\|\Delta\eta\|^2 ds.$

Finally, let us estimate I_5andI_6.

$I_5 = \int_0^\infty g(s)(\eta_s, u_t)ds = \int_0^\infty g(s)\frac{d}{ds}(\eta, u_t)ds = -\int_0^\infty g'(s)(\eta, u_t)ds \leq -\int_0^\infty g'(s)\|\eta\|\|u_t\|ds \leq$
$-\frac{c}{4\epsilon_7} \int_0^\infty g'(s)\|\Delta\eta\|^2 ds - \epsilon_7\|u_t\|^2 g(s)|_0^\infty = -\frac{c}{4\epsilon_7} \int_0^\infty g'(s)\|\Delta\eta\|^2 ds + \epsilon_7 g(0)\|u_t\|^2$

$I_6 = \gamma \int_0^\infty g(s)(\nabla\eta_s^t(s), \nabla u_t(t))$ (integrate by parts)

$I_6 = \gamma g(s)(\nabla\eta(s), \nabla u_t(t))|_0^\infty - \gamma \int_0^\infty g'(s)(\nabla\eta^t(s), \nabla u_t)ds \leq -\gamma \int_0^\infty g'(s)\|\nabla\eta^t\|\|\nabla u_t\|ds \leq$
$\epsilon_8\|\nabla u_t\|^2 + \frac{\gamma}{4\epsilon_8}(\int_0^\infty -g'(s)\|\nabla\eta^t\|ds)^2 \leq \epsilon_8\|\nabla u_t\|^2 + \frac{\gamma}{4\epsilon_8}(\int_0^\infty \sqrt{-g'(s)}\sqrt{-g'(s)}\|\nabla\eta^t\|ds)^2 \leq$
$\epsilon_8\|\nabla u_t\|^2 - \frac{\gamma g(0)}{4\epsilon_8} \int_0^\infty -g'(s)\|\nabla\eta^t\|^2 ds$

$I_7 = \mu_1 \int_0^\infty (\eta(s), u_t)ds \leq \frac{|\mu_1|}{4\epsilon_9}\| \int_0^\infty g(s)\eta(s)ds\|^2 + |\mu_1|\epsilon_9\|u_t\|^2 \leq \frac{c|\mu_1|}{4\epsilon_9} \int_0^\infty \frac{g(s)^2}{J_\delta(s)} s \int_0^\infty J_\delta(s)\|\Delta\eta\|^2 ds +$
$|\mu_1|\epsilon_9\|u_t\|^2$

$I_8 = \mu_2 \int_0^\infty g(s)(\eta(s), z(x,t,1))ds \leq |\mu_2|\| \int_0^\infty g(s)\eta(s)dds\|\|z(x,t,1)\| \leq \frac{\lambda\mu_2}{4\epsilon_{10}} \int_0^\infty \frac{g(s)^2}{J_\delta(s)} ds$
$\int_0^\infty J_\delta(s)\|\Delta\eta\|^2 ds + \epsilon_{10}\mu_2\|z(x,t,1)\|^2$

then

$\psi'(t) \leqslant (\frac{l}{4\epsilon_4} + \frac{c}{4\epsilon_5} + 1 + \frac{c}{4\epsilon_6} + \frac{\lambda\mu_1}{4\epsilon_9} + \frac{\lambda\mu_2}{4\epsilon_{10}}) \int_0^\infty \frac{g(s)^2}{J_\delta(s)} ds \int_0^\infty J_{\delta(s)} \|\Delta\eta\|^2 ds +$

$(\epsilon_4 l + c\epsilon_5 + c\epsilon_6)\|\Delta u\|^2 - (\int_0^\infty g(s)ds - \epsilon_7 g(0) - \mu_1\epsilon_9)\|u_t\|^2 -$

$(\frac{1}{4\epsilon_7} + \frac{\lambda g(0)\gamma}{4\epsilon_8}) \int_0^\infty g'(s)\|\Delta\eta\|^2 ds - (\gamma(1-l) - \epsilon_8)\|\nabla u_t\|^2 + \epsilon_{10}\mu_2\|z(x,t,1)\|^2$ □

A simple calculation yields that for $t \geqslant 0$,

$\Sigma(t) = \int_\Omega \int_0^1 e^{-2\tau\rho} z^2(x,t,\rho) d\rho dx$

we've got $z_t(x,t,\rho) = \frac{-1}{\tau} z_\rho(x,t,\rho)$

$\Sigma'(t) = \int_\Omega \int_0^1 e^{-2\tau\rho} 2zz_t d\rho dx = \frac{-2}{\tau} \int_\Omega \int_0^1 e^{-2\rho\tau} zz_\rho d\rho dx = \frac{-1}{\tau} \int_\Omega \int_0^1 e^{-2\tau\rho} \frac{d}{d\rho} z^2(x,t,\rho) d\rho dx = \frac{-1}{\tau} \int_\Omega [e^{-2\tau\rho} z^2(x,t,\rho)]_0^1 - \int_0^1 -2\tau e^{-2\tau\rho} z^2(x,t,\rho) d\rho dx = \frac{1}{\tau} \int_\Omega (z^2(x,t,0) - e^{-2\tau} z^2(x,t,1)) dx - 2\int_\Omega \int_0^1 e^{-2\tau\rho} z^2(x,t,\rho) d\rho dx = \frac{1}{\tau}\|u_t\|^2 - \frac{e^{-2\tau}}{\tau}\|z(x,t,1)\|^2 - 2\int_\Omega \int_0^1 e^{-2\tau\rho} z^2(x,t,\rho) d\rho dx$

$\rho \in]0,1[$ then $e^{-2\tau} < e^{-2\tau\rho}$

$\Sigma'(t) \leq \frac{1}{\tau}\|u_t\|^2 - \frac{e^{-2\tau}}{\tau}\|z(x,t,1)\|^2 - 2e^{-2\tau} \int_0^1 \|z(x,t,\rho)\|^2$

Theorem 3.1

Under the conditions $\int_0^\infty h(s)\|\Delta u(s)\|^2 ds < \infty$ and $|\mu_2| \leq \mu_1$ The energy of $(P1)$ has the estimate

$$E(t) \leq \frac{C}{1+t}, \ \forall t \geqslant 0$$

with a positive constant C.

Proof. Put

$L(t) = ME(t) + \epsilon_1\varphi(t) + \epsilon_2\psi(t) + \kappa R(t) + \epsilon|\mu_2|\Sigma(t)$

with positive constants $M, \epsilon_1, \epsilon_2, \epsilon, \kappa$ to be specified later.

$L'(t) = ME'(t) + \epsilon_1\varphi'(t) + \epsilon_2\psi'(t) + \kappa R'(t) + \epsilon|\mu_2|\Sigma'(t) \leq [(-\frac{M}{2} + \frac{\epsilon_2}{4\epsilon_7} + \frac{\epsilon_2 cg(0)\gamma}{4\epsilon_8}) + (\frac{\epsilon_1}{4\epsilon_3} + \frac{l\epsilon_2}{4\epsilon_4} + \frac{c\epsilon_2}{4\epsilon_5} +$

$\epsilon_2 + \frac{c\epsilon_2}{4\epsilon_6} + \frac{\lambda\mu_1\epsilon_2}{4\epsilon_9} + \frac{\lambda\mu_2\epsilon_2}{4\epsilon_{10}})G_\delta] \int_0^\infty J_\delta(s)\|\Delta\eta\|^2 ds + [(\frac{M}{2} - \frac{\epsilon_2}{4\epsilon_7} - \frac{\epsilon_2 cg(0)\gamma}{4\epsilon_8})\delta - \frac{\kappa}{2}] \int_0^\infty g(s)\|\Delta\eta^t(s)\|^2 ds$

$+(\epsilon_1 - \epsilon_2 \int_0^\infty g(s)ds + \epsilon_2\epsilon_7 g(0) + \frac{\epsilon|\mu_2|}{\tau} + \mu_1\epsilon_9)\|u_t(t)\|^2 - \epsilon_1\|\Delta v\|^2 + (-l\epsilon_1 + \epsilon_1\epsilon_3 + \epsilon_2\epsilon_4 l + c\epsilon_2\epsilon_5$

$+c\epsilon_2\epsilon_6 + 2h(0)\kappa)\|\Delta u(t)\|^2 - \epsilon_1 C_0 \int_\Omega F(u)dx + [\epsilon_1\gamma - (\gamma(1-l) - \epsilon_8)\epsilon_2]\|\nabla u_t(t)\|^2 + \frac{1}{2k''\lambda}\|\Delta u\|^2 + \frac{k''\mu_1^2}{2}\|u_t\|^2 +$

$\frac{1}{2k'\lambda}\|\Delta u\|^2 + (\frac{k'\mu_2^2}{2} + \epsilon_{10} - \frac{\epsilon e^{-2\tau}|\mu_2|}{\tau})\|z(x,t,1)\|^2 - 2\epsilon|\mu_2|e^{-2\tau} \int_0^\infty \|z(x,t,1)\|^2 d\rho$

where $G_\delta := \int_0^\infty \frac{g(s)^2}{J_\delta(s)} ds$.We will choose suitable positive constants $\epsilon_1, ..., \epsilon_8$ and κ, δ, M such that

$$\epsilon_1 - \epsilon_2 \int_0^\infty g(s)ds + \epsilon_7\epsilon_2 g(0) + \frac{\epsilon|\mu_2|}{\tau} + \mu_1\epsilon_9 + \frac{k''\mu_1^2}{2} < 0 \tag{3.3}$$

$$-l\epsilon_1 + \epsilon_3\epsilon_1 + \epsilon_4\epsilon_2 l + c\epsilon_5\epsilon_2 + c\epsilon_2\epsilon_6 + 2h(0)\kappa + \frac{1}{2k'\lambda} < 0 \tag{3.4}$$

$$(\frac{M}{2} - \frac{\epsilon_2}{4\epsilon_7} - \frac{\epsilon_2 cg(0)\gamma}{4\epsilon_8})\delta - \frac{\kappa}{2} < 0 \tag{3.5}$$

$$(-\frac{M}{2} + \frac{\epsilon_2}{4\epsilon_7} + \frac{\epsilon_2 cg(0)\gamma}{4\epsilon_8}) + (\frac{\epsilon_1}{4\epsilon_3} + \frac{l\epsilon_2}{4\epsilon_4} + \frac{c\epsilon_2}{4\epsilon_5} + \epsilon_2 + \frac{c\epsilon_2}{4\epsilon_6} + \frac{\lambda\mu_1\epsilon_2}{4\epsilon_9} + \frac{\lambda\mu_2\epsilon_2}{4\epsilon_{10}})G_\delta) < 0 \tag{3.6}$$

$$\epsilon_1\gamma - (\gamma(1-l) - \epsilon_8)\epsilon_2 < 0 \tag{3.7}$$

$$\epsilon_{10}|\mu_2| - \frac{\epsilon|\mu_2|e^{-2\tau}}{\tau} + \frac{k'\mu_2^2}{2} \tag{3.8}$$

To the end, we first take an $\epsilon_2 > 0$ fixed. Second, take ϵ_1, ϵ_7 and ϵ_8 small enough such that (3.3) and (3.7) are satisfied. Then take $\epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \kappa$ small enough such that (3.4) holds. For (3.5) and (3.6) to be true, we take M, δ satisfying $\delta_1 < \frac{M}{2} < \delta_2$, where

$\delta_1 := \frac{\epsilon_2}{4\epsilon_7} + \frac{\epsilon_2 cg(0)\gamma}{4\epsilon_8} + C(\epsilon_1, ..., \epsilon_6)G_\delta$, $\delta_2 := \frac{\kappa}{2\delta} + \frac{\epsilon_2}{4\epsilon_7} + \frac{\epsilon_2 cg(0)\gamma}{4\epsilon_8}$

and the symbol $C(\epsilon_1, ..., \epsilon_6)$ is self-evident. The existence of such M and δ relies on the observation: the assumption (2.2) ensures
$\lim\limits_{\delta \longrightarrow 0} \frac{\delta g^2(s)}{\delta g(s) - g'(s)} = 0$, a.e. in $(0, \infty)$,
and so an application of Lebesgue's dominated convergence theorem gives

$$\lim_{\delta \longrightarrow 0} \delta G_\delta = \lim_{\delta \longrightarrow 0} \int_0^\infty \frac{\delta g^2(s)}{\delta g(s) - g'(s)} ds = 0;$$

thus, $\delta_1 < \delta_2$ when δ small enough.

Accordingly, we infer

$$L'(t) \leqslant -C(\epsilon_1, ..., \epsilon_{10}, M, \delta, \kappa, \epsilon)E(t).$$

Note that $R(t) \geqslant 0$ and $\Sigma(t) \geqslant 0$, and ψ, φ can be bounded by $E(t)$.We deduce that, $\forall t \geqslant 0$,
$\int_0^t C(\epsilon_1, ..., \epsilon_{10}, M, \delta, \kappa, \epsilon)E(s)ds \leqslant L(0) - L(t)$
$= L(0) - ME(t) - \epsilon_1 \varphi(t) - \epsilon_2 \psi(t) - \kappa R(t) - \epsilon \Sigma(t) \leq C(M, \epsilon_1, \epsilon_2)E(0)$.
Thus $\int_0^\infty E(t)dt \leqslant C'(\epsilon_1, ..., \epsilon_{10}, M, \delta, \kappa, \epsilon)E(0)$. Since
$\frac{d}{dt}((t+1)E(t)) = (t+1)E'(t) + E(t) \leqslant E(t)$,
it follows that $(t+1)E(t) - E(0) \leqslant \int_0^\infty E(s)ds$. Therefore, $E(t) \leqslant \frac{c}{t+1}$.

□

Chapter 4

The von Kármán equations with Finite Memory and Discrete Time Delay

4.1 Setting of the Problem

The von Kármán equations with finite memory and time-varying delay are formulated as follows:

$$u_{tt}+\Delta^2 u-[u,\phi]-\int_0^t g(t-s)\Delta^2 u(s)ds+a_0 u_t(x,t)+a_1 u_t(x,t-\tau(t))=0 \text{ in } \Omega\times\mathbb{R}_+, \quad (4.1)$$

$$\Delta^2\phi=-[u,u] \text{ in } \Omega\times\mathbb{R}_+, \quad (4.2)$$

$$u=\partial_\nu u=0,\ \phi=\partial_\nu\phi=0 \quad on \quad \partial\Omega\times(0,\infty), \quad (4.3)$$

$$u(x,0)=u_0(x),\ u_t(x,0)=u_1(x) \quad on \quad \Omega, \quad (4.4)$$

$$u_t(x,t)=f_0(x,t) \text{ in } \Omega\times(-\tau(0),0), \quad (4.5)$$

where $\Omega\subset\mathbb{R}^2$ is a bounded domain with a suffciently smooth boundary $\partial\Omega,\nu=(\nu_1,\nu_2)$ is the outward unit normal vector to $\partial\Omega$, a_0 and a_1 are real numbers with $a_0>0,\tau(t)>0$ represents the time-varying delay, and u_0,u_1,f_0 are given functions.
Rivera and Menzala [24] studied the von Kármán equations with rotational inertia and

memory

$$u_{tt} - h\Delta u_{tt} + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(s)ds = [u, \phi] \text{ in } \Omega \times (0, \infty), \tag{4.6}$$

under the usual condition $-c_0 g(t) \leq g'(t) \leq -c_1 g(t),\ 0 \leq g''(t) \leq c_2 g(t)$ for some $c_i, i = 0, 1, 2$. Later, Raposo and Santos [25] generalized the decay result of [24]. They investigated the general decay of the solutions to the problem (4.6) under a more general condition for g such as

$$g'(t) \leq -\xi(t)g(t),\ \xi(t) > 0,\ \xi'(t) \leqslant 0,\ \forall t \geqslant 0, \tag{4.7}$$

where ξ is a nonincreasing and positive function. Kang [26] improved the decay result of [25] without imposing any restrictive asssumptions on the behavior of the relaxation function at infinity. The author proved the general stability for problem (4.6) under the condition

$$g'(t) \leqslant -H(g(t)), \tag{4.8}$$

where $H(0) = 0$ and H is a non-negative, strictly increasing and strictly convex function on (0,r], for some $r > 0$. When $a_0 = a_1 = 0$ in (4.1) and the memory kernel g satisfies (4.8), Cavalcanti et al. [27] studied the decay rate of energy. Recently, Mustafa [28] established the general decay for a viscoelastic wave equation under a more general condition

$$g'(t) \leq -\xi(t)H(g(t)). \tag{4.9}$$

This is a more general condition than (4.7) and (4.8) The stability of the solution to a viscoelastic wave equation when g satisfies the condition (4.9) has been studied in [29, 31] and the references thereien. Liu [32] investigated the general decay for a viscoelastic equation with time-varying delay under the condition $|a_1| \leq \sqrt{1 - da_0}$, where d is constant such that $\tau'(t) \leq d < 1$. For other related works, we refer the readers to [33, 34] and references therein. Inspired by these results, we prove the general decay result for problems (4.1)-(4.5). We allow more relaxed condition for the memory kernel g and delay. Using the multiplier method and some properties of convex function, we obtain an explicit decay rate of the energy.

Remark 4.1.1

If $\mathcal{L}$ is a convex function on $[a,b]$, $p:\Omega\longrightarrow[a,b]$ and q are integrable function on $\Omega, q(x)\geqslant 0$, and $\int_\Omega q(x)dx=q_0>0$, then Jensen's inequality states that

$$\mathcal{L}(\frac{1}{q_0}\int_\Omega p(x)q(x)dx)\leq\frac{1}{q_0}\int_\Omega \mathcal{L}(p(x)q(x))dx. \tag{4.10}$$

For the relaxation function g, we assume the following hypotheses:

(H_1) $g:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ is a differentiable function such that

$$1-\int_0^\infty g(s)ds=l>0 \tag{4.11}$$

(H_2) There exists a positive function $H\in C^1(\mathbb{R}^+)$, with $H''(0)=H'(0)=0$, and H and is a linear or it strictly increasing and strictly convex C^2 function on (0,k], $k\leq g(0)$, such that

$$g'(t)\leq-\xi(t)H(g(t)),\ \forall t\geqslant 0, \tag{4.12}$$

where ξ is a positive nonincreasing differentiable function. For the time-varying delay, we assume that a C^1 function τ satisfies

$$\tau(t)>0,\ 0<\tau'(t)<1,\quad \text{for}\quad t\geqslant 0, \tag{4.13}$$

and that a_0 and a_1 satisfy

$$|a_1|<a_0\sqrt{1-\tau'(t)}. \tag{4.14}$$

As in [32, 33], we introduce the following new function $z(x,\rho,t)=u_t(x,t-\tau(t)\rho)$, $x\in\Omega,\ \rho\in(0,1),\ t>0$.

Then problems (4.1)-(4.5) can be written as

$$\begin{cases} u_{tt} + \Delta^2 u - [u, \phi] - \int_0^t g(t-s)\Delta^2 u(s)ds + a_0 u_t(x,t) + a_1 z(x,t,1) = 0 \text{ in } \Omega \times \mathbb{R}_+, \\ \tau(t) z_t(x,\rho,t) + (1-\tau'(t)) z_\rho(x,\rho,t) = 0 \text{ in } \Omega \times (0,1) \times (0,\infty), \\ u = \frac{\partial u}{\partial \nu} = 0, \ \phi = \frac{\partial \phi}{\partial \nu} = 0 \text{ on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x) \text{ in } \Omega, \ z(x,\rho,0) = f_0(x, -\tau(0)\rho) \text{ in } \Omega \times (0,1). \end{cases} \tag{4.15}$$

Using the arguments of [32, 34], we can be establish a well-posedness result.

Theorem 4.1

Let (4.13) and (4.14) be satisfied and g satisfies (H_1). For $u_0(t) \in H_0^2(\Omega)$, $u_1 \in L^2(\Omega)$, $f_0 \in L^2(\Omega \times (0,1))$ and $T > 0$, the system (4.15) has a unique weak solution

$$u \in C^0([0,T]; H_0^2(\Omega)) \cap C^1([0,T]; L^2(\Omega)), \ u_t \in C([0,T]; L^2(\Omega)).$$

The energy functional associatedd to problem (4.15) is defined as

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}(1 - \int_0^t g(s)ds)\|\Delta u\|^2 + \frac{1}{2}(g\Box\Delta u)(t) + \frac{1}{4}\|\Delta\phi\|^2 + \frac{\zeta}{2}\int_0^1 \int_\Omega z^2(x,\rho,t)dxd\rho, \tag{4.16}$$

where $(g\Box u)(t) = \int_0^t g(t-s)\|u(t) - u(s)\|^2 ds$ and ζ is positive constant satisfying

$$\frac{\tau(t)a_1^2}{a_0(1-\tau'(t))} < \zeta < a_0\tau(t). \tag{4.17}$$

Note that the choice of ζ is possible from assumption (4.15).

Theorem 4.2

Assume that the hypotheses (4.11)-(4.14) hold. Then there exisit positive constants k_0 and k_1 such that the energy functional satisfies

$$E(t) \leqslant k_1 G^{-1}(k_0 \int_0^t \xi(s)ds) \tag{4.18}$$

where $G(t) = \int_t^k \frac{1}{sH'(s)}ds$ and G is strictly deceasing and convex on $(0,k]$, with $\lim_{t \longrightarrow 0} G(t) = +\infty$.

Lemma 4.1.1

Let (4.13) and (4.14) be satisfied and g satisfies (H_1). The energy satisfies

$$E'(t) \leqslant -C_0(\|u_t\|^2+\|z(x,1,t)\|^2+\int_0^1\int_\Omega z^2(x,\rho,t)dxd\rho)-\frac{g(t)}{2}\|\Delta u\|^2+\frac{1}{2}(g'\Box\Delta u)(t), \tag{4.19}$$

for some positive constant C_0.

Proof. Taking the deivative of $(g\Box\Delta u)(t)$, we can easily obtain

$\int_0^t g(t-s)(\Delta u(s),\Delta u_t(t))ds = \int_0^t g(t-s)\frac{d}{dt}(\Delta u(s),\Delta u(t))ds$

$= -\frac{1}{2}\int_0^t g(t-s)\frac{d}{dt}(\|\Delta u(t)-\Delta u(s)\|^2-\|\Delta u(s)\|^2-\|\Delta u(t)\|^2)ds$

$= -\frac{1}{2}\frac{d}{dt}(g\Box\Delta u)(t)+\frac{1}{2}(g'\Box\Delta u)(t)+\frac{1}{2}\int_0^t g(t-s)\frac{d}{dt}\|\Delta u(t)\|^2ds$

$\int_0^t g(t-s)ds = \int_0^t g(z)dz$ (using $z=t-s$)

$\int_0^t g(t-s)\frac{d}{dt}\|\Delta u(t)\|^2ds = \int_0^t g(t-s)ds\frac{d}{dt}\|\Delta u(t)\|^2 = \frac{d}{dt}\int_0^t g(s)ds\|\Delta u(t)\|^2 - \frac{g(t)}{2}\|\Delta u\|^2$

$\int_0^t g(t-s)(\Delta u(s),\Delta u_t(t))ds = -\frac{1}{2}\frac{d}{dt}(g\Box\Delta u)(t)+\frac{1}{2}\frac{d}{dt}\int_0^t g(s)ds\|\Delta u(t)\|^2 - \frac{g(t)}{2}\|\Delta u\|^2 + \frac{1}{2}(g'\Box\Delta u)(t)$.

Multiplying the first equation in (4.15) by u_t and using the above identity, we get

$$\begin{aligned}&\frac{d}{dt}(\frac{1}{2}\|u_t\|^2+\frac{1}{2}(1-\int_0^t g(s)ds)\|\Delta u\|^2+\frac{1}{2}(g\Box\Delta u)(t)+\frac{1}{4}\|\Delta\phi\|^2)\\ &= -a_0\|u_t\|^2 - a_1\int_\Omega z(x,1,t)u_tdx - \frac{g(t)}{2}\|\Delta u\|^2+\frac{1}{2}(g'\Box\Delta u)(t).\end{aligned} \tag{4.20}$$

Using the second equation in (4.15), we have

$$\begin{aligned}\int_0^t\int_\Omega z(x,\rho,t)dxd\rho &= -\int_0^1\int_\Omega \tfrac{1-\tau'(t)\rho}{\tau(t)}z(x,\rho,t)z_\rho(x,\rho,t)dxd\rho = -\tfrac{1}{2}\int_\Omega\int_0^1\tfrac{1-\tau'(t)\rho}{\tau(t)}\tfrac{d}{d\rho}z^2(x,\rho,t)d\rho dx\\ &= -\tfrac{1-\tau'(t)\rho}{2\tau(t)}\|z(x,1,t)\|^2+\tfrac{1}{2\tau(t)}\|u_t\|^2-\tfrac{\tau'(t)}{2\tau(t)}\int_0^1\|z(x,\rho,t)\|^2d\rho.\end{aligned} \tag{4.21}$$

Using Young's inequality (Lemma 1.1.1)

$$-a_1\int_\Omega z(x,1,t)u_tdx \leq |a_1|\|z(x,1,t)\|\|u_t\| \leq \frac{a_1^2}{2a_0}\|z(x,1,t)\|^2+\frac{a_0}{2}\|u_t\|^2. \tag{4.22}$$

Combining (4.16), (4.20), (4.21) and (4.22), we find that

$$\frac{dE}{dt} = \frac{d}{dt}(\frac{1}{2}\|u_t\|^2 + \frac{1}{2}(1 - \int_0^t g(s)ds)\|\Delta u\|^2 + \frac{1}{2}(g\Box\Delta u)(t) + \frac{1}{4}\|\Delta\phi\|^2 + \frac{\zeta}{2}\int_0^1\int_\Omega z^2(x,\rho,t)dxd\rho) =$$
$$-a_0\|u_t\|^2 - a_1\int_\Omega z(x,1,t)u_t dx - \frac{g(t)}{2}\|\Delta u\|^2 + \frac{1}{2}(g'\Box\Delta u)(t) + -\frac{1-\tau'(t)\rho}{2\tau(t)}\|z(x,1,t)\|^2 +$$
$$\frac{1}{2\tau(t)}\|u_t\|^2 - \frac{\tau'(t)}{2\tau(t)}\int_0^1 \|z(x,\rho,t)\|^2 d\rho \le -(\frac{a_0}{2} - \frac{\zeta}{2\tau(t)})\|u_t\|^2 - (\frac{(1-\tau'(t))\zeta}{2\tau(t)} - \frac{a_1^2}{2a_0})\|z(x,1,t)\|^2 -$$
$$\frac{\zeta\tau'(t)}{2\tau(t)}\int_0^1\int_\Omega z^2(x,\rho,t)dxd\rho - \frac{g(t)}{2}\|\Delta u\|^2 + \frac{1}{2}(g'\Box\Delta u)(t)$$
$$\le -C_0(\|u_t\|^2 + \|z(x,1,t)\|^2 + \int_0^1\int_\Omega z^2(x,\rho,t)dxd\rho)) - \frac{g(t)}{2}\|\Delta u\|^2 + \frac{1}{2}(g'\Box\Delta u)(t), \quad (4.23)$$

such that $C_0 = \min(\frac{a_0}{2} - \frac{\zeta}{2\tau(t)}, \frac{(1-\tau'(t))\zeta}{2\tau(t)} - \frac{a_1^2}{2a_0}, \frac{\zeta\tau'(t)}{2\tau(t)})$.

From (4.13), (4.17) and (4.23), we obtain the desired inequality (4.19). This implies that $E(t)$ is nonincreasing. □

For suitable choice of $N_1, N_2 > 0$, let us define the perturbed energy by

$$L(t) = N_1E(t) + N_2\Psi(t), \quad (4.24)$$

where $\psi(t) = \int_\Omega u_t u dx$. We easily have the following. For $N_1 > 0$ large enough, there exist positive constants α_1 and α_2 such that $\alpha_1 E(t) \le L(t) \le \alpha_2 E(t),\ \forall t \geqslant 0$

Lemma 4.1.2

Under the assumption (H_1), the functional $\psi(t)$ satisfies the estimate

$$\psi'(t) \leqslant -\frac{l}{2}\|\Delta u\|^2 + (1 + \frac{2a_0^2}{\lambda l})\|u_t\|^2 + \frac{2a_1^2}{\lambda l}\|z(x,1,t)\|^2 + \frac{C(\delta)}{l}(\chi\Box\Delta u)(t) - \|\Delta\phi\|^2 \quad (4.25)$$

for any $0 < \delta < 1$, where

$$C(\delta) = \int_0^\infty \frac{g^2(s)}{\chi(s)}ds \text{ and } \chi(t) = \delta g(t) - g'(t) > 0 \quad (4.26)$$

Proof. From (3.3), (4.11), (4.15) and Young's inequality (lemma 4.1.1), we have

$$\psi'(t) = \|u_t\|^2 + \int_\Omega u_{tt}u(t)dx = \|u_t\|^2 - \int_\Omega \Delta^2 u(t)u(t)dx + \int_\Omega\int_0^t g(t-s)\Delta^2 u(s)dsu(t)dx$$
$$+ \int_\Omega [u,\phi]u(t)dx - a_0\int_\Omega u_t(t)u(t)dx - a_1\int_\Omega z(x,1,t)u(t)dx.$$

$-a_0\int_\Omega u_t(t)u(t)dx \le a_0\|u_t\|\|u(t)\| \le \frac{2a_0^2}{\lambda l}\|u_t\|^2 + \frac{l\lambda}{8}\|u\|^2 \le \frac{2a_0^2}{\lambda l}\|u_t\|^2 + \frac{l}{8}\|\Delta u\|^2 (\varepsilon = \frac{4}{l\lambda})$

$-a_1\int_\Omega z(x,1,t)u(t)dx \le \frac{2a_1^2}{\lambda l}\|z(x,1,t)\|^2 + \frac{\lambda l}{8}\|u\|^2 \le \frac{2a_1^2}{\lambda l}\|z(x,1,t)\|^2 + \frac{l}{8}\|\Delta u\|^2$

$\int_\Omega\int_0^t g(t-s)\Delta^2u(s)dsu(t)dx = \int_0^t g(t-s)\int_\Omega \Delta^2 u(s)u(t)dxds = \int_0^t g(t-s)\int_\Omega \Delta u(s)\Delta u(t)dxds$

$= \int_0^t g(s)ds\|\Delta u\|^2 + \int_0^t g(t-s)(\Delta u(s)-\Delta u(t),\Delta u(t))ds$

$\le \int_0^t g(s)ds\|\Delta u\|^2 + \frac{1}{l}\|\int_0^t g(t-s)(\Delta u(s)-\Delta u(t))ds\|^2 + \frac{l}{4}\|\Delta u\|^2.$

Using Cauchy-Schwarz inequality:

$\|\int_0^t g(t-s)(\Delta u(s)-\Delta u(t))ds\|^2 = \int_\Omega(\int_0^t g(t-s)(\Delta u(s)-\Delta u(t)))^2dsdx =$

$\int_\Omega(\int_0^t \frac{g(t-s)}{\sqrt{\chi}}\sqrt{\chi}(\Delta u(s)-\Delta u(t)))^2dsdx \le \int_\Omega\int_0^t \frac{g^2(t-s)}{\chi(t-s)}ds\int_0^t \chi(t-s)(\Delta u(s)-\Delta u(t))^2dsdx$

$\le C(\delta)(\chi\Box\Delta u)(t)$ then

$\psi'(t) \le -\frac{l}{2}\|\Delta u\|^2 + (1+\frac{2a_0^2}{\lambda l})\|u_t\|^2 + \frac{2a_1^2}{\lambda l}\|z(x,1,t)\|^2 + \frac{C(\delta)}{l}(\chi\Box\Delta u)(t) - \|\Delta\phi\|^2.$ □

Lemma 4.1.3

Assume that (H_1) and (H_2) hold. Then for $N_1, N_2 > 0$, the functional L satisfies

$$L'(t) \le -C_1(\|u_t\|^2+\|z(x,1,t)\|^2+\int_0^1\int_\Omega z^2(x,\rho,t)dxd\rho+\|\Delta v\|^2)+3(1-l)\|\Delta u\|^2+\frac{1}{4}(g\Box\Delta u)(t), \tag{4.27}$$

where C_1 is some positive constant.

Proof. Combining (4.19), (4.24), (4.25), recalling that $g'(t) = \delta g(t) - \chi(t)$, we obtain

$L'(t) = N_1E'(t) + N_2\psi'(t) \le N_1(-C_0(\|u_t\|^2 + \|z(x,1,t)\|^2 + \int_0^1\int_\Omega z^2(x,\rho,t)dxd\rho) - \frac{g(t)}{2}\|\Delta u\|^2 + \frac{1}{2}(g'\Box\Delta u)(t)) + N_2(\frac{-l}{2}\|\Delta u\|^2 + (1+\frac{2a_0^2}{\lambda l})\|u_t\|^2 + \frac{2a_1^2}{\lambda l}\|z(x,1,t)\|^2 + \frac{C(\delta)}{l}(\chi\Box\Delta u)(t) - \|\Delta\phi\|^2) \le -(C_0N_1 - (1+\frac{2a_0^2}{\lambda l})N_2)\|u_t\|^2 - (C_0N_1 - \frac{2a_1^2}{\lambda l}N_2)\|z(x,1,t)\|^2 - (\frac{lN_2}{2} + \frac{g(t)N_1}{2})\|\Delta u\|^2 + \frac{N_1}{2}(g'\Box\Delta u)(t) - N_2\|\Delta\phi\|^2 + \frac{N_2C(\delta)}{l}(\chi\Box\Delta u)(t)$

$\le -(C_0N_1 - (1+\frac{2a_0^2}{\lambda l})N_2)\|u_t\|^2 - (C_0N_1 - \frac{2a_1^2}{\lambda l}N_2)\|z(x,1,t)\|^2 - (\frac{lN_2}{2} + \frac{g(t)N_1}{2})\|\Delta u\|^2 + \frac{\delta N_1}{2}(g\Box\Delta u)(t) - N_2\|\Delta\phi\|^2 - (\frac{N_1}{2} - \frac{C(\delta)N_2}{2})(\chi\Box\Delta u)(t).$

We first take N_2 large enough so that $N_2 > \frac{6(1-l)}{l}$. From (4.12) and (4.26), we see that

$$0 \le -g'(t) \Longrightarrow \delta g(t) \leqslant \delta g(t) - g'(t) \Longrightarrow \frac{\delta g(t)}{\chi(t)} \leqslant 1 \Longrightarrow \frac{\delta g^2(t)}{\chi(t)} \leqslant g(t). \tag{4.28}$$

Therefore, by (4.11), (4.26) and (4.28), we get

$$\delta C(\delta) = \delta \int_0^\infty \frac{g^2(t)}{\chi(t)} ds \leq \int_0^\infty g(s)ds = 1 - l.$$

Applying the Lebesgue dominated convergence theorem, we have $\delta C(\delta) \longrightarrow 0$ as $\delta \longrightarrow 0$. Hence, there is $0 < \delta_0 < 1$ such that if $\delta < \delta_0$, then $\frac{\delta C(\delta) N_2}{l} < \frac{1}{8}$. Choosing N_1 large enough so that

$$N_1 > \max\{(1 + \frac{2a_0^2}{\lambda l})\frac{N_2}{C_0}, \frac{2a_1^2 N_2}{C_0 \lambda l}\}.$$

Choosing $\delta = \frac{1}{2N_1} < \delta_0$, therefore $L'(t) \leq -C_1(\|u_t\|^2 + \|z(x,1,t)\|^2 + \int_0^1 \int_\Omega z^2(x,\rho,t)dxd\rho + \|\Delta v\|^2) + 3(1-l)\|\Delta u\|^2 + \frac{1}{4}(g\Box\Delta u)(t)$. □

Lemma 4.1.4

Under the assumption (H_1), the functional k satisfies

$$k'(t) \leq 2(1-l)\|\Delta u\|^2 - \frac{1}{2}(g\Box\Delta)(t), \tag{4.29}$$

where $k(t) = \int_0^t h(t-s)\|\Delta u(s)\|^2 ds$ and $h(t) = \int_t^\infty g(s)ds$.

Proof. Using the Young's inequality, (4.11) and $h'(t) = -g(s)$, we find that

$k'(t) = \int_0^t h'(t-s)\|\Delta u(s)\|^2 ds + h(0)\|\Delta u(t)\|^2 = -\int_0^t g(t-s)\|\Delta u\|^2 ds + \int_0^\infty g(s)ds\|\Delta u(t)\|^2$

we know the following:

$\forall a, b \in \mathbb{R} : (a^2 + b^2) \geqslant \frac{1}{2}(a-b)^2$

then

$\|\Delta u(s)\|^2 + \|\Delta u(t)\|^2 - \frac{1}{2}\|\Delta u(t) - \Delta u(s)\|^2 \geqslant 0$

$\int_0^t g(t-s)[\|\Delta u(s)\|^2 + \|\Delta u(t)\|^2 - \frac{1}{2}\|\Delta u(t) - \Delta u(s)\|^2]ds \geqslant 0$

for we conclude

$-\int_0^t g(t-s)\|\Delta u(s)\|^2 ds \leq \int_0^t g(t-s)\|\Delta u(t)\|^2 ds - \frac{1}{2}\int_0^t g(t-s)\|\Delta u(t) - \Delta u(s)\|^2 ds$

$\leq \int_0^\infty g(s)ds\|\Delta u(t)\|^2 - \frac{1}{2}(g\Box\Delta u)(t)$.

Consequently

$k'(t) \leq 2\int_0^\infty g(s)ds\|\Delta u(t)\|^2 - \frac{1}{2}(g\Box\Delta u)(t) \implies k'(t) \leq 2(1-l)\|\Delta u(t)\|^2 - \frac{1}{2}(g\Box\Delta u)(t)$ □

Proof. of Theorem 4.2. From (4.16) and (4.27), there exist constants α_3 and α_4 such that

$$L'(t) \leq -\alpha_3 E(t) + \alpha_4 (g\Box\Delta u)(t). \tag{4.30}$$

Indeed,

$L'(t) \leq -C_1(\|u_t\|^2 + \|z(x,1,t)\|^2 + \int_0^1 \int_\Omega z^2(x,\rho,t)dxd\rho + \|\Delta v\|^2) + 3(1-l)\|\Delta u\|^2 + \frac{1}{4}(g\Box\Delta u)(t) \leq -\min(2C_1, 2\zeta C_1, 4C_1, 1)E(t) + \frac{1}{2}(g\Box\Delta u)(t)$

such that $\alpha_3 = \min(2C_1, 2\zeta C_1, 4C_1, 1)$ and $\alpha_4 = \frac{1}{2}$.

First, we construct the functional $F(t) = L(t) + k(t)$. It is nonnegative. From (4.16), (4.27) and (4.29), we get $F'(t) \leq -C_2 E(t)$, where C_2 is some positive constant. Then, we obtain $C_2 \int_0^t E(s)ds \leq F(0) - F(t) \leq F(0) = L(0)$. Therefore, we deduce that

$$\int_0^\infty E(s)ds < \infty. \tag{4.31}$$

From (4.30), we define $\theta(t)$ by, for a constant $0 < \eta < 1$,

$\theta(t) := \eta \int_0^t \|\Delta u(t) - \Delta u(s)\|^2 ds \in (0,1).$

Since H is strictly convex on (0,k], then $H(\alpha x) \leq \alpha H(x)$ where $0 \leq \alpha \leq 1$ and $x \in (0,k]$. Using (4.10), (4.12), (4.19) and the fact that ξ is a positive nonincreasing function and $0 < \eta < 1$, we find that

$$\begin{aligned}(g\Box\Delta u)(t) &= \int_0^t g(t-s)\|\Delta u(t) - \Delta u(s)\|^2 ds \leq \int_0^t H^{-1}(\tfrac{-g'(t-s)}{\xi(t-s)})\|\Delta u(t) - \Delta u(s)\|^2 ds \\ &\leq \tfrac{\theta(t)}{\eta} \int_0^t H^{-1}(\tfrac{-g'(t-s)}{\xi(t-s)}) \tfrac{\eta\|\Delta u(t)-\Delta u(s)\|^2}{\theta(t)} ds \leq \tfrac{\theta(t)}{\eta} H^{-1}(\eta \int_0^t \tfrac{-g'(t-s)\|\Delta u(t)-\Delta u(s)\|^2}{\xi(t-s)\theta(t)} ds) \\ &\leq \tfrac{1}{\eta} H^{-1}(\eta \int_0^t \tfrac{-g'(t-s)\|\Delta u(t)-\Delta u(s)\|^2}{\xi(t-s)} ds) \leq \tfrac{1}{\eta} H^{-1}(\int_0^t \tfrac{-g'(t-s)\|\Delta u(t)-\Delta u(s)\|^2}{\xi(t-s)} ds) \\ &\leq \tfrac{1}{\eta} H^{-1}(\tfrac{-2E'(t)}{\xi(t)}).\end{aligned} \tag{4.32}$$

Combining (4.30) and (4.32), we get

$$L'(t) \leq -\alpha_3 E(t) + \frac{\alpha_4}{\eta} H^{-1}(-\frac{2E'(t)}{\xi(t)}). \tag{4.33}$$

Denote the conjugate function of the strictly convex function H by H^* using Legendre transformation, see [35], then

$$H^*(s) = s(H')^{-1}(s) - H((H')^{-1}(s)) \leq s(H')^{-1}(s), \tag{4.34}$$

and H^* satisfies the Young's inequality, $AB \leq H^*(A) + H(B)$. Now, for $\epsilon_0 < \frac{k}{E(0)}$, we define the functional

$$R(t) := L(t)H'(\epsilon_0 E(t)) + E(t),$$

which is equivalent to E. Using (4.33) and (4.34) and the fact that $H > 0$, $H' > 0$ and $H'' > 0$, we obtain

$R'(t) = L'(t)H'(\epsilon_0 E(t)) + \epsilon_0 E'(t)L(t)H''(\epsilon_0 E(t)) + E'(t) \leq L'(t)H'(\epsilon_0 E(t))$
$\leq -\alpha_3 E(t)H'(\epsilon_0 E(t)) + \frac{\alpha_4}{\eta}H'(\epsilon_0 E(t))H^{-1}(\frac{-2E'(t)}{\xi(t)}) \leq -\alpha_3 E(t)H'(\epsilon_0 E(t)) +$
$\frac{\alpha_4}{\eta}(\epsilon_0 E(t)H'(\epsilon_0 E(t)) - \frac{2E'(t)}{\xi(t)}) \leq -(\alpha_3 - \frac{\epsilon_0\alpha_4}{\eta})E(t)H'(\epsilon_0 E(t)) - \frac{2\alpha_4 E'(t)}{\eta\xi(t)}.$

Then, multiplying by $\xi(t)$, we see that

$$\xi(t)R'(t) \leq -(\alpha_3 - \frac{\epsilon_0\alpha_4}{\eta})\xi(t)E(t)H'(\epsilon_0 E(t)) - \frac{2\alpha_4 E'(t)}{\eta}. \tag{4.35}$$

We take $\Phi(t) = \xi(t)R(t) + \frac{2\alpha_4}{\eta}E(t)$, which satisfies

$$d_1 E(t) \leq \Phi(t) \leq d_2 E(t), \tag{4.36}$$

for some $d_1, d_2 > 0$. Consequently, with a suitable choice of ϵ_0, and using (4.35), (4.36) and the fact that $H'(t)$ is strictly increasing function, we find that

$$\Phi'(t) = \xi'(t)R(t) + \xi(t)R(t) + \frac{2\alpha_4 E'(t)}{\eta} \leq -(\alpha_3 - \frac{\epsilon_0\alpha_4}{\eta})\xi(t)E(t)H'(\epsilon_0 E(t)) \leq$$
$$-d_3\xi(t)\phi(t)H'(d_4\Phi(t)).$$

where $d_3 = (\alpha_3 - \frac{\epsilon_0\alpha_4}{\eta})\frac{1}{d_2} > 0$ and $d_4 = \frac{\epsilon_0}{d_2}$. A simple integration and a variable tansformation give

$$\int_0^t \frac{-\Phi'(s)}{\Phi(s)H'(d_4\Phi(s))}ds \geqslant d_3\int_0^t \xi(s)ds \Longrightarrow \int_{d_4\Phi(t)}^{d_4\Phi(0)} \frac{1}{sH'(s)}ds \geqslant d_3\int_0^t \xi(s)ds, \ \forall t \geqslant 0. \tag{4.37}$$

We take

$$G(t) = \int_t^k \frac{1}{sH'(s)} ds,$$

which is strictly decreasing function on (0,k]. From (4.37), we obtain

$$G(d_4\Phi(t)) = \int_{d_4\Phi(t)}^k \frac{1}{sH'(s)} ds \geqslant \int_{d_4\Phi(t)}^{d_4\Phi(0)} \frac{1}{sH'(s)} ds \geqslant d_3 \int_0^t \xi(s)ds \ \forall t \geqslant 0.$$

Consequently, we deduce that $\Phi(t) \leq \frac{1}{d_4} G^{-1}(d_3 \int_0^t \xi(s)ds)$.

Then $E(t) \leqslant \frac{1}{d_1 d_4} G^{-1}(d_3 \int_0^t \xi(s)ds)$ such that $k_1 = \frac{1}{d_1 d_4}$ and $k_0 = d_3$. □

Remark 4.1.2

It has to be noted that, in case $\int_0^\infty \xi(s)ds = \infty$, Theorem 4.2 ensures $\lim_{t\to\infty} E(t) = 0$. Moreover, the decay rate of $E(t)$ driven by (4.18) is optimal in the sense that it is consistent with the decay rate of $g(t)$ driven by (4.9).

Example

For $H(s) = s^p$, such that $1 \leq p < 2$, where (4.9) is given by

$$g'(t) \leq -\xi(t) g^p(t).$$

Then there are positive constants k, k_0, k_1 such that the decay rate of E is given by

$$E(t) \leq \begin{cases} k e^{-k_0 \int_0^t \xi(s)ds} & \text{if } p = 1 \\ k_1 (k_0 p(p-1) \int_0^t \xi(s)ds + k^{1-p})^{\frac{-1}{p-1}} & \text{if } 1 < p < 2 \end{cases}$$

ABSTRACT

In our work, we focus on studying the stability of nonlinear elastic plates of von Karman with memory and delay, using the energy method. Based on functional analysis techniques, we investigate the behavior of the system's energy. We show that this energy is decreasing and then determine the rate or speed of convergence, providing us with information about the system's stability. Our research work addresses three distinct problems. The first one concerns the study of energy decay for a von Karman plate hinged at the boundary, with infinite memory. The second problem extends the first one by adding a fixed time delay term. In both cases, the energy decays polynomially. The third problem deals with the energy decay of a von Karman plate clamped at the boundary, with finite memory and a variable time delay term. In this case, the system's energy decays exponentially. We observe that the type of boundary conditions and memory have an influence on the rate of energy decay of the system and its stability.

Key words: *von Karman plate, energy decay, stability, memory, time delay.*

Réusmé

Dans notre travail, nous nous intéressons à l'étude de la stabilité des plaques élastiques non linéaires de von Karman, avec mémoire et retard, en utilisant la méthode de l'énergie. En nous appuyant sur des techniques d'analyse fonctionnelle, nous examinons le comportement de l'énergie du système. Nous montrons que cette énergie décroît, puis nous déterminons la vitesse ou le taux de convergence, ce qui nous fournit des informations sur la stabilité du système. Notre travail de recherche aborde trois problèmes distincts. Le premier concerne l'étude de la décroissance de l'énergie d'une plaque de von Karman articulée sur le bord, avec une mémoire infinie. Le deuxième problème étend le premier en ajoutant un terme de retard temporel fixe. Dans les deux cas, l'énergie décroît de manière polynomiale. Le troisième problème traite la décroissance de l'énergie d'une plaque de von Karman encastrée sur le bord, avec une mémoire finie et un terme de retard temporel variable. Dans ce cas, l'énergie du système décroît de manière exponentielle. Nous constatons que le type de conditions aux limites et de mémoire a une influence sur la vitesse de décroissance de l'énergie du système et sa stabilité.

Mots clés: *plaque de von Karman, décroissance de l'énergie, stabilité, mémoire, retard temporel.*

ملخص

في عملنا, نركز على دراسة استقرار الصفائح المرنة غير الخطية لفون كارمان ذات الذاكرة و التأخير, باستخدام طريقة الطاقة. باستخدام تقنيات التحليل الدالي, نبحث في سلوك طاقة النظام. نثبت أن هذه الطاقة تنخفض, ثم نحدد معدل أو سرعة التناقص , مما يزودنا بمعلومات عن استقرار النظام.
يتناول عملنا البحثي ثلاثة مسائل متميزة. المسألة الأولى تتعلق بدراسة إنخفاض الطاقة لصفيحة فون كارمان معلقة على الحافة, مع ذاكرة لا نهائية. المسألة الثانية تمدد الحالة الأولى عن طريق إضافة تأخير زمني ثابت. في كلا الحالتين, تنخفض الطاقة بمعدل تضاؤل نوع كثير الحدود.
المسألة الثالثة تهتم بدراسة إنخفاض طاقة صفيحة فون كارمان مثبتة على الحافة, مع ذاكرة محدودة و تأخير زمني متغير. في هده الحالة تنخفض طاقة النظام بشكل متسارع أسي. نلاحظ أن نوع شروط الحدية و الذاكرة له تأثير على معدل إنخفاض طاقة النظام و إستقراره .

الكلمات المفتاحية: صفيحة فون كارمان, إضمحلال الطاقة, الاستقرار, الذاكرة, التأخير الزمني.

Bibliography

[1] Andrei D. Polyanin and Vsevolod G. Sorokin, Exact Solutions of Reaction–Diffusion PDEs with Anisotropic Time Delay, Mathematics (2023).

[2] Françoise Demengel, Gilbert Demengel, Functional Spaces for the Theory of Elliptic Partial Differential Equations Translated by Reinie, Springer (2012) .

[3] S. Salsa, Partial Differential Equations in Action: From Modelling to Theory, Springer International Publishing, 3rd ed., (2016).

[4] XIANG GAO, MEERA SITHARAM, ADRIAN E. ROITBERG, BOUNDS ON THE JENSEN GAP, AND IMPLICATIONS FOR MEAN-CONCENTRATED DISTRIBUTIONS, Australian Journal of Mathematical Analysis and Application, (2019).

[5] R. Datko, J. Lagnese and M. P. Polis, An Example on Effect of Time Delays in Boundary Feedback Stabilzation of Wave Equation, SIAM, (1986).

[6] C. M. Dafermos, Asymptotic stability in viscoelasticity, Arch. Rational Mech. Anal., 37 (1970), 297-308.

[7] S. Nicaise and C. Pignotti, Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks, SIAM J. Control Optim., 5 (2006), 1561-1585.

[8] S. Nicaise and C. Pignotti, Stabilization of the wave equation with boundary or internal distributed delay, Diff. Integral Equa., 9-10 (2008), 935-958.

[9] I. Chueshov, I. Lasiecka, Von Karman Evolution Equations: Well-Posedness and Long Time Dynamics, Springer, Berlin, (2010).

[10] T.G. Ha, S.H. Park, Existence and decay results for a von Karman equation with variable exponent nonlinearities, Math. Methods Appl. Sci. 44 (2021), 9475–9486.

[11] J.R. Kang, General decay for a von Karman equation with memory and time-varying delay, Appl. Math. Lett. 122 (2021) 107537, 7 pp.

[12] J.R. Kang, Existence and blow-up of solutions for von Karman equations with time delay and variable exponents, Appl. Math. Comput. 371 (2020) 124917, 15 pp.

[13] M.J. Lee, J.R. Kang, Blow-up results for a quasilinear von Karman equation of memory type with acoustic boundary conditions, Appl. Math. Lett. 112 (2021) 106693, 7 pp.

[14] S.H. Park, Attractors for a von Karman equation with memory, Sci. China Math. 58 (2015) 2505–2516.

[15] S.H. Park, Stability of a von Karman equation with infinite memory, Acta Math. Sci. 37 (2017) 965–973.

[16] S.H. Park, A general decay result for a von Karman equation with memory and acoustic boundary conditions, J. Appl. Anal. Comput. 12 (2022) 17–30.

[17] E.M. Bonotto, M.J.D. Nascimento, E.B. Santiago, Long-time behaviour for a non-autonomous Klein–Gordon-szakharov system, J. Math. Anal. Appl. 506 (2) (2022) 125670, 42 pp.

[18] K.P. Jin, J. Liang, T.J. Xiao, Coupled second order evolution equations with fading memory: Optimal energy decay rate, J. Differential Equations 257 (2014) 1501–1528.

[19] K.P. Jin, J. Liang, T.J. Xiao, Asymptotic behavior for coupled systems of second order abstract evolution equations with one infinite memory, J. Math. Anal. Appl. 475 (2019) 554–575.

[20] C. Li, J. Liang, T.J. Xiao, Long-term dynamical behavior of the wave model with locally distributed frictional and viscoelastic damping, Commun. Nonlinear Sci. Numer. Simul. 92 (2021) 105472, 1–22.

[21] Y.M. Qin, Z.Y. Ma, Global Well-Posedness and Asymptotic Behavior of the Solutions to Non-Classical Thermo(Visco)Elastic Models, Springer, Singapore, (2016).

[22] Y.M. Qin, S.Y. Sheng, Global existence, asymptotic behavior and uniform attractor for a non-autonomous Timoshenko system of type III with weak damping, Asymptot. Anal. 125 (2021) 133–157.

[23] T.J. Xiao, J. Liang, Coupled second order semilinear evolution equations indirectly damped via memory effects, J. Differential Equations 254 (5) (2013) 2128–2157.

[24] J.E.M. Rivera, G.P. Menzala, Decay rates of solutions to a von Karman system for viscoelastic plates with memory, Quart. Appl. Math. LVII (1999) 181–200.

[25] C.A. Raposo, M.L. Santos, General decay to a von Karman system with memory, Nonlinear Anal. 74 (2011) 937–945.

[26] J.R. Kang, A general stability for a von Karman system with memory, Bound. Value Probl. 2015 (2015) 204.

[27] M. Cavalcanti, D.D. Andre, Cavalcanti, I. Lasiecka, X. Wang, Existence and sharp decay rate estimates for a von Karman system with long memory, Nonlinear Anal. RWA 22 (2015) 289–306.

[28] M.I. Mustafa, Optimal decay rates for the viscoelastic wave equation, Math. Methods Appl. Sci. 41 (2018) 192–204.

[29] M.I. Mustafa, General decay result for nonlinear viscoelastic equations, J. Math. Anal. Appl. 457 (2018) 134–152.

[30] J.H. Hassan, S.A. Messaoudi, General decay rate for a class of weakly dissipative second-order systems with memory, Math. Methods Appl. Sci. 42 (8) (2019) 2842–2853.

[31] J.H. Hassan, S.A. Messaoudi, General decay results for a viscoelastic wave equation with a variable exponent nonlinearity, Asymptotic Anal. (2021) 1–24.

[32] W.J. Liu, General decay of the solution for a viscoelastic wave equation with a time-varying delay term in the internal feedback, J. Math. Physics. 54 (2013) 043504.

[33] S. Nicaise, C. Pignotti, Interior feedback stabilization of wave equations with time dependent delay, Electron. J. Differ. Equ. 2011 (41) (2011) 1–20.

[34] B.W. Feng, General decay for a viscoelastic wave equation with strong time-depenent delay, Boundary Value Probl. 2017 (2017) 57.

[35] V.I. Arnold, Mathematical Methods of Classical Mechanics, Springer-Verlag, New York, (1989).

[36] Ti-Jun Xiao and Rui-Han Zhou, Energy decay rate for a nonlinear von Karman equation with memory, (2022), 0893-9659.