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# Energy decay of thermopiezoelectric system with time delay

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# Dedication

I dedicate this humble work to my dear family,who have always been my constant support and encouragement.

To my first teachers in life, my pillars and sources of inspiration, my dear parents.

To my brothers and sisters.

To my esteemed teachers who supported me throughout my academic journey .

To my dear friends,and to everyone who played a role in my success.

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# Contents

# Notations and Conventions

## Notations

- $\partial_i = \frac{\partial}{\partial x_i}$: Partial differentiation with respect to $x_i$.
- $L^p(\Omega)$: Space of $p$-th integrable functions on $\Omega$ with respect to the LEBESGUE measure $dx$, for $p \in [1, +\infty[$.
- $L^\infty(\Omega)$: Space of bounded functions on $\Omega$.
- $H^m(\Omega)$: SOBOLEV space of order $m$, for $m \in \mathbb{N}$.
- $H_0^1(\Omega)$: Space of functions in $H^1(\Omega)$ vanishing on the boundary.
- $\|.\|_V$: The norm in the space V.
- $u_t = \frac{du}{dt}$, and $u_{tt} = \frac{d^2u}{dt^2}$: they symbolize the derivatives of function $u$ with respect to time $t$.

NOTATIONS AND CONVENTIONS

## Conventions

- Latin indices $\{i, j, k\}$ vary over the set $\{1, 2, 3\}$.
- Einstein summation convention is always used over repeated indices and exponents, i.e.
$$\begin{cases} a_i b_i = \sum_{i=1}^{3} a_i b_i, \qquad X_{ij}Y_{ij} = \sum_{i,j=1}^{3} X_{ij}Y_{ij}, \\ a_i b^i = \sum_{i=1}^{3} a_i b^i, \qquad X_{ij}Y^{ij} = \sum_{i,j=1}^{3} X_{ij}Y^{ij}, \\ a^i b^i = \sum_{i=1}^{3} a^i b^i, \qquad X^{ij}Y^{ij} = \sum_{i,j=1}^{3} X^{ij}Y^{ij}, \\ a_\alpha b_\alpha = \sum_{\alpha=1}^{2} a_\alpha b_\alpha, \qquad X_{\alpha\beta}Y_{\alpha\beta} = \sum_{\alpha,\beta=1}^{2} X_{\alpha\beta}Y_{\alpha\beta}. \end{cases}$$
- **Boldface** letters denote vector-valued functions or tensor-valued functions and their associated function spaces.
- $\mathbf{L}^2(\Omega)$ represent $(L^2(\Omega))^3$.
- $\mathbf{H}_0^1(\Omega)$ represent $(H_0^1(\Omega))^3$.
- $\mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)$ represent $(H^2(\Omega) \cap H_0^1(\Omega))^3$.
- $\mathbf{H}^1(\Omega)$ and $\mathbf{H}^2(\Omega)$ represent $(H^1(\Omega))^3$ and $(H^2(\Omega))^3$, respectively.

# Introduction

Thermopiezoelectric systems are materials or devices that exhibit both thermoelectric and piezoelectric effects. These systems have the ability to convert thermal energy into electrical energy (thermoelectric effect) and mechanical deformation into electrical energy (piezoelectric effect), or vice versa. Thermopiezoelectric systems find applications in various fields, including energy harvesting, sensing, actuation, and control systems. They can be used to convert waste heat into electricity, harvest energy from vibrations or mechanical motions, or generate electrical signals based on mechanical or thermal stimuli. These systems offer the advantage of dual functionality, enabling the simultaneous conversion of thermal and mechanical energy into electrical energy. To study the stability of these systems, we are concerned with analyzing the rate at which the energy dissipates over time. Also, we study the influence of time delay on the stability of the 1D-system. Recall that time delays arise in many applications because, in most instances, physical, chemical, biological, thermal, and economic phenomena naturally depend not only on the present state but also on some past occurrences. In recent years, the control of PDEs with time delay effects has become an active area of research. In many cases it was shown that delay is a source of instability and even an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable in the absence of delay unless additional conditions or control terms have been used. The stability issue of systems with delay is, therefore, of theoretical and practical importance.

The first chapter recalled some results in the theory of linear operators and semi-groups. In addition, some useful inequalities, the energy method and the notion of delay and its relation to the stability of a system.

In the second chapter we presented the piezoelectric three-dimensional system with dissipation term. Then we studied the existence and uniqueness by semigroup approach. We presented the stability analysis carried out by Miara [17], by examining the decay of energy using the energy method. Using these results, we investigated the stability of the same system when adding a constant time delay. This is our first result. The third chapter is based on the paper [6]. First, we adressed the well-posedness of a three-dimensional thermopizoelectric system. We presented the existence and uniqueness using the semigroup theory. Secondly, we examined the stability, the exponential decay, by using Weyl's theorem, the observability inequality and the decoupling method of Henry. Finally, the fourth chapter focused on the stability study of thermopizoelectric one-dimensional medium (rod) with time delay, we showed the exponential decay of energy using the energy method, where we considered the system presented in Chapter 3 in one dimension with Neumann-Neumann-Dirichlet boundary conditions. This is our second result. It's worth noting that article [12] helped us to complete this work.

Chapter 1

# Preliminaries

## 1.1 Basic concepts

**Definition 1.1.1** *[2]*
*Let $X$ be a Banach space. A one parameter family $T(t)$, $0 \leq t < \infty$, of bounded linear operators from $X$ into $X$ is a semigroup of a bounded linear operator on $X$ if*
*(i) $T(0) = I$, ($I$ is the identity operator on $X$).*
*(ii) $T(t+s) = T(t)T(s)$ for every $t, s \geq 0$ (the semi group property).*
*A semigroup of bounded linear operators, $T(t)$ is uniformly continuous if*

$$\lim_{t \to 0} \|T(t) - I\| = 0$$

*The linear operator $A$ defined by*

$$D(A) = \left\{ x \in X : \lim_{t \to 0} \frac{T(t)x - x}{t} \text{ exists} \right\}$$

*and*

$$Ax = \lim_{t \to 0} \frac{T(t)x - x}{t} = \left. \frac{d^{+}T(t)x}{dt} \right|_{t=0} \quad \text{for} \quad x \in D(A)$$

*is the infinitesimal generator of the semigroup $T(t)$, $D(A)$ is the domain of $A$.*

**Corollary 1.1.1** *[2]*
*Let $T(t)$ be a uniformly continuous semigroup of bounded linear operators. Then*
*a) There exists a constant $w \geq 0$ such that $\|T(t)\| \leq \exp(wt)$.*
*b) There exists a unique bounded linear operator $A$ such that $T(t) = \exp(tA)$.*
*c) The operator $A$ is the infinitesimal generator of $T(t)$.*
*d) $t \to T(t)$ is differentiable in norm and*

$$\frac{dT(t)}{dt} = AT(t) = T(t)A$$

**Definition 1.1.2** *[2]*
*A semigroup $T(t)$, $0 \leq t < \infty$ of bounded linear operators on $X$ is a strongly continuous semigroup ($C_0$ semi group) of bounded linear operators if*

$$\lim_{t \to 0} T(t)x = x \quad \text{for every} \quad x \in X$$

**Theorem 1.1.1** *[2]*
*Let $T(t)$ be a $C_0$ semigroup. There exist constants $w \geq 0$ and $M \geq 1$ such that*

$$\|T(t)\| \leq M \exp(wt) \quad \textit{for} \quad 0 \leq t < \infty.$$

In particular case M=1 and $w = 0$, we have

**Definition 1.1.3** *[2]*
*The semigroup $T(t)$ is called* ***a contraction semigroup of class*** *$C_0$ if*

$$\|T(t)\|_{L(X,X)} \leq 1 \ \textit{ for all } \ t \geq 0,$$

*where $L(X,X)$ is Banach algebra of all linear continuous operators from $X$ into itself.*

**Definition 1.1.4 (resolvent and spectrum)** *[9]*
*Let $A \in \mathcal{L}(E)$. The resolvent set, denoted by $\rho(A)$, is defined by*

$$\rho(A) = \{\lambda \in \mathbb{C} : (\lambda I - A) \textit{ is bijective from } E \textit{ onto } E\}.$$

*The spectrum of $A$, denoted by $\sigma(A)$, is the complement of the resolvent set, i.e, $\sigma(A) = \mathbb{C} \backslash \rho(A)$.*

**Definition 1.1.5** *[2] [13]*
*a)* ***The point spectrum***

$$\sigma_p(A) = \{\lambda \in \mathbb{C} : (\lambda I - A) \text{ is not one-to-one } \}.$$

*Thuse the point spectrum of $A$ is precisely the set of all eigenvalues of $A$.*
*b)* ***The continuous spectrum***

$$\sigma_c(A) = \left\{\lambda \in \mathbb{C} : N(\lambda I - A) = 0, \overline{R(\lambda I - A)} = X \text{ and } R(\lambda I - A) \neq X\right\}.$$

*c)* ***The residual spectrum***

$$\sigma_r(A) = \left\{\lambda \in \sigma(A) \diagdown \{\sigma_p(A) \cup \sigma_c(A)\}\right\} = \left\{\lambda \in \mathbb{C} : N(\lambda I - A) = 0 \text{ and } \overline{R(\lambda I - A)} \neq X\right\}.$$

*d)* ***The discrete spectrum***

*The discrete spectrum of $A$, $\sigma_d(A)$, is the set of all eigenvalues of $A$ with finite (algebraic) multiplicity and which are isolated points of $\sigma_d(A)$.*
*e)* ***The essential spectrum***

$$\sigma_{ess}(A) = \sigma(A) \diagdown \sigma_d(A).$$

**Remark 1.1.1** *The caracteristic property of the essential spectrum, is its robustness under various perturbations (it is stable under relatively compact perturbation of $A$).*

**Theorem 1.1.2 (Weyl's theorem)** *[11] [4]*
*Let $A$ and $B$ be unbounded self-adjoint operators in Hilbert space $\boldsymbol{H}$, then*

$$\sigma_{ess}(A + B) = \sigma_{ess}(A)$$

*if $B$ is compact.*

**Corollary 1.1.2** *If $\sigma_{ess}(A+B) = \sigma_{ess}(A)$ for all bounded self-adjoint operators $A$, then (the self-adjoint) $B$ is compact.*

**Theorem 1.1.3 ( Hille-Yosida)** *[2]*
*A linear (unbounded) operator $A$ is the infinitesimal generator of a $C_0$ semigroup of contractions $T(t), t \geq 0$ if and only if*
*(i) $A$ is closed and $D(A)$ is dense in $X$.*
*(ii) The resolvent set $\rho(A)$ of $A$ a contains $\mathbb{R}^+$ and for every $\lambda > 0$*

$$\|R(\lambda, A)\| \leq \frac{1}{\lambda}$$

*$R(\lambda, A) = (\lambda I - A)^{-1}$ is called the resolvant operator.*

**Remark 1.1.2** $R(\lambda, A) = (\lambda I - A)^{-1} = \int_0^\infty \exp(-\lambda t) T(t) dt.$

**Definition 1.1.6** *Let $H$ be a Hilbert space. An unbounded linear operator $A : D(A) \subset H \longrightarrow H$ is said to be* ***monotone*** *(or $-A$ is* ***dissipative****) if it satisfies*

$$(Av, v) \geq 0 \quad \forall v \in D(A).$$

*It is called* ***maximal monotone*** *if, in addition, $R(I + A) = H$, i.e.*

$$\forall f \in H \quad \exists U \in D(A) \text{ such that } U + AU = f.$$

**Proposition 1.1.1** *[9]*
*Let $A$ be a maximal monotone operator. Then*
*(a) $D(A)$ is dense in $H$,*
*(b) $A$ is a closed operator,*
*(c) For every $\lambda > 0$, $(I + \lambda A)$ is bijective from $D(A)$ onto $H$, $(I + \lambda A)^{-1}$ is a bounded operator, and $\|(I + \lambda A)^{-1}\|_{L(H)} \leq 1$.*

**Theorem 1.1.4** *(Hille-Yosida). [9]*
*Let $A$ be a maximal monotone operator. Then, given any $U^0 \in D(A)$ there exists a unique function*

$$U \in C^1([0,+\infty[;H) \cap C^0([0,+\infty[;D(A))$$

*satisfying*

$$\begin{cases} \dfrac{dU}{dt} + AU = 0 \quad on\ [0,+\infty[, \\ \qquad\qquad U(0) = U^0. \end{cases}$$

*Moreover,*

$$|U(t)| \leq \left|U^0\right| \quad and \quad \left|\frac{dU}{dt}(t)\right| = |AU| \leq \left|AU^0\right| \quad \forall t \geq 0.$$

**Definition 1.1.7** *[2]*
*Let $X$ be a Banach space and let $X^*$ be its dual. We denote the value $x^* \in X^*$ at $x \in X$ by $\langle x^*, x\rangle$ or $\langle x, x^*\rangle$. For every $x \in X$ we define the duality set $F(x) \subseteq X^*$ by*

$$F(x) = \left\{(x^* : x^* \in X^* and\, \langle x, x^*\rangle = ||x||^2 = ||x^*||^2)\right\}$$

**Definition 1.1.8** *[2]*
*A linear operator $A$ is dissipaiive if for every $x \in D(A)$ and $x^* \in F(x)$ such that $Re\ \langle Ax, x^*\rangle \leq 0$.*

**Theorem 1.1.5 (Lumer-Phillips)** *[2]*
*Let $A$ be a linear operator with dense domain $D(A)$ in $X$.*
*(a) If $A$ is dissipative and there is a $\lambda_0 > 0$ such that the range, $R(\lambda_0 I - A)$ of $(\lambda_0 I - A)$ is $X$, then $A$ is the infinitesimal generator of a $C_0$-semigroup of contractions on $X$.*
*(b) If $A$ is the infinitesimal generator of a $C_0$- semigroup of contractions on $X$ then $R(\lambda I - A) = X$ for all $\lambda > 0$ and $A$ is dissipative. Moreover, for every $x \in D(A)$ and every $x^* \in F(x)$, $Re\ \langle Ax, x^*\rangle \leq 0$ .*

**Corollary 1.1.3** *Let $A$ be a densely defined closed linear operator. If both $A$ and its adjoint operator $A^*$ are dissipative, then $A$ is the infinitesimal generator of a $C_0$-semigroup of contractions on $X$.*

**Definition 1.1.9** *[9]*
*A bounded operator $A \in \mathcal{L}(E,F)$ is said to be compact if $A(B_E)$ has compact closure in $F$(in the strong topology).*
*The set of all compact operators from $E$ into $F$ is denoted by $\mathcal{K}(E,F)$. For simplicity one writes $\mathcal{K}(E) = \mathcal{K}(E,E)$.*

**Theorem 1.1.6 (Young Inequality)** *[9]*
*Let $a$ and $b$ be two non-negative real numbers. If $p, q \in ]1, +\infty[$ with $\frac{1}{P} + \frac{1}{q} = 1$, then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

**Remark 1.1.3** *It is sometimes convenient to use the Young inequality in the form*

$$ab \leq \varepsilon a^p + C_\varepsilon b^q \text{ with } C_\varepsilon = \varepsilon / p^{-1/(p-1)}.$$

**Theorem 1.1.7 (Poincare Inequality)** *[9]*
*Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ bounded at least in one direction. There exists a positive constant $C_p$ such that, for every $v \in H_0^1(\Omega)$*

$$||v||_{L^2(\Omega)} \leq C_p ||\nabla v||_{L^2(\Omega)}.$$

**Theorem 1.1.8 (Korn Inequality with a Boundary Condition)** *[7]*

*Let $\Omega$ be a domain in $\mathbb{R}^3$ and $\Gamma$ be a mesurable subset of boundary $\Gamma$ such that area $(\Gamma) > 0$. Given a vector field $\mathbf{v} = (v_i) \in \mathbf{H}^1(\Omega)$, suppose that*

$$||\epsilon(\mathbf{v})||^2_{L^2(\Omega)} = \Sigma^3_{i,j=1} ||\epsilon_{ij}(\mathbf{v})||^2_{L^2(\Omega)}, where$$

$\epsilon_{ij}(\mathbf{v}) = \dfrac{1}{2} \left( \partial v_i / \partial x_j + \partial v_j / \partial x_i \right).$
*Then, there exists a positive constant $C_k$ such that*

$$||\mathbf{v}||_{\mathbf{H}^1} \leq C_k ||\epsilon(\mathbf{v})||_{L^2(\Omega)},$$

*for all $\mathbf{v}$ in $\mathbf{H}^1(\Omega)$ vanishing on $\Gamma$.*

**Theorem 1.1.9 (Cauchy-Schwarz inequality)** *[9]*

*Let $\mathbf{H}$ be a vector space with a scalar product $(u,v)$, then forall $u, v \in \mathbf{H}$.*

$$|(u,v)| \leq (u,u)^{\frac{1}{2}} \cdot (v,v)^{\frac{1}{2}}.$$

**Lemma 1.1.1** *[17]*
*We assine that $u \in C([0,T],\mathbb{R}), T \in (0,\infty)$, satisfies the differential inequality*

$$\frac{d}{dt}u \leq a(t)u + b(t) \quad on \ (0,T)$$

*for some $a, b \in L^1(0,T)$. Then $u$ satifies the pointwise estimate*

$$u(t) \leq \exp(A(t))u(0) + \int_0^t b(s)\exp(A(t)-A(s))ds, \forall t \in [0,T]$$

*where*

$$A(t) = \int_0^t a(s)ds.$$

## 1.2 Lyapunov stability method

In his famous doctoral dissertation in 1892, Aleksandr Mikhailovich Lyapunov developed the **stability theory** of dynamical systems determined by nonlinear time-varying ordinary differential equations. In doing so, he formulated his concepts of stability and instability and he developed two general methods for the stability analysis of an equilibrium: *Lyapunov's Direct Method*, also called The **Second Method of Lyapunov**, and *The Indirect Method of Lyapunov*, also called **The First Method**. The former involves the existence of scalar-valued auxiliary functions of the state space (called Lyapunov functions) to ascertain the stability properties of an equilibrium, whereas the latter seeks to deduce the stability properties of an equilibrium of a system described by a nonlinear differential equation from the stability properties of its linearization. In the process of discovering The First Method, Lyapunov established some important stability results for linear systems (involving the Lyapunov Matrix Equation). For more details see [1].

### 1.2.1 Lyapunov's Direct Method (for local stability)

Given a system $\frac{dx}{dt} = f(x)$, with $f$ continuous, and for some region $\mathfrak{R}$ around the origin (specifically an open subset of $\mathbb{R}^n$ containing the origin), if we can produce a scalar, continuously-differentiable function $V(x)$, such that

$$V(x) > 0,\ \forall x \in \mathfrak{R}\backslash\{0\},\ V(0) = 0, \quad \text{and}$$

$$\dot{V}(x) = \frac{d}{dt}V(x) = \frac{\partial V}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial V}{\partial x}f(x) \leq 0,\ \forall x \in \mathfrak{R}\backslash\{0\},\ \frac{d}{dt}V(0) = 0,$$

then the origin ($x = 0$) is **stable in the sense of Lyapunov**.

If, additionally, we have

$$\frac{d}{dt}V(x) = \frac{\partial V}{\partial x}f(x) < 0,\ \forall x \in \mathfrak{R}\backslash\{0\},$$

then the origin is (locally) **asymptotically stable**. And if we have

$$\frac{d}{dt}V(x) = \frac{\partial V}{\partial x}f(x) \leq -\alpha V(x),\ \forall x \in \mathfrak{R}\backslash\{0\},$$

for some $\alpha > 0$, then the origin is (locally) **exponentially stable**.

### 1.2.2 Lyapunov Theorems for Stability

The relations between **Lyapunov functions** and the **stability** of systems are made precise in a number of theorems in Lyapunov's direct method. Such theorems usually have local and global versions. The local versions are concerned with stability properties in the neighborhood of equilibrium point and usually involve a locally positive definite function. For more details see [10].

**Definition 1.2.1** *If, in a ball $\mathbf{B}_{R_O}$, the function $V(x)$ is positive definite and has continuous partial derivatives, and if its time derivative along any state trajectory of the system is negative semi-definite, i.e.,*

$$\dot{V}(x) \leq 0$$

*then $V(x)$ is said to be a **Lyapunov function** for the system.*

**Theorem 1.2.1** *(Local Stability). If, in a ball $\mathbf{B}_{R_O}$, there exists a scalar function $V(x)$ with continuous first partial derivatives such that*

- *$V(x)$ is positive definite (locally in $\mathbf{B}_{R_O}$)*
- *$\dot{V}(x)$ is negative semi-definite (locally in $\mathbf{B}_{R_O}$)*

*then the equilibrium point $0$ is **stable**.*
*If, actually, the derivative $\dot{V}(x)$ is locally negative definite in $\mathbf{B}_{R_O}$, then the **stability is asymptotic**.*

**Theorem 1.2.2** *(Global Stability). Assume that there exists a scalar function $V$ of the state $x$, with continuous first order derivatives such that*

- *$V(x)$ is positive definite*
- *$\dot{V}(x)$ is negative definite*
- *$V(x) \longrightarrow \infty$ as $\|x\| \longrightarrow \infty$*

*then the equilibrium at the origin is **globally asymptotically stable**.*

## 1.3 Energy Method

Consider a dynamical system defined on the set $\Omega \times \mathbb{R}^+$, where $u(x,t)$ is the solution.

Let $E$ denote the energy of the system, which is of the form

$$E(t) = \int_\Omega f(u, u_x, u_t, \ldots)dx \geq 0, \quad \forall t \geq 0.$$

There are two common properties of $E$:

1. If $\dfrac{dE}{dt} = 0$, then $E$ is called conserved and the system is conservative.
2. If $\dfrac{dE}{dt} \leq 0$, then $E$ is called dissipated and the system is dissipative.

**Example 1.3.1** *Consider the problem (wave problem)*

$$\left\{ \begin{array}{c} u_{tt} - u_{xx} = 0 \ \ in \ ]0, L[\times\mathbb{R}^+ \\ u(0,t) = u(L,t) = 0 \ , \ \ t > 0 \\ u(x,0) = u_0(x), \quad u_x(x,0) = u_1(x), \quad in \ ]0,L[ \end{array} \right. .$$

*The functional energy of this system defined by*

$$E(t) = \frac{1}{2}\int_0^L \left[(u_x)^2 + (u_t)^2\right] dx$$

*is conserved.*

*Indeed, using the wave equation and the boundary conditions we get*

$$\frac{dE}{dt} = \int_0^L \left[u_x u_{tx} + u_t u_{tt}\right] dx = \int_0^L \left[u_x u_{tx} + u_t u_{xx}\right] dx = \int_0^L \left[u_x u_{tx} - u_{tx} u_x\right] dx + \underbrace{\left[u_t u_x\right]_{x=0}^{x=L}}_{=0} = 0.$$

**Example 1.3.2** *Consider the problem (diffusion problem)*

$$\begin{cases} u_t - u_{xx} = 0 \ \ in \ ]0, L[\times\mathbb{R}^+ \\ u(0,t) = u(L,t) = 0 \ , \ \ t > 0 \\ u(x,0) = u_0(x) \ \ \ in \ ]0, L[ \end{cases}.$$

*The functional energy of this system defined by*

$$E(t) = \frac{1}{2}\int_0^L (u_x)^2 \, dx$$

*is dissipated.*

*Indeed, using the diffusion equation and the boundary conditions we get*

$$\frac{dE}{dt} = \int_0^L u_x u_{tx} dx = -\int_0^L u_{xx} u_t dx + \underbrace{[u_t u_x]_{x=0}^{x=L}}_{=0} \ = -\int_0^L (u_{xx})^2 \, dx < 0.$$

The stability of a system generally refers to its ability to return to its initial state when an external disturbance ceases. The Lyapunov stability theorem defines the stability of a system in terms of energy, the biggest advantage of which is that the stability can be determined without the need to solve the motion equation of the system.

If the system is dissipative, then the energy is decreasing. In order to study the stability of the system, it is interesting to know the decay rate of this energy.

◊ System stability is said to be **strong** if

$$\lim_{t\longrightarrow\infty} E(t) = 0.$$

◊ System stability is said to be **exponential** (or uniform) if

$$\exists c_1, c_2 > 0 : \ E(t) \leq c_1 \exp(-c_2 t), \ \ \forall t \geq 0.$$

◊ System stability is said to be **polynomial** if

$$\exists c_1, c_2 > 0 : \ E(t) \leq c_1 t^{-c_2}, \ \ \forall t > 0.$$

## 1.4 Time delay and stability

Time delay has two complementary, conterintuitive and almost contradicting facets. On the one hand, delay is able to induce instabilities, bifurcations of periodic and more complicated orbits, multi-stability and chaotic motion. On the other hand, delay can suppress instabilities, stabilize unstable stationary or periodic states and may control complex chaotic dynamics. Systems with delays arise in engineering, biology, physics, operations research, and economics.

The first contribution in the study of the effect of the time delay on the stabilization of the solution was made by Datko [15]. He considered the equation

$$u_{tt} - u_{xx} + 2u_t(x, t - \tau) = 0, \quad (x, t) \in (0, 1) \times (0, +\infty)$$

and showed that the presence of a time delay $\tau > 0$ in the damping given by the velocity term can destabilize the system. Also, Datko et al. [14] studied the problem

$$\begin{cases} u_{tt} - u_{xx} + 2au_t + a^2 u_t = 0, \quad (x, t) \in (0, 1) \times (0, +\infty) \\ u(0, t) = 0, \ u_x(1, t) = -ku_t(1, t - \tau), \ t \in (0, +\infty) \end{cases} \tag{1.1}$$

where the delay is acting on the boundary. They proved that for $a > 0$ and $k \geq \frac{1-e^{-2a}}{1+e^{-2a}}$ there exists a sequence of delay terms for which the solutions of (2.11) *can not be exponentially stable*.

Time delay is also considered in thermoelastic problems. Racke [16] studied the system

$$\begin{cases} u_{tt}(x, t) - au_{xx}(x, t - \tau_1) + b\theta_x(x, t) = 0, \quad (x, t) \in (0, 1) \times (0, +\infty), \\ \theta_t(x, t) - d\theta_{xx}(x, t - \tau_2) + bu_{tx}(x, t) = 0, \quad (x, t) \in (0, 1) \times (0, +\infty), \end{cases} \tag{1.2}$$

with two delay terms in the *displacement* and the *temperature* functions. He proved the **ill-posedness** and the **unstability** of the system (1.2), for $\tau_1 > 0$ or $\tau_2 > 0$. However, for $\tau_1 = \tau_2 = 0$, the system is **exponentially stable**. Mustafa and Kafini [12] considered the following thermoelastic system with internal distributed delay in the temperature function

$$\begin{cases} au_{tt}(x, t) - du_{xx}(x, t - \tau_1) + \beta\theta_x(x, t) = 0, \quad (x, t) \in (0, L) \times (0, +\infty), \\ b\theta_t(x, t) - k_1\theta_{xx}(x, t) - \int_{\tau_1}^{\tau_2} k_2(s)\theta_{xx}(x, t - s)ds + \beta u_{tx}(x, t) = 0, \quad (x, t) \in (0, L) \times (0, +\infty), \\ u(x, 0) = u_0(x), \ u_t(x, 0) = u_1(x), \quad \theta(x, 0) = \theta_0(x), \quad x \in (0, L), \\ \theta_x(x, -t) = f_0(x, t), \quad (x, t) \in (0, L) \times (0, \tau_2), \end{cases}$$

and established the **exponential stability** of the solution provided that $\int_{\tau_1}^{\tau_2} |k_2(s)|\, ds < k_2$

Chapter 2

# Energy decay of piezoelectric system with time delay

## 2.1 Piezoelectric system with dissipation term

The aim of this section is to study the influence of a dissipation term ($\gamma \mathbf{u_t}$) introduced into the equilibrium equation of a piezoelectric system on its stability, using the energy method. We begin by formulating the problem, then study the existence and uniqueness of the solution, and finally the energy decay rate of the system.

### 2.1.1 Setting of the problem

Let $\Omega$ be a bounded open set of $\mathbb{R}^3$ with regular boundary $\Gamma$. Let Q be the domain $\Omega \times ]0,T[, 0 < T < \infty$, and $\Sigma = \Gamma \times ]0,T[$ its boundary. We consider a piezoelectric body whose initial reference configuration (stress-free) is $\Omega$, it is characterized by a mass density $\rho > 0$ and a dissipation coefficient $\gamma > 0$. With applied mechanical volume forces $\mathbf{f} = (f_i) : \Omega \to \mathbb{R}^3$ and electric charges $g : \Omega \to \mathbb{R}$, this body undergoes a piezoelectric displacement made of an elastic displacement $\mathbf{u}(x,t) = (u_i(x,t)) : \mathrm{Q} \to \mathbb{R}^3$ and an electric potential $\varphi(x,t) : \mathrm{Q} \to \mathbb{R}$ given formally by the following evolution equations:

$$\begin{cases} \rho \mathbf{u}_{tt} + \gamma \mathbf{u}_t - \operatorname{div} \mathbf{T}(\mathbf{u}, \varphi) = \mathbf{f} \quad \text{in Q}, \\ - \operatorname{div} \mathbf{D}(\mathbf{u}, \varphi) = g \quad \text{in Q}, \\ \mathbf{u}(x,0) = \mathbf{u}^0(x), \mathbf{u}_t(x,0) = \mathbf{u}^1(x) \quad \text{in } \Omega \\ \mathbf{u} = \varphi = 0 \quad \text{on } \Sigma \end{cases} \tag{2.1}$$

The stress tensor $\mathbf{T} = (T^{ij})$ and the electric displacement $\mathbf{D} = (D^i)$ are related to $\mathbf{u}$ and $\varphi$ by the constitutive law given below.

$$\begin{cases} \mathbf{T}^{ij}(\mathbf{u}, \varphi) = C^{ijkl} \epsilon_{kl}(\mathbf{u}) + e^{kij} \partial_k \varphi, \\ \mathbf{D}^i(\mathbf{u}, \varphi) = -e^{ikl} \epsilon_{kl}(\mathbf{u}) + d^{ij} \partial_j \varphi, \end{cases} \tag{2.2}$$

where the characteristics of the material consist in three tensors, namely:
The fourth-order **elasticity tensor** ($C^{ijkl}$) is symmetric and positive definite, i.e.

$$C^{ijkl} = C^{jikl} = C^{klij} = C^{ijkl}$$

and there exists a positive constant $\alpha_c > 0$ such that

$$C^{ijkl} X_{kl} X_{ij} \geq \alpha_c X_{ij} X_{ij}, \quad \forall X_{ij} = X_{ji} \in \mathbb{R}.$$

The third-order **coupling tensor** ($e^{ijk}$) is partly symmetric, $e^{ijk} = e^{ikj}$.
The second-order **dielectric tensor** ($d^{ij}$) is symmetric and positive definite, i.e. $d^{ij} = d^{ji}$ and there exists a positive constant $\alpha_d$ such that

$$d^{ij} X_i X_j \geq \alpha_d X_i X_i, \quad \forall X_i \in \mathbb{R}.$$

The coefficients of the three tensors are assumed to satisfy

$$C^{ijkl}(x) \in L^\infty(\Omega),\ e^{ijk}(x) \in L^\infty(\Omega),\ d^{ij}(x) \in L^\infty(\Omega),$$

and for the sake of simplicity the mass density $\rho$ is taken, in the sequel, equal to 1.

***The functional framework***

Let us consider the Hilbert space $L^2(\Omega)$ endowed with the inner product

$$(u, v) = \int_\Omega u(x)v(x)dx$$

and its corresponding norm

$$|u|^2 = \int_\Omega u(x)^2 dx$$

We also consider the Sobolev spaces $H^1(\Omega)$ and $H^2(\Omega)$ and we define the subspace of $H^1(\Omega)$, denoted by $H^1_0(\Omega)$, as the closure of $C_0^\infty(\Omega)$ in the strong topology of $H^1(\Omega)$.
In $\mathbf{L}^2(\Omega), \mathbf{H}^1(\Omega)$ and $\mathbf{H}^2(\Omega)$ we consider the following inner products and norms, respectively:

$$(\mathbf{u}, \mathbf{v})_{\mathbf{L}^2(\Omega)} = \sum_{j=1}^{3}(u_j, v_j), \quad |\mathbf{u}|^2_{\mathbf{L}^2(\Omega)} = \sum_{j=1}^{3} |u_j|^2_{L^2(\Omega)}$$

$$((\mathbf{u}, \mathbf{v}))_{\mathbf{H}^1(\Omega)} = \sum_{j=1}^{3}((u_j, v_j))_{H^1(\Omega)}, \quad ||\mathbf{u}||^2_{\mathbf{H}^1(\Omega)} = \sum_{j=1}^{3} ||u_j||^2_{H^1(\Omega)}$$

$$((\mathbf{u}, \mathbf{v}))_{\mathbf{H}^2(\Omega)} = \sum_{j=1}^{3}((u_j, v_j))_{H^2(\Omega)}, \quad ||\mathbf{u}||^2_{\mathbf{H}^2(\Omega)} = \sum_{j=1}^{3} ||u_j||^2_{H^2(\Omega)}$$

### 2.1.2 Existence and regularity

In this section we establish the existence and regularity of solutions to the evolution system (2.1) using the semigroup approach . We study first the static problem with applied volume forces $\mathbf{f} = (f_i) : \Omega \to \mathbb{R}^3$ and electric charge $g : \Omega \to \mathbb{R}$

$$\begin{cases} -\operatorname{div} \mathbf{T}(\mathbf{u}, \varphi) = \mathbf{f} & \text{in } \Omega, \\ -\operatorname{div} \mathbf{D}(\mathbf{u}, \varphi) = g & \text{in } \Omega \\ \mathbf{u} = \varphi = 0 & \text{on } \Gamma. \end{cases} \tag{2.3}$$

Next, we express the elastic displacement $\mathbf{u}$ as a function $\mathbf{u}(\varphi)$. The evolution problem (2.1) can be written as

$$\begin{cases} \mathbf{U}_t(t) + \mathcal{L}\mathbf{U}(t) = 0 & \text{in Q}, \\ \mathbf{U}(0) = (\mathbf{u}^0, \mathbf{u}^1) & \text{in } \Omega, \\ \mathbf{U} = 0 & \text{on } \Sigma, \end{cases} \tag{2.4}$$

where $\mathbf{U} = (\mathbf{u}, \mathbf{u}_t)$ the solution of the evolution problem and the operator $\mathcal{L}$ generates a semigroup of contractions in an appropriate Hilbert space.

***Static problem***

Multipling the first equation in (2.3) by a test function $\mathbf{v} = (v_i) \in \mathbf{H}_0^1(\Omega)$, then using (2.2) and Green's formula, we get

$$\begin{aligned}
-\int_\Omega \operatorname{div} \mathbf{T}(\mathbf{u}, \varphi) \cdot \mathbf{v} dx &= \int_\Omega T^{ij}(\mathbf{u}, \varphi) \partial_i v_j dx - \int_\Gamma T^{ij}(\mathbf{u}, \varphi) v_j n_i d\Gamma, \\
&= \int_\Omega T^{ij}(\mathbf{u}, \varphi) \epsilon_{ij}(\mathbf{v}) dx - \int_\Gamma T^{ij}(\mathbf{u}, \varphi) v_j n_i d\Gamma, \\
&= \int_\Omega (C^{ijkl} \epsilon_{kl}(\mathbf{u}) + e^{kij} \partial_k \varphi) \epsilon_{ij}(\mathbf{v}) dx - \int_\Gamma T^{ij}(\mathbf{u}, \varphi) v_j n_i d\Gamma,
\end{aligned}$$

where $\mathbf{n} = (n_1, n_2, n_3)$ denotes the unit normal vector pointing the exterior of the boundary of $\Omega$ . We use the homogeneous Dirichlet boundary conditions. Similarly, multipling the second equation in (2.3) by a test function $\psi \in H_0^1(\Omega)$, then using (2.2) and Green's formula we have

$$\begin{aligned}
-\int_\Omega \operatorname{div} \mathbf{D}(\mathbf{u}, \varphi) \psi dx &= \int_\Omega D^i(\mathbf{u}, \varphi) \partial_i \psi dx - \int_\Gamma D^i(\mathbf{u}, \varphi) \psi n_i d\Gamma \\
&= \int_\Omega (-e^{ikl} \epsilon_{kl}(\mathbf{u}) + d^{ij} \partial_j \varphi) \partial_i \psi dx - \int_\Gamma D^i(\mathbf{u}, \varphi) \psi n_i d\Gamma.
\end{aligned}$$

Then

$$\begin{cases}
-\int_\Omega \operatorname{div} \mathbf{T}(\mathbf{u}, \varphi) \cdot \mathbf{v} dx = c(\mathbf{u}, \mathbf{v}) + e(\mathbf{v}, \varphi), \\
-\int_\Omega \operatorname{div} \mathbf{D}(\mathbf{u}, \varphi) \psi dx = -e(\mathbf{u}, \psi) + d(\varphi, \psi)
\end{cases} \tag{2.5}$$

such that

$$\begin{cases}
c : (\mathbf{u}, \mathbf{v}) \in \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \to \mathbb{R};\ c(\mathbf{u}, \mathbf{v}) = \int_\Omega C^{ijkl} \epsilon_{kl}(\mathbf{u}) \epsilon_{ij}(\mathbf{v}) dx, \\
e : \mathbf{v}, \psi \in \mathbf{H}^1(\Omega) \times H^1(\Omega) \to \mathbb{R};\ e(\mathbf{v}, \psi) = \int_\Omega e^{ijk} \epsilon_{jk}(\mathbf{v}) \partial_i \psi dx, \\
d : \varphi, \psi \in H^1(\Omega) \times H^1(\Omega) \to \mathbb{R};\ d(\varphi, \psi) = \int_\Omega d^{ij} \partial_i \varphi \partial_j \psi dx.
\end{cases}$$

**Theorem 2.1.1** *We assume that the domain $\Omega$ has a Lipschitz boundary. For applied force $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $g \in L^2(\Omega)$, the static problem (2.3) has a unique weak solution $(\mathbf{u}, \varphi) \in \mathbf{H}_0^1(\Omega) \times H_0^1(\Omega)$, by the Lax-Milgram lemma, which satisfies the following identity :*

$$\begin{cases}
c(\mathbf{u}, \mathbf{v}) + e(\mathbf{v}, \varphi) = \int_\Omega \mathbf{f} \cdot \mathbf{v} dx, \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \\
-e(\mathbf{u}, \psi) + d(\varphi, \psi) = \int_\Omega g \psi dx, \quad \forall \psi \in H_0^1(\Omega).
\end{cases}$$

*Moreover, this solution is a saddle-point of the static energy*

$$\frac{1}{2}[c(\mathbf{v}, \mathbf{v}) + 2e(\mathbf{v}, \psi) - d(\psi, \psi)] - \int_\Omega (\mathbf{f} \cdot \mathbf{v} - g\psi) dx.$$

**Remark 2.1.1** *The coerciveness of* $(C^{ijkl})$ *and* $(d^{ij})$*, imply that the bilinear forms* $c(\mathbf{v},\mathbf{v})$ *and* $d(\psi,\psi)$ *are norms equivalents to the classic norms on* $\mathbf{H}_0^1(\Omega)$ *and* $H_0^1(\Omega)$*, that is, there exist positive constants* $C_1$*,* $C_2$*,* $C_3$ *and* $C_4$ *such that*

$$C_1||\mathbf{v}||^2_{\mathbf{H}_0^1(\Omega)} \leq c(\mathbf{v},\mathbf{v}) \leq C_2||\mathbf{v}||^2_{\mathbf{H}_0^1(\Omega)}$$

*and*

$$C_3||\psi||^2_{H_0^1(\Omega)} \leq d(\psi,\psi) \leq C_4||\psi||^2_{H_0^1(\Omega)}.$$

***Evolution problem - Semigroup approach***

In this section, in addition to the symmetry and positivity conditions, it is assumed that the coefficients $C^{ijkl}(x), e^{ijk}(x), d^{ij}(x)$ are in $C^1(\overline{\Omega})$. Also, for simplicity, we suppose in the problem (2.1) that $\mathbf{f} = g = 0$.

**Lemma 2.1.1** *Let* $\mathbf{u} \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega)$ *be given. The problem*

$$(P_{\varphi(\mathbf{u})}) \left\{ \begin{array}{c} \textit{Find } \varphi \in H_0^1(\Omega) \textit{ such that} \\ d(\varphi,\psi) = e(\psi,\mathbf{u}) \ \forall \psi \in H_0^1(\Omega) \end{array} \right.$$

*admits one and only one solution by Lax-Milgram lemma.*
*Moreover, by regularity theory of elliptic equations, we have*

$$\varphi \in H_0^1(\Omega) \cap H^2(\Omega).$$

Let's define the operators

$$\left\{ \begin{array}{c} \mathcal{C} : \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega) \longrightarrow \mathbf{L}^2(\Omega) \\ \mathcal{C}\mathbf{u} = -\frac{\partial}{\partial x_j}\left[C^{ijkl}\frac{\partial u_k}{\partial x_l}\right] e_i, \end{array} \right.$$

$$\left\{ \begin{array}{c} \mathcal{E} : H_0^1(\Omega) \cap H^2(\Omega) \longrightarrow \mathbf{L}^2(\Omega) \\ \mathcal{E}\varphi = -\frac{\partial}{\partial x_j}\left[e^{kij}\frac{\partial \varphi}{\partial x_k}\right] e_i, \end{array} \right.$$

where $e = (e_1, e_2, e_3)$ is the canonical basis of $\mathbb{R}^3$,

$$\left\{ \begin{array}{c} \Re : \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega) \longrightarrow H_0^1(\Omega) \cap H^2(\Omega) \\ \varphi = \Re\mathbf{u}, \end{array} \right.$$

$$\left\{ \begin{array}{c} \mathcal{A} : D(\mathcal{A}) = \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega) \subseteq \mathbf{L}^2(\Omega) \longrightarrow \mathbf{L}^2(\Omega) \\ \mathcal{A} = \mathcal{C} + \mathcal{E}\Re, \end{array} \right.$$

and

$$\left\{ \begin{array}{c} \mathcal{L} : \mathbf{X} \longrightarrow \mathbf{X}, \text{ where } \mathbf{X} = \mathbf{H}_0^1(\Omega) \times \mathbf{L}^2(\Omega) \\ \mathcal{L} = \begin{bmatrix} 0 & -\mathcal{I} \\ \mathcal{A} & \gamma\mathcal{I} \end{bmatrix}, \text{ where } \mathcal{I} \text{ is the identity operator and } \gamma > 0. \end{array} \right.$$

We define on a Hilbert space $\mathbf{X} = \mathbf{H}_0^1(\Omega) \cap \mathbf{L}^2(\Omega)$ the scalar produit

$$\begin{cases} \text{For } \mathbf{U}_1 = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{w}_1 \end{pmatrix}, \ \mathbf{U}_2 = \begin{pmatrix} \mathbf{u}_2 \\ \mathbf{w}_2 \end{pmatrix}, \ \varphi_1 = \Re\mathbf{u}_1, \ \varphi_2 = \Re\mathbf{u}_2 : \\ (\mathbf{U}_1, \mathbf{U}_2)_{\mathbf{X}} = c(\mathbf{u}_1, \mathbf{u}_2) + d(\varphi_1, \varphi_2) + (\mathbf{w}_1, \mathbf{w}_2). \end{cases}$$

**Lemma 2.1.2** *The operator $\mathcal{L}$ satisfy the properties*
- $(\mathcal{L}\mathbf{U}, \mathbf{U})_{\mathbf{X}} \geq 0 \quad \forall \mathbf{U} \in D(\mathcal{L}) = (\mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega)) \times \mathbf{H}_0^1(\Omega)$.
- *Range* $(\mathcal{L} + \mathcal{I}) = \mathbf{X}$.

**Proof 2.1.1** *$\mathcal{L}$ is a linear unbounded operator and $\overline{D(\mathcal{L})} = (\mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega)) \times \mathbf{L}^2(\Omega)$.*

***Firstly**, let $\mathbf{U} = \begin{pmatrix} \mathbf{u} \\ \mathbf{w} \end{pmatrix} \in D(\mathcal{L})$, $\varphi = \Re\mathbf{u}$ be the solution of $(P_{\varphi(\mathbf{u})})$, and $\chi = \Re\mathbf{w}$ be the solution of $(P_{\chi(\mathbf{w})})$, then $\mathcal{L}\mathbf{U} = \begin{pmatrix} -\mathbf{w} \\ \mathcal{A}\mathbf{u} + \gamma\mathbf{w} \end{pmatrix}$ and*

$$(\mathcal{L}\mathbf{U}, \mathbf{U})_{\mathbf{X}} = c(-\mathbf{w}, \mathbf{u}) + d(-\Re\mathbf{w}, \Re\mathbf{u}) + (\mathcal{A}\mathbf{u} + \gamma\mathbf{w}, \mathbf{w}) = -c(\mathbf{w}, \mathbf{u}) + d(-\Re\mathbf{w}, \Re\mathbf{u}) + (\mathcal{A}\mathbf{u}, \mathbf{w}) + \gamma\,(\mathbf{w}, \mathbf{w})$$

$$= -c(\mathbf{w}, \mathbf{u}) - d(\Re\mathbf{w}, \Re\mathbf{u}) + (\mathcal{C}\mathbf{u}, \mathbf{w}) + (\mathcal{E}\Re\mathbf{u}, \mathbf{w}) + \gamma\,(\mathbf{w}, \mathbf{w})$$

$$= -c(\mathbf{w}, \mathbf{u}) + c(\mathbf{u}, \mathbf{w}) - d(\chi, \varphi) + (\mathcal{E}\varphi, \mathbf{w}) + \gamma\,(\mathbf{w}, \mathbf{w})$$

$$= -c(\mathbf{w}, \mathbf{u}) + c(\mathbf{u}, \mathbf{w}) - d(\chi, \varphi) + e\,(\varphi, \mathbf{w}) + \gamma\,(\mathbf{w}, \mathbf{w})\,.$$

*By choosing $\psi = \chi$ in $(P_{\varphi(\mathbf{u})})$ and $\psi = \varphi$ in $(P_{\chi(\mathbf{w})})$, we get*

$$\begin{cases} d(\varphi, \chi) = e(\chi, \mathbf{u}) \\ d(\chi, \varphi) = e(\varphi, \mathbf{w}). \end{cases} \tag{2.6}$$

*Using the symmetry of the bilinear form $c(.,.)$, we have*

$$(\mathcal{L}\mathbf{U}, \mathbf{U})_{\mathbf{X}} = \gamma\,(\mathbf{w}, \mathbf{w}) \geq 0.$$

***Secondly**, we know prove that $(\mathcal{L} + \mathcal{I})$ is surjective. Let $\mathbf{G} = \begin{pmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \end{pmatrix} \in \mathbf{X}$, the problem*

$$\begin{cases} \textit{Find } \mathbf{U} \in D(\mathcal{L}) \\ \mathcal{L}\mathbf{U} + \mathbf{U} = \mathbf{G} \end{cases} \tag{2.7}$$

*is equivalent to the system*

$$\begin{cases} \mathbf{u} \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega), \ \mathbf{w} \in \mathbf{H}_0^1(\Omega) : \\ -\mathbf{w} + \mathbf{u} = \mathbf{g}_1 \\ \mathcal{A}\mathbf{u} + (1 + \gamma)\,\mathbf{w} = \mathbf{g}_2 \end{cases}$$

*which implies that*

$$\left\{ \begin{array}{c} \mathbf{u} \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega) \\ \mathcal{C}\mathbf{u} + \mathcal{E}\Re\mathbf{u} + (1+\gamma)\,\mathbf{u} = (1+\gamma)\,\mathbf{g}_1 + \mathbf{g}_2. \end{array} \right. \tag{2.8}$$

*The variational formulation of (2.8) is*

$$\left\{ \begin{array}{c} \mathbf{u} \in \mathbf{H}_0^1(\Omega),\ \varphi \in H_0^1(\Omega): \\ c(\mathbf{u},\mathbf{v}) + e(\varphi,\mathbf{v}) + (1+\gamma)\,(\mathbf{u},\mathbf{v}) = ((1+\gamma)\,\mathbf{g}_1 + \mathbf{g}_2, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega) \\ d(\varphi,\psi) - e(\psi,\mathbf{u}) = 0,\ \forall \psi \in H_0^1(\Omega). \end{array} \right. \tag{2.9}$$

*Let* $\mathbf{Y} = \begin{pmatrix} \mathbf{u} \\ \varphi \end{pmatrix}$, $\mathbf{Z} = \begin{pmatrix} \mathbf{v} \\ \psi \end{pmatrix}$ *belong to* $\mathbf{H}_0^1(\Omega) \times H_0^1(\Omega)$. *Define the bilinear form*

$$\mathbf{\Lambda}(\mathbf{Y},\mathbf{Z}) = c(\mathbf{u},\mathbf{v}) + e(\varphi,\mathbf{v}) + d(\varphi,\psi) - e(\psi,\mathbf{u}) + (1+\gamma)\,(\mathbf{u},\mathbf{v})$$

*and the linear form*

$$\digamma(\mathbf{Z}) = ((1+\gamma)\,\mathbf{g}_1 + \mathbf{g}_2, \mathbf{v}).$$

*Then (2.9) is equivalent to the variational problem*

$$\left\{ \begin{array}{c} \textit{Find } \mathbf{Y} \in \mathbf{H}_0^1(\Omega) \times H_0^1(\Omega) \textit{ such} \\ \mathbf{\Lambda}(\mathbf{Y},\mathbf{Z}) = \digamma(\mathbf{Z}),\ \forall \mathbf{Z} \in \mathbf{H}_0^1(\Omega) \times H_0^1(\Omega). \end{array} \right. \tag{2.10}$$

*It is easy to check that* $\mathbf{\Lambda}(.,.)$ *is continuous and coercive and that* $\digamma$ *is continuous in* $\mathbf{H}_0^1(\Omega) \times H_0^1(\Omega)$. *Then by the Lax-Milgram lemma we obtain that the problem (2.10) admits a unique solution* $\mathbf{Y} \in \mathbf{H}_0^1(\Omega) \times H_0^1(\Omega)$. *Consequently, there exists a unique solution* $\mathbf{U} = \begin{pmatrix} \mathbf{u} \\ \mathbf{w} \end{pmatrix}$, *where* $\mathbf{w} = \mathbf{u} - \mathbf{g}_1$, *to the problem (2.7).*

**Theorem 2.1.2** *Assume that* $\mathbf{u}^0 \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega), \mathbf{u}^1 \in \mathbf{H}_0^1(\Omega)$, *then there exists one and only one solution* $(\mathbf{u},\varphi)$ *to the problem (2.1). Moreover,*

$$\left\{ \begin{array}{c} \mathbf{u} \in C([0,\infty[;\mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)) \cap C^1([0,\infty[;\mathbf{H}_0^1(\Omega)) \cap C^2([0,\infty[;\mathbf{L}^2(\Omega)) \\ \varphi \in C^1([0,\infty[;\mathbf{H}_0^1(\Omega)) \end{array} \right. .$$

**Proof 2.1.2** *We rewrite the problem (2.1) as a first order system:*

$$\left\{ \begin{array}{c} \dfrac{\partial \mathbf{u}}{\partial t} = \mathbf{w} \\ \dfrac{\partial \mathbf{w}}{\partial t} + \mathcal{A}\mathbf{u} + \gamma \mathbf{w} = \mathbf{0} \end{array} \right. . \tag{2.11}$$

*Let us note by* $\mathbf{U} = \begin{pmatrix} \mathbf{u} \\ \mathbf{w} \end{pmatrix}$, *then (2.11) becomes*

$$\frac{d\mathbf{U}}{dt} + \mathcal{L}\mathbf{U} = 0$$

*where*

$$\mathcal{L} = \begin{bmatrix} 0 & -\mathcal{I} \\ \mathcal{A} & \gamma\mathcal{I} \end{bmatrix}.$$

*Then we apply the Hille-Yosida theorem (1.1.4), by using the properties of the operator $\mathcal{L}$ in lemma 2.1.2.*

## 2.2 Exponential decay of the energy

We consider is this section the system (2.1) with $\mathbf{f} = g = 0$. The energy functional $E$ associated with the weak solution of system (2.1) is

$$E(t) = \frac{1}{2}[\int_\Omega |\mathbf{u}_t|^2 dx + c(\mathbf{u}, \mathbf{u}) + d(\varphi, \varphi)]$$

with initial condition

$$E(0) = \frac{1}{2}[\int_\Omega |\mathbf{u}^1|^2 dx + c(\mathbf{u}^0, \mathbf{u}^0) + d(\varphi^0, \varphi^0)]$$

where $\varphi^0 \in H_0^1(\Omega)$ is the unique weak solution to

$$d(\varphi^0, \psi) = e(\mathbf{u}^0, \psi), \quad \forall \psi \in H_0^1(\Omega).$$

In the first equation of (2.5) with test function $\mathbf{v} = \mathbf{u}_t \in \mathbf{H}_0^1(\Omega)$ and homogeneous boundary conditions we get

$$-\int_\Omega \operatorname{div} \mathbf{T}(\mathbf{u}, \varphi) \cdot \mathbf{u}_t dx = c(\mathbf{u}, \mathbf{u}_t) + e(\mathbf{u}_t, \varphi), \tag{2.12}$$

and the second equation of (2.5) with test function $\psi = \varphi_t \in H_0^1(\Omega)$ we have

$$0 = -\int_\Omega \operatorname{div} \mathbf{D}(\mathbf{u}_t, \varphi_t)\varphi_t dx = -e(\mathbf{u}_t, \varphi) + d(\varphi, \varphi_t). \tag{2.13}$$

Using (2.12) and (2.13) then

$$-\int_\Omega \operatorname{div} \mathbf{T}(\mathbf{u}, \varphi) \cdot \mathbf{u}_t dx = c(\mathbf{u}, \mathbf{u}_t) + d(\varphi, \varphi_t) = \frac{1}{2}\frac{d}{dt}[c(\mathbf{u}, \mathbf{u}) + d(\varphi, \varphi)].$$

Multiplying the first equation of (2.1) by $\mathbf{u_t}$, integrating over $\Omega$ and using the equality above, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathbf{u}_t|^2 dx + \frac{1}{2}\frac{d}{dt}[c(\mathbf{u}, \mathbf{u}) + d(\varphi, \varphi)] = -\gamma \int_\Omega |\mathbf{u}_t|^2 dx.$$

Then

$$\frac{d}{dt}E(t) = -\gamma \int_\Omega |\mathbf{u}_t|^2 dx \le 0$$

We conclude that the energy of the piezoelectric system (2.1) is decreasing .

We introduce the functional

$$\Lambda(t) = \int_\Omega \mathbf{u}_t \cdot \mathbf{u} dx + \frac{\gamma}{2} \int_\Omega |\mathbf{u}|^2 dx$$

with initial condition $\Lambda(0) = \int_\Omega \mathbf{u}^1 \cdot \mathbf{u}^0 dx + \frac{\gamma}{2} \int_\Omega |\mathbf{u}|^2 dx$.
We have

$$\frac{d}{dt}\Lambda(t) = \int_\Omega \mathbf{u}_{tt} \cdot \mathbf{u} dx + \int_\Omega |\mathbf{u}_t|^2 dx + \gamma \int_\Omega \mathbf{u}_t \cdot \mathbf{u} dx.$$

Multiplying the first equation of (2.1) by $\mathbf{u}$ and integrating over $\Omega$ we get

$$\int_\Omega \mathbf{u}_{tt} \cdot \mathbf{u} dx = \int_\Omega \operatorname{div} \mathbf{T} \cdot \mathbf{u} dx - \gamma \int_\Omega \mathbf{u}_t \cdot \mathbf{u} dx.$$

Using 2.5)$we get \int_\Omega \mathbf{u}_{tt} \cdot \mathbf{u} dx = -[c(\mathbf{u}, \mathbf{u}) + e(\mathbf{u}, \varphi)] - \gamma \int_\Omega \mathbf{u}_t \cdot \mathbf{u} dx. Then$

$$\frac{d}{dt}\Lambda(t) = \int_\Omega |\mathbf{u}_t|^2 dx - [c(\mathbf{u}, \mathbf{u}) + e(\mathbf{u}, \varphi)].$$

We prove the equivalence between the energy functional $E$ and the function $\Lambda(t)$ by the following lemma.

**Lemma 2.2.1** *For any $N$ large enough there exist two positive constants $\alpha_1 = \alpha_1(N, C_K(\Omega))$ and $\alpha_2 = \alpha_2(N, \gamma, C_K(\Omega))$ such that*

$$\alpha_1 E(t) \le NE(t) + \Lambda(t) \le \alpha_2 E(t)$$

**Proof 2.2.1** . ***Step1***. *From the definition of E(t) and $\Lambda(t)$ we have*

$$NE(t) + \Lambda(t) = NE(t) + \int_\Omega \mathbf{u}_t \cdot \mathbf{u} dx + \frac{\gamma}{2} \int_\Omega |\mathbf{u}|^2 dx.$$

*Using Young's inequality*

$$|\int_\Omega \mathbf{u}_t \cdot \mathbf{u} dx| \le \frac{1}{2} \int_\Omega (|\mathbf{u}_t|^2 + |\mathbf{u}|^2) dx$$

*we get*

$$NE(t) + \Lambda(t) \le (N+1)E(t) + \frac{\gamma+1}{2} \int_\Omega |\mathbf{u}|^2 dx.$$

*From Korn's inequality we have*

$$NE(t) + \Lambda(t) \le (N+1)E(t) + (\frac{\gamma+1}{2}) C_K(\Omega) c(\mathbf{u}, \mathbf{u}).$$

*Then we conclude that there exists a positive constant* $\alpha_2$ *such that*

$$NE(t) + \Lambda(t) \leq \alpha_2 E(t)$$

*with* $\alpha_2 = N + 1 + (\gamma + 1)C_K(\Omega)$.

***Step 2***. *We have*

$$\int_\Omega \mathbf{u}_t \cdot \mathbf{u} dx \geq -\frac{1}{2}\int_\Omega (|\mathbf{u}_t|^2 + |\mathbf{u}|^2) dx$$

*then*

$$\begin{aligned} NE(t) + \Lambda(t) = NE(t) + \int_\Omega \mathbf{u}_t \cdot \mathbf{u} dx + \frac{\gamma}{2}\int_\Omega |\mathbf{u}|^2 dx \geq NE(t) + \int_\Omega \mathbf{u}_t \cdot \mathbf{u} dx \\ \geq \frac{N}{2}E(t) - \frac{1}{2}\int_\Omega (|\mathbf{u}_t|^2 + |\mathbf{u}|^2) dx \\ \geq \frac{N-1}{2}\int_\Omega |\mathbf{u}_t|^2 dx + \frac{N}{2}[(c(\mathbf{u},\mathbf{u}) + d(\varphi,\varphi))] - \frac{1}{2}\int_\Omega |\mathbf{u}|^2 dx. \end{aligned}$$

*Using Korn's inequality we get*

$$NE(t) + \Lambda(t) \geq \frac{N-1}{2}\int_\Omega |\mathbf{u}_t|^2 dx + \frac{N - C_k(\Omega)}{2} c(\mathbf{u},\mathbf{u}) + \frac{N}{2} d(\varphi,\varphi).$$

*Choosing N large enough, that is* $N > max\{1, C_K(\Omega)\}$, *and* $\alpha_1 = min\{\frac{N-1}{2}, \frac{N-C_k(\Omega)}{2}, \frac{N}{2}\}$ *we conclude that*

$$\alpha_1 E(t) \leq NE(t) + \Lambda(t).$$

First, we prove the exponential decay of the $NE(t) + \Lambda(t)$.

Let $\phi(t) = NE(t) + \Lambda(t)$, from the above we get

$$\begin{aligned} \frac{d}{dt}\phi(t) &= -N\gamma \int_\Omega |\mathbf{u}_t|^2 dx - [c(\mathbf{u},\mathbf{u}) + d(\varphi,\varphi)] + \int_\Omega |\mathbf{u}_t|^2 dx \\ &= -(N\gamma - 1)\int_\Omega |\mathbf{u}_t|^2 dx - [c(\mathbf{u},\mathbf{u}) + d(\varphi,\varphi)]. \end{aligned}$$

Thus, by choosing $N > max(1, C_K(\Omega))$, (we recall that the dissipation coefficient $\gamma$ is positive) we obtain

$$\frac{d}{dt}\phi(t) \leq -inf\{(N\gamma - 1), 1\}[\int_\Omega |\mathbf{u}_t|^2 dx + c(\mathbf{u},\mathbf{u}) + d(\varphi,\varphi)]$$

$$\leq -2inf\{(N\gamma - 1), 1\}E(t)$$

and therefore

$$\frac{d}{dt}\phi(t) \leq -\lambda_0 E(t)$$

where $\lambda_0 = 2inf\{(N\gamma - 1), 1\}$ is a positive constant independent of t. It follows that

$$\phi(t) \leq \phi(0)\exp(-\frac{\lambda_0}{\alpha_2}t) \quad \forall t \geq 0$$

with $\phi(0) = NE(0) + \Lambda(0)$. Then

$$\alpha_1 E(t) \leq \alpha_2 E(0)\exp(-\frac{\lambda_0}{\alpha_2}t)$$

and therefore

$$E(t) \leq \beta \exp(-\zeta t)E(0)$$

where $\beta = \frac{\alpha_2}{\alpha_1}$ and $\zeta = \frac{\lambda_0}{\alpha_1}$.

## 2.3 Piezoelectric System with time delay

We consider in this section the homogeneous case ($\mathbf{f} = g = 0$) of the system (2.1) with constant time delay. Then the system is defined as follows

$$\begin{cases} \mathbf{u}_{tt} + \gamma\mathbf{u}_t - \operatorname{div}\mathbf{T}(\mathbf{u},\varphi) + \mu\mathbf{u}_t(x, t-\tau) = 0, \quad \text{in Q}, \\ -\operatorname{div}\mathbf{D}(\mathbf{u},\varphi) = 0, \quad \text{in Q}, \\ \mathbf{u}(x,0) = \mathbf{u}^0(x), \mathbf{u}_t(x,0) = \mathbf{u}^1(x), \quad \text{in } \Omega \\ \mathbf{u} = 0, \ \varphi = 0 \quad \text{on } \Sigma \\ \mathbf{u}_t(x, t-\tau) = \mathbf{f}_0(x, t-\tau), \quad (x,t) \in \Omega\times]0,\tau[, \end{cases} \tag{2.14}$$

where $\mathbf{f}_0$ is a history function defined in suitable functional space, and $\mu$ is a constant.

### 2.3.1 Study of the problem

Let's define the following new variable

$$\mathbf{z}(x,\rho,t) = \mathbf{u}_t(x, t-\tau\rho) \ \rho \in ]0,1[$$

where $\tau$ is a positive constant. Thus system (2.14) is equivalent to

$$\begin{cases} \mathbf{u}_{tt} + \gamma\mathbf{u}_t - \operatorname{div}\mathbf{T}(\mathbf{u},\varphi) + \mu\mathbf{z}(x,1,t) = 0, \quad \text{in Q}, \\ -\operatorname{div}\mathbf{D}(\mathbf{u},\varphi) = 0, \quad \text{in Q}, \\ \tau\mathbf{z}_t(x,\rho,t) + \mathbf{z}_\rho(x,\rho,t) = 0 \quad \text{in } \Omega\times]0,1[\times]0,T[ \\ \mathbf{z}(x,0,t) = \mathbf{u}_t(x,t) \quad \text{in Q}, \\ \mathbf{u}(x,0) = \mathbf{u}^0(x), \mathbf{u}_t(x,0) = \mathbf{u}^1(x), \quad \text{in } \Omega \\ \mathbf{u} = \varphi = 0 \quad \text{on } \Sigma \\ \mathbf{z}(x,\rho,0) = \mathbf{f}_0(x,-\tau\rho), \quad (x,\rho) \in \Omega\times]0,1[. \end{cases} \tag{2.15}$$

In the following, we consider $(\mathbf{u}, \varphi, \mathbf{z})$ to be a solution of system (2.15), and defined the energy of (2.15) by

$$E(t) = \frac{1}{2}(\int_\Omega \mathbf{u}_t^2 dx + c(\mathbf{u}, \mathbf{u}) + d(\varphi, \varphi) + \tau|\mu| \int_\Omega \int_0^1 \mathbf{z}^2 d\rho dx).$$

**Proposition 2.3.1** *Under the condition*

$$|\mu| < \gamma$$

*the energy function $E(t)$ is decreasing.*

**Proof 2.3.1** *Let us note*

$$F(t) = \frac{1}{2}[\int_\Omega \mathbf{u}_t^2 dx + c(\mathbf{u}, \mathbf{u}) + d(\varphi, \varphi)].$$

*Then*

$$E(t) = F(t) + \frac{1}{2}\tau|\mu| \int_\Omega \int_0^1 \mathbf{z}^2 d\rho dx$$

*and*

$$E'(t) = F'(t) + \frac{1}{2}\tau|\mu| \frac{d}{dt} \int_\Omega \int_0^1 \mathbf{z}^2 d\rho dx.$$

*We know from the previous section that*

$$F'(t) = -\gamma \int_\Omega \mathbf{u}_t^2 dx - \mu \int_\Omega \mathbf{u}_t \cdot \mathbf{z}(x, 1, t) dx$$

*and from the equation, $\tau \mathbf{z}_t(x, \rho, t) + \mathbf{z}_\rho(x, \rho, t) = 0$ we obtain*

$$\frac{d}{dt} \int_\Omega \int_0^1 \mathbf{z}^2 d\rho dx = 2 \int_\Omega \int_0^1 \mathbf{z}\mathbf{z}_t d\rho dx = -\frac{2}{\tau} \int_\Omega \int_0^1 \mathbf{z}\mathbf{z}_\rho d\rho dx = -\frac{1}{\tau} \int_\Omega [\mathbf{z}^2(x, 1, t) - \mathbf{u}_t^2(x, t)] dx.$$

*Then*

$$E'(t) = -\gamma \int_\Omega \mathbf{u}_t^2 dx - \mu \int_\Omega \mathbf{u}_t \cdot \mathbf{z}(x, 1, t) dx - \frac{|\mu|}{2} \int_\Omega [\mathbf{z}^2(x, 1, t) - \mathbf{u}_t^2(x, t)] dx.$$

*Using Cauchy-Schwarz and Young inequalities, we get*

$$-\mu \int_\Omega \mathbf{u}_t \cdot \mathbf{z}(x, 1, t) dx \leq \frac{|\mu|}{2}[\int_\Omega \mathbf{u}_t^2 dx + \int_\Omega \mathbf{z}^2(x, 1, t) dx].$$

*Therefore*

$$E'(t) \leq -(\gamma - |\mu|) \int_\Omega \mathbf{u}_t^2(x, t) dx.$$

*Consequently, the energy is decreasing if and only if $|\mu| < \gamma$.*

In order to justify the exponential decay of the energy $E(t)$, we define de Lyapunov functional $\Lambda$ as follows

$$\Lambda(t) = \int_\Omega \mathbf{u}_t \cdot \mathbf{u} dx + \frac{\gamma}{2}\int_\Omega |\mathbf{u}|^2 dx + |\mu| \int_\Omega \int_0^1 \tau \exp(-\tau\rho)\mathbf{z}^2(x,\rho,t) d\rho dx.$$

Therefore

$$\Lambda'(t) = \int_\Omega \mathbf{u}_t^2 dx - (c(\mathbf{u},\mathbf{u}) + d(\varphi,\varphi)) - |\mu| \int_\Omega \exp(-\tau)\mathbf{z}^2(x,1,t)dx + |\mu|\int_\Omega \mathbf{u}_t^2 dx \\ -|\mu|\tau \int_\Omega \int_0^1 \exp(-\tau\rho)\mathbf{z}^2 d\rho dx$$

such that

$$\begin{aligned}(|\mu| \int_\Omega \int_0^1 \tau \exp(-\tau\rho)\mathbf{z}^2(x,\rho,\tau)d\rho dx)' &= 2|\mu|\tau \int_\Omega \int_0^1 \exp(-\tau\rho)\mathbf{z}\cdot\mathbf{z}_t d\rho dx \\ &= -2|\mu| \int_\Omega \int_0^1 \exp(-\tau\rho)\mathbf{z}\cdot\mathbf{z}_\rho d\rho dx \\ &= -|\mu| \int_\Omega \int_0^1 \exp(-\tau\rho)\frac{\partial}{\partial\rho}\mathbf{z}^2 d\rho dx \\ &= -|\mu| \int_\Omega [\exp(-\tau)\mathbf{z}^2(x,1,t) - \mathbf{z}^2(x,0,t) + \tau\int_0^1 \exp(-\tau\rho)\mathbf{z}^2 d\rho]dx.\end{aligned}$$

**Lemma 2.3.1** *For any $N$ large enough there exist two positive constants $c_1 = max\{(N+1),(\gamma+1)C_k\}$ and $c_2 = min\{\frac{N-1}{2}, \frac{N-C_k}{2}, \frac{N}{2}, c^*\}$ such that*

$$c_1 E(t) \leq NE(t) + \Lambda(t) \leq c_2 E(t).$$

**Proof 2.3.2** *Te proof consists of two parts.*
***Part 1*** *: From the definition of $E(t)$ and $\Lambda(t)$ we have*

$$NE(t) + \Lambda(t) = NE(t) + \int_\Omega \mathbf{u}_t\cdot\mathbf{u}dx + \frac{\gamma}{2}\int_\Omega |\mathbf{u}|^2 dx + |\mu|\int_\Omega\int_0^1 \tau\exp(-\tau\rho)\mathbf{z}^2(x,\rho,t)d\rho dx.$$

*From Young inequality, we have*

$$NE(t) + \Lambda(t) \leq NE(t) + \frac{1}{2}\int_\Omega \mathbf{u}_t^2 dx + \frac{\gamma+1}{2}\int_\Omega |\mathbf{u}|^2 dx + |\mu|\int_\Omega\int_0^1 \tau\mathbf{z}^2(x,\rho,t)d\rho dx.$$

*Using Korn's inequality, we get*

$$\begin{aligned}NE(t) + \Lambda(t) &\leq NE(t) + \frac{1}{2}\int_\Omega \mathbf{u}_t^2 dx + \frac{\gamma+1}{2}C_k c(\mathbf{u},\mathbf{u}) + |\mu|\int_\Omega\int_0^1 \tau\mathbf{z}^2(x,\rho,t)d\rho dx \\ &\leq (N+1)E(t) + \frac{\gamma+1}{2}C_k c(\mathbf{u},\mathbf{u}) \\ &\leq c_1 E(t)\end{aligned}$$

*such that*

$$c_1 = max\{(N+1),(\gamma+1)C_k\}.$$

***Part 2 :***

$$NE(t)+\Lambda(t) = NE(t)+\int_\Omega \mathbf{u}_t\cdot\mathbf{u}dx+\frac{\gamma}{2}\int_\Omega|\mathbf{u}|^2dx+|\mu|\int_\Omega\int_0^1\tau\exp(-\tau\rho)\mathbf{z}^2(x,\rho,t)d\rho dx$$
$$\geq NE(t)+\int_\Omega \mathbf{u}_t\cdot\mathbf{u}dx+|\mu|\int_\Omega\int_0^1\tau\exp(-\tau\rho)\mathbf{z}^2(x,\rho,t)d\rho dx.$$

*Using the inequality*

$$\int_\Omega \mathbf{u}_t\cdot\mathbf{u}dx\geq -\frac{1}{2}\int_\Omega(\mathbf{u}_t^2+\mathbf{u}^2)dx$$

*then we have*

$$NE(t)+\Lambda(t)\geq\frac{N-1}{2}\int_\Omega\mathbf{u}_t^2dx+\frac{N}{2}(c(\mathbf{u},\mathbf{u})+d(\varphi,\varphi))-\frac{1}{2}\int_\Omega\mathbf{u}^2dx+|\mu|\int_\Omega\int_0^1\tau\exp(-\tau\rho)\mathbf{z}^2(x,\rho,t)d\rho dx.$$

*Using Korn's inequality, we obtain*

$$NE(t)+\Lambda(t)\geq\frac{N-1}{2}\int_\Omega\mathbf{u}_t^2dx+\frac{N-C_k}{2}c(\mathbf{u},\mathbf{u})+\frac{N}{2}d(\varphi,\varphi)+c^*|\mu|\tau\int_\Omega\int_0^1\mathbf{z}^2(x,\rho,t)d\rho dx$$
$$\geq c_2E(t)$$

*such that*

$$0<c^*<\exp(-\tau\rho),$$
$$N>max\{1,C_k\}$$

*and*

$$c_2=min\{\frac{N-1}{2},\frac{N-C_k}{2},\frac{N}{2},c^*\}.$$

**Theorem 2.3.1** *There exist two positive constants $\alpha$ and $\beta$ such that*

$$E(t)\leq\beta\exp(-\alpha t)E(0),\ \forall t\geq 0.$$

**Proof 2.3.3** *Let $\phi(t)=NE(t)+\Lambda(t)$.*
*Then*

$$\phi'(t)=-N(\gamma-\frac{|\mu|}{2})\int_\Omega\mathbf{u}_t^2dx-N\frac{|\mu|}{2}\int_\Omega\mathbf{z}^2(x,1,t)dx-N\mu\int_\Omega\mathbf{u}_t\cdot\mathbf{z}(x,1,t)dx+\int_\Omega\mathbf{u}_t^2dx$$
$$-(c(\mathbf{u},\mathbf{u})+d(\varphi,\varphi))-|\mu|\int_\Omega\exp(-\tau)\mathbf{z}^2(x,1,t)dx+|\mu|\int_\Omega\mathbf{u}_t^2dx-|\mu|\tau\int_\Omega\int_0^1\exp(-\tau\rho)\mathbf{z}^2d\rho dx.$$

*Using Young inequality, we have*

$$\phi'(t)\leq-N(\gamma-\frac{|\mu|}{2})\int_\Omega\mathbf{u}_t^2dx-N\frac{|\mu|}{2}\int_\Omega\mathbf{z}^2(x,1,t)dx+\frac{N\mu}{2}\int_\Omega\mathbf{u}_t^2dx+\frac{N\mu}{2}\int_\Omega\mathbf{z}^2(x,1,t)dx$$
$$+\int_\Omega\mathbf{u}_t^2dx-(c(\mathbf{u},\mathbf{u})+d(\varphi,\varphi))-c^*|\mu|\int_\Omega\mathbf{z}^2(x,1,t)dx+|\mu|\int_\Omega\mathbf{u}_t^2dx-\tau|\mu|c^*\int_\Omega\int_0^1\mathbf{z}^2d\rho dx$$
$$\leq-\gamma_0\int_\Omega\mathbf{u}_t^2dx-|\mu|c^*\int_\Omega\mathbf{z}^2(x,1,t)dx-(c(\mathbf{u},\mathbf{u})+d(\varphi,\varphi))-\tau|\mu|c^*\int_\Omega\int_0^1\mathbf{z}^2d\rho dx,$$

*such that*

$$\gamma_0 = N(\gamma - |\mu|) - (1 - |\mu|) > 0.$$

*Then*

$$\begin{aligned}\phi'(t) \leq -inf\{\gamma_0, 1, c^*\}(\int_\Omega \mathbf{u}_t^2 dx + c(\mathbf{u}, \mathbf{u}) + d(\varphi, \varphi) + |\mu|\tau \int_\Omega \int_0^1 \mathbf{z}^2 d\rho dx) \\ \leq -2inf\{\gamma_0, 1, c^*\}E(t) \\ \leq -\lambda_0 E(t)\end{aligned}$$

*such that* $\lambda_0 = 2inf\{\gamma_0, 1, c^*\}$.
*Therefore*

$$\phi'(t) \leq -\frac{\lambda_0}{c_2}\phi(t).$$

*So that*

$$\phi(t) \leq \phi(0)\exp(-\frac{\lambda_0}{c_2}t).$$

*Then*

$$\begin{aligned}E(t) &\leq \frac{c_1}{c_2}E(0)\exp(-\frac{\lambda_0}{c_2}t) \\ &\leq \beta E(0)\exp(-\alpha t)\end{aligned}$$

*such that* $\beta = \frac{c_1}{c_2}$ *and* $\alpha = \frac{\lambda_0}{c_2}$.

Chapter 3

# Energy decay of a thermopiezoelectric system

In this chapter, firstly we present the 3D model of our problem which represents the effect of temperature on piezoelectric body. Secondly, we study the well-posedness of the problem using a semigroup approach. Thirdly, we study the decay of the total energy of the system. Using the equivalence between the exponential decay of the total energy and an observability inequality. The idea is to apply a decoupling method introduced by Henry et al.[5], which shows that the difference between the semigroup generated by the thermopiezoelectric system and the decoupled system is compact in the energy space, then we apply the Weyl's theorem to conclude that the two semigroup have the same stability properties.

## 3.1 Thermopiezoelectric 3D model

Let $\Omega$ be a bounded region in $\mathbb{R}^3$ with smooth boundary. We consider the thermopiezoelectric model:

$$\begin{cases} \mathbf{u}_{tt} - \mathrm{div}\mathbf{T}(\mathbf{u},\varphi) + \alpha\nabla\theta = 0 & \\ \mathrm{div}\mathbf{D}(\mathbf{u},\varphi) = 0 & \text{in } \Omega\times\mathbb{R}^+ \\ \theta_t - \Delta\theta + \beta\mathrm{div}(\mathbf{u}_t) = 0 & \\ \mathbf{u} = 0, \quad \varphi = 0, \quad \theta = 0, & \text{on } \partial\Omega\times\mathbb{R}^+ \\ \mathbf{u}(x,0) = \mathbf{u}^0(x), \quad \mathbf{u}_t(x,0) = \mathbf{u}^1(x), \quad \theta_0(x,0) = \theta^0(x) & \text{in } \Omega. \end{cases} \tag{3.1}$$

We use the same notations as in Chapter 2, where $\mathbf{u}$, $\varphi$ and $\theta$ are the displacement vector, the electric potential and the temperature variation respectively, $\alpha$ and $\beta$ are positive constants, $\mathbf{T}$ is the mechanical stress tensor and $\mathbf{D}$ is the electric displacement vector already defined in **(2.2) chapter 2**.

### 3.1.1 Hypothesis I

We consider the same assumptions on the coefficients as in chapter 2. The fourth-order **elasticity tensor** $(C^{ijkl})$ is symmetric and positive definite, the third-order **coupling tensor** $(e^{ijk})$ is partly symmetric, $e^{ijk} = e^{ikj}$, the second-order **dielectric tensor** $(d^{ij})$ is symmetric and positive definite.

## 3.2 Well posedness: Functional setting

We use in this chapter some results of chapter 2.
Let us denote by $\mathbf{X}$ the set $\mathbf{X} = \mathbf{H}_0^1(\Omega)\times\mathbf{L}^2(\Omega)\times\mathbf{L}^2(\Omega)$. We define on $\mathbf{X}$ the scalar produit

$$\begin{cases} \text{For } \mathbf{U}_1 = \Big(\ \mathbf{u}_1, \mathbf{w}_1, \theta_1\ \Big),\ \mathbf{U}_2 = \Big(\ \mathbf{u}_2, \mathbf{w}_2, \theta_2\ \Big),\ \varphi_1 = \Re\mathbf{u}_1,\ \varphi_2 = \Re\mathbf{u}_2 : \\ \langle \mathbf{U}_1, \mathbf{U}_2\rangle_{\mathbf{X}} = c(\mathbf{u}_1,\mathbf{u}_2) + d(\varphi_1,\varphi_2) + (\mathbf{w}_1,\mathbf{w}_2) + \dfrac{\alpha}{\beta}(\theta_1,\theta_2). \end{cases}$$

Equipped with this scalar product $\mathbf{X}$ is a Hilbert space.

The evolution problem (3.1) can be written as

$$\frac{d}{dt}\mathbf{U}(t) = \mathcal{A}\mathbf{U}(t), \quad \mathbf{U}(0) = \mathbf{U}_0$$

where $\mathbf{U} = (\mathbf{u}, \mathbf{w}, \theta), \mathbf{U}_0 = (\mathbf{u}^0, \mathbf{u}^1, \theta^0)$ and the operator $\mathcal{A} : D(\mathcal{A}) \subset X \longrightarrow X$ is given by

$$\mathcal{A}(\mathbf{u}, \mathbf{w}, \theta) = (\mathbf{w}, \operatorname{div}\mathbf{T}(\mathbf{u}, \Re\mathbf{u}) - \alpha\nabla\theta, \Delta\theta - \beta\operatorname{div}(\mathbf{w}))$$

for any $(\mathbf{u}, \mathbf{w}, \theta) \in D(\mathcal{A})$ with domain

$$\mathcal{D}(\mathcal{A}) = \left\{(\mathbf{u}, \mathbf{w}, \theta) \in \mathbf{X}, \mathbf{u} \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega), \mathbf{w} \in \mathbf{H}_0^1(\Omega), \theta \in H^2(\Omega) \cap H_0^1(\Omega)\right\}.$$

We can easily verify that the adjoint operator $\mathcal{A}^*$ is given by

$$\mathcal{A}^*(\mathbf{u}, \mathbf{w}, \theta) = (-\mathbf{w}, -\operatorname{div}\mathbf{T}(\mathbf{u}, \Re\mathbf{u}) + \alpha\nabla\theta, \Delta\theta + \beta\operatorname{div}(\mathbf{w}))$$

with domain $\mathcal{D}(\mathcal{A}^*) = \mathcal{D}(\mathcal{A})$.

**Lemma 3.2.1** *Assume Hypothesis I and the above considerations.Then $\mathcal{A}$ and $\mathcal{A}^*$ are dissipative operators, that is*

$$a)\langle\mathcal{A}(\mathbf{u}, \mathbf{w}, \theta), (\mathbf{u}, \mathbf{w}, \theta)\rangle_X \leq 0 \quad \forall(\mathbf{u}, \mathbf{w}, \theta) \in \mathcal{D}(A)$$

*and*

$$b)\langle\mathcal{A}^*(\mathbf{u}, \mathbf{w}, \theta), (\mathbf{u}, \mathbf{w}, \theta)\rangle_X \leq 0 \quad \forall(\mathbf{u}, \mathbf{w}, \theta) \in \mathcal{D}(A^*).$$

**Proof 3.2.1** *Using **lemma 2.1.1** we have for a given $\mathbf{u}, \mathbf{w} \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)$ there exist unique $\varphi = \Re\mathbf{u}$ and $\psi = \Re\mathbf{w}$ in $H^2(\Omega) \cap H_0^1(\Omega)$ solutions of $(P_{\varphi(\mathbf{u})})$ and $(P_{\psi(\mathbf{w})})$ respectively.*

*Let $(\mathbf{u}, \mathbf{w}, \theta) \in \mathcal{D}(\mathcal{A})$. Then*

$$\begin{aligned}
&\langle\mathcal{A}(\mathbf{u}, \mathbf{w}, \theta), (\mathbf{u}, \mathbf{w}, \theta)\rangle_X = \\
&= \langle(\mathbf{w}, \operatorname{div}\mathbf{T}(\mathbf{u}, \Re\mathbf{u}) - \alpha\nabla\theta, \Delta\theta - \beta\operatorname{div}(\mathbf{w})), (\mathbf{u}, \mathbf{w}, \theta)\rangle_X \\
&= \textstyle\int_\Omega[\operatorname{div}\mathbf{T}(\mathbf{u}, \Re\mathbf{u}) - \alpha\nabla\theta] \cdot \mathbf{w}dx + c(\mathbf{w}, \mathbf{u}) \\
&\quad +d(\Re\mathbf{w}, \Re\mathbf{u}) + \tfrac{\alpha}{\beta}\textstyle\int_\Omega(\Delta\theta - \beta\operatorname{div}(\mathbf{w}))\,\theta dx.
\end{aligned}$$

*From **(2.5) chapter 2**, we have*

$$\int_\Omega \operatorname{div}\mathbf{T}(\mathbf{u}, \Re\mathbf{u}) \cdot \mathbf{w}dx = -c(\mathbf{u}, \mathbf{w}) - e(\mathbf{w}, \Re\mathbf{w}).$$

*Using Green's formula with boundary conditions, we get*

$$\frac{\alpha}{\beta}\int_\Omega(\Delta\theta - \beta\operatorname{div}(\mathbf{w}))\theta dx = -\frac{\alpha}{\beta}\int_\Omega|\nabla\theta|^2dx + \alpha\int_\Omega \mathbf{w} \cdot \nabla\theta dx.$$

*Therefore*

$$\langle \mathcal{A}(\mathbf{u},\mathbf{v},\theta),(\mathbf{u},\mathbf{v},\theta)\rangle_X = -e(\mathbf{w},\Re\mathbf{w}) + d(\Re\mathbf{u},\Re\mathbf{w}) - \frac{\alpha}{\beta}\int_\Omega |\nabla\theta|^2 dx = -\frac{\alpha}{\beta}\int_\Omega |\nabla\theta|^2 dx$$

*which proves a). In the same way we can prove b).*

**Theorem 3.2.1** *Under Hypothesis I and the above considerations we have:*
*a) The operator $\mathcal{A}$ is the infinitesimal generator of a $C_0$- semigroup, $\{\mathbf{S}(t)\}_{t\geq 0}$ of contractions in $X$.*
*b) For each $\mathbf{U}_0 = (\mathbf{u}^0, \mathbf{u}^1, \theta^0) \in \mathcal{D}(\mathcal{A})$ , the thermopiezoelectric system (3.1) has a unique (strong) solution $\mathbf{U} = (\mathbf{u},\mathbf{w},\theta) \in C([0,+\infty),\mathcal{D}(\mathcal{A}))$.*

**Proof 3.2.2** *Clearly $\mathcal{D}(\mathcal{A})$ is dense in $\mathbf{X}$ because contains $[C_0^\infty(\Omega)]^7$. Furthermore, $\mathcal{A}$ is a closed operator because an easy calculation shows that $\mathcal{A}^{**} = \mathcal{A}$. Indeed, define the linear bounded self adjoint operator $E_\theta$ on a Hilbert space $\mathbf{X}$ by $E_\theta(\mathbf{u},\mathbf{w},\theta) = (0,0,-\Delta\theta)$. Then we have*

$$\mathcal{A} + E_\theta = -\mathcal{A}^* - E_\theta.$$

*Therefore*

$$\mathcal{A}^{**} = (-\mathcal{A} - 2E_\theta)^* = -\mathcal{A}^* - 2E_\theta^* = \mathcal{A}.$$

*Using this information and the result that $\mathcal{A}$ and $\mathcal{A}^*$ are dissipative, we apply* ***corollary 1.1.3*** *to deduce a). Using the theory of semigroups then b) follows from a).*

## 3.3 Analyzing the system's stability

In order to study stability, we demonstrate exponential decay of the total energy of the system.
We know that the total energy $E(t)$ associated with (3.1) is

$$E(t) = \frac{1}{2}\|\mathbf{U}\|^2_{\mathbf{X}} = \frac{1}{2}[\int_\Omega |\mathbf{u_t}|^2 dx + c(\mathbf{u},\mathbf{u}) + d(\varphi,\varphi) + \frac{\alpha}{\beta}\int_\Omega |\theta|^2 dx]$$

for all $t \geq 0$ . It is easy to prove that $E(t)$ satisfy

$$\frac{d}{dt}E(t) = -\frac{\alpha}{\beta}\int_\Omega |\nabla\theta|^2 dx \leq 0$$

which indicates that $E(t)$ is decreasing.

### 3.3.1 Equivalence between the exponential decay as $t \to +\infty$ and an observability inequality

***Hypothesis II (Conditon on the region $\Omega$)***

If $\mathbf{\Phi} \in \mathbf{H}_0^1(\Omega)$ is such that

$$\begin{cases} \operatorname{div} \mathbf{T}(\mathbf{\Phi}, \Re(\mathbf{\Phi})) + \gamma\mathbf{\Phi} = 0 \text{ in } \Omega \\ \operatorname{div} \mathbf{\Phi} = 0 \text{ in } \Omega \end{cases} \tag{3.2}$$

for some $\gamma \in \mathbb{R}$, then $\mathbf{\Phi} = 0$ in $\bar{\Omega}$, where the tensor $\mathbf{T}$ is defined in **(2.2) chapter 2**.

**Theorem 3.3.1** *Let $\Omega$ be a bounded region of $\mathbb{R}^3$ with smooth boundary. Assume Hypotheses I and II. Then, there exist positive constants $M$ and $m$ such that*

$$E(t) \leq E(0) \exp(-mt) \quad \forall t > 0$$

*if and only if there exist $T > 0$ and $\lambda > 0$ such that*

$$||\mathbf{u}^0||^2_{\mathbf{L}^2(\Omega)} + ||\mathbf{u}^1||^2_{\mathbf{H}^{-1}(\Omega)} \leq \lambda \int_0^T || \operatorname{div} \mathbf{\Psi}(.,t)||^2_{H^{-1}(\Omega)} dt \tag{3.3}$$

*holds for every solution $\mathbf{\Psi}(x,t)$ of the associated system*

$$\begin{cases} \mathbf{\Psi}_{tt} - div\mathbf{T}(\mathbf{\Psi}, \Re(\mathbf{\Psi})) = 0 \;\; in \; \Omega \times (0,T) \\ \mathbf{\Psi} = 0 \quad on \; \partial\Omega \times (0,T) \\ \mathbf{\Psi}(x,0) = \mathbf{u}^0(x), \mathbf{\Psi}_t(x,0) = \mathbf{u}^1(x) \;\; in \; \Omega. \end{cases} \tag{3.4}$$

In order to prove the observability inequality implies the exponential decay we use the decoupling method to decouple our system into two problems. The first one a piezo-electric problem with dissipative term and the second is a heat equation with a source term. Following a similar strategy as the one used by G. Lebeau and E. Zuazua [8] we obtain $E(0) \leq \mathrm{c}_1 \int_0^{\mathrm{T}} \int_\Omega |\nabla\theta|^2 dxdt$ for initial data in terms of the norm of the gradient of temperature.

**Remark 3.3.1** *We define $\mathcal{P} \in \mathfrak{L}(\mathbf{L^2(\Omega)}, \mathbf{L^2(\Omega)})$ is the orthogonal projection from $\mathbf{L^2(\Omega)}$ into the closed subspace $\mathbf{S} = \{\nabla\phi$ where $\phi \in H_0^1(\Omega)\}$.*

*Clearly, $\mathcal{P} = \nabla(\Delta^{-1}) \operatorname{div}$, then $\mathcal{P}\mathbf{v} = \nabla\phi$ with $\phi \in H_0^1(\Omega)$ if and only if $\Delta\phi = \operatorname{div} \mathbf{v}$ in $\Omega$ and $\phi \in H_0^1(\Omega)$. This fact tells us that*

$$\begin{aligned}\int_\Omega \mathcal{P}\mathbf{v}\cdot\mathbf{v}dx = \int_\Omega \nabla\phi\cdot\mathbf{v}dx \quad &= -\int_\Omega \phi\,\mathrm{div}\,\mathbf{v}dx \\ = -\int_\Omega \phi\Delta\phi dx \qquad &= \int_\Omega |\nabla\phi|^2 dx \\ = \int_\Omega |\mathcal{P}\mathbf{v}|^2 dx. &\end{aligned} \tag{3.5}$$

We claim that $\int_\Omega |\mathcal{P}\mathbf{v}|^2 dx$ is equivalent to $||\,\mathrm{div}\,\mathbf{v}||^2_{H^{-1}(\Omega)}$.

***Decoupled system corresponding to system (3.1)***

We are going to introduce the decoupling system associated to the problem (3.1), for which we are able to prove exponential decay of energy. The idea is to use the method suggested by Henry and all. [5] for thermoelastic system.

$$\begin{cases} \mathbf{v}_{tt} - div\mathbf{T}(\mathbf{v},\Re(\mathbf{v})) + \alpha\beta\mathcal{P}\mathbf{v}_t = 0 \quad \text{in } \Omega\times(0,\infty) \\ \eta_t - \Delta\eta + \beta div\mathbf{v}_t = 0 \quad \text{in } \Omega\times(0,\infty) \\ \mathbf{v} = 0, \eta = 0 \text{ on } \partial\Omega\times(0,\infty) \\ \mathbf{v}(x,0) = \mathbf{u}^0(x), \mathbf{v}_t(x,0) = \mathbf{u}^1(x), \eta(x,0) = \theta^0(x) \quad \text{in } \Omega. \end{cases} \tag{3.6}$$

This system can be written as an abstract Cauchy problem

$$\frac{d}{dt}\mathbf{V}(t) = \mathcal{B}\mathbf{V}(t), \quad \mathbf{V}(0) = \mathbf{V}_0$$

where $\mathbf{V} = (\mathbf{v}, \mathbf{v}_t, \eta)$ and $\mathbf{V}(0) = (\mathbf{u}^0, \mathbf{u}^1, \theta^0)$
and the operator $\mathcal{B} : D(\mathcal{B}) \subset X \longrightarrow X$ is given by

$$\mathcal{B}(\mathbf{v}, \mathbf{v}_t, \eta) = (\mathbf{v}_t, div\mathbf{T}(\mathbf{v},\Re(\mathbf{v})) - \alpha\beta\mathcal{P}\mathbf{v}_t, \Delta\eta - \beta\,\mathrm{div}(\mathbf{v}_t))$$

for any $(\mathbf{v}, \mathbf{v}_t, \eta) \in \mathcal{D}(\mathcal{B}) = \mathcal{D}(\mathcal{A})$.

Under the same hypothesis of **theorem 3.2.1**, this system is also well posed in the Hilbert space $\mathbf{X}$. The operator $\mathcal{B}$ generate a semigroup $\{\mathbf{S_d}(t)\}_{t\geq 0}$.

**Lemma 3.3.1** *For any $T > 0$, the difference of the semigroups $\mathbf{S}(t) - \mathbf{S}_d(t)$ is compact from $\mathbf{X}$ into $C([0,T];\mathbf{X})$.*

**Proof 3.3.1** *see [6].*

**Remark 3.3.2** *The result of this lemma asserts that the difference between the generators of these two semigroups is compact, and by means of **Weyl's theorem** 1.1.2 the essential spectrum of these two operators is identical and consequently the stability of the system associated with the operator $\mathcal{A}$ is equivalent to the stability of the system associated with the operator $\mathcal{B}$. Therefore, the stability of the system (3.1) is equivalent to the stability of the system (3.6).*

To solve the decoupled system (3.6), we solve first the system

$$\begin{cases} \mathbf{v}_{tt} - div\mathbf{T}(\mathbf{v}, \Re(\mathbf{v})) + \alpha\beta\mathcal{P}\mathbf{v}_t = 0 \quad \text{in } \Omega \times (0, \infty) \\ \mathbf{v} = 0, \text{ on } \partial\Omega \times (0, \infty) \\ \mathbf{v}(x, 0) = \mathbf{u}^0(x), \mathbf{v}_t(x, 0) = \mathbf{u}^1(x) \quad \text{in } \Omega. \end{cases} \tag{3.7}$$

Afterwords, we solve the scalar equation

$$\begin{cases} \eta_t - \Delta\eta + \beta \operatorname{div} \mathbf{v}_t = 0 \quad \text{in } \Omega \times (0, \infty) \\ \eta = 0 \text{ on } \partial\Omega \times (0, \infty) \\ \eta(x, 0) = \theta^0(x) \quad \text{in } \Omega. \end{cases} \tag{3.8}$$

The energy associated to the system (3.7) is

$$E_2(t) = \frac{1}{2}[\int_\Omega |\mathbf{v_t}|^2 dx + c(\mathbf{v}, \mathbf{v}) + d(\Re(\mathbf{v}), \Re(\mathbf{v}))]$$

and the energy associated to the problem (3.8) is

$$E_3(t) = \frac{\alpha}{2\beta} \int_\Omega \eta^2 dx.$$

**Proof 3.3.2 (of theorem 3.3.1)** *The idea of the proof is to use a similar strategy used by G. Lebeau and E. Zuazua in [8] for linear thermoelasticity. We restrict ourselves to proving the inequality that leads to the stability of our system. The proof of the reciprocal inequality can be found in [6]. So we prove that the observability inequality (3.3) implies the exponential decay of the total energy $E(t)$ of problem (3.1).*
*Let $\mathbf{\Phi} = \mathbf{\Psi}_t$. Then we derive with respect to t the problem (3.4), we get*

$$\begin{cases} \mathbf{\Phi}_{tt} - div\mathbf{T}(\mathbf{\Phi}, R(\mathbf{\Phi})) = 0 \text{ in } \Omega \times (0, T) \\ \mathbf{\Phi}(x, 0) = \mathbf{\Phi}_0(x) = \mathbf{u}^1(x) \text{ in } \Omega \\ \mathbf{\Phi}_t(x, 0) = \mathbf{\Phi}_1(x) = div\mathbf{T}(\mathbf{u}^0, \Re(\mathbf{u}^0)) \text{ in } \Omega \\ \mathbf{\Phi} = 0 \text{ on } \partial\Omega \times (0, T). \end{cases} \tag{3.9}$$

*Using (3.3) we obtain*

$$||\mathbf{\Phi}_0||^2_{\mathbf{L}^2(\Omega)} + ||\mathbf{\Phi}_1||^2_{\mathbf{H}^{-1}(\Omega)} \leq \lambda \int_0^T || \operatorname{div} \mathbf{\Phi}||^2_{H^{-1}(\Omega)} dt \tag{3.10}$$

*since*

$$\begin{aligned} ||\Phi_1||^2_{\mathbf{H}^{-1}(\Omega)} &= ||div\mathbf{T}(\mathbf{u}^0, \Re(\mathbf{u}^0))||^2_{\mathbf{H}^{-1}(\Omega)} \\ &\geq C||\mathbf{u}^0||^2_{\mathbf{H}^1_0(\Omega)} \end{aligned} \tag{3.11}$$

where $C > 0$. Then, it follows from (3.4) together with (3.5) that

$$\begin{aligned}||\mathbf{u}^1||^2_{\mathbf{L}^2(\Omega)} + ||\mathbf{u}^0||^2_{\mathbf{H}^1_0(\Omega)} &\leq C\int_0^{\mathrm{T}} ||\operatorname{div}\mathbf{\Phi}||^2_{\mathbf{H}^{-1}(\Omega)}dt \\ &\leq C\int_0^{\mathrm{T}}\int_\Omega \mathcal{P}\mathbf{\Psi}_t\cdot\mathbf{\Psi}_t dxdt\end{aligned} \tag{3.12}$$

for some positive constant $C$.

Let's now decompose the solution $\mathbf{v}$ of the decoupled problem (3.6) into the sum of two functions $\mathbf{\Psi}$ and $\mathbf{\Upsilon}$ where $\mathbf{\Psi}$ is the solution of the problem (3.4) and $\mathbf{\Upsilon}$ is the solution of the problem

$$\begin{cases} \mathbf{\Upsilon}_{tt} - \operatorname{div}\mathbf{T}(\mathbf{\Upsilon}, \Re(\mathbf{\Upsilon})) = -\alpha\beta\mathcal{P}\mathbf{v}_t \quad \text{in } \Omega\times(0,T) \\ \mathbf{\Upsilon} = 0 \text{ on } \partial\Omega\times(0,T) \\ \mathbf{\Upsilon}(x,0) = 0, \mathbf{\Upsilon}_t(x,0) = 0 \quad \text{in } \Omega. \end{cases} \tag{3.13}$$

From (3.12) we get

$$\begin{aligned} E_2(0) &\leq ||\mathbf{u}^1||^2_{\mathbf{L}^2(\Omega)} + ||\mathbf{u}^0||^2_{\mathbf{H}^1_0(\Omega)} \\ &\leq C\int_0^{\mathrm{T}} ||\operatorname{div}\mathbf{\Phi}||^2_{\mathrm{H}^{-1}(\Omega)}dt \\ &= C\int_0^{\mathrm{T}} ||\operatorname{div}\mathbf{v}_t - \operatorname{div}\mathbf{\Upsilon}_t||^2_{\mathrm{H}^{-1}(\Omega)}dt \\ &\leq 2C\int_0^{\mathrm{T}} ||\operatorname{div}\mathbf{v}_t||^2_{\mathrm{H}^{-1}(\Omega)}dt + 2C\int_0^{\mathrm{T}} ||\operatorname{div}\mathbf{\Upsilon}_t||^2_{\mathrm{H}^{-1}(\Omega)}dt \\ &\leq 2C\int_0^{\mathrm{T}}\int_\Omega \mathcal{P}\mathbf{v}_t\cdot\mathbf{v}_t dxdt + 2C\int_0^{\mathrm{T}} ||\operatorname{div}\mathbf{\Upsilon}_t||^2_{\mathrm{H}^{-1}(\Omega)}dt. \end{aligned} \tag{3.14}$$

Next, we multiply (3.13) by $\mathbf{\Upsilon}_t$ and integrate over $\Omega$ we obtain

$$\frac{1}{2}\frac{d}{dt}[\int_\Omega |\mathbf{\Upsilon_t}|^2dx + c(\mathbf{\Upsilon},\mathbf{\Upsilon}) + d(\Re(\mathbf{\Upsilon}),\Re(\mathbf{\Upsilon}))] = -\alpha\beta\int_\Omega \mathcal{P}\mathbf{v}_t\cdot\mathbf{\Upsilon}_t dx \leq \alpha\beta||\mathcal{P}\mathbf{v}_t||||\mathbf{\Upsilon}_t||.$$

Then we integrate over $[0,T]$ give us

$$\int_\Omega |\mathbf{\Upsilon_t}|^2dx + c(\mathbf{\Upsilon},\mathbf{\Upsilon}) + d(\Re(\mathbf{\Upsilon}),\Re(\mathbf{\Upsilon})) \leq 2\alpha\beta\int_0^T ||\mathcal{P}\mathbf{v}_t||||\mathbf{\Upsilon}_t||ds. \tag{3.15}$$

Here $||.||$ denotes the norm in $\mathbf{L}^2(\Omega)$. and from (3.15) we obtain

$$\begin{aligned} ||\mathbf{\Upsilon}_t||^2_{\mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega))} &= sup_{0<t<T}||\mathbf{\Upsilon}_t||^2_{\mathbf{L}^2(\Omega)} \\ &\leq 2\alpha\beta\int_0^T ||\mathcal{P}\mathbf{v}_t||||\mathbf{\Upsilon}_t||ds \\ &\leq 2\alpha\beta||\mathbf{\Upsilon}_t||_{\mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega))}\int_0^T ||\mathcal{P}\mathbf{v}_t||ds. \end{aligned} \tag{3.16}$$

Using Cauchy-Schwarz inequality, we get

$$||\boldsymbol{\Upsilon}_t||_{\mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega))} \leq 2\alpha\beta\sqrt{T}||\mathcal{P}\mathbf{v}_t||_{\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega))}. \tag{3.17}$$

Finally, using (3.5), (3.17) in (3.14) we get the estimate

$$E_2(0) \leq C\int_0^{\mathrm{T}}\int_\Omega \mathcal{P}\mathbf{v}_t \cdot \mathbf{v}_t dxdt. \tag{3.18}$$

The semigroup property together with (3.18) implies the existence of positive constant $C > 0$ and $\omega > 0$ such that

$$E_2(t) \leq CE_2(0)\exp(-\omega t)\ for\ any\ t \geq 0.$$

Next, we consider the energy $E_3(t)$.
Using, from (3.6), the equation

$$\eta_t = \Delta\eta - \beta\,\mathrm{div}\,\mathbf{v}_t,$$

we have

$$\frac{d}{dt}E_3(t) = \frac{d}{dt}[\frac{\alpha}{2\beta}\int_\Omega \eta^2 dx] = \frac{\alpha}{\beta}\int_\Omega \eta\eta_t dx \leq -\frac{\alpha}{\beta}\int_\Omega |\nabla\eta|^2 dx + \alpha\int_\Omega \mathbf{v}_t\cdot\nabla\eta dx.$$

Using Cauchy-Schwarz and Young inequalities, we get

$$\begin{aligned}\alpha\int_\Omega \mathbf{v}_t\cdot\nabla\eta dx = \frac{\alpha}{\beta}\int_\Omega(\beta\mathbf{v}_t)\cdot\nabla\eta dx \leq \frac{\alpha}{\beta}\|\beta\mathbf{v}_t\|\|\nabla\eta\| \leq \\ \frac{\alpha}{2\beta}[\beta^2\|\mathbf{v}_t\|^2 + \|\nabla\eta\|^2] = \frac{\alpha\beta}{2}\|\mathbf{v}_t\|^2 + \frac{\alpha}{2\beta}\|\nabla\eta\|^2.\end{aligned}$$

Then, using Poincare inequality, we get

$$\begin{aligned}\frac{d}{dt}E_3(t) &\leq -\frac{\alpha}{2\beta}\|\nabla\eta\|^2 + \frac{\alpha\beta}{2}\|\mathbf{v}_t\|^2 \\ &\leq -\frac{\alpha}{2\beta}\int_\Omega|\nabla\eta|^2 dx + \alpha\beta E_2(t) \\ &\leq -\frac{\alpha}{2\beta C_p}||\eta||^2 + \alpha\beta E_2(t) \\ &\leq -\frac{1}{C_p}E_3(t) + CE_2(0)\exp(-\omega t).\end{aligned}$$

Using Gronwall's inequality (**lemma 1.1.1**), we get

$$\begin{aligned} E_3(t) &\leq E_3(0)\exp(-\frac{1}{C_p}t) + CE_2(0)\exp(-\omega t) \\ &\leq C(E_3(0)+E_2(0))\exp(-\omega_3 t) \; for \; all \; t \geq 0. \end{aligned} \tag{3.19}$$

Using the decay of $E_2(t)$ and $E_3(t)$ we conclude that the decoupled system ( 3.7), (3.8) decays exponentially as $t \to +\infty$ .
Let us note

$$E_4(t) = E_2(t) + E_3(t).$$

Thus, $E_4(t) \leq C_4 E_4(0)\exp(-\omega_4 t)$ for all $t \geq 0$. In particular, we can choose $T > 0$ suficiently large to obtain

$$E_4(T) \leq \gamma E_4(0) \; for \; some \; 0 < \gamma < 1. \tag{3.20}$$

By **Lemma 3.3.1** we know the existence of a linear compact map $\mathbf{K}(t) : \mathbf{X} \to \mathbf{X}$

$$\mathbf{K}(t) = \mathbf{S}(t) - \mathbf{S}_d(t).$$

We write the solution of system (3.1) as

$$(\mathbf{u}, \theta) = (\mathbf{v}, \eta) + (\mathbf{w}, \psi)$$

where $(\mathbf{v}, \eta)$ solves the decoupled system (3.6) and

$$(\mathbf{w}, \mathbf{w}_t, \psi) = \mathbf{K}(t)(\mathbf{u}^0, \mathbf{u}^1, \theta^0)$$

where $(\mathbf{u}^0, \mathbf{u}^1, \theta^0)$ is the initial data of problem (1.3). Therefore

$$(\mathbf{u}, \mathbf{u}_t, \theta) = \mathbf{S}(t)(\mathbf{u}^0, \mathbf{u}^1, \theta^0)$$

and

$$(\mathbf{v}, \mathbf{v}_t, \eta) = \mathbf{S}_d(t)(\mathbf{u}^0, \mathbf{u}^1, \theta^0).$$

We have

$$||\mathbf{S}(T)(\mathbf{u}^0, \mathbf{u}^1, \theta^0)||_{\mathbf{X}} \leq ||\mathbf{S}_d(T)(\mathbf{u}^0, \mathbf{u}^1, \theta^0)||_{\mathbf{X}} + ||\mathbf{K}(T)(\mathbf{u}^0, \mathbf{u}^1, \theta^0)||_{\mathbf{X}}$$

where $T > 0$ is as in (3.20). Consequently

$$\begin{aligned} E(T) &\leq E_4(T) + ||\mathbf{K}(T)(\mathbf{u}^0, \mathbf{u}^1, \theta^0)||^2_{\mathbf{X}} \\ &\leq \gamma E_4(0) + ||\mathbf{K}(T)(\mathbf{u}^0, \mathbf{u}^1, \theta^0)||^2_{\mathbf{X}} \\ &\leq \gamma E(0) + ||\mathbf{K}(T)(\mathbf{u}_0, \mathbf{u}_1, \theta_0)||^2_{\mathbf{X}}. \end{aligned} \tag{3.21}$$

We have from the above

$$\frac{d}{dt}E(t) = -\frac{\alpha}{\beta}\int_\Omega |\nabla\theta|^2 dx \leq 0 \tag{3.22}$$

integration over $[0, T]$ give us

$$\begin{aligned} E(T) &= -\frac{\alpha}{\beta}\int_0^T \int_\Omega |\nabla\theta|^2 dxdt + E(0) \\ &\leq \gamma E(0) + ||\mathbf{K}(t)(\mathbf{u}^0, \mathbf{u}^1, \theta^0)||^2_{\mathbf{X}} \end{aligned} \tag{3.23}$$

then

$$E(0) \leq \frac{1}{1-\gamma} \ [\frac{\alpha}{\beta}\int_0^T \int_\Omega |\nabla\theta|^2 dxdt + ||\mathbf{K}(t)(\mathbf{u}^0, \mathbf{u}^1, \theta^0)||^2_{\mathbf{X}} \ ].$$

We claim that

$$||\mathbf{K}(t)(\mathbf{u}^0, \mathbf{u}^1, \theta^0)||^2_{\mathbf{X}} \leq C\int_0^T \int_\Omega |\nabla\theta|^2 dxdt \tag{3.24}$$

for the proof see [6].
We conclude

$$E(0) \leq C\int_0^T \int_\Omega |\nabla\theta|^2 dxdt \tag{3.25}$$

for some positive constant $C$. It is well known that (3.25) together with (3.22) implies the exponential decay of $E(t)$. This proves that (3.3) implies the exponential decay in Theorem 3.3.1.

Chapter 4

# Energy decay of thermopiezoelectric rod with time delay

In this chapter, we study the decay of the energy in Thermopiezoelectric rod with time delay, the one-dimensional version of the system (3.1). We use the energy method to prove the stability of the system.

## 4.1 Thermopiezoelectric Systems with time delay

We consider a Thermopiezoelectric $1D$ medium, a rod of lenght $L$, with time delay associated in the function $\theta$, with initial data $u_0$, $u_1$, $\theta_0$ and history function $h$ in suitable function spaces. Here $c$, $p$, $\alpha$, $d$, $\beta$, $\tau_2$ are positive constants, $\tau_1$ is a nonnegative constant with $\tau_1 < \tau_2$, $u$ is the displacement, $\theta$ is the temperature difference from a reference value, $\phi$ is the electric potential, and $k : [\tau_1, \tau_2] \to \mathbb{R}$ is a bounded function. Then the problem is defined as follows

$$\begin{cases} u_{tt}(x,t) - cu_{xx}(x,t) + p\phi_{xx} + \alpha\theta_x(x,t) = 0 \quad \text{in}]0,L[\times\mathbb{R}^+ \\ pu_{xx}(x,t) - d\phi_{xx}(x,t) = 0 \quad \text{in}]0,L[\times\mathbb{R}^+ \\ \theta_t(x,t) - \theta_{xx}(x,t) - \int_{\tau_1}^{\tau_2} k(s)\theta_{xx}(x,t-s)ds + \beta u_{tx}(x,t) = 0 \quad \text{in}]0,L[\times\mathbb{R}^+ \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \phi(x,0) = \phi_0(x), \quad \theta_0(x,0) = \theta_0(x), \quad x \in ]0,L[ \\ u_x(0,t) = u_x(L,t) = \phi_x(0,t) = \phi_x(L,t) = \theta(0,t) = \theta(L,t) = 0, \quad \forall t \geq 0 \\ \theta_x(x,-t) = h(x,t), \quad (x,t) \in ]0,L[\times]0,\tau_2[. \end{cases} \tag{4.1}$$

### 4.1.1 Study of the problem

Let us introduce the following new variable

$$z(x,\rho,s,t) = \theta_x(x,t-\rho s), \quad (x,\rho,s,t) \in ]0,L[\times]0,1[\times(\tau_1,\tau_2)\times\mathbb{R}^+. \tag{4.2}$$

Then, system (4.1) is equivalent to

$$\begin{cases} u_{tt}(x,t) - cu_{xx}(x,t) + p\phi_{xx} + \alpha\theta_x(x,t) = 0 \quad \text{in}]0,L[\times\mathbb{R}^+ \\ pu_{xx}(x,t) - d\phi_{xx}(x,t) = 0 \quad \text{in}]0,L[\times\mathbb{R}^+ \\ \theta_t(x,t) - \theta_{xx}(x,t) - \int_{\tau_1}^{\tau_2} k(s)z_x(x,1,s,t)ds + \beta u_{tx}(x,t) = 0 \quad \text{in}]0,L[\times\mathbb{R}^+ \\ sz_t(x,\rho,s,t) + z_\rho(x,\rho,s,t) = 0, \quad \text{in}]0,L[\times]0,1[\times(\tau_1,\tau_2)\times\mathbb{R}^+ \\ z(x,0,s,t) = \theta_x(x,t) \quad \text{in}]0,L[\times(\tau_1,\tau_2)\times\mathbb{R}^+ \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \phi(x,0) = \phi_0(x), \quad \theta_0(x,0) = \theta_0(x), \quad x \in ]0,L[ \\ u_x(0,t) = u_x(L,t) = \phi_x(0,t) = \phi_x(L,t) = \theta(0,t) = \theta(L,t) = 0, \quad \forall t \geq 0 \\ z(x,1,s,0) = h(x,s), \quad (x,s) \in ]0,L[\times]0,\tau_2[. \end{cases} \tag{4.3}$$

We consider $(u, \phi, \theta, z)$ to be a solution of system (4.3), and defined the energy of (4.3) by

$$E(t) = \frac{1}{2}\int_0^L (u_t^2 + (c - \frac{p^2}{d})u_x^2 + \frac{\alpha}{\beta}\theta^2)dx + \frac{1}{2}\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s(|k(s)| + \xi)z^2(x, \rho, s, t)dsd\rho dx$$

where $\xi$ is positive constant.

**Proposition 4.1.1** *Under the conditions*

$$(c - \frac{p^2}{d}) > 0,$$
$$\int_{\tau_1}^{\tau_2} |k(s)|ds < \frac{\beta}{\alpha}$$

*and*

$$\int_{\tau_1}^{\tau_2} |k(s)|ds + \xi(\tau_2 - \tau_1) < \frac{\alpha}{\beta}.$$

*The energy function $E(t)$ is exponentially decaying.*

**Proof 4.1.1** *We have*

$$E'(t) = -[\frac{\alpha}{\beta} - \frac{1}{2}\int_{\tau_1}^{\tau_2} |k(s)|ds - \frac{\xi}{2}(\tau_2 - \tau_1)]\int_0^L \theta_x^2 dx - \frac{\alpha}{\beta}\int_0^L \theta_x^2 dx$$
$$+\frac{\alpha}{\beta}\int_0^L \theta\int_{\tau_1}^{\tau_2} |k(s)|z_x(x, 1, s, t)dsdx - \frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}(|k(s)| + \xi)z^2(x, 1, s, t)dsdx$$

*such that*

$$\int_0^L \theta\int_{\tau_1}^{\tau_2} |k(s)|z_x(x, 1, s, t)dsdx = \int_0^L \theta \cdot \frac{d}{dx}(\int_{\tau_1}^{\tau_2} |k(s)|z(x, 1, s, t)ds)dx$$
$$= -\int_0^L \theta_x \cdot (\int_{\tau_1}^{\tau_2} |k(s)|z(x, 1, s, t)ds)dx + [\theta \cdot \int_{\tau_1}^{\tau_2} |k(s)|z(x, 1, s, t)ds]_0^L$$
$$= -\int_0^L \theta_x \cdot (\int_{\tau_1}^{\tau_2} |k(s)|z(x, 1, s, t)ds)dx.$$

*Then*

$$E'(t) = -[\frac{\alpha}{\beta} - \frac{1}{2}\int_{\tau_1}^{\tau_2} |k(s)|ds - \frac{\xi}{2}(\tau_2 - \tau_1)]\int_0^L \theta_x^2 dx - \frac{\alpha}{\beta}\int_0^L \theta_x^2 dx$$
$$-\frac{\alpha}{\beta}\int_0^L \theta_x(\int_{\tau_1}^{\tau_2} |k(s)|z(x, 1, s, t)ds)dx - \frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2}(|k(s)| + \xi)z^2(x, 1, s, t)dsdx.$$

*Let*

$$A(t) = -\frac{\alpha}{\beta}\int_0^L \theta_x(\int_{\tau_1}^{\tau_2} |k(s)|z(x,1,s,t)ds)dx, B(t) = -\frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} |k(s)|z^2(x,1,s,t)dsdx$$

*and* $H(x,t) = \int_{\tau_1}^{\tau_2} |k(s)|z(x,1,s,t)ds$. *Then* $A(t) = -\frac{\alpha}{\beta}\int_0^L \theta_x(x,t)H(x,t)dx$.

*Using Cauchy-Schwarz and young inequalities, we get :*

$$A(t) \leq \frac{\alpha}{\beta}\|\theta_x\|_2 \cdot \|H\|_2 \leq \frac{\alpha}{2\beta}[\|\theta_x\|_2^2 + \|H\|_2^2] \tag{4.4}$$

and

$$\begin{aligned} |H(x,t)| &\leq \int_{\tau_1}^{\tau_2} |k(s)||z(x,1,s,t)|ds = \int_{\tau_1}^{\tau_2} (|k(s)|)^{\frac{1}{2}}(|k(s)|^{\frac{1}{2}}|z(x,1,s,t)|)ds \\ &\leq (\int_{\tau_1}^{\tau_2} |k(s)|ds)^{\frac{1}{2}}(\int_{\tau_1}^{\tau_2} |k(s)||z(x,1,s,t)|^2 ds)^{\frac{1}{2}}. \end{aligned}$$

Then

$$|H(x,t)|^2 \leq (\int_{\tau_1}^{\tau_2} |k(s)|ds)(\int_{\tau_1}^{\tau_2} |k(s)|z^2(x,1,s,t)ds) \tag{4.5}$$

and

$$\|H\|_2^2 = \int_0^L |H(x,t)|^2 dx \leq (\int_{\tau_1}^{\tau_2} |k(s)|ds)\int_0^L \int_{\tau_1}^{\tau_2} |k(s)|z^2(x,1,s,t)dsdx. \tag{4.6}$$

Using (4.4) and (4.6) , we have

$$A(t) \leq \frac{\alpha}{2\beta}[\|\theta_x\|_2^2 + (\int_{\tau_1}^{\tau_2} |k(s)|ds)\int_0^L \int_{\tau_1}^{\tau_2} |k(s)|z^2(x,1,s,t)dsdx].$$

Therefore

$$\begin{aligned} A(t) + B(t) &\leq \frac{\alpha}{2\beta}\|\theta_x\|_2^2 + \frac{1}{2}(\frac{\alpha}{\beta}\int_{\tau_1}^{\tau_2} |k(s)|ds - 1)\int_0^L \int_{\tau_1}^{\tau_2} |k(s)|z^2(x,1,s,t)dsdx \\ &\leq \frac{\alpha}{2\beta}\|\theta_x\|_2^2. \end{aligned}$$

So, we have

$$\begin{aligned} E'(t) &\leq -[\frac{\alpha}{\beta} - \frac{\alpha}{2\beta} - \frac{1}{2}\int_{\tau_1}^{\tau_2} |k(s)|ds - \frac{\xi}{2}(\tau_2 - \tau_1)]\|\theta_x\|_2^2 - \frac{\xi}{2}\int_0^L \int_{\tau_1}^{\tau_2} z^2(x,1,s,t)dsdx \\ &\leq -\frac{1}{2}[\frac{\alpha}{\beta} - \int_{\tau_1}^{\tau_2} |k(s)|ds - \xi(\tau_2 - \tau_1)]\|\theta_x\|_2^2 - \frac{\xi}{2}\int_0^L \int_{\tau_1}^{\tau_2} z^2(x,1,s,t)dsdx. \end{aligned}$$

Under the above assumptions, we get

$$E'(t) \leq 0$$

which implies that the energy is decreasing.

Integrate the first equation in (4.3) between 0 and $L$ and using boundary conditions, we get

$$\frac{d^2}{dt^2}\int_0^L u(x,t)dx - c\int_0^L u_{xx}(x,t)dx + p\int_0^L \phi_{xx}(x,t)dx + \alpha\int_0^L \theta_x(x,t)dx = 0.$$

Using the second equation in (4.3) and the boundary conditions, we have

$$\begin{aligned}\frac{d^2}{dt^2}\int_0^L u(x,t)dx &= (c-\frac{p^2}{d})\int_0^L u_{xx}(x,t)dx - \alpha\int_0^L \theta_x(x,t)dx \\ &= (c-\frac{p^2}{d})[u_x(L,t)-u_x(0,t)] - \alpha[\theta(L,t)-\theta(0,t)] \\ &= 0.\end{aligned}$$

Then

$$\int_0^L u(x,t)dx = at + b$$

where $a$ and $b$ are constants to be determined by the initial conditions.
We have

$$b = \int_0^L u(x,0)dx = \int_0^L u_0(x)dx,$$

and

$$a = \frac{d}{dt}(at+b) = \frac{d}{dt}\int_0^L u(x,t)dx = \int_0^L u_t(x,t)dx,\ \forall t.$$

In particular when t = 0, we have

$$a = \int_0^L u_t(x,0)dx = \int_0^L u_1(x)dx.$$

Define

$$w(x,t) = u(x,t) - \frac{1}{L}(at+b)$$

we have

$$\begin{aligned}\int_0^L w(x,t)dx &= \int_0^L u(x,t)dx - \frac{1}{L}\int_0^L (at+b)dx \\ &= \int_0^L u(x,t)dx - (at+b) = 0\end{aligned}$$

and

$$\int_0^L w_t(x,t)dx = \int_0^L u_t(x,t)dx - \int_0^L \frac{a}{L}dx = \int_0^L u_t(x,t)dx - a = 0.$$

Then

$$\int_0^L w(x,t)dx = \int_0^L w_t(x,t)dx = 0.$$

It is easy to verify that

$$w_{tt} = u_{tt}\ , w_{tx} = u_{tx}\ , w_{xx} = u_{xx}$$

and

$$\begin{cases} w_x(0,t) = u_x(0,t) \\ w_x(L,t) = u_x(L,t) \end{cases}$$

$$\begin{cases} w(x,0) = u(x,0) - \dfrac{b}{L} = u_0(x) - \dfrac{1}{L}\displaystyle\int_0^L u_0(x)dx \\ w_t(x,0) = u_t(x,0) - \dfrac{a}{L} = u_1(x) - \dfrac{1}{L}\displaystyle\int_0^L u_1(x)dx. \end{cases}$$

Then $u$ and $w$ verify the same equations and the same boundary conditions, but with different initial conditions.
So $(u, \phi, \theta, z)$ and $(w, \phi, \theta, z)$ are solutions of the same problem (4.3) with different initial conditions.
Thus the uniqueness of the solution to problem (4.3) requires that $\int_0^L u_0(x)dx = \int_0^L u_1(x)dx = 0$
with imply that

$$\int_0^L u_t(x,t)dx = 0. \tag{4.7}$$

In order to prove the exponential decay of the energy, we know the Lyapunov functional $\Lambda$ and we prove that it is equivalent to the energy functional $E$. Let

$$\Lambda(t) = N_1 E(t) + F_1(t) + N_2 F_2(t) + F_3(t)$$

such that

$$F_1(t) = \int_0^L u u_t dx,$$

$$F_2(t) = -\int_0^L \theta(\int_0^x u_t(y,t)dy)dx$$

and

$$F_3(t) = \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s\exp(-s\rho)(|k(s)| + \xi)z^2(x,\rho,s,t)dsd\rho dx$$

where $N_1$ and $N_2$ are positive constants to be determined.
Using Young and Poincare inequalities we obtain the following estimates.
**Estimation of $F_1'(t)$ :**

$$\begin{aligned} F_1'(t) &= \int_0^L u_t^2 dx + \int_0^L u u_{tt} dx \\ &= \int_0^L u_t^2 dx + \int_0^L u(cu_{xx} - p\phi_{xx} - \alpha\theta_x)dx \\ &= \int_0^L u_t^2 dx - (c - \frac{p^2}{d})\int_0^L u_x^2 dx - \alpha\int_0^L u\theta_x dx \\ &\leq \int_0^L u_t^2 dx - (c - \frac{p^2}{d})\int_0^L u_x^2 dx + \alpha\varepsilon_0 C_p \int_0^L u_x^2 dx + \alpha C_{\varepsilon_0}\int_0^L \theta_x^2 dx, \end{aligned}$$

where $\varepsilon_0$, $C_{\varepsilon_0}$ are the Young constants and $C_p$ the Poincare constant.
**Estimation of $F_2'(t)$ :**

$$F_2'(t) = -\int_0^L \theta_t(\int_0^x u_t(y,t)dy)dx - \int_0^L \theta(\int_0^x u_t(y,t)dy)dx.$$

Using (4.3) we get

$$\begin{aligned}
F_2'(t) = &-\int_0^L (\theta_{xx} + \int_{\tau_1}^{\tau_2} |k(s)| z_x(x,1,s,t)ds - \beta u_{xt}) \int_0^x u_{tt}(y,t)dydx \\
&- \int_0^L \theta \int_0^x ((c - \frac{p^2}{d})u_{xx} - \alpha\theta_x)dydx \\
= &\int_0^L \theta_x u_t dx + \int_0^L u_t \int_{\tau_1}^{\tau_2} |k(s)| z(x,1,s,t)dsdx - \beta \int_0^L u_t^2 dx - (c - \frac{p^2}{d}) \int_0^L u_x \theta dx \\
&+ \alpha \int_0^L \theta^2 dx.
\end{aligned}$$

Let

$$g(x,t) = \int_{\tau_1}^{\tau_2} |k(s)| z(x,1,s,t)ds$$

by Holder inequality we get

$$|g(x,t)| \leq (\int_{\tau_1}^{\tau_2} |k(s)|^2 ds)^{\frac{1}{2}} (\int_{\tau_1}^{\tau_2} z^2(x,1,s,t)ds)^{\frac{1}{2}}.$$

Then

$$\begin{aligned}
\|g(x,t)\|^2 = \int_0^L |g(x,t)|^2 dx &\leq (\int_{\tau_1}^{\tau_2} |k(s)|^2 ds) \int_0^L (\int_{\tau_1}^{\tau_2} z^2(x,1,s,t)ds)dx \\
&\leq C \int_0^L (\int_{\tau_1}^{\tau_2} z^2(x,1,s,t)ds)dx.
\end{aligned}$$

Using Young inequality, we have

$$\begin{aligned}
F_2'(t) \leq -\beta \int_0^L u_t^2 dx + (c - \frac{p^2}{d})C_{\varepsilon_1} \int_0^L \theta^2 dx + (c - \frac{p^2}{d})\varepsilon_1 \int_0^L u_x^2 dx + C_{\varepsilon_2} \int_0^L \theta_x^2 dx + \varepsilon_2 \int_0^L u_t^2 dx \\
+ \varepsilon_3 \int_0^L u_t^2 dx + C_{\varepsilon_3} C \int_0^L \int_{\tau_1}^{\tau_2} z^2(x,1,s,t)dsdx + \alpha \int_0^L \theta^2 dx.
\end{aligned}$$

Using the Poincare inequality, we get

$$\begin{aligned}
F_2'(t) \leq (\varepsilon_2 + \varepsilon_3 - \beta) \int_0^L u_t^2 dx + [\alpha C_p + C_{\varepsilon_2} + C_p(c - \frac{p^2}{d})C_{\varepsilon_1}] \int_0^L \theta_x^2 dx + (c - \frac{p^2}{d})\varepsilon_1 \int_0^L u_x^2 dx \\
+ C_{\varepsilon_3} C \int_0^L \int_{\tau_1}^{\tau_2} z^2(x,1,s,t)dsdx.
\end{aligned}$$

**Estimation of $F_3'(t)$ :**

$$
\begin{aligned}
F_3'(t) &= 2\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s\exp(-s\rho)(|k(s)|+\xi)zz_t\,dsd\rho dx\\
&= -2\int_0^L\int_0^1\int_{\tau_1}^{\tau_2}\exp(-s\rho)(|k(s)|+\xi)zz_\rho\,dsd\rho dx\\
&= -\int_0^L\int_{\tau_1}^{\tau_2}(|k(s)|+\xi)\int_0^1\exp(-s\rho)\frac{\partial}{\partial\rho}z^2\,d\rho dsdx\\
&= -\int_0^L\int_{\tau_1}^{\tau_2}(|k(s)|+\xi)[\exp(-s)z^2(x,1,s,t)-z^2(x,0,s,t)+s\int_0^1\exp(-s\rho)z^2d\rho]dsdx\\
&\leq M_3\int_0^L\theta_x^2dx-\gamma_0\int_0^L\int_0^1\int_{\tau_1}^{\tau_2}s(|k(s)|+\xi)z^2dsd\rho dx
\end{aligned}
$$

such that $\gamma_0=\exp(-\tau_2)$ and $M_3=\int_{\tau_1}^{\tau_2}|k(s)|ds+\xi(\tau_2-\tau_1)$ .
We conclude from the above and using Poincare inequality that

$$
\begin{aligned}
\Lambda'(t) &= N_1E'(t)+F_1'(t)+N_2F_2'(t)+F_3'(t)\\
&\leq -(N_2m_1-1)\int_0^L u_t^2dx-(N_1m_2-m_3)\int_0^L\theta_x^2dx-(m_0-\delta_1)\int_0^L u_x^2dx\\
&-(\frac{N_1\xi}{2}-N_2M_2)\int_0^L\int_{\tau_1}^{\tau_2}z^2(x,1,s,t)dsdx-\gamma_0\int_0^L\int_0^1\int_{\tau_1}^{\tau_2}s(|k(s)|+\xi)z^2dsd\rho dx
\end{aligned}
$$

such that

$$
\begin{aligned}
m_0 &= (c-\frac{p^2}{d})-\alpha\varepsilon_0C_p,\\
\delta_1 &= (c-\frac{p^2}{d})\varepsilon_1,\\
m_1 &= \beta-\varepsilon_2-\varepsilon_3,\\
M_1 &= [\alpha+(c-\frac{p^2}{d})C_pC_{\varepsilon_1}+C_{\varepsilon_2}],\\
m_2 &= \frac{\alpha}{2\beta}-\frac{1}{2}M_3,\\
M_2 &= C_{\varepsilon_3}C,\\
m_3 &= M_1N_2+\alpha C_{\varepsilon_0}+M_3.
\end{aligned}
$$

We choose $N_2$ large enough, $\varepsilon_2$ and $\varepsilon_3$ small enough such that

$$\gamma_1=(N_2m_1-1)>0.$$

We choose $N_1$ large enough such that

$$\gamma_2=(N_1m_2-m_3)>0$$

and

$$\gamma_4=(\frac{N_1\xi}{2}-N_2M_2)>0.$$

We choose $\varepsilon_0$ and $\delta_1$ small enough such that

$$\gamma_3 = (m_0 - \delta_1) > 0.$$

Then

$$\Lambda'(t) \leq -\gamma_1 \int_0^L u_t^2 dx - \gamma_2 \int_0^L \theta_x^2 dx - \gamma_3 \int_0^L u_x^2 dx - \gamma_0 \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s(|k(s)| + \xi) z^2 ds d\rho dx.$$

Using Poincare inequality we get

$$\Lambda'(t) \leq -C' E(t). \tag{4.8}$$

We have

$$|\Lambda(t) - N_1 E(t)| \leq |F_1(t)| + N_2 |F_2(t)| + |F_3(t)|$$
$$\leq \int_0^L |uu_t| dx + N_2 \int_0^L \theta(\int_0^x u_t(s,t) ds) dx + \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s \exp(-s\rho)(|k(s)| + \xi) z^2(x,\rho,s,t) ds d\rho dx.$$

Using Young and Poincare inequalities, we obtain

$$\begin{aligned} |\Lambda(t) - N_1 E(t)| &\leq C_1 \int_0^L (u_x^2 + u_t^2 + \theta^2) dx + \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s(|k(s)| + \xi) z^2(x,\rho,s,t) ds d\rho dx \\ &\leq C_2 E(t). \end{aligned}$$

Therefore

$$\Lambda(t) \backsim E(t). \tag{4.9}$$

Hence, we conclude from (4.8) and (4.9) that

$$\Lambda'(t) \leq -C'' \Lambda(t).$$

Then

$$E(t) \leq C_3 \exp(-C_4 t).$$

# Conclusion

In this master's thesis, we started by studying the mathematical analysis (existence, uniqueness) of a 3D piezoelectric evolution system with a damping term using semigroup approach. We then investigated the stability of this system without delay and with constant time delay using the energy method.
Next, we examined a 3D thermopiezoelectric evolution system. We analyzed the existence and uniqueness of this system using the semigroup method. Subsequently, we explored the exponential stability of this system using the observability inequality.
Finally, we considered the stability of a 1D thermopiezoelectric evolution system with a time delay acted on the temperature equation. We demonstrated that the energy decreases exponentially using the energy method.

We can suggest **some perspectives** for this work :

⧫ Energy decay of thermopiezoelectric thin structures (beam, plate, shallow, shell).

⧫ Energy decay of thermoviscoelastic systems.

⧫ Energy decay of piezoelectric systems with memory (finite - infinite) and delay.

⧫ Energy decay of thermopiezoelectric systems with memory and delay.

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**Abstract**

The objective of this master's thesis is to study the mathematical analysis of selected mathematical systems originating from mechanics. Firstly, we focus on the analysis of a piezoelectric system with damping, considering both non-delayed and constant time-delay cases. Next, we examine a 3D thermopiezoelectric system, studying its well-posedness and stability through the use of an observability inequality. Lastly, we investigate the stability of a 1D thermopiezoelectric evolution system with a time delay applied to the temperature. We demonstrate the exponential decay of energy using the energy method.

**Keywords :**
semigroups method , energy method, exponential decay, observability inequality, thermopizoelectric, time delay.

**Résumé**

L'objectif de ce mémoire de master est d'étudier l'analyse mathématique de certains systèmes mathématiques issus de la mécanique. Tout d'abord, nous nous concentrons sur l'analyse d'un système piézoélectrique avec amortissement, en considérant à la fois les cas sans retard et avec un retard temporel constant. Ensuite, nous examinons un système thermopiézoélectrique en 3D, étudiant l'existence et l'unicité du système à l'aide de la méthode des semi-groupes et sa stabilité grâce à l'utilisation d'une inégalité d'observabilité. Enfin, nous étudions la stabilité d'un système d'évolution thermopiézoélectrique 1D avec un retard temporel appliqué à la température. Nous démontrons la décroissance exponentielle de l'énergie en utilisant la méthode de l'énergie.



**Mots clés :**
méthode des semi-groupes, méthode de l'énergie, décroissance exponentielle, inégalité d'observabilité, thermopiézoélectrique, retard temporel



***الملخص:***

الهدف من رسالة الماستر هذه هو دراسة التحليل الرياضي لنظم رياضية محددة مصدرها الميكانيكا.

في البداية, نركز على تحليل نظام كهروضغطي مع تخميد, مع النظر في الحالات بدون تأخير وبتأخير زمني ثابت. بعد ذلك, ندرس نظام كهروضغطي حراري ثلاثي الأبعاد, ندرس وجود ووحدانية الحل باستخدام طريقة أنصاف الزمر, واستقراره باستعمال متباينه قابلية المراقبة.

أخيرا, نبحث في استقرار نظام تطور الحرارة -الكهروضغطي في البعد الواحد مع تأخير زمني يطبق على درجة الحرارة.

نبرهن تناقص الطاقة بشكل تسارع أسي باستخدام طريقة الطاقة.



***الكلمات المفتاحية:***

طريقة أنصاف الزمر, طريقة الطاقة, تناقص أسي, متباينة المراقبة, كهروضغطي حراري, تأخير زمني.