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Dedicate

To All Who Are Humble In Seeking Science.

With respect, Yasmina

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large *First of all, I would like to thank* **Allah** *for all His blessings and favors.*

Then, I would like to express my gratitude to **My family** *for the invaluable support they provided during my work journey.*

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الملخص

في هذه الدراسة، نركز على فحص وجود، وتفرّد، واستقرار الحلول لجملة معادلات التفاضلية كسرية تحت شروط حدودية في فضاء بناخ. لتحقيق ذلك، نستخدم تقنية تحويل مشكلتنا إلى البحث عن نقطة ثابتة لمعادلات التكامل. النتائج التي نحصل عليها ترتبط بنظرية بناخ للنقطة الثابتة لدراسة وجود وتفرّد الحلول، ومن ثم ندرس استقرار أولام-هييرس لمشكلتنا.

الكلمات المفتاحية: جمل معادلات، المعادلات التفاضلية الكسرية، نظرية نقطة الثابتة

Résumé

Dans cette étude, nous nous concentrons sur l'examen de l'existence, de l'unicité et de la stabilité des solutions pour un système couplé d'équations différentielles fractionnaires soumis à des conditions aux limites dans un espace de Banach. Pour ce faire, nous utilisons une technique qui consiste à transformer notre problème en une recherche d'un point fixe pour des équations intégrales. Les résultats que nous obtenons sont liés à la théorie du point fixe de Banach pour étudier l'existence et l'unicité des solutions, et ensuite, nous analysons la stabilité d'Ulam-Hyers pour notre problème.

Mots clés : système couplé, équations différentielles fractionnaires, théorie du point fixe

Abstract

In this study, we focus on investigating the existence, uniqueness, and stability of solutions for a coupled system of fractional differential equations subject to boundary conditions in a Banach space. To achieve this, we employ a technique that involves transforming our problem into the search for a fixed point of integral equations. The results we obtain are linked to Banach's fixed-point theory for studying the existence and uniqueness of solutions, and subsequently, we analyze Ulam-Hyers stability for our problem.

Keywords: coupled system, fractional differential equations , fixed point theory

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Introduction

Fractional differential equations have garnered significant interest across various application domains, including economics, signal processing, image recognition, optical systems, aerodynamics, biophysics, materials science, mechanical systems, and control theory.

Most applied phenomena and processes can be modeled as coupled systems of classical or non-integer order differential equations. Consequently, many researchers have focused on establishing the existence theory of solutions for these systems. A substantial body of literature is dedicated to this topic, including key studies such as [1, 2, 3, 4, 5, 6, 7].

Recently, Sudsutad and Tariboon [8] investigated the sufficient conditions for the existence of solutions for the following class with three-point integral boundary conditions:

$$\begin{cases} {}^{C}\mathfrak{D}^{\ell_1}\mu(\mathfrak{t}) = K(\mathfrak{t}, v(\mathfrak{t})), \quad \mathfrak{t} \in \mathcal{O} := [0,1], \\ \mu(0) = 0, \quad v(1) = \frac{1}{\Gamma(\theta)} \int_0^{\nu} (\nu - s)^{\theta} \mu(\mathfrak{t}) ds, \end{cases}$$

Kamal Shah et al. [9] aimed to establish the existence theory for the following movable type boundary value problem:

$$\begin{cases} {}^{C}\mathfrak{D}^{\ell_1}\mu(\mathfrak{t}) = K\left(\mathfrak{t}, v(\mathfrak{t})\right), \quad \mathfrak{t} \in \mathcal{O}, \\ {}^{C}\mathfrak{D}_2^{\ell_2} v(\mathfrak{t}) = M\left(\mathfrak{t}, \mu(\mathfrak{t})\right), \quad \mathfrak{t} \in \mathcal{O}, \\ \mu(0) = \mu(1) = \int_0^{\nu} \mu(\mathfrak{t}) d\mathfrak{t}, \\ v(0) = v(1) = \int_0^{\xi} v(\mathfrak{t}) d\mathfrak{t}. \end{cases}$$

Inspired by the aforementioned work, this thesis aims to establish the existence and uniqueness theory for the following movable type boundary value problem:

This study is focused on establishing the existence, uniqueness and stability of solutions for a coupled system of fractional differential equations

$$\begin{cases} {}^{RL}\mathcal{D}_{0^+}^{\varrho_1}\mathbf{x}(\mathfrak{z}) & = f(\mathfrak{z}, \mathbf{x}(\mathfrak{z}), \mathbf{y}(\mathfrak{z})), \quad \mathfrak{z} \in \mathcal{O} = [0,1] \\ {}^{RL}\mathcal{D}_{0^+}^{\varrho_2}\mathbf{y}(\mathfrak{z}) & = g(\mathfrak{z}, \mathbf{x}(\mathfrak{z}), \mathbf{y}(\mathfrak{z})), \quad \mathfrak{z} \in \mathcal{O} = [0,1], \end{cases} \tag{0.0.1}$$

subjected with the following mixed boundary conditions

$$\begin{cases} \mathbf{x}(0) = 0, \quad \mathbf{x}(1) = \int_0^1 \psi(\mathfrak{z})\mathbf{x}(\mathfrak{z})d(\mathfrak{z}), \\ \mathbf{y}(0) = 0, \quad \mathbf{y}(1) = \int_0^1 \psi(\mathfrak{z})\mathbf{y}(\mathfrak{z})d(\mathfrak{z}), \end{cases} \tag{0.0.2}$$

where ${}^{RL}\mathcal{D}_{0^+}^{\theta}$ denote the standard Riemann-Liouville fractional derivatives of order θ, $1 < \varrho_1, \varrho_2 < 2$, $f, g \in \mathcal{C}\big([0,1] \times \mathbb{R}^2, \mathbb{R}\big)$.

This work is structured into three chapters:

The first chapter is dedicated to introducing the fundamental concepts and fractional tools utilized throughout this study. We provide an overview of the essential notions and preliminary properties associated with the two primary approaches of fractional differentiation: the Riemann-Liouville and Caputo approaches.

In the second chapter, we focus on a coupled system of fractional differential equations and establish results concerning their existence and uniqueness. These findings are derived through the application of the Banach fixed-point theorem. We conclude this chapter with an illustrative example.

The final chapter revisits the same coupled system of fractional differential equations examined in Chapter 2. Here, we concentrate on proving the stability of solutions. Ulam-Hyers type stability is checked. We conclude this chapter with another illustrative example.

Chapter 1

Preliminaries

In this chapter, we recall some basic results needed for our investigations. For more details, see the following references [10, 11, 12, 13, 14, 15, 17, 18, 19]

1.1 Useful Functions

1.1.1 The Gamma Function

The Gamma function was first introduced by Swiss mathematician Leonhard Euler in the 18th century as a generalization of the factorial function to non-integer values. It was later studied and extended by other mathematicians, including Adrien-Marie Legendre, who gave it the name "Gamma function." This function plays a crucial role in various fields of mathematics and physics due to its deep connections with factorials, integrals, and special functions.

Definition 1.1.1 *The Gamma function $\Gamma(z)$ is defined for complex numbers z with a real part greater than zero ($\Re(z) > 0$) by the integral*

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt. \tag{1.1.1}$$

Example 1 .

- *For any positive integer n, $\Gamma(n) = (n-1)!$. This demonstrates how the Gamma function extends the factorial function, which is only defined for non-negative integers.*

- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. *This is a remarkable result that connects the Gamma function to the value of* π.

Characteristics

The Gamma function has several important properties:

1. **Recurrence Relation:** $\Gamma(z+1) = z\Gamma(z)$. This property is useful for evaluating the Gamma function for arguments that differ by integers.

2. **Analytic Continuation:** Although initially defined only for $\Re(z) > 0$, the Gamma function can be extended to all complex numbers except the non-positive integers, where it has simple poles.

3. **Euler's Reflection Formula:** For any z not a non-positive integer,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}. \tag{1.1.2}$$

   This formula establishes a deep connection between the Gamma function and trigonometric functions.

1.1.2 The Beta Function

The Beta function, also known as the Euler integral of the first kind, was first introduced by the Swiss mathematician Leonhard Euler in the 18th century. It arose from his work on integrals and was extensively studied in connection with the Gamma function. The Beta function is significant in mathematics due to its applications in calculus, probability, and the theory of special functions.

Definition 1.1.2 *The Beta function, denoted as* $B(x, y)$*, for real numbers* $x, y > 0$*, is defined by the integral*

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt. \tag{1.1.3}$$

This function serves as a continuous analogue of the binomial coefficients and can also be expressed in terms of the Gamma function Γ as

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{1.1.4}$$

Proposition 1.1.1 *The Beta function possesses several important properties:*

1. ***Symmetry:*** $B(x, y) = B(y, x)$. *This property reflects the symmetry of the function with respect to its arguments.*

2. ***Relation to Gamma Function:*** *As previously mentioned, the Beta function can be expressed in terms of the Gamma function, establishing a link between these two fundamental functions in analysis.*

3. ***Reduction Formula:*** *For any positive integers m and n,*
$$B(m, n) = \frac{(m-1)!(n-1)!}{(m+n-1)!}. \tag{1.1.5}$$
This illustrates the relationship between the Beta function and factorials, providing a combinatorial interpretation.

1.2 Fixed point

Definition 1.2.1 *(Fixed point)*
Let T be an application of a set E in it itself. We call fixed point of T any point $e \in E$ such that $T(e) = e$.

Theorem 1.1 *(Banach contraction principle)*
Let E be a complete metric space and let $T : E \to E$ be a contracting application, i.e. there exists $0 < k < 1$ such that $d(Tx, Ty) \leq k(x, y), \forall x, y \in E$. Then T admits a single fixed point $e \in E$.

1.3 Fractional integral

Definition 1.3.1 *Let $f : [a, b) \to \mathbb{R}$ be a continuous function on the interval $[a, b)$ where $b \in \mathbb{R}$. The n-th primitive (or n-th iterated integral) of f is defined as follows:*

$$\mathcal{I}^n f(t) = \int_a^t dt_1 \int_a^{t_1} dt_2 \cdots \int_a^{t_{n-1}} f(t_n)\, dt_n = \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s)\, ds \quad \textit{for } n \in \mathbb{N}^*.$$

This formula is known as Cauchy's formula for iterated integrals.

Definition 1.3.2
The Riemann-Liouville fractional integral of order $\alpha \in \mathbb{R}_+$ of the continuous function $f : [a, b) \to \mathbb{R}$ defined by

$$\mathcal{I}_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds. \quad (-\infty \leq a < t < \infty)$$

Example 2 *Consider the function $f(x) = (x-a)^\beta$, Then*

$$\mathcal{I}_a^\alpha (x-a)^\beta = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} (t-a)^\beta dt.$$

We get to

$$\mathcal{I}_a^\alpha (x-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\alpha)} (x-a)^{\beta+\alpha} \tag{1.3.1}$$

we can see that this is a generalization of the case $\alpha = 1$ where we have

$$\begin{aligned} \mathcal{I}_a^1 (x-a)^\beta =& \frac{\Gamma(\beta+1)}{\Gamma(\beta+2)} (x-a)^{\beta+1} \\ =& \frac{(x-a)^{\beta+1}}{\beta+1} \end{aligned}$$

Properties:

1. $\mathcal{I}_a^\alpha f(t)$ exists almost everywhere for $t \in [a, b]$ if $f \in L^1([a, b])$ and $\alpha > 0$.

2. For $f \in L^1([a, b])$, the Riemann-Liouville fractional integral has the property of a semi-group:

$$\mathcal{I}_a^\alpha [\mathcal{I}_a^\beta f(t)] = \mathcal{I}_a^{\alpha+\beta} f(t) = \mathcal{I}_a^\beta [\mathcal{I}_a^\alpha f(t)] \quad \text{for} \quad \alpha > 0, \beta > 0. \tag{1.3.2}$$

This holds almost everywhere for $t \in [a, b]$.

3. For any function $f \in L^1([a, b])$, the fractional integral has the property of linearity, i.e.,

$$\mathcal{I}^\alpha (\lambda f(t) + g(t)) = \lambda \mathcal{I}^\alpha f(t) + \mathcal{I}^\alpha g(t), \quad \alpha \in \mathbb{R}_+, \ \lambda \in \mathbb{C}. \tag{1.3.3}$$

1.4 Fractional derivation in the sense of Rimann-Liouville

Definition 1.4.1
Let f be a function that can be integrated on $[a,b]$ and $\alpha \in]n-1,n[$ with $n \in \mathbb{N}^*$. We call the fractional order α derivative of a function f in the sense of Riemann-Liouville ***left*** *and* ***right*** *is defined by :*

$$\mathcal{D}_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau. \tag{1.4.1}$$

and

$$\mathcal{D}_{b^-}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d^n}{dt^n}\right) \int_t^b (\tau-t)^{n-\alpha-1} f(\tau) d\tau. \tag{1.4.2}$$

respectively.
where $n = [\alpha]+1$ and $[\alpha]$ designates the integer part of the real number α.

The relation between the fractional derivative and the ordinary derivative, we have:

$$\mathcal{D}_{a^+}^{\alpha} f(t) = \mathcal{D}^n(\mathcal{I}_{a^+}^{n-\alpha} f(t)), \tag{1.4.3}$$

and

$$\mathcal{D}_{b^-}^{\alpha} f(t) = (-D)^n(I_{b^-}^{n-\alpha} f(t)). \tag{1.4.4}$$

In particular , when $\alpha = n \in \mathbb{N}$ we get :

$$\mathcal{D}_{a^+}^{n} f(t) = f^{(n)}(t) \quad and \quad \mathcal{D}_{b^-}^{n} f(t) = (-1)^n f^{(n)}(t).$$

Example 3 *1- The derivative of $f(t) = (t-a)^{\beta}$ in the sense of Riemann-Liouville*
Let α be non-integer and $0 \leq n-1 < \alpha < n$ and $\beta > -1$ then we have :

$$\mathcal{D}^{\alpha}(t-a)^{\beta} = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int (t-\tau)^{n-\alpha-1} d\tau \tag{1.4.5}$$

After simple calculation we find:

$$\mathcal{D}^{\alpha}(t-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} (t-a)^{\beta-\alpha}$$

Then

$$\mathcal{D}^{\alpha}(t-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}, \tag{1.4.6}$$

For $\alpha = 1.5$ *and* $\beta = 1.5$ *we have :*

$$\mathcal{D}^{1.5}t^{1.5} = \frac{\Gamma(2.5)}{\Gamma(1)} = \Gamma(2.5) \tag{1.4.7}$$

2- The derivative of $f(t) = C$

$$\mathcal{D}^{\alpha}C = \frac{C}{\Gamma(1-\alpha)}(t-a)^{-\alpha} \tag{1.4.8}$$

For the demonstration it is enough to take $\beta = 0$.

1.5 Fractional derivation in the sense of Caputo

Definition 1.5.1
Let f *a function such that* $\frac{d^n}{dt^n}f \in L_1([a,b])$ *and* $\alpha \in]n-1, n[$ *with* $n \in \mathbb{N}^*$. *The fractional derivative of order* α *of* f *in the Caputo sense on the* ***left*** *and on the* ***right*** *are defined by:*

$$^{C}\mathcal{D}^{\alpha}_{a^+}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-\tau)^{n-\alpha-1}f^{(n)}(\tau)d\tau \tag{1.5.1}$$

and

$$^{C}\mathcal{D}^{\alpha}_{b^-}f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_t^b (\tau-t)^{n-\alpha-1}f^{(n)}(\tau)d\tau \tag{1.5.2}$$

respectively.

The relation between the fractional derivative and the ordinary:

$$^{C}\mathcal{D}^{\alpha}_{a^+}f(t) = (I^{n-\alpha}_{a^+}D^n f)(t) \tag{1.5.3}$$

and

$$^{C}\mathcal{D}^{\alpha}_{b^-}f(t) = (I^{n-\alpha}_{b^-}D^n f)(t) \tag{1.5.4}$$

If $\alpha = n \in \mathbb{N}$ then:

$$^{C}\mathcal{D}^{n}_{a^+} = f^{(n)}(t) \quad \textit{and} \quad ^{C}\mathcal{D}^{n}_{b^-} = f^{(n)}(t). \tag{1.5.5}$$

- The fractional derivative operator is linear, let f and g betwo functions, for λ and $\mu \in \mathbb{R}$, then: $\mathcal{D}^\alpha(\lambda f + \mu g)$ exists, and we have:

$$\mathcal{D}^\alpha(\lambda f + \mu g)(t) = \lambda \mathcal{D}^\alpha f(t) + \mu \mathcal{D}^\alpha g(t) \tag{1.5.6}$$

Example 4

1- The derivative of a constant function

$$^C\mathcal{D}^\alpha C = 0 \tag{1.5.7}$$

2- The derivative of $f(t) = (t-a)^\beta$
Let α *be an integer and* $0 \le n-1 < \alpha < n$ *with* $\beta > n-1$*, then we have*

$$^C\mathcal{D}^\alpha (t-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}$$

1.6 The composition of a fractional derivative with a fractional integral

Lemma 1.6.1 *If* $\mathcal{D}^\alpha f = 0$*,* $\mathcal{D}^\alpha$ *is fractional derivation in the sense of Rimann-Liouville. Then*

$$f(t) = \sum_{j=1}^{n-1} k_j (t-a)^{j+\alpha-n}. \tag{1.6.1}$$

Where k_j *are constants*

Proof. according to the definition, we have

$$(\mathcal{D}_a^\alpha f)(t) = \mathcal{D}^n[\mathcal{I}^{n-\alpha} f](t) = 0$$

So, first we have

$$[\mathcal{I}^{n-\alpha} f](t) = \sum_{j=0}^{n-1} c_j (t-a)^j$$

and by the application of $\mathcal{I}_a^\alpha$ we get

$$[\mathcal{I}^n f](t) = \sum_{j=0}^{n-1} c_j \mathcal{I}^\alpha[(t-a)^j]$$

Taking into account the relationship (2.2), we will have

$$[\mathcal{I}^n f](t) = \sum_{j=0}^{n-1} c_j \frac{\Gamma(j+1)}{\Gamma(j+1+\alpha)}(t-a)^{j+\alpha}$$

Then using the classical derivation and the fact that

$$\mathcal{D}^n(t-a)^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-n)}(t-a)^{\lambda-n}$$

one finds

$$f(t) = \sum_{j=0}^{n-1} c_j \frac{\Gamma(j+1)}{\Gamma(j+1+\alpha-n)}(t-a)^{j+\alpha-n}$$

■

Proposition 1.6.1 *The fractional derivation and the classical derivation (integer order) only switch that if $f^{(k)}(a) = 0$ for all $k = 0, 1, 2, \cdots, n-1$*

$$\frac{d^n}{dt^n}(\mathcal{D}^\alpha f(t)) = \mathcal{D}^{n+\alpha} f(t). \tag{1.6.2}$$

But

$$\mathcal{D}^\alpha \left(\frac{d^n}{dt^n} f(t)\right) = \mathcal{D}^{n+\alpha} f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(t-a)^{k-\alpha-n}}{\Gamma(k-\alpha-n+1)} \tag{1.6.3}$$

Theorem 1.2
Let $\alpha, \beta > 0$ *and* $n = [\alpha] + 1, m = [\beta] + 1$ *such that* $(n, m \in \mathbb{N}^*)$, *then :*

1. *If* $\alpha > \beta > 0$, *then for* $f \in L^1([a, b])$ *equality:*

$$\mathcal{D}^{\beta}(\mathcal{I}^{\alpha} f)(t) = \mathcal{I}^{\alpha-\beta} f(t) \tag{1.6.4}$$

is true of almost everything about $[a, b]$.

2. *If there is a function* $\varphi \in L^1([a, b])$ *such that* $f = \mathcal{I}^{\alpha}\varphi$ *then :*

$$\mathcal{D}^{\alpha}(\mathcal{I}^{\alpha} f(t)) = f(t). \tag{1.6.5}$$

is true of almost everything $t \in [a, b]$.

3. *If the fractional derivative of order* α, *of a function* $f(t)$ *is Integrable, then*

$$\mathcal{I}^{\alpha}_{a^+}(\mathcal{D}^{\alpha}_{a^+} f(t)) = f(t) - \sum_{j=1}^{n} c_j (t-a)^{\alpha-j}. \tag{1.6.6}$$

where c_i *is constant*

1.6.1 Properties of the fractional derivation in the sense of Caputo

Theorem 1.3
Let $\alpha > 0$ *and* $n = [\alpha] + 1$ *such that* $n \in \mathbb{N}^*$ *then the following equals*

1.

$$^{C}\mathcal{D}^{\alpha}\mathcal{I}^{\alpha}_{a} f = f \tag{1.6.7}$$

2.

$$\mathcal{I}^{\alpha}_{a}(^{C}\mathcal{D}^{\alpha} f(t)) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(t-a)^k}{k!} \tag{1.6.8}$$

are true for almost everything $t \in [a, b]$.

Theorem 1.4 *Let f and g be two functions whose fractional derivatives of Caputo exist, for* λ *and* $\mu \in \mathbb{R}$*, then:* ${}^{C}\mathcal{D}^{\alpha}(\lambda f + \mu g)$ *exists, and we have :*

$$ {}^{C}\mathcal{D}^{\alpha}(\lambda f(t) + \mu g(t)) = \lambda^{C}\mathcal{D}^{\alpha} f(t) + \lambda^{C}\mathcal{D}^{\alpha} g(t) $$

Chapter 2

Existence and uniqueness of solutions for coupled system of fractional differential equation

2.1 Introduction

In this chapter, we will study the existence and uniqueness of solutions for a coupled system of fractional differential equations with boundary condition

$$\begin{cases} {}^{RL}\mathcal{D}_{0^+}^{\varrho_1}\mathbf{x}(\mathfrak{z}) & = f(\mathfrak{z}, \mathbf{x}(\mathfrak{z}), \mathbf{y}(\mathfrak{z})), \quad \mathfrak{z} \in \mathcal{O} = [0,1] \\ {}^{RL}\mathcal{D}_{0^+}^{\varrho_2}\mathbf{y}(\mathfrak{z}) & = g(\mathfrak{z}, \mathbf{x}(\mathfrak{z}), \mathbf{y}(\mathfrak{z})), \quad \mathfrak{z} \in \mathcal{O} = [0,1], \end{cases} \tag{2.1.1}$$

subjected with the following mixed boundary conditions

$$\begin{cases} \mathbf{x}(0) = 0, \quad \mathbf{x}(1) = \int_0^1 \psi(\mathfrak{z})\mathbf{x}(\mathfrak{z})d(\mathfrak{z}), \\ \mathbf{y}(0) = 0, \quad \mathbf{y}(1) = \int_0^1 \psi(\mathfrak{z})\mathbf{y}(\mathfrak{z})d(\mathfrak{z}), \end{cases} \tag{2.1.2}$$

where ${}^{RL}\mathcal{D}_{0^+}^{\varrho_1}$ denote the standard Riemann-Liouville fractional derivatives of order ϱ_1, $1 < \varrho_1, \varrho_2 < 2$, $f, g \in \mathcal{C}\big([0,1] \times \mathbb{R}^2, \mathbb{R}\big)$. To investigate the existence and uniqueness of a solution to our problem, we will employ Banach's fixed point theorem.

2.2 Integral equation

Lemma 2.2.1 *Let* $\boldsymbol{h}_i \in C([a,b],\mathbb{R})$. *Then the pair* $(\boldsymbol{x},\boldsymbol{y})$ *is a solution of the linear version of the problem*

$$\begin{cases} {}^{RL}\mathcal{D}_{0^+}^{\varrho_1}\boldsymbol{x}(\mathfrak{z}) = h_1(\mathfrak{z}), \quad \mathfrak{z}\in\mathcal{O}=[0,1], \\ {}^{RL}\mathcal{D}_{0^+}^{\varrho_2}\boldsymbol{y}(\mathfrak{z}) = h_2(\mathfrak{z}), \quad \mathfrak{z}\in\mathcal{O}=[0,1], \\ \boldsymbol{x}(0)=0, \quad \boldsymbol{x}(\imath) = \int_0^1 \psi(\mathfrak{z})\boldsymbol{x}(\mathfrak{z})d(\mathfrak{z}), \\ \boldsymbol{y}(0)=0, \quad \boldsymbol{y}(1) = \int_0^1 \psi(\mathfrak{z})\boldsymbol{y}(\mathfrak{z})d(\mathfrak{z}), \end{cases} \tag{2.2.1}$$

if and only if

$$\begin{cases} \boldsymbol{x}(\mathfrak{z}) & = \dfrac{1}{\Gamma(\varrho_1)}\int_0^t(\mathfrak{z}-s)^{\varrho_1-1}h_1(s)ds + \mathfrak{z}^{\varrho_1-1}\Big(\int_0^1\psi(\mathfrak{z})\boldsymbol{x}(\mathfrak{z})d(\mathfrak{z}) - I_{0+}^{\varrho_1}h_1(1)\Big) \\ \boldsymbol{y}(\mathfrak{z}) & = \dfrac{1}{\Gamma(\varrho_2)}\int_0^t(\mathfrak{z}-s)^{\varrho_2-1}h_2(s)ds + \mathfrak{z}^{\varrho_2-1}\Big(\int_0^1\psi(\mathfrak{z})\boldsymbol{y}(\mathfrak{z})d(\mathfrak{z}) - I_{0+}^{\varrho_2}h_2(1)\Big) \end{cases} \tag{2.2.2}$$

Proof. (2.2.1) $\Rightarrow$ (2.2.2)

Assume that the pair $(\mathbf{x},\mathbf{y})$ is the solution of the system (2.2.1. Operating fractional integrals $I_{0+}^{\varrho_1}$ and $I_{0+}^{\varrho_2}$ on the first and second fractional differential equations in system (2.2.1), respectively, we obtain

$$\mathbf{x}(\mathfrak{z}) = I_{0+}^{\varrho 1}h_1(\mathfrak{z}) + c_1\mathfrak{z}^{\varrho_1-1} + c_2\mathfrak{z}^{\varrho_1-2} \tag{2.2.3}$$

$$\mathbf{y}(\mathfrak{z}) = I_{0+}^{\varrho 2}h_2(\mathfrak{z}) + c_3\mathfrak{z}^{\varrho_2-1} + c_4\mathfrak{z}^{\varrho_2-2} \tag{2.2.4}$$

For $\mathfrak{z}=0$, we find $c_2 = 0$, $c_4 = 0$.
Then the Equations (2.2.2) and (2.2.3) becomes

$$\mathbf{x}(\mathfrak{z}) = I_{0+}^{\varrho 1}h_1(\mathfrak{z}) + c_1\mathfrak{z}^{\varrho_1-1} \tag{2.2.5}$$

$$\mathbf{y}(\mathfrak{z}) = I_{0+}^{\varrho 2}h_2(\mathfrak{z}) + c_3\mathfrak{z}^{\varrho_2-1}. \tag{2.2.6}$$

By using second condition $\mathbf{x}(\imath) = \int_0^1\psi(\mathfrak{z})\mathbf{x}(\mathfrak{z})d(\mathfrak{z})$ and $\mathbf{y}(\imath) = \int_0^1\psi(\mathfrak{z})\mathbf{y}(\mathfrak{z})d(\mathfrak{z})$, we have

$$c_1 = \int_0^1 \psi(\mathfrak{z})\mathbf{x}(\mathfrak{z})d(\mathfrak{z}) - I_{0^+}^{\varrho 1} h_1(1) \tag{2.2.7}$$

$$c_3 = \int_0^1 \psi(\mathfrak{z})\mathbf{y}(\mathfrak{z})d(\mathfrak{z}) - I_{0^+}^{\varrho 2} h_2(1) \tag{2.2.8}$$

We insert c_1 and c_2 into Equations (2.2.4) and (2.2.5), respectively, then we obtain the following

$$\mathbf{x}(\mathfrak{z}) = I_{0^+}^{\varrho_1} h_1(\mathfrak{z}) + \mathfrak{z}^{\varrho_1 - 1}\Big(\int_0^1 \psi(\mathfrak{z})\mathbf{x}(\mathfrak{z})d(\mathfrak{z}) - I_{0^+}^{\varrho_1} h_1(1)\Big)$$

$$\mathbf{y}(\mathfrak{z}) = I_{0^+}^{\varrho_2} h_2(\mathfrak{z}) + \mathfrak{z}^{\varrho_2 - 1}\Big(\int_0^1 \psi(\mathfrak{z})\mathbf{y}(\mathfrak{z})d(\mathfrak{z}) - I_{0^+}^{\varrho_2} h_2(1)\Big)$$

(2.2.2) $\Rightarrow$ (2.2.1)

we have

$$\mathbf{x}(\mathfrak{z}) = I_{0^+}^{\varrho_1} h_1(\mathfrak{z}) + \mathfrak{z}^{\varrho_1 - 1}\Big(\int_0^1 \psi(\mathfrak{z})\mathbf{x}(\mathfrak{z})d(\mathfrak{z}) - I_{0^+}^{\varrho_1} h_1(1)\Big) \tag{2.2.9}$$

$$\mathbf{y}(\mathfrak{z}) = I_{0^+}^{\varrho_2} h_2(\mathfrak{z}) + \mathfrak{z}^{\varrho_2 - 1}\Big(\int_0^1 \psi(\mathfrak{z})\mathbf{y}(\mathfrak{z})d(\mathfrak{z}) - I_{0^+}^{\varrho_2} h_2(1)\Big) \tag{2.2.10}$$

By using the Riemann-Liouville fractional derivative of order ϱ_1 and ϱ_2 on equations (2.2.9) and (2.2.10) respectively, we obtain

$$\begin{cases} {}^{LR}D_{0^+}^{\varrho_1}\mathbf{x}(\mathfrak{z}) = h_1(\mathfrak{z}), \\ {}^{RL}D_{0^+}^{\varrho_2} = \mathbf{y}(\mathfrak{z})h_2(\mathfrak{z}). \end{cases} \tag{2.2.11}$$

Now we verify that the integral equations satisfy the three conditions.

1. From equation (2.2.1), we find $\mathbf{x}(0) = 0$
2. For second condition

$$
\begin{aligned}
\mathbf{x}(\imath) &= I_{0^+}^{\varrho_1} h_1(\imath) + \imath^{\varrho_1 - 1}\left(\int_0^1 \psi(\mathfrak{z})\mathbf{x}(\mathfrak{z})d(\mathfrak{z}) - I_{0^+}^{\varrho_1} h_1(1)\right) \\
&= I_{0^+}^{\varrho_1} h_1(\imath) + \int_0^1 \psi(\mathfrak{z})\mathbf{x}(\mathfrak{z})d(\mathfrak{z}) - I_{0^+}^{\varrho_1} h_1(1) \\
&= \int_0^1 \psi(\mathfrak{z})\mathbf{x}(\mathfrak{z})d(\mathfrak{z})
\end{aligned}
$$

3. Similarly, we obtain $\mathbf{y}(0) = 0, \quad \mathbf{y}(1) = \int_0^1 \psi(\mathfrak{z})\mathbf{y}(\mathfrak{z})d(\mathfrak{z}),$ Hence the proof is completed.

■

Lemma 2.2.2 *Let $f, g \in \mathcal{C}\big([0,1] \times \mathbb{R}^2, \mathbb{R}\big)$. Then the pair $(\boldsymbol{x}, \boldsymbol{y})$ is a solution of the linear version of th problem*

$$
\begin{cases}
{}^{RL}\mathcal{D}_{0^+}^{\varrho_1} \boldsymbol{x}(\mathfrak{z}) = f(\mathfrak{z}, \boldsymbol{x}(\mathfrak{z}), \boldsymbol{y}(\mathfrak{z})), \quad \mathfrak{z} \in \mathcal{O} = [0,1] \\
{}^{RL}\mathcal{D}_{0^+}^{\varrho_2} \boldsymbol{y}(\mathfrak{z}) = g(\mathfrak{z}, \boldsymbol{x}(\mathfrak{z}), \boldsymbol{y}(\mathfrak{z})), \quad \mathfrak{z} \in \mathcal{O} = [0,1], \\
\boldsymbol{x}(0) = 0, \quad \boldsymbol{x}(\imath) = \int_0^1 \psi(\mathfrak{z})\boldsymbol{x}(\mathfrak{z})d(\mathfrak{z}), \\
\boldsymbol{y}(0) = 0, \quad \boldsymbol{y}(1) = \int_0^1 \psi(\mathfrak{z})\boldsymbol{y}(\mathfrak{z})d(\mathfrak{z}),
\end{cases}
\tag{2.2.12}
$$

if and only if

$$
\begin{cases}
\boldsymbol{x}(\mathfrak{z}) &= \dfrac{1}{\Gamma(\varrho_1)} \int_0^t (\mathfrak{z} - s)^{\varrho_1 - 1} f(s, \boldsymbol{x}(s), \boldsymbol{y}(s))ds + \mathfrak{z}^{\varrho_1 - 1}\left(\int_0^1 \psi(\mathfrak{z})\boldsymbol{x}(\mathfrak{z})d(\mathfrak{z}) - I_{0^+}^{\varrho_1} f(\imath, \boldsymbol{x}(\imath), \boldsymbol{y}(\imath))\right). \\
\boldsymbol{y}(\mathfrak{z}) &= \dfrac{1}{\Gamma(\varrho_2)} \int_0^t (\mathfrak{z} - s)^{\varrho_2 - 1} g(s, \boldsymbol{x}(s), \boldsymbol{y}(s))ds + \mathfrak{z}^{\varrho_2 - 1}\left(\int_0^1 \psi(\mathfrak{z})\boldsymbol{y}(\mathfrak{z})d(\mathfrak{z}) - I_{0^+}^{\varrho_2} f(\imath, \boldsymbol{x}(\imath), \boldsymbol{y}(\imath))\right).
\end{cases}
$$

2.3 Results about the existence of solutions

Further, the Banach space is defined by $(Z, \|.\|)$ with the norm $\|\mathbf{x}\| = \max_{z \in [0,1]} |\mathbf{x}|$. Consequently, $U = Z \times Z$ is a Banach space with norms denoted by $\|(\mathbf{x}, \mathbf{y})\| = \|\mathbf{x}\| + \|\mathbf{y}\|$

In light of Lemma 2.2.2, the solution of system (2.1.1) and (2.1.2) is given as follows:

$$\begin{cases} \mathbf{x}(\mathfrak{z}) = & \dfrac{1}{\Gamma(\varrho_1)} \int_0^t (\mathfrak{z}-s)^{\varrho_1-1} f(s,\mathbf{x}(s),\mathbf{y}(s))ds \\ & +\mathfrak{z}^{\varrho_1-1}\Big(\int_0^1 \psi(\mathfrak{z})\mathbf{x}(\mathfrak{z})d(\mathfrak{z}) - I_{0^+}^{\varrho_1} f(1,\mathbf{x}(1),\mathbf{y}(1))\Big) \\ \mathbf{y}(\mathfrak{z}) & = \dfrac{1}{\Gamma(\varrho_2)} \int_0^t (\mathfrak{z}-s)^{\varrho_2-1} g(s,\mathbf{x}(s),\mathbf{y}(s))ds \\ & +\mathfrak{z}^{\varrho_2-1}\Bigg(\int_0^1 \psi(\mathfrak{z})\mathbf{y}(\mathfrak{z})d(\mathfrak{z}) - I_{0^+}^{\varrho_2} g(\mathfrak{z},\mathbf{x}(1),\mathbf{y}(1))\Bigg) \end{cases}$$

Let $\mathbf{K}_1, \mathbf{K}_2 : U \to U$ such that

$$\begin{cases} \mathbf{K}_1(\mathbf{x},\mathbf{y}) & = \dfrac{1}{\Gamma(\varrho_1)} \int_0^t (\mathfrak{z}-s)^{\varrho_1-1} f(s,\mathbf{x}(s),\mathbf{y}(s))ds \\ & +\mathfrak{z}^{\varrho_1-1}\Big(\int_0^1 \psi(\mathfrak{z})\mathbf{x}(\mathfrak{z})d(\mathfrak{z}) - I_{0^+}^{\varrho_1} f(1,\mathbf{x}(1),\mathbf{y}(1))\Big) \\ \mathbf{K}_2(\mathbf{x},\mathbf{y}) & = \dfrac{1}{\Gamma(\varrho_1)} \int_0^t (\mathfrak{z}-s)^{\varrho_2-1} g(s,\mathbf{x}(s),\mathbf{y}(s))ds \\ & +\mathfrak{z}^{\varrho_2-1}\Bigg(\int_0^1 \psi(\mathfrak{z})\mathbf{y}(\mathfrak{z})d(\mathfrak{z}) - I_{0^+}^{\varrho_2} g(\mathfrak{z},\mathbf{x}(1),\mathbf{y}(1))\Bigg) \end{cases}$$

and $\mathbf{K}(\mathbf{x},\mathbf{y}) = (\mathbf{K}_1(\mathbf{x},\mathbf{y}), \mathbf{K}_2(\mathbf{x},\mathbf{y}))$. Thus, solutions of (2.1.1) and (2.1.2) are fixed points of $\mathbf{K}$.

We will set the following conditions:

($\mathbb{V}_1$) For all $\mathbf{x}, \bar{\mathbf{x}}, \mathbf{y}, \bar{\mathbf{y}} \in U$ and for every $\mathfrak{z} \in [0,1]$, $\exists L_f > 0$

$$|f(\mathfrak{z},\mathbf{x}(\mathfrak{z}),\mathbf{y}(\mathfrak{z})) - f(\mathfrak{z},\bar{\mathbf{x}}(\mathfrak{z}),\bar{\mathbf{y}}(\mathfrak{z}))| \leq L_f(|\mathbf{x}-\bar{\mathbf{x}}| + |\mathbf{y}-\bar{\mathbf{y}}|)$$

($\mathbb{V}_2$) For all $\mathbf{x}, \bar{\mathbf{x}}, \mathbf{y}, \bar{\mathbf{y}} \in U$ and for every $\mathfrak{z} \in [0,1]$, $\exists L_g > 0$

$$|g(\mathfrak{z},\mathbf{x}(\mathfrak{z}),\mathbf{y}(\mathfrak{z})) - g(\mathfrak{z},\bar{\mathbf{x}}(\mathfrak{z}),\bar{\mathbf{y}}(\mathfrak{z}))| \leq L_g(|\mathbf{x}-\bar{\mathbf{x}}| + |\mathbf{y}-\bar{\mathbf{y}}|)$$

($\mathbb{V}_3$)

$$|\psi| \leq Q$$

($\mathbb{V}_4$) For some positive real numbers c_f, d_f, m_f, c_g, d_g and m_g

$$|f(\mathfrak{z},\mathbf{x}(\mathfrak{z}),\mathbf{y}(\mathfrak{z}))| \leq c_f|\mathbf{x}| + d_f|\mathbf{y}| + m_f,$$

$$|g(\mathfrak{z}, \mathbf{x}(\mathfrak{z}), \mathbf{y}(\mathfrak{z}))| \leq c_g|\mathbf{x}| + d_g|\mathbf{y}| + m_g.$$

Theorem 2.1 *Under assumptions $\mathbb{V}_3$ and $\mathbb{V}_4$, and if the functions $f, g : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ are continuous, then $\mathbf{K}(U) \subset U$.*

Proof. Let

$$Q_1 = \frac{2c_f + 2d_f}{\Gamma(\varrho_1 + 1)} + Q$$

$$Q_2 = \frac{2m_f}{\Gamma(\varrho_1 + 1)}$$

$$Q_3 = \frac{2c_g + 2d_g}{\Gamma(\varrho_2 + 1)} + Q$$

$$Q_4 = \frac{2m_g}{\Gamma(\varrho_2 + 1)}$$

We define $\mathbf{B}$ a closed subset of U:

$$\mathbf{B} = \{(\mathbf{x}, \mathbf{y}) \in U : \|(\mathbf{x}, \mathbf{y})\| \leq \mathbf{r}\}$$

we have $\|(\mathbf{x}, \mathbf{y})\| \leq \mathbf{r}$ implies $\|\mathbf{x}\| \leq \mathbf{r}$ and $\|\mathbf{y}\| \leq \mathbf{r}$,
we put

$$\mathbf{r} > \frac{Q_2 + Q_4}{1 - Q_1 - Q_3}.$$

For arbitrary $(\mathbf{x}, \mathbf{y}) \in \mathbf{B}$, we have

$$
\begin{aligned}
|\mathbf{K}_1(\mathbf{x},\mathbf{y})| &= \left| I_{0^+}^{\varrho_1} f(\mathfrak{z},\mathbf{x}(\mathfrak{z}),\mathbf{y}(\mathfrak{z})) + \mathfrak{z}^{\varrho_1-1}\Big(\int_0^1 \psi(\mathfrak{z})\mathbf{x}(\mathfrak{z})d(\mathfrak{z}) - I_{0^+}^{\varrho_1} f(1,\mathbf{x}(1),\mathbf{y}(1))\Big)\right| \\
&\le I_{0^+}^{\varrho_1} |f(\mathfrak{z},\mathbf{x}(\mathfrak{z}),\mathbf{y}(\mathfrak{z}))| + \mathfrak{z}^{\varrho_1-1}\Big(\int_0^1 |\psi(\mathfrak{z})||\mathbf{x}(\mathfrak{z})|d(\mathfrak{z}) + I_{0^+}^{\varrho_1} |f(1,\mathbf{x}(1),\mathbf{y}(1)|)\Big) \\
&\le I_{0^+}^{\varrho_1} |f(1,\mathbf{x}(1),\mathbf{y}(1))| + \Big(\int_0^1 |\psi(\mathfrak{z})||\mathbf{x}(\mathfrak{z})|d(\mathfrak{z}) + I_{0^+}^{\varrho_1} |f(1,\mathbf{x}(1),\mathbf{y}(1)|)\Big) \\
&\le \frac{2}{\Gamma(\varrho_1)}\int_0^1 (1-s)^{\varrho_1-1} f(s,\mathbf{x}(s),\mathbf{y}(s))ds + + \int_0^1 |\psi(\mathfrak{z})| \max_{z\in[0,1]} |\mathbf{x}(\mathfrak{z})| d(\mathfrak{z}) \\
&\le \frac{2}{\Gamma(\varrho_1)}\int_0^1 (1-s)^{\varrho_1-1} (c_f|\mathbf{x}(s)| + d_f|\mathbf{y}(s)| + m_f)ds + Q \max_{z\in[0,1]} |\mathbf{x}(\mathfrak{z})| \int_0^1 1d(\mathfrak{z}) \\
&\le \frac{2(c_f \max_{s\in[0,1]} |\mathbf{x}(s)| + d_f \max_{s\in[0,1]} |\mathbf{y}(s)| + m_f)}{\Gamma(\varrho_1)}\int_0^1 (1-s)^{\varrho_1-1} ds + Q\|\mathbf{x}\| \\
&\le \frac{2(c_f\|\mathbf{x}\| + d_f\|\mathbf{y}\| + m_f)}{\Gamma(\varrho_1)}\int_0^1 (1-s)^{\varrho_1-1} ds + Q\|\mathbf{x}\| \\
&\le \frac{2(c_f r + d_f r + m_f)}{\Gamma(\varrho_1+1)} + Qr \\
&\le \big(\frac{2c_f + 2d_f}{\Gamma(\varrho_1+1)} + \mathcal{Q}\big)r + \frac{2m_f}{\Gamma(\varrho_1+1)}
\end{aligned}
$$

$$
|\mathbf{K}_1(\mathbf{x},\mathbf{y})| \le Q_1\,\mathbf{r} + Q_2 \tag{2.3.1}
$$

Then

$$
\|\mathbf{K}_1(\mathbf{x},\mathbf{y})\| \le Q_1\,\mathbf{r} + Q_2
$$

Similarly,

$$
\|\mathbf{K}_2(\mathbf{x},\mathbf{y})\| \le Q_3\,\mathbf{r} + Q_4 \tag{2.3.2}
$$

Therefore, from (2.3.1) and (2.3.2), one has

$$\begin{aligned}
\|\mathbf{K}(\mathbf{x},\mathbf{y})\| &= \|(\mathbf{K}_1(\mathbf{x},\mathbf{y}),\mathbf{K}_2(\mathbf{x},\mathbf{y}))\| \\
&\leq \|\mathbf{K}_1(\mathbf{x},\mathbf{y})\| + \|\mathbf{K}_2(\mathbf{x},\mathbf{y})\| \\
&\leq Q_1\,\mathbf{r} + Q_2 + Q_3\,\mathbf{r} + Q_4 \leq \mathbf{r}
\end{aligned}$$

The proof is completed.

■

Theorem 2.2 *Under assumptions* $(\mathbb{V}_1)$, $(\mathbb{V}_2)$ $(\mathbb{V}_3)$ *and* $(\mathbb{V}_4)$ *and if* $\mathfrak{J} < 1$, *then systems (2.1.1) and (2.1.2) has a unique solution, with*

$$\mathfrak{J} = \frac{2}{\Gamma(\varrho_1+1)}L_f + \frac{2}{\Gamma(\varrho_2+1)}L_g + 2Q$$

Proof. From Theorem 2.1, we show that $K(Z) \subset Z$.

In the next stage, let us assume that $\mathbf{x}, \bar{\mathbf{x}}, \mathbf{y}, \bar{\mathbf{y}} \in \mathscr{X}$ and for every $\mathfrak{z} \in [0,1]$, let

$$
\begin{aligned}
|\mathbf{K}_1(\mathbf{x},\mathbf{y}) - \mathbf{K}_1(\bar{\mathbf{x}},\bar{\mathbf{y}})| = &\Bigg| \frac{1}{\Gamma(\varrho_1)} \int_0^t (\mathfrak{z}-s)^{\varrho_1-1} f(s,\mathbf{x}(s),\mathbf{y}(s))ds \\
&+ \mathfrak{z}^{\varrho_1-1}\Big(\int_0^1 \psi(\mathfrak{z})\mathbf{x}(\mathfrak{z})d(\mathfrak{z}) - I_{0^+}^{\varrho_1} f(\imath,\mathbf{x}(\imath),\mathbf{y}(\imath))\Big) \\
&- \frac{1}{\Gamma(\varrho_1)} \int_0^t (\mathfrak{z}-s)^{\varrho_1-1} f(s,\bar{\mathbf{x}}(s),\bar{\mathbf{y}}(s))ds + \\
&- \mathfrak{z}^{\varrho_1-1}\Big(\int_0^1 \psi(\mathfrak{z})\bar{\mathbf{x}}(\mathfrak{z})d(\mathfrak{z}) - I_{0^+}^{\varrho_1} f(\imath,\bar{\mathbf{x}}(\imath),\bar{\mathbf{y}}(\imath))\Big) \Bigg| \\
&\leq \Bigg| \frac{1}{\Gamma(\varrho_1)} \int_0^t (\mathfrak{z}-s)^{\varrho_1-1}\big(f(s,\mathbf{x}(s),\mathbf{y}(s)) - f(\mathfrak{s},\bar{\mathbf{x}}(\mathfrak{s}),\bar{\mathbf{y}}(\mathfrak{s}))\big)ds \Bigg| \\
&+ \mathfrak{z}^{\varrho_1-1}\Big(\int_0^1 \psi(\mathfrak{z})|\mathbf{x}(\mathfrak{z}) - \bar{\mathbf{x}}(\mathfrak{z})|d(\mathfrak{z}) + I_{0^+}^{\varrho_1} f(1,\mathbf{x}(1),\mathbf{y}(1)) - f(1,\bar{\mathbf{x}}(1),\bar{\mathbf{y}}(1))\Big) \\
&\leq \Bigg| \frac{1}{\Gamma(\varrho_1)} \int_0^t (\mathfrak{z}-s)^{\varrho_1-1}\big(L_f(|\mathbf{x}(s) - \bar{\mathbf{x}}(s)| + |\mathbf{y}(s) - \bar{\mathbf{y}}(s)|)\big)ds \Bigg| \\
&+ \mathfrak{z}^{\varrho_1-1}\Big(\int_0^1 \psi(\mathfrak{z})|\mathbf{x}(\mathfrak{z}) - \bar{\mathbf{x}}(\mathfrak{z})|d(\mathfrak{z}) \\
&+ \frac{1}{\Gamma(\varrho_1)} \int_0^1 (1-s)^{\varrho_1-1}\big(L_f(|\mathbf{x}(s) - \bar{\mathbf{x}(s)}| + |\mathbf{y}(s) - \bar{\mathbf{y}(s)}|)\big)ds\Big) \\
&\leq \frac{1}{\Gamma(\varrho_1+1)} L_f(\|\mathbf{x}-\bar{\mathbf{x}}\| + \|\mathbf{y}-\bar{\mathbf{y}}\|) + \mathcal{Q}\|\mathbf{x}-\bar{\mathbf{x}}\| \\
&+ L_f \frac{1}{\Gamma(\varrho_1+1)} (\|\mathbf{x}-\bar{\mathbf{x}}\| + \|\mathbf{y}-\bar{\mathbf{y}}\|) \\
&= \frac{2}{\Gamma(\varrho_1+1)} L_f(\|\mathbf{x}-\bar{\mathbf{x}}\| + \|\mathbf{y}-\bar{\mathbf{y}}\|) + Q\|\mathbf{x}-\bar{\mathbf{x}}\| \\
&\leq \Big(\frac{2}{\Gamma(\varrho_1+1)} L_f + Q\Big)\|\mathbf{x}-\bar{\mathbf{x}}\| + \frac{2}{\Gamma(\varrho_1+1)} L_f \|\mathbf{y}-\bar{\mathbf{y}}\| \\
&\leq \Big(\frac{2}{\Gamma(\varrho_1+1)} L_f + Q\Big)\|\mathbf{x}-\bar{\mathbf{x}}\| + \frac{2}{\Gamma(\varrho_1+1)} L_f \|\mathbf{y}-\bar{\mathbf{y}}\| + Q\|\mathbf{y}-\bar{\mathbf{y}}\| \\
&\leq \Big(\frac{2}{\Gamma(\varrho_1+1)} L_f + Q\Big)\Big(\|\mathbf{x}-\bar{\mathbf{x}}\| + \|\mathbf{y}-\bar{\mathbf{y}}\| \Big). \qquad (2.3.3)
\end{aligned}
$$

Then

$$\|\mathbf{K}_1(\mathbf{x},\mathbf{y})-\mathbf{K}_1(\bar{\mathbf{x}},\bar{\mathbf{y}})\| \le \Big(\frac{2}{\Gamma(\varrho_1+1)}L_f+Q\Big)\Big(\|\mathbf{x}-\bar{\mathbf{x}}\|+\|\mathbf{y}-\bar{\mathbf{y}}\|\Big)$$

Similarly,

$$\|\mathbf{K}_2(\mathbf{x},\mathbf{y})-\mathbf{K}_2(\bar{\mathbf{x}},\bar{\mathbf{y}})\| \le \Big(\frac{2}{\Gamma(\varrho_2+1)}L_g+Q\Big)\Big(\|\mathbf{x}-\bar{\mathbf{x}}\|+\|\mathbf{y}-\bar{\mathbf{y}}\|\Big) \tag{2.3.4}$$

Hence, from (2.3.3) and (2.3.4), one has

$$\begin{aligned}
\|\mathbf{K}(\mathbf{x},\mathbf{y})-\mathbf{K}(\bar{\mathbf{x}},\bar{\mathbf{y}})\| &= \Big\|\Big(\mathbf{K}_1(\mathbf{x},\mathbf{y}),\mathbf{K}_2(\mathbf{x},\mathbf{y})\Big)-\Big(\mathbf{K}_1(\bar{\mathbf{x}},\bar{\mathbf{y}}),\mathbf{K}_2(\bar{\mathbf{x}},\bar{\mathbf{y}})\Big)\Big\| \\
&= \Big\|\Big(\mathbf{K}_1(\mathbf{x},\mathbf{y})-\mathbf{K}_1(\bar{\mathbf{x}},\bar{\mathbf{y}}),\mathbf{K}_2(\mathbf{x},\mathbf{y})-\mathbf{K}_2(\bar{\mathbf{x}},\bar{\mathbf{y}})\Big)\Big\| \\
&= \Big\|\mathbf{K}_1(\mathbf{x},\mathbf{y})-\mathbf{K}_1(\bar{\mathbf{x}},\bar{\mathbf{y}})\Big\|+\Big\|\mathbf{K}_2(\mathbf{x},\mathbf{y})-\mathbf{K}_2(\bar{\mathbf{x}},\bar{\mathbf{y}})\Big\| \\
&\le \big(\frac{2}{\Gamma(\varrho_1+1)}L_f+Q\big)\Big(\|\mathbf{x}-\bar{\mathbf{x}}\|+\|\mathbf{y}-\bar{\mathbf{y}}\|\Big) \\
&+ \big(\frac{2}{\Gamma(\varrho_2+1)}L_g+Q\big)\Big(\|\mathbf{x}-\bar{\mathbf{x}}\|+\|\mathbf{y}-\bar{\mathbf{y}}\|\Big) \\
&\le \big(\frac{2}{\Gamma(\varrho_1+1)}L_f+\frac{2}{\Gamma(\varrho_2+1)}L_g+2Q\big)\Big(\|\mathbf{x}-\bar{\mathbf{x}}\|+\|\mathbf{y}-\bar{\mathbf{y}}\|\Big)
\end{aligned}$$

Then

$$\|\mathbf{K}(\mathbf{x},\mathbf{y})-\mathbf{K}(\bar{\mathbf{x}},\bar{\mathbf{y}})\| \le \big(\frac{2}{\Gamma(\varrho_1+1)}L_f+\frac{2}{\Gamma(\varrho_2+1)}L_g+2Q\big)\Big(\|\mathbf{x}-\bar{\mathbf{x}}\|+\|\mathbf{y}-\bar{\mathbf{y}}\|\Big)$$

where

$$J=\frac{2}{\Gamma(\varrho_1+1)}L_f+\frac{2}{\Gamma(\varrho_2+1)}L_g+2Q<1$$

Hence, $\mathbf{K}$ is a contraction, thus $\mathbf{K}$ has a unique fixed point, which implies that the concerned system (0.0.1) and (0.0.2) has a unique solution. ■

2.4 Example

Example 5 *We take the system :*

$$\begin{cases} {}^{RL}\mathcal{D}_{0^+}^{1.99}\boldsymbol{x}(\mathfrak{z}) & = \frac{\sin(\mathfrak{z})}{\mathfrak{z}+2025} + \frac{1}{\pi^4}\arctan(\boldsymbol{x}(\mathfrak{z}) + \boldsymbol{y}(\mathfrak{z})), \quad \mathfrak{z} \in \mathcal{O} = [0,1] \\ {}^{RL}\mathcal{D}_{0^+}^{1.22}\boldsymbol{y}(\mathfrak{z}) & = \exp(-z^2 - \pi^4) + \dfrac{|\boldsymbol{x}|}{20(|\boldsymbol{x}|+1)} + \dfrac{|\boldsymbol{y}|}{10|\boldsymbol{y}|+10}, \quad \mathfrak{z} \in \mathcal{O} = [0,1], \end{cases} \tag{2.4.1}$$

subjected with the following mixed boundary conditions

$$\begin{cases} \boldsymbol{x}(0) = 0, \quad \boldsymbol{x}(1) = \displaystyle\int_0^1 \frac{\mathfrak{z}^2}{44}\boldsymbol{x}(\mathfrak{z})d(\mathfrak{z}), \\ \boldsymbol{y}(0) = 0, \quad \boldsymbol{y}(1) = \displaystyle\int_0^1 \frac{\mathfrak{z}^2}{44}\boldsymbol{y}(\mathfrak{z})d(\mathfrak{z}), \end{cases} \tag{2.4.2}$$

$$f(\mathfrak{z}, \boldsymbol{x}(\mathfrak{z}), \boldsymbol{y}(\mathfrak{z})) = \frac{\sin(\mathfrak{z})}{\mathfrak{z}+2025} + \frac{1}{\pi^4}\arctan(\boldsymbol{x}(\mathfrak{z}) + \boldsymbol{y}(\mathfrak{z}))$$

$$g(\mathfrak{z}, \boldsymbol{x}(\mathfrak{z}), \boldsymbol{y}(\mathfrak{z})) = \exp(-z^2 - \pi^4) + \frac{|\boldsymbol{x}|}{10(|\boldsymbol{x}|+1)} + \frac{|\boldsymbol{y}|}{10|\boldsymbol{y}|+10}$$

$$\psi(\mathfrak{z}) = \frac{\mathfrak{z}^2}{44}$$

After the calculation, we have

$$\psi(\mathfrak{z}) \leq \frac{\mathit{1}}{44} = Q$$

$$|f(\mathfrak{z}, \boldsymbol{x}(\mathfrak{z}), \boldsymbol{y}(\mathfrak{z})) - f(\mathfrak{z}, \bar{\boldsymbol{x}}(\mathfrak{z}), \bar{\boldsymbol{y}}(\mathfrak{z}))| = |\frac{1}{\pi^4}\arctan(\boldsymbol{x}(\mathfrak{z}) + \boldsymbol{y}(\mathfrak{z})) - \frac{1}{\pi^4}\arctan(\bar{\boldsymbol{x}}(\mathfrak{z}) + \bar{\boldsymbol{y}}(\mathfrak{z}))|$$
$$\leq \frac{1}{\pi^4}(|\boldsymbol{x}(\mathfrak{z}) - \bar{\boldsymbol{x}}(\mathfrak{z})| + |\boldsymbol{y}(\mathfrak{z})| - \bar{\boldsymbol{y}}(\mathfrak{z})|)$$

$$L_f = \frac{1}{\pi^4}$$

$$|g(\mathfrak{z}, \boldsymbol{x}(\mathfrak{z}), \boldsymbol{y}(\mathfrak{z})) - g(\mathfrak{z}, \bar{\boldsymbol{x}}(\mathfrak{z}), \bar{\boldsymbol{y}}(\mathfrak{z}))| \leq \frac{1}{10}(|\boldsymbol{x}(\mathfrak{z}) - \bar{\boldsymbol{x}}(\mathfrak{z})| + |\boldsymbol{y}(\mathfrak{z})| - \bar{\boldsymbol{y}}(\mathfrak{z})|)$$

$$L_g = \frac{1}{10}$$

After the calculation, we have

$$\mathfrak{J} = \frac{2}{\Gamma(\varrho_1 + 1)} L_f + \frac{2}{\Gamma(\varrho_2 + 1)} L_g + 2Q = 0.2354$$

As $\mathfrak{J} = 0.2354 < 1$. *According to Theorem 2.2, it can be deduced that the system (2.4.1)-(2.4.2) has a unique solution*

Chapter 3

The stability analysis of Ulam-Hyers type

3.1 Introduction

In this chapter, we will study the stability of solutions for a coupled system of fractional differential equations with boundary condition

$$\begin{cases} {}^{RL}\mathcal{D}_{0+}^{\varrho_1}\mathbf{x}(\mathfrak{z}) &= f(\mathfrak{z},\mathbf{x}(\mathfrak{z}),\mathbf{y}(\mathfrak{z})), \quad \mathfrak{z}\in\mathcal{O}=[0,1] \\ {}^{RL}\mathcal{D}_{0+}^{\varrho_2}\mathbf{y}(\mathfrak{z}) &= g(\mathfrak{z},\mathbf{x}(\mathfrak{z}),\mathbf{y}(\mathfrak{z})), \quad \mathfrak{z}\in\mathcal{O}=[0,1], \end{cases} \tag{3.1.1}$$

subjected with the following mixed boundary conditions

$$\begin{cases} \mathbf{x}(0)=0, \quad \mathbf{x}(1)=\int_0^1\psi(\mathfrak{z})\mathbf{x}(\mathfrak{z})d(\mathfrak{z}), \\ \mathbf{y}(0)=0, \quad \mathbf{y}(1)=\int_0^1\psi(\mathfrak{z})\mathbf{y}(\mathfrak{z})d(\mathfrak{z}), \end{cases} \tag{3.1.2}$$

where ${}^{RL}\mathcal{D}_{0+}^{\theta}$ denote the standard Riemann-Liouville fractional derivatives of order θ, and $1<\varrho_1,\varrho_2<2$ and $f,g\in\mathcal{C}\big([0,1]\times\mathbb{R}^2,\mathbb{R}\big)$.

To investigate the stability of a solution to our problem, we will employ the definition and theorem, which is defined as follows:

Definition 3.1.1 *We say that the problem 3.1.1 and 3.1.2 is Ulam-Hyers stable if there exists a real number $K_f,K_g>0$ such that for $\epsilon_1,\epsilon_1>0$ and for each $(\bar{\boldsymbol{x}},\bar{\boldsymbol{y}})$ solution to the* *problem*

$$|{}^{RL}\mathcal{D}^{\varrho_1}\bar{\boldsymbol{x}}(\boldsymbol{z}) - f(\boldsymbol{z}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})| \leq \epsilon_1, \tag{3.1.3}$$

$$|{}^{RL}\mathcal{D}^{\varrho_2}\bar{\boldsymbol{y}}(\boldsymbol{z}) - g(\boldsymbol{z}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})| \leq \epsilon_2, \tag{3.1.4}$$

there exists a solution $(\boldsymbol{x}, \boldsymbol{y})$ *to the problem 3.1.1 and 3.1.2 with*

$$|\bar{\boldsymbol{x}}(\boldsymbol{z}) - \boldsymbol{x}(\boldsymbol{z})| \leq K_f\epsilon_1, \tag{3.1.5}$$

$$|\bar{\boldsymbol{y}}(\boldsymbol{z}) - \boldsymbol{y}(\boldsymbol{z})| \leq K_g\epsilon_2, \tag{3.1.6}$$

for all $\boldsymbol{z} \in [a, b]$.

Definition 3.1.2 *[20] If* β_i $(i = 1, 2, 3, ..., n)$ *are eigenvalues of a matrix* $\mathcal{M}$ *of order* $n \times n$*, with spectral radius of* $\rho(\mathcal{M})$ *defined by*

$$\rho(\mathcal{M}) = \max\{|\beta| \text{ for } i = 1, 2, ..., n\}.$$

In addition, if $\rho(\mathcal{M}) < 1$ *, then* $\mathcal{M}$ *converges to* 0.

Theorem 3.1 *[20] For the two operators* $S_1, S_2 : U \to Z$

$$\begin{cases} \|S_1(\boldsymbol{x}, \boldsymbol{y}) - S_1(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})\| \leq Q_1\|\boldsymbol{x} - \bar{\boldsymbol{x}}\| + Q_2\|\boldsymbol{y} - \bar{\boldsymbol{y}}\| \\ S_2(\boldsymbol{x}, \boldsymbol{y}) - S_2(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})\| \leq Q_3\|\boldsymbol{x} - \bar{\boldsymbol{x}}\| + Q_4\|\boldsymbol{y} - \bar{\boldsymbol{y}}\| \\ \forall(\boldsymbol{x}, \boldsymbol{y})(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in U \end{cases} \tag{3.1.7}$$

and if the matrix

$$\mathcal{M} = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix}$$

goes to zero, Consequently, then the solutions of the system described by

$$\begin{cases} \boldsymbol{x}(\mathfrak{z}) & = S_1(\boldsymbol{x}, \boldsymbol{y})(\mathfrak{z}), \\ \boldsymbol{y}(\mathfrak{z}) & = S_2(\boldsymbol{x}, \boldsymbol{y})(\mathfrak{z}), \end{cases} \tag{3.1.8}$$

are then determined to exhibit Hyers-Ulam-type stability.

3.2 Integral equation

Lemma 3.2.1 *Let* $f, g \in \mathcal{C}\big([0,1]\times\mathbb{R}^2,\mathbb{R}\big)$. *Then the pair* $(\boldsymbol{x}, \boldsymbol{y})$ *is a solution of the problem given by*

$$\begin{cases} {}^{RL}\mathcal{D}^{\varrho_1}_{0^+}\boldsymbol{x}(\mathfrak{z}) = f(\mathfrak{z}, \boldsymbol{x}(\mathfrak{z}), \boldsymbol{y}(\mathfrak{z})), \quad \mathfrak{z} \in \mathcal{O} = [0,1] \\ {}^{RL}\mathcal{D}^{\varrho_2}_{0^+}\boldsymbol{y}(\mathfrak{z}) = g(\mathfrak{z}, \boldsymbol{x}(\mathfrak{z}), \boldsymbol{y}(\mathfrak{z})), \quad \mathfrak{z} \in \mathcal{O} = [0,1], \\ \boldsymbol{x}(0) = 0, \quad \boldsymbol{x}(1) = \int_0^1 \psi(\mathfrak{z})\boldsymbol{x}(\mathfrak{z})d(\mathfrak{z}), \\ \boldsymbol{y}(0) = 0, \quad \boldsymbol{y}(1) = \int_0^1 \psi(\mathfrak{z})\boldsymbol{y}(\mathfrak{z})d(\mathfrak{z}), \end{cases} \tag{3.2.1}$$

if and only if

$$\begin{cases} \boldsymbol{x}(\mathfrak{z}) &= \dfrac{1}{\Gamma(\varrho_1)}\int_0^t (\mathfrak{z}-s)^{\varrho_1-1} f(s, \boldsymbol{x}(s), \boldsymbol{y}(s))ds + \mathfrak{z}^{\varrho_1-1}\Bigg(\int_0^1 \psi(\mathfrak{z})\boldsymbol{x}(\mathfrak{z})d(\mathfrak{z}) - I^{\varrho_1}_{0^+} f(1, \boldsymbol{x}(1), \boldsymbol{y}(1))\Bigg). \\ \boldsymbol{y}(\mathfrak{z}) &= \dfrac{1}{\Gamma(\varrho_2)}\int_0^t (\mathfrak{z}-s)^{\varrho_2-1} g(s, \boldsymbol{x}(s), \boldsymbol{y}(s))ds + \mathfrak{z}^{\varrho_2-1}\Bigg(\int_0^1 \psi(\mathfrak{z})\boldsymbol{y}(\mathfrak{z})d(\mathfrak{z}) - I^{\varrho_2}_{0^+} f(1, \boldsymbol{x}(1), \boldsymbol{y}(1))\Bigg). \end{cases}$$

3.3 Results regarding stability

Theorem 3.2 *If the matrix* $\mathscr{M}$ *converges to* 0 *and assumptions* $\mathbb{V}_1$, $\mathbb{V}_2$ *and* $\mathbb{V}_3$ *hold, then the results of (3.1.1) and (3.1.2) are Hyers Ulam type stability*

Proof. Taking $(\mathbf{x},\mathbf{y}),(\bar{\mathbf{x}},\bar{\mathbf{y}})\in U$ and for every $\mathfrak{z}\in[0,1]$, let

$$
\begin{aligned}
|\mathbf{K}_1(\mathbf{x},\mathbf{y})-\mathbf{K}_1(\bar{\mathbf{x}},\bar{\mathbf{y}})| = &\Bigg|\frac{1}{\Gamma(\varrho_1)}\int_0^t(\mathfrak{z}-s)^{\varrho_1-1}f(s,\mathbf{x}(s),\mathbf{y}(s))ds\\
&+\mathfrak{z}^{\varrho_1-1}\Bigg(\int_0^1\psi(\mathfrak{z})\mathbf{x}(\mathfrak{z})d(\mathfrak{z})-I_{0^+}^{\varrho_1}f(\mathfrak{1},\mathbf{x}(\mathfrak{1}),\mathbf{y}(\mathfrak{1}))\Bigg)\\
&-\frac{1}{\Gamma(\varrho_1)}\int_0^t(\mathfrak{z}-s)^{\varrho_1-1}f(s,\bar{\mathbf{x}}(s),\bar{\mathbf{y}}(s))ds+\\
&-\mathfrak{z}^{\varrho_1-1}\Bigg(\int_0^1\psi(\mathfrak{z})\bar{\mathbf{x}}(\mathfrak{z})d(\mathfrak{z})-I_{0^+}^{\varrho_1}f(\mathfrak{1},\bar{\mathbf{x}}(\mathfrak{1}),\bar{\mathbf{y}}(\mathfrak{1}))\Bigg)\Bigg|\\
&\le\Bigg|\frac{1}{\Gamma(\varrho_1)}\int_0^t(\mathfrak{z}-s)^{\varrho_1-1}\big(f(s,\mathbf{x}(s),\mathbf{y}(s))-f(\mathfrak{s},\bar{\mathbf{x}}(\mathfrak{s}),\bar{\mathbf{y}}(\mathfrak{s}))\big)ds\Bigg|\\
&+\mathfrak{z}^{\varrho_1-1}\Big(\int_0^1\psi(\mathfrak{z})|\mathbf{x}(\mathfrak{z})-\bar{\mathbf{x}}(\mathfrak{z})|d(\mathfrak{z})+I_{0^+}^{\varrho_1}f(1,\mathbf{x}(1),\mathbf{y}(1))-f(1,\bar{\mathbf{x}}(1),\bar{\mathbf{y}}(1))\Big)\\
&\le\Bigg|\frac{1}{\Gamma(\varrho_1)}\int_0^t(\mathfrak{z}-s)^{\varrho_1-1}\big(L_f(|\mathbf{x}(s)-\bar{\mathbf{x}}(s)|+|\mathbf{y}(s)-\bar{\mathbf{y}}(s)|)\big)ds\Bigg|\\
&+\mathfrak{z}^{\varrho_1-1}\Big(\int_0^1\psi(\mathfrak{z})|\mathbf{x}(\mathfrak{z})-\bar{\mathbf{x}}(\mathfrak{z})|d(\mathfrak{z})\\
&+\frac{1}{\Gamma(\varrho_1)}\int_0^1(1-s)^{\varrho_1-1}\big(L_f(|\mathbf{x}(s)-\mathbf{x}\bar{(s)}|+|\mathbf{y}(s)-\mathbf{y}\bar{(s)}|)\big)ds\Big)\\
&\le\frac{1}{\Gamma(\varrho_1+1)}L_f(\|\mathbf{x}-\bar{\mathbf{x}}\|+\|\mathbf{y}-\bar{\mathbf{y}}\|)+\mathcal{Q}\|\mathbf{x}-\bar{\mathbf{x}}\|\\
&+L_f\frac{1}{\Gamma(\varrho_1+1)}(\|\mathbf{x}-\bar{\mathbf{x}}\|+\|\mathbf{y}-\bar{\mathbf{y}}\|)\\
&=\frac{2}{\Gamma(\varrho_1+1)}L_f(\|\mathbf{x}-\bar{\mathbf{x}}\|+\|\mathbf{y}-\bar{\mathbf{y}}\|)+Q\|\mathbf{x}-\bar{\mathbf{x}}\|\\
&\le\Big(\frac{2}{\Gamma(\varrho_1+1)}L_f+Q\Big)\|\mathbf{x}-\bar{\mathbf{x}}\|+\frac{2}{\Gamma(\varrho_1+1)}L_f\|\mathbf{y}-\bar{\mathbf{y}}\|\\
&\le\mathfrak{Q}_1\|\mathbf{x}-\bar{\mathbf{x}}\|+\mathfrak{Q}_2\|\mathbf{y}-\bar{\mathbf{y}}\| \qquad (3.3.1)
\end{aligned}
$$

$$\mathfrak{Q}_1 = \frac{2}{\Gamma(\varrho_1+1)} L_f + Q$$

$$\mathfrak{Q}_2 = \frac{2}{\Gamma(\varrho_1+1)} L_f$$

Similarly,

$$\|\mathbf{K}_2(\mathbf{x},\mathbf{y}) - \mathbf{K}_2(\bar{\mathbf{x}},\bar{\mathbf{y}})\| \le \mathfrak{Q}_3\|\mathbf{x}-\bar{\mathbf{x}}\| + \mathfrak{Q}_4\|\mathbf{y}-\bar{\mathbf{y}}\| \tag{3.3.2}$$

$$\mathfrak{Q}_3 = \frac{2}{\Gamma(\varrho_2+1)} L_g + Q$$

$$\mathfrak{Q}_4 = \frac{2}{\Gamma(\varrho_2+1)} L_g$$

So, from (3.3.1) and (3.3.2), we get

$$\begin{aligned} \|\mathbf{K}_1(\mathbf{x},\mathbf{y}) - \mathbf{K}_1(\bar{\mathbf{x}},\bar{\mathbf{y}})\| &\le \mathfrak{Q}_1\|\mathbf{x}-\bar{\mathbf{x}}\| + \mathfrak{Q}_2\|\mathbf{y}-\bar{\mathbf{y}}\| \\ \|\mathbf{K}_2(\mathbf{x},\mathbf{y}) - \mathbf{K}_2(\bar{\mathbf{x}},\bar{\mathbf{y}})\| &\le \mathfrak{Q}_3\|\mathbf{x}-\bar{\mathbf{x}}\| + \mathfrak{Q}_4\|\mathbf{y}-\bar{\mathbf{y}}\| \end{aligned} \tag{3.3.3}$$

From (3.3.3), we can get the matrix $\mathscr{M}$ as follows:

$$\mathscr{M} = \begin{bmatrix} \mathfrak{Q}_1 & \mathfrak{Q}_2 \\ \mathfrak{Q}_3 & \mathfrak{Q}_4 \end{bmatrix}$$

Because it is given that the matrix tends to zero, by the conclusion of Theorem 3.1 the solution of (3.1.1) and (3.1.2) is UH-type stable. ■

3.4 Example

Example 6 *We take the FBVP :*

$$\begin{cases} {}^{RL}\mathcal{D}_{0^+}^{1.88}\boldsymbol{x}(\mathfrak{z}) &= \dfrac{|\boldsymbol{x}(\mathfrak{z})+\boldsymbol{y}(\mathfrak{z})|}{2023(|\boldsymbol{x}(\mathfrak{z})+\boldsymbol{y}(\mathfrak{z})|+1)}, \quad \mathfrak{z}\in\mathcal{O}=[0,1] \\ {}^{RL}\mathcal{D}_{0^+}^{1.77}\boldsymbol{y}(\mathfrak{z}) &= \dfrac{1}{2000}\sin(|\boldsymbol{x}(\mathfrak{z})+\boldsymbol{y}(\mathfrak{z})|), \quad \mathfrak{z}\in\mathcal{O}=[0,1], \end{cases} \tag{3.4.1}$$

with

$$\begin{cases} \boldsymbol{x}(0) = 0, \quad \boldsymbol{x}(1) = \int_0^1 \frac{\mathfrak{z}}{20}\boldsymbol{x}(\mathfrak{z})d(\mathfrak{z}), \\ \boldsymbol{y}(0) = 0, \quad \boldsymbol{y}(1) = \int_0^1 \frac{\mathfrak{z}}{20}\boldsymbol{y}(\mathfrak{z})d(\mathfrak{z}), \end{cases} \tag{3.4.2}$$

$$f(\mathfrak{z}, \boldsymbol{x}(\mathfrak{z}), \boldsymbol{y}(\mathfrak{z})) = \frac{|\boldsymbol{x}(\mathfrak{z}) + \boldsymbol{y}(\mathfrak{z})|}{2023(|\boldsymbol{x}(\mathfrak{z}) + \boldsymbol{y}(\mathfrak{z})| + 1)}$$

$$g(\mathfrak{z}, \boldsymbol{x}(\mathfrak{z}), \boldsymbol{y}(\mathfrak{z})) = \frac{1}{2000}\sin(|\boldsymbol{x}(\mathfrak{z}) + \boldsymbol{y}(\mathfrak{z})|)$$

$$\psi(\mathfrak{z}) = \frac{\mathfrak{z}}{20} \leq \frac{\boldsymbol{1}}{20} = Q$$

After the calculation, we have

$$\psi(\mathfrak{z}) \leq \frac{\boldsymbol{1}}{20}$$

$$\begin{aligned} |f(\mathfrak{z}, \boldsymbol{x}(\mathfrak{z}), \boldsymbol{y}(\mathfrak{z})) - f(\mathfrak{z}, \bar{\boldsymbol{x}}(\mathfrak{z}), \bar{\boldsymbol{y}}(\mathfrak{z}))| &= |\frac{|\boldsymbol{x}(\mathfrak{z}) + \boldsymbol{y}(\mathfrak{z})|}{2023(|\boldsymbol{x}(\mathfrak{z}) + \boldsymbol{y}(\mathfrak{z})| + 1)} - \frac{|\bar{\boldsymbol{x}}(\mathfrak{z}) + \bar{\boldsymbol{x}}(\mathfrak{z})|}{2023(|\bar{\boldsymbol{x}}(\mathfrak{z}) + \bar{\boldsymbol{y}}(\mathfrak{z})| + 1)}| \\ &\leq \frac{1}{2023}|\frac{|\boldsymbol{x}(\mathfrak{z}) + \boldsymbol{y}(\mathfrak{z})| - |\bar{\boldsymbol{x}}(\mathfrak{z}) + \bar{\boldsymbol{y}}(\mathfrak{z})|}{(|\boldsymbol{x}(\mathfrak{z}) + \boldsymbol{y}(\mathfrak{z})| + 1)(|\bar{\boldsymbol{x}}(\mathfrak{z}) + \bar{\boldsymbol{y}}(\mathfrak{z})| + 1)}| \\ &\leq \frac{1}{2023}(|\boldsymbol{x}(\mathfrak{z}) + \boldsymbol{y}(\mathfrak{z})| - |\bar{\boldsymbol{x}}(\mathfrak{z}) + \bar{\boldsymbol{y}}(\mathfrak{z})|) \\ &\leq \frac{1}{2023}(|\boldsymbol{x}(\mathfrak{z}) - \bar{\boldsymbol{x}}(\mathfrak{z})| + |\boldsymbol{y}(\mathfrak{z})| - \bar{\boldsymbol{y}}(\mathfrak{z})|) \end{aligned}$$

$$L_f = \frac{1}{2023}$$

$$|g(\mathfrak{z}, \boldsymbol{x}(\mathfrak{z}), \boldsymbol{y}(\mathfrak{z})) - g(\mathfrak{z}, \bar{\boldsymbol{x}}(\mathfrak{z}), \bar{\boldsymbol{y}}(\mathfrak{z}))| \leq \frac{1}{2000}(|\boldsymbol{x}(\mathfrak{z}) - \bar{\boldsymbol{x}}(\mathfrak{z})| + |\boldsymbol{y}(\mathfrak{z})| - \bar{\boldsymbol{y}}(\mathfrak{z})|)$$

$$L_g = \frac{1}{2000}$$

$$\mathfrak{Q}_1 = \frac{2}{\Gamma(\varrho_1 + 1)}L_f + Q = 0.0506$$

$$\mathfrak{Q}_2 = \frac{2}{\Gamma(\varrho_1+1)} L_f = 0.00055$$

$$\mathfrak{Q}_3 = \frac{2}{\Gamma(\varrho_2+1)} L_g + Q = 0.0506$$

$$\mathfrak{Q}_4 = \frac{2}{\Gamma(\varrho_2+1)} L_g = 0.0006115$$

$$\mathcal{M} = \begin{bmatrix} 0.0506 & 0.00055 \\ 0.0506 & 0.0006115 \end{bmatrix}$$

Upon performing the necessary computations, we find that the eigenvalues of the system are determined to be $\beta_1 = 0.0512$ *and* $\beta_2 = 0.0001$*. Consequently, it can be observed that* $\Upsilon(\mathcal{M}) = 0.0512 < 1$*. As a result, according to Theorem, we can conclude that the given system, incorporating a delay term and corresponding to the specified fractional order, is HU-stable.*

Conclusion

In this study, we established the existence, uniqueness, and stability of solutions for a coupled system of fractional differential equations with boundary conditions. Using Banach's fixed-point theorems to prove the existence and uniqueness of solutions, we also demonstrated that the solutions to the problem are stable. We rigorously proved our results and provided illustrative examples to demonstrate their validity. Our research contributes to the theoretical understanding of fractional differential equations, highlighting the effectiveness of these mathematical tools in solving complex systems.

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