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Option:Modelisations and Numerical Analysis

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On the study of local existence and blow up result for vescoelastic wave equation with variable exponents

In front of the jury composed of :

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Dedications

It's the fun of endings, I come gladly When I receive it, I say:"My Lord has spoken the truth"

Praise be to God for the joy of achievement Thank God in the beginning and in the end......

To my **"father"**, who illuminated my path, my path, and my role model in every step I took.To my pure angel and my strength after God, my first and eternal support, **"my mother"**And to those about whom it was said: "We will support you through your brother."My brothers**"Mouhmed said ,Hicham,Ahmed wassim"** and sisters **"soundous,wajdan,Rania"** who have always and forever stood with me and supported me in my educational career,

I dedicate this humble work to you all, which is the result of my efforts, and God is the Grantor of success.

Acknowledgement

Despite the fatigue on the way to the dream... upon arrival, you will forget that fatigue

Praise be to God who granted me graduation He rewarded me with the best reward for my patience and diligence He honored me with this great joy after years of toil, hard work and perseverance Praise be to God, then With my mother and father and everyone who supported me from near or far. Today I am the first graduate in my family ,On this occasion, I thank all my professors at the university for their support for me,I thank my supervisor, **Sadok Otmani**, who guided me step by step to complete this work until it comes into being. God reward you. , I thank my friends **Rim,khawla ,khadija ,hadjira,Rabab,Chaima ....** in university residence for standing with me in difficult times, I thank everyone who prayed for me, and I pray to God for more success.

"Fardous Debba"

WHOEVER SAID " Iam hers "Got it.

And Iam hers,and if she refuses against her will,I will bring her to her Brought it.

ملخص

نسلط الضوء في موضوعنا حول دراسة وجود وانفجار حلول المعادلة دات حد المرونة واللزوجة وذات الأسس المتغيرة والتي تعتبر من المعادلات التفاضلية الجزئية، حيث في الفصل الأول تطرقنا الى بعض التعريفات والمبرهنات والنظريات الأساسية والتي تستعمل في الفصل الثاني والثالت، أما في الفصل الثاني فقمنا بدراسة وجود الحل المحلي بطريقة غاليركين ومن ثم وحدانية الحل، وفي الأخير أثبتنا أن الحل يؤول الى المالانهاية عند ما يبلغ الزمن لحظة زمنية معينة.

كلمات مفتاحية: فضاءات سوبولوف، انفجار ،الأسس المتغيرة، الوجود المحلي .

Abstract

Our study focuses on the existence and blow-up of solutions to the partial differential equation with elastic and viscous terms and variable exponents, In the first chapter, we covered some essential definitions, theorems, and inequalities that will be needed in the second and third chapters, as for the second chapter we studied the existence of the solution using the faedo-Galerkin method and proved its uniqueness through various methods. In the final chapter, we examined the energy associated with this equation and found that when the time reaches a critical point, a blow-up occurs.

Key words : Sobolev space, Blow-up , variable exponents, local existence.

Notation

Ω : bounded domain in $\mathbb{R}^2$

∇u : gradient of u.

Δu : Laplacien of u.

D (Ω) : space of infinity differentiable functions with compact support in Ω.

$D'(\Omega)$: distribution space.

$C^k(\Omega)$: space of functions k-times continuously differentiable in Ω.

$L^p(\Omega)$: space of functions p-th power integrated on measure of dx.

$\|f\|_p = \left(\int_\Omega |f|^p\right)^{\frac{1}{p}}$.

$W^{1,p} = \{u \in L^p(\Omega), \nabla u \in L^p(\Omega)\}$.

H: Hilbert space.

$H_0^1(\Omega) = W_0^{1,2}(\Omega)$.

$\|v\|_{H^1(\Omega)} = \left(\int_\Omega \sum_{i=1}^n \|v^i(x)\|^2 dx + \int_\Omega \sum_{i=1,j}^n \|\partial_j v^i(x)\|^2 dx\right)^{1/2}$.

Contents

Introduction

In recent decades, viscoelastic wave equations with acoustic boundary conditions have garnered significant attention from many researchers. It is well-known that viscoelastic materials exhibit memory effects, where their mechanical response is influenced by the history of the materials themselves. Mathematically, these damping effects are modeled using integro-differential operators. Consequently, differential equations that incorporate memory effects have become a vibrant area of research in recent years. We can mention some works [7, 8, 10, 13, 27, 28, 2].

In this work see [41] , we show the details of local existence and the blow up result of viscoelastic wave equation:

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + a|u_t|^{m(x)-2}u_t = b|u|^{p(x)-2}u, \text{ in } \Omega \times (0,T) \tag{1}$$

with the boundary condition:

$$u = \frac{\partial u}{\partial \nu} = 0, \text{ in } \partial\Omega \times (0,\infty) \tag{2}$$

and the initial condition:

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \text{ in } \Omega. \tag{3}$$

where $0 < T < \infty$ and Ω is a bounded domain in $\mathbb{R}^n, n \geq 2$, with a smooth boundary $\partial\Omega = \Gamma$, ν is the unit outer normal to $\partial\Omega$. a, b are two no-negative constants. g is a positive nonincreasing function defined on $\mathbb{R}_+$. $(u_0, u_1,)$ are the initial data belonging to a suitable function space. The exponents $p(.)$ and $m(.)$ is given measurable functions on Ω satisfying:

$$2 \leq r_1 \leq r(x) \leq r_2 < \infty, \tag{4}$$

with

$$r_1 = ess \inf_{x\in\Omega} r(x), \qquad r_2 = ess \sup_{x\in\Omega} r(x),$$

we also assume that $r(.)$ be log-continuous in Ω such that

$$\forall (x,y) \in \Omega^2, \ |r(x) - r(y)| \leq -\frac{C}{log|x-y|}, \quad \text{with } |x-y| < \delta, \tag{5}$$

where $C > 0, \ 0 < \delta < \frac{1}{2}$. In [30] Messaoudi studied the following equation

$$u_{tt} + \Delta^2 u + |u_t|^{m-2}u_t = |u|^{p-2}u, \qquad \text{in } \Omega \times (0,T) \tag{6}$$

with the boundary condition:

$$u = \frac{\partial u}{\partial \nu} = 0, \text{ in } \partial\Omega \times (0, \infty) \tag{7}$$

and the initial condition:

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \text{ in } \Omega. \tag{8}$$

Messaoudi established an existence result and showed that the solution continues to exist globally if $m \geq p$, and blows up in finite time if $m < p$ and the initial energy is negative. Santos and Junior in [38] studied the following system:

$$\begin{cases} u_{tt} + \Delta^2 u = 0, & \text{in } \Omega \times (0, \infty), \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \Gamma_0 \times (0, \infty), \\ -u + \int_0^t g_1(t-s)\beta_1 u(s)ds = 0, & \text{on } \Gamma_1 \times (0, \infty), \\ \frac{\partial u}{\partial \nu} + \int_0^t g_2(t-s)\beta_2 u(s)ds = 0, & \text{on } \Gamma_2 \times (0, \infty), \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), & \text{in } \Omega, \end{cases} \tag{9}$$

where

$$\beta_1 u = \Delta u + (1-\mu)B_1 u \quad \text{and} \quad \beta_2 u = \frac{\partial \Delta u}{\partial \mu} + (1-\mu)\frac{\partial B_2 u}{\partial \eta}$$

with

$$B_1 u = 2\nu_1\nu_2 u_{xy} - \nu_1^2 u_{yy} - \nu_2^2 u_{xx} \quad \text{and} \quad B_2 u = (\nu_1 - \nu_2)\, u_{xy} + \nu_1\nu_2\,(u_{yy} - u_{xx})\,.$$

Liu and Sun in [27] considered the equation

$$u_{tt} - \Delta u + \alpha(t)\int_0^t g(t-s)\Delta u(s)ds = 0, \qquad \text{in } \Omega \times (0, \infty)$$

with a homogeneous Dirichlet condition on a portion of the boundary and acoustic boundary conditions on the rest of the boundary. The authors established a general decay result, which depends on the behavior of both α and g, by using the perturbed energy functional technique. Cavalcanti et al. [11] considered the equation

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)ds - \gamma\Delta u_t = 0, \qquad \text{in } \Omega \times (0, \infty)$$

subject to Dirichlet boundary conditions. Taking $0 \leq \rho \leq \frac{2}{n-2}$ if $n \geq 3$ or $\rho > 0$ if $n = 1, 2$ and assuming that the kernel g decays exponentially, the authors obtained global existence of solutions in the case $\gamma \geq 0$. They also proved that the energy decays exponentially when $\gamma > 0$. Messaoudi et al in [31] showed the existence and blow-up of solutions for the nonlinear damped wave equation with variable exponents: :

$$u_{tt} - \Delta u + a|u_t|^{m(.)-2}u_t = b|u|^{p(.)-2}u, \qquad \in \Omega, t \in [0, T),$$

with the boundary condition

$$u(x, t) = 0, \qquad \text{on } \partial\Omega \times (0, T)$$

and the initial condition

$$u(x,0) = u_0, \quad u_t(x,0) = u_1 \qquad \text{in } \Omega,$$

where a, b are positive constants and the exponents $m()$ and $p()$ are given measurable functions. They proved that the solution with negative initial energy blows up in finite time. Messaoudi and Talahmeh in [32] studied the blow-up in solutions of a quasilinear wave equation with variable exponent nonlinearities:

$$u_{tt} - \operatorname{div}\left(|\nabla u|^{r(\cdot)-2}\nabla u\right) + a\,|u_t|^{m(\cdot)-2}\,u_t = b|u|^{p(\cdot)-2}u \quad \text{in} \quad \Omega \times (0,T).$$

They obtained the blow-up result for the solutions with negative initial energy and for certain solutions with positive energy. However, to our knowledge, there is no blow-up result of solutions for the viscoelastic hyperbolic equations with variable exponents. We prove a finite time blow-up result of solutions with positive initial energy for the problem (1)-(3).

For $p(\cdot)$ and $m(\cdot)$ are constants, Messaoudi in [33] considered the following viscoelastic wave equation with a nonlinear damping:

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + |u_t|^{m-2}\,u_t = |u|^{p-2}u, \quad \text{in} \quad \Omega \times (0,\infty).$$

He showed a blow-up result of solutions with negative initial energy and $p > m$. The body of this thesis is organized as follows:

- In chapter 1, we present some notations and material needed for our work.
- In chapter 2, we show the local existence of weak solutions by using Faedo-Galerkin method.
- In chapter 3, we state and prove a blow-up result of solutions for the problem (1)-(3) when the initial energy lies in positive as well as nonpositive.

Chapter 1

Preliminary

In this chapter, we give some basic definitions, theorems, lemmas, and inequalities that will be useful in the work see ([9], [14], [17], [19], [20], [36], [41]).

1.1 Banach Spaces and Banach fixed-point theorem

We first review some basic facts from calculus in the most important class of linear spaces the "Banach spaces".

Definition 1.1.1 *A Banach space is a complete normed linear space X. Its dual space X' is the linear space of all continuous linear functional $f : X \longrightarrow \mathbb{R}$.*

Proposition 1.1.2 *X' equipped with the norm*

$$\boxed{\|f\|_{X'} = \sup \{|f(u)| : \|u\|_X \leq 1\},}$$

is also a Banach space.

Definition 1.1.3 *Let X be a Banach space, and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in X. Then u_n converges strongly to u in X if and only if*

$$\boxed{\lim_{n\longrightarrow\infty} \|u_n - u\|_X = 0,}$$

and this is denoted by $u_n \longrightarrow u$, or $\lim\limits_{n\longrightarrow\infty} u_n = u$.

Definition 1.1.4 *A sequence (u_n) in X is weakly convergent to u if and only if*

$$\boxed{\lim_{n\longrightarrow\infty} f(u_n) = f(u),}$$

for every $f \in X'$ and this is denoted by $\boxed{u_n \rightharpoonup u.}$

1.1.1 Banach fixed-point theorem

Definition 1.1.5 *Let (X, d) be a Banach space. Then a map $T : X \to X$ is called a contraction mapping on X if there exists $q \in [0, 1)$ such that*

$$\boxed{\|T(x) - T(y)\|_X \leqslant q\|x - y\|_X,}$$

for all x, y in X.

Theorem 1.1.6 *Let (X, d) be a non-empty complete metric space with a contraction mapping $T : X \to X$. Then T admits a unique fixed-point x^* in X (i.e. $T(x^*) = x^*$). Furthermore, x^* can be found as follows:*
start with an arbitrary element x^0in X and define a sequence $\{x_n\}$ by $x_n = T(x_{n-1})$. Then $x_n \longrightarrow x^*$.

The Mean Value Theorem
In the next two theorems a and b are real numbers such that $a < b$.

Theorem 1.1.7 ***(Rolle's theorem)***
Suppose that $f \in C([a, b], \mathbb{R})$ is differentiable on (a, b). If $f(a) = f(b)$, then there is some $\xi \in (a, b)$ such that $f'(\xi) = 0$.

Theorem 1.1.8 ***(mean value theorem)***
if $f \in C([a, b], \mathbb{R})$ is differentiable on (a, b), then there is some $\xi \in$ (a,b) such that

$$\boxed{f(b) = f(a) + f'(\xi)(b - a).}$$

1.2 Hilbert spaces

1.2.1 Definitions and Elementary Properties

Definition 1.2.1 *Let H be a vector space. A scalar product (u, v) is a bilinear form on $H \times H$ with values in $\mathbb{R}$ (i.e., a map from $H \times H$ to R that is linear in both variables) such that*
(u, v) = (v, u) $\forall$ u, v $\in$ H (symmetry),
(u, u)$\geq$ 0 $\forall$ u $\in$ H (positive),
(u, u) $\neq$ 0 $\forall$ u $\neq 0$ (definite).
Let us recall that a scalar product satisfies the Cauchy Schwarz inequality

$$\boxed{|(u, v)| \leq (u, u)^{1/2}(v, v)^{1/2}, \qquad \forall u, v \in H.}$$

[It is sometimes useful to keep in mind that the proof of the Cauchy Schwarz in-equality does not require the assumption (u, u) $\neq$ 0 $\forall$u $\neq$ 0.] It follows from the Cauchy Schwarz inequality that the quantity

$$\boxed{\|u\| = (u, u)^{1/2}}$$

is norm arising from scalar products. Indeed, we have

$$\|u+v\|^2 = (u+v,u+v) = \|u\|^2 + (u,v) + (v,u) + \|v\|^2 \leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2,$$

and thus $\|u+v\| \leq \|u\| + \|v\|$ *let us recall the classical parallelogram law:*

$$\left\|\frac{a+b}{2}\right\|^2 + \left\|\frac{a-b}{2}\right\|^2 = \frac{1}{2}(\|a\|^2 + \|b\|^2), \qquad \forall a,b \in H.$$

1.3 Functional spaces

1.3.1 The L^p spaces

Definition 1.3.1 *Let* $1 \leq p < \infty$, *and let* Ω *be an open domain in* $\mathbb{R}^n$, $n \in \mathbb{N}^*$. *Define the standard Lebesgue space* $L^p(\Omega)$, *by*

$$L^p(\Omega) = \left\{ f : \Omega \longrightarrow \mathbb{R} \text{ is measurable; } \int_\Omega |f(x)|^p \ dx < \infty \right\}.$$

with the norm

$$\|u\|_{L^p} = [\int_\Omega |f(x)|^p dx]^{\frac{1}{p}}. \tag{1.1}$$

Definition 1.3.2 *We define* $L^\infty(\Omega)$ *by:*

$$L^\infty(\Omega) = \{f : \Omega \longrightarrow \mathbb{R} f \text{ is measurable and exists the constant } C \text{ such that } \|f(x)\| \leq c \text{ a.e on } \Omega\}$$

equipped with the norm

$$\|f\|_{L^\infty} = \inf\{C; \|f(x)\| \leqslant C \ \text{a.e on } \Omega\}$$

Theorem 1.3.3 $\left(L^p(\Omega), \|.\|_p\right), (L^\infty(\Omega), \|.\|_\infty)$ *are a Banach spaces.*

Remark 1.3.4 *In particularly, when* $p = 2$, $L^2(\Omega)$ *equipped with the inner product*

$$(f,\ g)_\Omega = \int_\Omega f(x).g(x)\,dx,$$

is a Hilbert space.

1.3.2 Sobolev spaces

Definition 1.3.5 *For $k \in \mathbb{N}$ and $1 \leq p \leq \infty$. We define the Sobolev space*

$$\boxed{W^{k,p}(\Omega) = \{u \in L^p(\Omega), D^\alpha u \in L^p(\Omega) \forall \alpha \in \mathbb{N}^n \text{ with } |\alpha| \leq k\},}$$

equipped with the norm

$$\boxed{\|u\|_{k,p} = \Big(\sum_{\|\alpha\| \leq k} \|D^\alpha u\|_p^p \Big)^{\frac{1}{p}}, 1 \leq p < \infty}$$

$$\boxed{\|u\|_{k,\infty} = \max \|D^\alpha u\|_\infty}$$

where $D^\alpha u$ is the α-th weak derivative of u which is defined as

$$\boxed{\int_\Omega u(x) D^\alpha \varphi(x) = -1^{\|\alpha\|} \int_\Omega \upsilon(x)\varphi(x), \forall \varphi \in C_c^\infty(\Omega),}$$

$|\alpha| = \alpha_1 + ... + \alpha_n$ *and*

$$\boxed{\upsilon = D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial^{\alpha_1} x_1 ... \partial^{\alpha_n} x_n}}$$

The space $W^{k,2}(\Omega)$ is denoted by $H^k(\Omega)$, which is a Hilbert space with respect to the inner product

$$\boxed{(u, \upsilon)_{H^k} = \int_\Omega \sum_{\alpha \leq k} D^\alpha u(x) D^\alpha \upsilon(x) dx, \forall u, \upsilon \in H^k(\Omega)}$$

Definition 1.3.6 *[20] We denote by $W_0^{k,p}(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$*

The Sobolev Space $W^{1,p}$(I)

[9]Let I = (a, b) be an open interval, possibly unbounded, and let p ∈ ℝ with $1 \leq p \leq \infty$.

Definition 1.3.7 *The Sobolev space $W^{1,p}(I)^1$ is defined to be*

$$\boxed{W^{1,p}(I) = \left\{ u \in L^p(I); \exists \varphi \in L^p(I) \text{ such that } \int_I u\varphi' = -\int_I u\varphi \qquad \forall \varphi \in C_c^1(I) \right\}}$$

We set

$$\boxed{H^1(I) = W^{1,2}(I)}.$$

Notation 1.3.8 *The space $W^{1,p}(I)$ is equipped with the norm*

$$\boxed{\|u\|_{W^{1,p}} = \|u\|_{L^p} + \|u'\|_{L^p}},$$

or sometimes, if 1 <p< ∞, with the equivalent norm $(\|u\|_{L^p}^p + \|u'\|_{L^p}^p)^{1/p}$. *The space* H^1 *is equipped with the scalar product*

$$(u,v)_{H^1} = (u,v)_{L^2} + (u',v')_{L^2} = \int_a^b (uv + u'v')$$

and with the associated norm

$$\|u\|_{H^1}(= \|u\|_{L^2}^2 + \|u'\|_{L^2}^2)^{1/2}$$

The Sobolev Spaces $W^{m,p}$

Definition 1.3.9 *Given an integer* $m \geq 2$ *and a real number* $1 \leq p \leq \infty$ *we define by induction the space*

$$W^{m,p}(I) = u \in W^{m-1,p}(I); u' \in W^{m-1,p}(I).$$

We also set

$$H^m(I) = W^{m,2}(I),$$

It is easily shown that $u \in W^{m,p}(I)$ *if and only if there exist m functions* $g_1, g_2, \ldots$, $g_m \in L^p(I)$ *such that*

$$\int_I u\, D^j\varphi = (-1)^j \int_I g_j\varphi \qquad \forall\varphi \in C_c^\infty(I), \qquad \forall j = 1, 2...m,$$

where $D^j\varphi$ *denotes the j th derivative of* φ *when* $u \in W^{m,p}(I)$ *we may thus consider the successive derivatives of* u : $u' = g_1$, $(u')' = g_2, \ldots$, *up to order m. They are denoted by* Du, D^2u, . . . , $D^m u$. *The space* $W^{m,p}(I)$ *is equipped with the norm*

$$\|u\|_{W^{m,p}(I)} = \|u\|_p + \sum_{\alpha=1}^{m} \|D^\alpha u\|_p,$$

and the space $H^m(I)$ *is equipped with the scalar product*

$$(u,v)_{H^m} = (u,v)_{L^2} + \sum_{\alpha=1}^{m} (D^\alpha u, D^\alpha v)_{L^2} = \int_I uv + \sum_{\alpha=1}^{m} \int_I D^\alpha u\, D^\alpha v.$$

The Space $W_0^{1,p}$

Definition 1.3.10 *Given* $1 \leq p < \infty$, *denote by* $W_0^{1,p}(I)$ *the closure of* $C_c^1(I)$ *in* $W^{1,p}(I)$[10] . *Set*

$$H_0^1(I) = W_0^{1,2}(I).$$

space $W_0^{1,p}(I)$ *is equipped with the norm of* $W^{1,p}(I)$, *and the space*H_0^1 *is equipped with the scalar product of* H^1.
The space $W_0^{1,p}(I)$ *is a separable Banach space. Moreover, it is reflexive for p > 1.* H_0^1 *The space is a separable Hilbert space.*

Theorem 1.3.11 . *Let* $u \in W^{1,p}(I)$. *Then* $u \in W_0^{1,p}(I)$ *if and only if* $u = 0$ *on* ∂I.

1.4 Some inequalities

Theorem 1.4.1 ***(Cauchy-Schwarz inequality)*** *Let u, $v \in L^2(\Omega)$ and, then $uv \in L^1(\Omega)$ and*

$$\boxed{\|uv\|_1 \leq \|u\|_2 \|v\|_2,}$$

Theorem 1.4.2 ***(Hölder's inequality)*** *Let $f \in L^p$ and $g \in L^{p'}$ with $1 \leqslant p \leqslant \infty$, so, $f.g \in L^1$ and*

$$\boxed{\int |fg| \leq \|f\|_{L^p} \|g\|_{L^{p'}}.}$$

Theorem 1.4.3 ***(Young's inequality)*** *Let $1 \leq p \leq \infty$. then a, b > 0, Then for any $\epsilon > 0$, we have*

$$\boxed{ab \leq \epsilon a^p + C_\epsilon b^{p'}}$$

where $C_\epsilon = \dfrac{1}{p'(\epsilon p)^{\frac{p'}{p}}}$. *For $p = p' = 2$, we have*

$$\boxed{ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}.}$$

1.4.1 Some results about Sobolev spaces

In this Section, we list a few pertinent qualities that Sobolev space-related functions benefit from without providing any supporting evidence

Theorem 1.4.4 ***(Trace theorem)*** *Let Ω be a bounded open set with Lipchitz continuous boundary and let $s > 1/2$.*
1.There exists a unique linear continuous map γ_0: $H^s(\Omega) \to H^{s-1/2}(\partial\Omega)$ such that $\gamma_0 v = v|_{\partial\Omega}$ for each $v \in H^s(\Omega) \cap C^0(\overline{\Omega})$

2.There exists a linear continuous map $\mathbf{R}_0 : H^{s-1/2}(\partial\Omega) \to H^s(\Omega)$such that $\gamma_0 \circ R_0\phi = \phi$ for each $\phi \in H^{s-1/2}(\partial\Omega)$. Analogous results also hold true if we consider the trace $\gamma_{\sum}$ over a Lipschitz continuous subset $\sum$ of the boundary $(\partial\Omega)$

The so-called poincaré inequality is a crucial finding that will be widely applied in the sequel.

Theorem 1.4.5 ***(Poincaré inequality)***
We suppose that I is bounded .
Then there exists a constant C (depends on $|I|$) such that

$$\boxed{\|u\|_{W^{1,p}} \leq C\|u'\|_{L^p}, \qquad \forall u \in W_0^{1,p}(I)}$$

In other words ,on $W_0^{1,p}(I)$the quantity $\|u'\|_{L^p}$ is a norm equivalent to the norm of $W^{1,p}$.

Lemma 1.4.6 ***(Sobolev poincaré inequality)*** *Let q be a number with*

$$2 \leq q < \infty, (n = 1, 2), 2 \leq q \leq \frac{2n}{n-2}(n \geq 3),$$

then there exists a constant $C_s = C_s(\Omega, q)$ such that

$$\boxed{\|u\|_q \leq c\|\nabla u\|_2, \ \ for \ \ u \in H_0^1(\Omega)}$$

Theorem 1.4.7 ***(Sobolev embedding theorem)*** *Assume that Ω is a (bounded or unbounded) open set of $\mathbb{R}^d$ with a Lipschitz continuous boundary. Then the following continuous embedding hold:*
1.If $1 \leq p < d, then\, W^{s,p}(\Omega) \subset L^{P*}$ with $P = dp/(d - sp)$.*
2.If $sp = d, then\, W^{s,p}(\Omega) \subset L^q$ for any q such that $p \leq q < \infty$
3. If $sp > d, then\, W^{s,p}(\Omega) \subset C^0(\overline{\Omega})$

Lemma 1.4.8 ***(Korn's inequality)*** *Let Ω be an open, connected domain in n-dimensional Euclidean space $\mathbb{R}^n$, $n \geq 2$. Let $H^1(\Omega)$ be the Sobolev space of all vector fields $v = (v^1...v^n)$on Ω that, along with their (first) weak derivatives, lie in the Lebesgue space $L^1(\Omega)$.*

Then there is a constant $C \geq 0$, known as the Korn constant of Ω, such that, for all $v \in H^1(\Omega)$

$$\boxed{\|v\|^2_{H^1(\Omega)} \leq C \int_\Omega \sum_{i,j=1}^{n} \left(\|v^i(x)\|^2 + \|(e_{ij}v)(x)\|^2\right) dx}$$

where e denotes the symmetrized gradient given by

$$\boxed{e_{ij}v = \frac{1}{2}(\partial_i v^j + \partial_j v^i).}$$

Lemma 1.4.9 ***([4] Gronwall inequality)*** *Let $C > 0$,$u(t)$ and $y(t)$ be continuous non negative functions defined for $0 \leq t < \infty$ satisfying the inequality*

$$\boxed{u(t) \leq C + \int_0^t u(s)y(s)ds, \qquad 0 \leq t < \infty}$$

then

$$\boxed{u(t) \leq Cexp\Big(\int_0^t y(s)ds\Big), \qquad 0 \leq t < \infty}$$

1.4.2 Green's formula

Proposition 1.4.10 *Let Ω be an open subset of $\mathbb{R}^d$, with a Lipschitz boundary. Then for all u, $v \in H^1(\Omega)$, we have*

$$\boxed{\int_\Omega (\frac{\partial u}{\partial x_i} v + \frac{\partial v}{\partial x_i} u)dx = \int_{\partial\Omega} \gamma_0(u)\gamma_0(v)\eta_i ds, \qquad i = 1, ..., d.}$$

where η_i is the i-th component of the outward normal vector η

1.5 Logarithmic Hölder Continuity

In this section we introduce the most important condition on the exponent in the study of variable exponent spaces, the log-Hölder continuity condition.

Definition 1.5.1 *We say that the function $\alpha : \Omega \to R$ is locally log-Hölder continuous on Ω if there exists $C_1 > 0$ such that*

$$|\alpha(x) - \alpha(y)| \leq \frac{C_1}{log(e + 1/|x+y|)},$$

for all x,y $\in \Omega$ we say that α satisfies the log-Hölder decay condition if there exist $\alpha_\infty \in \mathbb{R}$ and constant $C_2 > 0$ such that

$$|\alpha(x) - \alpha_\infty| \leq \frac{C_2}{log(e + |x|)},$$

for all $x \in \Omega$ we say that α is globally log-Hölder continuous in Ω if it is locally log-Hölder continuous and satisfies the log-Hölder decay condition.
The constant C_1 and C_2 are called the local log-Hölder constant and the log-Hölder decay constant, respectively. The maximum $\max\{C_1, C_2\}$ is just called the log-Hölder constant of α.

1.6 Spaces with variable exponents

In this section,we provide some preliminary facts about Lebesgue space with variable exponent. Let $p : \Omega \longrightarrow [1, \infty]$ be a measurable function. we introduce the Lebesgue space with a variable exponent $p(.)$.

1.6.1 $L^{p(.)}, W^{1,p(.)}$ spaces

We define the space

$$C^+(\overline{\Omega}) = \{ \text{ continuous function } p(.) : \overline{\Omega} \to \mathbb{R}_+ \text{ such that } 2 < p^- < p^+ < \infty\}$$

where

$$p^- = min_{x\in\overline{\Omega}} p(x) \text{ and } p^+ = max_{x\in\overline{\Omega}} p(x).$$

We define the Lebesgue space with variable exponent

$$L^{p(.)} = \left\{ u : \Omega \to \mathbb{R} \text{ measurable } : \int_\Omega |u(x)|^{p(x)} dx \right\},$$

endowed with Luxembourg norm :

$$\|u\|_{p(.)} = \|u\|_{L^{p(.)}} = inf\left\{ \epsilon > 0, \int_\Omega | \frac{u(x)}{\epsilon} | dx \leq 1 \right\}.$$

The space $(L^{p(.)}(\Omega), \|.\|_{p(.)})$ is a reflexive Banach space, uniformly convex and its dual space is isomorphic to $(L^{q(.)}(\Omega), \|.\|_{q(x)})$ where

$$\boxed{\frac{1}{p(x)} + \frac{1}{q(x)} = 1,}$$

and

$$\boxed{W^{1,p(x)}(\Omega) = \Big\{u \in L^{p(x)}(\Omega), |\nabla u| \in L^{p(x)}(\Omega)\Big\},}$$

with the norm

$$\boxed{\|u\| = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}, u \in W^{1,p(x)}(\Omega).}$$

Remark 1.6.1 *We denote by $W_0^{1,p(x)}(\Omega)$ the closure of C_0^∞ in $W^{1,p(x)}(\Omega)$.*

1.6.2 $L^p(0,T;X)$*spaces*

Definition 1.6.2 *Let X be a Banach space, denote by $L^p(0,T;X)$ the space of measurable functions*

$$f :]0,T[\longrightarrow X$$
$$t \longrightarrow f(t),$$

such that

$$\boxed{\int_0^T (\|f(t)\|_X^p)^{\frac{1}{p}} dt = \|f\|_{L^p(0,T;X)} < \infty.}$$

If $p = \infty$

$$\boxed{\|f\|_{L^\infty(0,T;X)} = \sup_{t \in]0,T[} \|f(t)\|_X,}$$

Theorem 1.6.3 *The space $L^p(0,T;X)$ is a Banach space.*

Lemma 1.6.4 *Let $f \in L^p(0,T;X)$ and $\frac{\partial f}{\partial t} \in L^p(0,T;X), (1 \le p \le \infty)$ then, the function f is continuous from [0,T] to X.i.e.$f \in C^1(0,T;X)$*

1.7 Results in spaces with exponents variable

Lemma 1.7.1 *[22] If p is a measurable function on Ω satisfying (4), then the embedding $H_0^1(\Omega) \hookrightarrow L^{P(.)}(\Omega)$ is continuous and compact . Therefore, there exist positive constant B satisfying*

$$\boxed{\|u\|_{p(.)} \le B\|\nabla u\|_{H_0^1}, \qquad for \qquad u \in H_0^1(\Omega)} \tag{1.2}$$

.

Lemma 1.7.2 *[22]If p is a measurable function on Ω satisfying (4) ,then for a.e.x ∈ Ω,we have*

$$\boxed{\|u\|_{p(.)}\leq 1 \qquad \text{if and only if} \qquad \rho_{p(.)}(u) \leq 1,}$$

and

$$\boxed{\min\left\{\|u\|^{p_1}_{p(.)}, \|u\|^{p_2}_{p(.)}\right\} \leq p_{p(.)}(u) \leq \max\left\{\|u\|^{p_1}_{p(.)}, \|u\|^{p_2}_{p(.)}\right\},} \tag{1.3}$$

for any $u \in L^{p()}(\Omega)$ with $\rho_{p(.)}(u) = \int_\Omega |u|^{p(x)}dx$.

Conditions on the function g ***in the problem*** $(1) - (3)$

Let g:[0,∞) ⟶ (0,∞) be a nonicreasing and differentiable function satisfying

$$\boxed{g(0) > 0, \qquad 1 - \int_0^\infty g(s)ds := L > 0,} \tag{1.4}$$

and

$$\boxed{\int_0^\infty g(s)ds < \frac{\frac{p_1}{2} - 1}{\frac{p_1}{2} - 1 + \frac{1}{2p_1}}} \tag{1.5}$$

By using the direct calculation, we get

$$\int_0^t g(t-s)(\nabla u(s), \nabla u_t(t))ds =$$
$$-\frac{1}{2}g(t)\|\nabla u(t)\|^2 + \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}\frac{d}{dt}\Big\{(g \circ \nabla u)(t) - (\int_0^t g(s)ds)\|\nabla u(t)\|^2\Big\}, \tag{1.6}$$

where

$$\boxed{(g \circ \nabla u)(t) = \int_0^t g(t-s)\|\nabla u(t) - \nabla u(s)\|^2 ds}. \tag{1.7}$$

Chapter 2

Existence of weak solution

2.1 Existence of solution

In this chapter, we study the local existence of solution of the problem (1)-(3) by using Faedo-Galerkin method.

2.1.1 Part 1

We consider the following problem:

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + a|u_t|^{m(x)-2}u_t = h(x,t), \\ u(x,t) = 0, \qquad \text{on} \qquad \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \qquad \text{in} \qquad \Omega, \end{cases} \tag{2.1}$$

where a>0 is a constant.

Theorem 2.1.1 *Suppose that* (1.4) *hold and* $h \in L^2(\Omega \times (0,T))$, *the exponent* $m(x)$ *satisfies* (5). *Then, for every* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, *the problem* (2.1) *has a unique local solution for some* $T > 0$

$$u \in L^\infty((0,T); H_0^1((\Omega)), \qquad u_t \in L^\infty((0,T); L^2(\Omega)) \cap L^{m(.)}(\Omega \times (0,T)).$$

Proof. *Let* $\{v_i\}_1^\infty$ *be an orthonormal basis of* $H_0^1(\Omega)$, *with*

$$-\Delta v_i = \lambda_i v_i \text{ in } \Omega, v_i = 0 \text{ on } \partial\Omega,$$

and define the finite-dimensional subspace $V_k = span\{v_1...v_k\}$, *we have* $\|v_i\| = 1$, *we define*

$$u^k(x,t) = \sum_{i=1}^{k} c_i(t)v_i, \tag{2.2}$$

where $u^k(x,t)$ is a solution of the following approximate problem

$$\int_\Omega u_{tt}^k(x,t)v_i(x)dx + \int_\Omega \nabla u^k(x,t)\nabla v_i(x)dx - \int_\Omega\int_0^t g(t-s)\nabla u(x,s)\nabla v_i(x)dsdx$$
$$+ a\int_\Omega |u_t^k(x,t)|^{m(x)-2}u_t^k(x,t)v_i(x)dx \tag{2.3}$$
$$= \int_\Omega h(x,t)v_i(x)dx,$$

$$u^k(x,0) = u_0^k \qquad u_1^k(x,0) = u_1^k \qquad \forall i = 1,...,k. \tag{2.4}$$

we have

$$u_t^k(x,t) = \sum_{i=1}^k c_i'(t)v_i,$$

where

$$u_0^k = \sum_{i=1}^k (u_0, v_i)v_i \longrightarrow u_0 \; in \; H_0^1(\Omega),$$

and

$$u_1^k = \sum_{i=1}^k (u_1, v_i)v_i \longrightarrow u_1 \; in \; L^2(\Omega),$$

respectively.

By the standard theory of ODE the system (2.3)-(2.4) *admits a local solution in* $[0, t_k)$, $0 < t_k < T$ *for an arbitrary* $T > 0$*. Next, we have to prove that* $t_k = T$, $\forall k \geq 1$ *,multiplying equation (2,3) by* $c_i'(t)$ *we obtain*

$$\int_\Omega u_{tt}^k(x,t)v_i(x)c_i'(t)dx + \int_\Omega \nabla u^k(x,t)\nabla v_i(x)c_i'(t)dx - \int_\Omega\int_0^t g(t-s)\nabla u(x,s)\nabla v_i(x)c_i'(t)dsdx$$
$$+ a\int_\Omega |u_t^k(x,t)|^{m(x)-2}u_t^k(x,t)v_i(x)c_i'(t)dx$$
$$= \int_\Omega h(x,t)v_i(x)c_i'(t)dx,$$

and summing with respect to i, we get

$$\int_\Omega u_{tt}^k(x,t)\sum_{i=1}^k v_i(x)c_i'(t)dx + \int_\Omega \nabla u^k(x,t)\nabla\sum_{i=1}^k v_i(x)c_i'(t)dx -$$
$$\int_\Omega\int_0^t g(t-s)\nabla u(x,s)\nabla\sum_{i=1}^k v_i(x)c_i'(t)dsdx + a\int_\Omega |u_t^k(x,t)|^{m(x)-2}u_t^k(x,t)\sum_{i=1}^k v_i(x)c_i'(t)dx$$
$$= \int_\Omega h(x,t)\sum_{i=1}^k v_i(x)c_i'(t)dx,$$

so

$$\int_\Omega u_{tt}^k(x,t)u_t^k(x,t)dx + \int_\Omega \nabla u^k(x,t)\nabla u_t^k(x,t)dx - \int_\Omega \int_0^t g(t-s)\nabla u(x,s)\nabla u_t^k(x,t)dsdx$$
$$+ a\int_\Omega |u_t^k(x,t)|^{m(x)-2}u_t^k(x,t)u_t^k(x,t)dx$$
$$= \int_\Omega h(x,t)u_t^k(x,t)dx,$$

then, the equation (1.6) *implies*

$$-\int_0^t g(t-s)(\nabla u(s)\cdot\nabla u_t(t))ds =$$
$$+\frac{1}{2}g(t)\|\nabla u(t)\|^2 - \frac{1}{2}(g'\circ\nabla u)(t) + \frac{1}{2}\frac{d}{dt}\Big\{(g\circ\nabla u)(t) - \Big(\int_0^t g(s)ds\Big)\|\nabla u\|^2\Big\}. \quad (2.5)$$

By using (2.5)*, we obtain*

$$\int_\Omega u_{tt}^k(x,t)u_t^k(x,t)dx + \int_\Omega \nabla u^k(x,t)\nabla u_t^k(x,t)dx - \int_\Omega \int_0^t g(t-s)\nabla u(x,s)\nabla u_t^k(x,t)dsdx$$
$$+ a\int_\Omega |u_t^k(x,t)|^{m(x)-2}u_t^k(x,t)u_t^k(x,t)dx$$
$$= \int_\Omega h(x,t)u_t^k(x,t)dx,$$

then

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_t^k|^2dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u^k|^2dx + \frac{1}{2}g(t)\int_\Omega \|\nabla u^k(t)\|^2 - \frac{1}{2}(g'\circ\nabla u)(t) +$$
$$\frac{1}{2}\frac{d}{dt}\Big\{(g\circ\nabla u^k)(t) - \Big(\int_0^t g(s)ds\Big)\int_\Omega \|\nabla u^k(t)\|^2\Big\} + a\int_\Omega |u_t^k(x,t)|^{m(x)}dx$$
$$= \int_\Omega h(x,t)u_t^k dx,$$

so

$$\frac{1}{2}\frac{d}{dt}\Big\{\int_\Omega |u_t^k|^2dx + \Big(1-\int_0^t g(s)ds\Big)\int_\Omega |\nabla u^k(t)|^2dx + (g\circ\nabla u^k)(t)\Big\} + a\int_\Omega |u_t^k(x,t)|^{m(x)}dx \quad (2.6)$$
$$= -\frac{1}{2}g(t)\int_\Omega \|\nabla u^k(t)\|^2 + \frac{1}{2}(g'\circ\nabla u^k)(t) + \int_\Omega h(x,t)u_t^k dx.$$

Integrating (2.6) *over* $(0,t)$ *and using initial condition, we get*

$$\frac{1}{2}\Big\{\int_\Omega |u_t^k|^2dx + (1-\int_0^t g(s)ds)\int_\Omega |\nabla u^k(t)|^2dx + (g\circ\nabla u^k)(t)\Big\} + a\int_0^t\int_\Omega |u_t^k(x,t)|^{m(x)}dsdx$$
$$= \frac{1}{2}\int_\Omega |u_1^k|^2dx + \frac{1}{2}\int_\Omega |\nabla u^k(0)|^2dx + \int_0^t\int_\Omega h(x,t)u_t^k dxds - \int_0^t \frac{1}{2}g(s)\int_\Omega |\nabla u(t)|^2dxds$$
$$+ \int_0^t \frac{1}{2}(g'\circ\nabla u^k)(t)ds,$$

by (1.4), *we get*

$$\frac{1}{2}\int_\Omega |u_1^k|^2dx + \frac{1}{2}\int_\Omega |\nabla u^k(0)|^2dx + \int_0^t\int_\Omega h(x,t)u_t^kdxds - \int_0^t \frac{1}{2}g(s)\int_\Omega |\nabla u^k(t)|^2dxds$$
$$+\int_0^t \frac{1}{2}(g'\circ\nabla u^k)(t)ds \leq \frac{1}{2}\int_\Omega |u_1^k|^2dx + \frac{1}{2}\int_\Omega |\nabla u^k(0)|^2dx + \int_0^t\int_\Omega h(x,t)u_t^kdxds,$$

using young's inequality, we obtain

$$\frac{1}{2}\int_\Omega |u_1^k|^2dx + \frac{1}{2}\int_\Omega |\nabla u^k(0)|^2dx + \int_0^t\int_\Omega h(x,t)u_t^kdxds$$
$$\leq \frac{1}{2}\int_\Omega |u_1^k|^2dx + \frac{1}{2}\int_\Omega |\nabla u^k(0)|^2dx + \frac{1}{4}\int_0^t\int_\Omega |u_t^k(x,s)|^2dxds + \int_0^t\int_\Omega |h(x,s)|^2dxds$$
$$\leq C + \frac{1}{4}sup\int_\Omega |u_t^k(x,t)|^2dx, \qquad \forall t\in[0,t_k)$$

then, we have

$$\begin{cases} \sup\int_\Omega |u_t^k(t)|^2dx \leq C, \\ \sup\int_\Omega |\nabla u^k(t)|^2dx \leq C, \\ a\int_0^{t_k}\int_\Omega |u_t^k(x,s)|^{m(x)}dxds \leq C. \end{cases} \tag{2.7}$$

So we arrive at

$$\sup\int_\Omega |u_t^k(t)|^2dx + \sup l\int_\Omega |\nabla u^k(t)|^2dx + a\int_0^{t_k}\int_\Omega |u_t^k(x,s)|^{m(x)}dxds \leq C,$$

then the solution can be extended to (0,T) and we obtain

$$\begin{cases} (u^k) \text{ is a bounded sequence in } L^\infty((0,T);H_0^1(\Omega)), \\ (u_t^k) \text{ is a bounded sequence in } L^\infty((0,T);L^2(\Omega))\cap L^{m(.)}(\Omega\times(0,T)), \end{cases}$$

hence, there exist a sub-sequence (u^μ) *of* (u_t^k) *such that*

$$\begin{cases} u^\mu \longrightarrow u \text{ weak star in } L^\infty((0,T);H_0^1(\Omega)), \\ u_t^\mu \longrightarrow u_t \text{ weak star in } L^\infty((0,T);L^2(\Omega))\cap L^m(\Omega\times(0,T)), \end{cases}$$

on the other hand, from Lions lemma, we deduce that $u\in C((0,T);L^2(\Omega))$. *Since* (u_t^μ) *is bounded in* $L^m(\Omega\times(0,T))$. *then* $|u_t^\mu|^{m(x)-2}u_t^\mu$ *is a bounded in* $L^{\frac{m(.)}{m(.)-1}}(\Omega\times(0,T))$ *similar as in [33], we have*

$$|u_t^\mu|^{m(x)-2}u_t^\mu \longrightarrow |u_t|^{m(x)-2}u_t \text{ weakly in } L^{\frac{m(.)}{m(.)-1}}(\Omega\times(0,T)),$$

$$\int_\Omega (u_{tt}v + \nabla u\nabla v - \int_0^t g(t-s)\nabla u(s)\nabla v ds + a|u_t|^{m(x)-2}u_t v)dx = \int_\Omega hvdx,$$

which gives

$$u_{tt} - \triangle u - \int_0^t g(t-s)\triangle u(s) + a|u_t|^{m(x)-2}u_t = h \qquad \text{in } D'(\Omega\times(0,T)),$$

■

2.1.2 Part 2

Lemma 2.1.2 *For $x \in \Omega$ and $p(.)$ satisfying*

$$2 < p_1 \leq p(x) \leq p_2 < \infty$$

the function $q(s) = b|s|^{p(x)-2}s$ is differentiable and $|q'(s)| = b|p(x)-1||s|^{p(x)-2}$.

Theorem 2.1.3 *Suppose that (1.4) hold and $m(x)$ satisfies (4) and $p(x)$ satisfying*

$$2 < p_1 \leq p(x) \leq p_2 < \frac{2(n-1)}{n-2}, \qquad n \geq 3$$

then, for every $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, problem has s a unique local solution for some $T > 0$.

$$u \in C([0,T]; H_0^1(\Omega)), \qquad u_t \in C([0,T]; L^2(\Omega)) \cap L^{m(.)}(\Omega \times (0,T)),$$

Proof. let $v \in L^\infty\ ((0,t); H_0^1(\Omega))$. Since $2(P_1 - 1) \leqslant 2(P_1 - 2) \leqslant \dfrac{2n}{n-2}$, then

$$\|q(v)\|^2 = |b|^2 \int_\Omega |v|^{2(p(x)-1)}dx \leq |b|^2 \Big\{ \int_\Omega |v|^{2(p_1-1)}dx + \int_\Omega |v|^{2(p_2-1)}dx \Big\} < \infty, \tag{2.8}$$

so we have

$$q(v) \in L^\infty((0,T); L^2(\Omega) \subset L^2(\Omega) \times (0,T)).$$

From Theorem (2.1.1), for each $v \in L^\infty((0,T); H_0^1(\Omega))$ there exists a unique

$$u \in L^\infty((0,T); H_0^1(\Omega)), \qquad u_t \in L^\infty((0,T); L^2(\Omega) \cap L^{m(.)}(\Omega \times (0,T))),$$

satisfying the nonlinear problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + a|u_t|^{m(x)-2}u_t = q(v) \qquad \text{in} \qquad \Omega \times (0,T) \\ u(x,t) = 0, \qquad \text{on} \qquad \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \qquad \text{in} \qquad \Omega, \end{cases} \tag{2.9}$$

we define a map $H : X_t \longrightarrow X_t$ by $H(v) = u$, where

$$X_t := C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)),$$

equipped with the norm

$$\|w\|_{X_t}^2 = \max\Big\{ \int_\Omega |w_t(t)|^2 dx + \int_\Omega l|\nabla w(t)|^2 dx \Big\}, \tag{2.10}$$

multiplying equation (2.9) by u_t and integration over Ω get

$$\int_\Omega u_{tt}u_t dx - \int_\Omega \Delta u u_t dx + \int_\Omega \int_0^t g(t-s)\Delta u(s) u_t ds dx + \int_\Omega a|u_t|^{m(x)-2}u_t u_t dx = \int_\Omega q(v)u_t dx,$$

(1.6) ,we obtain

$$\int_\Omega u_{tt}u_t dx - \int_\Omega \Delta u u_t dx + \int_\Omega \int_0^t g(t-s)\Delta u(s) u_t ds dx + \int_\Omega a|u_t|^{m(x)-2}u_t u_t dx$$
$$= \int_\Omega q(v) u_t dx,$$

implies

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_t|^2 dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}g(t)\int_\Omega \|\nabla u(t)\|^2 dx - \frac{1}{2}(g' \circ \nabla u)(t)$$
$$+ \frac{1}{2}\frac{d}{dt}\Big\{(g\circ \nabla u)(t) - \Big(\int_0^t g(s)ds\Big) \int_\Omega \|\nabla u(t)\|^2\Big\} + a\int_\Omega |u_t|^{m(x)} dx$$
$$= \int_\Omega q(v) u_t dx,$$

integration over $(0,t)$ and using initial condition, we obtain

$$\int_0^t \Big[\frac{1}{2}\frac{d}{dt}\int_\Omega |u_t|^2 dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}g(t)\int_\Omega \|\nabla u(t)\|^2 dx - \frac{1}{2}(g' \circ \nabla u)(t) + \frac{1}{2}\frac{d}{dt}\Big\{(g\circ \nabla u)(t)$$
$$- \Big(\int_0^t g(s)ds\Big) \int_\Omega \|\nabla u(t)\|^2\Big\}\Big] + a\int_0^t\int_\Omega |u_t|^{m(x)} dxds$$
$$= \int_\Omega\int_0^t q(v) u_t dx,$$

then

$$\frac{1}{2}\int_\Omega |u_t|^2 dx + \frac{1}{2}(1 - \Big(\int_0^t g(s)ds)\Big) \int_\Omega |\nabla u(t)|^2 dx + \frac{1}{2}(g\circ \nabla u)(t) - \frac{1}{2}\int_0^t (g'\circ \nabla u)(s) ds,$$
$$+ \frac{1}{2}\int_0^t g(s)\int_\Omega |\nabla u(s)|^2 dx + a\int_0^t\int_\Omega |u_t(s)|^{m(x)} dxds,$$
$$= \frac{1}{2}\int_\Omega u_1^2 dx + \frac{1}{2}\int_\Omega |\nabla u_0|^2 + b\int_0^t\int_\Omega |v|^{p(x)-2} v u_t(s) dxds, \tag{2.11}$$

applying young inequality $ab \le \frac{\epsilon}{4}a^2 + \frac{1}{\epsilon}b^2$, obtain

$$\Big|\int_\Omega |v|^{p(x)-2} v u_t(s) dx\Big| \le \frac{\epsilon}{4}\int_\Omega |u_t(s)|^2 dx + \frac{1}{\epsilon}\int_\Omega \Big|v|^{2(p(x)-1)} dx,$$

by (2.8), we obtain

$$\frac{\epsilon}{4}\int_\Omega |u_t(s)|^2 dx + \frac{1}{\epsilon}\int_\Omega |v|^{2(p(x)-1)} dx,$$
$$\le \frac{\epsilon}{4}\int_\Omega |u_t(s)|^2 dx + \frac{1}{\epsilon}\Big\{\int_\Omega |v|^{2(p_1-1)} + \int_\Omega |v|^{2(p_2-1)}\Big\},$$

using poincaré inequality $|u|\leq c|\nabla u|$
we get

$$\frac{\epsilon}{4}\int_\Omega |u_t(s)|^2dx + \frac{1}{\epsilon}\Big\{\int_\Omega |v|^{2(p_1-1)} + \int_\Omega |v|^{2(p_2-1)}\Big\},$$
$$\leq \frac{\epsilon}{4}\int_\Omega |u_t(s)|^2dx + \frac{c_e}{\epsilon}\Big\{\int_\Omega |\nabla v|^{2(p_1-1)} + \int_\Omega |\nabla v|^{2(p_1-2)}\Big\},$$

so,

$$|\int_\Omega |v|^{p(x)-2}vu_t(s)dx| \leq \frac{\epsilon}{4}\int_\Omega |u_t(s)|^2dx + \frac{c_e}{\epsilon}\Big\{\|\nabla v\|^{2(p_1-1)} + \|\nabla v\|t^{2(p_1-2)}\Big\}, \tag{2.12}$$

by (1.4), we get

$$\begin{cases} \frac{1}{2}(g\circ\nabla u)(t) \geq 0 \\ -\frac{1}{2}(g'\circ\nabla u)(s)ds \geq 0 \\ \frac{1}{2}\int_0^t g(s)\int_\Omega |\nabla u(s)|^2dxds \geq 0. \end{cases} \tag{2.13}$$

So by (2.13), we obtain

$$\frac{1}{2}(g\circ\nabla u)(t) - \frac{1}{2}(g'\circ\nabla u)(s)ds + \frac{1}{2}\int_0^t g(s)\int_\Omega |\nabla u(s)|^2dxds \geq 0, \tag{2.14}$$

by (2.11)and (2.14) ,we get

$$\frac{1}{2}\int_\Omega |u_t|^2dx + \frac{l}{2}\int_\Omega |\nabla u(t)|^2dx \leq \frac{1}{2}\int_\Omega u_1^2dx + \frac{1}{2}\int_\Omega |\nabla u_0|^2 + b\int_0^t\int_\Omega |v|^{p(x)-2}vu_tdxds, \tag{2.15}$$

and by(2.12) (2.15), we get

$$\frac{1}{2}\int_\Omega |u_t|^2dx + \frac{l}{2}\int_\Omega |\nabla u(t)|^2dx,$$
$$\leq \frac{1}{2}\int_\Omega u_1^2dx + \frac{1}{2}\int_\Omega |\nabla u_0|^2dx + \frac{\epsilon bT}{4}\int_\Omega |u_t(t)|^2dx + \frac{c_eb}{\epsilon}\Big\{\int_0^T \|\nabla v\|^{2(p_1-1)}ds + \int_0^T \|\nabla v\|^{2(p_1-2)}ds\Big\},$$

using (2.10), we get $\int_\Omega |\nabla u(t)|^2 \leq \frac{1}{L}\|u\|_{X_T}^2$
so we obtain

$$\frac{1}{2}\sup\int_\Omega |u_t|^2dx + \frac{l}{2}\sup\int_\Omega \|\nabla u(t)\|^2dx,$$
$$\leq \frac{1}{2}\int_\Omega u_1^2dx + \frac{1}{2}\int_\Omega |\nabla u_0|^2dx + \frac{\epsilon bT}{4}\sup\int_\Omega |u_t(t)|^2dx + \frac{c_ebT}{\epsilon l}\Big\{\int_0^T |v|_{X_T}^{2(p_1-1)}ds + \int_0^T |v|_{X_T}^{2(p_1-2)}ds\Big\}, \tag{2.16}$$

multiplying equation (2.16) by 4, we find

$$2\sup\int_\Omega |u_t|^2dx + 2\sup\int_\Omega \|\nabla u(t)\|^2dx - \epsilon bT\sup\int_\Omega |u_t(t)|^2dx,$$
$$\leq 2\int_\Omega u_1^2dx + 2\int_\Omega |\nabla u_0|^2dx + \frac{4c_ebT}{\epsilon l}\Big\{\|v\|_{X_T}^{2(p_1-1)}ds + \|v\|_{X_T}^{2(p_2-1)}ds\Big\},$$

(2.10) we get

$$\|u\|_{X_T}^2 \leq 2\int_\Omega |u_t|^2dx + 2\int_\Omega \|\nabla u(t)\|^2dx + C_0T\Big\{\|v\|_{X_T}^{2(p_1-1)} + \|v\|_{X_T}^{2(p_1-1)}\Big\},$$

such that $C_0 = \dfrac{4c_eb}{\epsilon l}$ for $M > 0$ large and $T > 0$, then we suppose $\|v\|_{X_T} \leq M$ so that

$$\int_\Omega u_1^2dx + \int_\Omega |\nabla u_0|^2dx \leq \frac{M^2}{4}$$

and T sufficiently small so that

$$T \leq \frac{1}{2C_0(M^{2p_1-4}M^{2p_2-4})},$$

then,

$$\| u \|_{X_T}^2 \leq \frac{M^2}{2} + C_0\frac{1}{2C_0(M^{2p_1-4}M^{2p_2-4})}\Big\{M_{X_T}^{2(p_1-1)} + M_{X_T}^{2(p_1-1)}\Big\} \tag{2.17}$$

so we have

$$\| u \|_{X_T}^2 \leqslant M^2$$

this shows that $H : B(M,T) \longrightarrow B(M,T)$ where

$$B(M,T) = \{w \in C([0,T]; H_0^1(\Omega)) \qquad w_t \in C([0,T]; L^2(\Omega)) \qquad \text{such that} \qquad \|w\|_{X_T} \leq M\}.$$

Next, we verify that **H is contraction**.for this purpose,let $H(v_1) = u_1$ and $H(v_2) = u_2$ and set $u = u_2 - u_2$ then satisfies

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + a|u_{1t}|^{m(x)-2}u_t - a|u_{2t}|^{m(x)-2}u_{2t} = b|v_1|^{p(x)-2}v_1 - b|v_2|^{p(x)-2}v_2, & \text{in } \Omega\times[0,T], \\ u(x,t) = 0, \qquad \text{on} \qquad \partial\Omega\times(0,T), \\ u(x,0) = 0, u_t(x,0) = 0, \qquad \text{in} \qquad \Omega, \end{cases} \tag{2.18}$$

multiplying equation (2.18) by u_t and integrating over $\Omega\times[0,T]$ we get

$$\int_\Omega u_{tt}u_tdx - \int_\Omega \Delta uu_t + \int_\Omega\int_0^t g(t-s)\Delta u(s)u_tds + \int_\Omega a|u_{1t}|^{m(x)-2}u_{1t}u_t - \int_\Omega a|u_{2t}|^{m(x)-2}u_{2t}u_t$$
$$= \int_\Omega b|v_1|^{p(x)-2}v_1u_t - \int_\Omega b|v_2|^{p(x)-2}v_2u_t,$$

the same calculation of (2.11) and $u_t = u_{1t} - u_{2t}$

$$\frac{1}{2}\int_\Omega |u_t|^2 dx + \frac{1}{2}\Big(\ 1 - \Big(\ \int_0^t g(s)ds\Big)\ \Big)\ \int_\Omega \|\nabla u(t)\|^2 dx + \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{2}\int_0^t (g' \circ \nabla u)(s)ds + \tag{2.19}$$

$$\frac{1}{2}\int_0^t g(s)\int_\Omega \|\nabla u(s)\|^2 dx, ds$$

$$+ a\int_0^t \int_\Omega |(u_{1t}(s)|^{m(x)-2}u_{1t}(s) - |u_{2t}(s)|^{m(x)-2}u_{2t}(s))(u_{1t}(s) - u_{2t}(s))dxds,$$

$$= \int_0^t \int_\Omega q(v_1) - q(v_2)u_t(s)dxds, \tag{2.20}$$

where $q(v) = b|v|^{p(x)-2}v$, and by (1.4) and (2.23),we arrive at

$$\frac{1}{2}\int_\Omega |u_t|^2 dx + \frac{l}{2}\int_\Omega \|\nabla u(t)\|^2 dx \leqslant \int_0^t \int_\Omega q(v_1) - q(v_2)u_t(s)dxds$$

using mean value theorem so $\dfrac{q(v_1) - q(v_2)}{|v_1 - v_2|} = q'(\varepsilon)$with $v = v_1 - v_2$

$$\int_\Omega |q(v_1) - q(v_2)||u_t(s)|dx = \int_\Omega |q'(\varepsilon)||v||u_t(s)|dx$$

two, using inequality the young

$$\int_\Omega |q'(\varepsilon)||v||u_t(s)|dx \leqslant \frac{\delta}{2}\int_\Omega |u_t(s)|^2 dx + \frac{1}{2\delta}\int_\Omega |q'(\varepsilon)|^2|v|^2 dx$$

we have $|q'(s)| = b|p(x) - 1||s|^{p(x)-2}$ and $p(x) \leqslant p_2$ so

$$\frac{\delta}{2}\int_\Omega |u_t(s)|^2 dx + \frac{1}{2\delta}\int_\Omega |q'(\varepsilon)|^2|v|^2 dx \leqslant \frac{\delta}{2}\int_\Omega |u_t(s)|^2 dx + \frac{b^2(p_2 - 1)^2}{2\delta}\int_\Omega |\xi|^{2(p(x)-2)}|v|^2 dx$$

using (2.8) from ξ, and using hölder inequality, suppose $p = \dfrac{2}{n}$ and $q = \dfrac{n-2}{n}$, we get

$$\frac{\delta}{2}\int_\Omega |u_t(s)|^2 dx + \frac{b^2(p_2 - 1)^2}{2\delta}\int_\Omega |\xi|^{2(p(x)-2)}|v|^2 dx,$$

$$\leq \frac{\delta}{2}\int_\Omega |u_t(s)|^2 dx + \frac{b^2(p_2 - 1)^2}{2\delta}\Big(\ |v|^{\frac{2n}{n-2}}dx\Big)^{\frac{n-2}{n}}\Big[\ \Big(\ \int_\Omega |\ \xi|^{n(p_1-2)}dx\Big)^{\frac{2}{n}} + \Big(\ \int_\Omega |\xi|^{n(p_2-2)}dx\Big)^{\frac{2}{n}}\Big]$$

using poincaré inequality

$$\frac{\delta}{2}\int_\Omega |u_t(s)|^2 dx + \frac{b^2(p_2 - 1)^2}{2\delta}\Big(\ |v|^{\frac{2n}{n-2}}dx\Big)^{\frac{n-2}{n}}\Big[\ \Big(\ \int_\Omega |\ \xi|^{n(p_1-2)}dx)^{\frac{2}{n}} + \Big(\ \int_\Omega |\xi|^{n(p_2-2)}dx\Big)^{\frac{2}{n}}\Big]$$

$$\leq \frac{\delta}{2}\int_\Omega |u_t(s)|^2 dx + \frac{b^2(p_2 - 1)^2 C_e}{2\delta}\ \|\ \nabla v\ \|^2\ \Big[\ \ \|\ \nabla\xi\ \|^{2(p_1-2)} + \|\ \nabla\xi\ \|^{2(p_2-2)}\ \Big]$$

we have by (2.10) $\int_\Omega |\nabla\xi(t)|^2 \leq \frac{1}{L}\|\xi\|^2_{X_T}$, and we have $\|\xi\|_{X_T} \leq M$ so

$$\begin{aligned}&\frac{\delta}{2}\int_\Omega |u_t(s)|^2dx + \frac{b^2(p_2-1)^2C_e}{2\delta} \parallel \nabla v \parallel^2 \Big[\parallel \nabla\xi \parallel^{2(p_1-2)} + \parallel \nabla\xi \parallel^{2(p_2-2)} \Big] \\ &\leq \frac{\delta}{2}\int_\Omega |u_t(s)|^2dx + \frac{b^2(p_2-1)^2C_e}{\delta L^{p_2-2}}\Big(M^{2(p_1-2)} + M^{2(p_2-2)}\Big) \|\nabla v\|^2,\end{aligned}$$

(2.17), obtain

$$\|u\|^2_{X_T} \leqslant \frac{4b^2(p_2-1)^2C_eT}{\delta L^{p_2-1}}(M^{2(p_1-2)} + M^{2(p_2-2)}) \parallel v \parallel^2_{X_T} \tag{2.21}$$

the same of

$$\parallel H(v) \parallel^2 \leqslant \alpha \parallel v \parallel^2,$$

since

$$\frac{4b^2(p_2-1)^2C_eT}{\delta L p_2 - 1}(M^{2(p_1-2)} + M^{2(p_2-2)}) < 1,$$

so

$$\alpha < 1$$

we have $H(v) \parallel^2 \leqslant \alpha \parallel v \parallel^2$ and $\alpha < 1$ so H is a **contraction** mapping then, Banach fixed point theorem infer that H has a unique $u \in B(M,T)$ satisfying $H(u) = u$. obviously, it is a solution of (1)-(3). ■

2.2 Uniqueness of solution

2.2.1 part 1

Suppose that (2.1) has two solutions u and z then $w = u - z$ satisfies

$$\begin{cases} w_{tt} - \Delta w + \int_0^t g(t-s)\Delta w(s)ds + a|u_t|^{m(x)-2}u_t - a|z_t|^{m(x)-2}z_t = 0, \\ w(x,t) = 0, \qquad \text{on } \partial\Omega \times (0,T), \\ w(x,0) = 0, w_t(x,0) = 0, \qquad \text{in } \Omega, \end{cases} \tag{2.22}$$

multiply equation (2.22) by w_t and integrate over Ω to get

$$\int_\Omega w_{tt}w_tdx - \int_\Omega \Delta w w_tdx + \int_\Omega\int_0^t g(t-s)\Delta w(s)w_tdsdx + \int_\Omega a|u_t|^{m(x)-2}u_tw_tdx - \int_\Omega a|z_t|^{m(x)-2}z_tw_tdx = 0,$$

by(1.6) we get

$$\begin{aligned}&\int_\Omega w_{tt}w_tdx - \int_\Omega \Delta w w_tdx - \int_\Omega\int_0^t g(t-s)\nabla w(s)\nabla w_tdsdx + \int_\Omega a|u_t|^{m(x)-2}u_tw_tdx \\ &- \int_\Omega a|z_t|^{m(x)-2}z_tw_tdx = 0,\end{aligned}$$

$\Longrightarrow$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |w_t|^2dx + \frac{1}{2}g(t)\int_\Omega \|\nabla w(t)\|^2dx - \frac{1}{2}(g'\circ\nabla w)(t) + \frac{1}{2}\frac{d}{dt}(g\circ\nabla w)(t),$$
$$-\frac{1}{2}\frac{d}{dt}\Big(\int_0^t g(s)ds\Big)\int_\Omega \|\nabla w(t)\|^2dx + \int_\Omega \nabla|w|^2dx + \int_\Omega a|u_t|^{m(x)-2}u_tw_tdx - \int_\Omega a|z_t|^{m(x)-2}z_tw_t = 0,$$

and $w_t = u_t - z_t$ obtain

$$\frac{1}{2}\frac{d}{dt}\Big\{\int_\Omega |w_t|^2dx + \Big(1-\int_0^t g(s)ds\Big)\int_\Omega \|\nabla w(t)\|^2dx + g\circ\nabla w(t)\Big\} + \frac{1}{2}g(t)\int_\Omega \|\nabla w(t)\|^2dx,$$
$$= -a\int_\Omega\Big(|u_t|^{m(x)-2}u_t - |z_t|^{m(x)-2}z_t\Big)\Big(u_t(t)-z_t(t)\Big)dx + \frac{1}{2}(g'\circ\nabla w)(t),$$

from the inequality

$$|a_t|^{m(x)-2}a_t - |b_t|^{m(x)-2}b_t)(a_t(t)-b_t(t)) \geqslant 0, \tag{2.23}$$

we get

$$|u_t|^{m(x)-2}u_t - |z_t|^{m(x)-2}z_t)(u_t(t)-z_t(t)) \geqslant 0,$$

in the problem (2.1) from initial condition we have $(u(x,0),z(x,0)) = (u_0(x),u_0(x))and(u_t(x,0),z_t(x,0)) = (u_1(x),u_1(x))$, this means that $w(x,0) = w_t(x,0) = 0$, then we get

$$\int_\Omega(|w_t(t)|^2 + l|\nabla w(t)|^2)dx = 0$$

with $l = (1-\int_0^t g(s)ds)$, this gives $w = 0$ a.e $u = z$.

2.2.2 pat 2

Proof. Suppose that (1)-(3) have two solution u and z. then $w = u - z$ satisfies

$$\begin{cases} w_{tt} - \Delta w + \int_0^t g(t-s)\Delta w(s)ds + a|u_t|^{m(x)-2}u_t - a|z_t|^{m(x)-2}z_t = b|u|^{p(x)-2}u - b|z|^{p(x)-2}z, \\ w(x,t) = 0, \qquad \text{on } \partial\Omega\times(0,T), \\ w(x,0) = 0, w_t(x,0) = 0, \qquad \text{in } \Omega, \end{cases} \tag{2.24}$$

multiply equation (2.24) by w_t and integrate over Ω to get

$$\int_\Omega w_{tt}w_tdx - \int_\Omega \Delta ww_tdx + \int_\Omega\int_0^t g(t-s)\Delta w(s)w_tdsdx + \int_\Omega a|u_t|^{m(x)-2}u_tw_tdx$$
$$-\int_\Omega a|z_t|^{m(x)-2}z_tw_tdx$$
$$= \int_\Omega b|u|^{p(x)-2}uw_tdx - \int_\Omega b|z|^{p(x)-2}zw_tdx,$$

by(1.6) we have

$$\int_\Omega w_{tt}w_t dx - \int_\Omega \Delta w w_t dx + \int_\Omega \int_0^t g(t-s)\Delta w(s) w_t ds dx + \int_\Omega a|u_t|^{m(x)-2}u_t w_t dx,$$
$$- \int_\Omega a|z_t|^{m(x)-2} z_t w_t dx,$$
$$= \int_\Omega b|u|^{p(x)-2}uw_t dx - \int_\Omega b|z|^{p(x)-2}zw_t dx,$$

$\Longrightarrow$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |w_t|^2 dx + \frac{1}{2}g(t)\int_\Omega \|\nabla w(t)\|^2 dx - \frac{1}{2}(g'\circ \nabla w)(t),$$
$$+ \frac{1}{2}\frac{d}{dt}(g\circ\nabla w)(t) - \frac{1}{2}\frac{d}{dt}\Big(\int_0^t g(s)ds\Big)\int_\Omega \|\nabla w(t)\|^2 dx,$$
$$+ \int_\Omega |\nabla w|^2 dx + \int_\Omega a|u_t|^{m(x)-2}u_t w_t dx - \int_\Omega a|z_t|^{m(x)-2}z_t w_t,$$
$$= \int_\Omega b|u|^{p(x)-2}uw_t dx - \int_\Omega b|z|^{p(x)-2}zw_t dx.$$

So

$$\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega w_t dx + \Big(1-\int_0^t g(s)ds\Big)\int_\Omega \|\nabla w(t)\|^2 dx + g\circ\nabla w(t)\Big) + \frac{1}{2}g(t)\int_\Omega \|\nabla w(t)\|^2 dx,$$
$$+ a\int_\Omega \Big(|u_t|^{m(x)-2}u_t - |z_t|^{m(x)-2}z_t\Big)\Big(u_t(t)-z_t(t)\Big)dx - \frac{1}{2}(g'\circ\nabla w)(t),$$
$$= \int_\Omega b|u|^{p(x)-2}uw_t dx - \int_\Omega b|z|^{p(x)-2}zw_t dx.$$

By integration over $(0,t)$ we obtain

$$= \frac{1}{2}\int_\Omega w_t dx + \frac{1}{2}\Big(1-\int_0^t g(s)ds\Big)\int_\Omega \|\nabla w(t)\|^2 dx + \frac{1}{2}g\circ\nabla w(t) + \frac{1}{2}\int_0^t g(s)\int_\Omega \|\nabla w(s)\|^2 dx ds,$$
$$+ a\int_\Omega\int_0^t \Big(|u_t|^{m(x)-2}u_t - |z_t|^{m(x)-2}z_t\Big)\Big(u_t(t)-z_t(t)\Big)dx ds - \frac{1}{2}(g'\circ\nabla w)(s)ds,$$
$$= \int_\Omega\int_0^t \Big(b|u|^{p(x)-2}udx - b|z|^{p(x)-2}z\Big)w_t dx ds,$$

by using Gronwall's inequality, we obtain

$$\int_\Omega (|w_t(t)|^2 + |\nabla w(t)|^2)dx \leqslant C\int_0^t\int_\Omega (|w_t(t)|^2 + |\nabla w(t)|^2)dx ds$$

such that $u = \int_\Omega (|w_t(t)|^2 + |\nabla w(t)|^2)dx$
in the problem (1)-(3) from the initial conditions we have $u(x,0)=u_0$ and $u_t(x,0)=u_1$ and u,z solutions of the problem then we have $w=u-z \Rightarrow w(0)=0$, it means that

$$\int_\Omega (|w_t(0)|^2 + |\nabla w(0)|^2)dx = 0,$$

so, we obtain

$$\int_\Omega (|w_t(t)|^2 + l|\nabla w(t)|^2)dx = 0,$$

thus, $w = 0$, the proof is completed. ■

Chapter 3

BLOW-UP RESULT

In this chapter we show that the solution for the problem (1)-(3) blow-up in finite time when the initial energy lies in positive as non-positive.

3.1 Some fundamental lemmas

Lemma 3.1.1 *Let u solution of the problem* (1)-(3)*, then the modified energy functional of the this problem is :*

$$E(t) = \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}(1 - \int_0^t g(s)ds)\|\nabla u(t)\|^2 + \frac{1}{2}(g \circ \nabla u)(t) - b\int_\Omega \frac{|u(t)|^{p(x)}}{p(x)}dx. \quad (3.1)$$

Proof. First, multiplying the equation (1) by u_t and integrating over Ω we obtain

$$\int_\Omega u_{tt}u_t dx - \int_\Omega \Delta u u_t dx + \int_\Omega\int_0^t g(t-s)\Delta u(s)u_t ds dx + \int_\Omega a|u_t|^{m(x)-2}u_t u_t dx$$
$$= \int_\Omega b|u|^{p(x)-2}u u_t dx, \quad (3.2)$$

so

$$\int_\Omega u_{tt}u_t dx - \int_\Omega \Delta u u_t dx + \int_\Omega\int_0^t g(t-s)\Delta u(s)u_t ds dx + \int_\Omega a|u_t|^{m(x)-2}u_t u_t dx$$
$$= \int_\Omega b|u|^{p(x)-2}u u_t dx,$$

$\Longrightarrow$

$$\frac{1}{2}\frac{d}{dt}\|u_t(t)\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2 + \int_\Omega\int_0^t g(t-s)\Delta u(s)u_t ds dx + \int_\Omega a|u_t|^{m(x)}$$
$$= b\frac{d}{dt}\int_\Omega \frac{|u|^{p(x)}dx}{p(x)}dx,$$

by (1.6) we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_t(t)\|^2+\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2+\int_\Omega\int_0^t g(t-s)\Delta u(s)u_t dsdx+\int_\Omega a|u_t|^{m(x)}, \quad (3.3)$$
$$=b\frac{d}{dt}\int_\Omega\frac{|u|^{p(x)}dx}{p(x)}dx,$$

implies

$$\frac{1}{2}\frac{d}{dt}\|u_t(t)\|^2+\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2+\frac{1}{2}g(t)\|\nabla u(t)\|^2-\frac{1}{2}(g'\circ\nabla u)(t),$$
$$+\frac{1}{2}\frac{d}{dt}\Big\{(g\circ\nabla u)(t)-\Big(\int_0^t g(s)ds\Big)\|\nabla u\|^2\Big\}+\int_\Omega a|u_t|^{m(x)},$$
$$=b\frac{d}{dt}\int_\Omega\frac{|u|^{p(x)}dx}{p(x)}. \quad (3.4)$$

■

Lemma 3.1.2 *The energy $E(t)$ is nonincreasing for all $t>0$.*

Proof. By (3.4) we conclude that

$$E'(t)=-a\int_\Omega|u_t(t)|^{m(x)}dx-\frac{1}{2}g(t)\|\nabla u(t)\|^2+\frac{1}{2}(g'\circ\nabla u)\leq 0, \text{for} \qquad t\geq 0.$$

as, we have

$$\begin{cases}-\frac{1}{2}\int_0^t g(t)\|\nabla u(t)\|^2\leq 0,\\ +\frac{1}{2}\int_0^t(g'\circ\nabla u)(t)\leq 0,\end{cases} \quad (3.5)$$

and

$$-\int_\Omega\int_0^t a|u_t|^{m(x)}\leq 0. \quad (3.6)$$

However (3.5) and (3.6) show that $E'(t)\leq 0, \qquad \forall t>0$ ■

Now, we set

$$\boxed{B_1=\max\{1,\frac{B}{\frac{1}{2}},(\frac{1}{b})^{\frac{1}{2}}\},\lambda_1=(\frac{1}{bB_1^{p1}})^{\frac{1}{p1-2}},E_1=(\frac{1}{2}-\frac{1}{P1})\lambda_1^2,}$$

and we define the functional G by:

$$\boxed{G(t)=E_2-E(t),} \quad (3.7)$$

where the constant $E_2\in(E(0),E_1)$ will be chosen later.

Lemma 3.1.3 *The functional G is increasing.*

Proof. We derivative the functional G in (3.7), we obtain

$$G'(t)=-E'(t)\geqslant a\int_\Omega|u_t(t)|^{m(x)}dx\geqslant 0. \quad (3.8)$$

■

Lemma 3.1.4 *suppose that* (1.4) *hold and the exponents p(x) and m(x) satisfy condition*(5) *.assume further that*

$$E(0) < E_1 \qquad and \qquad \lambda_1 < \lambda(0) = L^{\frac{1}{2}}\|\nabla u_0\| \leqslant B_1^{-1},$$

then there exists a constant $\lambda_2 > \lambda_1$*such that*

$$L\|\nabla u(t)\|^2 \geq \lambda_2^2, \qquad t \geq 0 \tag{3.9}$$

and

$$\int_\Omega |u(t)|^{p(x)}dx \geq B_1^{p_1}\lambda_2^{p_1}, t \geq 0 \tag{3.10}$$

Proof. Using (1.4), we find that for $0 < \lambda(t) \leqslant B_1^{-1}$

$$E(t) \geq \frac{1}{2}(1 - \int_0^t g(s)ds)\|\nabla u(t)\|^2 - b\int_\Omega \frac{|u(t)|^{p(x)}}{p(x)}dx,$$

we have $-\frac{1}{p(x)} \geq -\frac{1}{p_1}$ so

$$\begin{aligned}&\frac{1}{2}(1 - \int_0^t g(s)ds)\|\nabla u(t)\|^2 - b\int_\Omega \frac{|u(t)|^{p(x)}}{p(x)}dx,\\ &\geq \frac{L}{2}\|\nabla u(t)\|^2 - \frac{b}{p_1}\int_\Omega |u(t)|^{p(x)}dx,\end{aligned}$$

then by using the equation (1.3), we find

$$\begin{aligned}&\frac{L}{2}\|\nabla u(t)\|^2 - \frac{b}{p_1}\int_\Omega |u(t)|^{p(x)}dx,\\ &\geq \frac{L}{2}\|\nabla u(t)\|^2 - \frac{b}{p_1}\ max\Big\{\|u(t)\|_{p(.)}^{p_1}, \|u(t)\|_{p(.)}^{p_2}\Big\},\end{aligned}$$

using (1.2)

$$\begin{aligned}&\frac{L}{2}\|\nabla u(t)\|^2 - \frac{b}{p_1}\ max\Big\{\|u(t)\|_{p(.)}^{p_1}, \|u(t)\|_{p(.)}^{p_2}\Big\},\\ &\geq \frac{L}{2}\|\nabla u(t)\|^2 - \frac{b}{p_1}\ max\Big\{(B\|\nabla u(t)\|^{p_1}), (B\|\nabla u(t)\|^{p_2})\Big\},\end{aligned}$$

using lemma the (3.1.4) we obtain

$$\begin{aligned}&\lambda(t)^{p_1} = L^{\frac{1}{2}}\|\nabla u(t)\|^{p_1}\\ &\Rightarrow \frac{B^{p_1}}{L^{\frac{p_1}{2}}}\lambda(t)^{p_1} = B^{p_1}\|\nabla u(t)\|^{p_1}\\ &\Rightarrow (\frac{B}{L^{\frac{1}{2}}})^{p_1}\lambda(t)^{p_1} = B^{p_1}\|\nabla u(t)\|^{p_1}\\ &\Rightarrow B_1^{p_1}\lambda(t)^{p_1} = (B\|\nabla u(t)\|)^{p_1},\end{aligned}$$

so

$$\frac{L}{2}\|\nabla u(t)\|^2 - \frac{b}{p_1}\ max\Big\{(B\|\nabla u(t)\|^{p_1}),(B\|\nabla u(t)\|^{p_2})\Big\}$$
$$= \frac{1}{2}\lambda^2(t) - \frac{b}{p_1}\max\Big\{B_1^{p_1}\lambda^{p_1}, B_1^{p_2}\lambda^{p_2}\Big\},$$

we have $-p_1 > -p_2 \Rightarrow -\max p_1 > -\max p_2$ we obtain

$$\frac{1}{2}\lambda^2(t) - \frac{b}{p_1}\max\Big\{B_1^{p_1}\lambda^{p_1}, B_1^{p_2}\lambda^{p_2}\Big\}$$
$$= \frac{1}{2}\lambda^2(t) - \frac{b}{p_1}B_1^{p_1}\lambda^{p_1}(t)$$
$$:= f(\lambda(t)), \tag{3.11}$$

where $\lambda(t) = L\frac{1}{2}\|\nabla u(t)\|$. It is easy to verify that $f(\lambda(t))$ has a maximum at $\lambda_1 > 0$, and the maximum value is $f(\lambda_1) = E_1$. from the definition of $f(\lambda(t))$

$$f(\lambda(t)) = \frac{1}{2}\lambda^2(t) - \frac{b}{p_1}B_1^{p_1}\lambda^{p_1}(t)$$
$$f'(\lambda(t)) = \lambda - \frac{b}{p_1}p_1 B_1^{p_1}\lambda^{p_1-1}$$
$$= \lambda - bB_1^{p_1}\lambda^{p_1-1}$$
$$= \lambda(1 - bB_1^{p_1}\lambda^{p_1-2})$$

then,

$$f'(\lambda) = \lambda(1 - bB_1^{p_1}\lambda^{p_1-2}).$$

we can show that

$$\begin{cases} f'(\lambda(t)) > 0, \lambda \in (0,\lambda_1) \\ f'(\lambda(t)) < 0, \lambda \in (\lambda_1, +\infty), \end{cases}$$

which implies that

$$\begin{cases} f(\lambda) = \text{is strictly increasing in}(0,\lambda_1) \\ f(\lambda) = \text{is strictly decreasing in}(\lambda_1, +\infty), \end{cases}$$

because $E(0) < E_1 = f(\lambda_1)$,there exists a positive constant $\lambda_2 \in (\lambda_1, \infty)$ such that $f(\lambda_2) = E(0)$.by(3.11),we see that $f(\lambda(0)) \leq E(0) = f(\lambda_2)$.It implies that $\lambda(0) \geq \lambda_2$,since$\lambda(0) \in (0,\lambda_1)$ the strictly increasing and $\lambda_2 \in (\lambda_1, +\infty)$ the strictly decreasing .

Now ,to show (3.9), we suppose on the contrary that

$$\lambda^2(t_0) = L\|\nabla u(t_0)\|^2 < \lambda_2^2,$$

for some $t_0 > 0$.then there exists $t_1 > 0$ such that $\lambda_1 < \lambda(t_1) = L^{\frac{1}{2}}|\nabla u(t_1)| < \lambda_2$ $(\lambda(t_1) = L^{\frac{1}{2}}\|\nabla u(t_1)\|)$and

$$L\|\nabla u(t_0)\|^2 < \lambda_2^2$$
$$= L^{\frac{1}{2}}\|\nabla u(t_0)\| < \lambda_2$$
$$= L^{\frac{1}{2}}\|\nabla u(t_1)\| < \lambda_2,$$

using (3.11) , we obtain $E(t_1) \geqslant f(\lambda(t_1))$ and we have to $f(\lambda_2) < f(\lambda_1)$ and $f(\lambda(t_1)) > f(\lambda_1)$ so $f(\lambda(t_1)) > f(\lambda_2) = E(0)$ consequently

$$E(t_1) \geq f(\lambda(t_1)) > f(\lambda_2) = E(0),$$

which contradicts $E(t) < E(0)$, for all $t \in (0, T)$. hence(3.9) holds, so $E(t) < E(0)$ and

$$\begin{aligned} E(t) &\geq \frac{1}{2}(1 - \int_0^t g(s)ds)\|\nabla u(t)\|^2 - b\int_\Omega \frac{|u(t)|^{p(x)}}{p(x)}dx \\ &\geq \frac{L}{2}\|\nabla u(t)\|^2 - \frac{b}{p_1}\int_\Omega |u(t)|^{p(x)}dx, \end{aligned}$$

So,

$$\begin{aligned} &- E(t) \\ &\leq -\frac{1}{2}(1 - \int_0^t g(s)ds)\|\nabla u(t)\|^2 + b\int_\Omega \frac{|u(t)|^{p(x)}}{p(x)}dx \\ &\leq -\frac{L}{2}\|\nabla u(t)\|^2 + \frac{b}{p_1}\int_\Omega |u(t)|^{p(x)}dx, \end{aligned}$$

$\Longrightarrow$

$$\begin{aligned} &\frac{b}{p_1}\int_\Omega |u(t)|^{p(x)}dx \\ &\geq b\int_\Omega \frac{|u(t)|^{p(x)}}{p(x)}dx \\ &\geq \frac{1}{2}(1 - \int_0^t g(s)ds)\|\nabla u(t)\|^2 - E(t), \end{aligned}$$

then (3.9) we have $L\|\nabla u(t)\|^2 \geq \lambda_2^2$ and $E(0) = f(\lambda_2)$

$$\begin{aligned} &\frac{1}{2}(1 - \int_0^t g(s)ds)\|\nabla u(t)\|^2 - E(t) \\ &\geq \frac{L}{2}\|\nabla u(t)\|^2 - E(0) \\ &\geq \frac{1}{2}\lambda_2^2 - E(0) = \frac{1}{2}\lambda_2^2 - f(\lambda_2) \end{aligned} \tag{3.12}$$

then,

$$\begin{aligned} &\frac{1}{2}\lambda_2^2 - f(\lambda_2) \\ &= \frac{1}{2}\lambda_2^2 - \frac{1}{2}\lambda_2^2(t) + \frac{b}{p_1}B_1^{p_1}\lambda_2^{p_1}(t) \\ &= \frac{b}{p_1}B_1^{p_1}\lambda_2^{p_1}(t), \end{aligned}$$

the proof is completed. ■

Lemma 3.1.5 *Suppose that $p(x)$ is a measurable function on Ω satisfying(4). Then there exists a positive constant $C_1 = \max\{1, B^2\}$ such that*

$$\rho_{p(.)}^{\frac{s}{p_1}}(u) \leq C_1(|\nabla u(t)|^2 + p_{p(.)}(u)), \tag{3.13}$$

for any $u \in H_0^1(\Omega)$ and $2 \leq s \leq p_1$.

Lemma 3.1.6 *Suppose the conditions of lemma* (3.1.4) *hold. Then there exists a positive constantC_2 ,which depends on b,L and C_1,such that*

$$\rho_{p(.)}^{\frac{s}{p_1}}(u) \leq C_2(-G(t) - |u_t(t)|^2 + \rho_{p(.)}(u)), \tag{3.14}$$

for any $u \in H_0^1(\Omega)$ and $2 \leq s \leq p_1$.
proof. *Form* (3.10).

$$\int_\Omega \|u(t)\|^{p(x)}dx \geq B_1^{p_1}\lambda_2^{p_1},$$

and the relation $\lambda_2 > \lambda_1$, we find that

$$B_1^{p_1}\lambda_2^{p_1} > B_1^{p_1}\lambda_1^{p_1},$$

we have $B_1 = \max\{1, \frac{B}{L^{\frac{1}{2}}}, (\frac{1}{b})^{\frac{1}{2}}\}$ so

$$B_1^{p_1}\lambda_1^{p_1} = \max\{1, \frac{B}{L^{\frac{1}{2}}}, (\frac{1}{b})^{\frac{1}{2}}\}^{p_1}\lambda_1^{p_1} = \lambda_1^{p_1}(\frac{1}{b})^{\frac{p_1}{2}},$$

and we have $p_1 > 2$ so

$$\lambda_1^{p_1}(\frac{1}{b})^{\frac{p_1}{2}} > \frac{\lambda_1^2}{b},$$

and from it we find

$$B_1^{p_1}\lambda_1^{p_1} = \frac{\lambda_1^2}{b},$$

which implies

$$\begin{aligned} E_1 &= (\frac{1}{2} - \frac{1}{p_1})\lambda_1^2 \\ &\leq (\frac{1}{2} - \frac{1}{p_1})b\int_\Omega \|u(t)\|^{p(x)}dx, \end{aligned} \tag{3.15}$$

(3.7) *and* (3.1) *we have*

$$G(t) = E_2 - E(t), \tag{3.16}$$

implies

$$G(t) = E_2 - \frac{1}{2}\|u_t(t)\|^2 - \frac{1}{2}(1 - \int_0^t g(s)ds)\|\nabla u(t)\|^2 - \frac{1}{2}(g \circ \nabla u)(t) + b\int_\Omega \frac{|u(t)|^{p(x)}}{p(x)}dx,$$

it's clear that

$$\frac{l}{2}\|\nabla u(t)\|^2 \leq E_2 - G(t) - \frac{1}{2}\|u_t(t)\|^2 - \frac{1}{2}(g \circ \nabla u)(t) + b\int_\Omega \frac{|u(t)|^{p(x)}}{p(x)}dx,$$

as $E_2 \leq E_1$ *and* $-\frac{1}{2}(g \circ \nabla u)(t) \leq 0$ *we get*

$$\begin{aligned}
&\frac{l}{2}\|\nabla u(t)\|^2 \\
&\leq E_2 - G(t) - \frac{1}{2}|u_t(t)|^2 - \frac{1}{2}(g \circ \nabla u)(t) + b\int_\Omega \frac{|u(t)|^{p(x)}}{p(x)}dx \\
&\leq E_1 - G(t) - \frac{1}{2}\|u_t(t)\|^2 + \frac{b}{p_1}\int_\Omega |u(t)|^{p(x)}dx,
\end{aligned}$$

then (3.15) , $E_1 \leq (\frac{1}{2} - \frac{1}{p_1})b\int_\Omega \|u(t)\|^{p(x)}dx$ *obtain*

$$\begin{aligned}
&E_1 - G(t) - \frac{1}{2}\|u_t(t)\|^2 + \frac{b}{p_1}\int_\Omega |u(t)|^{p(x)}dx \\
&\leq -G(t) - \frac{1}{2}\|u_t(t)\|^2 + (\frac{1}{2} - \frac{1}{p_1} + \frac{1}{p_1})b\int_\Omega \|u(t)\|^{p(x)}dx \\
&\leq -G(t) - \frac{1}{2}|u_t(t)|^2 + \frac{b}{2}|u(t)|^{p(x)}dx,
\end{aligned} \tag{3.17}$$

Inserting (3.17) *into* (3.13), *with* (3.14) *holds.*
As a special case , we obtain the following.

Corollary 3.1.7 *Let the assumption in lemma* (3.1.4) *be satisfied then we get*

$$\|u_t(t)\|_{p_1}^s \leq C_2(-G(t) - \|u_t(t)\|^2 + \|u_t(t)\|_{p_1}^{p_1}),$$

for any $u \in H_0^1(\Omega)$ *and* $2 \leq s \leq p_1$.

Lemma 3.1.8 *Let the assumption in lemma* (3.1.4) *be satisfied. Then we have*

$$0 < G(0) \leq G(t) \leq \frac{b}{p_1}p_{p(.)}(u). \tag{3.18}$$

Proof. *We have of* (3.7)

$$E_2 - \|u_t(t)\|^2 - \frac{1}{2}(1 - \int_0^t g(s)ds)\|\nabla u(t)\|^2 - \frac{l}{2}(g \circ \nabla u)(t),$$

$E_2 \leq E_1$ *and by* (3.12), *we obtain*

$$\begin{aligned}
&E_2 - \|u_t(t)\|^2 - \frac{1}{2}(1 - \int_0^t g(s)ds)\|\nabla u(t)\|^2 - \frac{1}{2}(g \circ \nabla u)(t) \\
&\leq E_1 - \frac{l}{2}\|\nabla u(t)\|^2,
\end{aligned}$$

by using (3.9), *we get*

$$E_1 - \frac{l}{2}\|\nabla u(t)\|^2 \leq E_1 - \frac{1}{2}\lambda_2^2,$$

as $-\lambda_2 < -\lambda_1$ *obtain*

$$E_1 - \frac{1}{2}\lambda_2^2 < E_1 - \frac{1}{2}\lambda_1^2,$$

in (3.15), *we have* $E_1 = (\frac{1}{2} - \frac{1}{p_1})\lambda_1^2$ *so*

$$\begin{aligned} E_1 - \frac{1}{2}\lambda_1^2 &= (\frac{1}{2} - \frac{1}{p_1})\lambda_1^2 - \frac{1}{2}\lambda_1^2 \\ &= -\frac{1}{p_1}\lambda_1^2 < 0 \qquad \forall t \geq 0, \end{aligned} \tag{3.19}$$

because $G(t)$ *is increasing function, then we have*

$$0 < G(0) \leq G(t), \ \forall t > 0$$

we remind that

$$G(t) = E_2 - E(t),$$

and

$$-E(t) = -\frac{1}{2}\|u_t(t)\|^2 - \frac{1}{2}(1 - \int_0^t g(s)ds)\|\nabla u(t)\|^2 - \frac{1}{2}(g \circ \nabla u)(t) + \frac{|u(t)|^{p(x)}}{p(x)}dx,$$

with

$$\frac{|u(t)|^{p(x)}}{p(x)}dx > 0,$$

by (3.19) *we have*

$$E_2 - \frac{1}{2}\|u_t(t)\|^2 - \frac{1}{2}(1 - \int_0^t g(s)ds)\|\nabla u(t)\|^2 - \frac{1}{2}(g \circ \nabla u)(t) < 0,$$

then from the last inequality, we get

$$G(t) \leq b\int_\Omega \frac{|u(t)|^{p(x)}}{p(x)}dx,$$

and

$$\frac{1}{p(x)} \leq \frac{1}{p_1},$$

so

$$b\int_\Omega \frac{|u(t)|^{p(x)}}{p(x)}dx \leq \frac{b}{p_1}\int_\Omega |u(t)|^{p(x)}dx, \qquad \forall t \geq 0,$$

the proof is completed ■

Lemma 3.1.9 *Let the assumption in lemma* (3.1.4) *be satisfied. then we have, for some* $C_3 > 0$,

$$\rho_{p(.)}(u) \geq C_3 \|u(t)\|_{p_1}^{p_1}.$$

By lemma (3.1.9) and the Sobolev embedding $L^{p_1}(\Omega) \hookrightarrow L^{m_2}(\Omega)$, we obtain the following lemma.

Lemma 3.1.10 *Let the assumption of lemma* (3.1.4) *hold and* $m_2 < p_1$, *then we get, for some* $C_4 > 0$,

$$\int_\Omega |u(t)|^{m(x)} dx \leq C_4 (p_{p(.)}(u)^{\frac{m_1}{p_1}} + p_{p(.)}(u)^{\frac{m_2}{p_1}}). \tag{3.20}$$

3.2 blow up main theorem

Theorem 3.2.1 *suppose that the exponents* $p(x)$ *and* $m(x)$ *satisfy condition* (5) *and* $m_2 < p_1$, *assume that* g *satisfies* (1.4) *and* (1.5). *Then the solution of problem* (1)-(3) *blows up in finite time if*

$$E(0) < (1 - \frac{1-L}{p_1(p_1-2)}) E_1 \text{ and } \lambda_1 < L^{\frac{1}{2}} \|\nabla u_0\| \leq B_1^{-1}.$$

Proof. Let us define

$$F(t) = G^{1-\sigma}(t) + \epsilon(u(t), u_t(t)),$$

where $\epsilon > 0$ to be chosen later and

$$0 < \sigma < \min\{\frac{p_1 - 2}{2p_1}, \frac{p_1 - m_2}{p_1(m_2 - 1)}\}, \tag{3.21}$$

we have

$$F(t) = G^{1-\sigma}(t) + \epsilon(u(t), u_t(t)),$$

we have

$$\epsilon(u(t), u_t(t))' = \epsilon \|u_t\|^2 + \epsilon(u, u_{tt}),$$

and from the equation (1)

$$u_{tt} = \Delta u - \int_0^t g(t-s) \Delta u(s) ds - a|u_t|^{m(x)-2} u_t + b|u|^{p(x)-2} u.$$

So

$$\begin{aligned} \epsilon(u, u_{tt}) &= -\epsilon \|\nabla u(t)\|^2 + \epsilon \int_0^t g(t-s)(\nabla u(s), \nabla u(t)) ds \\ &- a\epsilon \int_\Omega |u_t(t)|^{m(x)-2} u_t(t) u(t) dx + b\epsilon \int_\Omega |u(t)|^{p(x)} dx, \end{aligned}$$

so

$$\begin{aligned} \epsilon(u(t), u_t(t))' &= \epsilon \|u_t(t)\|^2 - \epsilon \|\nabla u(t)\|^2 + \epsilon \int_0^t g(t-s)(\nabla u(s), \nabla u(t)) ds \\ &- a\epsilon \int_\Omega |u_t(t)|^{m(x)-2} u_t(t) u(t) dx + b\epsilon \int_\Omega |u(t)|^{p(x)} dx. \end{aligned}$$

We obtain

$$F'(t) = (1-\sigma)G^{-\sigma}(t)G'(t) + \epsilon\|u_t(t)\|^2 - \epsilon\|\nabla u(t)\|^2 + \epsilon\int_0^t g(t-s)(\nabla u(s), \nabla u(t))ds \tag{3.22}$$

$$- a\epsilon\int_\Omega |u_t(t)|^{m(x)-2}u_t(t)u(t)dx + b\epsilon\int_\Omega |u(t)|^{p(x)}dx.$$

We estimate $\int_0^t g(t-s)(\nabla u(s), \nabla u(t))ds$,

$$\begin{aligned}&\int_0^t g(t-s)(\nabla u(s), \nabla u(t))ds\\ &= \int_0^t g(t-s)(\nabla u(s) - \nabla u(t) + \nabla u(t), \nabla u(t)),\end{aligned}$$

then

$$\begin{aligned}&\int_0^t g(t-s)(\nabla u(s) - \nabla u(t) + \nabla u(t), \nabla u(t))\\ &= \int_0^t g(t-s)(\nabla u(s) - \nabla u(t), \nabla u(t)) + \int_0^t g(t-s)(\nabla u(t), \nabla u(t))ds\\ &= \int_0^t g(t-s)(\nabla u(s) - \nabla u(t), \nabla u(t)) + \int_0^t g(t-s)\|\nabla u(t)\|^2 ds,\end{aligned}$$

the equation (1.7) and young's inequality ($ab \le \frac{1}{2\epsilon_1}a^2 + \frac{\epsilon_1}{2}b^2$), we obtain

$$\begin{aligned}&\int_0^t g(t-s)(\nabla u(s) - \nabla u(t), \nabla u(t)) + \int_0^t g(t-s)\|\nabla u(t)\|^2 ds\\ &\ge \int_0^t g(t-s)(\nabla u(s) - \nabla u(t), \nabla u(t))\\ &\ge (1 - \frac{1}{2p_1(1-\epsilon_1)})\int_0^t g(s)ds\|\nabla u(t)\|^2 - \frac{p_1(1-\epsilon_1)}{2}(g\circ\nabla u)(t),\end{aligned} \tag{3.23}$$

combining (3.23) and (3.22), we get

$$\begin{aligned}F'(t) \ge{}& (1-\sigma)G^{-\sigma}(t)G'(t) + \epsilon\|u_t(t)\|^2 - \epsilon\|\nabla u(t)\|^2\\ &+ \epsilon(1 - \frac{1}{2p_1(1-\epsilon_1)})\int_0^t g(s)ds\|\nabla u(t)\|^2 - \frac{\epsilon p_1(1-\epsilon_1)}{2}(g\circ\nabla u)(t)\\ &- a\epsilon\int_\Omega |u_t(t)|^{m(x)-2}u_t(t)u(t)dx + b\epsilon\int_\Omega |u(t)|^{p(x)}dx.\end{aligned} \tag{3.24}$$

from (3.7) we have

$$\epsilon p_1(1-\epsilon_1)G(t) = \epsilon p_1(1-\epsilon_1)E_2 - \epsilon p_1(1-\epsilon_1)E(t),$$

and then (3.1) we have

$$- \epsilon p_1(1-\epsilon_1)E(t) = -\epsilon p_1(1-\epsilon_1)\frac{1}{2}\|u_t(t)\|^2 - \epsilon p_1(1-\epsilon_1)\frac{1}{2}(1 - \int_0^t g(s)ds)\|\nabla u(t)\|^2 -$$

$$\epsilon p_1(1-\epsilon_1)\frac{1}{2}(g\circ\nabla u)(t) + \epsilon p_1(1-\epsilon_1)b\int_\Omega \frac{|u(t)|^{p(x)}}{p(x)}dx.$$

Adding $\epsilon p_1(1-\epsilon_1)E_2$ to the equation (3.24) becomes:

$$F'(t) \geq (1-\sigma)G^{-\sigma}(t)G'(t) + \epsilon\|u_t(t)\|^2 - \epsilon\|\nabla u(t)\|^2 + \epsilon(1-\frac{1}{2p_1(1-\epsilon_1)})\int_0^t g(s)ds\|\nabla u(t)\|^2$$
$$+\epsilon p_1(1-\epsilon_1)E_2 - \epsilon p_1(1-\epsilon_1)E_2 - \frac{\epsilon p_1(1-\epsilon_1)}{2}(g\circ\nabla u)(t)$$
$$-a\epsilon\int_\Omega |u_t(t)|^{m(x)-2}u_t(t)u(t)dx + b\epsilon\int_\Omega |u(t)|^{p(x)}dx. \tag{3.25}$$

We have

$$\epsilon p_1(1-\epsilon_1)\frac{1}{2}\|u_t(t)\|^2 + \epsilon p_1(1-\epsilon_1)\frac{1}{2}(1-\int_0^t g(s)ds)\|\nabla u(t)\|^2 + \tag{3.26}$$
$$\epsilon p_1(1-\epsilon_1)\frac{1}{2}(g\circ\nabla u)(t) - \epsilon p_1(1-\epsilon_1)b\int_\Omega \frac{|u(t)|^{p(x)}}{p(x)}dx.$$

Adding (3.26) the equation (3.25) becomes:

$$F'(t) \geq (1-\sigma)G^{-\sigma}(t)G'(t) + \epsilon\|u_t(t)\|^2 - \epsilon\|\nabla u(t)\|^2 + \epsilon(1-\frac{1}{2p_1(1-\epsilon_1)})\int_0^t g(s)ds\|\nabla u(t)\|^2$$
$$+\epsilon p_1(1-\epsilon_1)E_2 - \epsilon p_1(1-\epsilon_1)E_2 - \frac{\epsilon p_1(1-\epsilon_1)}{2}(g\circ\nabla u)(t) - a\epsilon\int_\Omega |u_t(t)|^{m(x)-2}u_t(t)u(t)dx$$
$$+b\epsilon\int_\Omega |u(t)|^{p(x)}dx. + \epsilon p_1(1-\epsilon_1)\frac{1}{2}\|u_t(t)\|^2$$
$$-\epsilon p_1(1-\epsilon_1)\frac{1}{2}\|u_t(t)\|^2 + \epsilon p_1(1-\epsilon_1)\frac{1}{2}(1-\int_0^t g(s)ds)\|\nabla u(t)\|^2$$
$$-\epsilon p_1(1-\epsilon_1)\frac{1}{2}(1-\int_0^t g(s)ds)\|\nabla u(t)\|^2 + \epsilon p_1(1-\epsilon_1)\frac{1}{2}(g\circ\nabla u)(t) - \epsilon p_1(1-\epsilon_1)\frac{1}{2}(g\circ\nabla u)(t)$$
$$-\epsilon p_1(1-\epsilon_1)b\int_\Omega \frac{|u(t)|^{p(x)}}{p(x)}dx, \tag{3.27}$$

by using (1.7) and (3.27), obtain

$$F'(t) \geq (1-\sigma)G^{-\sigma}(t)G'(t) + \epsilon(1+\frac{p_1(1-\epsilon_1)}{2})\|u_t(t)\|^2 + \epsilon p_1(1-\epsilon_1)G(t)$$
$$+\epsilon\{(\frac{p_1(1-\epsilon_1)}{2}-1)(1-\int_0^t g(s)ds) - \frac{1}{2p_1(1-\epsilon_1)\int_0^t}g(s)ds\}\|\nabla u(t)\|^2 - \epsilon p_1(1-\epsilon_1)E_2$$
$$-a\epsilon\int_\Omega |u_t(t)u(t)dx + b\epsilon\epsilon_1\int_\Omega |u(t)|^{p(x)}dx, \tag{3.28}$$

by (1.4) : $L = 1-\int_0^t g(s)ds > 0 \Rightarrow (1-L) = \int_0^t g(s)ds > 0$, we obtain

$$(\frac{p_1}{2}-1)L - \frac{1}{2p_1}(1-L) > 0, \tag{3.29}$$

from (3.29) and $\lambda_2 > \lambda_1$, it is easy to find that there exist $\epsilon_1^* > 0$ and$T_0 > 0$ such that for $0 < \epsilon_1 < \epsilon_1^*$ and $t < T_0$,

$$\{(\frac{p_1(1-\epsilon_1)}{2}-1)L - \frac{1}{2p_1(1-\epsilon_1)}(1-L)\}\lambda_2^2 > \{(\frac{p_1(1-\epsilon_1)}{2}-1)L - \frac{1}{2p_1(1-\epsilon_1)}(1-L)\}\lambda_1^2, \tag{3.30}$$

by

$$E(0) < (1 - \frac{1-L}{p_1(p_1-2)L})E_1 = \frac{(\frac{p_1}{2}-1)L - \frac{1}{2p_1}(1-L)}{P_1 L}\lambda_1^2 < E_1,$$

and

$$\frac{(\frac{p_1(1-\epsilon_1)}{2}-1)L - \frac{1}{2p_1(1-\epsilon_1)}(1-L)}{p_1 L}\lambda_1^2 < \frac{(\frac{p_1}{2}-1)L - \frac{1}{2p_1}(1-L)}{p_1 L}\lambda_1^2,$$

we can take $\epsilon_1 > 0$ sufficiently small and $E_2 \in (E(0), E_1)$ such that

$$\frac{(\frac{p_1(1-\epsilon_1)}{2}-1)L - \frac{1}{2p_1(1-\epsilon_1)}(1-L)}{p_1 L}\lambda_1^2 \geq (1-\epsilon_1)E_2, \tag{3.31}$$

by (3.31) we get

$$\frac{(\frac{p_1(1-\epsilon_1)}{2}-1)L - \frac{1}{2p_1(1-\epsilon_1)}(1-L)}{p_1 L}\lambda_1^2 - (1-\epsilon_1)E_2 \geq 0,$$

using (1.4) : $L = 1 - \int_0^t g(s)ds > 0 \Rightarrow (1-L) = \int_0^t g(s)ds > 0$, we obtain

$$\frac{(\frac{p_1(1-\epsilon_1)}{2}-1)(1-\int_0^t g(s)ds) - \frac{1}{2p_1(1-\epsilon_1)}\int_0^t g(s)ds}{p_1 L}\lambda_1^2 - (1-\epsilon_1)E_2 \geq 0, \tag{3.32}$$

by (3.30)

$$\frac{\{(\frac{p_1(1-\epsilon_1)}{2}-1)(1-\int_0^t g(s)ds) - \frac{1}{2p_1(1-\epsilon_1)}\int_0^t g(s)ds\}}{p_1 L}\lambda_2^2 >$$

$$\frac{\{(\frac{p_1(1-\epsilon_1)}{2}-1)(1-\int_0^t g(s)ds) - \frac{1}{2p_1(1-\epsilon_1)}\int_0^t g(s)ds\}}{p_1 L}\lambda_1^2,$$

by (3.9) obtain

$$\frac{\{(\frac{p_1(1-\epsilon_1)}{2}-1)(1-\int_0^t g(s)ds) - \frac{1}{2p_1(1-\epsilon_1)}\int_0^t g(s)ds\}}{p_1 L}\lambda_2^2 \leq$$

$$L\|\nabla u(t)\|^2 \frac{\left\{(\frac{p_1(1-\epsilon_1)}{2}-1)(1-\int_0^t g(s)ds) - \frac{1}{2p_1(1-\epsilon_1)}\int_0^t g(s)ds\right\}}{p_1 L},$$

Are the same

$$\|\nabla u(t)\|^2 \frac{\{(\frac{p_1(1-\epsilon_1)}{2}-1)(1-\int_0^t g(s)ds)-\frac{1}{2p_1(1-\epsilon_1)}\int_0^t g(s)ds\}}{p_1},$$

the inequality (3.32) rewritten as follows

$$\Big\{(\frac{p_1(1-\epsilon_1)}{2}-1)(1-\int_0^t g(s)ds)-\frac{1}{2p_1(1-\epsilon_1)}\int_0^t g(s)ds\Big\}\|\nabla u(t)\|^2 - p_1(1-\epsilon_1)E_2 \geq 0, \tag{3.33}$$

inserting (3.33) into (3.28) ,we get

$$F'(t) \geq (1-\sigma)G^{-\sigma}(t)G'(t) + \epsilon(1+\frac{p_1(1-\epsilon_1)}{2})\|u_t(t)\|^2 + \epsilon p_1(1-\epsilon_1)G(t) + b\epsilon\epsilon_1\int_\Omega |u(t)|^{p(x)}dx -$$

$$a\epsilon\int_\Omega |u_t(t)|^{m(x)-2}u_t(t)u(t)dx, \qquad for \qquad t \geq t_0. \tag{3.34}$$

by inequality the young $ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$ such that $p = m(x), q = \frac{m(x)}{m(x)-1}$,

$$\int_\Omega |u_t(t)|^{m(x)-1}u(t)dx \leq \frac{1}{m_1}\int_\Omega |u(t)|^{m(x)}dx + \frac{m_2-1}{m_2}\int_\Omega |u_t(t)|^{m(x)}dx,$$

multiply by η and using the inequality $ab \leqslant \frac{\eta a^2}{2} + \frac{1}{2\eta}b^2$ obtain

$$\int_\Omega |u_t(t)|^{m(x)-1}u(t)dx \leq \frac{1}{m_1}\int_\Omega \eta^{m(x)}|u(t)|^{m(x)}dx + \frac{m_2-1}{m_2}\int_\Omega \eta^{-\frac{m(x)}{m(x)-1}}|u_t(t)|^{m(x)}dx, \qquad \forall\eta > 0. \tag{3.35}$$

By taking η so that

$$\eta^{-\frac{m(x)}{m(x)-1}} = \beta G^{-\sigma}(t)$$

for a large constant β to be specified later, and substituting in (3.35)

$$\int_\Omega |u_t(t)|^{m(x)-1}u(t)dx$$
$$\leq \frac{1}{m_1}\int_\Omega \beta^{1-m(x)}G^{\sigma(m(x)-1)}(t)|u(t)|^{m(x)}dx + \frac{(m_2-1)\beta}{m_2}G^{-\sigma}(t)|u_t(t)|^{m(x)}.$$

by (3.8) we obtain

$$\int_\Omega |u_t(t)|^{m(x)-1}u(t)dx \tag{3.36}$$
$$\leq \frac{1}{m_1}\int_\Omega \beta^{1-m(x)}G^{\sigma(m(x)-1)}(t)|u(t)|^{m(x)}dx + \frac{(m_2-1)\beta}{am_2}G^{-\sigma}(t)G'(t). \tag{3.37}$$

we have

$$\frac{1}{m_1}\int_\Omega \beta^{1-m(x)}G^{\beta(m(x)-1)}(t)|u(t)|^{m(x)}dx \leq \frac{\beta^{1-m_1}}{m_1}\int_\Omega G^{\sigma(m(x)-1)}(t)|u(t)|^{m(x)}dx,$$

by (1.3) we have $G(t)^m \leq \max\{G^{(m)_1}(t), G^{(m)_2}\}$

$$\frac{\beta^{1-m_1}}{m_1}\int_\Omega G^{\sigma(m(x)-1)}(t)|u(t)|^{m(x)}dx$$
$$\leq \frac{\beta^{1-m_1}}{m_1}\max\left\{G^{\sigma(m_1-1)}(t), G^{\sigma(m_2-1)}(t)\right\}\int_\Omega |u(t)|^{m(x)}dx,$$

by using (3.18) we find

$$\frac{\beta^{1-m_1}}{m_1}\max\left\{G^{\sigma(m_1-1)}(t), G^{\sigma(m_2-1)}(t)\right\}\int_\Omega |u(t)|^{m(x)}dx$$
$$\leq \frac{\beta^{1-m_1}C_4}{m_1}\max\left\{G^{\sigma(m_1-1)}(t), G^{\sigma(m_2-1)}(t)\right\}(p_{p(.)}(u)^{\frac{m_1}{p_1}} + p_{p(.)}(u)^{\frac{m_2}{p_1}}),$$

by (3.18) obtain

$$\frac{\beta^{1-m_1}C_4}{m_1}\max\{G^{\sigma(m_1-1)}(t), G^{\sigma(m_2-1)}(t)\}(p_{p(.)}(u)^{\frac{m_1}{p_1}} + p_{p(.)}(u)^{\frac{m_2}{p_1}})$$
$$\leq \frac{\beta^{1-m_1}C_4}{m_1}\max\left\{(\frac{b}{p_1}p_{p(.)}(u))^{\sigma(m_1-1)}, (\frac{b}{p_1}p_{p(.)}(u))^{\sigma(m_2-1)}\right\}(p_{p(.)}(u)^{\frac{m_1}{p_1}} + p_{p(.)}(u)^{\frac{m_2}{p_1}}),$$

by hölder inequality, we get

$$\frac{\beta^{1-m_1}C_4}{m_1}\max\left\{(\frac{b}{p_1}p_{p(.)}(u))^{\sigma(m_1-1)}, (\frac{b}{p_1}p_{p(.)}(u))^{\sigma(m_2-1)}\right\}(p_{p(.)}(u)^{\frac{m_1}{p_1}} + p_{p(.)}(u)^{\frac{m_2}{p_1}})$$
$$\leq \frac{\beta^{1-m_1}C_4}{m_1}\max\left\{(\frac{b}{p_1})^{\sigma(m_1-1)}(p_{p(.)}(u)^{\sigma(m_1-1)+\frac{m_1}{p_1}} + p_{p(.)}(u)^{\sigma(m_1-1)+\frac{m_2}{p_1}}),\right. \tag{3.38}$$
$$\left.(\frac{b}{p_1})^{\sigma(m_2-1)}(p_{p(.)}(u)^{\sigma(m_2-1)+\frac{m_1}{p_1}} + p_{p(.)}(u)^{\sigma(m_2-1)+\frac{m_2}{p_1}})\right\},$$

from $2 \leq m_1 \leq m_2 < p_1$,we obtain

$$\max\{\sigma(m_1-1)+\frac{m_1}{p_1}, \sigma(m_1-1)+\frac{m_2}{p_1}, \sigma(m_2-1)+\frac{m_1}{p_1}, \sigma(m_2-1)+\frac{m_2}{p_1}\} = \sigma(m_2-1)+\frac{m_2}{p_1},$$

by the inequality (3.38), we have $s = m_2 + \sigma p_1(m_2-1) \leq p_1$

$$\frac{1}{m_1}\int_\Omega \beta^{1-m(x)}G^{\beta(m(x)-1)}(t)|u(t)|^{m(x)}dx$$
$$\leq \frac{\beta^{1-m_1}C_4}{m_1}\max\left\{(\frac{b}{p_1})^{\sigma(m_1-1)}(p_{p(.)}(u)^{\sigma(m_1-1)+\frac{m_1}{p_1}} + p_{p(.)}(u)^{\sigma(m_1-1)+\frac{m_2}{p_1}}),\right.$$
$$\left.(\frac{b}{p_1})^{\sigma(m_2-1)}(p_{p(.)}(u)^{\sigma(m_2-1)+\frac{m_1}{p_1}} + p_{p(.)}(u)^{\sigma(m_2-1)+\frac{m_2}{p_1}})\right\},$$

and by lemma (3.14) we get

$$\frac{1}{m_1}\int_\Omega \beta^{1-m(x)}G^{\beta(m(x)-1)}(t)|u(t)|^{m(x)}dx$$
$$\leq \frac{\beta^{1-m_1}C_4}{m_1}\max\{(\frac{b}{p_1})^{\sigma(m_1-1)}, (\frac{b}{p_1})^{\sigma(m_1-1)}\}(\leq C_2\Big(-G(t) - |u_t(t)|^2 + p_{p(.)}(u)\Big)$$

so

$$\frac{1}{m_1}\int_\Omega \beta^{1-m(x)}G^{\beta(m(x)-1)}(t)|u(t)|^{m(x)}dx$$
$$\leq \frac{\beta^{1-m_1}C_4}{m_1}\max\Big\{(\frac{b}{p_1})^{\sigma(m_1-1)},(\frac{b}{p_1})^{\sigma(m_1-1)}\Big\}\Big(\ \leq 2C_2(-G(t)-|u_t(t)|^2+p_{p(.)}(u))\Big)$$

and from the last inequality

$$\frac{1}{m_1}\int_\Omega \beta^{1-m(x)}G^{\beta(m(x)-1)}(t)|u(t)|^{m(x)}dx$$
$$\leq \frac{2C_2\beta^{1-m_1}C_4}{m_1}\max\Big\{(\frac{b}{p_1})^{\sigma(m_1-1)},(\frac{b}{p_1})^{\sigma(m_1-1)}\Big\}\Big(G(t)-|u_t(t)|^2+p_{p(.)}(u)\Big)$$

from it we get

$$\frac{1}{m_1}\int_\Omega \beta^{1-m(x)}G^{\sigma(m(x)-1)}(t)|u(t)|^{m(x)}dx$$
$$\leq C_5\beta^{1-m_1}(-G(t)-\|u_t(t)\|^2+p_{p(.)}(u)), \tag{3.39}$$

where

$$C_5=\frac{2C_2C_4}{m_1}\max\Big\{(\frac{b}{p_1})^{\sigma(m_1-1))},(\frac{b}{p_1})^{\sigma(m_2-1))}\Big\},$$

by equation (3.19) we have

$$F'(t)\geq(1-\sigma)G^{-\sigma}(t)G'(t)+\epsilon(1+\frac{p_1(1-\epsilon_1)}{2})\|u_t(t)\|^2+\epsilon p_1(1-\epsilon_1)G(t)+b\epsilon\epsilon_1\int_\Omega|u(t)|^{p(x)}dx-$$
$$a\epsilon\int_\Omega|u_t(t)|^{m(x)-2}u_t(t)u(t)dx,$$

and using the equation (3.36) obtain

$$F'(t)\geq(1-\sigma)G^{-\sigma}(t)G'(t)+\epsilon(1+\frac{p_1(1-\epsilon_1)}{2})\|u_t(t)\|^2+\epsilon p_1(1-\epsilon_1)G(t)+b\epsilon\epsilon_1\int_\Omega|u(t)|^{p(x)}dx-$$
$$a\epsilon(\frac{1}{m_1}\int_\Omega\beta^{1-m(x)}G^{\sigma(m(x)-1)}(t)|u(t)|^{m(x)}dx)+\frac{(m_2-1)\beta}{am_2}G^{-\sigma}(t)G'(t)),$$

by using (3.39) obtain

$$F'(t)\geq(1-\sigma)G^{-\sigma}(t)G'(t)+\epsilon(1+\frac{p_1(1-\epsilon_1)}{2})\|u_t(t)\|^2+\epsilon p_1(1-\epsilon_1)G(t)+b\epsilon\epsilon_1\int_\Omega|u(t)|^{p(x)}dx-$$
$$a\epsilon(C_5\beta^{1-m_1}(-G(t)-\|u_t(t)\|^2+p_{p(.)}(u)+\frac{(m_2-1)\beta}{am_2}G^{-\sigma}(t)G'(t)),$$

and by (1.1) obtain

$$F'(t)\geq[(1-\sigma)-\frac{(m_2-1)\epsilon\beta}{m_2}]G^{-\sigma}(t)G'(t)+\epsilon(1+aC_5\beta^{1-m_1}+\frac{p_1(1-\epsilon_1)}{2})\|u_t(t)\|^2+$$
$$\epsilon(p_1(1-\epsilon_1)+aC_5\beta^{1-m_1})G(t)+\epsilon(b\epsilon_1-aC_5\beta^{1-m_1})p_{p(.)(u)}\qquad for\qquad t\geq T_0.$$

First, we take $\beta > 0$ large enough such that

$$b\epsilon_1 - aC_5\beta^{1-m_1} > 0,$$

then β is fixed, we select $\epsilon > 0$ small enough so that

$$(1-\sigma) - \frac{(m_2-1)\epsilon\beta}{m_2} > 0, \qquad G^{1-\sigma}(T_0) + \epsilon\int_\Omega u(T_0)u_t(T_0)dx > 0,$$

and we have $G'(t) \geq 0$ from then obtain

$$F'(t) \geq C(\|u_t(t)\|^2 + \|u(t)\|_{p_1}^{p_1} + G(t)), \qquad for \qquad t \geq T_0. \tag{3.40}$$

Here and in the sequel ,C denotes a generic positive constant. Hence , we find that

$$F'(t) \geq F(T_0) > 0 \qquad for \qquad t \geq T_0.$$

on the other hand, using the similar arguments in Massaoudi et al, we have

$$F^{\frac{1}{1-\sigma}}(t) \leq C(\|u_t(t)\|^2 + \|u(t)\|_{p_1}^{p_1} + G(t)), \qquad for \qquad t \geq T_0. \tag{3.41}$$

Indeed ,we first note that from the embedding $L^{p_1}(\Omega) \hookrightarrow L^2(\Omega)$

$$\left|\int_\Omega u(t)u_t(t)dx\right| \leq \|u(t)\| + \|u(t)_t\| \leq C\|u(t)\|_{p_1} + \|u(t)_t\|,$$

using young inequality with $\frac{1-2\sigma}{2(1-\sigma)} + \frac{1}{2(1-\sigma)} = 1$,we obtain

$$\left|\int_\Omega u(t)u_t(t)dx\right|^{\frac{1}{1-\sigma}} \leq C\|u(t)\|_{P_1}^{\frac{1}{1-\sigma}} + \|u(t)_t\|^{\frac{1}{1-\sigma}} \leq C(\|u(t)\|_{p_1}^{\frac{2}{1-2\sigma}} + \|u(t)_t\|^2).$$

By exploiting (3.21) and Corollary (3.1.7), for $s = \frac{2}{1-2\sigma} \leq p_1$, we find that

$$\left|\int_\Omega u(t)u_t(t)dx\right|^{\frac{1}{1-\sigma}} \leq C(\|u_t(t)\|^2 + \|u(t)\|_{p_1}^{p_1} + G(t)),$$

using (3.40) and (3.41) obtain

$$F'(t) \geq CF^{\frac{1}{1-\sigma}}(t), \qquad fort \geq T_0 \tag{3.42}$$

by (3.42) obtain

$$\int_{T_0}^t F'(t)F^{-\frac{1}{1-\sigma}}(t) \geq \int_{T_0}^t C, \tag{3.43}$$

Using the fact that $u'u^n = \frac{1}{n+1}u^{n+1} + c$, then (3.43) became

$$\left[\frac{1-\sigma}{-\sigma}F^{\frac{-\sigma}{1-\sigma}}(t)\right]_{T_0}^t. \geq \int_{T_0}^t C,$$

$$\frac{1-\sigma}{-\sigma}F^{\frac{-\sigma}{1-\sigma}}(t)-\frac{1-\sigma}{-\sigma}F^{\frac{-\sigma}{1-\sigma}}(T_0)\geq\int_{T_0}^{t}C. \tag{3.44}$$

Multiply equation (3.44) by $\dfrac{-\sigma}{1-\sigma}$

$$(\frac{-\sigma}{1-\sigma})(\frac{1-\sigma}{-\sigma}F^{\frac{-\sigma}{1-\sigma}}(t)-\frac{1-\sigma}{-\sigma}F^{\frac{-\sigma}{1-\sigma}}(T_0))\leq(\frac{-\sigma}{1-\sigma})(\int_{T_0}^{t}C),$$

so

$$F^{\frac{-\sigma}{1-\sigma}}(t)\leq F^{\frac{-\sigma}{1-\sigma}}(T_0)+(\frac{-\sigma Ct}{1-\sigma})-\frac{-\sigma CT_0}{1-\sigma}.$$

Then

$$F^{\frac{-\sigma}{1-\sigma}}(t)\leq F^{\frac{-\sigma}{1-\sigma}}(T_0)-(\frac{\sigma Ct}{1-\sigma})+\frac{\sigma CT_0}{1-\sigma},$$

we obtain

$$\frac{1}{F^{\frac{-\sigma}{1-\sigma}}(t)}\geq\frac{1}{F^{\frac{-\sigma}{1-\sigma}}(T_0)-(\frac{\sigma Ct}{1-\sigma})+\frac{\sigma CT_0}{1-\sigma}},$$

so

$$F^{\frac{\sigma}{1-\sigma}}(t)\geq\frac{1}{F^{\frac{-\sigma}{1-\sigma}}(T_0)-(\frac{\sigma Ct}{1-\sigma})+\frac{\sigma CT_0}{1-\sigma}},$$

consequently, the solution of problem (1)-(3) blow up in finite time

$$T^{*}\leq\frac{1-\sigma}{C\sigma}\frac{C\sigma T_0}{\sigma}+\frac{1-\sigma}{C\sigma}F^{\frac{\sigma}{\sigma-1}}(T_0).$$

which established the proof of main theorem. ■

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