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By:Sakina khenfer

Theme

Iterative methods for some variational inequalities

The jury members:

Chacha ahmed Djamel	Professor	Kasdi Merbah University- Ouargla	President
Merabet Ismail	Professor	Kasdi Merbah University- Ouargla	Supervisor
Choutri Abd Elaziz	Professor	Higher Normal School of Kouba	Examiner
Merad Ahcene	Professor	Larbi Ben M'hidi University - Oum El Bouaghi	Examiner
Bensayah Abdallah	Professor	Kasdi Merbah University - Ouargla	Examiner
Ghezal Abderrezak	Professor	Kasdi Merbah University - Ouargla	Examiner

Dedication

In the name of Allah, the Most Gracious, the Most Merciful,
I dedicate this thesis to my loved ones.
To the soul of my dear father, whom I know, if alive, would take pride in my achievements, for him I say: "Insha'Allah, your wishes became true, and may the heavens not withhold their satisfaction upon you. May Allah have mercy on your days and grant you a place in his spacious gardens."
To my wonderful mother and my dear husband, you supported me at every stage of my life, I thank you for your boundless love, immeasurable support, and great encouragement. To the lights of my life, My dear son and daughter, this thesis is dedicated to you, as you represent the spark of motivation I feel every day.
To my entire family, my parents in law Hacini, my brothers, sisters, and faithful friends, and my teachers in the Department of Mathematics, I dedicate this work as a symbol of my gratitude for all the support and encouragement you have provided.
To everyone who contributed in any way to my academic journey, I extend my heartfelt thanks and appreciation.

Acknowledgement

After overcoming numerous challenges, I have reached the culmination of my academic endeavor. This page serves as a space for expressing sincere gratitude to all those who played a pivotal role in the successful completion of this work.

I am deeply thankful to **Prof. Merabet Ismail** for his exceptional mentorship and dedicated supervision during the course of my doctoral research. For his guidance, motivation, patience, and support throughout the years. His expertise, constructive feedback, and assistance significantly enhanced my experience.

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Finally a warm and heartfelt thank you is extended to all my teachers for their unwavering assistance and support throughout this academic journey. I am truly grateful for the knowledge and inspiration you have shared with me.

الملخص

الهدف من هذه الأطروحة هو تكييف الطرق التكرارية للمتباينات المتغيراتية من نوع العوائق. قمنا بتحليل طريقة العناصر المنتهية لنسختين متكافئتين من مسألة العوائق في صدفة نجدي. تقدم النسخة الثانية منظورًا جديدًا للمشكلة المستمرة من خلال التركيز على المساحة غير المقيدة لمجال الإزاحة والدوران. يتم استخدام اثنين من معاملي لاغرانج لفرض شرط التماس شعاع الدوران وتقييد عدم الاختراق. بالإضافة إلى ذلك، نقدم تقديرات خطأ مسبقة ونثبت وجود ووحدانية الحلول لكل من المسائل المستمرة والمتقطعة. علاوة على ذلك، فقد أثبتنا أن طريقة أوزاوا، التي تتعامل مع المتراجحة المتغيرة، متقاربة. والطريقة المقترحة مدعومة بأمثلة من الاختبارات العددية التي توضح فعاليتها في حل مسألة التلامس الأحادي الجانب بين الأصداف المرنة والأجسام الصلبة.

كلمات مفتاحية : مسألة التصادم, صدفة نجدي, العنصر المنتهي, تحليل الخطأ القبلي, الطريقة التراجعية.

Abstract

The aim of this thesis is to adapt iterative methods for variational inequalities of obstacle-type. We analyze the finite element approach for two versions of an obstacle problem in a Naghdi shell, which was equivalent. The second version presents a novel perspective on the continuous problem by focusing on the unconstrained space of the displacement field and rotation. Two Lagrange multipliers are employed to enforce the tangency requirement on rotation and the inequality restriction. Additionally, we provide a priori error estimates and prove the existence and uniqueness of solutions for both continuous and discrete problems. Furthermore, we demonstrate that the Uzawa method, which deals with this variational inequality, exhibits even greater convergence. The suggested method is supported by example numerical tests, illustrating its efficacy in solving the problem of unilateral contact between elastic shells and rigid objects.

Key words : Contact problem, Naghdi shell, Finite element, a priori error analysis, iterative method.

Résumé

L'objectif de cette thèse est d'adapter les méthodes itératives aux inégalités variationnelles de type obstacle. Nous analysons l'approche par éléments finis pour deux versions équivalentes d'un problème d'obstacle pour une coque de Naghdi. La deuxième version présente une nouvelle perspective sur le problème continue en se concentrant sur l'espace relaxé du champ de déplacement et de rotation. Deux multiplicateurs de Lagrange sont utilisés pour forcer la rotation d' être tangentielle au plan tangent et la contrainte d'inégalité. De plus, nous fournissons des estimations d'erreur a priori et prouvons l'existence et l'unicité de solution pour le problèmes continue et discrete. De plus, nous démontrons que la méthode d'Uzawa, qui traite cette inégalité variationnelle, est convergente. La méthode proposée est suivie par des tests numériques, illustrant son efficacité pour résoudre le problème du contact unilatéral entre des coques élastiques et des objets rigides.

Mots clés : Problème de contact, coque Naghdi, éléments finis, analyse d'erreurs a priori, méthode itérative.

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Notations

Throughout this thesis, we will use the following conventions:

➤ ω: An open subset with boundary $\partial\omega$ of $\mathbb{R}^2$.

➤ $\bar{\omega}$: The closure of ω.

➤ δ: The dirac distribution.

➤ S : Midsurface of the shell.

➤ λ , μ: The homogeneous and isotropic material's *Lamé* moduli that make up the shell, respectively.

➤ $\Gamma^{\rho}_{\alpha\beta}$: The Cristofel symbols of the surface.

➤ E , ν : Coefficient of the young of material and Poisson ratio, respectivly.

➤ $W^{s,p}(\omega)$: The Soboleve space with integer $s \;\geq\; 0$, $p \geq 0$.

➤ $\|\cdot\|_{s,p,\omega}$ and $|\cdot|_{s,p,\omega}$: The Sobolev $\mathscr{W}^{s,p}(\omega, \mathbb{R}^{\ell})$ typical norm and semi-norm, respectively, where $\ell \in \mathbb{N}$.

➤ $C^k(\omega)$:The space of functions over ω which are continuously differentiable k times.

➤ $D(\omega)$: The space of all indefinitely derivable function with compact support in (ω).

➤ $(\cdot,\cdot)_\omega$: The $L^2(\omega)$-inner product.

➤ The estimate $A \leq C\ B$ is obtained using $A \lesssim B$, where C is a generic constant independent of A and B.

➤ $B \lesssim A$ and $A \lesssim B$ both hold when $A \sim B$.

Introduction

Many numerical techniques have been developed and used recently to find approximations for variational inequalities. The theory of variational inequalities is a relatively modern mathematical field that found its roots in Fichera's paper [35], concerning unilateral problems and further contributions by J.Lions and G. Stampacchia [53] [67].
Variational inequalities provide precise and concise descriptions for various phenomena, often surpassing variational equalities in accuracy. According to [43], they have applications in a wide range of domains, such as control theory, flows through porous media, lubrication theory, mechanics (contact between deformable elastic bodies), and financial mathematics. This theory represents a crucial class of nonlinear problems with origins in various sources, encompassing the realms of physical and mechanical phenomena [31] [42].
When it comes to approximating variational inequalities, we acknowledge the significant contributions made by Mosco [55], Glowinski [69].
As a first model of variational inequality there exists obstacle problems which involving the interaction of stiff barriers with elastic solids, constitute a significant category within free boundary problems, finding diverse applications from physics to finance. The most common numerical methods for solving contact problems in general, and obstacle problems in particular, are mathematical programming approaches and schemes based on

penalty and Lagrange multiplier formulations. As a result, the discretization of obstacle problems without constraints using the finite element technique, either in its primal formulation or using Lagrange multipliers, has been a long-standing area of discussion. The literature on the finite element approximation of such formulations is extensive, with notable references including [45], [19], [34], [43],[40], [20].

Variational inequalities in thin structures with state restrictions, particularly thin shells, have received little attention in academia. which is the main motivation of this thesis.

A thin shell is defined as a three-dimensional entity where the thickness dimension is significantly smaller than the other dimensions. Renowned for their efficiency in supporting loads over extensive areas with minimal material usage, these structures indeed demonstrate their effectiveness. Shells, with their diverse configurations, play a pivotal role in various elastic structures, including bodies and ship hulls,etc. As evidenced in natural examples like the eggshell.

In engineering applications, shell structures serve many purposes, spanning aerospace, automotive, civil and naval engineering. The shared objective across these fields is to design thin structures for optimal lightness, minimal material usage, safety assurance, and when applicable, aesthetically pleasing designs (see Figure 1).

Figure 1: Bosjes Chapel (Steyn Studio) in south africa

A continuum shell formulation is a mathematical framework used in computational mechanics to model thin structures, like shells or membranes. It is a simple method of representing and analysing material behaviour. Shells are often used to model materials like fabrics, see figure 2, where thickness is much smaller compared to other dimensions.

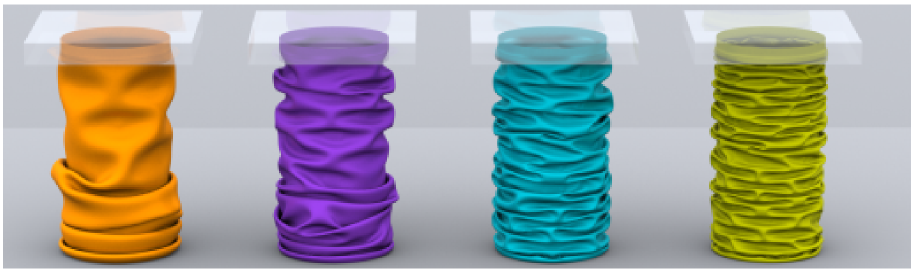


Figure 2: Sleeves: stiffness increases from right to left [28]

Mathematical modelling and numerical analysis of three-dimensional elastic body problems are well-established. However, when dealing with thin structures like shells, numerical methods well-suited to the three-dimensional case encounter challenges due to the relatively small thickness parameter. It is natural to contemplate replacing the three-dimensional models with two-dimensional models placed on the hull's average surface, mostly for computational costs. Pioneers of this approach include Kirchhoff in 1876 and Love in 1934. The derivation of hull models is a longstanding subject that saw extensive development in the 1950 in the Soviet Union and the United States in the 1960 and 1970, with a strong resurgence of interest in France following the work of Sanchez-Palencia on 1989. Under sufficiently small loads, the shell deforms according to the usual laws of three-dimensional elasticity.

Shell theories can be categorized into two main families of : Koiter family [49], which builds upon the work of Kirchhoff-Love. Koiter developed a two-dimensional model based on certain hypotheses about applied forces and the stress tensor, with the unknown being the displacement of the points on the midsurface, which provides an approximation of the displacement of the hull points.

The second family of the shell theory is that of Reissner family, derived from the work of the Cosserat brothers [29] on surfaces, which addresses problems ranging from thin to thick shells. This method was predominantly developed by Naghdi [56], [57].

In the context of this theory, the problem's unknowns are the displacement of points on the average surface and the rotation of the normal to that same surface. The inclusion of the latter unknown allows for the consideration of transverse shear deformations in the thickness. The mathematical analysis of the Naghdi model was carried out for the first time by Coutris on 1978 then improved by Ciarlet and Miara on 1994.

In this thesis we concentrate on the latter family, the Naghdi shell model which has the following variational form:

$$\text{Find } U = (u, r) \in \mathcal{V} \quad \text{such that} \quad a(U, V) = \mathcal{L}(V), \quad \forall V = (v, s) \in \mathcal{V},$$

where

$$\mathcal{V} = \left\{ V = (v, s) \in H^1(\omega, \mathbb{R}^3) \times H^1(\omega, \mathbb{R}^3) \,|\, s \cdot a_3 = 0 \text{ in } \omega, \quad v_{|\gamma_0} = s_{|\gamma_0} = 0 \right\},$$

To solve numerical problems in shell theory, the choice of iterative methods is essential, especially for addressing complex geometries, material behaviors, and boundary conditions. These methods encompass:

- Nonlinear Solvers: like Newton's method or its variants.
- Linearization Techniques:to utilize linear iterative solvers such as conjugate gradient or GMRES.
- Preconditioning: like the Schur complement preconditioner.

These some methods empower researchers and engineers to tackle computational challenges in shell theory, facilitating accurate simulations of complex shell structures.

In this thesis we are interested application of an iterative methods for the unilateral contact of shell with an obstacle on which it is clamped. In this context, we are examining the Naghdi's shell model in the free-basis formulation, as introduced by Blouza [15] and further refined by Blouza and Le Dret [11]. This formulation is based on the notion of using Cartesian coordinates for the unknowns, which facilitates the treatment of the shell's interaction with an obstruction in numerical execution. This formulation demonstrates the capability to handle shells with a $W^{2,\infty}$ midsurface, thus allowing for

curvature discontinuities, as opposed to C^3 in the classical formalism, see[26]. Relying on Naghdi model, we present a contact model with a rigid body based on the fundamental laws of elasticity. In contrast to membrane contacts, one significant problem here consists of the shell is predominantly modelled by its midsurface, whereas the contact occurs on a piece of the shell's physical surface. Nonetheless, by defining the physical surface appropriately, we establish a mathematically reasonable and mechanically realistic model. The resultant system is equivalent to a double mixed problem, combining variational equalities and inequalities. Similar systems have been addressed in [20] and [65].
Let the relaxed functions space: $\mathbb{X} = H^1_{\gamma 0}(\omega)^3 \times H^1_{\gamma 0}(\omega)^3$, and $\mathbb{M} = H^1_{\gamma 0}(\omega)^3$. Thus the variational problem of the contact Naghdi is:

$$\begin{cases} \text{Find } (U, \psi, \lambda) \in \mathbb{X} \times \mathbb{M} \times \Lambda \text{ such that :} \\ \qquad a(U, V) + b(V, \psi) - c(V, \lambda) = \mathcal{L}(V), \qquad \forall V \in \mathbb{X} \\ \qquad b(U, \chi) = 0, \qquad \forall \chi \in \mathbb{M} \\ \qquad c(U, \mu - \lambda) \geq \langle \Phi, \mu - \lambda \rangle . \qquad \forall \mu \in \Lambda \end{cases}$$

where the cone of nonnegative distributions in the dual space of $H^1_{\gamma_0}(\omega)$:

$$\Lambda = \{\mu \in \mathbb{M}'; \forall \sigma \in \mathbb{M}, \sigma \geq 0, \quad \langle \sigma, \mu \rangle \geq 0\},$$

The objectives of this thesis:
Firstly, we aim to revisit the derivation of the contact problem model associated with the Naghdi model, which detailled in first time in [5]. Secondly, we present and analyze new several variational problems, outlining the necessary assumptions for their well-posedness. Another important component of our study is the examination of the finite element approximation of two similar formulations of a Naghdi shell contact problem, as reported in [47]. The first formulation, known as the reduced problem, includes a variational inequality and a variational equality. The second formulation, referred to as the entire problem, includes a variational inequality and two variational equalities. The latter is a new formulation of the continuous problem defined on the unconstrained space of the displacement field and rotation. To enforce the tangency requirement on the rotation (a state constraint) and the inequality constraint, two multipliers called Lagrange are

used. We suggest a non-conforming approximation to the reduced problem, inspired by [43]. Simultaneously, we explore a conforming finite element approximation of the whole issue by inserting elementwise $\mathbb{P}_3$ bubble functions into $\mathbb{P}_1$ elements to approximate the displacement field. This addition is necessary to ensure a discrete stability estimate (see Theorem 3.1 in [43] and Lemma 4.10 below). We prove the existence and uniqueness of solutions for both continuous and discrete problems and develop an apriori error estimate. Furthermore, we show that the iterative methods: Uzawa algorithme converges with this variational inequality. While its convergence speed is slow, we choose it for its ease of implementation and low memory requirements.

Finally, we validate and illustrate our approach through numerical tests.
The initial phase of numerical modeling requires a quantitative representation of the actual geological scenario. Following this, due to the complexity of solving the governing equations directly, discretization and numerical methods are applied to approximate these equations. This process includes breaking down the problem into manageable elements, and algorithms executed on computers are employed to calculate the approximate solutions. The concluding step involves interpreting the obtained solutions to derive meaningful insights or conclusions. See figure 3 below.
The outline of the thesis, excluding the introduction, consists of five chapters.

- Basic Concepts of Surface Theory: This chapter is crucial for understanding various expressions used in the theory of thin shells. In addition, we review the Naghdi model's formulation and key features. Furthermore, it derivates the mathematical framework for the contact problem, presents many variational problems, and specifies the assumptions required for their well-posedness.

- New Constrained Continuous problem of a Contact Naghdi Shell Model: This part introduces a new constrained continuous problem for a contact Naghdi shell model, demonstrating its well-posedness using perturbation techniques. The convergence of the solution from the perturbed problem to the original one is also proven.

- Approximation of the Contacts model using the finite element approach: This chap-

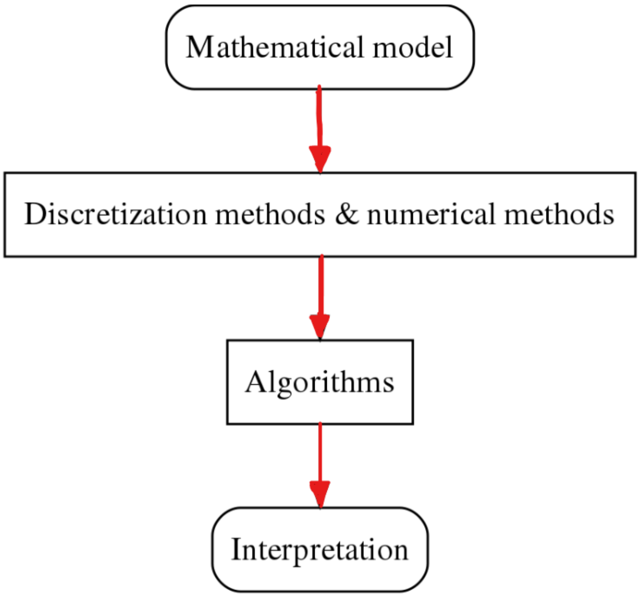


Figure 3: Steps in numerical modeling

ter discusses how to approximate the contact model using the finite element approach for both reduced and full contact problems. Both situations provide evidence of a priori inaccuracy.

- Double Saddle Point and Their Application with appropriate iterative methods: This chapter discusses the double saddle point and its application with an appropriate iterative method for problem resolution.

- Uzawa Method for solving the approximate problem: Finally, to solve the approximate problem, we propose the Uzawa method, considered as a projection technique.

These five chapters cover various stages of the thesis, ranging from theoretical foundations to numerical methods and resolution techniques, providing a comprehensive approach to addressing the contact problem in the context of thin shells.

Chapter 1

Geometrical preliminaries

When an elastic body is exposed to applied forces and proper boundary conditions, nonlinear or linearized three-dimensional elasticity equations can be used to describe it. These equations simplify the three-dimensional elasticity equations based on shell thickness. It is normal to consider replacing three-dimensional models with two-dimensional models placed on the shell's midsurface for computational cost concerns. For further details, see [15], [12], and [51].

In this chapter we will first review several key concepts in differential geometry of surfaces that are relevant to the thesis's theory of usable surfaces. In the second part, we define the shell and its reference configuration as the set of points in $\mathbb{R}^3$ that are within a distance $\leq \varepsilon$ from a specified surface in $\mathbb{R}^3$. Next, we show both undeformed and deformed shells, i.e. shells before and after applying forces. As an example, consider the linear Naghdi shell model and its several formulations. The existence and uniqueness of solutions to the linear Naghdi shell equations are demonstrated using a fundamental Korn inequality on a surface and an infinitesimal stiff displacement lemma.

1.1 The differential geometry of surfaces

Greek indices and exponents take values in $\{1,2\}$, whereas Latin indices and exponents take values in $\{1,2,3\}$. Except as otherwise noted, we follow the Einstein summation convention.

Let there be given a mapping $\varphi \in W^{2,\infty}(\omega, \mathbb{R}^3)$ such that:

$$a_\alpha(x) = \partial_\alpha \varphi(x) = \frac{\partial \varphi}{\partial x_\alpha}(x)$$

are linearly independent at all point x of ω.

The vectors a_1 and a_2 represent the plane that is tangent to the surface S at any point $p = \varphi(x)$, denoted as T_pS.The vector a_3 represents the unit normal as specified by

$$a_3(x) = \frac{a_1(x) \wedge a_2(x)}{|a_1(x) \wedge a_2(x)|}$$

The triplet (a_1, a_2, a_3) denotes the covariant local basis at point $\varphi(x)$, and (a^1, a^2, a^3) denote the contravariant basis which defined by the relation: $a_i \cdot a^j = \delta_i^j$, where: $\delta_i^j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

In particular,

$$a^3(x) = a_3(x)$$

The surface's first fundamental form can be expressed in covariant components as

$$a_{\alpha\beta} = a_\alpha \cdot a_\beta$$

The contravariant components of the metric are given in [8] by:

$$a^{\alpha\beta} = a^\alpha \cdot a^\beta = (a_{\alpha\beta})^{-1} = \frac{1}{a} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{12} & a_{11} \end{pmatrix}$$

where $a(x) = |a_1(x) \wedge a_2(x)|^2 = \det(a_{\alpha\beta}) = a_{11}a_{22} - (a_{12})^2$ such that the area of the midsurface in the chart φ is represented by $\sqrt{a(x)}$. Likewise, the length component ℓ on the boundary $\partial\omega$ is expressed as $\sqrt{a_{\alpha\beta}\varsigma_\alpha\varsigma_\beta}$, with the usual summation convention for repeated indices and coefficients, where $(\varsigma_1, \varsigma_2)$ are the covariant coordinates of the unit vector tangent to $\partial\omega$.

We further introduce the components of the second fundamental form

$$b_{\alpha\beta} = a_3 \cdot \partial_\beta a_\alpha = -a_\alpha \cdot \partial_\beta a_3.$$

The mixed components are specified as

$$b_\alpha^\beta = a^{\beta\rho} b_{\rho\alpha}.$$

Finally the surfac with the Christoffel symbols $\Gamma^\rho_{\alpha\beta}$ takes the form

$$\Gamma^\rho_{\alpha\beta} = \Gamma^\rho_{\beta\alpha} = a^\rho \cdot \partial_\beta a_\alpha = -\partial_\beta a^\rho \cdot a_\alpha.$$

The Christoffel symbols are used to calculate the covariant derivative vectors for surface tensors:

$$\begin{cases} \partial_\beta a_\alpha = \Gamma^\rho_{\alpha\beta} a_\rho + b_{\alpha\beta} a_3. \\ \partial_\beta a^\alpha = -\Gamma^\alpha_{\beta\rho} a^\rho + b^\alpha_\beta a_3. \\ \partial_\beta a_3 = \partial_\beta a^3 = -b_{\beta\rho} a_\rho = -b^\rho_\beta a_\rho. \end{cases}$$

1.2 Geometry of shell

The canonical orthogonal basis of $\mathbb{R}^3$ is denoted by (e_1, e_2, e_3). We define the reference configuration of a shell ω as a bounded connected domain in $\mathbb{R}^2$ with a Lipschitz boundary $\partial\omega$.

Consider a shell that has a midsurface $\mathcal{S} = \varphi(\bar{\omega})$ where:

$$\begin{aligned} \varphi : \quad & \bar{\omega} \subset \mathbb{R}^2 \longrightarrow S \subset \mathbb{R}^3 \\ & x = (x_1, x_2) \longmapsto \varphi(x) \end{aligned}$$

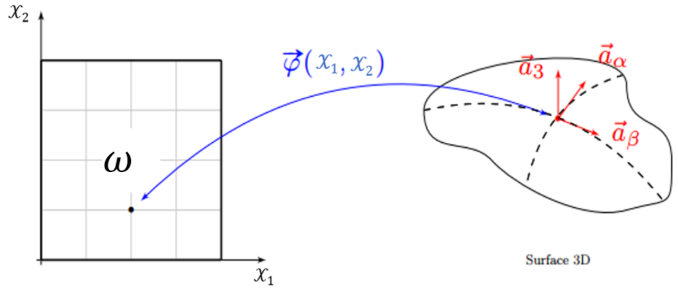


Figure 1.1: A geometry of thin shell and the function φ.

an injective mapping, considered to be sufficiently regular (at least C^1 class on $\bar{\omega}$). The figure (1.1) illustrates that every point on the midsurface is regular.

In the context of shell problems, there equations are not expressed on the surface in three dimensions but are brought back to the reference domain ω. The function φ called the shape function, contains the parameterization that links the reference domain to the three-dimensional shell. The equations are written on the reference domain using several entities of differential geometry, see examples in figures (1.2) and (1.3).

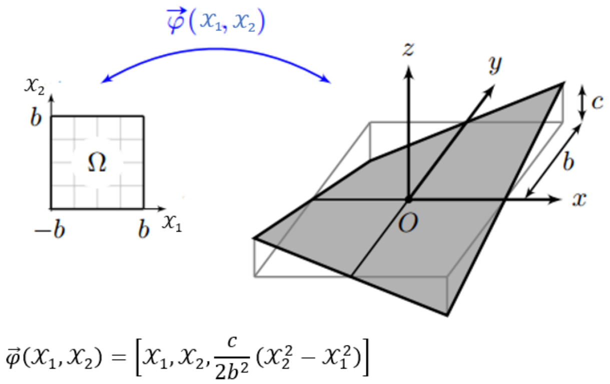


Figure 1.2: Parameterization of the shape function (A hyperbolic paraboloid).

Now we can define undeformed shell, or the reference configuration of a shell with a midsurface $\mathcal{S}$ and thickness ε,which is a subset of $\mathbb{R}^3$ such that :

$$\bar{\mathcal{C}} = \left\{ \phi(x, z) = \varphi(x) + z a_3(x), x \in \bar{\omega} \text{ and } -\frac{1}{2}\varepsilon \leq z \leq \frac{1}{2}\varepsilon \right\}.$$

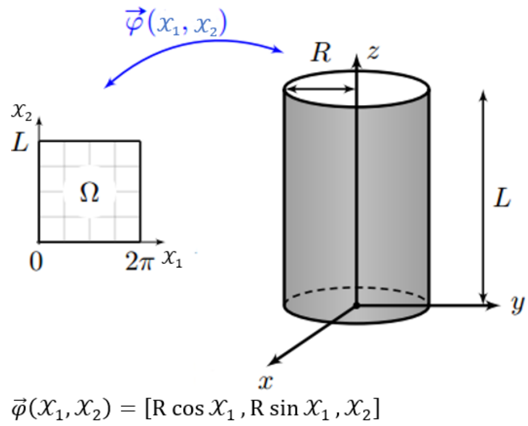


Figure 1.3: Parameterization of the shape function (A cylinder).

and the function $\varepsilon : \omega \to \mathbb{R}_+^*$ defines the measurement of thickness of the shell. The shell is considered thin if $\varepsilon(x)$ is modest in comparison to the midsurface's smallest radius of curvature and external dimensions.

The shell deforms when it is under the action of these following loads,

1. Clamped on the part $\partial S_0 = \varphi(\partial\gamma_0) \times \left[-\frac{1}{2}\varepsilon, \frac{1}{2}\varepsilon\right]$ of its boundary, where $\gamma_0 = \partial\omega_0$ is assumed to have strictly positive measure, i.e. $\text{mes}(\gamma_0) > 0$.

2. Loaded by a force distribution with a resultant force f on the average surface.

3. Loaded on the complementary part $\partial S_1 = \varphi(\gamma_1) = \partial S - \partial S_0$ of its boundary by a force distribution with a resultant force N on γ_1 and a resultant moment M.

With suitable a priori assumptions, different models of shells can be obtained. Subsequently, we mention two distinct models that are representative of the two major classes of thin shell modeling. The first, by Koiter, relies on an assumption of norm conservation during deformation. This model is representative of the family of classical Kirchhoff and Love models.The second, by Naghdi, accounts for the consequences of transverse shear. This model belongs to the family of E. and F. Cosserats. As the Koiter model can be obtained as a particular case of the Naghdi model, we'll stick with the description of the latter model.

1.3 Modeling of Naghdi's shell

1.3.1 Geometric aspects of deformation

First, we present in the following paragraph: the asumptions of Kirchhoff-Love:

- A1: The distance between a location on the shell and the midsurface is constant during deformation.
- A2: The vector a_3 is converted into a vector a_3^* orthogonal to the deformed surface. i.e.:
$$a_3^* = \frac{a_1^* \wedge a_2^*}{|a_1^* \wedge a_2^*|}$$
where $a_\alpha^* = a_\alpha + \partial_\alpha u$ is the covariant base of the deformed surface. After linearization, it can be verified that:
$$a_3^*(x) = a_3(x) - (\partial_\alpha u \cdot a_3) a^\alpha.$$
- A3: The stresses are approximately planar to the middle surface.

Our Naghdi model considers membrane deformations, bending distortions, and transverse shear impacts for homogeneous and isotropic shells. while adhering to the assumption of plane stress (A3) and that of the conservation of distance between a point and the midsurface during deformation (A1).
Therefore, the following points explain the deformed shell $\mathcal{C}^*$:

$$\bar{\mathcal{C}}^* = \left\{ \phi^*(x,z) = \varphi^*(x) + z a_3^*(x), x \in \bar{\omega} \text{ and } -\frac{1}{2}\varepsilon(x) \leq z \leq \frac{1}{2}\varepsilon(x) \right\}.$$

where

$$a_3^* = a_3(x) + r_\alpha a^\alpha$$

$$\varphi^*(x) = \varphi(x) + u$$

where $r_\alpha a^\alpha$ represents the linear covariant components of the rotation field associated with the normal a_3, and u indicates the change in position of the point $\varphi(x)$ on the surface S. The change in position of the point (the displacement) on the shell $U(x, z)$ is expressed as:

$$\begin{aligned} U(x, z) &= \phi^*(x, z) - \phi(x, z) \\ &= \varphi^*(x) + za_3^*(x) - (\varphi(x) + za^3(x)) \\ &= \varphi(x) + u(x) + z(a^3(x) + r_\alpha a^\alpha) - (\varphi(x) + za^3(x)) \\ &= u(x) + zr_\alpha a^\alpha \end{aligned}$$

then

$$U(x, z) = u(x) + zr(x)$$

So, to determine the displacement U through the dimension of the shell (thickness), it suffices to find the displacement $u(x)$ of the average surface as well as the components r_α of the rotation of the normal a_3.

1.3.2 The strain tensors of deformation

This subsection aims to introduce novel definitions for different strain tensors as presented in Blouza's work from 1997 [11]. Given $\varphi \in C^2(\omega; \mathbb{R}^3)$, let u represent the movement of the midsurface, and r denote a rotation of the normal vector a_3.
In the traditional method, rotation and displacement are continuous functions of ω in $\mathbb{R}^3$, represented by the triplet $(u)_i$, $i = 1, 2, 3$ and the pair $(r)_\alpha$, $\alpha = 1, 2$ of their covariant elements such that:

$$\text{if} \quad u_i = u \cdot a_i \quad \text{then} \quad u(x) = u_i(x)a^i(x)$$

and

$$\text{if} \quad r_\alpha = r \cdot a_\alpha \quad \text{then} \quad r = r_\alpha(x)a^\alpha(x)$$

The tangential components of u and the components of r have covariant derivatives defined as:

$$u_\alpha|_\beta = \partial_\beta u_\alpha - \Gamma^\rho_{\alpha\beta} u_\rho$$

and

$$r_\alpha|_\beta = \partial_\beta r_\alpha - \Gamma^\rho_{\alpha\beta} r_\rho.$$

We review the linearized tensors representing deformation, transverse shearing deformation, and curvature change of the average surface of the shell. The definitions are provided within the functional framework established in the classical method of Bernardou, Ciarlet, and Miara. The classical linearized strain tensor is defined as

$$\gamma_{\beta\alpha}(u) = \frac{1}{2}(u_\beta|_\alpha + u_\alpha|_\beta) - b_{\beta\alpha} u_3. \tag{1.1}$$

The classical linearized change of curvature tensor is

$$\chi_{\alpha\beta}(U) = \frac{1}{2}(r_\alpha|_\beta + r_\beta|_\alpha) - \frac{1}{2} b^\rho_\alpha (u_\rho|_\beta - b^\rho_\beta u_3) - \frac{1}{2} b^\sigma_\beta (u_\sigma|_\alpha - b^\sigma_\alpha u_3). \tag{1.2}$$

Finally, the classical linearized transverse shear strain is noted by

$$\delta_{\alpha 3}(U) = \frac{1}{2}(\partial_\alpha u_3 + b^\rho_\alpha u_\rho + r_\alpha). \tag{1.3}$$

Such that $U = (u, r)$.

Now, we recall alternative formulations for the different strain tensors that are more basic and inherent than equations (1.1)-(1.3).

We may demonstrate the existence and uniqueness of general shells with potentially discontinuous curvatures by utilising these equations. Double backslash In this context, we consider the displacement and rotation u, r as mappings from ω to $\mathbb{R}^3$, rather than identifying them with their covariant components. then, we obtain the following symmetrized expressions when using the same idea as that used in [14].

The updated covariant elements of the metric tensor transformation are

$$\gamma^n_{\beta\alpha}(u) = \frac{1}{2}(\partial_\beta u \cdot a_\alpha + \partial_\alpha u \cdot a_\beta), \tag{1.4}$$

The new covariant elements of the change in the transverse shear tensor are:

$$\delta^n_{\beta 3}(U) = \frac{1}{2}(\partial_\beta u \cdot a_3 + r \cdot a_\beta), \tag{1.5}$$

The new covariant elements of the curvature tensor change are:

$$\chi^n_{\alpha\beta}(U) = \frac{1}{2}(\partial_\alpha u \cdot \partial_\beta a_3 + \partial_\beta u \cdot \partial_\alpha a_3 + \partial_\alpha r \cdot a_\beta + \partial_\beta r \cdot a_\alpha). \tag{1.6}$$

Following [12], These quantities are applicable to little regularity shells and can be straightforwardly represented using the Cartesian coordinates of the unknowns and geometric data.

We consider the case of an isotropic and homogeneous material with a Young modulus E greater than 0 and a Poisson ratio ν, where 0 is less than or equal to ν and ν is less than $\frac{1}{2}$.

The elasticity tensor $a^{\alpha\beta\rho\sigma} \in L^\infty(\omega)$ with contravariant components is determined by the expression:

$$a^{\alpha\beta\rho\sigma} = \frac{E}{(2+2\nu)}(a^{\alpha\rho}a^{\beta\sigma} + a^{\alpha\sigma}a^{\beta\rho}) + \frac{E\nu}{1-\nu^2}a^{\alpha\beta}a^{\rho\sigma}. \tag{1.7}$$

This tensor adheres to the typical symmetry properties and is uniformly strictly positive, indicating the existence of a positive constant c such that:

$$a^{\alpha\beta\rho\sigma}\gamma^n_{\alpha\beta}\gamma_{\rho\sigma} \geq c\sum_{\alpha,\beta}|\gamma^n_{\alpha\beta}|^2, \tag{1.8}$$

for all 2×2 symmetric real-valued matrices. The symbol γ represents a matrix with elements $\gamma^n_{\alpha\beta}$, for $1 \leq \alpha, \beta \leq 2$.

1.3.3 Constrained variational formulation

It is assumed that the boundary of the chart, denoted as $\partial\omega$ is composed of two components: γ_0 which represents a strictly positive one-dimensional measure and serves as the clamping point for the shell, and $\gamma_1 = \partial\omega \setminus \gamma_0$,which represents the boundary across which the shell is subjected to applied tractions and moments. In order to satisfy these boundary conditions, we establish the space $H^1_{\gamma_0}(\omega)$ as the collection of functions μ within $H^1(\omega)$ that have a value of zero on γ_0.

$$H^1_{\gamma_0}(\omega) = \{\mu \in H^1(\omega);\ \mu = 0\ on\ \gamma_0\}$$

Now, let's present the space introduced in reference [15]:

$$\mathcal{V} = \left\{V = (v, s) \in H^1(\omega)^3 \times H^1(\omega)^3 \,|\, s \cdot a_3 = 0 \text{ in } \omega, \quad v_{|\gamma_0} = s_{|\gamma_0} = 0\right\}, \tag{1.9}$$

endowed with the natural Hilbert norm

$$\|V\|_{\mathcal{V}} = \|(v, s)\|_{\mathcal{V}} = \left(\|v\|^2_{H^1(\omega)^3} + \|s\|^2_{H^1(\omega)^3}\right)^{\frac{1}{2}} \tag{1.10}$$

Lemma 1.3.1 *The space $\mathcal{V}$ is a Hilbert space.*

Proof. The proof of this lemma is detailed in [10] ■

The Naghdi shel model is represented by the variational form given below:

$$\begin{cases} \text{Find } U = (u, r) \in \mathcal{V} \quad \text{ such that} \\ \boldsymbol{a}(U, V) = \mathcal{L}(V), \quad \forall V = (v, s) \in \mathcal{V}, \end{cases} \tag{1.11}$$

where

$$\begin{aligned} a(U, V) := &\int_\omega (\{\varepsilon a^{\alpha\beta\rho\sigma}\left[\gamma^n_{\alpha\beta}(u)\gamma^n_{\rho\sigma}(v) + \frac{\varepsilon^2}{12}\chi^n_{\alpha\beta}(U)\chi^n_{\rho\sigma}(V)\right] \\ &+ \frac{2\varepsilon E}{1+\nu} a^{\alpha\beta}\delta^n_{\alpha 3}(U)\delta^n_{\beta 3}(V))\}\sqrt{a}dx, \end{aligned} \tag{1.12}$$

and

$$\mathcal{L}(V) := \int_\omega f \cdot v\sqrt{a}dx + \int_{\gamma_1} (N \cdot v + M \cdot s)\ell\, d\varsigma. \tag{1.13}$$

The data $f \in L^2(\omega)^3, N \in L^2(\gamma_1)^3$ and $M \in L^2(\gamma_1)^3$ represent a given resultant force density, an applied traction density and an applied moment density, respectively.

Theorem 1.3.2 *For any data $(f, N, M) \in L^2(\omega)^3 \times L^2(\gamma_1)^3 \times L^2(\gamma_1)^3$, problem (1.11) admits a unique solution U in $\mathcal{V}(\omega)$. Moreover, this solution satisfies:*

$$\|U\|_{\mathcal{V}(\omega)} \leq c\|\mathcal{L}\|$$

The following form of the infinitesimal rigid displacement lemma is a key part of the proof. It works for a $W^{2,\infty}$ shell and relies on expressions (1.4), (1.5), and (1.6).

Lemma 1.3.3 (Rigid Displacement Lemma [15]) *Let $u \in H^1(\omega, \mathbb{R}^3)$ and $r \in H^1(\omega, \mathbb{R}^3)$ such that $r \cdot a_3 = 0$ with $\varphi \in W^{2,\infty}(\omega, \mathbb{R}^3)$.*

- *In the case where u is such that $\gamma^n(u) = 0$, then a distinct $\tilde{\psi} \in L^2(\omega, \mathbb{R}^3)$ exists ans satisfy:*

$$\partial_\alpha u = \tilde{\psi} \times a_\alpha, \quad \alpha = 1, 2. \tag{1.14}$$

- *If, in addition, u and r satisfy $\delta^3_\alpha(u, r) = 0$, then $\partial_\alpha u = -r \times a_\alpha$ belong to $H^1(\omega)$. Moreover, $r \cdot a_\alpha = -e_{\alpha\beta} a^\beta \cdot \tilde{\psi}$ with $e_{11} = e_{22} = 0$ and $e_{12} = -e_{21} = \sqrt{a}$.*

- *If, in addition, $\chi(u, r)^n = 0$, then $\tilde{\psi}$ is identified with a constant vector in $\mathbb{R}^3$, and for all $x \in \omega$:*

$$u(x) = c + \tilde{\psi} \times \varphi(x),$$

where c is a constant in $\mathbb{R}^3$ and

$$r(x) = -(e_{\alpha\beta} a_\beta \cdot) a_\alpha(x).$$

Proof.

- For a proof of the existence and uniqueness of the infinitesimal rotation vector $\tilde{\psi}$ such that (1.14) holds true, refer to [13].

- If we assume $\delta^3_\alpha(u, r) = 0$, then

$$\partial_\alpha u \cdot a_3 = -r \cdot a_\alpha \in H^1(\omega), \tag{1.15}$$

since $r \in H^1(\omega, \mathbb{R}^3)$ and $a_\alpha \in W^{1,\infty}(\omega, \mathbb{R}^3)$. Therefore, we have

$$r \cdot a_\alpha = (a_\alpha \wedge a_3) \cdot \tilde{\psi} = -\varepsilon_{\alpha\beta} a^\beta \cdot \tilde{\psi}.$$

- Let us first note that under the previous hypotheses, we have, because of the formula (1.15),

$$\partial_{\alpha\beta} u \cdot a_3 = \partial_\beta(\partial_\alpha u \cdot a_3) - \partial_\alpha u \cdot \partial_\beta a_3 = -\partial_\beta(r \cdot a_\alpha) - \partial_\alpha u \cdot \partial_\beta a_3 \in L^2(\omega). \tag{1.16}$$

because $\partial_\beta a_3 \in L^\infty(\omega, \mathbb{R}^3)$. It follows, by (1.16), that

$$\begin{aligned}\chi^n_{\alpha\beta}(u,r) &= \frac{1}{2}(\partial_\alpha u \cdot \partial_\beta a_3 + \partial_\beta u \cdot \partial_\alpha a_3 + \partial_\alpha r \cdot a_\beta + \partial_\beta r \cdot a_\alpha) \\ &= \frac{1}{2}\left(\partial_\alpha u \cdot \partial_\beta a_3 + \partial_\beta u \cdot \partial_\alpha a_3 + \partial_\alpha (r \cdot a_\beta) + \partial_\beta (r \cdot a_\alpha) - 2r \cdot \partial_\alpha a_\beta\right) \\ &= \frac{1}{2}(-2\partial_{\alpha\beta} u \cdot a_3 - 2r \cdot \partial_\alpha a_\beta) \\ &= -(\partial_{\alpha\beta} u \cdot a_3 + \Gamma^\rho_{\alpha\beta} r \cdot a_\rho),\end{aligned}$$

since $\partial_\alpha a_\beta = \Gamma^\rho_{\alpha\beta} a_\rho$. Then, using (1.15), we see that

$$\chi_{\alpha\beta}(u,r) = -(\partial_{\alpha\beta} u - \varepsilon_{\alpha\beta\rho}\partial_\rho u) \cdot a_3. \tag{1.17}$$

If we assume now that $\chi^n(u,r) = 0$, we just need to use the infinitesimal rigid displacement lemma valid for $W^{2,\infty}$-Koiter shell given in [14] to complete the proof.

■

Now we require the following lemmas to prove the V-ellipticity of the bilinear form of (1.11).

Lemma 1.3.4 *There is a constant $C > 0$ so that*

$$a(V,V) \geq C\left(\|\gamma^n(v)\|^2_{L^2(\omega)} + \|\chi^n(V)\|^2_{L^2(\omega)} + \|\delta^n_{\alpha 3}(V)\|^2_{L^2(\omega)}\right)^{1/2}$$

for all $V = (v,s) \in (H^1(\omega, \mathbb{R}^3))^2$.

Proof. This is clear in view of inequality (1.8) and the fact that $a_{\alpha\beta}(x)\eta_\alpha\eta_\beta \geq C\gamma^n_\alpha(\eta_\alpha)^2$ for all $x \in \omega$. ■

Lemma 1.3.5 *The bilinear form of the problem (1.11) is $\mathcal{V}$-elliptic.*

Proof. Given the previous Lemma and the assumptions made about the chart φ, the elasticity tensor, and the thickness of the shell, it suffices to demonstrate that

$$\|(V)\| = \left(\|\gamma^n(v)\|^2_{L^2(\omega)} + \|\chi^n(V)\|^2_{L^2(\omega)} + \|\delta^n_{\alpha 3}(V)\|^2_{L^2(\omega)}\right)^{1/2}$$

is a norm on $\mathcal{V}$ that is bounded from below by a multiple of the natural norm (1.10) of $\mathcal{V}$, see [15] to complete the proof. ■

Proof. (Proof of Theorem (1.3.2))
It's clear that the bilinear and linear forms in (1.11) are continuous on $\mathcal{V}$. Along with Rellich's theorem and the two-dimensional Korn inequality, we just used a contradiction argument to show that the bilinear form is $\mathcal{V}$-elliptic. The Lax-Milgram lemma applied to problem (1.11) shows that there is existence and uniqueness. ■

Remark 1.3.1 *We can represent the problem described in equation (1.11) as a set of partial differential equations according to reference [5]. Let's establish the operator:*

$$\bar{\mathcal{A}} : H^1(\omega)^3 \times H^1(\omega)^3 \longrightarrow H^{-1}(\omega)^3 \times H^{-1}(\omega)^3$$

by duality as follows: $\langle \bar{\mathcal{A}}\, U, V\rangle = a(U,V)$
The Neumann operator associated with it, denoted as

$$\bar{\mathcal{N}} : H^1(\omega)^3 \times H^1(\omega)^3 \longrightarrow H^{1/2}_{00}(\gamma_1)^3 \times H^{1/2}_{00}(\gamma_1)^3$$

defined by $\langle \bar{\mathcal{N}} U, V\rangle = a(U,V) - \langle \bar{\mathcal{A}}(U,V))\rangle$
Note that this necessitates an additional regularity property (for example, if $\bar{\mathcal{A}}U \in L^2(\omega)^3$*) which we assume here. Therefore, in terms of distributions, it can be readily verified that the solution to problem (1.11) produces the subsequent system:*

$$\begin{cases} \bar{\mathcal{A}}U = \begin{pmatrix} f\sqrt{a} \\ 0 \end{pmatrix} & in\ \omega \\ r \cdot a_3 = 0 & in\ \omega \\ u = r = 0 & on\ \gamma_0 \\ \bar{\mathcal{N}}U = \begin{pmatrix} Nl \\ Ml \end{pmatrix} & in\ \gamma_1 \end{cases} \tag{1.18}$$

1.3.4 Mixed variational formulation

For the general shell situation, the constraint $s \cdot a_3$ which appears in the definition of $\mathcal{V}$ cannot be implemented in astandard conforming way. This amounts to say that the problem (1.11) cannot be approximated by conforming methods for a general shell [12]. The approach used here consists in introducing a Lagrange multiplier in order to handle the tangency requirement on the rotation.

$$\mathbb{X} = H^1_{\gamma_0}(\omega)^3 \times H^1_{\gamma_0}(\omega)^3,$$

Still possessing the norm specified in Equation (1.10), now represented as $\|\cdot\|_{\mathbb{X}}$.
We also established

$$\mathbb{M} = H^1_{\gamma_0}(\omega) = \left\{\mu \in H^1(\omega); \quad \mu = 0 \text{ on } \gamma_0\right\}.$$

So, the mixed variational problem is as follows :

$$\begin{cases} \text{Find } (U,\psi) \ in \ \mathbb{X}\times\mathbb{M} \text{ such that} \\ a(U,V) + b(V,\psi) = \mathcal{L}(V) \qquad \forall V \in \mathbb{X} \\ \qquad b(U,\chi) = 0 \qquad \forall \chi \in \mathbb{M} \end{cases} \tag{1.19}$$

where

$$\boldsymbol{b}(V,\chi) = \int_\omega \partial_\alpha(s\cdot a_3)\partial_\alpha\chi \, dx. \quad \forall (V,\chi) \in \mathbb{X}\times\mathbb{M}, \tag{1.20}$$

The form $b(\cdot,\cdot)$ is continuous on $\mathbb{X}\times\mathbb{M}$ because a_3 in $W^{1,\infty}(\omega)^3$.
In addition, the following description is accurate:

$$\mathcal{V} = \{V \in \mathbb{X} \mid \forall \chi \in \mathbb{M}, \quad b(V,\chi) = 0\}.$$

Lemma 1.3.6 *There exists a positive constant c_* such that*

$$\forall \chi \in \mathbb{M} \ \ \sup_{V\in\mathbb{X}} \frac{\boldsymbol{b}(V,\chi)}{\|V\|_{\mathbb{X}}} \geq c_*\|\chi\|_{\mathbb{M}} \tag{1.21}$$

Proof. The inf-sup condition for the form $b(\cdot,\cdot)$ may be obtained by setting $V = (0, \chi a_3)$.
■

Theorem 1.3.7 *[58] Let $f \in L^2(\omega)$ be a given force resultant density. Then the mixed variational problem (1.19) has a unique solution (U,ψ) in $\mathbb{X}\times\mathbb{M}$, which is such that U is the solution of Naghdi's problem (1.11). Moreover, this solution satisfies:*

$$\|U\|_{\mathbb{X}} + \|\psi\|_{\mathbb{M}} \leq \|f\|$$

Proof. (See [39] lemma 4.1.) ■

A similar as (1.18), the mixed problem (1.19) can be written as a system of partial differential equations. This ones using an explicit form of the operators $\bar{\mathcal{A}}$ and $\bar{\mathcal{N}}$, see [9].

$$\begin{cases} -\partial_\rho\left((n^{\rho\sigma}(u)a_\sigma + m^{\rho\sigma}(U)\partial_\sigma a_3 + q^\rho(U)a_3)\sqrt{a}\right) = f\sqrt{a} \text{ in } \omega, \\ -\partial_\rho(m^{\rho\sigma}(U)a_\sigma\sqrt{a}) + q^\beta(U)a_\beta\sqrt{a} + \partial_{\rho\rho}\psi a_3 = 0 \quad \text{in } \omega, \\ r\cdot a_3 = 0 \quad \text{in } \omega, \\ u = r = 0 \quad \text{on } \gamma_0, \\ \psi = 0 \quad \text{on } \gamma_0, \\ \nu_\rho\left((n^{\rho\sigma}(u)a_\sigma + m^{\rho\sigma}(U)\partial_\sigma a_3 + q^\rho(U)a_3)\sqrt{a}\right) = Nl \quad \text{on } \gamma_1, \\ \nu_\rho(m^{\rho\sigma}(U)a_\sigma\sqrt{a} + \partial_\rho\psi a_3) = Ml \quad \text{on } \gamma_1, \end{cases} \tag{1.22}$$

Where

$$\begin{aligned} &n^{\alpha\beta}(u) = \varepsilon a^{\alpha\beta\rho\sigma}\gamma_{\alpha\beta}(u), \quad \text{the stress resultant} \\ &m^{\alpha\beta}(U) = \frac{\varepsilon}{3}a^{\alpha\beta\rho\sigma}\chi_{\alpha\beta}(U), \quad \text{the stress couple} \\ &q^\beta(U) = \varepsilon\frac{E}{1+\nu}a^{\alpha\beta}\delta_{\alpha3}(U). \quad \text{the transverse shear force.} \end{aligned}$$

Chapter 2

Contact Problem of a Naghdi's shell

In this chapter, we present a contact Naghdi's shell model proposed in [47]. We adopt the geometric preliminaries and notations introduced in pervious sections.
In this context, we are considering Naghdi's shell model in the free-basis formulation. This formulation is based on the idea of using Cartesian coordinates for the unknowns, which facilitates the handling of contact between the shell and an obstacle. The construction and the implementation of conforming finite element methods in FreeFem++ software for this formulation is easier unlike the curvilinear coordinates, despite of adventage of absence the constraint imposed on the rotation field (which must be tangential to the middle surface) if we use the last coordonates. The contact model is predicated on the assumption that:

- During the deformation of the shell, the distance between a given point and its orthogonal projection onto the midsurface remains constant.

- Although the nodes remain on a line that is normal to the midsurface, this line is no longer normal to the midsurface that has been deformed
- The two-dimensional model incorporates two unknown variables: the displacement u of the shell midsurface points and the linearized rotation field r, which characterises the midsurface's normal straight fibre rotation.

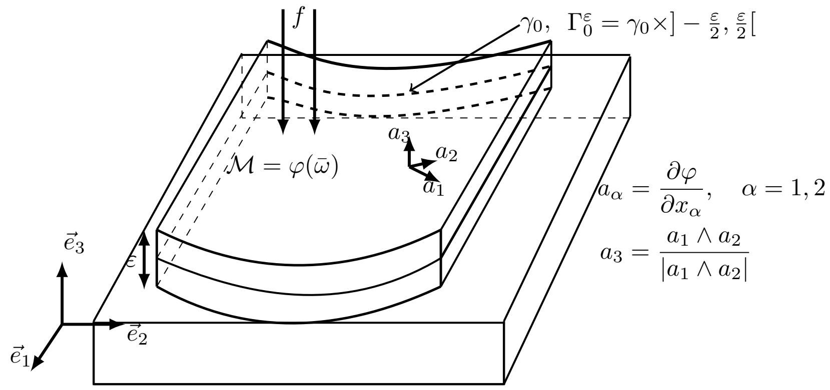


Figure 2.1: A shell in contact with a rigid obstacle

2.1 The derivation of model

The midsurface $\mathcal{S} = \varphi(\bar{\omega})$ of the shell $\mathcal{C}$ and thickness ε is defined as follows:

$$\bar{\mathcal{C}} = \left\{ \varphi(x) + z a_3(x), x \in \bar{\omega} \text{ and } -\frac{1}{2}\varepsilon(x) \le z \le \frac{1}{2}\varepsilon(x) \right\},$$

which z is the distance of a point on the shell to its midsurface. Here, we are interested in studying the contact of this shell with a rigid obstacle contained in the half-space

$$z \cdot e_3 < 0$$

and such that its boundary occupies the whole plane $z \cdot e_3 = 0$. Therefore, from now on, we assume without restriction that the function φ satisfies:

$$\varphi \cdot e_3 > 0, \quad \text{for all } x \in \omega$$

Thus, the contact occurs on part of the lower surface of the shell, specifically on the surface

$$\left\{\varphi(x) - \frac{\varepsilon}{2}a_3, \quad x \in \bar{\omega}\right\}$$

As standard for contact models, the contact condition combines three equations or inequalities:

2.1.1 The contact conditions

- **The positions of the shell and of the obstacle**:

  As a result of the shell model, the deformed shell $\mathcal{C}^*$ has its midsurface

  $$\mathcal{S}^* = \varphi^*(\bar{\omega}), \quad \varphi^* = \varphi + u$$

  Now, we set: $a_3^* = a_3 + r$

  $$\bar{\mathcal{C}}^* = \left\{\varphi^*(x) + za_3^*(x), x \in \bar{\omega} \text{ and } -\frac{1}{2}\varepsilon(x) \leq z \leq \frac{1}{2}\varepsilon(x)\right\}.$$

  Therefore, the fact that the shell is above the obstacle can be expressed as:

  $$\forall x \in \bar{\omega} \quad -\frac{1}{2}\varepsilon(x) \leq z \leq \frac{1}{2}\varepsilon(x) \quad ((\varphi^*(x) + za_3^*(x))) \cdot e_3 \geq 0.$$

  If we choose $z = -\frac{\varepsilon}{2}$ then

  $$\forall x \in \bar{\omega} \quad \left(\varphi(x) - \frac{\varepsilon}{2}a_3 + u(x) - \frac{\varepsilon}{2}r(x)\right) \cdot e_3 \geq 0.$$

  Let's set

  $$\Phi(x) = \left(\frac{\varepsilon}{2}a_3(x) - \varphi(x) \cdot e_3\right) \tag{2.1}$$

  thus $\Phi \in W^{1,\infty}(\omega)$, and from now on, we assume that

  $$\Phi(x) \leq 0 \text{ for a. e } x \; in \; \omega. \tag{2.2}$$

Due to the non-negativeness of $a_3 \cdot e_3$ and the positivity of $\varphi \cdot e_3$, it follows that the contact model is cleary useless without this condition. If the shell is not flat and the thickness ε is not too big, then (2.2) is true,i.e:

$$\frac{\varepsilon}{2} \max_{x \in \bar{\omega}} a_3(x) \cdot e_3 \leq \min_{x \in \bar{\omega}} \varphi(x) \cdot e_3. \tag{2.3}$$

Therefore, if the shell is flat, meaning if $a_3 \cdot e_3 = 0$ on ω, then (2.3) holds (and hence (2.2)), $\forall\ \varepsilon > 0$.

Conversely, if $\max_{x \in \bar{\omega}} a_3(x) \cdot e_3 > 0$, then (2.3) holds if and only if

$$\varepsilon \leq \varepsilon_0 := 2 \frac{\min_{x \in \bar{\omega}} \varphi(x) \cdot e_3}{\max_{x \in \bar{\omega}} a_3(x) \cdot e_3}.$$

Therefore (2.2) holds under this last constraint on ε.

The first contact inequality takes the form:

$$\left(u - \frac{\varepsilon}{2} r\right) \cdot e_3 \geq \Phi \quad \text{a.e } in\ \omega.$$

Let ω_c represent the contact zone, defined as the set of points x in ω satisfying:

$$\left(u - \frac{\varepsilon}{2} r\right) \cdot e_3 = \Phi \quad \text{a.e } in\ \omega \tag{2.4}$$

- **Reaction of the obstacle:**

In the situation that we consider, the reaction of the obstacle due to the presence of the shell is of the form λe_3 for a scalar function λ. Thus, in the right-hand side of the equation, the term

$$\int_\omega f \cdot v \sqrt{a}\ dx$$

must be replaced by:

$$\int_\omega f \cdot v \sqrt{a}\ dx + \int_\omega \lambda e_3 \cdot (v - \frac{\varepsilon}{2} s)\ dx$$

Moreover, since the shell is above the obstacle, we have:

$$\lambda \geq 0 \quad in\ \omega.$$

- **Location of the reaction**

  Naturally, the reaction of the obstacle is confined to the contact zone ω_c defined by (2.4). This results in the complementarity equation

$$\lambda\left(\left(u-\frac{\varepsilon}{2}r\right)\cdot e_3-\Phi\right)=0 \quad \text{in} \quad \omega.$$

2.1.2 System of inequalities

Taking into account all of above contact conditions, we derive the model for the contact of the shell, where the unknowns are the deformation of the shell u, its rotation r and the reaction coefficient λ:

$$\begin{cases} \bar{\mathcal{A}}U-\begin{pmatrix}\lambda e_3\\ -\frac{\varepsilon}{2}\lambda e_3\end{pmatrix}=\begin{pmatrix}f\sqrt{a}\\ 0\end{pmatrix} & \textit{in } \omega\\ r\cdot a_3=0 & \textit{in } \omega\\ \left(u-\frac{\varepsilon}{2}r\right)\cdot e_3\geq\Phi,\ \lambda\geq 0,\ \ \lambda\left(\left(u-\frac{\varepsilon}{2}r\right)\cdot e_3-\Phi\right)=0 & \textit{in } \omega\\ u=r=0 & \textit{on } \gamma_0\\ \bar{\mathcal{N}}U=\begin{pmatrix}Nl\\ Ml\end{pmatrix} & \textit{in } \gamma_1\end{cases} \tag{2.5}$$

2.1.3 Variational inequalities formulation

First, we introduce the cone of nonnegative distributions in the dual space of $H^1_{\gamma_0}(\omega)$:

$$\Lambda=\{\mu\in\mathbb{M}';\ \forall\sigma\in\mathbb{M},\quad \sigma\geq 0,\quad \langle\sigma,\mu\rangle\geq 0\},$$

with the bilinear form $c:\mathbb{X}\times\mathbb{M}'\to\mathbb{R}$ defined by

$$c(V,\mu)=\left\langle\left(v-\frac{\varepsilon}{2}s\right)\cdot e_3,\mu\right\rangle,\quad \forall(V,\mu)\in\mathbb{X}\times\mathbb{M}'. \tag{2.6}$$

Following [5], the variational problem of the contact Naghdi is:

$$\begin{cases} \text{Find } (U,\psi,\lambda) \in \mathbb{X}\times\mathbb{M}\times\Lambda \text{ such that :} \\ \qquad a(U,V)+b(V,\psi)-c(V,\lambda)=\mathcal{L}(V), \qquad \forall V\in\mathbb{X} \\ \qquad (U,\chi)=0, \qquad \forall\chi\in\mathbb{M} \\ \qquad c(U,\mu-\lambda)\geq\langle\Phi,\mu-\lambda\rangle\,. \qquad \forall\mu\in\Lambda \end{cases} \tag{2.7}$$

Proposition 2.1.1 *Assume that the data are sufficiently smooth. Hence any triplet* (U,ψ,λ) *in* $\mathbb{X}\times\mathbb{M}\times\Lambda$ *is a solution of Problem (2.7) if and only if it satisfies the system (2.5).*

Proof. The equivalence between the problem (2.7) and (2.5) is done in [5] ■

A closely related yet simpler system is examined in [20] [65].

Nevertheless, our approach in this analysis differs slightly. We are introducing an initial simplified version of the problem.

$$\begin{cases} \text{Find } (U,\lambda)\in\mathcal{V}\times\Lambda \text{ such that :} \\ \qquad a(U,V)-c(V,\lambda)=\mathcal{L}(V), \qquad \forall V=(v,s)\in\mathcal{V}(\omega), \\ \qquad c(U,\mu-\lambda)\geq\langle\Phi,\mu-\lambda\rangle. \quad \forall\mu\in\Lambda. \end{cases} \tag{2.8}$$

Proposition 2.1.2 *Problems (2.7) and (2.8) are equivalent, in the following sense:*

1. *Given that* (U,ψ,λ) *solves problem (2.7), then the pair* (U,λ) *also solves Problem (2.8).*

2. *If* (U,λ) *is a solution to problem (2.8), then there is a unique function* ψ *in* $\mathbb{M}$ *such that the triple* (U,ψ,λ) *is a solution to problem (2.7).*

However, this is not sufficient to prove the existence of a solution for problem (2.7). We now introduce the convex set

$$\mathcal{K}_\Phi=\left\{V\in\mathcal{V};\left(v-\frac{\varepsilon}{2}s\right)\cdot e_3\geq\Phi,\ \text{a.e. in }\omega\right\} \tag{2.9}$$

We consider the problem:

$$\begin{cases} Find\ \ U\in\mathcal{K}_\Phi \ \text{ such that :} \\ a(U,V-U)\geq\mathcal{L}(V-U) \quad \forall V=(v,s)\in\mathcal{K}_\Phi. \end{cases} \tag{2.10}$$

Therefore, we can now state the following proposition:

Proposition 2.1.3 *For any data* (f, N, M) *in* $L^2(\omega)^3 \times L^2(\gamma_1)^3 \times L^2(\gamma_1)^3$ *problem (2.10) has a unique solution* U *in* $\mathcal{K}_\Phi$

Proof.

1. $\mathcal{K}_\Phi$ is closed, convex set.

2. $\Phi \leq 0$ then the set $\mathcal{K}_\Phi$ is not empty.

3. $a(\cdot, \cdot)$ is $\mathcal{V}$-elliptic.

Then, the existence and uniqueness of a solution to problem (2.10) follows directly from the Lions-Stampacchia theorem [53]. ■

Remark 2.1.1 *To establish the existence of a solution for problem (2.8), and therefore for problem (2.7), we assume: In order to ascertain the presence of a solution for problem (2.8), and consequently problem (2.7), the following assumptions are made:*

- $c(\cdot, \cdot)$ *satisfies the inf-sup condition belongs to* $\mathcal{V}$.

- *The function* Φ *satisfies*

$$\Phi(x) = 0 \text{ for a.e } x \in \gamma_0. \tag{2.11}$$

then the Problem (2.8) has at least a solution $(U, \lambda) \in \mathcal{V} \times \Lambda$*. Consequently from [5] the problem (2.7) has a unique solution using the proposition (2.1.2).*

2.2 New well posdness analysis

In what follows, we are mainly interested in the discretization of problem (2.10). It must be noted from Proposition (2.1.3) that the well-posedness of this problem requires no further assumption. However, if assumption (2.11) is not satisfied, there is no link between this problem and the contact model in the subsequent discussion.

Furthermore, from a numerical point of view, the constraint $r \cdot a_3 = 0$ is not easy to handle since the direction of a_3 is not constant. For this reason the paper [5] proposed new variational model of Naghdi's contact defined by:

Problem 1 *Find* (U, ψ) *in* $\mathbb{N}_\Phi \times \mathbb{M}$ *such that*

$$\begin{cases} a(U, V-U) + b(V-U, \psi) \geq \mathcal{L}(V-U), \ \forall V \in \mathbb{N}_\Phi, \\ b(U, \chi) = 0, \quad \forall \chi \in \mathbb{M}, \end{cases} \tag{2.12}$$

where $\mathbb{N}_\Phi$ is a closed convex set of $\mathbb{X}$ given by

$$\mathbb{N}_\Phi := \left\{ V \in \mathbb{X}; \ \left(v - \frac{\varepsilon}{2} s \right) \cdot e_3 \geq \Phi \text{ a.e. in } \omega \right\}. \tag{2.13}$$

The primary objective of this chapter is to give another proof in the spirit of [65] based on a perturbation technique. Since $r \cdot a_3 = 0$ a.e. in ω, verifying that Problem 1 is equivalent to the subsequent problem is straightforward:

Problem 2

$$\begin{cases} \text{Find } (U, \psi) \in \mathbb{N}_\Phi \times \mathbb{M} \text{ such that} \\ \boldsymbol{a}_\rho(U, V-U) + \boldsymbol{b}(V-U, \psi) \geq \mathcal{L}(V-U), \ \forall V \in \mathbb{N}_\Phi, \\ \boldsymbol{b}(U, \chi) = 0, \quad \forall \chi \in \mathbb{M}, \end{cases} \tag{2.14}$$

For any real parameter $\rho > 0$*, we define:*

$$\boldsymbol{a}_\rho(U, V) = \boldsymbol{a}(U, V) + \rho \int_\omega \partial_\alpha (r \cdot a_3) \partial_\alpha (s \cdot a_3)\, dx, \quad \forall U, V \in \mathbb{X}.$$

Remark 2.2.1 *Note that the bilinear form* $\boldsymbol{a}(\cdot, \cdot)$ *is not* $\mathbb{X}$*-elliptic (see [12, Lemma 3.3]). Replacing the bilinear form* $\boldsymbol{a}((\cdot,\cdot),(\cdot,\cdot))$ *by* $\boldsymbol{a}_\rho((\cdot,\cdot),(\cdot,\cdot))$ *allows us to recover the ellipticity over the space* $\mathbb{X}$*, where as soon as* $\rho > 0$*.*

2.2.1 A compact formulation of reduced problem

It is clear that neither Problem reduced nor Problem 2 is in the "standard" form of variational inequalities, i.e., a single variational inequality. In this subsection following [43], we first rewrite Problem 2 in a compact form involving a single variational inequality set in a closed convex set.

$$\mathcal{H} = \mathbb{X} \times \mathbb{M} \tag{2.15}$$

and

$$\mathcal{K} = \mathbb{N}_\Phi \times \mathbb{M}.$$

which is a closed convex set. The bilinear form is defined as follows: $\mathcal{A}_\rho : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$ through

$$\mathcal{A}_\rho((W,\mu);(V,\chi)) := a_\rho(W,V) + b(V,\mu) + b(W,\chi),$$

Subsequently, expressing Problem 2 concisely can be formulated as:

Problem 3 *Find (U,ψ) in $\mathcal{K}$ such that*

$$\mathcal{A}_\rho((U,\psi);V-U,\chi)) \geq \mathcal{L}(V-U), \quad \forall (V,\chi) \in \mathcal{K}. \tag{2.16}$$

The next lemma is useful for proving the existence result.

Lemma 2.2.1 ([47]) [1]

The bilinear form $b(\cdot,\cdot)$ fulfills the subsequent inf-sup condition:

$$\exists c_* > 0 \text{ such that } \sup_{W\in\mathbb{W}(\omega)} \frac{\boldsymbol{b}(W,\chi)}{\|W\|_{\mathbb{X}}} \geq c_* \|\chi\|_{\mathbb{M}}, \forall \chi \in \mathbb{M}, \tag{2.17}$$

Where $\mathbb{W}$ denotes the closed subspace of $\mathbb{X}$, which is encompassed within $\mathbb{N}_\Phi$, as defined by:

$$\mathbb{W} := \{(v,s) \in \mathbb{X}; \quad (v - \frac{\varepsilon}{2}s)\cdot e_3 = 0\}.$$

Befor making the proof, we first recall the general result of ([39] Lemma 4.1):

Let's X and M denotes two Hilbert spaces with the normes $\|\cdot\|_X$ and $\|\cdot\|_M$ respectively. Let $b(\cdot,\cdot) : X \times M \to \mathbb{R}$ continuous belinear form, and

$$\mathscr{V} = \{v \in X \quad | \quad b(v,\mu) = 0, \quad \forall \mu \in M\}$$

We introduce $B' : X \to M'$ the dual operator of B i.e:

$$\langle Bv, \mu\rangle = b(v,\mu) = \langle v, B'\mu\rangle, \quad \forall (v,\mu) \in X \times M \tag{2.18}$$

[1] S. Khenfar, S. Nicaise, and I. Merabet. On the finite element approximation of the obstacle problem of a naghdi shell.

Lemma 2.2.2 ([39]) *The three following properties are equivalnet:*

(i) There exists a constant $\beta > 0$ such that

$$\inf_{\mu \in M} \sup_{v \in X} \frac{b(v,\mu)}{\|\mu\|_M \|v\|_X} \geq \beta \tag{2.19}$$

(ii) The operator B' is an isomorphism from M onto the polar set $\mathscr{V}^\circ$ of $\mathscr{V}$ and

$$\|B'\mu\|_{X'} \geq \beta \|\mu\|_M \quad \forall \mu \in M \tag{2.20}$$

(iii) The operator B is an isomorphism from $\mathscr{V}^\perp$ onto M' and

$$\|Bv\|_{M'} \geq \beta \|v\|_X \quad \forall v \in \mathscr{V}^\perp \tag{2.21}$$

Proof.

1. Let us show that (i) $\Leftrightarrow$ (ii): by (2.18) the first statment is equivalent to

   $$\sup_{v \in X} \frac{\langle B'\mu, v\rangle}{\|\mu\|_M \|v\|_X} \geq \beta$$

   that is (2.19) is equinalent to (2.20) it remains to prove that B' is an isomorphism. Clearly, (2.20) implies that B' is a one to one operator from M onto its range $\mathcal{R}(B')$. Thus we are led to prove that $\mathcal{R}(B') = \mathscr{V}^\circ$. For this, we remark that $\mathcal{R}(B')$ is a closed subspace of X', since B' is an isomorphism. Therefore, we can apply the closed range theorem of Banach which says that $\mathcal{R}(B') = (Ker(B))^\circ = \mathscr{V}^\circ$. the first part of proof is complete.

2. (ii) $\Leftrightarrow$ (iii): We observe that $\mathscr{V}^\circ$ can be identified isometrically with $(\mathscr{V}^\perp)'$. Indeed, for $v \in X$, let $v^\perp$ denote the orthogonal projection of v on $V^\perp$. Then, with each $z \in (\mathscr{V}^\perp)'$ we associate the element $\tilde{z}$ of X' defined by $\langle \tilde{z}, v\rangle = \left\langle z, v^\perp \right\rangle \quad \forall v \in X$. Obviously $\tilde{z} \in V^\circ$ and it is easy to check that the correspondence $z \longrightarrow \tilde{z}$ maps isometrically $(\mathscr{V}^\perp)'$ onto $\mathscr{V}^\circ$. This permits to identify $(\mathscr{V}^\perp)'$ and $\mathscr{V}^\circ$. So, statement (ii) and (iii) are equivalent.

■

Proof. (of Lemma (2.2.1)) If we consider $\chi \ \in \mathbb{M}$, then $\tilde{W} = (\frac{2}{\varepsilon}\chi e_3, \chi e_3) \ \in \mathbb{W}$ and meets the condition

$$\|\tilde{W}\|_{\mathbb{X}} \sim \|\chi\|_{\mathbb{M}}.$$

As a result, one obtains

$$\sup_{W)\in\mathbb{W}} \frac{\boldsymbol{b}(W),\chi)}{\|W\|_{\mathbb{X}}} \geq \frac{\boldsymbol{b}(\tilde{W},\chi)}{\|\tilde{W}\|_{\mathbb{X}}} \sim \frac{|\chi|^2_{1,\omega}}{\|\chi\|_{\mathbb{M}}} \gtrsim \|\chi\|_{\mathbb{M}}.$$

for each $W = (w, \tilde{r})$ ■

Theorem 2.2.3 ([47]) *For any data For $(f, N, M) \in L^2(\omega)^3 \times L^2(\gamma_1)^3 \times L^2(\gamma_1)^3$, Problem 3 has a unique solution.*

Proof. The proof is done in four steps: **First step:**
It is noteworthy to mention that the bilinear form $\mathcal{A}(\cdot,\cdot)$ does not exhibit coercivity across the entire $\mathbb{X} \times \mathbb{M}$ space. As a result, it is not possible to explicitly deduce the existence and uniqueness of Problem 2 using Stampacchia's theorem. Undoubtedly, a perturbed bilinear form $\mathcal{A}^p$ is introduced, which is dependent on a minor positive parameter p.
The form is described as follows:

$$\mathcal{A}^p((W,\mu);(V,\chi)) := a_\rho(W,) + b(V,\mu) + b(W,\chi) + p(W,V)_{\mathbb{X}} + p(\mu,\chi)_{\mathbb{M}},$$

With $(\cdot,\cdot)_{\mathbb{X}}$ and $(\cdot,\cdot)_{\mathbb{M}}$ representing the inner products in $\mathbb{X}$ and $\mathbb{M}$ respectively, we then investigate the ensuing perturbed problem:

Problem 4 *Find (U_p, ψ_p) in $\mathcal{K}$ such that*

$$\mathcal{A}^p((U_p,\psi_p);(V-U_p,\chi)) \geq \mathcal{L}(V-U_p), \quad \forall (V,\chi) \in \mathcal{K}. \tag{2.22}$$

Given the coercivity of the bilinear form $\mathcal{A}^p$ on the space $\mathcal{H}$, Stampacchia's theorem guarantees a unique solution for Problem 4. It is imperative to establish the connection between the solution of Problem 4 and Problem 3.

Second step

In this step, we show that the sequence (U_p, ψ_p) is uniformly bounded in p.

The solution (U_p, ψ_p) of **Problem** 4 satisfies

$$\begin{cases} a_\rho(U_p, V - U_p) + b(V - U_p, \psi_p) + p(U_p, (V - U_p) \geq \mathcal{L}(V - U), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall V \in \mathbb{N}_\Phi, \\ \qquad\qquad b(U_p, \chi) - p(\psi_p, \chi)_{\mathbb{M}} = 0, \qquad \forall \chi \in \mathbb{M}(\omega), \end{cases} \tag{2.23}$$

Since $(\mathbf{0}_{\mathbb{X}}, \mathbf{0}_{\mathbb{M}}) \in \mathcal{K}$. Then, we can establish the following bound:

$$p\|U_p\|_{\mathbb{X}} + p\|\psi_p\|_{\mathbb{M}} \leq \|\mathcal{L}\|_{\mathbb{X}'}. \tag{2.24}$$

Third step:

From the last step,possibly passing to a sub-sequence, we can assume that (U_p, ψ_p) converges weakly to an element $(U^*, \psi^*) \in \mathcal{K} \times \Lambda$.

Our aim now is to show that the limit (U_*, ψ_*) satisfies problem (3). Now, we introduce the functional $G : \mathbb{M} \to \mathbb{R}$, defined as follows:

$$\forall \chi \in \mathbb{M}, \quad G(\chi) = p(\psi_p, \chi).$$

It's evident that $G \in \mathbb{M}'$, and according to (2.24), we obtain the subsequent bound:

$$\|G\|_{\mathbb{M}'} \lesssim \|\mathcal{L}\|_{\mathbb{X}'}. \tag{2.25}$$

By Lemma (2.2.2) there exists a unique solution $Q_p \in \mathbb{W}$ of

$$b(Q_p, \chi) = G(\chi), \quad \forall \chi \in \mathbb{M},$$

such that

$$\|Q_p\|_{\mathbb{X}} \lesssim \|G\|_{\mathbb{M}'}. \tag{2.26}$$

Therefore $U_p - Q_p \in \mathbb{N}_\Phi$ and by the second line in (2.23) we have

$$b(U_p - Q_p, \chi) = 0 \quad \forall \chi \in \mathbb{M}$$

whence,

$$U_p - Q_p \in \mathbb{N}_\Phi \cap \mathcal{V}.$$

Substituting $V = Q_p$ into the first line of (2.23), we obtain:

$$a_\rho(U_p, Q_p - U_p) + b(Q_p - U_p, \psi_p) + p(U_p, Q_p - U_p) \geq \mathcal{L}(Q_p - U_p),$$

and subtracting $\boldsymbol{a}_\rho(Q_p, Q_p - U_p)$ from both sides we obtain

$$a_\rho(U_p - Q_p, Q_p - U_p) + p(U_p, Q_p - U_p) \geq \mathcal{L}(Q_p - U_p) - \boldsymbol{a}_\rho(Q_p, Q_p - U_p).$$

Then using (2.24), (2.25), (2.26) and the Cauchy-Schwarz inequality we get

$$\begin{aligned} a_\rho(U_p - Q_p, U_p - Q_p) &\leq \mathcal{L}(U_p - Q_p) - \boldsymbol{a}_\rho(Q_p, U_p - Q_p) + p(U_p, Q_p - U_p) \\ &\lesssim \|\mathcal{L}\|_{\mathbb{X}'} \|U_p - Q_p\|_{\mathbb{X}}. \end{aligned}$$

Using the fact that $U_p - Q_p \in \mathcal{V}$ and Considering the coercivity of the bilinear form $a(\cdot, \cdot)$ over the space $\mathcal{V}$, we obtain:

$$\|U_p - Q_p\|_{\mathbb{X}} \lesssim \|\mathcal{L}\|_{\mathbb{X}'}. \tag{2.27}$$

Applying the triangle inequality, (2.27) and (2.26) collectively imply:

$$\|U_p\|_{\mathbb{X}} \lesssim \|\mathcal{L}\|_{\mathbb{X}'}. \tag{2.28}$$

It's important to note that the hidden constant in this final estimate is independent of p. Now, we aim to bound $\|\psi_p\|_{\mathbb{M}}$. Since $\forall (v, s) \in \mathbb{N}_\Phi$

$$\boldsymbol{a}_\rho(U_p, V - U_p) + \boldsymbol{b}(V - U_p, \psi_p) + p(U_p, V - U_p) \geq \mathcal{L}(V - U),$$

Subsequently (considering that $\mathbb{W}$ is a closed subspace included in $\mathbb{N}_\Phi$)

$$\forall (v, s) \in \mathbb{W}, \;\; \boldsymbol{a}_\rho(U_p, V) + \boldsymbol{b}(V, \psi_p) + p(U_p, V) = \mathcal{L}(V)$$

This can be expressed as

$$\boldsymbol{b}(V, \psi_p) = \mathcal{L}(V) - \boldsymbol{a}_\rho(U_p, V) - p(U_p, V), \quad \forall (v, s) \in \mathbb{W}.$$

Once more, the inf-sup inequality (2.17), along with the Cauchy-Schwarz inequality, and (2.28), indicate that

$$\|\psi_p\|_{\mathbb{M}} \lesssim \|\mathcal{L}\|_{\mathbb{X}'}. \tag{2.29}$$

Using the estimate (2.29) and the second line in (2.23), we infer that

$$\lim_{p\to 0} \boldsymbol{b}(U_p,\chi) = 0, \quad \forall \chi \in \mathbb{M} \quad \text{and} \quad \lim_{p\to 0} \boldsymbol{b}(U_p,\psi_p) = 0. \tag{2.30}$$

Collecting (2.28) and (2.29) we conclude that the sequence $((U_p,\psi_p))_p$ is uniformly bounded in the Hilbert space $\mathbb{X}\times\mathbb{M}$.
Consequently, there exists $(U^*,\psi^*)\in\mathcal{K}$ ((bearing in mind that $(U_p,\psi_p)\in\mathcal{K}$ for all p) such that

$$(U_p,\psi_p) \rightharpoonup (U^*,\psi^*) \in \mathbb{X}\times\mathbb{M} \text{ weakly as } p\to 0.$$

For any $V\in\mathbb{N}_\Phi$ we have,

$$\mathcal{L}(V-U_p) \leq \mathcal{A}^p((U_p,\psi_p);(V-U_p,\chi))$$

Since

$$\begin{aligned}
\mathcal{A}^p((U_p,\psi_p);(V-U_p,\chi)) &= \mathcal{A}_\rho((U_p,\psi_p);(V-U)_p,\chi)) \\
&+ p(U_p,V-U_p) + p(\psi_p,\chi) = \boldsymbol{a}_\rho(U_p,V-U_p) + b(U_p,\chi) \\
&+ b(V-U_p,\psi_p) + p(U_p,V-U_p) + p(\psi_p,\chi) = \boldsymbol{a}_\rho(U_p,(v,s)) + b(V,\psi_p) \\
&+ p(U_p,(v,s)) - \boldsymbol{a}(U_p,U_p) - p(U_p,U_p) + p(\psi_p,\chi) + b(U_p,\chi) - b(U_p,\psi_p) \\
&= \boldsymbol{a}_\rho(U_p,V) + b(V,\psi_p) - \boldsymbol{a}(U_p,U_p) + p(U_p,V) - p(U_p,U_p) \\
&+ b(U_p,\chi) - b(U_p,\psi_p) + p(\psi_p,\chi),
\end{aligned}$$

we then have,

$$\begin{aligned}
\mathcal{L}(V-U_p) \leq \boldsymbol{a}_\rho(U_p,(v,s)) + b(V,\psi_p) - \boldsymbol{a}(U_p,U_p) + p(U_p,V) \\
- p(U_p,U_p) + b(U_p,\chi) - b(U_p,\psi_p) + p(\psi_p,\chi).
\end{aligned}$$

By allowing p to 0 (utilizing (2.30)), we obtain

$$\boldsymbol{a}_\rho(U^*,V) + \boldsymbol{b}(V,\psi^*) - \lim_{p\to 0}\boldsymbol{a}_\rho(U_p,U_p) \geq \mathcal{L}(V-U_*).$$

Since,

$$\lim_{p\to 0}\boldsymbol{a}_\rho(U_p-U^*,U_p-U^*) \geq 0$$

then

$$\lim_{p\to 0} \boldsymbol{a}_\rho(U_p, U_p) \geq \boldsymbol{a}(U^*, U^*).$$

Combining the aforementioned inequalities with (2.30), we can express

$$\boldsymbol{a}_\rho(U^*, V - U^*) + \boldsymbol{b}(V, \psi^*) \geq \mathcal{L}(V - U^*), \quad \forall (v, s) \in \mathbb{N}_\Phi,$$
$$\boldsymbol{b}(U^*, \chi) = 0, \quad \forall \chi \in \mathbb{M}.$$

Therefore (U^*, ψ^*) is a solution of Problem 3.

Fourth step:

Now, let's demonstrate the uniqueness. Consider $(U_1, \psi_1) \in \mathcal{K}$ and $(U_2, \psi_2) \in \mathcal{K}$ as two solutions of Problem 3. Then

$$\boldsymbol{a}_\rho(U_1, U_2 - U_1) + \boldsymbol{b}(U_2 - U_1, \psi_1) \geq \mathcal{L}(U_2 - U_1),$$
$$\boldsymbol{b}(_1, \chi) = 0 \quad \forall \chi \in \mathbb{M}.$$

and

$$\boldsymbol{a}_\rho(U_2, U_1 - U_2) + \boldsymbol{b}(U_1 - U_2, \psi_2) \geq \mathcal{L}(U_1 - U_2),$$
$$\boldsymbol{b}(U_2, \chi) = 0 \quad \forall \chi \in \mathbb{M}.$$

Since U_1 and $U_2 \in \mathcal{V}$, Consequently, we have

$$\begin{cases} \boldsymbol{a}(U_1, U_2 - U_1) \geq \mathcal{L}(U_2 - U_1), \\ \boldsymbol{a}(U_2, U_1 - U_2) \geq \mathcal{L}(U_1 - U_2). \end{cases} \tag{2.31}$$

Hence

$$\boldsymbol{a}(U_1 - U_2, U_1 - U_2) \leq 0,$$

which implies that $U_1 = U_2$ using the coercivity of $\boldsymbol{a}(\cdot, \cdot)$ on $\mathcal{V}$.

From the inf-sup condition (2.17) we can get the uniqueness of ψ. Also, since,

$$\boldsymbol{a}(U, (V - U) + \boldsymbol{b}(V - U, \psi) \geq \mathcal{L}(V - U), \quad \forall (v, s) \in \mathbb{N}_\Phi,$$

therefore

$$\boldsymbol{a}(U,W)+\boldsymbol{b}(W,\psi)=\mathcal{L}(W),\quad \forall W\in\mathbb{W}. \tag{2.32}$$

Suppose $(U,\psi_1)\in\mathcal{K}$ and $(U,\psi_2)\in\mathcal{K}$ are two solutions of Problem 3, then the inf-sup condition (2.17) and (2.32) imply that

$$\|\psi_1-\psi_2\|_{\mathbb{M}}\le \sup_{W\in W(\omega)}\frac{\boldsymbol{b}(W,\psi_1-\psi_2)}{\|W\|_{\mathbb{X}}}=0,$$

which leads to $\psi_1=\psi_2$. ■

Now, let $c^{\#}$ and $c_{\#}$ denote the coercivity and the continuity constants of the form $\boldsymbol{a}_\rho(\cdot,\cdot)$ on $\mathbb{X}$, respectively.

2.2.2 First stability result

Based on the following stability condition, we can built our error analysis.

Lemma 2.2.4 ([47]) *For any $(W,\xi)\in\mathbb{X}\times\mathbb{M}$ there exists $V\in\mathbb{X}$ such that:*

$$\mathcal{A}_\rho((W,\xi);(V,-\eta))\ge C_1\left(\|W\|_{\mathbb{X}}+\|\xi\|_{\mathbb{M}}\right)^2, \tag{2.33}$$

$$\|V\|_{\mathbb{X}}+\|\xi\|_{\mathbb{M}}\le C_2\|W\|_{\mathbb{X}}+\|\xi\|_{\mathbb{M}}. \tag{2.34}$$

where C_1 and C_2 are two positive constants depending only on the constants $c^{\#},c_{\#}$ and c_*.

Proof. Let $(W,\eta)\in\mathbb{X}\times\mathbb{M}$ and let $Q\in\mathbb{X}$ be the unique solution of the following problem:

$$\begin{cases}\text{Find } Q\in\mathbb{X} \text{ such that}\\ \boldsymbol{a}_\rho(Q,Z)+(Q,Z)_{\mathbb{X}}=\boldsymbol{b}(Z,\eta),\ \forall Z\in\mathbb{X}.\end{cases} \tag{2.35}$$

By substituting $Z=Q$ in (2.35) we obtain

$$\|Q\|^2_{\mathbb{X}}\le\boldsymbol{b}(Q,\eta).$$

The Cauchy-Schwarz inequality implies

$$\|Q\|^2_{\mathbb{X}}\lesssim\|Q\|_{\mathbb{X}}\|\xi\|_{\mathbb{M}}.$$

This simplifies to

$$\|Q\|_{\mathbb{X}} \lesssim \|\xi\|_{\mathbb{M}}.$$

Moreover, due to the inf-sup condition, we have

$$\|\xi\|_{\mathbb{M}} \lesssim \sup_{Z\in\mathbb{X}} \frac{\boldsymbol{b}(Z,\eta)}{\|Z\|_{\mathbb{X}}}.$$

Using (2.35) and Cauchy-Schwarz's inequality, we get

$$\|\xi\|_{\mathbb{M}} \lesssim \|Q\|_{\mathbb{X}}.$$

Thus $\|Q\|_{\mathbb{X}} \sim \|\xi\|_{\mathbb{M}}$.
We now take $(v,s) = W + \delta Q$, where $0 < \delta < \dfrac{c^{\#}}{c_{\#}^2}$, and get

$$\begin{aligned}
\mathcal{A}_\rho((W,\xi);(v,-\xi)) &= \mathcal{A}_\rho((W,\xi);(W+\delta Q,-\xi)) \\
&= \boldsymbol{a}_\rho(W, W+\delta Q) + \boldsymbol{b}(W+\delta Q,\xi) + \boldsymbol{b}(W,-\xi) \\
&= \boldsymbol{a}_\rho(W,W) + \delta\boldsymbol{a}_\rho(W,Q) + \boldsymbol{b}(W,\xi) + \delta\boldsymbol{b}(Q,\xi) - \boldsymbol{b}(W,\xi) \\
&= \boldsymbol{a}_\rho(W,W) + \delta\boldsymbol{a}_\rho(W,Q) + \delta\boldsymbol{b}(Q,\xi) \\
&\geq c^{\#}\|W\|_{\mathbb{X}}^2 - \frac{\delta c_{\#}^2}{2}\|W\|_{\mathbb{X}}^2 - \frac{\delta}{2}\|\|_{\mathbb{X}}^2 + \delta\|Q\|_{\mathbb{X}}^2 \\
&\gtrsim \|W\|_{\mathbb{X}}^2 + \|\xi\|_{\mathbb{M}}^2.
\end{aligned}$$

■

Recalling the following bilinear form $\boldsymbol{c} : \mathbb{X} \times \mathbb{M}' \to \mathbb{R}$ defined by

$$\boldsymbol{c}(V,\mu) = \left\langle \left(v - \frac{\varepsilon}{2}s\right)\cdot e_3, \mu \right\rangle, \forall (V,\mu) \in \mathbb{X}\times\mathbb{M}'. \tag{2.36}$$

Now, we want to prove the connection between the reduced problem (2.12) and the full problem (2.7). Hence, we require the following lemma:

Lemma 2.2.5 *There exists a positive constant C_c such that*

$$\inf_{\chi\in\mathbb{M}'} \sup_{v\in H^1_{\gamma_0}(\omega,\mathbb{R}^3)} \frac{\boldsymbol{c}((v,0),\mu)}{\|\mu\|_{\mathbb{M}'}\|(v,0)\|_{\mathbb{X}}} \geq C_c. \tag{2.37}$$

Proof. Building upon the proof of [5, Lemma 4.4], we establish the continuity of the form $c(\cdot,\cdot)$ through its definition and the continuity of the mapping: $V \mapsto (v - \frac{\varepsilon}{2}s)\cdot e_3$ from $\mathbb{X}$ into $\mathbb{M}$.
Conversely, for any $\mu \in \mathbb{M}$ the Lax-Milgram lemma combined with the Poincaré-Friedrichs inequality implies that the problem:
Find σ in $\mathbb{M}$ such that

$$\forall \varsigma \in \mathbb{M} \int_\omega (grad\ \sigma)\cdot(grad\ \varsigma)(x)dx = \langle \varsigma, \mu\rangle \tag{2.38}$$

has a unique solution denoted by σ. Additionally the description of the norm of $\mathbb{M}'$ implies

$$\|\mu\|_{\mathbb{M}'} \leq |\sigma|_{\mathbb{M}}$$

Next, we note that the previous mapping is onto: By taking $V = (v, 0)$, with $v = (0, 0, \sigma)$, we obtain

$$\boldsymbol{c}(V,\mu) = |\sigma|^2_{\mathbb{M}}, \quad \|V\|_{\mathbb{X}} = \|\sigma\|_{\mathbb{M}}.$$

Furthermore, this V belongs to $\mathcal{V}$. Combining this with the Poincaré-Friedrichs inequality yields the desired condition. Clearly, the preceding result implies that

$$\inf_{\chi\in\mathbb{M}'}\sup_{V\in\mathbb{X}} \frac{\boldsymbol{c}(V,\mu)}{\|\mu\|_{\mathbb{M}'}\|V\|_{\mathbb{X}}} \geq C_c. \tag{2.39}$$

■

Proposition 2.2.1 *Let (U,ψ) be the solution of Problem 1. Then there exists a unique $\lambda \in \Lambda$ such that*

$$\boldsymbol{c}(V,\lambda) = \boldsymbol{a}_\rho(U,(V) + \boldsymbol{b}(V,\psi) - \mathcal{L}(V), \quad \forall V \in \mathbb{X}. \tag{2.40}$$

Moreover, the multiplier λ satisfies the following bound

$$\|\lambda\|_{\mathbb{M}'} \leq C\|\mathcal{L}\|_{\mathbb{X}'}.$$

Proof. See [5]. ■

2.2.3 A compact formulation of the full problem

Now we consider the "full" problem

Problem 5 *Find* $(U, \psi, \lambda) \in \mathbb{X} \times \mathbb{M} \times \Lambda$ *such that:*

$$\begin{cases} a_\rho(U, V) + b(V, \psi) - c(V, \lambda) = \mathcal{L}(V), \ \forall (v, s) \in \mathbb{X}, \\ b(U, \chi) = 0, \ \forall \chi \in \mathbb{M} \\ c(U, \mu - \lambda) \geq \langle \Phi, \mu - \lambda \rangle \, , \ \forall \mu \in \bar{\Lambda}, \end{cases} \tag{2.41}$$

Proposition 2.2.2 *The full problem 5 and the reduced problem 1 are equivalent, in the following sense: If* (U, ψ, λ) *is a solution of full problem, then* (U, ψ) *is a solution of the reduced. Conversely, if* (U, ψ) *is a solution of the reduced problem then there exists a unique* $\lambda \in \Lambda$ *such that* (U, ψ, λ) *is a solution of the full problem.*

Proof. The proof can be carried out similarly to the one of [5, Proposition 4.2]. ■

Let's introduce the following forms:

$$\mathscr{B}(U, \psi, \lambda; V, \chi, \mu) := a_\rho(U, V) + b(V, \psi) + b(U, \chi) - c(V, \lambda) + c(U, \mu),$$
$$\mathscr{L}(V, \chi, \mu) := \mathcal{L}(V) + \langle \Phi, \mu \rangle \, .$$

Now, Problem 5 can be expressed in the following concise form:

Problem 6 *Find* $(U, \psi, \lambda) \in \mathbb{X} \times \mathbb{M} \times \Lambda$ *such that:*

$$\mathscr{B}(U, \psi, \lambda; V, \chi, \mu - \lambda) \geq \mathscr{L}(V, \chi, \mu - \lambda), \quad \forall (V, \chi, \mu) \in \mathbb{X} \times \mathbb{M} \times \Lambda.$$

2.2.4 Second stability result

Theorem 2.2.6 *(Continuous stability) For any* $(V, \chi, \mu) \in \mathbb{X} \times \mathbb{M} \times \mathbb{M}'$ *there exists* $W \in \mathbb{X}$ *such that*

$$\mathscr{B}(V, \chi, \mu; W, -\chi, \mu) \gtrsim \left(\|W\|_{\mathbb{X}} + \|\chi\|_{\mathbb{M}} + \|\mu\|_{\mathbb{M}} \right)^2 , \tag{2.42}$$

$$\|W\|_{\mathbb{X}} \lesssim \|V\|_{\mathbb{X}}. \tag{2.43}$$

In order to prove Theorem 2.2.6, we require the following lemma.

Lemma 2.2.7 ([47]) *There exists a constant $\beta^{\#} > 0$ such that:*

$$\inf_{(\chi,\mu)\in\mathbb{M}\times\mathbb{M}'} \sup_{V\in\mathbb{X}} \frac{\boldsymbol{c}(V,\mu)-\boldsymbol{b}(V,\chi)}{\|(\chi,\mu)\|_{\mathbb{M}\times\mathbb{M}'}\|V\|_{\mathbb{X}}} \geq \beta^{\#}. \tag{2.44}$$

Proof. Suppose (χ,μ) be in $\mathbb{M}\times\mathbb{M}'$, then there exists σ in $H^1_{\gamma_0}(\omega)$ such that

$$\forall\varsigma\in H^1_{\gamma_0}(\omega), \int_\omega (\mathbf{grad}\ \sigma)\cdot(\mathbf{grad}\ \varsigma)\ dx = \langle\varsigma,\mu\rangle\,. \tag{2.45}$$

From this, we directly deduce that

$$\|\sigma\|_{1,\omega} \lesssim \|\mu\|_{\mathbb{M}'}.$$

Furthermore

$$\|\mu\|_{\mathbb{M}'} = \sup_{\varsigma\in H^1_{\gamma_0}(\omega)} \frac{\langle\varsigma,\mu\rangle}{\|\varsigma\|_{1,\omega}} = \sup_{\varsigma\in H^1_{\gamma_0}(\omega)} \frac{\int_\omega(\mathbf{grad}\ \sigma)\cdot(\mathbf{grad}\ \varsigma)\ dx}{\|\varsigma\|_{1,\omega}} \lesssim \|\sigma\|_{1,\omega}. \tag{2.46}$$

Hence $\|\sigma\|_{1,\omega}\sim\|\mu\|_{\mathbb{M}'}$. In (2.45), take $\tilde{V}=(\tilde{v},\tilde{s})$ with

$$\tilde{v} = (-\sigma+\frac{\varepsilon}{2}\chi a_3\cdot e_3)e_3 \quad \text{and} \quad \tilde{s}=\chi a_3,$$

then we have

$$\begin{aligned}
\boldsymbol{b}(\tilde{V},\chi) &= \int_\omega \partial_\alpha(\tilde{s}\cdot a_3)\partial_\alpha\chi\ dx = |\chi|^2_{1,\omega},\\
\boldsymbol{c}(\tilde{V},\mu) &= -\langle\sigma,\mu\rangle = -|\chi|^2_{1,\omega},\\
\|\tilde{V}\|_{\mathbb{X}} &\lesssim \|\sigma\|_{1,\omega}+\|\chi\|_{1,\omega} \lesssim \|\mu\|_{\mathbb{M}'}+\|\chi\|_{1,\omega}.
\end{aligned}$$

These properties directly imply

$$\begin{aligned}
\sup_{V\in\mathbb{X}}\frac{\boldsymbol{b}(V,\chi)-\boldsymbol{c}(V,\mu)}{\|V\|_{\mathbb{X}}} \geq \frac{\boldsymbol{b}(\tilde{V},\chi)-\boldsymbol{c}(\tilde{V},\mu)}{\|\tilde{V}\|_{\mathbb{X}}} &= \frac{\|\chi\|^2_{1,\omega}+\|\sigma\|^2_{1,\omega}}{\|\tilde{V}\|_{\mathbb{X}}}\\
&\gtrsim (\|\chi\|^2_{1,\omega}+\|\mu\|^2_{\mathbb{M}'})^{1/2}.
\end{aligned}$$

■

Proof. (of Theorem 2.2.6) Let $(V, \chi, \mu) \in \mathbb{X} \times \mathbb{M} \times \mathbb{M}'$, and consider the following variational problem:

$$\begin{cases} \text{Find } Q \text{ in } \mathbb{X} \text{ such that} \\ \boldsymbol{a}_\rho(Q, Z) + (Q, Z)_\mathbb{X} = \boldsymbol{b}(Z, \chi) - \boldsymbol{c}(\mu, Z), \quad \forall Z \in \mathbb{X}. \end{cases} \tag{2.47}$$

Given that the bilinear form $\boldsymbol{a}_\rho(\cdot,\cdot)+(\cdot,\cdot)_\mathbb{X}$ is $\mathbb{X}$-elliptic, problem (2.47) possesses a unique solution $Q \in \mathbb{X}$.
Moreover, because $\boldsymbol{c}(\cdot,\cdot) - \boldsymbol{b}(\cdot,\cdot)$ satisfies the inf-sup condition, the Cauchy-Schwarz inequality yields

$$\|\chi\|_\mathbb{M} + \|\mu\|_{\mathbb{M}'} \lesssim \sup_{Z\in\mathbb{X}} \frac{\boldsymbol{c}(\mu, Z) - \boldsymbol{b}(\chi, Z)}{\|Z\|_\mathbb{X}} = \sup_{Z\in\mathbb{X}} \frac{\boldsymbol{a}(Q, Z) + (Q, Z)}{\|Z\|_\mathbb{X}} \lesssim \|Q\|_\mathbb{X}.$$

Consider $W = V + \delta Q$ where $(v, s) = V$ and δ is a positive constant to be determined later, then we obtain:

$$\begin{aligned} \mathscr{B}(V, \chi, \mu; W, -\chi, \mu) &= \boldsymbol{a}_\rho(V, V + \delta Q) + \boldsymbol{b}(V + \delta Q, \chi) + \boldsymbol{b}(V, -\chi) - \boldsymbol{c}(V + \delta Q, \mu) + \boldsymbol{c}(\mu, V) \\ &= \boldsymbol{a}_\rho(V, V) + \delta \boldsymbol{a}_\rho(V, Q) + \delta \boldsymbol{b}(Q, \chi) - \delta \boldsymbol{c}(\mu, Q) \\ &\geq \boldsymbol{a}_\rho(V, V) - \delta c^\# \|V\|_\mathbb{X} \|Q\|_\mathbb{X} + \delta \|Q\|^2_\mathbb{X} \\ &\geq (c_\# - \frac{\delta (c^\#)^2}{2}) \|V\|^2_\mathbb{X} + \frac{\delta}{2} \|Q\|^2_\mathbb{X} \end{aligned}$$

Therefore, it suffices to choose $0 < \delta < \dfrac{c_\#^2}{2(c^\#)^2}$ to obtain

$$\mathscr{B}(V, \chi, \mu; W, -\chi, \mu) \gtrsim (\|V\|^2_\mathbb{X} + \|\chi\|^2_1 + \|\mu\|^2_{-1}).$$

■

Chapter 3

Finit element discretisation

The finite element method (FEM) stands out as a powerful numerical technique with various advantages for solving engineering and physical problems. One of its primary strengths lies in its versatility and broad applicability across a range of scientific domains. FEM's ability to handle complex geometries and solve problems with intricate boundary conditions is a key advantage, allowing for realistic modeling of physical structures and systems. It excels in solving partial differential equations that describe phenomena like heat transfer, fluid dynamics, and structural mechanics. Additionally, FEM is effective in addressing nonlinear problems, including large deformations, material nonlinearities, and contact problems, providing a more accurate representation of real-world scenarios. Properly implemented, the method ensures stable and convergent numerical solutions, particularly as the mesh is refined.

In this chapter, we are interested in finding an approximative solution to the contact problem using finit element method. We shall prove existence and uniqueness of a solution to this finitedimensional problem. Furthermore, we shall study the behavior of the approximative solutions, as the parameter of discretization tends to zero.

3.1 Statement of discret problem

We assume that ω is a polygonal domain such that $\bar{\omega}$ can be precisely triangulated as $\bar{\omega} = \cup \mathcal{T}_i$ where $\mathcal{T}_i$ is a triangle such that $\mathcal{T}_i \cap \mathcal{T}_j = \varnothing$ or a vertex for $i \neq j$. We denote by s_i the vertices of the triangles. The mesh-size is characterized by

$$h = \max h_{\mathcal{T}} = diam\ \mathcal{T},$$

where $h_{\mathcal{T}}$ is the triangle size defined by: $h_{\mathcal{T}} = \max_{s_i, s_j \in \mathcal{T}} |s_i - s_j|$ Then $\mathbb{T}_h$ is noted as the affine triangulation formed by the triangles $\mathcal{T}_i$ which cover the domain ω. Later on, we will consider a regular family of triangulations $\mathbb{T}_h$, i.e., there existe σ such that

$$\sigma_{\mathcal{T}} = \frac{h_{\mathcal{T}}}{\varrho_{\mathcal{T}}} \leq \sigma\ \forall \mathcal{T} \in \mathbb{T}_h$$

where $\varrho_T = \sup\{\text{the diameter of B, where } B \text{ is a ball contained in} \mathcal{T}\}$

For a positive integer k, $\mathbb{P}_k(\mathcal{T})$ represents the set of functions on $\mathcal{T}$ that are restrictions of polynomials of degree less than or equal to k.
For $\mathcal{T} \in \mathbb{T}_h$, b_T represents the bubble function defined as $b_{\mathcal{T}} = \frac{\lambda_1 \lambda_2 \lambda_3}{27}$, where λ_i,be the barycentric coordinates of $\mathcal{T}$. It's worth noting that $b_{\mathcal{T}} \in H_0^1(\mathcal{T}) \cap \mathbb{P}_3(\mathcal{T})$ and has a maximum value of one. As [68], we further establish

$$B_3(\mathcal{T}) = \{v \in H_0^1(\mathcal{T}); v = b_{\mathcal{T}} w, \quad w \in \mathbb{P}_0(\mathcal{T})\}. \tag{3.1}$$

Let us to define the finite-dimensional spaces as follows:

$$\begin{aligned}
\mathbb{M}_h :=& \{\chi_h \in H^1_{\gamma_0}(\omega) \quad | \quad \chi_{h|\mathcal{T}} \in \mathbb{P}_1(\mathcal{T}) \oplus B_3(T),\ \forall \mathcal{T} \in \mathbb{T}_h\}.\\
\mathbb{Q}_h :=& \{\mu_h \in L^2(\omega) \quad | \quad \mu_{h|\mathcal{T}} \in \mathbb{P}_0(\mathcal{T}),\ \forall \mathcal{T} \in \mathbb{T}_h\}.\\
\mathbb{X}_h :=& (\mathbb{M}_h)^3 \times (\mathbb{M}_h)^3.\\
\mathbb{W}_h :=& \left\{ (v_h, s_h) \in \mathbb{X}_h; \left(v_h - \frac{\varepsilon}{2} s_h \right) \cdot e_3 = 0 \right\}
\end{aligned}$$

Next, we introduce the discrete convex cone.

$$\mathbb{N}_h = \left\{ (v_h, s_h) \in \mathbb{X}_h; \left(v_h - \frac{\varepsilon}{2} s_h \right) \cdot e_3 \geq \Phi_h \right\}, \tag{3.2}$$

where $\Phi_h := \mathcal{I}_h \Phi$.

$\mathcal{I}_h$ represents the standard Lagrange interpolant operator, defined as $(\mathcal{I}_h\Phi)_{\mathcal{T}} \in \mathbb{P}_1(\mathcal{T})$, with $(\mathcal{I}_h\Phi)\mathcal{T}(x) = \Phi(x)$ for all vertices x of $\mathcal{T}$. It's evident that $\mathbb{X}_h \subset \mathbb{X}$, $\mathbb{M}_h \subset \mathbb{M}$, and $\mathbb{W}_h \subset \mathbb{N}_h$, but $\mathbb{N}_h$ is not automatically include in $\mathbb{N}_\Phi$.

Initially, we examine the discrete version of **Problem** 2, which is given by:

Problem 7 *Find* $(U_h, \psi_h) \in \mathbb{N}_h \times \mathbb{M}_h$ *such that:*

$$\begin{cases} \forall V_h \in \mathbb{N}_h, \quad \boldsymbol{a}_\rho(U_h, V_h - U_h) + \boldsymbol{b}(V_h - U_h, \psi_h) \geq \mathcal{L}(V_h - U_h), \\ \qquad \forall \chi_h \in \mathbb{M}_h, \quad \boldsymbol{b}(U_h, \chi_h) = 0. \end{cases} \tag{3.3}$$

As in the continuous case, we can express **Problem** 7 in the following compact form:

Problem 8 *Find* $(U_h, \psi_h) \in \mathbb{N}_h \times \mathbb{M}_h$ *such that:*

$$\mathcal{A}_\rho((U_h, \psi_h); (V_h - U_h, \chi_h) \geq \mathcal{L}(V_h - U_h), \quad \forall (V_h, \chi_h) \in \mathbb{N}_h \times \mathbb{M}_h. \tag{3.4}$$

3.2 The well posedness of approximate solution

Our objective here is to establish the existence and uniqueness of the discrete solution. Hence, we present the following lemma:

Lemma 3.2.1 ([47]) *If the mesh size h is sufficiently small, then there exists a positive constant C_b such that*

$$\inf_{\chi_h \in \mathbb{M}_h} \sup_{V_h \in \mathbb{W}_h} \frac{\boldsymbol{b}(V_h, \chi_h)}{\|\chi_h\|_{\mathbb{M}} \|V_h\|_{\mathbb{X}}} \geq C_b. \tag{3.5}$$

Proof. Recalling that $\mathcal{I}_h$ represents the standard Lagrange interpolation operator, for any $\chi_h \in \mathbb{M}_h$, we take

$$V_h = (\frac{2}{\varepsilon} \mathcal{I}_h(\chi_h a_3), \mathcal{I}_h(\chi_h a_3)),$$

then clearly, $V_h \in \mathbb{W}_h$ and

$$\boldsymbol{b}(V_h, \chi_h) \gtrsim \|\chi_h\|_{\mathbb{M}}$$

since the inverse estimate $\|\nabla v_h\|_{\infty,\omega} \lesssim h^{-1}\|v_h\|_{\infty,\omega}$, this holds true for all $v_h \in \mathbb{M}_h$, see [17, Lemma 3.3] and [12, Lemma 5.6]. ■

Theorem 3.2.2 *If the mesh size h is sufficiently small, then Problem 7 admits a unique solution.*

Proof. Since $\mathbb{W}_h$ is a closed subspace of $\mathbb{N}_h$ (see for instance [65]), the proof can be done by using the same perturbation technique as for the continuous problem. ■

Now, we introduce the closed convex cone

$$\Lambda_h = \{\mu_h \in \mathbb{Q}_h; \mu_h \geq 0\}, \tag{3.6}$$

This is evidently a subspace of Λ. We then proceed to address the "full" discrete problem (compare with Problem 5).

Problem 9 *Find (U_h, ψ_h, λ_h) in $\mathbb{X}_h(\omega) \times \mathbb{M}_h(\omega) \times \Lambda_h$ such that:*

$$\begin{cases} \forall V_h \in \mathbb{X}_h, \ \boldsymbol{a}_\rho(U_h, V_h) + \boldsymbol{b}(V_h, \psi_h) - \boldsymbol{c}(V_h, \lambda_h) = \mathcal{L}(V_h), \\ \forall \chi_h \in \mathbb{M}_h, \qquad \boldsymbol{b}(U_h, \chi_h) = 0, \\ \forall \mu_h \in \bar{\Lambda}_h, \qquad \boldsymbol{c}(U_h, \mu_h - \lambda_h) \geq \langle \mathcal{I}_h \Phi, \mu_h - \lambda_h \rangle . \end{cases} \tag{3.7}$$

Let us also introduce the following h-dependent norm:

$$\|\chi_h\|_h^2 = \sum_{\mathcal{T} \in \mathbb{T}_h} h_{\mathcal{T}}^2 \|\chi_h\|_{\mathcal{T}}^2, \forall \chi_h \in \mathbb{Q}_h. \tag{3.8}$$

Lemma 3.2.3 *There exist two positive constants C_1 and C_2 which are independent of h such that*

$$\forall \chi_h \in \mathbb{Q}_h, \sup_{V_h \in \mathbb{X}_h \cap \ker b} \frac{\boldsymbol{c}(V_h, \chi_h)}{\|V_h\|_{\mathbb{X}_h}} \geq C_1 \|\chi_h\|_{\mathbb{M}'} - C_2 \|\chi_h\|_h. \tag{3.9}$$

Proof. Suppose $\chi_h \in \mathbb{Q}_h \subset \mathbb{M}'$. Then, according to the inf-sup condition (2.37), there exists $v \in H^1_{\gamma_0}(\omega, \mathbb{R}^3)$ and $C_1 > 0$ such that

$$\boldsymbol{c}((v, 0), \chi_h) \geq C_1 \|(v, 0)\|_{\mathbb{X}} \|\chi_h\|_{\mathbb{M}'}.$$

Let V_h be the Clément interpolant of $(v,0)$ (hence V_h is in the form $(v_h,0)$ and belongs to $\ker b$), then we have

$$\begin{aligned}
\boldsymbol{c}(V_h,\chi_h) &= \boldsymbol{c}(V_h-V,\chi_h)+\boldsymbol{c}(V,\chi_h)\\
&= \sum_{\mathcal{T}\in\mathbb{T}_h}((v_h-v)\cdot e_3,\chi_h)_{\mathcal{T}}+\boldsymbol{c}(V,\chi_h)\\
&\geq \sum_{\mathcal{T}\in\mathbb{T}_h}((v_h-v)\cdot e_3,\chi_h)_{\mathcal{T}}+C_1\|V\|_{\mathbb{X}}\|\chi_h\|_{\mathbb{M}'}\\
&\geq -\sum_{\mathcal{T}\in\mathbb{T}_h}\|\,(v_h-v)\cdot e_3\|_{\mathcal{T})}\|\chi_h\|_{\mathcal{T}}+C_1\|V\|_{\mathbb{X}}\|\chi_h\|_{\mathbb{M}'}\\
&= -\sum_{\mathcal{T}\in\mathbb{T}_h}h_{\mathcal{T}}^{-1}\|\,(v_h-v)\cdot e_3\|_{\mathcal{T}}h_{\mathcal{T}}\|\chi_h\|_{\mathcal{T}}+C_1\|V\|_{\mathbb{X}}\|\chi_h\|_{\mathbb{M}'}.
\end{aligned} \tag{3.10}$$

With the properties of the Cl'ement interpolant and considering that $\mathbb{T}_h$ is quasi-uniform, we obtain:

$$\mathcal{T}\left(\sum_{\mathcal{T}\in\mathbb{T}_h}h_{\mathcal{T}}^{-2}\|\,(v_h-v)\cdot e_3\|_{\mathcal{T}}^2\right)^{1/2}\leq C_2\|V\|_{\mathbb{X}}\quad\text{and}\quad\|V_h\|_{\mathbb{X}}\lesssim\|V\|_{\mathbb{X}},$$

consequently

$$\sum_{\mathcal{T}\in\mathbb{T}_h}h_{\mathcal{T}}^{-1}\|\,(v_h-v)\cdot e_3\|_{\mathcal{T}}h_{\mathcal{T}}\|\chi_h\|_{\mathcal{T}}\leq C_2\|\chi_h\|_h\|V\|_{\mathbb{X}},$$

This, together with (3.10), indicates that

$$\boldsymbol{c}(V_h,\chi_h)\geq C_1\|V\|_{\mathbb{X}}\|\chi_h\|_{\mathbb{M}'}-C_2\|V\|_{\mathbb{X}}\|\chi_h\|_h. \tag{3.11}$$

Now if $C_1\|\chi_h\|_{\mathbb{M}'}-C_2\|\chi_h\|_h\geq 0$, then (3.11) implies that

$$C_1\|\chi_h\|_{\mathbb{M}'}-C_2\|\chi_h\|_h\leq\frac{\boldsymbol{c}(V_h,\chi_h)}{\|V\|_{\mathbb{X}}}\lesssim\frac{\boldsymbol{c}(V_h,\chi_h)}{\|V_h\|_{\mathbb{X}}},$$

since $c(V_h,\chi_h)\geq 0$ because the left-hand side of this estimate is positive.
Alternatively, if $C_1\|\chi_h\|_{\mathbb{M}'}-C_2\|\chi_h\|_h\leq 0$, then clearly

$$\sup_{V_h\in\mathbb{X}_h}\frac{\boldsymbol{c}(V_h,\chi_h)}{\|V_h\|_{\mathbb{X}}}\gtrsim C_1\|\chi_h\|_{\mathbb{M}'}-C_2\|\chi_h\|_h,$$

Considering $W_h=(0,0,\chi_h b_{\mathcal{T}},0,0,0)$ for some interior triangle $\mathcal{T}$, we find: $\boldsymbol{c}(W_h,\chi_h)\geq 0$.

■

Lemma 3.2.4 *We have the following inf-sup condition for the mesh-dependent norm (3.8): namely there exists a positive constant C_3 (independent of h) such that*

$$\forall \chi_h \in \mathbb{Q}_h, \qquad \sup_{V_h \in \mathbb{X}_h \cap \ker b} \frac{\boldsymbol{c}(V_h, \chi_h)}{\|V_h\|_{\mathbb{X}}} \geq C_3 \|\chi_h\|_h. \tag{3.12}$$

Proof. Let $\chi_h \in \mathbb{Q}_h$, we define $V_h \in \mathbb{X}_h$ as follow:

$$V_h = (0, 0, \sigma_h, 0, 0, 0), \qquad \text{with } (\sigma_h)_{|\mathcal{T}} = h_{\mathcal{T}}^2 \chi_h b_{\mathcal{T}}, \forall \mathcal{T} \in \mathbb{T}_h.$$

Then clearly $V_h \in \mathbb{X}_h \cap \ker b$, and we have

$$\boldsymbol{c}(V_h, \chi_h) = \sum_{\mathcal{T} \in \mathbb{T}_h} \left(\left(v_h - \frac{\varepsilon}{2} s_h \right) \cdot e_3, \chi_h \right) = \sum_{\mathcal{T} \in \mathbb{T}_h} \int_{\mathcal{T}} h_{\mathcal{T}}^2 \chi_h^2 b_{\mathcal{T}} \gtrsim \|\chi_h\|_h^2$$

and

$$\|V_h\|_{\mathbb{X}}^2 = \|\sigma_h\|_{1,\mathcal{T}}^2 \lesssim \sum_{\mathcal{T} \in \mathbb{T}_h} h_{\mathcal{T}}^{-2} \|\sigma_h\|_{0,\mathcal{T}}^2 \leq \sum_{\mathcal{T} \in \mathbb{T}_h} h_{\mathcal{T}}^2 \|\chi_h\|_{0,\mathcal{T}}^2 = \|\chi_h\|_h^2$$

From this, we conclude that

$$\boldsymbol{c}(V_h, \chi_h) \gtrsim \|V_h\|_{\mathbb{X}} \|\chi_h\|_h.$$

■

Lemma 3.2.5 *It holds*

$$\sup_{V_h \in \mathbb{X} \cap \ker b} \frac{\boldsymbol{c}(V_h, \chi_h)}{\|V_h\|_{\mathbb{X}}} \gtrsim \|\chi_h\|_{\mathbb{M}'}. \tag{3.13}$$

Proof. For $\delta \in (0, 1)$, based on (3.9) and (3.12), we find

$$\begin{aligned}
\sup_{V_h \in \mathbb{X} \cap \ker b} \frac{\boldsymbol{c}(V_h, \chi_h)}{\|V_h\|_{\mathbb{X}}} &= \delta \sup_{V_h \in \mathbb{X} \cap \ker b} \frac{\boldsymbol{c}(V_h, \chi_h)}{\|V_h\|_{\mathbb{X}}} + (1 - \delta) \sup_{V_h \in \mathbb{X} \cap \ker b} \frac{\boldsymbol{c}(V_h, \chi_h)}{\|V_h\|_{\mathbb{X}}} \\
&\geq \delta \left(C_1 \|\chi_h\|_{\mathbb{M}'} - C_2 \|\chi_h\|_h \right) + (1 - \delta) C_3 \|\chi_h\|_h \\
&\geq \delta C_1 \|\chi_h\|_{\mathbb{M}'} + \left((1 - \delta) C_3 - \delta C_2 \right) \|\chi_h\|_h.
\end{aligned}$$

By selecting δ such that $(1 - \delta) C_3 - \delta C_2 = 0$ or equivalently $\delta = \frac{C_3}{C_2 + C_3}$ which indeed falls within the range $(0, 1)$, we derive (3.13). ■

Proposition 3.2.1 *The full problem* (3.7) *and the reduced problem* (3.3) *are equivalent in the following sense: if* (U_h, ψ_h, λ_h) *is a solution of the full problem, then* (U_h, ψ_h) *is a solution of the reduced problem. Conversely, if* (U_h, ψ_h) *is a solution of the reduced problem, then there exists* $\lambda_h \in \Lambda_h$ *such that* (U_h, ψ_h, λ_h) *is a solution of the full problem.*

Proof. Let's assume that (U_h, ψ_h, λ_h) satisfies (3.7), then from the first equation of (3.7) we have

$$\boldsymbol{a}_\rho(U_h, V_h) + \boldsymbol{b}(V_h, \psi_h) = \mathcal{L}(V_h) + \boldsymbol{c}(V_h, \lambda_h), \quad \forall V_h = (v, s)_h \in \mathbb{X}_h. \tag{3.14}$$

Taking $\mu_h = 0$ and $\mu_h = 2\lambda_h$ in the third line of (3.7) gives

$$\boldsymbol{c}(U_h, \lambda_h) = \langle \mathcal{I}_h \Phi, \lambda_h \rangle, \tag{3.15}$$

and since $\mathcal{I}_h \Phi \leq 0$ in ω, for any $V_h \in \mathbb{N}_h$ we have

$$\boldsymbol{c}(V_h, \lambda_h) \geq \langle \mathcal{I}_h \Phi, \lambda_h \rangle. \tag{3.16}$$

So, combining (3.15) and (3.16) we get

$$\boldsymbol{c}(V_h - U_h, \lambda_h) \geq 0, \quad \forall V_h \in \mathbb{N}_h.$$

Therefore, (3.10) and the second line in (3.7) can be expressed as:

$$\begin{aligned} a_\rho(U)_h, V_h - U_h) + \boldsymbol{b}(V_h - U_h, \psi_h) \geq \mathcal{L}(V_h - U_h), \qquad & \forall V_h \in \mathbb{N}_h, \\ \boldsymbol{b}(U_h, \chi_h) = 0. \qquad & \forall \chi_h \in \mathbb{M}_h. \end{aligned}$$

Conversely, if (U_h, ψ_h) is a solution of Problem 7, we aim to demonstrate the existence of $\lambda_h \in \Lambda_h$ such that (U_h, ψ_h, λ_h) is a solution of Problem 9.
Let's start by recalling that the first line of (3.3) with $V_h = 0$ and $V_h = 2U_h$ results in:

$$\boldsymbol{a}_\rho(U_h, U_h) + \boldsymbol{b}(U_h, \psi_h) = \mathcal{L}(U_h). \tag{3.17}$$

Given that the bilinear form $c(\cdot, \cdot)$ satisfies the inf-sup condition (refer to Lemma 3.2.5), there exists a unique $\lambda_h \in \mathbb{Q}_h$ such that

$$\boldsymbol{c}(V_h, \lambda_h) = \boldsymbol{a}_\rho\left(U_h, V_h\right) + \boldsymbol{b}\left(V_h, \psi_h\right) - \mathcal{L}\left(V_h\right), \forall V_h \in \mathbb{X}_h. \tag{3.18}$$

Now we need to prove that $\lambda_h \in \Lambda_h$.

First let $\mathcal{T} \in \mathbb{T}_h$ be an arbitrary triangle and let $V_h \in \mathbb{X}_h$ be chosen such that

$$V_h = (u_h + (0, 0, b_{\mathcal{T}}), r_h),$$

Then since $b_{\mathcal{T}} \in H_0^1(\mathcal{T}), b_{\mathcal{T}} \geq 0$, it is clear that $V_h \in \mathbb{N}_h$ and

$$\boldsymbol{c}(V_h - U_h, \lambda_h) = \int_{\mathcal{T}} \lambda_h b_{\mathcal{T}} = \boldsymbol{a}_\rho (U_h, V_h - U_h) + \boldsymbol{b} (V_h - U_h, \psi_h) - \boldsymbol{\mathcal{L}} (V_h - U_h) \geq 0$$

Hence $\lambda_h \geq 0$ in ω, which means that $\lambda_h \in \Lambda_h$.

We now need to establish the final property of Problem 9. Firstly, since U_h belongs to $\mathbb{N}_h$, we have:

$$\left(u_h - \frac{\varepsilon}{2} r_h\right) \cdot e_3 \geq \mathcal{I}_h \Phi, \tag{3.19}$$

which directly implies that

$$\boldsymbol{c}(U_h, \mu_h) \geq \langle \mathcal{I}_h \Phi, \mu_h \rangle \quad \forall \mu_h \in \Lambda_h. \tag{3.20}$$

The last inequality of (3.7) then holds if we show that

$$\left\langle \left(u_h - \frac{\varepsilon}{2} r_h\right) \cdot e_3 - \mathcal{I}_h \Phi, \lambda_h \right\rangle = 0, \tag{3.21}$$

Since $U_h \in \mathbb{N}_h$ and $\lambda_h \in \Lambda_h$ then we have

$$\left\langle \left(u_h - \frac{\varepsilon}{2} r_h\right) \cdot e_3 - \mathcal{I}_h \Phi, \lambda_h \right\rangle \geq 0. \tag{3.22}$$

On the other hand, by (3.17) and (3.18), we have

$$\boldsymbol{c}(U_h, \lambda_h) = 0,$$

while the fact that $\lambda_h \in \Lambda_h$ and that $\mathcal{I}_h \Phi \leq 0$ lead to

$$\langle \mathcal{I}_h \Phi, \lambda_h \rangle \geq 0.$$

This directly implies

$$\left\langle \left(u_h - \frac{\varepsilon}{2} r_h\right) \cdot e_3 - \mathcal{I}_h \Phi, \lambda_h \right\rangle \leq 0. \tag{3.23}$$

Hence, (3.22) and (3.23) imply that (3.21) holds.

■

3.3 A Priori Error Analysis

3.3.1 A priori error estimation of the reduced problem

The objective of this subsection is to conduct the a priori error analysis of Problem 2. We assume that the mesh size h is sufficiently small such that Lemma 3.2.1 holds.

Lemma 3.3.1 *For any $(W_h, \xi_h) \in \mathbb{X}_h \times \mathbb{M}_h$ there exists $V_h \in \mathbb{X}_h$ such that $V_h - W_h \in \mathbb{W}_h$ and satisfying*

$$\mathcal{A}_\rho((W_h, \xi_h); (V_h, -\xi_h)) \gtrsim (\|W_h\|_{\mathbb{X}} + \|\xi_h\|_{\mathbb{M}})^2 \tag{3.24}$$

$$\|V_h\|_{\mathbb{X}} + \|\chi_h\|_{\mathbb{M}} \lesssim \|W_h\|_{\mathbb{X}} + \|\xi_h\|_{\mathbb{M}}. \tag{3.25}$$

Proof. The proof can proceed in the same manner as in Lemma 2.2.4 for the continuous problem. Consider $(W_h, \xi_h) \in \mathbb{X}_h \times \mathbb{M}_h$ and let $Q_h \in \mathbb{W}_h$ be the unique solution of

$$\begin{cases} \text{Find } Q_h \in \mathbb{W}_h \text{ such that} \\ \boldsymbol{a}_\rho(Q_h, Z_h) + (Q_h, Z_h)_{\mathbb{X}} = \boldsymbol{b}(Z_h, \eta_h), \ \forall Z_h \in \mathbb{W}_h. \end{cases} \tag{3.26}$$

By taking $Z_h = Q_h$ in (3.26) we get

$$\|Q_h\|_{\mathbb{X}}^2 \leq \boldsymbol{b}(Q_h, \eta_h)$$

This, by Cauchy-Schwarz's inequality, results in:

$$\|Q_h\|_{\mathbb{X}} \lesssim \|\eta_h\|_{\mathbb{M}}.$$

Additionally, Lemma 3.2.1 implies:

$$C_b\|\xi_h\|_{\mathbb{M}} \leq \sup_{V_h \in \mathbb{W}_h} \frac{\boldsymbol{b}(V_h, \chi_h)}{\|V_h\|_{\mathbb{X}}},$$

and by (3.26) and Cauchy-Schwarz's inequality, we obtain

$$\|\xi_h\|_{\mathbb{M}} \lesssim \|Q_h\|_{\mathbb{X}}.$$

This implies that

$$\|Q_h\|_{\mathbb{X}} \sim \|\xi_h\|_{\mathbb{M}}.$$

Let's now consider $V_h = W_h + \delta Q_h$ where $0 < \delta < \dfrac{c^{\#}}{c_{\#}^2}$, and get

$$\begin{aligned}
\mathcal{A}_\rho((W_h, \xi_h); (V_h, -\xi_h)) &= \mathcal{A}_\rho((W_h, \xi_h); (W_h + \delta Q_h, -\xi_h)) \\
&= \boldsymbol{a}_\rho(W_h, W_h + \delta Q_h) + \boldsymbol{b}(W_h + \delta Q_h, \xi_h) + \boldsymbol{b}(W_h, -\xi_h) \\
&= \boldsymbol{a}_\rho(W_h, W_h) + \delta \boldsymbol{a}_\rho(W_h, Q_h) + \boldsymbol{b}(W_h, \xi_h) + \delta \boldsymbol{b}(Q_h, \xi_h) - \boldsymbol{b}(W_h, \xi_h) \\
&= \boldsymbol{a}_\rho(W_h, W_h) + \delta \boldsymbol{a}_\rho(W_h, Q_h) + \delta \boldsymbol{b}(Q_h, \xi_h) \\
&\geq c^{\#}\|W_h\|_{\mathbb{X}}^2 - \frac{\delta c_{\#}^2}{2}\|W_h\|_{\mathbb{X}}^2 - \frac{\delta}{2}\|Q_h\|_{\mathbb{X}}^2 + \delta\|Q_h\|_{\mathbb{X}}^2 \\
&\gtrsim \|W_h\|_{\mathbb{X}}^2 + \|\xi_h\|_{\mathbb{M}}^2.
\end{aligned}$$

Let us finally notice that $V_h - W_h = \delta Q_h$, which indeed belongs to $\mathbb{W}_h$. ■

Theorem 3.3.2 ([47]) *Suppose (U, ψ) and (U_h, ψ_h) are the solutions of Problem 2 and Problem 7, respectively. Then,*

$$\|U - U_h\|_{\mathbb{X}} + \|\psi - \psi_h\|_{\mathbb{M}} \lesssim \inf_{V_h \in \mathbb{N}_h} \left(\|U - V_h\|_{\mathbb{X}} + \sqrt{\boldsymbol{c}(V_h - U, \lambda) - \langle \mathcal{I}_h\Phi - \Phi, \lambda\rangle}\right) + \inf_{\chi_h \in \mathbb{M}_h} \|\psi - \chi_h\|_{\mathbb{M}}$$

Proof. Let $(V_h, \chi_h, \mu_h) \in \mathbb{N}_h \times \mathbb{M}_h \times \Lambda_h$ and let $Q_h \in \mathbb{W}_h$ be the solution of

$$\boldsymbol{a}_\rho(Q_h, Z_h) + (Q_h, Z_h)_{\mathbb{X}_h} = \boldsymbol{b}(\chi_h, Z_h) - \boldsymbol{c}(\mu_h, Z_h), \quad \forall Z_h \in \mathbb{W}_h$$

For an arbitrary $\chi_h \in \mathbb{M}_h$, we apply Lemma 3.3.1 to the pair $(V_h - (u, r)_h, \chi_h - \psi_h)$, hence there exists $W_h \in \mathbb{X}_h$ such that $D_h := W_h - (V_h - U_h) \in \mathbb{W}_h$ and satisfying

$$\begin{aligned}
(\|V_h - U_h\|_{\mathbb{X}} + \|\chi_h - \psi_h\|_{\mathbb{M}})^2 \lesssim& \mathcal{A}_\rho((V_h - U_h, \chi_h - \psi_h); (W_h, \psi_h - \chi_h)) \\
=& \mathcal{A}_\rho((V_h - U, \chi_h - \psi); (W_h, \psi_h - \chi_h)) + \mathcal{A}_\rho((U, \psi); (W_h, \psi_h - \chi_h)) \\
&- \mathcal{A}_\rho((U_h, \psi_h); (W_h, \psi_h - \chi_h)) \qquad (3.27)
\end{aligned}$$

Taking V_h as the test function in the first line of problem 7, with $U_h + W_h = D_h + V_h$ that belongs to $\mathbb{N}_h$ because D_h is in $\mathbb{W}_h$ and V_h is in $\mathbb{N}_h$, we find that

$$\boldsymbol{a}_\rho(U_h, W_h) + \boldsymbol{b}(W_h, \psi_h) \geq \mathcal{L}(W_h).$$

Since

$$\boldsymbol{b}(U_h, \psi_h - \chi_h) = 0,$$

we obtain

$$\boldsymbol{a}_\rho(U_h, W_h) + \boldsymbol{b}(W_h, \psi_h) + \boldsymbol{b}(U_h, \psi_h - \chi_h) \geq \mathcal{L}(W_h),$$

or equivalently,

$$\mathcal{A}_\rho((U_h, \psi_h); (W_h, \psi_h - \chi_h)) \geq \mathcal{L}(W_h), \tag{3.28}$$

recalling that

$$\mathcal{A}_\rho((U_h, \psi_h); (W_h, \psi_h - \chi_h)) = \boldsymbol{a}_\rho(U_h, W_h) + \boldsymbol{b}(W_h, \psi_h) + \boldsymbol{b}((U_h, \psi_h - \chi_h).$$

Then (3.28) and (3.27) amount to write

$$\begin{aligned}(\|V_h - U_h\|_{\mathbb{X}} + \|\chi_h - \psi_h\|_{\mathbb{M}})^2 \lesssim\ & \mathcal{A}_\rho((V_h - U, \chi_h - \psi); (W_h, \psi_h - \chi_h)) \\ & + \mathcal{A}_\rho((U, \psi); (W_h, \psi_h - \chi_h)) - \mathcal{L}(W_h).\end{aligned}$$

But by Proposition (2.2.2), we may write

$$\mathcal{A}_\rho((U, \psi); (W_h, \psi_h - \chi_h)) - \mathcal{L}(W_h) = \boldsymbol{c}(W_h, \lambda).$$

Then we get

$$(\|V_h - U_h\|_{\mathbb{X}} + \|\chi_h - \psi_h\|_{\mathbb{M}})^2 \lesssim \mathcal{A}_\rho((V_h - U, \chi_h - \psi); (W_h, \psi_h - \chi_h)) + \boldsymbol{c}(W_h, \lambda). \tag{3.29}$$

As $W_h - (V_h - U_h) \in \mathbb{W}_h$, we directly obtain

$$\boldsymbol{c}(W_h, \lambda) = \boldsymbol{c}(V_h - U_h, \lambda) = \boldsymbol{c}(V_h - U, \lambda) + \boldsymbol{c}(U - U_h, \lambda).$$

Now let's recall that

$$\boldsymbol{c}(U, \lambda) = \langle \Phi, \lambda \rangle \quad \text{and} \quad \boldsymbol{c}(U_h, \lambda) \geq \langle \mathcal{I}_h \Phi, \lambda \rangle\,,$$

hence,

$$\boldsymbol{c}(U - U_h, \lambda) + \langle \mathcal{I}_h \Phi - \Phi, \lambda \rangle \leq 0.$$

Putting it all together, we then have

$$\boldsymbol{c}(W_h, \lambda) \leq \boldsymbol{c}(V_h - U, \lambda) - \langle \mathcal{I}_h \Phi - \Phi, \lambda \rangle .$$

The estimate in (3.29) implies

$$\begin{aligned}(\|V_h - U_h\|_{\mathbb{X}} + \|\chi_h - \psi_h\|_{\mathbb{M}})^2 \lesssim & \mathcal{A}_\rho((V_h - U, \chi_h - \psi); (W_h, -\eta_h)) + \boldsymbol{c}(V_h - U, \lambda) \\ & - \langle \mathcal{I}_h \Phi - \Phi, \lambda \rangle .\end{aligned}$$

The continuity of the bilinear form $\mathcal{A}_\rho$ implies

$$\begin{aligned}(\|V_h - U_h\|_{\mathbb{X}} + \|\chi_h - \psi_h\|_{\mathbb{M}})^2 \lesssim & \|(V_h - U, \chi_h - \psi)\|_{\mathcal{H}} \|(W_h, -\eta_h)\|_{\mathcal{H}} + \boldsymbol{c}(V_h - U, \lambda) \\ & - \langle \mathcal{I}_h \Phi - \Phi, \lambda \rangle .\end{aligned}$$

Note that

$$\boldsymbol{c}(V_h - U, \lambda) - \langle \mathcal{I}_h \Phi - \Phi, \lambda \rangle \geq 0, \quad \forall V_h \in \mathbb{N}_h.$$

Applying Young's inequality to the first term on the right-hand side and completing the square gives

$$\|V_h - U_h\|_{\mathbb{X}} + \|\chi_h - \psi_h\|_{\mathbb{M}} \lesssim \|V_h - U\|_{\mathbb{X}} + \|\chi_h - \psi\|_{\mathbb{M}} + \sqrt{\boldsymbol{c}(V_h - U, \lambda) - \langle \mathcal{I}_h \Phi - \Phi, \lambda \rangle}$$

The last inequality, along with the triangule inequality, leads to the required estimate. ■

The following a priori error estimate is a direct consequence of the Lagrange interpolant properties. In particular, we can observe that if U belongs to $\mathbb{N}$, then $\mathcal{I}_h U$ belongs to $\mathbb{N}_h$.

Corollary 3.3.3 ([47]) *Suppose the solution (U, ψ) of Problem 1 belongs to $(H^2(\omega, \mathbb{R}^3))^2 \times H^2(\omega)$, and the function Φ belongs to $H^2(\omega)$. Then,*

$$\|U - U_h\|_{\mathbb{X}} + \|\psi - \psi_h\|_{\mathbb{M}} \lesssim \sqrt{h}\ [|U|_{2,\omega} + |\psi|_{2,\omega} + |\Phi|_{2,\omega}].$$

3.3.2 A Priori Error Estimation of the Full Problem

In this section, we conduct a priori error analysis for problem 5. Recall that it involves finding $(U, \psi, \lambda) \in \mathbb{X} \times \mathbb{M} \times \Lambda$ such that

$$\begin{cases} \boldsymbol{a}_\rho(U, V) + \boldsymbol{b}(V, \psi) - \boldsymbol{c}(V, \lambda) = \mathcal{L}(V), \ \forall V = (v, s) \in \mathbb{X}, \\ \boldsymbol{b}(U, \chi) = 0, \ \forall \chi \in \mathbb{M}, \\ \boldsymbol{c}(U, \mu - \lambda) \geq \langle \Phi, \mu - \lambda \rangle, \ \forall \mu \in \bar{\Lambda}. \end{cases} \tag{3.30}$$

while its discrete approximation (Problem 9) consists of finding $(U_h, \psi_h, \lambda_h) \in \mathbb{X}_h \times \mathbb{M}_h \times \Lambda_h$ such that

$$\begin{cases} \forall V_h \in \mathbb{X}_h, \ \boldsymbol{a}_\rho(U_h, V_h) + \boldsymbol{b}(V_h, \psi_h) - \boldsymbol{c}(V_h, \lambda_h) = \mathcal{L}(V_h), \\ \forall \chi_h \in \mathbb{M}_h, \quad \boldsymbol{b}(U_h, \chi_h) = 0 \\ \forall \mu_h \in \Lambda_h, \quad \boldsymbol{c}(U_h, \mu_h - \lambda_h) \geq \langle \Phi_h, \mu_h - \lambda_h \rangle . \end{cases} \tag{3.31}$$

Firstly, we observe that (3.31) can now be written in a more compact form as follows:

Problem 10 *Find* $(U_h, \psi_h, \lambda_h) \in \mathbb{X}_h \times \mathbb{M}_h \times \Lambda - h$ *such that:*

$$\mathscr{B}(U_h, \psi_h, \lambda_h; V_h, \chi_h, \mu_h - \lambda_h) \geq \mathscr{L}_h(V_h, \mu_h - \lambda_h), \quad \forall (V_h, \chi_h, \mu_h) \in \mathbb{X}_h \times \mathbb{M}_h \times \Lambda_h,$$

where

$$\mathscr{B}(U_h, \psi_h, \lambda_h; V_h, \chi_h, \mu_h) := \boldsymbol{a}_\rho(U_h, V_h) + \boldsymbol{b}(V_h, \psi_h) + \boldsymbol{b}(U_h, \chi_h) - \boldsymbol{c}(V_h, \lambda_h) + \boldsymbol{c}(U_h, \mu_h)$$
$$\mathscr{L}_h(V_h, \chi_h, \mu_h) := \mathcal{L}(V) + \langle \Phi_h, \mu_h \rangle$$

Lemma 3.3.4 ([47]) *There exists a constant* $\tilde{\beta}^{\#} > 0$ *such that:*

$$\inf_{(\chi_h, \mu_h) \in \mathbb{M}_h \times \mathbb{Q}_h} \sup_{Z_h = (z_h, t_h) \in \mathbb{X}_h} \frac{\boldsymbol{c}(Z_h, \mu_h) - \boldsymbol{b}(Z_h, \chi_h)}{\|(\chi_h, \mu_h)\|_{\mathbb{M}_h \times \mathbb{M}'} \|Z_h\|_{\mathbb{X}}} \geq \tilde{\beta}^{\#}. \tag{3.32}$$

Proof. Let us fix $(\chi_h, \mu_h) \in \mathbb{M}_h \times \mathbb{Q}_h$ such that $(\chi_h, \mu_h) \neq (0, 0)$. Initially, note that by Lemma 3.2.5, there exists $V_h \in \mathbb{X}_h \cap \ker \boldsymbol{b}$ with $|V_h|_{\mathbb{X}} = 1$, such that

$$\|\mu_h\|_{\mathbb{M}'} \lesssim \boldsymbol{c}(V_h, \mu_h) = \boldsymbol{c}(V_h, \mu_h) - \boldsymbol{b}(V_h, \chi_h),$$

while by Lemma 3.2.1 there exists $W_h \in \mathbb{W}_h = \ker \boldsymbol{c}$ with $\|W_h\|_{\mathbb{X}} = 1$ such that:

$$\|\chi_h\|_{\mathbb{M}} \lesssim -\boldsymbol{b}(W_h, \chi_h) = \boldsymbol{c}(W_h, \mu_h) - \boldsymbol{b}(W_h, \chi_h).$$

Now, observe that $|V_h + W_h|_{\mathbb{X}}$ is positive. If $V_h + W_h = 0$, then $W_h = -V_h$, implying that U_h and W_h belong to $\ker b \cap \ker c$. From the previous estimates, we would have $\chi_h = \mu_h = 0$, which contradicts our assumption. Therefore, by the triangular inequality, we have $0 < |V_h + W_h|_{\mathbb{X}} \leq 2$, and consequently $1 \leq \frac{2}{|V_h+W_h|_{\mathbb{X}}}$. Using all these estimates, we get:

$$\begin{aligned}\|\chi_h\|_{\mathbb{M}} + \|\mu_h\|_{\mathbb{M}'} &\lesssim \boldsymbol{c}(V_h + W_h, \mu_h) - \boldsymbol{b}(V_h + W_h, \chi_h)\\ &\lesssim \frac{\boldsymbol{c}(V_h + W_h, \mu_h) - \boldsymbol{b}(V_h + W_h, \chi_h)}{\|V_h + W_h\|_{\mathbb{X}}}.\end{aligned}$$

This obviously implies that

$$\|\chi_h\|_{\mathbb{M}} + \|\mu_h\|_{\mathbb{M}'} \lesssim \sup_{Z_h \in \mathbb{X}_h} \frac{\boldsymbol{c}(Z_h, \mu_h) - \boldsymbol{b}(Z_h, \chi_h)}{\|Z_h\|_{\mathbb{X}}}. \tag{3.33}$$

■

Lemma 3.3.5 *For any $(W_h, \chi_h, \mu_h) \in \mathbb{X}_h \times \mathbb{M}_h \times \mathbb{Q}_h$ there exists $Y_h \in \mathbb{X}_h$ such that:*

$$\mathscr{B}(W_h, \chi_h, \mu_h; Y_h, -\chi_h, \mu_h) \gtrsim \left(\|W_h\|_{\mathbb{X}} + \|\chi_h\|_{\mathbb{M}} + \|\mu_h\|_{\mathbb{M}'}\right)^2, \tag{3.34}$$

$$\|Y_h\|_{\mathbb{X}} + \|\chi_h\|_{\mathbb{M}} + \|\mu_h\|_{\mathbb{M}'} \lesssim \|W_h\|_{\mathbb{X}} + \|\chi_h\|_{\mathbb{M}} + \|\mu_h\|_{\mathbb{M}'}. \tag{3.35}$$

Proof. The proof follows the same lines as that of Theorem 2.2.6, utilizing the previous lemma. We present it here for completeness. Let $(W_h, \chi_h, \mu_h) \in \mathbb{X}_h \times \mathbb{M}_h \times \mathbb{Q}_h$. We consider the following variational problem:

$$\begin{cases} \text{Find } Q_h \text{ in } \mathbb{X}_h \text{ such that} \\ \boldsymbol{a}_\rho(Q_h, Z_h) + (Q_h, Z_h)_{\mathbb{X}} = \boldsymbol{b}(Z_h, \chi_h) - \boldsymbol{c}(\mu_h, Z_h), \quad \forall Z_h \in \mathbb{X}_h \end{cases} \tag{3.36}$$

As the bilinear form $\boldsymbol{a}\rho(\cdot,\cdot) + (\cdot,\cdot)\mathbb{X}$ is $\mathbb{X}$-elliptic and $\mathbb{X}_h \subset \mathbb{X}$, problem (3.36) possesses a unique solution $Q_h \in \mathbb{X}_h$.

Furthermore, since $\boldsymbol{c}(\cdot,\cdot) - \boldsymbol{b}(\cdot,\cdot)$ satisfies the inf-sup condition (refer to Lemma 3.3.4), and the Cauchy-Schwarz inequality implies:

$$\|\chi_h\|_{\mathbb{M}} + \|\mu_h\|_{\mathbb{M}'} \lesssim \sup_{Z_h \in \mathbb{X}_h} \frac{\boldsymbol{c}(\mu_h, Z_h) - \boldsymbol{b}(\chi_h, Z_h)}{\|Z_h\|_{\mathbb{X}}} = \sup_{Z_h \in \mathbb{X}_h} \frac{\boldsymbol{a}(Q_h, Z_h) + (Q_h, Z_h)_{\mathbb{X}}}{\|Z_h\|_{\mathbb{X}}} \lesssim \|Q_h\|_{\mathbb{X}}.$$

Take $Y_h = W_h + \delta Q_h$ such δ be a positive constant to be determined later, then we have:

$$\begin{aligned}\mathscr{B}(W_h, \chi_h, \mu_h; Y_h, -\chi_h, \mu_h) &= \boldsymbol{a}_\rho(W_h, W_h + \delta Q_h) + \boldsymbol{b}(W_h + \delta Q_h, \chi_h) + \boldsymbol{b}(W_h, -\chi_h) - \boldsymbol{c}(W_h + \delta Q_h, \mu_h) \\ &+ \boldsymbol{c}(\mu_h, W_h) \\ &= \boldsymbol{a}_\rho(W_h, W_h) + \delta \boldsymbol{a}_\rho(W_h, Q_h) + \delta \boldsymbol{b}(Q_h, \chi_h) - \delta \boldsymbol{c}(\mu_h, Q_h) \\ &\geq \boldsymbol{a}_\rho(W_h, W_h) - \delta c^\# \|W_h\|_{\mathbb{X}} \|Q_h\|_{\mathbb{X}} + \delta \|Q_h\|^2_{\mathbb{X}} \\ &\geq (c_\# - \frac{\delta (c^\#)^2}{2}) \|W_h\|^2_{\mathbb{X}} + \frac{\delta}{2} \|Q_h\|^2_{\mathbb{X}}.\end{aligned}$$

It suffices to choose $0 < \delta < \frac{c_\#^2}{2(c^\#)^2}$ to get

$$\mathscr{B}(W_h, \chi_h, \mu_h; Y_h, -\chi_h, \mu_h) \gtrsim (\|W_h\|^2_{\mathbb{X}} + \|\chi_h\|^2_{\mathbb{M}} + \|\mu_h\|^2_{\mathbb{M}'}). \tag{3.37}$$

■

Theorem 3.3.6 *Let (U, ψ, λ) and (U_h, ψ_h, λ_h) be the solution of Problem 5 and Problem 9 respectively. Then*

$$\begin{aligned}\|U - U_h\|_{\mathbb{X}} + \|\psi - \psi_h\|_{\mathbb{M}} + \|\lambda - \lambda_h\|_{\mathbb{M}'} \lesssim & \inf_{V_h \in \mathbb{N}_h} \|U - V_h\|_{\mathbb{X}} + \inf_{\chi_h \in \mathbb{M}_h} \|\psi - \chi_h\|_{\mathbb{M}} \\ &+ \inf_{\mu_h \in \Lambda_h} (\|\mu_h - \lambda\|_{\mathbb{M}'} + \sqrt{\boldsymbol{c}(U, \mu_h - \lambda) - \langle \Phi, \mu_h - \lambda \rangle}) \\ &+ \|\Phi - \Phi_h\|_{\mathbb{M}}.\end{aligned}$$

Proof. Let $V_h \in \mathbb{N}_h$ and let $Q_h \in \mathbb{W}_h$ be the solution of

$$\boldsymbol{a}_\rho(Q_h, Z_h) + (Q_h, Z_h)_{\mathbb{X}_h} = \boldsymbol{b}(\chi_h - \psi_h, Z_h) - \boldsymbol{c}(\mu_h - \lambda_h, Z_h), \quad \forall Z_h \in \mathbb{W}_h.$$

Using Lemma 3.3.5 with $W_h = V_h - U_h, \chi_h = \chi_h - \psi_h$ and $\mu_h = \mu_h - \lambda_h$, there exists $Y_h \in \mathbb{X}_h$ satisfying (3.34) and (3.35), namely

$$\|Y_h\|_{\mathbb{X}} \lesssim \|V_h - U_h\|_{\mathbb{X}}. \tag{3.38}$$

as well as

$$\begin{aligned}(\|V_h - U_h\|_{\mathbb{X}} + \|\chi_h - \psi_h\|_{\mathbb{M}} + \|\mu_h - \lambda_h\|)^2 \lesssim & \mathscr{B}(V_h - U_h, \chi_h - \psi_h, \mu_h - \lambda_h; Y_h, \psi_h - \chi_h, \mu_h - \lambda_h) \\ =& \mathscr{B}(V_h - U, \chi_h - \psi, \mu_h - \lambda; Y_h, \psi_h - \chi_h, \mu_h - \lambda_h) \\ &+ \mathscr{B}(U), \psi, \lambda; Y_h, \psi_h - \chi_h, \mu_h - \lambda_h) \\ &- \mathscr{B}(U_h, \psi_h, \lambda_h; Y_h, \psi_h - \chi_h, \mu_h - \lambda_h).\end{aligned}$$

Considering the definition of Problem 10,

$$-\mathscr{B}(U_h,\psi_h,\lambda_h;Y_h,\psi_h-\chi_h,\mu_h-\lambda_h)\leq-\mathscr{L}_h(Y_h,\mu_h-\lambda_h).$$

For the second term, since $\mathbb{X}_h\subset\mathbb{X}$, the definition of the bilinear form $\boldsymbol{c}(\cdot,\cdot)$ implies that:

$$\begin{aligned}\mathscr{B}(U,\psi,\lambda;Y_h,\psi_h-\chi_h,\mu_h-\lambda_h)&=\boldsymbol{a}_\rho(U,Y_h)+\boldsymbol{b}(Y_h,\psi)+\underbrace{\boldsymbol{b}(U,\psi_h-\chi_h)}_{=0}-\boldsymbol{c}(Y_h,\lambda)+\boldsymbol{c}(U,\mu_h-\lambda_h)\\&=\mathcal{L}(Y_h)+\boldsymbol{c}(U,\mu_h-\lambda_h),\end{aligned}$$

we then get

$$\begin{aligned}(\|V_h-U_h\|_{\mathbb{X}}+\|\chi_h-\psi_h\|_{\mathbb{M}}+\|\mu_h-\lambda_h\|)^2\lesssim&\mathscr{B}(V_h-U,\chi_h-\psi,\mu_h-\lambda;Y_h,\psi_h-\chi_h,\mu_h-\lambda_h)\\&+\boldsymbol{c}(U,\mu_h-\lambda_h)+\mathcal{L}(Y_h)-\mathscr{L}_h(Y_h,\mu_h-\lambda_h)\\=&\mathscr{B}(V_h-U,\chi_h-\psi,\mu_h-\lambda;Y_h,\psi_h-\chi_h,\mu_h-\lambda_h)\\&+\boldsymbol{c}(U,\mu_h-\lambda_h)-\langle\Phi_h,\mu_h-\lambda_h\rangle.\end{aligned}$$

Since $\Lambda_h\subset\Lambda$ then,

$$\boldsymbol{c}(U,\lambda_h)\geq\langle\Phi,\lambda_h\rangle \tag{3.39}$$

On the other hand, we have,

$$\begin{aligned}\boldsymbol{c}(U,\mu_h-\lambda_h)-\langle\Phi_h,\mu_h-\lambda_h\rangle=&\boldsymbol{c}((u,r),\mu_h-\lambda)+\boldsymbol{c}(U,\lambda-\lambda_h)-\langle\Phi_h-\Phi,\mu_h-\lambda_h\rangle-\langle\Phi,\mu_h-\lambda\rangle\\&-\langle\Phi,\lambda-\lambda_h\rangle\\\leq&\boldsymbol{c}(U,\mu_h-\lambda)-\langle\Phi_h-\Phi,\mu_h-\lambda_h\rangle-\langle\Phi,\mu_h-\lambda\rangle\end{aligned}$$

whence we obtain

$$\begin{aligned}(\|V_h-U_h\|_{\mathbb{X}}+\|\chi_h-\psi_h\|_{\mathbb{M}}+\|\mu_h-\lambda_h\|)^2\lesssim&\mathscr{B}(V_h-U,\chi_h-\psi,\mu_h-\lambda;Y_h,\psi_h-\chi_h,\mu_h-\lambda_h)\\&+\boldsymbol{c}(U,\mu_h-\lambda)-\langle\Phi,\mu_h-\lambda\rangle-\langle\Phi_h-\Phi,\mu_h-\lambda_h\rangle\end{aligned}$$

Applying Young's inequality to the first term on the right-hand side, utilizing the estimate (3.38), and completing the square yields:

$$\begin{aligned}\|V_h-U_h\|_{\mathbb{X}}+\|\chi_h-\psi_h\|_{\mathbb{M}}+\|\mu_h-\lambda_h\|_{\mathbb{M}'}\lesssim&\|V_h-U\|_{\mathbb{X}}+\|\chi_h-\psi\|_{\mathbb{M}}+\|\mu_h-\lambda\|_{\mathbb{M}'}\\&+\sqrt{\boldsymbol{c}(U,\mu_h-\lambda)-\langle\Phi,\mu_h-\lambda\rangle}+\|\Phi-\Phi_h\|_{\mathbb{M}}\end{aligned}$$

Finally, from this last inequality and the triangle inequality, we obtain the desired estimate. ■

Corollary 3.3.7 ([47]) *Suppose the solution (U, ψ, λ) of Problem 5 belongs to $(H^2(\omega, \mathbb{R}^3))^2 \times H^2(\omega) \times L^2(\omega)$, and the function Φ belongs to $H^2(\omega) \cap H^1_{\gamma 0}(\omega)$. Then,*

$$\|U - U_h\|_{\mathbb{X}} + \|\psi - \psi_h\|_{\mathbb{M}} + \|\lambda - \lambda_h\|_{\mathbb{M}'} \lesssim \sqrt{h}\ [|U|_{2,\omega} + |\psi|_{2,\omega} + |\Phi|_{2,\omega} + \|\lambda\|_{\omega}].$$

Proof. The proof relies on the a priori error estimate shown in Theorem 3.3.6. The estimates for the terms $\|V_h - U\|_{\mathbb{X}}$, $\|\chi_h - \psi_h\|_{\mathbb{M}}$, $\|\mu_h - \lambda\|_{\mathbb{M}'}$, and $\|\Phi - \Phi_h\|_{\mathbb{M}}$ can be easily obtained by standard interpolation procedures. To establish the result, we need to prove the estimate for the term $\sqrt{\boldsymbol{c}(U, \mu_h - \lambda) - \langle \Phi, \mu_h - \lambda \rangle}$.
As $U \in H^1_{\gamma_0}(\omega; \mathbb{R}^3))^2$ and $\Phi \in H^1_{\gamma 0}(\omega)$, we have:

$$|\boldsymbol{c}(U, \mu_h - \lambda) - \langle \Phi, \mu_h - \lambda \rangle| \lesssim (\|U\|_{1,\omega} + \|\Phi\|_{1,\omega})\|\mu_h - \lambda\|_{\mathbb{M}'}. \tag{3.40}$$

Now, let's take μ_h as the weighted Clément type interpolation operator of λ [24], denoted by $\mu_h = Q_h\lambda$, defined as:

$$Q_h\lambda = \sum_{x \in \mathcal{N}_h} \pi_x(\lambda)\lambda_x,$$

For any $\varphi \in L^1(\omega)$, define:

$$\pi_x(\varphi) = \begin{cases} \frac{\int_{\omega_x} \varphi\lambda_x}{\int_{\omega_x} \lambda_x} & \text{if } x \notin \bar{\gamma}_0, \\ 0 & \text{if } x \in \bar{\gamma}_0. \end{cases}$$

Therefore, as

$$\|\lambda - Q_h\lambda\|_{\mathbb{M}'} = \sup_{\varphi \in H^1_{\gamma_0}(\omega), \varphi \neq 0} \frac{\int_\omega (\lambda - Q_h\lambda)\varphi}{\|\varphi\|_{1,\omega}},$$

and as we directly verify that

$$\int_\omega (\lambda - Q_h\lambda)\varphi = \int_\omega \lambda(\varphi - Q_h\varphi),$$

we obtain

$$\|\lambda - Q_h\lambda\|_{\mathbb{M}'} = \sup_{\varphi \in H^1_{\gamma_0}(\omega), \varphi \neq 0} \frac{\int_\omega \lambda(\varphi - Q_h\varphi)}{\|\varphi\|_{1,\omega}}.$$

By Cauchy-Schwarz's inequality and Lemma 6.2 of [24], we obtain:

$$\|\lambda - Q_h\lambda\|_{\mathbb{M}'} \lesssim h\|\lambda\|_{\omega}.$$

Inserting this estimate into (3.40), we obtain:

$$|\boldsymbol{c}(U, \mu_h - \lambda)| \lesssim h(\|U)\|_{1,\omega} + \|\Phi\|_{1,\omega})\|\lambda\|_{\omega} \lesssim h(\|(U\|_{1,\omega}^2 + \|\Phi\|_{1,\omega}^2 + \|\lambda\|_{\omega}^2).$$

■

3.4 Comments on the Regularity of the Solution

The regularity of the solutions plays an important role in the error analysis. The a priori error analysis carried out in Section 3.3 requires additional regularity on the solution of the continuous problem. Our contact problem takes the following complementarity system:

$$\begin{cases} -\partial_\varrho((n^{\varrho\sigma}(u)a_\sigma + m^{\varrho\sigma}(U)\partial_\sigma a_3 + q^\varrho(u,r)a_3)\sqrt{a}) - \lambda \boldsymbol{e}_3 = f\sqrt{a} & \text{in } \omega, \\ -\partial_\varrho(m^{\varrho\sigma}(U)a_\sigma\sqrt{a}) + q^\beta(U)a_\beta\sqrt{a} + \dfrac{\varepsilon}{2}\lambda\boldsymbol{e}_3 = 0 & \text{in } \omega, \\ r\cdot a_3 = 0 & \text{in } \omega, \\ \left(u - \dfrac{\varepsilon}{2}r\right)\cdot\boldsymbol{e}_3 \geq \Phi, \quad \lambda \geq 0, \quad \lambda\left(\left(u-\dfrac{\varepsilon}{2}r\right)\cdot\boldsymbol{e}_3 - \Phi\right) = 0 & \text{in } \omega, \\ u = r = 0 & \text{on } \partial\omega, \end{cases} \tag{3.41}$$

with coefficients which are in $L^\infty(\omega)$, and the function Φ belongs only to $\mathscr{W}^{1,\infty}(\omega)$ when the chart $\varphi \in \mathscr{W}^{2,\infty}(\omega,\mathbb{R}^3)$ and therefore, the translations (or finite difference quotients) method of Nirenberg ([59]) can not be applied here. The coefficients of the system satisfy the ellipticity condition and in the non contact set the system is a "standard" second order elliptic system. But the famous De Giorgi's counter-example (see [38, p.205]) indicates that the regularity problem for systems of equations (or vectorial case) can not be treated as the case of a single elliptic equation (or scalar case), so the Stampacchia-Brezis [18] technique can not used here. However, if we assume that the chart φ is more regular, namely $\varphi \in C^3(\omega,\mathbb{R}^3)$, then the formulation (expressions of the tensors (1.4), (1.5) and (1.6)) used in this paper coincides with the classical formulation of thin shell theory (see [12]). For sufficiently smooth surfaces, recent papers (see [27],[61], [62]) improved interior regularity of the solution of elastic shell in the presence of obstacles, by using the Nirenberg method. The main difficulty for this approach is the construction of admissible displacement field in term of the finite difference quotient satisfying the inequality constraint.

Chapter 4

Iterative methods for double saddle point system

The relationship between variational inequalities and saddle points is interesting and involves concepts from convex analysis and optimization. A saddle point is a concept often encountered in game theory and optimization. In fact, variational inequalities can be seen as a special case of saddle point problems. They share similarities in their mathematical formulations, and the existence of solutions in one context often implies the existence of solutions in the other.

The chapter focuses on the analysis and numerical solution of double saddle-point systems, specifically those with a block-3×3 structure. These systems are common in multiphysics problems, implying the need for effective numerical solutions. The numerical solution of double saddle-point systems is gaining importance and interest in the research community. Solving several saddle-point issues iteratively has recently attracted a lot of attention. For instance, [66], [60], and [22] are recent articles that offer intriguing analysis. In this context, we first provide a mathematical problem statement and discuss the connections between double and classical (block-2×2) saddle-point systems. We then provide a brief overview of iterative solution methods.

4.1 Problem Statement:

We consider the double saddle-point system

$$\mathscr{A}u = b$$

such that:

$$\mathscr{A} \equiv \begin{bmatrix} A_1 & A_2^T & A_3^T \\ A_2 & 0 & 0 \\ A_3 & 0 & -A_4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{4.1}$$

where $A_1 \in \mathbb{R}^{n\times n}$ is symmetric positive definite (SPD), $A_2 \in \mathbb{R}^{m\times n}$, $A_3 \in \mathbb{R}^{p\times n}$, and $A_4 \in \mathbb{R}^{p\times p}$ is symmetric positive semidefinite (SPS) and possibly zero. Throughout the section we assume that $n \geq m + p$.

We further mention the large linear systems with coefficient matrices of the form:

$$\mathscr{B} \equiv \begin{bmatrix} A_1 & A_3^T & A_2^T \\ A_3 & -A_4 & 0 \\ A_2 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \mathscr{C} \equiv \begin{bmatrix} -A_4 & A_3 & 0 \\ A_3^T & A_1 & A_2^T \\ 0 & A_2 & 0 \end{bmatrix}$$

see, e.g., [1] and [25], respectively. Using symmetric permutations (row and column interchanges), it is clear that $\mathscr{B}$ and $\mathscr{C}$ may be transformed into the same form as matrix $\mathscr{A}$ in (4.1).

Generalization:

- The matrix $\mathscr{A}$ is considered a generalization of the block-2×2 or "classical" saddle-point matrix $\tilde{A} = \begin{bmatrix} A_1 & A_2 \\ A_2^T & -A_4 \end{bmatrix}$.

The first-order optimality requirements for quadratic programming problems with equality constraints are classical saddle-point matrices:

$$\min_{x} \frac{1}{2} x^T A_1 x - F^T x \tag{4.2}$$

$$\text{subject to} \quad A_2 x = g. \tag{4.3}$$

Letting y denote the vector of Lagrange multipliers, an optimal solution of (4.2) and (4.3) is a saddle point for the Lagrangian:

$$L(x, y) = \frac{1}{2} x^T A_1 x - F^T x + (A_2 x - g)^T y.$$

This is equivalent to the solution (x, y) of the linear system:

$$\begin{bmatrix} A_1 & A_2 \\ A_2^T & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} g \\ F \end{bmatrix}. \tag{4.4}$$

Here, the coefficient matrix in (4.19), denoted as $\tilde{A}_0$ is a matrix in classical saddle-point form with $A_4 = 0$. The matrix A_4 is often associated with regularization or stabilization, and matrices with $A_4 = 0$ are referred to as unregularized or unstabilized. Block-3×3 matrices in tridiagonal form as in (4.1) can occur when a physical problem with constraints, such as the contact problem of Nagdi'shell with a rigide body (3.7) which mention in previous chapters.

Remark 4.1.1 *Equatrical features, including invertibility, spectral characteristics, conditioning, existence, and different factorizations, were present in the saddle point matrix. The development of solution algorithms relies on knowing these qualities. For further information, go to [6].*

4.2 Double Saddle-Point Systems

Double saddle-point systems of the form (4.1) arise in several applications. We provide an overview of some these applications as described in [16] in the following section.

4.2.1 Examples Arising from Optimization

PDE-Constrained Optimization

Consider a discretized linear-quadratic optimization problem of the form:

$$\begin{aligned} \min_{y,u} \quad & \frac{1}{2} y^T A_3 y - y^T w + \frac{\beta}{2} u^T R u \\ \text{subject to} \quad & Ky + Lu = d, \end{aligned} \tag{4.5}$$

where $K \in \mathbb{R}^{n \times n}$ is a stiffness matrix corresponding to a partial differential equation (PDE); $L \in \mathbb{R}^{n \times m}$ is a control matrix; and $A_3 \in \mathbb{R}^{n \times n}$ is a positive semidefinite (sometimes positive definite) observation matrix. Consider a linear-quadratic optimization problem with partial differential equation (PDE) constraints:

$$\begin{aligned} \min_{y,u} \quad & J(y,u) = \frac{1}{2} y^T A_3 y - y^T w + \frac{\beta}{2} u^T R u \\ \text{subject to} \quad & Ky + Lu = d, \end{aligned} \tag{4.6}$$

where:

- $K \in \mathbb{R}^{n \times n}$ is a stiffness matrix corresponding to a PDE.
- $L \in \mathbb{R}^{n \times m}$ is a control matrix.
- $A_3 \in \mathbb{R}^{n \times n}$ is a positive semidefinite (sometimes positive definite) observation matrix.
- $R \in \mathbb{R}^{m \times m}$ is a positive definite regularization matrix.

- $\beta > 0$ is a regularization parameter (often around 10^{-2} in practice).
- The vector $y \in \mathbb{R}^n$ denotes the state variables, $u \in \mathbb{R}^m$ the control variables, and $\lambda \in \mathbb{R}^m$ represents Lagrange multipliers.

The associated Karush-Kuhn-Tucker (KKT) system can be written as a classical saddle-point system:

$$\begin{bmatrix} A_3 & 0 & K^T \\ 0 & \beta R & L^T \\ K & L & 0 \end{bmatrix} \begin{bmatrix} y \\ u \\ \lambda \end{bmatrix} = \begin{bmatrix} w \\ 0 \\ d \end{bmatrix}$$

We can also reorder the unknowns so that the coefficient matrix is in double saddle-point form (4.1):

$$\begin{bmatrix} A_3 & K^T & 0 \\ K & 0 & L \\ 0 & L^T & \beta R \end{bmatrix} \begin{bmatrix} y \\ \lambda \\ u \end{bmatrix} = \begin{bmatrix} w \\ d \\ 0 \end{bmatrix}$$

Interior point techniques saddle point systems

In this paper, we demonstrate the emergence of saddle point systems as a result of solving limited optimisation issues using interior point techniques. The excellent summary provided in [7] forms the basis of our presentation. Think about a nonlinear programming issue that is convex.

$$\min F(x) \tag{4.7}$$

$$\text{subject to } c(x) \leq 0. \tag{4.8}$$

Think about dual differentiable convex functions $F : \mathbb{R}^n \to \mathbb{R}$ and $c : \mathbb{R}^n \to \mathbb{R}^m$. The inequality constraint may be expressed as the system of equalities $c(x) + z = 0$ after introducing a nonnegative slack variable $z \in \mathbb{R}^m$. After that, we may define the related barrier:

$$\min F(x) - \eta \sum_{i=1}^{m} \ln z_i \tag{4.9}$$

$$\text{subject to } c(x) + z = 0. \tag{4.10}$$

The corresponding Lagrangian is given by:

$$\mathbb{L}(x, y, z; \eta) = F(x) + y^T(c(x) + z) - \eta \sum_{i=1}^{m} \ln z_i. \tag{4.11}$$

To find a stationary point of the Lagrangian, we set the following partial derivatives equal to zero:

$$\nabla_x \mathbb{L}(x, y, z; \eta) = \nabla F(x) + \nabla c(x)^T y = 0, \tag{4.12}$$

$$\nabla_y \mathbb{L}(x, y, z; \eta) = c(x) + z = 0, \tag{4.13}$$

$$\nabla_z \mathbb{L}(x, y, z; \eta) = y - \eta Z^{-1} e = 0. \tag{4.14}$$

Where $Z = \text{diag}(z_1, z_2, \ldots, z_m)$ and $e = [1, 1, \ldots, 1]^T$.

Introducing the diagonal matrix $Y = \text{diag}(y_1, y_2, \ldots, y_m)$, the first-order optimality conditions for the barrier problem become:

$$\nabla F(x) + \nabla c(x)^T y = 0, \tag{4.15}$$

$$c(x) + z = 0, \tag{4.16}$$

$$YZe = \eta e, \tag{4.17}$$

$$y, z \geq 0. \tag{4.18}$$

This is a system of equations with nonlinearity and nonnegativity requirements that can be resolved using Newton's technique. The barrier parameter η is decreased progressively to guarantee convergence of the iterates to the optimal solution of issue (4.7)–(4.8). At each iteration of Newton's method, a linear system of equations must be solved.

$$\begin{bmatrix} H(x,y) & A_2(x)^T & Q \\ A_2(x) & Q & I \\ Q & Z & Y \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix} = \begin{bmatrix} -\nabla F(x) - A_2(x)^T y \\ -c(x) - z \\ \eta e - YZe \end{bmatrix}, \tag{4.19}$$

Where

$$H(x, y) = \nabla^2 F(x) + \sum_{i=1}^{m} y_i \nabla^2 c_i(x) \in \mathbb{R}^{n \times n}$$

and

$$A_2(x) = \nabla c(x) \in \mathbb{R}^{m \times n}.$$

Here, $\nabla^2 F(x)$ denotes the Hessian of F evaluated at x. The matrix in (4.19) can be symmetrized using a diagonal similarity transformation:

$$\mathscr{A} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & Z^{1/2} \end{bmatrix}^{-1} \begin{bmatrix} H(x,y) & A_2(x)^T & 0 \\ A_2(x) & 0 & I \\ 0 & Z & Y \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & Z^{1/2} \end{bmatrix} \tag{4.20}$$

$$= \begin{bmatrix} H(x,y) & A_2(x)^T & 0 \\ A_2(x) & 0 & Z^{1/2} \\ 0 & Z^{1/2} & Y \end{bmatrix}. \tag{4.21}$$

The matrix in (4.21) is equivalent to the double saddle-point formulation (4.1) up to a difference in sign.

Constrained weighted least-squares

Linear systems of saddle-point type frequently arise when addressing least squares problems. Let's examine the least squares problem with linear equality requirements.

$$\min_x \|c - \mathcal{G}y\|^2 \tag{4.22}$$

$$\text{subject to} \quad \mathcal{E}y = d, \tag{4.23}$$

where $c \in \mathbb{R}^p$, $\mathcal{G} \in \mathbb{R}^{p\times m}$, $y \in \mathbb{R}^m$, $\mathcal{E} \in \mathbb{R}^{q\times m}$, $d \in \mathbb{R}^q$, and $q < m$.
Issues of this nature arise, particularly in curve or surface fitting scenarios where the curve needs to interpolate specific data points.

The optimality conditions for problem (4.22) are:

$$\begin{bmatrix} I_p & Q & \mathcal{G} \\ Q & Q & \mathcal{E}^T \\ \mathcal{G}^T & \mathcal{E} & Q \end{bmatrix} \begin{bmatrix} r \\ \lambda \\ y \end{bmatrix} = \begin{bmatrix} c \\ d \\ 0 \end{bmatrix} \tag{4.24}$$

where I_p is the $p \times p$ identity matrix and $\lambda \in \mathbb{R}^q$ is a vector of Lagrange multipliers. Clearly, (4.24) is a special case of the symmetric saddle point problem (4.19).

4.2.2 Examples arising from the numerical solution of PDE

Second application on double saddle point systems belongs to the numerical solution of partial differential equations for example: Dual-dual fnite element formulations [37] and Magma mantle dynamics Rhebergen et al, [63]. Also Darcy-Stokes following the formulation of [23].
In other hand thers exist some application deal with classical saddle-point systems with singular leading blocks. Some of these examples are in fact, double saddle-point systems, where the systems are re-ordered and partitioned into block-2×2 systems.

4.3 Iterative solution of sparse linear systems

Double saddle-point systems, as expressed in the form of (4.1), commonly exhibit substantial sparsity. In the context of solving such systems, direct solvers utilizing methods like Gaussian elimination (as detailed in [30], [41]) may introduce a notable amount of fill-in. This phenomenon occurs when the matrix decompositions required for precise inversion result in a considerable increase in nonzero entries compared to the original matrix.Furthermore, in addressing problems rooted in the modeling of physical phenomena, the pursuit of an extremely accurate solution, as provided by direct solvers, is often unnecessary.
Discretization processes, whether in space [54],[32] time in [6], or continuous processes like differentiation and integration, inherently introduce errors.

4.3.1 Stationary Iterative Methods

Stationary iterative methods tackle a linear system using a simplified matrix that approximates the original one, often based on a splitting of the original matrix ($A = M - N$). In each step, the iterate x_{k+1} updates based on the residual at step k, defined by $r_k =$

$b - Ax_k$. Specifically, considering a splitting of the matrix $A = M - N$, a stationary iteration takes the form $x_{k+1} = x_k + M^{-1}r_k$.
Examples of methods in this category include the Richardson method, Jacobi method, Gauss-Seidel. In fluid dynamics, a popular iterative approach for solving the linear system (4.1) is Uzawa's method, which is similar to a relaxed block SOR iteration (see [64], Chapter 4). In the remainder of the paper we will always assume that A is nonsingular.

Uzawa-type methods

Uzawa-like iterative methods have long been among the most popular algorithms for solving linear systems in saddle point form ([6] Sect 8.1). In this paragraph we study variants of Uzawa's algorithm [4]. We discuss the case where the matrix D in (4.1) is zero.

First choice ; We consider the following splittings for $\mathscr{A}$

$$\mathscr{A} = \mathcal{M}_1 - \mathcal{N}_1$$

where

$$\mathcal{M}_1 = \begin{bmatrix} A_1 & 0 & 0 \\ B & \frac{-1}{\alpha}I & 0 \\ A_3 & 0 & \frac{-1}{\beta}I \end{bmatrix}, \qquad \mathcal{N}_1 = \begin{bmatrix} 0 & -A_2^T & -A_3^T \\ 0 & \frac{-1}{\alpha}I & 0 \\ 0 & 0 & \frac{-1}{\beta}I \end{bmatrix}. \qquad \text{and } \alpha, \beta \neq 0$$

The corresponding iterative scheme for solving (4.1) are given by

$$x_{k+1} = \mathscr{G}_1 x_k + \mathcal{M}_1^{-1} b, \qquad k = 0, 1, 2..., \tag{4.25}$$

where x_0 is arbitrary,and $\mathscr{G}_1 = \mathcal{M}_1^{-1}\mathcal{N}_1$ the iteration matrix .

Now we analyze the convergence properties of iterative methods (4.25). The following useful lemma is needed, which is a special case of Weyl's Theorem [[46], Theorem 4.3.1]

Lemma 4.3.1 *Let s_i be the eignvalues. If A_1 and A_2 are two Hermitian matrices. Then*

$$s_{max}(A_1 + A_2) \leq s_{max}(A_1) + s_{max}(A_2),$$

$$s_{min}(A_1 + A_2) \geq s_{min}(A_1) + s_{min}(A_2),$$

Corollary 4.3.2 *[4] Let $\mathscr{A}$ is nonsingular with $A_1 > 0$, A_2^T and A_3^T have full column rank, and range$(A_2^T) \cap$ range$(A_3^T) = 0$. If the parametres $\alpha > 0$ and $\beta > 0$ satisfy*

$$\alpha s_{max}(A_2 A_1^{-1} A_2^T) + \beta s_{max}(A_3 A_1^{-1} A_3^T) < 2 \tag{4.26}$$

then the iterative scheme (4.25) is convergent for any initial guss, i.e:$\rho(\mathscr{G}_1) < 1$.

Proof. we gave the proof in the following steps:

- $\mathscr{A}$ is nonsungular lead us that the solvability condition in [4] are satisfied.
- Note that if the assumptions on A_1, A_2 and A_3 are satisfied, Then all of the eigenvalues of the following matrix are real and positive for positive parametersα and β:

$$\mathscr{S}_{\alpha,\beta} = \begin{bmatrix} \alpha A_2 A_1^{-1} A_2^T & \alpha A_2 A_1^{-1} A_3^T \\ \beta A_3 A_1^{-1} A_2^T & \beta A_3 A_1^{-1} A_3^T \end{bmatrix}$$

- observe that the nonzero eigenvalues of $\mathscr{S}_{\alpha,\beta}$ are the same as those of

$$\mathscr{S}_1 = A_1^{-1} \begin{bmatrix} A_2^T & A_3^T \end{bmatrix} \begin{bmatrix} \alpha A_2 \\ \beta A_3 \end{bmatrix} = \alpha A_1^{-1} A_2^T A_2 + \beta A_1^{-1} A_3^T A_3.$$

  then we can say

$$s_{min}(\mathscr{S}_1) \leq s(\mathscr{S}_{\alpha,\beta}) \leq s_{max}(\mathscr{S}_1),$$

- From tha following iteration matrix :

$$\mathscr{G}_1 = \mathcal{M}_1^{-1} \mathcal{N}_1 = \begin{bmatrix} 0 & -A_1^{-1} A_2^T & -A_1^{-1} A_3^T \\ 0 & I - \alpha A_2 A_1^{-1} A_2^T & -\alpha A_2 A_1^{-1} A_3^T \\ 0 & -\beta A_3 A_1^{-1} A_2^T & I - \beta A_3 A_1^{-1} A_3^T \end{bmatrix} \tag{4.27}$$

  the iterative schemme (4.25) is convergent if and only if $\rho(I - \mathscr{S}_{\alpha,\beta}) < 1$.

- Using lemma (4.3.1) and noting $S_B = A_2 A_1^{-1} A_2^T$ and $S_C = A_3 A_1^{-1} A_3^T$ we get

$$1 - (\alpha s_{max}(S_B) + \beta s_{max}(S_C)) \leq s(\mathscr{S}_1) \leq 1 - (\alpha s_{min}(S_B) + \beta s_{min}(S_C)). \tag{4.28}$$

as a result:

$$-1 < 1 - (\alpha s_{max}(S_B) + \beta s_{max}(S_C))$$

which complet the proof.

■

Second choice :

$$\mathscr{A} = \mathcal{M}_2 - \mathcal{N}_2,$$

where

$$\mathcal{M}_2 = \begin{bmatrix} A_1 & A_2^T & 0 \\ A_2 & 0 & 0 \\ A_3 & 0 & \frac{-1}{\alpha} I \end{bmatrix}, \text{ and } \mathcal{N}_2 = \begin{bmatrix} 0 & 0 & -A_3^T \\ 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{\alpha} I \end{bmatrix}.$$

The corresponding iterative scheme for solving (4.1) are given by

$$x_{k+1} = \mathscr{G}_2 x_k + \mathcal{M}_2^{-1} b, \qquad k = 0, 1, 2..., \tag{4.29}$$

where

$$\mathscr{G}_2 = \mathcal{M}_2^{-1}\mathcal{N}_2 = \begin{bmatrix} 0 & 0 & -\tilde{A}A_3^T \\ 0 & 0 & -S_B^{-1}A_2A_1^{-1}A_3^T \\ 0 & 0 & I - \alpha A_3\tilde{A}A_3^T \end{bmatrix} \tag{4.30}$$

with $\tilde{A} = A^{-1} - A^{-1}A_2^T S_B^{-1} A_2 A_1^{-1}$ and $S_B = A_2 A_1^{-1} A_2^T$.

We finalize this subsection with a brief exploration of the convergence properties of the iterative method (4.29). To achieve this, we initially require the following two propositions.

Proposition 4.3.1 *Assume that $A_1 > 0$ and A_2^T has full column rank. Then*

$$\tilde{A} = A_1^{-1} - A_1^{-1}A_2^T S_B^{-1} A_2 A_1^{-1} \geq 0$$

.

Proof. Since A_1 is SPD, we can write

$$\tilde{A} = A_1^{-1/2}(I - A_1^{-1/2} A_2^T S_B^{-1} A_2 A_1^{-1/2}) A_1^{-1/2}.$$

The nonzero eigenvalues of

$$A_1^{-1/2} A_2^T S_B^{-1} A_2 A_1^{-1/2}$$

are the same as those of

$$A_2 A_1^{-1/2} A_1^{-1/2} A_2^T S_B^{-1} = S_B S_B^{-1} = I$$

and therefore are all equal to 1. Hence, $I - A_1^{-1/2} A_2^T S_B^{-1} A_2 A_1^{-1/2} \geq 0$ as claimed. ■

Proposition 4.3.2 *Suppose that $A_1 > 0$, A_2^T,A_3^T have full comumn rank and*

$$\tilde{A} = A_1^{-1} - A_1^{-1} A_2^T S_B^{-1} A_2 A_1^{-1} \geq 0.$$

If $z^T(A_3 \tilde{A} A_3^T)z = 0$ for some nonzero vector z, then $range(A_2^T) \cap range(A_3^T) \neq 0$.

Proof. Suppose that z is a nonzero vector such that $z^T(A_3 \tilde{A} A_3^T)z = 0$.
Setting $y = A_3^T z$ and invoking Proposition (4.3.1), we obtain that $y^T \tilde{A} y = 0$ where $\tilde{A} \geq 0$. Note that A_3^T has full column rank, hence $y \neq 0$. From [9, Page 400], we obtain that $\tilde{A}y = 0$, or $y \in ker(\tilde{A})$. On the other hand, $\tilde{A}y = 0$ implies that $y = A_2^T S^{-1} A_2 A_1^{-1} y$, which shows that $y \in range(A_2^T)$.
Consequently, in view of the definition of y, we have that that $y \in range(A_2^T) \cap range(A_3^T)$ as claimed. ■

The following proposition provides a necessary and sufficient condition under which $\rho(\mathscr{G}_2) < 1$.

Proposition 4.3.3 *Assume that $\mathscr{A}$ is invertible, with $A_1 > 0$ and $A_4 = 0$. A necessary and sufficient condition for the iterative scheme (4.29)to be convergent is*

$$0 < \alpha < \frac{2}{s_{max}(\hat{S}_C)},$$

where $\hat{S}_C = A_3 \tilde{A} A_3^T$ and $\tilde{A}$ is defined as before. The minimum value od the spectral radius $\rho(\mathscr{G}_2)$ is attained for

$$\alpha^* = \frac{2}{s_{max}(\hat{S}_C) + s_{min}(\hat{S}_C)}$$

Proof. Since $\mathscr{A}$ is assumed to be nonsingular, and let the matrices A_2^T and A_3^T have full column rank and $range(A_2^T) \cap range(A_3^T) \neq 0$. Therefore, Proposition (4.3.2) implies that $S_C > 0$.
From the structure of the iteration matrix $\mathscr{G}_2$ given by (4.30), it is clear that a necessary and sufficient condition for the convergence of (4.29) is that $\rho(I - \alpha \hat{S}_C) < 1$.
The asymptotic convergence rate is equivalent to Richardson's approach for solving a linear system of equations with coefficient matrix $\hat{S}_C$. The function of α is seen in Figure (4.1).

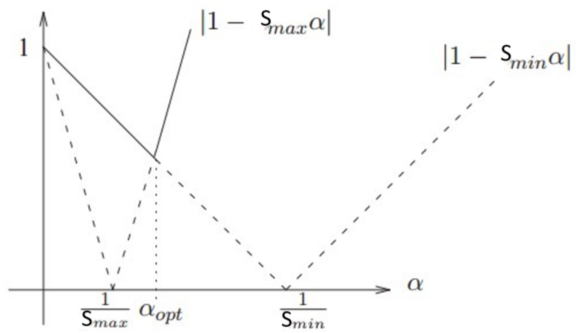


Figure 4.1: The curve $\rho(\mathscr{G}_\alpha)$ as a function of α.

The optimal α is achieved at the intersection point of the curve $|1 - s_{max}\alpha|$ with positive slope and the curve $|1 - s_{min}\alpha|$ with negative slope, as seen in the graph.

Now the conclusions follow from the results in [[64], Chapter 4] on the convergence properties of Richardson's method applied to a linear system with an SPD coefficient matrix. ■

Chapter 5

Numerical solution for some variationl inequalities

In this chapter we analyze the iterative solution algorithms like Uzawa method for solving some variational inequalities. The Uzawa method's primary appeal lies in its ease of implementation and its minimal memory footprint. Despite its potentially sluggish convergence rate, we've opted for it precisely because of these characteristics. Alternatively, a primal-dual active set method could be employed. During each iteration of the Uzawa method, an elliptic solver is necessary to compute the inverse of a large sparse matrix. The convergence of this method for saddle point systems has been explored by various authors.(see for instance [50], [3]). Here we establish the convergence of the Uzawa algorithm through two distinct approaches. First, we leverage the convergence results derived from the analysis of the Richardson iteration on simple conatct with singular saddle point. In the second approach, we present the uzawa method for double saddle point system and give their convergence proof in the context of a variational inequality and equality .

5.1 Obstacle problem

As a model example of a problem formulated in a **variational inequality** we choose the so-called Obstacle problem. We consider the problem where an elastic membrane ($\Omega \subset \mathbb{R}^2$) is fixed at the boundary, subjected to an external force g in the vertical direction. The membrane is in contact with a rigid body (the obstacle) positioned above it.

Several mathematical frameworks are available for this problem : linear complementarity, free boundary, variational inequality, restricted convex minimization, and others. Optimal control, financial mathematics, elasto-plasticity, fluid filtration in porous media, and other areas find mathematical treatments of this topic useful. Numerous studies have conducted thorough explorations of the barrier issue and its formulations, as well as any solutions that may exist. for instance, refer to [48],[21], and [36].

Let

$$\mathscr{K} = \{v \in H_0^1(\Omega) \,|\, v \geq \varphi \text{ in } \Omega\}$$

The corresponding minimization problem reads:

$$u = \arg\min_{v \in \mathscr{K}} \mathcal{J} \quad \text{with} \quad \mathcal{J}(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - (g, v)\, dx$$

The formulation as a variational inequality takes the form:

$$\begin{cases} \text{Find } u \in \mathscr{K} \text{ such that} \\ \int_\Omega \nabla u \cdot \nabla(v-u)\, dx \geq \int_\Omega g(v-u)\, dx, \quad \forall v \in \mathscr{K} \end{cases}$$

The classical solution u of this model problem is determined by the system in $\Omega \subset \mathbb{R}^2$ (see Evans [33], for instance):

$$\begin{aligned} -\Delta u - g &\geq 0 \quad \text{in } \Omega, \\ u - \varphi &\geq 0 \quad \text{in } \Omega, \\ (u-\varphi)(-\Delta u - g) &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Introducing a non-negative Lagrange multiplier gunction $\lambda : \Omega \to \mathbb{R}$, we can rewrite the obstacle problem as:

$$\begin{aligned} -\Delta u - \lambda = g &\quad \text{in } \Omega, \\ u - \varphi \geq 0 &\quad \text{in } \Omega, \\ \lambda \geq 0 &\quad \text{in } \Omega, \\ (u-\varphi)\lambda = 0 &\quad \text{in } \Omega, \\ u = 0 &\quad \text{on } \partial\Omega. \end{aligned} \tag{5.1}$$

The Lagrange multiplier is in the dual space $Q = H^{-1}(\Omega)$ with the norm

$$\|\eta\|_{-1} = \sup_{v \in \ H_0^1(\Omega)} \frac{\langle v, \eta \rangle}{\|v\|_1},$$

The corresponding mixed problem in variational formulation becomes:

$$\begin{cases} \text{find } ((u,\lambda) \in V \times \Lambda \text{ such that} \\ \qquad (\nabla u, \nabla v) - \langle \lambda, v \rangle = (g, v) \quad \forall v \in V, \\ \qquad \langle u - \varphi, \mu - \lambda \rangle \geq 0 \quad \forall \mu \in \Lambda, \end{cases} \tag{5.2}$$

where

$$V = H_0^1(\Omega) \ \ \textit{and} \ \ \Lambda = \{\mu \in Q : \langle \mu, v \rangle \geq 0 \quad \forall v \in V, v \geq 0 \text{ a.e. in } \Omega\}$$

5.1.1 Uzawa method

Now, our goal is to construct an algorithm that can approximate the solution of the primal-dual problem (5.2) . In general, the Gradient method is used, but the projection of this problem is not explicitly known and very difficult to determine. Note that we have significant a priori information on the Lagrange multiplier $\lambda = u - \varphi$ which is always positive based on the mixed formulation. Building upon this observation, we propose **the Uzawa algorithm**, which is essentially a saddle point search method utilizing a very simple projection operator given by:

$$P_\Lambda(\lambda) = \max(\lambda_i, 0)$$

Therefore, we apply the Uzawa algorithm to problem (5.2). We can see that the previous mixed formulation when discretized can be rewritten as a system of equations and inequalities.

By letting $\theta_j, j \in \{1, \dots, N\}$, be the basis for V_h, and writing $u_h = \sum_{j=1}^{N} u_j \theta_j$,

$$\begin{cases} Au + B^T \lambda := g \\ (\mu - \lambda)^T Bu \leq (\mu - \lambda)^T \varphi \quad \forall \mu \in \Lambda \end{cases} \tag{5.3}$$

where: $A \in \mathbb{R}^{N \times N}$, $(A)_{ij} = (\nabla_i, \nabla_j)$; $B \in \mathbb{R}^{M \times N}$, $(B)_{ij} = (\eta_i, \theta_j)$; $g \in \mathbb{R}^N$, $(g)_i = (g, \theta_i)$; $\varphi \in \mathbb{R}^M$, $(\varphi)_i = (\varphi, \eta_i)$; $u \in \mathbb{R}^N$, $(u)_i = u_i$; $\lambda \in \mathbb{R}^M$, $(\lambda)_i = \lambda_i$.

Starting with initial guesses u_0 and λ_0, Uzawa's method [6] consists of the following coupled iteration

$$\begin{cases} Au^{k+1} = g - B^T \lambda^k \\ \lambda^{k+1} = \lambda^k + \alpha(Bu^{k+1} - \varphi) \end{cases} \tag{5.4}$$

where $\alpha > 0$ is a relaxation parameter. This iteration can be written in terms of a matrix splitting $\mathscr{A} = \mathcal{P} - \mathcal{Q}$ where

$$\mathscr{A} = \begin{bmatrix} A & 0 \\ B & -\frac{1}{\alpha} I \end{bmatrix}, \quad \mathcal{Q} = \begin{bmatrix} 0 & -B^T \\ 0 & -\frac{1}{\alpha} I \end{bmatrix} \tag{5.5}$$

The corresponding iterative schemes is

$$\mathcal{P} U^{k+1} = \mathcal{Q} U^k + b \quad \text{where} \quad U^k = \begin{bmatrix} u^k \\ \lambda^k \end{bmatrix} \tag{5.6}$$

then the iteration matrix is

$$\mathcal{P}^{-1} \mathcal{Q} = \begin{bmatrix} 0 & A^{-1} B^T \\ 0 & I - \alpha B A^{-1} B^T \end{bmatrix} \tag{5.7}$$

we note that the eignvalues of $\mathcal{P}^{-1}\mathcal{Q}$ are all real. On the other hand we can eliminate $u^{(k+1)}$ from the construction of $\lambda^{(k+1)}$. This means that we compute $\lambda^{(k+1)}$ directly from $\lambda^{(k)}$ without finding $u^{(k+1)}$.

From the first step of the kth iteration, we obtain

$$u^{k+1} = A^{-1}(g - B^T \lambda^k) \tag{5.8}$$

after inserting this into the second equation in (5.4) we compute λ^{k+1} from λ^k in two steps:

$$\tilde{\lambda}^{k+1} = \lambda^k - \alpha(BA^{-1}B^T\lambda^k - BA^{-1}f + \varphi)$$
$$\lambda^{k+1} = \mathcal{P}_\Lambda(\tilde{\lambda}^{k+1}). \qquad k \geq 1$$

Our goal now is to analyze the convergence of Uzawa's method of the obstacle problem.

Theorem 5.1.1 *Let (u, λ) be a solution to the system (5.4) and s_1, s_2 denote the smallest and the largest eigenvalues of $BA^{-1}B^T$. Then there exists a constant $\alpha > 0$ such that for each choice $\alpha > 0$ there holds $u^k \longrightarrow u$ and $\lambda^k \longrightarrow \lambda$.*

Proof.

$$\tilde{\lambda}^{k+1} = \lambda^k - \alpha(BA^{-1}B^T\lambda^k - BA^{-1}g + \varphi) \tag{5.9}$$

we set: $S_B = BA^{-1}B^T$ *and* $h = BA^{-1}g + \varphi$. Then

$$\tilde{\lambda}^{(k+1)} := \lambda^{(k)} - \alpha(S_B\lambda^{(k)} - h) = (I - \alpha S_B)\lambda^{(k)} + \alpha h.$$

One then degines $\lambda^{(k+1)} := P_\Lambda\left((I - \alpha S_B)\lambda^{(k)} + \alpha h\right)$. Similarly, if $\lambda \in \Lambda$ is a solution to the mixed problem, then one has $\lambda = P_\Lambda\left((I - \alpha S_B)\lambda + \alpha h\right)$. Then, the errors satisfy

$$\lambda^{(k+1)} - \lambda = P_\Lambda\left((I - \alpha S_B)\lambda^{(k)} + \alpha h\right) - P_\Lambda\left((I - \alpha S_B)\lambda + \alpha h\right).$$

By the contraction property of projections, we have

$$\|\lambda^{(k+1)} - \lambda\| \leq \|(I - \alpha S_B)\|\|\lambda^{(k)} - \lambda\|.$$

Let us denote $\|\lambda^{(k+1)} - \lambda\|$ by $e^{(k+1)}$ and $\|\lambda^{(k)} - \lambda\|$ by $e^{(k)}$, then the errors satisfy

$$e^{(k+1)} \leq \|(I - \alpha S_B)\|e^{(k)}.$$

Thus $(e^{(k+1)}, e^{(k+1)}) \leq ((I - \alpha S_B)e^{(k)}, (I - \alpha S_B)e^{(k)})$. Since S_B is symmetric, it follows that $\rho(I - \alpha S_B) = \|I - \alpha S_B\|$, so that the error norm satisfies

$$\|e^{(k+1)}\| \leq \rho((I - \alpha S_B))\|e^{(k)}\|.$$

μ_i denote the eigenvalues of $(I - \alpha S_B)$. Then $1 - \alpha s_2 \leq \mu_i \leq 1 - \alpha s_1$, and the Uzawa's Method is convergent provided $\rho(I - \alpha S_B) < 1$, i.e.,

$$0 < \alpha < \frac{2}{s_2}$$

as u_k depends continuously on λ_k, we can conclude that $u_k \to u$. ■

5.2 Contact Probem of Naghdi's shell

The mathematical description of this model is presented in second chapter. We will suffice here with a reminder of the variational form (**Problem** 9) which consists in finding $(U_h, \psi_h, \lambda_h) \in \mathbb{X}_h \times \mathbb{M}_h \times \Lambda_h$ such that:

$$\begin{cases} \forall V_h \in \mathbb{X}_h, \ \boldsymbol{a}_\rho(U_h, V_h) + \boldsymbol{b}(V_h, \psi_h) - \boldsymbol{c}(V_h, \lambda_h) = \mathcal{L}(V_h), \\ \quad \forall \chi_h \in \mathbb{M}_h, \qquad\qquad \boldsymbol{b}(U_h, \chi_h) = 0 \\ \quad \forall \mu_h \in \Lambda_h, \qquad\qquad \boldsymbol{c}(U_h, \mu_h - \lambda_h) \geq \langle \Phi_h, \mu_h - \lambda_h \rangle . \end{cases} \tag{5.10}$$

5.2.1 Uzawa-type stationary methods

In this section, we will analyze the Uzawa method to solve Problem 9, which is regarded as an illustrative example of a double saddle point. It's notable that **Problem** 9, when expressed in its matrix form, can be interpreted as a 2×2 block matrix in two distinct manners, depending on the chosen partitioning strategy.

Firstly, by seeking $(U_h^{k+1}, \psi_h^{k+1})$ for a given $\lambda_h^k \in \Lambda_h$, one emphasizes that **Problem** 9 can essentially be treated as a standard saddle point problem to determine U_h^{k+1}, ψ_h^{k+1}, followed by a projection procedure to compute λ_h^{k+1}.

Alternatively, the second approach involves finding U_h^{k+1} for a given $(\psi_h^k, \lambda_h^k) \in \mathbb{M}_h \times \Lambda_h$. To clarify, this means considering iterative methods for solving large, sparse linear systems of equations of a particular form.

$$\mathscr{A} x = b, \qquad \text{with } \mathscr{A} \equiv \begin{bmatrix} A & B^T & C^T \\ B & 0 & 0 \\ C & 0 & 0 \end{bmatrix}, \tag{5.11}$$

In our context, the matrix $\mathscr{A}$ can be seen as a 2×2 block matrix in two different ways, depending on the chosen partitioning strategies. This is pertinent when $A \in \mathbb{R}^{n \times n}$ is symmetric positive definite, $B \in \mathbb{R}^{m \times n}$, and $C \in \mathbb{R}^{p \times n}$. In iterative methods, we commonly employ a splitting $\mathscr{A} = \mathcal{M} - \mathcal{N}$, where $\mathcal{M}$ is an invertible matrix. The iteration scheme follows the pattern:

$$x_{k+1} = \mathcal{M}^{-1}\mathcal{N}x_k + \mathcal{M}^{-1}b,$$

For our considered case, the matrix $\mathscr{A}$ can be regarded as a 2×2 block matrix in two different ways, according to which of the following partitioning strategies is used:

$$\mathscr{A} \equiv \left[\begin{array}{c|cc} A & B^T & C^T \\ \hline B & 0 & 0 \\ C & 0 & 0 \end{array}\right] \quad \text{or} \quad \mathscr{A} \equiv \left[\begin{array}{cc|c} A & B^T & C^T \\ B & 0 & 0 \\ \hline C & 0 & 0 \end{array}\right]. \tag{5.12}$$

Then if the splitting $\mathscr{A} = \mathcal{M}_1 - \mathcal{N}_1$ is used, we need to initialize our iterative method with a given $(\psi_h^0, \lambda_h^0) \in \mathbb{M}_h \times \Lambda_h$ and two parameters α and β. While for the second choice we need only one parameter and an initial guess $\lambda_h^0 \in \Lambda_h$.
In this thesis we will make use only the second type with only one parameter α, but we are especially interested in studying the convergence of the considered method when the third line of the system (5.11) is replaced by an inequality. Here we prove the convergence of the Uzawa algorithm for the case of variational inequality.

The Uzawa algorithm for **Problem** 9 in its variational form can be expressed as follows:

$$\begin{cases} \boldsymbol{a}_\rho(U_h^{k+1}, V_h) + \boldsymbol{b}(V_h, \psi_h^{k+1}) = \mathcal{L}(V_h) + \boldsymbol{c}(V_h, \lambda_h^k), \quad \forall V_h \in \mathbb{X}_h, \\ \boldsymbol{b}(U_h^{k+1}, \chi_h) = 0, \quad \forall \chi_h \in \mathbb{M}_h, \\ (\tilde{\lambda}_h^{k+1}, \tilde{\mu}_h) = (\lambda_h^k, \tilde{\mu}_h) + \alpha \boldsymbol{c}(U_h^{k+1}, \tilde{\mu}_h) - \alpha(\Phi_h, \tilde{\mu}_h), \ \forall \tilde{\mu}_h \in \mathbb{M}_h, \\ \lambda_h^{k+1} = \mathcal{P}_{\Lambda_h}(\tilde{\lambda}_h^{k+1}). \end{cases} \tag{5.13}$$

Now we outline the algorithm for this method

Algorithm 1 Uzawa methods

1. give some initial value $\lambda_h^{(0)}$
2. $k = 0$
3. repeat :
4. compute $(U^{(k+1)}, \psi^{(k+1)})$ from the equations

$$\begin{cases} \boldsymbol{a}_\rho(U_h^{k+1}, V_h) + \boldsymbol{b}(V_h, \psi_h^{k+1}) = \mathcal{L}(V_h) + \boldsymbol{c}(V_h, \lambda_h^k), & \forall V_h \in \mathbb{X}_h, \\ \boldsymbol{b}(U_h^{k+1}, \chi_h) = 0, & \forall \chi_h \in \mathbb{M}_h, \end{cases} \tag{5.14}$$

5.
$$(\tilde{\lambda}_h^{k+1}, \tilde{\mu}_h) = (\lambda_h^k, \tilde{\mu}_h) + \alpha \boldsymbol{c}(U_h^{k+1}, \tilde{\mu}_h) - \alpha(\Phi_h, \tilde{\mu}_h), \ \forall \tilde{\mu}_h \in \mathbb{M}_h, \tag{5.15}$$

   $\{\alpha$ is some given, small positive constant $\}$

6. take $\lambda^{(k+1)}$as the projection of $\tilde{\lambda}^{(k+1)}$ on $\bar{\Lambda}_h$:
   $\lambda_h^{k+1} = \mathcal{P}_{\bar{\Lambda}_h}(\tilde{\lambda}_h^{k+1})$
7. $k = k + 1$
8. until: $\|\lambda^{(k+1)} - \lambda^{(k)}\| / \|\lambda^{(k+1)}\|\} \leq \epsilon$

The study of convergence:

For an appropriate choice of α one can show that the sequence $(U^{(k)}, \psi^{(k)}, \lambda^{(k)})$ converges to (U, ψ, λ) of the contact naghdi problem (2.41). For $k \geq 1$, let's introduce the following notation:

$$\boldsymbol{E}_h^k = U_h - U_h^k, \quad E_h^k = \psi_h - \psi_h^k, \quad \text{and} \quad e_h^k = \lambda_h - \lambda_h^k,$$

then we can easily write

$$\boldsymbol{a}_\rho(\boldsymbol{E}_h^{k+1}, V_h) + \boldsymbol{b}(V_h, E_h^{k+1}) = \boldsymbol{c}(V_h, e_h^k), \quad \forall V_h \in \mathbb{X}_h$$
$$\boldsymbol{b}(\boldsymbol{E}_h^{k+1}, \chi_h) = 0, \quad \forall \chi_h \in \mathbb{M}_h$$

Utilizing the coercivity of $\boldsymbol{a}_\rho(\cdot,\cdot)$, the continuity of $\boldsymbol{c}(\cdot,\cdot)$, and the inf-sup condition of $\boldsymbol{b}(\cdot,\cdot)$, we obtain:

$$\|\boldsymbol{E}_h^{k+1}\|_{\mathbb{X}} + \|E_h^{k+1}\|_{\mathbb{M}} \lesssim \|e_h^k\|$$

Hence, the convergence to zero of the sequence $|e_h^k|$ will immediately imply the convergence of $|\boldsymbol{E}_h^{k+1}|_{\mathbb{X}}$ and $|E_h^{k+1}|_{\mathbb{M}}$.

We conclude these preliminary remarks with the following observation:

$$\lambda_h^{k+1} = \mathcal{P}_{\bar{\Lambda}_h}(\tilde{\lambda}_h^{k+1}),$$

i.e, λ_h^{k+1} is the projection of $\tilde{\lambda}_h^{k+1}$ onto the closed convex set $\bar{\Lambda}_h$. Therefore

$$(\tilde{\lambda}_h^{k+1} - \lambda_h^{k+1}, \mu_h - \lambda_h^{k+1}) \leq 0, \quad \forall \mu_h \in \bar{\Lambda}_h \text{ and } \|\lambda_h^{k+1}\| \leq \|\tilde{\lambda}_h^{k+1}\|.$$

Since $(\tilde{\lambda}_h^{k+1} - \lambda_h^{k+1}, \lambda_h) \leq 0$, we then have

$$(\tilde{\lambda}_h^{k+1} - \lambda_h^{k+1}, \lambda_h) + \|\lambda_h^{k+1}\|^2 \leq \|\tilde{\lambda}_h^{k+1}\|^2,$$

which is equivalent to

$$(\lambda_h - \lambda_h^{k+1}, \lambda_h - \lambda_h^{k+1}) \leq (\lambda_h - \tilde{\lambda}_h^{k+1}, \lambda_h - \tilde{\lambda}_h^{k+1}).$$

This amounts to write

$$\|e_h^{k+1}\| \leq \|\tilde{e}_h^{k+1}\|, \quad \text{where } \tilde{e}_h^k = \lambda_h - \tilde{\lambda}_h^k. \tag{5.16}$$

Therefore, the convergence to zero of the sequence $\|\tilde{e}_h^{k+1}\|$, will imply the convergence to zero of the sequence $\|e_h^{k+1}\|$.

Theorem 5.2.1 ([47]) *Let's define $K_0 = \dfrac{c_\#}{\tilde{c}_c}$, where $\tilde{c}_c$ represents the inf-sup constant of Lemma 3.2.4. Now, let $c_{c,\#}$ denote the continuity constant of the bilinear form c in $\mathbb{X} \times L^2(\omega)$. This constant is defined as the smallest positive constant such that*

$$|c(V,\mu)| \leq c_{c,\#}\|U\|_{\mathbb{X}}\|\mu\|, \forall V \in \mathbb{X}, \mu \in L^2(\omega). \tag{5.17}$$

Should the parameter α be selected such that

$$0 < 1 + \alpha(\alpha c_{c,\#}^2 - 2c^\#)K_0^{-2}h^2 < 1 \tag{5.18}$$

then

$$\lim_{k\to+\infty} \|\tilde{e}_h^{k+1}\| = 0.$$

Proof. First we have

$$\boldsymbol{a}_\rho(\boldsymbol{E}_h^{k+1}, V_h) + \boldsymbol{b}(V_h, E_h^{k+1}) - \boldsymbol{c}(V_h, e_h^k) = 0, \quad \forall V_h \in \mathbb{X}_h. \tag{5.19}$$

Take $V_h = \boldsymbol{E}_h^{k+1}$, we get

$$\boldsymbol{a}_\rho(\boldsymbol{E}_h^{k+1}, \boldsymbol{E}_h^{k+1}) = \boldsymbol{c}(\boldsymbol{E}_h^{k+1}, e_h^k) - \underbrace{\boldsymbol{b}(\boldsymbol{E}_h^{k+1}, E_h^{k+1})}_{=0} = \boldsymbol{c}(\boldsymbol{E}_h^{k+1}, e_h^k). \tag{5.20}$$

The fifth line of the algorithm (1) and (3.31) amount to write

$$\begin{aligned}(\tilde{\lambda}_h^{k+1}, \tilde{\mu}_h) &= (\lambda_h^k, \tilde{\mu}_h) + \alpha\boldsymbol{c}(U_h^{k+1}, \tilde{\mu}_h) + \alpha(\Phi_h, \tilde{\mu}_h),\\ \boldsymbol{c}(U_h, \lambda_h^k - \lambda_h) &\geq \left\langle \Phi_h, \lambda_h^k - \lambda_h \right\rangle.\end{aligned}$$

For $\tilde{\mu}_h = e_h^k$, we get

$$(\tilde{\lambda}_h^{k+1}, e_h^k) = (\lambda_h^k, e_h^k) + \alpha\boldsymbol{c}(U^{k+1}, e_h^k) + \alpha(\Phi_h, e_h^k),$$
$$(\lambda_h, e_h^k) \geq (\lambda_h, e_h^k) + \alpha\boldsymbol{c}(U_h, e_h^k) + \alpha(\Phi_h, e_h^k).$$

Then,

$$(\tilde{e}^{k+1}, e_h^k) \geq (e_h^k, e_h^k) + \alpha\boldsymbol{c}(\boldsymbol{E}_h^{k+1}, e_h^k)$$

or equivalently,

$$\alpha\boldsymbol{c}(\boldsymbol{E}_h^{k+1}, \mu_h) \leq (\tilde{e}^{k+1} - e_h^k, e_h^k).$$

Using (5.20) we find

$$\boldsymbol{a}_\rho(\boldsymbol{E}_h^{k+1}, \boldsymbol{E}_h^{k+1}) \leq -\frac{1}{\alpha}(\tilde{e}^{k+1} - e_h^k, e_h^k) = -\frac{1}{2\alpha}\left(\|\tilde{e}_h^{k+1}\|^2 - \|e_h^k\|^2 - \|\tilde{e}_h^{k+1} - e_h^k\|^2\right), \tag{5.21}$$

In the final equality, we utilized the identity:

$$\begin{aligned}(d-c, d-c) - (d,d) + (c,c) &= (d, d-c) - (c, d-c) - (d,d) + (c,c)\\ &= (d,d) - (d,c) - (c, d-c) - (d,d) + (c,c)\\ &= (c, c-d) - (c, d-c)\\ &= 2(c, c-d).\end{aligned}$$

The estimate (5.21) suggests that

$$2\alpha c^{\#}\|\boldsymbol{E}_h^{k+1}\|_{\mathbb{X}}^2 + \|\tilde{e}_h^{k+1}\|^2 \le \|e_h^k\|^2 + \|\tilde{e}_h^{k+1} - e_h^k\|^2, \tag{5.22}$$

Here, $c^{\#}$ represents the coercivity constant of $\boldsymbol{a}_\rho$. The subsequent step involves computing $|\tilde{e}_h^{k+1} - e_h^k|^2$.

Let $\tilde{\mu}_h \in \mathbb{M}_h$ and $\mu_h \in \bar{\Lambda}_h$ be defined as follows:

$$\tilde{\mu}_h = \tilde{e}_h^{k+1} - e_h^k, \qquad \mu_h = \lambda_h - (\lambda_h^k - \tilde{\lambda}_h^{k+1})$$

then,

$$\tilde{\mu}_h = \mu_h - \lambda_h.$$

We substitute this $\tilde{\mu}_h$ into the fifth line of above algorithm to obtain:

$$(\tilde{\lambda}_h^{k+1}, \tilde{\mu}_h) = (\lambda_h^k, \tilde{\mu}_h) + \alpha\boldsymbol{c}(U_h^{k+1}, \tilde{\mu}_h) - \alpha(\Phi_h, \tilde{\mu}_h),$$

while by the third line of (3.7) one has

$$(\lambda_h, \mu_h - \lambda_h) \le (\lambda_h, \mu_h - \lambda_h) + \alpha\boldsymbol{c}(U_h, \mu_h - \lambda_h) - \alpha(\Phi_h, \mu_h - \lambda_h).$$

So by taking the difference

$$(\tilde{e}_h^{k+1}, \tilde{e}_h^{k+1} - e_h^k) \le (e_h^k, \tilde{e}_h^{k+1} - e_h^k) + \alpha\boldsymbol{c}(\boldsymbol{E}_h^{k+1}, \tilde{e}_h^{k+1} - e_h^k) \tag{5.23}$$

Then the estimate (5.17) yields

$$\|\tilde{e}_h^{k+1} - e_h^k\|^2 \le \alpha\boldsymbol{c}(\boldsymbol{E}_h^{k+1}, \tilde{e}_h^{k+1} - e_h^k) \le \alpha c_{c,\#}\|\boldsymbol{E}_h^{k+1}\|_{\mathbb{X}}\|\tilde{e}_h^{k+1} - e_h^k\|.$$

Ultimately, this estimate is equivalent to:

$$\|\tilde{e}_h^{k+1} - e_h^k\|^2 \le \alpha^2 c_{c,\#}^2\|\boldsymbol{E}_h^{k+1}\|_{\mathbb{X}}^2.$$

By (5.22), we deduce that,

$$(2\alpha c^{\#} - \alpha^2 c_{c,\#}^2)\|\boldsymbol{E}_h^{k+1}\|_{\mathbb{X}}^2 + \|\tilde{e}_h^{k+1}\|^2 \le \|e_h^k\|^2,$$

or equivalently

$$\|\tilde{e}_h^{k+1}\|^2 \le \|e_h^k\|^2 + \alpha(\alpha c_{c,\#}^2 - 2c^{\#})\|\boldsymbol{E}_h^{k+1}\|_{\mathbb{X}}^2. \tag{5.24}$$

We therefore chose $\alpha > 0$ small enough such that $\alpha c_{c,\#}^2 - 2c^\# < 0$. So that we will conclude if one can show that

$$\|e_h^k\| \leq K_0 h^{-1} \|\boldsymbol{E}_h^{k+1}\|_{\mathbb{X}}. \tag{5.25}$$

Indeed if this estimate is valid then (5.24) becomes

$$\|\tilde{e}_h^{k+1}\|^2 \leq (1 + \alpha(\alpha c_{c,\#}^2 - 2c^\#) K_0^{-2} h^2) \|e_h^k\|^2,$$

and by (5.16)

$$\|e_h^{k+1}\|^2 \leq (1 + \alpha(\alpha c_{c,\#}^2 - 2c^\#)) K_0^{-2} h^2) \|e_h^k\|^2.$$

Through iteration, we eventually find:

$$\|e_h^k\| \leq (1 + \alpha(\alpha c_{c,\#}^2 - 2c^\#)) K_0^{-2} h^2)^{\frac{k}{2}} \|e_h^0\|,$$

and establishes the convergence of $\|e_h^k\|$ to zero if $0 < 1 + \alpha(\alpha c_{c,\#}^2 - 2c^\#) K_0^{-2} h^2 < 1$.

To prove (5.25), we utilize the identity (5.19), which states that

$$\boldsymbol{c}(V_h, e_h^k) = \boldsymbol{a}_\rho(\boldsymbol{E}_h^{k+1}, V_h) + \boldsymbol{b}(V_h, E_h^{k+1}), \quad \forall V_h \in \mathbb{X}_h,$$

which reduces to

$$\boldsymbol{c}(V_h, e_h^k) = \boldsymbol{a}_\rho(\boldsymbol{E}_h^{k+1}, V_h) \quad \forall V_h \in \mathbb{X}_h \cap \ker b,$$

Therefore, employing Lemma 3.2.4, we conclude that

$$\tilde{c}_c \|e_h^k\|_h \leq c_\# \|\boldsymbol{E}_h^{k+1}\|_{\mathbb{X}}.$$

By applying the definition of the norm $|\cdot|_h$, we obtain (5.25) with $K_0 = \dfrac{c\#}{\tilde{c}_c}$. ■

5.3 Numerical experiments

In this section, we provide several numerical tests designed to demonstrate the effectiveness of the suggested solvers, particularly Uzawa. Detailed results are provided exclusively forthe methods examined in the theoretical sections, accompanied by brief comments on the remaining ones. We address two distinct types of variational inequality problems, each arising from vastly different contact applications. One involves a singular saddle point, while the other features a double saddle point. The tests and codes illustrated in this chapter are developed using the FreeFem++-cs 15.2 32 software, which was developed at the Jacques-Louis Lions Laboratory at Pierre and Marie Curie University (Paris 6). Indeed, this software is an open-source finite element tool that provides a flexible environment for solving partial differential equations (PDEs). It is particularly designed for numerical simulations and computations related to scientific and engineering problems. FreeFem++ supports a wide range of applications, including but not limited to fluid dynamics, solid mechanics, heat transfer, and electromagnetism. We initiate the application of the Uzawa methods to model simple contact, as represented by the obstacle problem. Following that, we delve into our contribution in this thesis to the thin shell domain, specifically addressing the contact in the obstacle problem for Naghdi's shell model.

First test

We choose the mesh Ω as a disk, such that

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$$

and consider the problem (5.1) with

$$\begin{cases} f = -4 \\ \varphi = -x^2 + y^2 - 0.3 \end{cases}$$

That is we suppose that the membrane is attached at a point $y = 0.7$ and is loaded by a force $f = -4$. In this case the problem is fully radial and $u = u(r)$. Thus

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = 1; \quad a < r < 1; \quad u(1) = 0; \quad u(a) = \varphi(a); \quad u'(a) = \varphi'(a); \tag{5.26}$$

where a represents the contact area. when solving (5.26) we obtain

$$u(r) = r^2 - 4a^2 ln(r) - 2a^2 + 0.3 + 4a^2 ln(a) \quad and \quad a \simeq 0.29$$

If $r > a$ then $\lambda = 0$ and when $r < a$, the solution (u, λ) satisfies:

$$\lambda = 1 - \Delta\varphi \quad \text{and} \quad u = \varphi$$

The Uzawa method is employed to solve the given problem. The table below provides some information on the convergence between the exact solution and the result of the Uzawa method iteratioin using the L^2 norm.

step size h_T	0.80	0.40	0.20	0.10
iteration number	69	1907	2999	2999
$\|\lambda^{k+1} - \lambda^k\|_{L^2(\omega)}$	9.295 e-5	9.9834 e-6	7.5953 e-5	0.0002023

Table 5.1: Convergence outcomes for the Uzawa scheme using $\mathbb{P}_1 \oplus B_3 - \mathbb{P}_0$

step size h_T	0.80	0.40	0.20	0.10
iteration number	38	416	1412	2999
$\|\lambda^{k+1} - \lambda^k\|_{L^2(\omega)}$	9.0111 e-6	9.9358 e-6	9.98504 e-6	1.75612 e-5

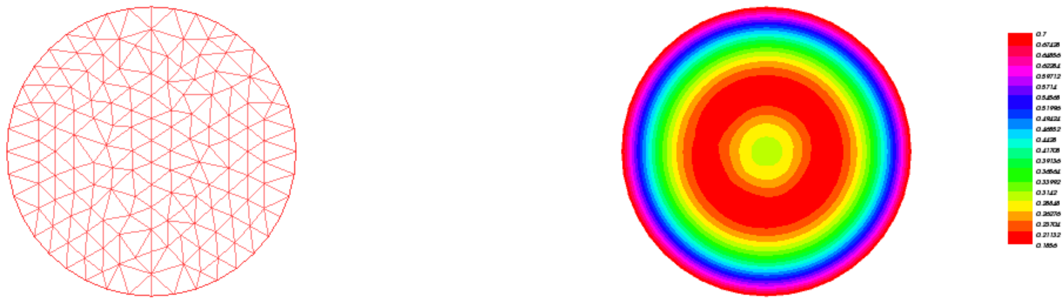
Table 5.2: Convergence results for the Uzawa scheme using $\mathbb{P}_2 - \mathbb{P}_1$

The finit element	iteration number	$\|u_h - u\|_{L^2(\omega)}$	$\|\lambda^{k+1} - \lambda^k\|_{L^2(\omega)}$	Convergence factor (rate)
$\mathbb{P}_1 \oplus B_3 - \mathbb{P}_0$	2999	6.16032 e-5	7.5953 e-5	0.98
$\mathbb{P}_2 - \mathbb{P}_1$	1412	0.0002783	9.98 e-6	1.59

Table 5.3: Convergence rate when $\alpha = 1.9$

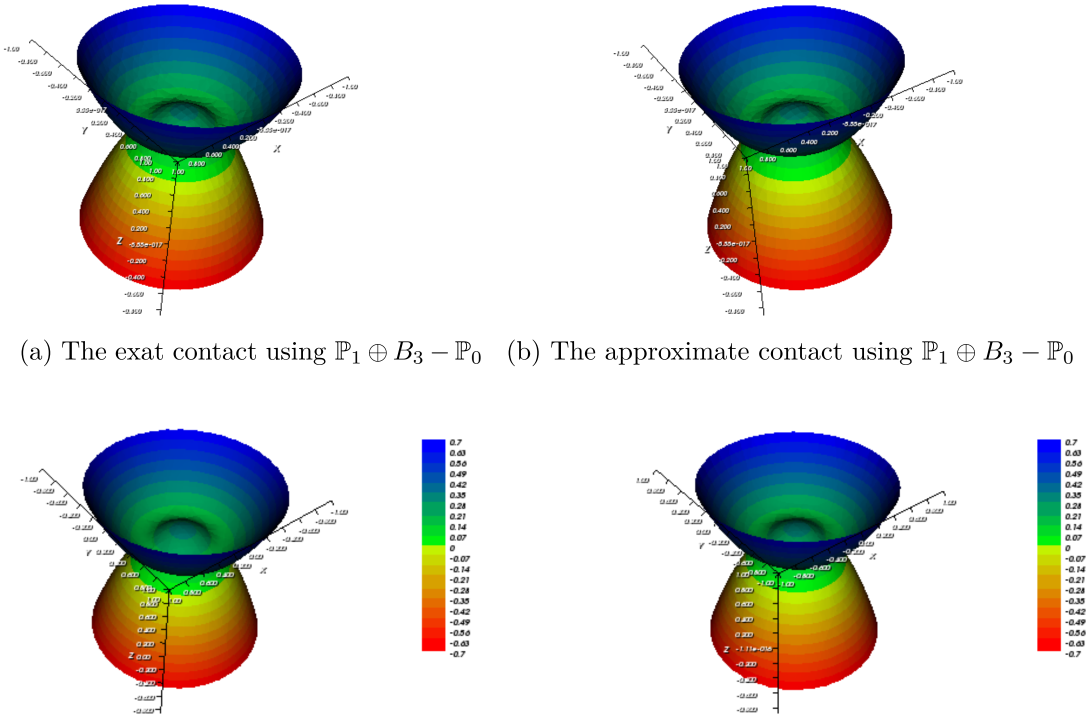
The finit element	iteration number	$\|u_h - u\|_{L^2(\omega)}$	Convergence factor (rate)
$\mathbb{P}_1 \oplus B_3 - \mathbb{P}_0$	2999	0.000153	0.98
$\mathbb{P}_2 - \mathbb{P}_1$	2559	9.99 e-6	1.37

Table 5.4: Convergence rate when $\alpha = 0.9$



(a) The mesh (b) Isovalues for u_h

Figure 5.1: The mesh and isovalues of u_h



(a) The exat contact using $\mathbb{P}_1 \oplus B_3 - \mathbb{P}_0$ (b) The approximate contact using $\mathbb{P}_1 \oplus B_3 - \mathbb{P}_0$

(c) The exact contact using $\mathbb{P}_2 - \mathbb{P}_1$ (d) The approximate contact using $\mathbb{P}_2 - \mathbb{P}_1$

Figure 5.2: The comparison between the eaxat and approximate contact

Second test

In this subsection, we examine the efficacy of the Uzawa algorithm outlined in the preceding section for addressing the discrete problem **Problem** 9. We focus on the hyperbolic paraboloid shell $S = \varphi(\omega)$, with the reference domain ω being.

$$\omega = \left\{(x, y) \in \mathbb{R}^2, \mid x \mid + \mid y \mid < 50\sqrt{2}\right\} \tag{5.27}$$

and the chart is defined by (Figure 5.3)

$$\varphi(x, y) = (x, y, \frac{x^2 - y^2}{R^2} + 1.4), \quad \text{with } R = 50\sqrt{2}. \tag{5.28}$$

The shell is clamped on $\partial\omega$, meaning we select $\gamma_0 = \partial\omega$ (thus γ_1 is empty) and it is subjected to a uniform pressure $f_3 = q = -0.25kp/cm2$.

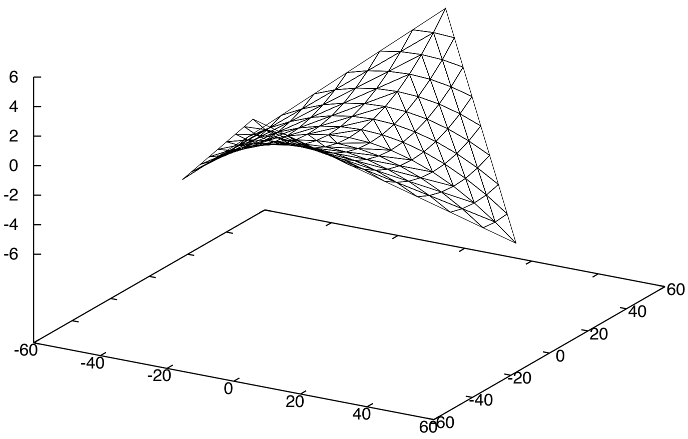


Figure 5.3: The surface $S = \varphi(\omega)$.

Alternatively, in equation (1.13), we set $f = (0, 0, q)$. Young's modulus and Poisson's ratio are $E = 2.85 \times 10^4$ kp/cm2, $\nu = 0.4$ respectively, with the shell thickness being $\varepsilon = 0.8$ cm.

Then the function Φ defined by (2.1) is expressed as

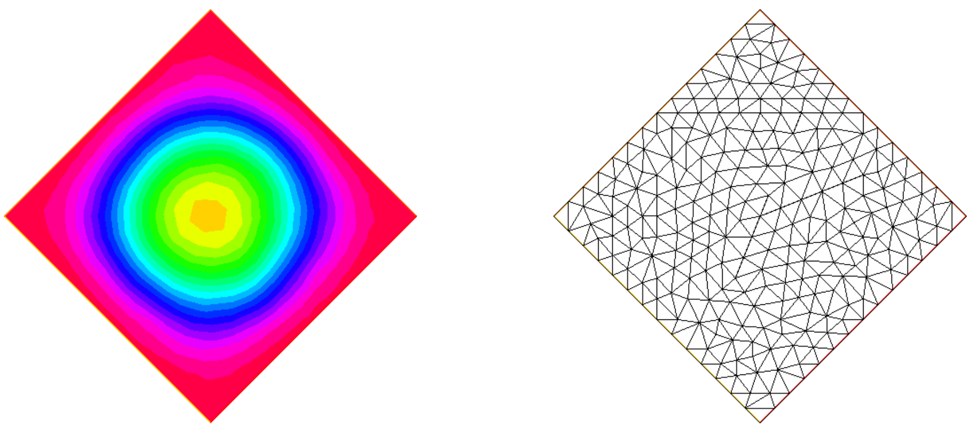
$$\Phi(x, y) = \frac{0.4R^2}{\sqrt{4(x^2 + y^2) + R^4}} - \frac{x^2 - y^2}{R^2} - 1.4.$$

It's worth noting that the function Φ (refer to Figure 5.6 (a)) meets the condition (2.2), ensuring that the surface fulfills the necessary criteria discussed in the introductory section. The numerical experiments presented here were conducted using the finite element software FreeFem++ [44]. We investigate the convergence behavior of the Uzawa method concerning the number of iterations.

The Uzawa method is widely recognized as being equivalent to a fixed-parameter first-order Richardson iteration, achieved through the elimination of the unknowns U and ψ and employing the Schur complement for the unknown λ (refer to [6]). Thus, we can employ a stopping criterion where we halt the process when $\|\lambda^{k+1} - \lambda^k\|_{L^\infty}$ becomes sufficiently small.

In our problem, the contact zone is delineated as the set of points $(x, y) \in \omega$ that satisfy $(\left(u - \frac{\varepsilon}{2}r\right) \cdot \boldsymbol{e}_3)(x, y) = \Phi(x, y)$. In contact problems, both the contact zone and the free boundary are a priori unknowns.

Nonetheless, in the considered example we have noticed that at the origin $(0, 0)$ we approximate $(\left(u_h - \frac{\varepsilon}{2}r_h\right) \cdot e_3)(0, 0) \approx \Phi_h(0, 0)$ as the number of iterations grows large. Given that the analytical expression of the function Φ_h is accessible, and through analytical calculations, we ascertain $\Phi_h(0, 0) = -1$, we have monitored the quantity $|\left(u_h - \frac{\varepsilon}{2}r_h\right) \cdot e_3)(0, 0) - \Phi_h(0, 0)|$ at various steps. The results suggest that this quantity diminishes relatively to zero at a comparable rate to $\|\lambda^{k+1} - \lambda^k\|_{L^\infty}$ (see Figure 5.8). This observation could be interpreted as follows: the nature of the applied loading and the position of the function Φ relative to the shell suggest that the origin $(0, 0)$ is part of the contact zone, at least in the context of discrete problems.

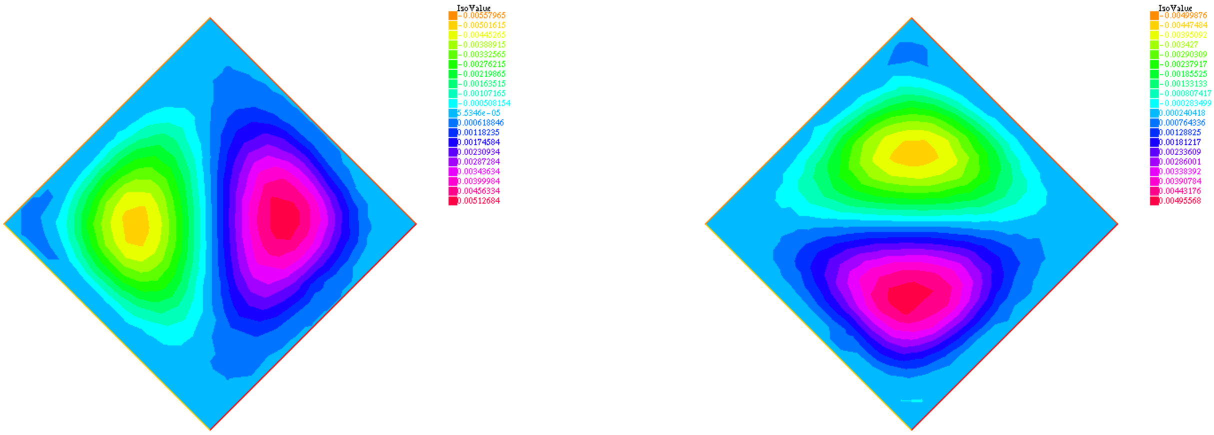


(a) isovalues for u_3 (b) The mesh

Figure 5.4: The mesh and isovalues for u_3

In Figure 5.4 (a) the isovalues for u_3 are plotted using the quasi uniform mesh shown in Figure 5.4 (b). Due to the form of the considered loading we can expect that the displacement u_3 will be larger than the tangential displacement $u_\beta, \beta = 1, 2$.

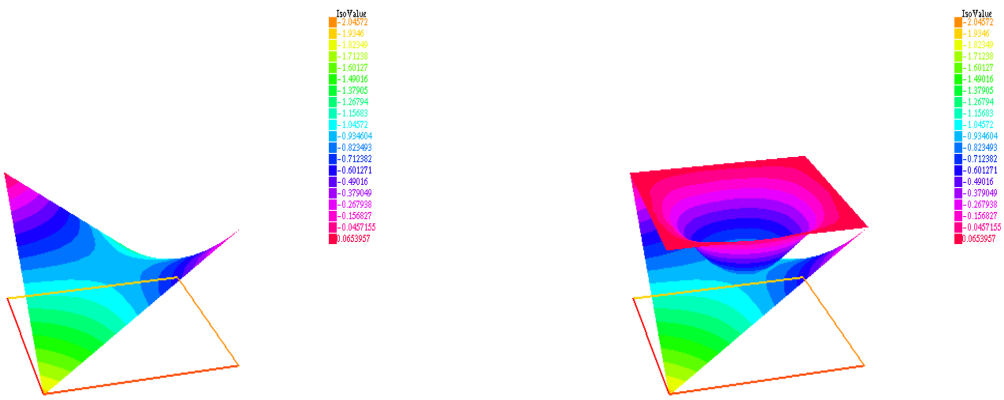
Figure 5.5 shows this significant difference between u_3 and $u_\beta, \beta = 1, 2$. Indeed the



(a) Isovalues for u_1 (b) Isovalues for u_2

Figure 5.5: isovalues for $u_\beta, \beta = 1, 2$

range of u_3 is between 0.029 and -1.09 while the values of u_1 and u_2 varies between -0.005 and 0.005.



(a) The function Φ (b) $\left(u_h - \frac{\varepsilon}{2} r_h\right) \cdot e_3$ and Φ_h

Figure 5.6: The constraint $\left(u_h - \frac{\varepsilon}{2} r_h\right) \cdot e_3$ and the functions Φ and Φ_h

The constraint $\left(u_h - \frac{\varepsilon}{2} r_h\right) \cdot e_3$ and the function Φ_h are depicted in Figure 5.6 (b).

There, we notice that the function Φ_h can be regarded as an obstacle for the unknown $\left(u_h - \frac{\varepsilon}{2} r_h\right) \cdot e_3$.

In Figure 5.7, we plot the "contact zone" and the free boundary after 350 iterations. It appears to be a connected and non-convex subset of ω containing the origin.

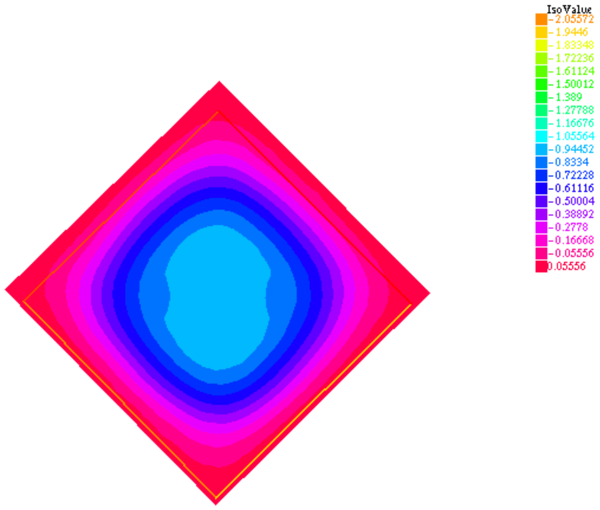
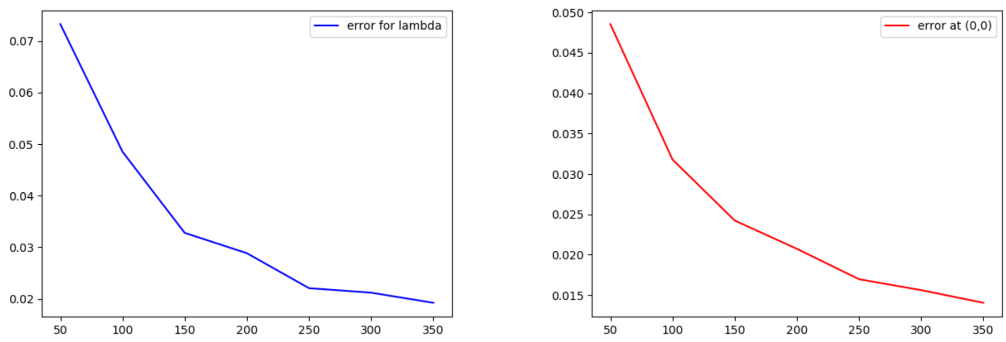


Figure 5.7: The contact zone $\{(x,y) \in \omega; \left(\left(u_h - \frac{\varepsilon}{2} r_h\right) \cdot e_3\right)(x,y) = \Phi_h(x,y)\}$.

iteration	100	150	200	250	300	350
$\|\lambda^{k+1} - \lambda^k\|_{L^2(\omega)}$	0.0485238	0.0327921	0.0288491	0.0220641	0.021193	0.0192298
Value of the constraint at the point (0,0)	0.0317884	0.024239	0.0207364	0.0169701	0.0156183	0.0140509

Table 5.5: Convergence outcomes for the Uzawa scheme using $\mathbb{P}_1 \oplus B_3 - \mathbb{P}_0$

We present in Table 5.5 and Figure 5.8, the evolution of $\|\lambda^{k+1} - \lambda^k\|_{L^\infty(\omega)}$ and of $|\left(u_h - \frac{\varepsilon}{2} r_h\right) \cdot e_3)(0,0) - \Phi_h(0,0)|$ at different iterations. Note that the number of iterations to stop the algorithm for some reasonable stopping criteria is huge. Indeed, we have



(a) $\|\lambda^{k+1} - \lambda^k\|_{L^\infty(\omega)}$ (b) $|((u_h - \frac{\varepsilon}{2}r_h) \cdot e_3)(0,0) - \Phi_h(0,0)|$

Figure 5.8: The error as a function of the number of iterations

observed that in order to obtain $\|\lambda^{k+1} - \lambda^k\|_{L^\infty(\omega)} < 10^{-6}$ or $|\left(u_h - \frac{\varepsilon}{2}r_h\right) \cdot e_3)(0,0) - \Phi_h(0,0)| < 10^{-6}$ more then 100000 iterations is needed but no pathological behavior is observed. Therefore we have preferred to present the evolution of the errors up to 350 iterations.

The convergence of the Uzawa method depends strongly on the value of the parameter α. The optimal choice of this parameter depends on the eigenvalues of the system. Based on the inequality (5.18), it must satisfy

$$0 < \alpha < \frac{2c^{\#}}{c_{c,\#}^2}$$

but the coercivity of the bilinear form depends on the parameter ρ.

Indeed, for $\rho = 0$, we have observed that the method does not converge choosing $\rho > 0$ big enough gives a large range of α for which the method converges. In order to show the influence of the parameter ρ on the performance of the algorithm we have performed numerical experiments on a mesh consists of 512 triangles, 6119 degree of freedom with fixed value of α and different values of $\rho = 10^1, 10^3, 10^5, 10^7$ and 10^9.

The results are listed in tables 5.6 and 5.8. We observe that the augmenting the value of ρ with α fixed reduces slightly the number of iterations for a given stopping criteria which is chosen $\|\lambda^{k+1} - \lambda^k\|_{L^\infty(\omega)} < 0.01$ (see table 5.8). For a fixed (large enough) value of ρ

ρ	10^3	10^5	10^7	10^9
$\|\lambda_h^{k+1} - \lambda_h^k\|_{L^2(\omega)}$	0.0119165	0.0119167	0.0119348	0.0087216
$\|((u_h - \frac{\varepsilon}{2} r_h) \cdot e_3 - \Phi_h)(0,0)\|$	0.0635194	0.0636253	0.0637142	0.0621998

Table 5.6: Convergence results for the Uzawa sheme for $\alpha = 0.01$ and differents values of ρ

we have performed numerical tests with different values of α. The results are shown in Table 5.7. Contrary to the previous case, we observe that changing α and fixing ρ large enough may affect significantly the convergence.

α	10^{-1}	10^{-2}	10^{-3}	10^{-4}
$\|\lambda_h^{k+1} - \lambda_h^k\|_{L^2(\omega)}$	0.01851140	0.01191650	0.00732018	0.00148822

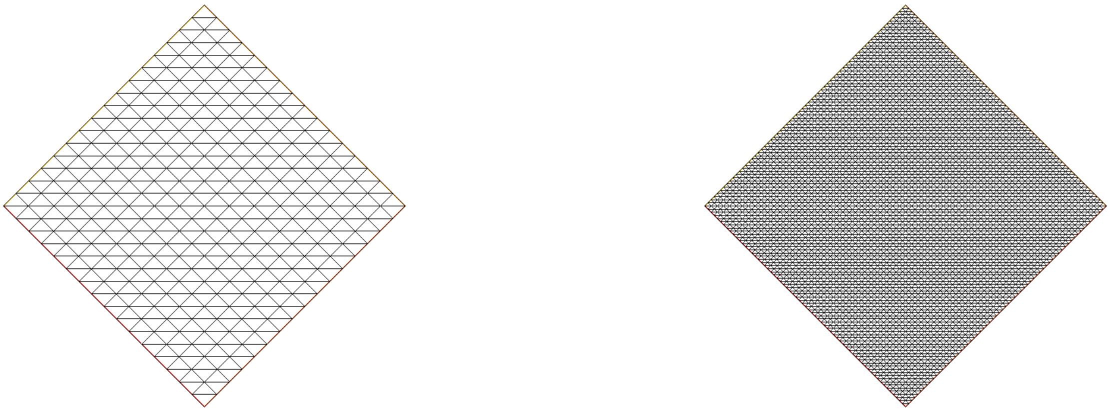
Table 5.7: Convergence results for the Uzawa sheme for $\rho = 10^3$ and differents values of α

ρ	1	10^1	10^3	10^5	10^7	10^9
Number of iterations	701	688	686	686	686	684

Table 5.8: Comparison of the number of iteration for $\alpha = 0.01$ and differents values of ρ

In the context of finite element approximation of PDEs, the rate of convergence depends strongly on the regularity of the solution of the exact solution and the degree of the used polynomials, an inverse theorem also exists (see [2]). It should be noticed that for contact problems, the limited regularity of the solution due to the unknown contact boundary limits the convergence rate. For our problem the exact solution and the a priori regularity are unknown. In order to overcome this issue, we follow the algorithm purposed in [52, Sec. 6]. Indeed, the prescribed numerical test is solved by our mixed formulation discretised using $(\mathbb{P}_1 \oplus B_3, \mathbb{P}_1 \oplus B_3, \mathbb{P}_0)$ and the Uzawa algorithm with fixed parameters $\rho = 10^3$ and $\alpha = 0.01$. The mesh is refined uniformly and the mesh size is of the form

$h_{\max} = 50\sqrt{2}/2^{(n+4)}$, $n = 0, 1, 2, 3$ (see figure 5.9). Since the different components have very different order of magnitude, we prefer to use the relative error instead of the absolute error. The resulting convergence curves are visualized in Figure 5.10 as a function of the mesh parameter. The parameter κ stands for the rate of convergence $\mathcal{O}(h^{\kappa})$.



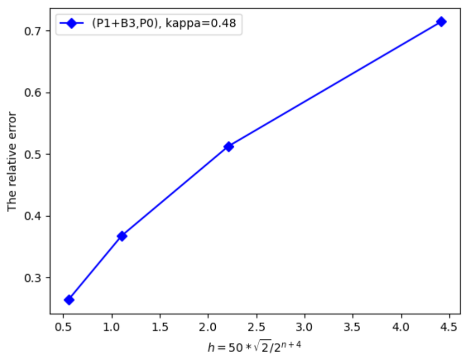
(a) $n = 0$ (b) $n = 3$

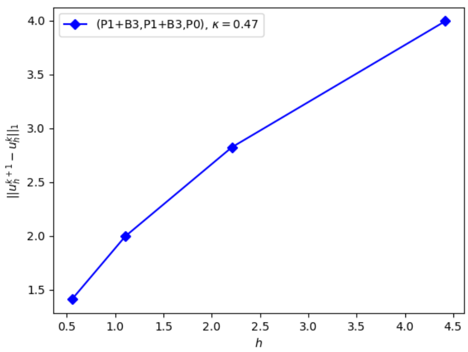
Figure 5.9: The first and the fourth meshes

The numerical test shows that the algorithm converges with a "global" rate $\kappa \approx 0.5$ and it is in good agreement with the theoretical results obtained in Corollary 3.3.7 and 3.3.3. Note that the experimental rate of convergence is the not the same for different components.

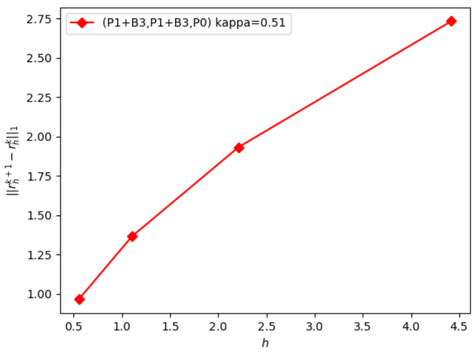
We have observed that it is smaller when comparing it the rate of convergence for of the Lagrange multipliers ψ and λ, (or even for the tangential components u_β, r_β), this is due to the fact that the regularity of components u_3 and r_3 is limited by the presence of the obstacle Φ.

It seems to be a very interesting work to investigate whether the rate of convergence can be improved by an automatic adaptive refinement strategy using a reliable and efficient a posteriori error indicator together with high order polynomial spaces like $P_k + B_{k+1}, P_{k-2}$, $k \geq 2$.

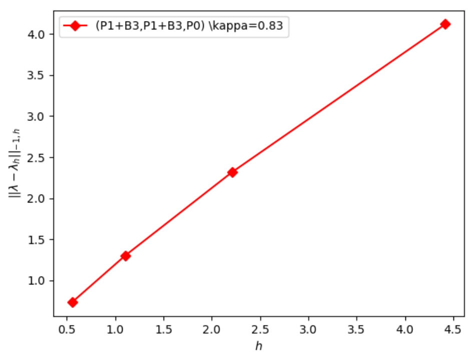




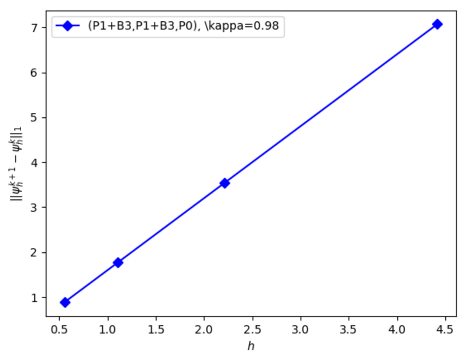
(a) $\|u_h^{k+1} - u_h^k\|_{H^1(\omega,\mathbb{R}^3)}$



(b) $\|r_h^{k+1} - r_h^k\|_{H^1(\omega,\mathbb{R}^3)}$



(c) $\|\lambda_h^{k+1} - \lambda_h^k\|_{-1,h}$



(d) $\|\psi_h^{k+1} - \psi_h^k\|_{H^1(\omega)}$

Figure 5.10: The rate of the convergence as a function of the mesh size

Conclusion

In conclusion, this research has delved into the contribution: application of iterative methods to address variational inequalities arising from the finite element approximation of the obstacle problem in Naghdi's shell formulation within Cartesian coordinates. The focus has been on finding solutions within a convex, non-linear subset, subject to additional constraints, particularly the tangency requirement on the rotation field.

Throughout the investigation, the finite element approximation has been employed to address unilateral contact problems involving elastic shells and rigid obstacles. The research has achieved significant milestones, including establishing existence and uniqueness results, stability estimates, and optimal a priori error estimates for both continuous and discrete problems.

Efficient iterative solution methods, such as the Uzawa algorithm associated with the variational inequality, have been proposed and validated through numerical tests. The study extends its scope by considering two equivalent formulations of the obstacle problem for a Naghdi shell, introducing a new perspective on the continuous problem and deriving Lagrange multipliers to enforce tangency requirements and inequality constraints.

In summary, the proposed approach has undergone rigorous validation through a series of numerical tests, encompassing various aspects such as convergence results, the behavior of the approximate solution, characteristics of the contact zone, and the rate of convergence. These tests have collectively demonstrated the effectiveness of the developed methodology in addressing unilateral contact problems within the realm of structural mechanics.

This research represents a valuable contribution, extending beyond theoretical foundations to provide practical methodologies. The comprehensive validation process undertaken serves to strengthen the reliability and applicability of the proposed approach. By successfully navigating through the complexities of structural mechanics, this work stands as a significant advancement in the field, offering solutions to intricate problems and paving the way for future developments in the understanding and resolution of similar challenges.

Future Perspectives

Several extensions and future directions emerge from this work. Potential areas of exploration include:

- High Order Finite Element Methods: Investigate the application of high-order finite element methods to enhance the accuracy and efficiency of the proposed approach.
- Adaptive Mesh Refinement: Implementing adaptive mesh refinement techniques based on a posteriori error estimates to improve the accuracy and efficiency of numerical solutions.
- Stabilized Finite Element Method: Explore the integration of stabilized finite element methods to address stability concerns and improve the robustness of the proposed algorithms.
- Iterative Solution Algorithms: Study alternative iterative solution algorithms such as inexact Uzawa methods, interior point methods, multigrid, and preconditioned Krylov subspace methods for potential improvements in computational efficiency.

- Thin Shells with Signorini Boundary Conditions: Extend the research to incorporate thin shells with Signorini boundary conditions, providing a more comprehensive understanding of the problem.

- Extension to the Koiter Shell Model: Explore the application of the developed methodology to the Koiter shell model, expanding the scope of the research to different shell models.

  These future perspectives aim to enhance the versatility, efficiency, and applicability of the proposed methods, contributing to advancements in the field of structural mechanics and variational inequalities.

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