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الإهداء:

إلى الطلاب الشرفاء طلاب العلم النبلاء إلى كل من جد و إجتهد و إستوفى الإجتهاد. إلى أحرار الأقصى حملة اللواء.

Dedication:

To the honorable students, the noble seekers of knowledge, to all those who have exerted effort and diligence. To the free ones of Al-Aqsa, bearers of the flag.

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Abbreviations and Notations

The different abbreviations and notations used throughout this dissertation are explained below :

$\mathbb{E}[X]$	Expectation at X.
$\mathbb{E}\left[X \vert \mathcal{B}\right]$	Conditional expectation of X given the σ-field $\mathcal{B}$.
$var(X)$	Variance of the random variable X.
EDS	Stochastic differential equations.
$SFDEs$	Stochastic functional differential equations.
$PSDERB$	Perturbed stochastic differential equations with reflecting boundary.
Ω	Fundamental space of a random experiment.
$\mathbb{P}$	Propability.
$(\Omega, \mathcal{F}, \mathbb{P})$	Propability space.
$\{\mathcal{F}_t\}$	Filtration.
$(\Omega, \mathcal{A}, \mathbb{P})$	Probability space.
$a.s.$	Almost surely.
$i.e$	For every integer.
$\mathbb{P}-a.s$	Almost surely in propability.
$a.e.$	Almost everywhere.
$r.v$	Random variable.
$r.c.l.l.$	Right continuous with left limits.
$\Vert f \Vert_p$	$\mathbb{L}^p$-norm of the function f.

$\mathbb{R}^d$	Euclidean real space of dimension d
$\mathbb{1}_B$	indicator function of the set B.
lim	Limit
lim ↑	Limit from the right
inf	Infinity.
sup	Supremum.
min	Minimum
max	Maximum

Table des matières

Introduction

In recent years, there has been significant interest in the Carathéodory approximation method ([9]). Researchers have explored various aspects of this technique for stochastic differential equations (SDEs). For instance, Bell and Mohammed ([1]) extended the Carathéodory approximation to SDEs and demonstrated the convergence of the approximate solutions. Mao ([6],[7]) investigated a class of SDEs with variable delays, focusing on the Carathéodory approximation of delayed SDEs. Turo ([14]) addressed the Carathéodory approximation for stochastic functional differential equations (SFDEs) and established an existence theorem for SFDEs. Liu ([5]) studied semilinear stochastic evolution equations with time delays and proved convergence of the Carathéodory approximate solutions to the solutions of stochastic delay evolution equations. Additionally, Mao ([8]) examined the Carathéodory approximation scheme for doubly perturbed stochastic differential equations (DPSDEs) and established an existence theorem for DPSDEs with non-Lipschitz coefficients. In this work, we will study the Carathéodory approximate scheme of a class of one-dimensional perturbed stochastic differential equations with reflecting boundary, This paper is divided as follows : In chaptre one, we will explain the theory of stochastic calculus by providing definitions and properties. The second chapter focuses on the study of existence and uniqueness of solution for stochastic differential equations. Finally in chaptre three we prove that (PSDERB) have a unique solution and show that the Carathéodory approximate solution converges to the solution of (PSDERB) whose both drift and diffusion coefficients are non-Lipschitz.

Chapitre 1

Stochastic analysis

1.1 Conditional Expectation [4]

1.1.1 Integrable Random Variables

Théoréme 1.1 *Let $\mathcal{B}$ be a sub-σ-field of $\mathcal{A}$, and let $X \in \mathbb{L}^1(\Omega, \mathcal{A}, \mathbb{P})$. There exists a unique element of $\mathbb{L}^1(\Omega\ , \mathcal{B}, \mathbb{P})$, which is denoted by $\mathbb{E}\left[X|\mathcal{B}\right]$, such that*

$$\forall B \in \mathcal{B},\ \mathbb{E}\left[X 1_B\right] = \mathbb{E}[\mathbb{E}[X|\mathcal{B}] 1_B]. \tag{1.1}$$

We have more generally, for every bounded $\mathcal{B}$-measurable real random variable Z,

$$\mathbb{E}[XZ] = \mathbb{E}[\mathbb{E}\left[X|B\right] Z]. \tag{1.2}$$

If $X \geq 0$, we have $\mathbb{E}\left[X|\mathcal{B}\right] \geq 0$ a.s.

The crucial point is the fact that $\mathbb{E}\left[X|\mathcal{B}\right]$ is $\mathcal{B}$-measurable. Either of properties (1.1) and (1.2) characterzes the conditional expectation $\mathbb{E}\left[X|\mathcal{B}\right]$ among random variables of $\{X\acute{} > X"\}$. In what follows, we will refer to (1.1) or (1.2) as the characteristic property of $\mathbb{E}\left[X|\mathcal{B}\right]$.

Proof. Let us start by proving uniqueness. Let X' and X' be two random variables in $\mathbb{L}^1(\Omega, \mathcal{B}, \mathbb{P})$ such that

$$\forall B \in \mathcal{B}, \mathbb{E}\left[X\text{'}1_B\right] = \mathbb{E}\left[X1_B\right] = \mathbb{E}\left[X\text{"}1_B\right].$$

Taking $B = \{X\text{'} > X\text{"}\}$ (which is in $\mathcal{B}$ since both X' and X" are $\mathcal{B}$-measurable), weget

$$\mathbb{E}\left[(X\text{'} - X\text{"})1_{\{X\text{'}>X\text{"}\}}\right] = 0,$$

which implies that $\{X\text{'} \leq X\text{"}\}$ a.s., and we have similarly $\{X\text{'} \geq X\text{"}\}$ a.s. Thus $\{X\text{'} = X\text{"}\}$ a.s., which means that X' and X" are equal as elements of $\mathbb{L}^1(\Omega, \mathcal{B}, \mathbb{P})$. Let us now turn to existence. We first assume that $X \geq 0$, and we let Q be the finite measure on $(\Omega, \mathcal{B})$ defined by

$$\forall B \in \mathcal{B}, Q(B) := \mathbb{E}\left[X1_B\right].$$

Let us emphasize that we define $Q(B)$ only for $B \in \mathcal{B}$. We may also view $\mathbb{P}$ as a probability measure on $(\Omega, \mathcal{B})$, by restricting the mapping $B \mapsto \mathbb{P}(B)$ to $B \in \mathcal{B}$, and it is immediate that $Q \ll \mathbb{P}$. The Radon-Nikodym theorem applied to the probability measures $\mathbb{P}$ and Q on the measurable space $(\Omega, \mathcal{B})$ yields the existence of a nonnegative $\mathcal{B}$-measurable random variable $\tilde{X}$ such that

$$\forall B \in \mathcal{B},\ \mathbb{E}\left[X1_B\right] = Q(B) = \mathbb{E}\left[\tilde{X}1_B\right].$$

Taking $B = \Omega$, we have $\mathbb{E}\left[\tilde{X}\right] = \mathbb{E}\left[X\right] < \infty$, and thus $\tilde{X} \in \mathbb{L}^1(\Omega, \mathcal{B}, \mathbb{P})$. The random variable $\mathbb{E}\left[X|\mathcal{B}\right] = \tilde{X}$ satisfies (1.1). When X is of arbitrary sign, we just have to take

$$\mathbb{E}\left[X|\mathcal{B}\right] = \mathbb{E}\left[X^+|\mathcal{B}\right] - \mathbb{E}\left[X^-|\mathcal{B}\right],$$

and it is clear that (1.1) also holds in that case.

Finally, to see that (1.1) implies (1.2), we rely on the usual measure-theoretic arguments. (1.2) follows from (1.1) when Z is a simple random variable (taking only finitely many values), and in the general case Proposition (3.1) allows us to write Z as the pointwise limit of a sequence

$(Z_n)_{n\in\mathbb{N}}$ of simple $\mathcal{B}-$measurable random variables that are uniformly bounded by the same constant K (such that $|Z| \leq K$) and the dominated convergence theorem yields the desired result. ■

1.1.2 Properties of Conditional Expectation

(a) If $X \in \mathbb{L}^1(\Omega, \mathcal{A}, \mathbb{P})$ and X is $\mathcal{B}$-measurable, then $\mathbb{E}[X|\mathcal{B}] = X$.

(b) The mapping $X \longmapsto \mathbb{E}[X|\mathcal{B}]$ is linear on $\mathbb{L}^1(\Omega, \mathcal{A}, \mathbb{P})$.

(c) If $X \in \mathbb{L}^1(\Omega, \mathcal{A}, \mathbb{P}), \mathbb{E}[\mathbb{E}[X|\mathcal{B}]] = \mathbb{E}[X]$.

(d) If $X \in \mathbb{L}^1(\Omega, \mathcal{A}, \mathbb{P})$, then $|\mathbb{E}[X|\mathcal{B}]| \leq \mathbb{E}[|X|\,|\mathcal{B}]$a.s. and, consequently, $\mathbb{E}[|\mathbb{E}[X|\mathcal{B}]|] \leq \mathbb{E}[|X|]$. Therefore the mapping $X \longmapsto \mathbb{E}[X|\mathcal{B}]$ is a contraction of $\mathbb{L}^1(\Omega, \mathcal{A}, \mathbb{P})$.

(e) If $X, X' \in \mathbb{L}^1(\Omega, \mathcal{A}, \mathbb{P})$ and $X \geq X'$, then $\mathbb{E}[X|\mathcal{B}] \geq \mathbb{E}[X'|\mathcal{B}]$ a.s

Proof. (a) immediately folllows from uniqueness in Theorem (1.1). Similarly, for (b), we observe that, if $X, X' \in \mathbb{L}^1(\Omega, \mathcal{A}, \mathbb{P})$ and α, $\alpha' \in \mathbb{R}$ the random variable

$$\alpha\mathbb{E}[X|\mathcal{B}] + \alpha'\mathbb{E}[X'|\mathcal{B}]$$

satisfies the characteristic property (1.1) for the conditional expectation of $\alpha X + \alpha' X'$. Property (c) is the special case $B = \Omega$ in (1.1). As for (d), using the fact that $X \geq 0$ implies, $\mathbb{E}[X|\mathcal{B}] \geq 0$ we have

$$|\mathbb{E}[X|\mathcal{B}]| = \left|\mathbb{E}\left[X^+|\mathcal{B}\right] - \mathbb{E}\left[X^-|\mathcal{B}\right]\right| \leq \mathbb{E}\left[X^+|\mathcal{B}\right] + \mathbb{E}\left[X^-|\mathcal{B}\right] = \mathbb{E}[|X|\,|\mathcal{B}]$$

Finally, (e) is immediate by linearity. ■

1.1.3 Nonnegative Random Variables

Théoréme 1.2 *Let X be a random variable with values in $[0, \infty]$. There exists a $\mathcal{B}$-measurable random variable with values in $[0, \infty]$, which is denoted by $\mathbb{E}[X|\mathcal{B}]$ and is such that, for every* *nonnegative $\mathcal{B}$-measurable random variable Z,*

$$\mathbb{E}[XZ] = \mathbb{E}[\mathbb{E}[X|\mathcal{B}]\, Z]. \tag{1.3}$$

Furthermore $\mathbb{E}[X|\mathcal{B}]$ is unique up to a $\mathcal{B}$-measurable set of probability zero.

Proof. We define $\mathbb{E}[X|\mathcal{B}]$ by setting

$$\mathbb{E}[X|\mathcal{B}] := \lim_{n\to\infty} \uparrow \mathbb{E}[X \wedge n|\mathcal{B}] \text{ a.s.}$$

This definition makes sense because, for every $n \in \mathbb{N}$, $X \wedge n$ is bounded hence integrable and thus $\mathbb{E}[X \wedge n|\mathcal{B}]$ is well defined by the previous section. Furthermore, the fact that the sequence $(\mathbb{E}[X \wedge n|\mathcal{B}])_{n\in\mathbb{N}}$ is $(a.s.)$ increasing follows from property (e) above. Then, if Z is nonnegative and $\mathcal{B}$-measurable, the monotone convergence theorem implies that

$$\mathbb{E}[\mathbb{E}[X|\mathcal{B}]\, Z] = \lim_{n\to\infty} \mathbb{E}[\mathbb{E}[X \wedge n|\mathcal{B}]\,(Z \wedge n)] = \lim_{n\to\infty} \mathbb{E}[(X \wedge n)(Z \wedge n)] = \mathbb{E}[XZ].$$

It remains to establish uniqueness. Let X' and X' be two $\mathcal{B}$-measurable random variables with values in $[0, \infty]$, such that

$$\mathbb{E}[X\text{'}Z] = \mathbb{E}[X\text{"}Z]$$

for every nonnegative $\mathcal{B}$-measurable random variable Z. Let us fix two nonnegative rationals $a < b$, and take

$$Z = 1_{\{X' \leq a < b \leq X\text{"}\}}.$$

If follows that

$$\begin{aligned} a\mathbb{P}(X\text{'} \leq a < b \leq X\text{"}) &\geq \mathbb{E}\left[1_{\{X\text{'}\leq a<b\leq X\text{"}\}}\, X\text{'}\right] \\ &\geq \mathbb{E}\left[1_{\{X\text{'}\leq a<b\leq X\text{"}\}}\, X\text{"}\right] \\ &\geq b\mathbb{P}(X\text{'} \leq a < b \leq X\text{"}) \end{aligned}$$

which is only possible if $\mathbb{P}(X\text{'}\leq a < b \leq X\text{''}) = 0$. Hence,

$$\mathbb{P}(\bigcup_{\substack{a,b\in Q_+\\ a<b}} \{X\text{'} \leq a < b \leq X\text{''}\}) = 0$$

which implies $X\text{'}\geq X\text{''}$ a.s., and interchanging the roles of X' and X" also gives $X\text{''}\geq X\text{'}$ a.s.

■

1.1.4 Properties

In the statement of the following properties, "nonnegative" means "with values in $[0, \infty]$".

(a) If X' and X' are nonnegative random variables, and $a, b \geq 0$,

$$\mathbb{E}\left[aX + bX'|\mathcal{B}\right] = a\mathbb{E}\left[X|\mathcal{B}\right] + b\mathbb{E}\left[X'|\mathcal{B}\right].$$

(b) If X is nonnegative and $\mathcal{B}$-measurable, $\mathbb{E}\left[X|\mathcal{B}\right] = X$.

(c) For any nonnegative random variable X, $\mathbb{E}\left[\mathbb{E}\left[X|\mathcal{B}\right]\right] = \mathbb{E}\left[X|\right]$.

(d) If $(X_n)_{n\in\mathbb{N}}$ is an increasing sequence of nonnegative random variables,and $X = \lim_{n\to\infty} \uparrow X_n$, *then*

$$\mathbb{E}\left[X|\mathcal{B}\right] = \lim_{n\to\infty} \uparrow \mathbb{E}\left[X_n|\mathcal{B}\right], \text{a.s.}$$

As a useful consequence, if $(Y_n)_{n\in\mathbb{N}}$ is a sequence of nonnegative random variables, we have

$$\mathbb{E}\left[\sum_{n\in\mathbb{N}} Y_n|\mathcal{B}\right] = \sum_{n\in\mathbb{N}} \mathbb{E}\left[Y_n|\mathcal{B}\right].$$

(e) If $(X_n)_{n\in\mathbb{N}}$ is any sequence of nonnegative random variables

$$\mathbb{E}\left[\liminf X_n|\mathcal{B}\right] \leq \liminf \mathbb{E}\left[X_n|\mathcal{B}\right], \text{a.s.}$$

(f) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of integrable random variables that converges a.s. to X. Assume that there exists a nonnegative random variable Z such that $|X_n| \leq Z$ a.s. for every $n \in \mathbb{N}$, and $\mathbb{E}[Z] < \infty$. Then,

$$\mathbb{E}[X|\mathcal{B}] = \lim_{n\to\infty} \mathbb{E}[X_n|\mathcal{B}], \text{a.s. and in } \mathbb{L}^1.$$

(g) (Jensen's Inequality for Conditional Expectations) If $f : \mathbb{R} \to \mathbb{R}_+$ is convex,and if $X \in \mathbb{L}^1$

$$\mathbb{E}[f(X)|\mathcal{B}] \geq f(\mathbb{E}[X|\mathcal{B}]).$$

Proof. (a) and (b) are easy using the characteristic property (1.3), and (c) is the special case $Z = 1$ in (1.3). (d) It follows from (a) that we have $\mathbb{E}[X|\mathcal{B}] \geq \mathbb{E}[Y|\mathcal{B}]$ if $X \geq Y \geq 0$. Under the assumptions of (d), we can therefore set $X' = \lim \uparrow \mathbb{E}[X_n|\mathcal{B}]$, which is a -$\mathcal{B}$-measurable random variable with values in $[0, \infty]$. Then, for every nonnegative $\mathcal{B}$-measurable random variable Z, the monotone convergence theorem gives

$$\mathbb{E}[ZX'] = \lim \uparrow \mathbb{E}[Z\mathbb{E}[X_n|\mathcal{B}]] = \lim \uparrow \mathbb{E}[ZX_n] = \mathbb{E}[ZX]$$

which implies $X' = \mathbb{E}[X|\mathcal{B}]$ thanks to the characteristic property (1.3). The second assertion in (d) follows by applying the first one to $X_n = Y_1 + ... + Y_n$. (e) Using (d), we have

$$\begin{aligned}
\mathbb{E}[\liminf X_n|\mathcal{B}] &= \mathbb{E}\left[\lim_{k\uparrow\infty} \uparrow (\inf_{n\geq k} X_n)|\mathcal{B}\right] \\
&= \lim_{k\uparrow\infty} \uparrow \mathbb{E}\left[\inf_{n\geq k} X_n|\mathcal{B}\right] \\
&\leq \lim_{k\uparrow\infty} (\inf_{n\geq k} \mathbb{E}[X_n|\mathcal{B}]) \\
&= \liminf \mathbb{E}[X_n|\mathcal{B}].
\end{aligned}$$

(f) It suffices to apply (e) twice :

$$\mathbb{E}[Z - X|\mathcal{B}] = \mathbb{E}[\liminf(Z - X_n)|\mathcal{B}] \leq \mathbb{E}[Z|\mathcal{B}] - \limsup \mathbb{E}[X_n|\mathcal{B}],$$

$$\mathbb{E}[Z + X|\mathcal{B}] = \mathbb{E}[\liminf(Z + X_n)|\mathcal{B}] \leq \mathbb{E}[Z|\mathcal{B}] + \liminf \mathbb{E}[X_n|\mathcal{B}],$$

which leads to

$$\mathbb{E}[X|\mathcal{B}] \leq \liminf \mathbb{E}[X_n|\mathcal{B}] \leq \limsup \mathbb{E}[X_n|\mathcal{B}] \leq \mathbb{E}[X|\mathcal{B}],$$

giving the desired almost sure convergence. The convergence in $\mathbb{L}^1$ is now a consequence of the dominated convergence theorem, since we have $|\mathbb{E}[X_n|\mathcal{B}]| \leq \mathbb{E}[|X_n||\mathcal{B}] \leq \mathbb{E}[Z|\mathcal{B}]$ and $\mathbb{E}[\mathbb{E}[Z|\mathcal{B}]] = \mathbb{E}[Z] < \infty$.
(g) set

$$E_f = \left\{(a,b) \in \mathbb{R}^2 : \forall x \in \mathbb{R}, f(x) \geq ax + b\right\}.$$

Then,

$$\forall x \in \mathbb{R}^2,\ f(x) = \sup_{(a,b)\in E_f} (ax+b) = \sup_{(a,b)\in E_f\cap\mathbb{Q}^2} (ax+b).$$

We can take advantage of the fact that $\mathbb{Q}^2$ is countable to disgard a countable collection of sets of probability zero and to get that, a.s.,

$$\begin{aligned}\mathbb{E}[f(X)|\mathcal{B}] &= \mathbb{E}\left[\sup_{(a,b)\in E_f\cap\mathbb{Q}^2} (aX+b)|\mathcal{B}\right] \\ &\geq \sup_{(a,b)\in E_f\cap\mathbb{Q}^2} \mathbb{E}[aX+b|\mathcal{B}] = f(\mathbb{E}[X|\mathcal{B}])\end{aligned}$$

■

1.1.5 The Special Case of Square Integrable Variables

Théoréme 1.3 *If $X \in \mathbb{L}^2(\Omega, \mathcal{A}, \mathbb{P})$, then $\mathbb{E}[X|\mathcal{B}]$ is the orthogonal projection of X on $X \in \mathbb{L}^2(\Omega, \mathcal{A}, \mathbb{P})$.*

Proof. Jensen's inequality shows that $\mathbb{E}[X|\mathcal{B}]^2 \leq \mathbb{E}[X^2|\mathcal{B}]$, a.s. This implies that $\mathbb{E}\left[\mathbb{E}[X|\mathcal{B}]^2\right] \leq \mathbb{E}[\mathbb{E}[X^2|\mathcal{B}]] = \mathbb{E}[X^2] < \infty$, and thus the random variable $\mathbb{E}[X|\mathcal{B}]$ belongs to $\mathbb{L}^2(\Omega, \mathcal{A}, \mathbb{P})$. On the other hand, for every bounded $\mathcal{B}$-measurable random variable Z,

$$\mathbb{E}[Z(X - \mathbb{E}[X|\mathcal{B}])] = \mathbb{E}[ZX] - \mathbb{E}[Z\mathbb{E}[X|\mathcal{B}]] = 0,$$

by the characteristic property (1.2). Hence $X - \mathbb{E}[X|\mathcal{B}]$ is orthogonal to the space of all bounded $\mathcal{B}$-measurablo random variables, and the latter space is dense in $\mathbb{L}^2(\Omega, \mathcal{A}, \mathbb{P})$ (for instance by Theorem(3.7)). It follows that $X - \mathbb{E}[X|\mathcal{B}]$ is orthogonal to $\mathbb{L}^2(\Omega, \mathcal{A}, \mathbb{P})$, which gives the desired result. ■

1.2 Martingals

Définition 1.1 ***(filtration)*** *A filtration on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*is an increasing family of sub-*σ*-fields*$(\mathcal{F}_n) = \{\mathcal{F}_n : n \geq 0\}$ *of* $\mathcal{F}$ *indexed by* $n \in \{0, 1, 2, ..., m, ...\}$.

Définition 1.2 M_t *is a continuous-time martingale with respect to the filtration* $\{\mathcal{F}_t\}$ *and the probability measure* $\mathbb{P}$ *if*

1. $\mathbb{E}|M_t| < \infty$ for each t;
2. $M_t$is$\mathcal{F}_t$ measurable for each t;
3. $\mathbb{E}[M_t|\mathcal{F}_s] = M_s, a.s.$,if $s < t$.

Part (2) of the definition can be rephrased as saying M_t is adapted to $\mathcal{F}_t$. If in part (3) "=" is replaced by "$\geq$" then M_t is a submartingale, and if it is replaced by "$\leq$" then we have a supermartingale.

Définition 1.3 *A sequence of random variables* $M_0, M_{1,}...$ *is called a martingale with respect to the filtration* $\{\mathcal{F}_n\}$ *if :*

– For each n, M_n is an $\mathcal{F}_n$-measurable random variable with $\mathbb{E}[|M_n|]$.

– If $m < n$, then

$$\mathbb{E}[M_n|\mathcal{F}_m] = M_m. \tag{1.4}$$

We can also write (1.4) as

$$\mathbb{E}[M_n - M_m|\mathcal{F}_m] = 0.$$

If we think of M_n as the winnings of a game, then this implies that no matter what has happened up to time m, the expected winnings in the next $n - m$ games is 0. Sometimes one just says “M_0, M_1,... is a martingale” withoutreference to the filtration. In this case, the assumed filtration is $\mathcal{F}_n$, the information in $M_0, ..., M_n$. In order to establish (1.4) it suffices to show for all n,

$$\mathbb{E}\left[M_{n+1}|\mathcal{F}_n\right] = M_n. \tag{1.5}$$

In order to see this, we can use the tower property (1.5) for conditional expectation to see that

$$\mathbb{E}\left[M_{n+2}|\mathcal{F}_n\right] = \mathbb{E}\left[\mathbb{E}\left[M_{n+2}|\mathcal{F}_{n+1}\right]|\mathcal{F}_n\right] = \mathbb{E}\left[M_{n+1}|\mathcal{F}_n\right] = M_n,$$

and so forth. Also note that if M_n is a martingale, then

$$\mathbb{E}\left[M_n\right] = \mathbb{E}\left[\mathbb{E}\left[M_n|\mathcal{F}_0\right]\right] = \mathbb{E}\left[M_0\right].$$

1.2.1 Martingale convergence theorem [3]

The martingale convergence theorem describes the behavior of a martingale M_n as $n \to \infty$.

Théoréme 1.4 *(**Martingale Convergence Theorem**). Suppose M_n is a martingale with respect to $\{\mathcal{F}_n\}$ and there exists $C < \infty$ such that $\mathbb{E}\left[|M_n|\right] \leq C$ for all n. Then there exists a random variable M_∞ such that with probability one*

$$\lim_{n\to\infty} M_n = M_\infty.$$

It does not follow from the theorem that $\mathbb{E}\left[M_\infty\right] = \mathbb{E}\left[M_0\right]$. For example, the martingale betting strategy satisfies the conditions of the theorem since

$$\mathbb{E}\left[|W_n|\right] = \left(1 - 2^{-n}\right) \cdot 1 + \left(2^n - 1\right) \cdot 2^{-n} \leq 2.$$

However, $W_\infty = 1$ and $W_0 = 0$.

We will prove the martingale convergence theorem. The proof uses a well-known financial strategy—buy low, sell high. Suppose $M_0, M_1, ...$ is a martingale such that

$$\mathbb{E}\left[|W_n|\right] \leq C \leq \infty$$

for all n. Suppose $a < b$ are real numbers. We will show that it is impossible for the martingale to fluctuate infinitely often below a and above b. Define a sequence of stopping times by

$$S_1 = \min\left\{n : M_n \leq a\right\}, \; T_1 = \min\left\{n > S_1 : M_n \geq b\right\}$$

and for $j > 1$,

$$S_j = \min\left\{n > T_{j-1} : M_n \leq a\right\},$$

$$T_j = \min\left\{n > S_j : M_n \geq b\right\}.$$

We set up the discrete stochastic integral

$$W_n = \sum B_k \left[M_k - M_{k-1}\right],$$

with $B_n = 0$ if $n - 1 < S_1$ and

$$B_n = 1 \text{ if } S_j \leq n - 1 < T_j,$$

$$B_n = 0 \text{ if } T_j \leq n - 1 < S_{j+1}.$$

In other words, every time the "price" drops below a we buy a unit ofthe asset and hold onto it until the price goes above b at which time we sell. Let Un denote the number of times by time n that we have seen a fluctuation ; that is,

$$U_n = j \text{ if } T_j \leq n \leq T_{j+1}.$$

We call U_n the number of upcrossings by time n. Eery upcrossing results in a profit of at least $b - a$. From this we see that

$$W_n \geq U_n (b - a) + (M_n - a).$$

The term $a - M_n$ represents a possible loss caused by holding a share of the asset at the current time. Since Wn is a martingale, we know that $\mathbb{E}[W_n] = \mathbb{E}[W_0] = 0$, and hence

$$\mathbb{E}[U_n] \leq \frac{\mathbb{E}[a - M_n]}{b - a} \leq \frac{|a| + \mathbb{E}[|M_n|]}{b - a} \leq \frac{|a| + C}{b - a}.$$

This holds for every n, and hence

$$\mathbb{E}[U_n] \leq \frac{|a| + C}{b - a} < \infty.$$

In particular with probability one, $U_n < \infty$, and hence there are only a finite number of fluctuations. We now allow a, b to run over all rational numbers to see that with probability one,Therefore, the limit

$$M_n = \lim_{n \to \infty} M_n$$

exists. We have not yet ruled out the possibility that M_∞ is $\pm\infty$, but it is not difficult to see that if this occurred with positive probability, then $\mathbb{E}[|M_n|]$ would not be uniformly bounded. To illustrate the martingale convergence theorem, we will consider an other example of a martingale called Polya's urn. Suppose we have an urn with red and green balls. At time $n = 0$, we start with one red ball and one green ball. At each positive integer time we choose a ball at random from the urn (with each ball equally likely to be chosen), look at the color of the ball, and then put the ball back in with another ball of the same color. Let R_n, G_n denote the number of red and green balls in the urn after the draw at time n so that

$$R_0 = G_0 = 1, R_n + G_n = n + 2,$$

and let

$$M_n = \frac{R_n}{R_n + G_n} = \frac{R_n}{n+2}$$

be the fraction of red balls at this time. Let $\mathcal{F}_n$ denote the information in the data $M_1, ..., M_n$,, which one can check is the same as the information in $R_1, R_2, ..., R_n$. Note that the probability that a red ball is chosen at time n depends only on the number (or fraction) of red balls in the urn before choosing. It does not depend on what order the red and green balls were putin. This is an example of the Markov property. This concept will appear a number of times for us, so let us define it.

1.2.2 Square intrgrable martingales [3]

Définition 1.4 *A martingale M_n is called square integrable if for each $n, \mathbb{E}\left[M_n^2\right] < \infty$. Note that this condition is not as strong. We do not require that there exists a $c < \infty$ such that for each n. Random variables X, Y are orthogonal if $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$. Independent random variables are orthogonal, but orthogonal random variables need not be independent. If $X_1, ..., X_n$ are pairwise orthogonal random variables with mean zero, thenand by expanding the square we can see that*

$$\mathbb{E}\left[(X_1, ..., X_n)^2\right] = \sum_{j=1}^{n} \mathbb{E}\left[X_j^2\right].$$

This can be thought of as a generalization of the Pythagorean theorem $a^2 + b^2 = c^2$ for right triangles. The increments of a martingale are not necessarily independent, but for square integrable martingales they are orthogonal as we now show.

Proposition 1.1 *Suppose that M_n is a square integrable martingale with respect to $\{\mathcal{F}_n\}$. Then if $m < n$,*

$$\mathbb{E}\left[(M_{n+1} - M_n)(M_{m+1} - M_m)\right] = 0.$$

Moreover, for all n,

$$\mathbb{E}\left[M_n^2\right] = \mathbb{E}\left[M_0^2\right] + \sum_{j=1}^{n} \mathbb{E}\left[(M_j - M_{j-1})^2\right].$$

Proof. If $m < n$, $\mathcal{F}_n$-measurable, and hence

$$\mathbb{E}\left[(M_{n+1} - M_n)(M_{m+1} - M_m) \mid \mathcal{F}_n\right] = (M_{m+1} - M_m)\,\mathbb{E}\left[M_{n+1} - M_n \mid \mathcal{F}_n\right] = 0$$

Hence

$$\mathbb{E}\left[(M_{n+1} - M_n)(M_{m+1} - M_m)\right] = \mathbb{E}\left[\mathbb{E}\left[(M_{n+1} - M_n)(M_{m+1} - M_m) \mid \mathcal{F}_n\right]\right] = 0$$

Also, if we set $M_{-1} = 0$,

$$M_n^2 = \left[M_0 + \sum_{j=1}^{n} \mathbb{E}\left[(M_j - M_{j-1})\right]\right]^2 = M_0^2 + \sum_{j=1}^{n} (M_j - M_{j-1})^2 + \sum_{j \neq k} (M_j - M_{j-1})(M_k - M_{k-1}).$$

Taking expectations of both sides gives the second conclusion. ■

1.3 Brownian motion [3]

Définition 1.5 *Brownian motion or the Wiener process is a model of random continuous motion. We will start by making the assumptions that underlie the phrase "random continuous motion". Let $B_t = B(t)$ be the value at time t. For each t, B_t is a random variable. A collection of random variables indexed by time is called a stochastic process. We can view the process in two different ways :*

– For each t, there is a random variable B_t,and there are correlations between the values at different times.

– The function $t \mapsto B(t)$ is a random function. In other words, it is a random variable whose value is a function.

There are three major assumptions about the random variables B_t.

– **Stationary increments**. If $s < t$, then the distribution of $B_t - B_s$ is the same as that of $B_{t-s} - B_0$.

- **Independent increments**. If $s < t$, the random variable $B_t - B_s$ is independent of the values B_r for $r \leq s$.
- **Continuous paths**. The function $t \mapsto B_t$ is a continuous function of t.

We often assume $B_0 = 0$ for convenience, but we can also take other initial conditions. All of the assumptions above are very reasonable for a model of random continuous motion. However, it is not obvious that these are enough assumptions to characterize our process uniquely. It turns out that they do up to two parameters. that if B_t is a process satisfying the three conditions above, then the distribution of B_t for each t must be normal. Suppose B_t is such a process, and let m, σ^2be the mean and variance of B_1. If $s < t$, then independent, identically distributed increments imply that

$$\mathbb{E}\left[B_t\right] = \mathbb{E}\left[B_s\right] + \mathbb{E}\left[B_t - B_s\right] = \mathbb{E}\left[B_s\right] + \mathbb{E}\left[B_{t-s}\right],$$

$$Var\left[B_t\right] = Var\left[B_s\right] + Var\left[B_t - B_s\right] = Var\left[B_s\right] + Var\left[B_{t-s}\right].$$

Using this relation, we can see that $\mathbb{E}\left[B_t\right] = tm, Var\left[B_t\right] = t\sigma^2$. At this point, we have only shown that if a process exists, then the increments must have a normal distribution. We will show that such a process exists. It will be convenient to put the normal distribution in the defintion.

Définition 1.6 *A stochastic process B_t is called a (one-dimensional) Brownian motion with drift m and variance (parameter) σ^2 starting at the origin if it satisfies the following*

- $B_0 = 0$.
- For $s < t$, the distribution of $B_t - B_s$ is normal with mean $m(t-s)$ and variance $\sigma^2(t-s)$.
- If $s < t$, the random variable $B_t - B_s$ is independent of the values B_r for $r \leq s$.
- With probability one, the function $t \mapsto B_t$ is a continuous function of t.

  If $m = 0, \sigma^2 = 1$, then B_t is called a standard Brownian motion.

  Recall that if Z has a $N(0,1)$distribution and $Y = \sigma Z + m$, then Y has a $N(m, \sigma^2)$ distribution. Given that it is easy to show the following.

– If B_t is a standard Brownian motion and

$$Y_t = \sigma B_t + mt.$$

then Y_t is a Brownian motion with drift m and variance σ.

Indeed, one just checks that it satisfies the conditions above. Hence, in or der to establish the existence of Brownian motion, it suffices to construct a standard Brownian motion.

There is a mathematical challenge in studying stochastic processes in dexed by continuous time. The problem is that the set of positive real num bers is uncountable, that is, the elements cannot be enumerated $t_1, t_2, ...$ The major axiom of probability theory is the fact that if $A_1, A_2, ...$ is a countable sequence of disjoint events, then

$$\mathbb{P}\left[\bigcup_{n=1}^{\infty} A_n\right] = \sum_{n=1}^{\infty} \mathbb{P}\left[A_n\right].$$

This rule does not hold for uncountable unions. An example that we have all had to deal with arises with continuous random variables. Suppose, for instance, that Z has a $N(0, 1)$ distribution. Then for each

$$x \in \mathbb{R}, \mathbb{P}\left\{z = x\right\} = 0.$$

However,

$$1 = \mathbb{P}\left\{z \in \mathbb{R}\right\} = \mathbb{P}\left[\bigcup_{x\in\mathbb{R}} A_x\right],$$

where A_x denotes the event $\{x = z\}$. The events A_x are disjoint, each with probability zero, but it is not the case that

$$\mathbb{P}\left[\bigcup_{x\in\mathbb{R}} A_x\right] = \sum \mathbb{P}(A_x) = 0.$$

In constructing Brownian motion, we use the fact that if $g : [0, \infty) \to \mathbb{R}$ is a continuous function and we know the value of g on a countable, dense set, such as the dyadic rationals

$$\left\{ \frac{k}{2^n} : k = 0, 1, ...; n = 0, 1, ... \right\},$$

then we know the value at every t. Indeed, we need only find a sequence of dyadic rationals tn that converge to t, and let

$$g(t) = \lim_{t_n \to t} g(t_n).$$

This is fine if a priori we know the function g is continuous. Our strategy formconstructing Brownian motion will be :

- First define the process B_t when t is a dyadic rational.
- Prove that with probability one, the function $t \to B_t$ is continuous on the dyadics (this is the hardest step, and we need some care in the definition of continuity).
- Extend B_t to other t by continuity.

The next section shows that one can construct a Brownian motion. Thereader can skip this section and just have faith that such a process exists.

Proposition 1.2 *With probability one, for all $\alpha < \frac{1}{2}$,*

$$\lim_{n \to \infty} 2^{\alpha n} k_n = 0. \tag{1.6}$$

In particular, $K_n \to 0$.

In order to prove this proposition, it is easier to consider another sequence of random variables

$$J_n = \max_{j=1,\dots,2^n} Y(j, n)$$

where $Y(j, n)$ equals

$$\sup \left\{ \left| B - B_{(j-1)2^{-1}} \right| : q \in D, (j-1)2^{-n} \leq q \leq j2^{-n} \right\}.$$

A simple argument using the triangle inequality shows that $K_n \leq 3J_n$. It turns out J_n is *easier to analyze. For any* $\varepsilon > 0$,

$$\mathbb{P}\{J_n \geq \varepsilon\} \leq \sum_{j=1}^{2^n} \mathbb{P}\{Y(j,n) \geq \varepsilon\} = 2^n \mathbb{P}\{Y(1,n) \geq \varepsilon\}.$$

Also the distribution of

$$Y(1,n) = \sup\left\{|B_q| : q \in D, q \leq j2^{-n}\right\}.$$

is the same as that of $2^{-n/2Y}$ *where*

$$Y = Y(1,0) = \sup\{|B_q| : q \in D\}.$$

Using this we see that

$$\mathbb{P}\left\{J_n \geq C\sqrt{n}2^{-n/2}\right\} \leq 2^n \mathbb{P}\left\{Y \geq C\sqrt{n}\right\}.$$

We will show below that if $C > \sqrt{2\log 2}$, *then*

$$\sum_{n=1}^{\infty} 2^n \mathbb{P}\left\{Y \geq C\sqrt{n}\right\} < \infty. \tag{1.7}$$

The Borel-Cantelli lemma then implies that with probability one, the event $\left\{J_n \geq C\sqrt{n}2^{-n/2}\right\}$ *happens for only finitely many values of n. In particular,*

$$\lim_{n\to\infty} 2^{n/2} n^{-1} J_n = 0,$$

which implies. To get our estimate, we will need the following lemma which is a form of the "reflection principle" for Brownian motion.

Théoréme 1.5 *With probability one, for all* $\alpha < 1/2$, B_t *is Hölder continuous of order* α *but it is not Hölder continuous of order* $1/2$.

We will be using Brownian motion and functions of Brownian motions to model prices of assets. In all of the Brownian models, the functions will have Hôlder exponent $1/2$.

Hence, we could find a positive integer $M < \infty$ *such that for all sufficiently large integers* n, *there exists* $K \leq n$ *such that* $Y_{k,n} \leq M/n$, *where* $Y_{k,n}$ *is*

$$\max\left\{\left|B(\frac{k+1}{n}) - B(\frac{k}{n})\right|, \left|B(\frac{k+2}{n}) - B(\frac{k+1}{n})\right|, \left|B(\frac{k+3}{n}) - B(\frac{k+2}{n})\right|\right\}.$$

Let $Y_n = \min\{Y_{k,n} : k = 0, 1, ..., n-1\}$ *and let* A_M *be the event that for all* n *sufficiently large,* $Y_n \leq M/n$. *For each positive integer* M,

$$\begin{aligned}
\mathbb{P}\{Y_{k,n} \leq M/n\} &= [\mathbb{P}\{|B(1/n)| \leq M/n\}]^3 \\
&= \left[\mathbb{P}\left\{n^{-1/2}|B_1| \leq M/n\right\}\right]^3 \\
&= \left[\int \frac{1}{\sqrt{2\pi}} \exp(-y^2/2)dy\right]^3 \\
&\leq \left[\frac{2M}{\sqrt{n}} \frac{1}{\sqrt{2\pi}}\right]^3 \leq \frac{M^3}{n^{3/2}},
\end{aligned}$$

and hence,

$$\mathbb{P}\{Y_n \leq M/n\} \leq \sum_{k=0}^{n-1} \mathbb{P}\{Y_{k,n} \leq M/n\} \leq \frac{M^3}{n^{1/2}} \to 0.$$

This shows that $\mathbb{P}(A_M)$ *for each* M, *and hence*

$$\mathbb{P}\left[\bigcup_{M=1}^{\infty} A_M\right] = 0$$

But our first remark shows that the event that B_t *is differentiable at some point is contained in* $\cup_M A_M$.

1.3.1 Brownian motion as a continuous martingale

The definition of a martingale in continuous time is essentially the same as in discrete time. Suppose we have an increasing filtration $\{\mathcal{F}_t\}$ of information and integrable random variables M_t such that for each t, M_t is $\mathcal{F}_t$-measurable. (We say that Mt is adapted to the filtration if M_t is $\mathcal{F}_t$-measurable for each t.) Then, M_t is a martingale with respect to $\{\mathcal{F}_t\}$ if for each $s < t$,

$$\mathbb{E}\left[M_t \mid \mathcal{F}_t\right] = M_s$$

When one writes an equality as above there is an implicit "up to an event of probability zero". In discrete time this presents no problem because there are only a countable number of pairs of times (s, t) and hence there can be only a countable number of sets of measure zero. For continuous time, there are instances where some care is needed but we will not worry about this at the moment. As in the discrete case, if the filtration is not mentioned explicitly then one assumes that $\mathcal{F}_t$ is the information contained in $\{M_s : s \leq t\}$. In that case, if B_t is a standard Brownian motion and $s < t$,

$$\mathbb{E}\left[B_t \mid \mathcal{F}_s\right] = \mathbb{E}\left[B_s \mid \mathcal{F}_s\right] + \mathbb{E}\left[B_t - B_s \mid \mathcal{F}_s\right] = B_s + \mathbb{E}\left[B_t - B_s\right] = B_s. \tag{1.8}$$

Often we will have more information at time t than the values of the Brownian motion so it is useful to extend our definition of Brownian motion. We say that B_t is Brownian motion with respect to the filtration $\{\mathcal{F}_t\}$ if each B_t is $\mathcal{F}_t$-measurable and B_t satisfies the conditions to be a Brownian motion with the third condition being replaced by

– If $s < t$, the random variable $B_t - B_s$ is independent of $\mathcal{F}_s$.

In other words, although we may have more information at time s than the value of the Brownian motion, there is nothing useful for predicting the future increments. Under these conditions, (1.8) holds and B_t is a martingale with respect to $\{\mathcal{F}_t\}$.

If M_s, $0 \leq s \leq t$ is a martingale, then by definition, for each $s \leq t$,

$$\mathbb{E}\left(Y \mid \mathcal{F}_s\right) = M_s$$

where $Y = M_t$.Conversely, if Y is an integrable random variable that is measurable with respect to $\mathcal{F}_t$, we can define a martingale M_s, $0 \leq s \leq t$ by

$$\mathbb{E}(Y \mid \mathcal{F}_s) = M_s.$$

Indeed, if we define M_t as above and $r < s$, then the tower property for conditional expectation implies that

$$\mathbb{E}(M_s \mid \mathcal{F}_r) = \mathbb{E}(\mathbb{E}(Y \mid \mathcal{F}_s) \mid \mathcal{F}_s) = \mathbb{E}(Y \mid \mathcal{F}_r) = M_r.$$

A martingale M_t is called a continuous martingale if with probability one the function $t \to M_t$ is a continuous function. The word continuous in continuous martingale refers not just to the fact that time is continuous but also to the fact that the paths are continuous functions of t. One can have martingales in continuous time that are not continuous martingales.

1.3.2 Brownian motion as a Gaussian process

A process $\{X_t\}$ is called a Gaussian process if each finite subcollection

$$(X_{t_1}, ..., X_{t_n})$$

has a joint normal distribution. Recall that to describe a joint normal distri bution one needs only give the means and the covariances. Hence the finite dimensional distributions of a Gaussian process are determined by the num bers

$$m_t = \mathbb{E}[X_t], \Gamma_{st} = \mathbb{COV}(X_t, X_t).$$

Is a standard Brownian motion and $t_1, t_2, ..., t_n$, then we can write B_{t_1},...,B_{t_n} as linear combinations of the independent standard normal random variables

$$z_j = \frac{B_{t_j} - B_{t_{j-1}}}{\sqrt{t_j - t_{j-1}}}, j = 1, ..., n$$

Hence B_t is a Gaussian process with mean zero. If $s < t$,

$$\begin{aligned}\mathbb{E}\left[B_s B_t\right] &= \mathbb{E}\left[B_s(B_s + B_t - B_s)\right] \\ &= \mathbb{E}\left[B_s^2\right] + \mathbb{E}\left[B_s\left(B_t - B_s\right)\right] \\ &= s + \mathbb{E}\left[B_s\right]\mathbb{E}\left[B_t - B_s\right] = s,\end{aligned}$$

which gives the general rule

$$\mathbb{COV}(B_s, B_t) = \min\left\{s, t\right\}.$$

The description of Brownian motion as a Gaussian process describes only the finite-dimensional distributions but our definition includes some aspects that depend on more than finite-dimensional distributions. In particular, one cannot tell from the finite-dimensional distributions alone whether or not the paths are continuous.

1.4 The Itô's Integral [13]

1.4.1 Quadratic Variation of Brownian Motion

Let (W_t) denote a one-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We have seen that W is a martingale W, its natural filtration $(\mathcal{F}_\cdot^W)$ and with $M_t = W_t^2 - t$, M is also $(\mathcal{F}_\cdot^W)$-martingale. These properties are easy consequence of the independent increment property of Brownian motion.

Wiener and Ito's realized the need to give a meaning to limit of what appeared to be Riemann–Stieltjes sums for the integral

$$\int_0^t f_s dW_s \tag{1.9}$$

in different contexts—while in case of Wiener, the integrand was a deterministic function, Itô's needed to consider a random process (f_s) that was a non-anticipating function of W—i.e. f is adapted to $(\mathcal{F}_\cdot^W)$.

It is well known that paths $s \mapsto W_s(\omega)$ are nowhere differentiable for almost all ω, and hence we cannot interpret the integral in (1.9) as a path-by-path Riemann–Stieltjes integral. We will deduce the later from the following result that is relevant for stochastic integration.

Théoréme 1.6 *Let (W_t) be a Brownian motion. Let $t_1^n = i2^{-2}$. Let $V_t^n = \sum_{i=0}^{\infty} \left| W_{t_{i+1}^n \wedge t} - W_{t_i^n \wedge t} \right|$, $Q_t^n = \sum_{i=0}^{\infty} (W_{t_{i+1}^n \wedge t} - W_{t_i^n \wedge t})^2$. Then for all $t > 0$,*

(a) $V_t^n \to \infty$ *a.s.*

(b) $Q_t^n \to t$ *a.s.*

Proof. We will first prove (b). Let us fix $t < \infty$ and let

$$X_i^n = W_{t_{i+1}^n \wedge t} - W_{t_i^n \wedge t}, Z_i^n = (W_{t_{i+1}^n \wedge t} - W_{t_i^n \wedge t})^2 - (t_{i+1}^n \wedge t - t_i^n \wedge t).$$

then from properties of Brownian motion it follows that $\{X_i^n, i \geq 0\}$ are independent random variables with normal distribution and $\mathbb{E}(X_i^n) = 0$, $\mathbb{E}(X_i^n) = (t_{i+1}^n \wedge t - t_i^n \wedge t)$. So, $\{Z_i^n, i \geq 0\}$ are independent random variables with $\mathbb{E}(Z_i^n) = 0$ and $\mathbb{E}(Z_i^n)^2 = (t_{i+1}^n \wedge t - t_i^n \wedge t)^2$. Now

$$\begin{aligned} \mathbb{E}(Q_t^n - t)^2 &= \mathbb{E}(\sum_{i=0}^{\infty} Z_t^n)^2 \\ &= \sum_{i=0}^{\infty} \mathbb{E}(Z_t^n)^2 \\ &= 2\sum_{i=0}^{\infty} (t_{i+1}^n \wedge t - t_i^n \wedge t)^2 \\ &\leq 2^{-2+1} \sum_{i=0}^{\infty} (t_{i+1}^n \wedge t - t_i^n \wedge t) \\ &= 2^{-2+1} t. \end{aligned} \tag{1.10}$$

Note that each of the sum appearing above is actually a finite sum. Thus

$$\mathbb{E}\sum_{i=0}^{\infty} (Q_t^n - t)^2 \leq t < \infty$$

so that $\sum_{i=0}^{\infty} (Q_t^n - t)^2 < \infty$. and hence $Q_t^n \to t$ a.s.

For (a), let $\alpha(\delta, \omega, t) = \sup \{ |W_u(\omega) - W_v(\omega)| : |u - v| \leq \delta, u, v \in [0, \infty] \}$.

Then uniform continuity of $u \mapsto W_u(\omega)$ implies that for all t finite and for each

$$\lim_{\delta \downarrow 0} \alpha(\delta, \omega, t) = 0. \tag{1.11}$$

Now note that for any ω,

So if $\liminf_n V_t^n(\omega)$ for some ω, then $\liminf_n Q_t^n(\omega) = 0$ in view of (**??**) and (**??**). For $t > 0$, since $Q_t^n \to t$ a.s., we must have $V_t^n \to \infty$ a.s. ∎

1.4.2 Itô's Integral

Let $\mathbb{S}$ be the class of stochastic processes f of the form

$$f_s(\omega) = a_0(\omega)\imath_{\{0\}}(s) + \sum_{j=0}^{m} a_{j+1}(\omega)\imath_{(s_j, s_{j+1}]}(s) \tag{1.12}$$

where $0 = s_0 < s_1 < s_2 < ... < s_{m+1} < \infty$, a_j is bounded $\mathcal{F}_{s_{j-1}}$ measurable random variable for $1 \leq j \leq (m+1)$, and a_0 is bounded $\mathcal{F}_0$-measurable. Elements of $\mathbb{S}$ will be called simple processes. For an f given by (1.12), we define $X = \int f dW$ by

$$X_t(\omega) = \sum_{j=0}^{m} a_{j+1}(\omega)(W_{s_{j+1} \wedge t}(\omega) - W_{s_j \wedge t}(\omega)). \tag{1.13}$$

a_0 does not appear on the right side because $W_0 = 0$. It can be easily seen that $\int f dW$ defined via (1.12) and (1.13) for $f \in \mathbb{S}$ does not depend upon the representation(1.12). In other words, if g is given by

$$g_t(\omega) = b_0(\omega)\imath_{\{0\}}(s) + \sum_{j=0}^{n} b_{j+1}(\omega)\imath_{(r_j, r_{j+1}]}(t) \tag{1.14}$$

where $0 = r_0 < r_1 < r_2 < ... < r_{n+1}$ and b_j is $\mathcal{F}_{r_{j-1}}$-measurable bounded random variable,$1 \leq j \leq (n+1)$, and b_0 is bounded $\mathcal{F}_0$-measurable and $f = g$, then$\int fdW = \int gdW$, i.e.

$$\sum_{j=0}^{m} a_{j+1}(\omega)(W_{s_{j+1} \wedge t}(\omega) - W_{s_j \wedge t}(\omega)) = \sum_{j=0}^{n} b_{j+1}(\omega)(W_{r_{j+1} \wedge t}(\omega) - W_{r_j \wedge t}(\omega)). \tag{1.15}$$

By definition, X is a continuous adapted process. We will denote X_t as $\int_0^t f dW$.

We will obtain an estimate on the growth of the integral defined above for simple $f \in \mathbb{S}$ and then extend the integral to an appropriate class of integrands—those that can be obtained as limits of simple processes. This approach is different from the one adopted by Itô's, and we have adopted this approach with an aim to generalize the same to martingales.

We first note some properties of $f dW$ for $f \in \mathbb{S}$ and obtain an estimate.

Lemme 1.1 *Let $f, g \in \mathbb{S}$ and let $a, b \in \mathbb{R}$. Then*

$$\int_0^t (af + bg) dW = a \int_0^t f dW + b \int_0^t g dW. \tag{1.16}$$

Proof. Let f, g have representations (1.12) and (1.14), respectively. Easy to see that $0 = t_0 < t_1 < ... < t_k$ such that

$$\{t_j : 0 \leq j \leq k\} = \{s_j : 0 \leq j \leq m\} \cup \{r_j : 0 \leq j \leq n\}$$

and then represent both f, g over common time partition. Then the result (1.16) follows easily. ■

Lemme 1.2 *Let $f, g \in \mathbb{S}$, and let $Y_t = \int_0^t f dW$, $Z_t = \int_0^t g dW$ and $A_t = \int f_s g_s ds$, $M_t = Y_t Z_t - A_t$. Then Y, Z, M are $(\mathcal{F})$-martingales.*

Proof. By linearity property (1.16) and the fact that sum of martingales is a martin gale, suffices to prove the lemma in the following two cases :

Case 1: $0 \leq s < r$ and

$$f_t = a \imath_{(s,r]}(t), \;\; g_t = b \imath_{(s,r]}(t), \;\; a, \;\; b \text{ are } \mathcal{F}_s\text{-measurable.}$$

Case 2: $0 \leq s < r \leq u < v$ and

$$f_t = a \,1_{(s,r]}\,(t), \;\; g_t = b \,1_{(u,v]}\,(t), \;\; a \text{ is } \mathcal{F}_s\text{-measurable and } b \text{ is } \mathcal{F}_u\text{-measurable.}$$

Here in both cases, a, b are assumed to be bounded. In both the cases, $Y_t = a(W_{t\wedge r} - W_{t\wedge s})$. That Y is a martingale follows from Theorem(3.8). Thus in both cases, Y is an $(\mathcal{F}_\cdot)$-martingale and similarly, so is Z. Remains to show that M is a martingale. In case 1, writing $N_t = W_t^2 - t$

$$\begin{aligned} M_t &= ab((W_{t\wedge r} - W_{t\wedge s})^2 - (t\wedge r - t\wedge s)) \\ &= ab((W_{t\wedge r}^2 - W_{t\wedge s}^2) - (t\wedge r - t\wedge s) - 2W_{t\wedge s}(W_{t\wedge r} - W_{t\wedge s})) \\ &= ab(N_{t\wedge r} - N_{t\wedge s}) - 2abW_{t\wedge s}(W_{t\wedge r} - W_{t\wedge s}) \end{aligned}$$

Recalling that N, W are martingales, it follows from Theorem (3.8) that M is amartingale as

$$abW_{t\wedge s}(W_{t\wedge r} - W_{t\wedge s}) = abW_s(W_{t\wedge r} - W_{t\wedge s})$$

In case 2, recalling$0 \leq s \leq r \leq u \leq v$, note that

$$\begin{aligned} M_t &= a(W_{t\wedge r} - W_{t\wedge s})b(W_{t\wedge v} - W_{t\wedge u}) \\ &= a(W_r - W_s)b(W_{t\wedge v} - W_{t\wedge u}) \end{aligned}$$

as $M_t = 0$ if $t \leq u$. Proof is again completed using Theorem (3.8).

Théoréme 1.7 *Let $f \in \mathbb{S}$, $M_t = \int_0^t f dW$ and $N_t = M_t^2 - \int_0^t f^2 ds$. Then M and N are $(\mathcal{F}_\cdot)$-martingales. Further, for any $T < \infty$,*

$$\mathbb{E}\left[\sup_{t\leq T}\left|\int_o^t f dW\right|^2\right] \leq 4\mathbb{E}\left[\int_0^T f_s^2 ds\right]. \tag{1.17}$$

■

Proof. The fact that M and N are martingales follows from Lemma (1.2). As a consequence $\mathbb{E}[N_t]$ and hence

$$\mathbb{E}\left[(\int_o^t f dW)^2\right] = \mathbb{E}\left[\int_0^T f_s^2 ds\right]. \tag{1.18}$$

Now the growth inequality (1.17) follows from Doob's maximal inequality, Theorem applied to M and using (1.18). ■

Lemme 1.3 *Let f be a predictable process such that*

$$\mathbb{E}\left[\int_0^T f_s^2 ds\right] < \infty \ \forall T < \infty. \tag{1.19}$$

Then there exists a continuous adapted process Y such that for all simple predictable processes $h \in \mathbb{S}$,

$$\mathbb{E}\left[\left(\sup_{0\le t\le T}\left|Y_t - \int_o^t hdW\right|\right)^2\right] \le 4\mathbb{E}\left[\int_0^T (f_s - h_s)ds\right] \ \forall T < \infty. \tag{1.20}$$

Further, Y and Z are $(\mathcal{F}_t)$-martingales where

$$Z_t = Y_t - \int_0^T f_s^2 ds$$

Proof. For $r > 0$, let μ_r be the measure on $(\tilde{\Omega}, \mathbb{P})$ defined as follows : for $\mathcal{P}$ measur able bounded functions g

$$\int_{\tilde{\Omega}} g d\mu_r = \mathbb{E}\left[\int_0^r g_s ds\right],$$

and let us denote the $\mathbb{L}^2$ norm on μ_r. By Theorem(3.9), $\mathbb{S}$ is dense in μ_r for every $r > 0$ and hence for integers $m \ge 1$, we can get $f^m \in \mathbb{S}$ such that

$$\|f - f^m\|_{2,\mu_m} \le 2^{-m-1}. \tag{1.21}$$

Using $\|.\|_{2,\mu_r} \le \|.\|_{2,\mu_s}$ for $r \le s$ it follows that for $k \ge 1$

$$\left\|f^{m+k} - f^m\right\|_{2,\mu_m} \le 2^{-m}. \tag{1.22}$$

Denoting the $\mathbb{L}^1(\Omega\ ,\mathcal{B},\ \mathbb{P})$ norm by $\|.\|_{2,p}$, the growth inequality (1.17) can be rewrit ten as, for $g \in \mathbb{S}, m \geq 1$,

$$\left\| \sup_{0\leq t\leq m} \left| \int_0^t g dW \right| \right\|_{2,p} \leq 2 \|g\|_{2,\mu_m}.$$

Recall that $f^k \in \mathbb{S}$ and hence $\int_0^t f^k dW$ is already defined. Let $Y_t^k = \int_0^k f dW$. Now using (1.22) and (**??**), we conclude that for $k \geq 1$

$$\left\| \sup_{0\leq t\leq m} \left| Y_t^{m+k} - Y_t^m \right| \right\|_{2,p} \leq 2^{-m+1}. \tag{1.23}$$

Fix an integer n. For m $\geq$ n, using (1.23) for k = 1 we get

$$\left\| \sup_{0\leq t\leq n} \left| Y_t^{m+1} - Y_t^m \right| \right\|_{2,p} \leq 2^{-m+1}. \tag{1.24}$$

and hence

$$\begin{aligned} \left\| \left[\sum_{m=n}^{\infty} \sup_{0\leq t\leq n} \left| Y_t^{m+1} - Y_t^m \right| \right] \right\|_{2,p} &\leq \sum_{m=n}^{\infty} \left\| \left[\sup_{0\leq t\leq n} \left| Y_t^{m+1} - Y_t^m \right| \right] \right\|_{2,p} \\ &\leq \sum_{m=n}^{\infty} 2^{-m+1} \\ &< \infty. \end{aligned}$$

Hence,

$$\sum_{m=n}^{\infty} \left[\sup_{0\leq t\leq n} \left| Y_t^{m+1} - Y_t^m \right| \right]_{2,p} < \infty \text{ a.s.}\mathbb{P}. \tag{1.25}$$

So let

$$N_n = \left\{ \omega : \sum_{m=n}^{\infty} \left[\sup_{0\leq t\leq n} \left| Y_t^{m+1} - Y_t^m \right| \right]_{2,p} = \infty \right\}$$

and let $N = \cup_{n=1}^{\infty} N_n$. Then N is a $\mathbb{P}$ null set. For $\omega \notin N$, let us define

$$Y_t(\omega) = \lim_{m\to\in} Y_t^m(\omega),$$

and for $\omega \in N$, let $Y_t(\omega) = 0$. It follows from (1.25) that for all $T < \infty$, $\omega \notin N$

$$\sup_{0 \le t \le T} |Y_t^m(\omega) - Y_t(\omega)| \to 0. \tag{1.26}$$

Thus$Y$ is a process with continuous paths. Now using (**??**) for $f - h \in \mathbb{S}$ we get

$$\mathbb{E}\left[(\sup_{0 \le t \le T} \left| Y_t^m - \int_0^t h dW \right|)^2 \right] \le 4\mathbb{E}\left[\int_0^T (f^m - h)^2 ds \right]. \tag{1.27}$$

In view of (1.21), the right-hand side above converges to

$$\mathbb{E}\left[\int_0^T (f_s - h_s)^2 ds \right].$$

Using Fatou's lemma and (1.26) along with $\mathbb{P}(N) = 0$, taking lim inf in (1.27) we conclude that (1.20) is true. From these observations, it follows that Y_t^m converges to Y_t in $\mathbb{L}^2(\mathbb{P})$ for each fixed t. The observation $\|.\|_{2,\mu_r} \le \|.\|_{2,\mu_s}$ for $r \le s$ and (1.21) implies that for all r, $\|f - f^m\|_{2,\mu_r} \to 0$ and hence for all t

$$\mathbb{E}\left[\int_0^t (f_s - f_s^m)^2 ds \right] \to 0.$$

As a consequence,

$$\mathbb{E}\left[\int_0^t \left| (f_s)^2 - (f_s^m)^2 \right| ds \right] \to 0$$

and hence

$$\mathbb{E}\left[\int_0^t (f_s^m)^2 ds - \int_0^t f_s^2 ds \right] \to 0.$$

By Theorem (1.7), we have Y^n and Z^n which are martingales where $Z_t^n = (Y_t^n)^2 - \int_0^t (f_s^n)^2 ds$. As observed above Y_t^n converges in $\mathbb{L}^1(\mathbb{P})$ to Y_t, and Z_t^n converges in $\mathbb{L}^1(\mathbb{P})$ to Z_t for each $t$and hence $Y$and Z are martingales. ■

Définition 1.7 *For a predictable process f such that $\mathbb{E}\left[\int\limits_0^T f_s^2 ds\right] < \infty \ \forall T < \infty$, we define the Itô's integral $\int\limits_0^t f dW$; to be the process Y that satisfies (1.20).*

The next result gives the basic properties of the Ito's integral $\int f dW$; most of them have essentially been proved above.

Théoréme 1.8 *Let f, g be predictable processes satisfying (1.19). Then*

$$\int\limits_0^t (af + bg) dW = a \int\limits_0^t f dW = b \int\limits_0^t g dW. \tag{1.28}$$

Let $M_t = \int\limits_0^t f dW$ and $N_t = M_t^2 - \int\limits_0^t f_s^2 ds$. Then M and N are $(\mathcal{F})$-martingales. Further, for any $T < \infty$,

$$\mathbb{E}\left[\sup_{t \leq T} \left| \int\limits_0^t f dW \right|^2\right] \leq 4\mathbb{E}\left[\int\limits_0^T f_s^2 ds\right]. \tag{1.29}$$

Proof. The linearity (1.28) follows by linearity for the integral for simple functions observed in Lemma and then for general predictable processes via approximation. That M, N are martingales has been observed in Lemma (1.3). The growth inequality (1.29) follows from (1.20) with $h = 0$. ■

1.5 Itô Formula [11]

1.5.1 Formula for Brownian motion

We want a rule to "differentiate" expressions of the form $f(W(t))$, where $f(x)$ is a differentiable function and $W(t)$ is a Brownian motion. If $W(t)$ were also differentiable, then the chain rule from ordinary calculus would give

$$\frac{d}{dt} f(W(t)) = f'(W(t)) W'(t) dt,$$

which could be written in differential notation as

$$df(W(t)) = f\text{'}(W(t))W\text{'}(t)dt = (W(t))dW(t).$$

Because W has nonzero quadratic variation, the correct formula has an extra term, namely,

$$df(W(t)) = f\text{'}(W(t))dW(t) + \frac{1}{2}f\text{''}(W(t))dt. \tag{1.30}$$

This is the ltô-Doeblin formula in differential form. Integrating this, we obtain the ltô-Doeblin formula in integral form :

$$f(W(t)) - f(W(0)) = \int_0^t f\text{'}(W(u))dW(u) + \int_o^t f\text{''}(W(u))du. \tag{1.31}$$

1.5.2 Formula for Itô processes

Définition 1.8 *Let $W(t), t \geq 0$, be a Brownian motion, and let :$F(t)$, $t \geq 0$, be an associated filtration. An Itô process is a stochastic process of the form*

$$X(t) = X(0) + \int_0^t \Delta(u)dW(u) + \int_0^t \theta(u)du, \tag{1.32}$$

where $X(0)$ is nonrandom and $\Delta(u)$ and $\theta(u)$ are adapted stochastic processes.
In order to understand the volatility associated with Itô processes, we must determine the rate at which they accumulate quadratic variation.

Lemme 1.4 *The quadratic variation of the Itô process (1.32) is*

$$[X, X] = \int_0^t \Delta^2(u)du. \tag{1.33}$$

Proof. We introduce the notation $I(t) = \int_0^t \Delta(u)dW(u), R(t)\int_0^t \theta(u)du$. Both these processes are continuous in their upper limit of integration t. To determine the quadratic variation of

X on $[0,t]$, we choose a partition $\Pi = \{t_0, t_1, ..., t_n\}$ of $[0,t]$ (i.e.,$0 = t_0 < t_1 < ... < t_n = t$) and we write the sampled quadratic variation.

$$\sum_{j=0}^{n-1} [X(t_{j+1}) - X(t_j)]^2 = \sum_{j=0}^{n-1} [I(t_{j+1}) - I(t_j)]^2 + \sum_{j=0}^{n-1} [R(t_{j+1}) - R(t_j)]^2$$
$$+ 2\sum_{j=0}^{n-1} [I(t_{j+1}) - I(t_j)] [R(t_{j+1}) - R(t_j)] .$$

As $\|\Pi\| \to 0$, the first term on the right-hand side, $\sum_{j=0}^{n-1} [I(t_{j+1}) - I(t_j)]^2$, converges to the quadratic variation of I on $[0,t]$, which according to Theorem (3.10) (vi) is $[I,I](t) = \int_0^t \Delta^2(u)du$. The absolute value of the second term is bounded above by

$$\max_{0\le k\le n-1} |R(t_{k+1}) - R(t_k)| \cdot \sum_{j=0}^{n-1} |R(t_{j+1}) - R(t_j)| = \max_{0\le k\le n-1} |R(t_{k+1}) - R(t_k)| \cdot \sum_{j=0}^{n-1} \left| \int_{t_j}^{t_{j+1}} \theta(u)du \right|$$
$$\le \max_{0\le k\le n-1} |R(t_{k+1}) - R(t_k)| \cdot \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} |\theta(u)|\, du$$
$$= \max_{0\le k\le n-1} |R(t_{k+1}) - R(t_k)| \cdot \sum_{j=0}^{n-1} \int_0^t |\theta(u)|\, du,$$

and as $\|\Pi\| \to 0$, this has limit $0 \cdot \int_0^t |\theta(u)|\, du = 0$ because $R(t)$ is continuous. The absolute value of the third term is bounded above by

$$2 \max_{0\le k\le n-1} |I(t_{k+1}) - I(t_k)| \cdot \sum_{j=0}^{n-1} |R(t_{j+1}) - R(t_j)| \le 2 \max_{0\le k\le n-1} |I(t_{k+1}) - I(t_k)| \cdot \int_0^t |\theta(u)|\, du,$$

and this has limit $0 \cdot \int_0^t |\theta(u)|^2\, du = 0$ as $\|\Pi\| \to 0$ because $I(t)$ is continuous. We conclude that $[X,X](t) = [I,I](t) = \int_0^t \Delta^2(u)du$.

■

Théoréme 1.9 ***(Itô's formula for an Itô process).*** *Let $X(t), t \geq 0$, be an Itô process as described and let $f(t,x)$ be a function for which the partial derivatives $f_t(t,x)$, $f_x(t,x)$, and $f_{xx}(t,x)$ are defined and continuous. Then, for every $T \geq 0$,*

$$f(T,X(T)) = f(O,X(O)\} + \int_0^T f_t(t,X(t))dt + \int_0^T f_x(t,X(t))dX(t) + \frac{1}{2}\int_0^T f_{xx}(t,X(t))d[X,X](t) \tag{1.34}$$

$$= f(O,X(O)) + \int_0^T f_t(t,X(t))dt + \int_0^T f_x(t,X(t))\Delta(t)dW(t)$$
$$+ \int_0^T f_x(t,X(t))\theta(t)dt + \frac{1}{2}\int_0^T f_{xx}(t,X(t))\Delta^2(t) \tag{1.35}$$

Proof. However, it is easier to remember and use the result of this theorem if we recast it in differential notation. We may rewrite (1.34) as

$$df(t,X(t)) = f_t(t,X(t))dt + f_s(t,X(t))dX(t) + \frac{1}{2}f_{xx}(t,X(t))dX(t)dX(t). \tag{1.36}$$

The guiding principle here is that we write out the Taylor series expansion of $f(t,X(t))$ with respect to all its arguments, which in this case are t and $X(t)$. We take this Taylor series expansion out to first order for every argument that has zero quadratic variation, which in this case is t, and we take the expansion out to second order for every argument that has nonzero quadratic variation, which in this case is $X(t)$.

We may reduce (1.36) to an expression that involves only dt and by using the differential form of the Itô process (i.e., $dX(t) = \Delta(t)dW(t) + \theta(t)dt$) and the formula

$$dX(t)dX(t) = \Delta^2(t)dW(t)dW(t) + 2\Delta(t)\theta(t)dW(t)dt + \theta^2(t)dtdt$$
$$= \Delta^2(t)dt.$$

For the rate at which $X(t)$ accumulates quadratic variation (i.e., $dX(t)dX(t) = \Delta^2(t)dt.$).

This is obtained by squaring the formula for $dX(t)$ and using the multiplication table

$$dW(t)dW(t) = dt,\ dtdW(t) = dW(t)dt = 0,\ dtdt = 0.$$

Making these substitutions in (1.36), we obtain

$$df(t, X(t)) = f_t(t, X(t))dt + f_x(t, X(t))\Delta(t)dW(t) + f_x(t, X(t))\theta(t)dW(t)\frac{1}{2}f_{xx}(t, X(t))\Delta^2(t)dt.$$

Itô calculus is little more than repeated use of this formula in a variety of situations. ∎

Chapitre 2

Stochastic Differential Equations

2.1 Existence and uniqueness solution of stochastic differential equation

We are going to consider stochastic differential equations (**SDE**) of the type

$$dX_t = \sigma(t, X_t)dW_t + b(t, X_t)dt. \tag{2.1}$$

Equation (2.1) is to be interpreted as an integral equation :

$$X_t = X_0 + \int_0^t \sigma(s, X_s)dW_s + \int_o^t b(s, X_s)ds. \tag{2.2}$$

Here W is an $\mathbb{R}^d$ -valued Brownian motion, X_0 is an $\mathbb{R}^d$ -valued $\mathcal{F}_0$ measurable ran dom variable, $\sigma : [o, \infty) \times \mathbb{R}^m \mapsto L(m, d)$ and $b : [o, \infty) \times \mathbb{R}^m \mapsto \mathbb{R}^m$ are given functions, and one is seeking a process X such that (2.2) is true. The solution X to the **SDE** (2.1), when it exists, is called a diffusion process with diffusion coefficient $\sigma\sigma^*$ and drift coefficient b. We shall impose the following conditions on σ,b :

$$\begin{aligned} \sigma &: [o,\infty) \times \mathbb{R}^m \mapsto L(m,d) \text{ is a continuous function} \\ b &: [o,\infty) \times \mathbb{R}^m \mapsto \mathbb{R}^m \text{ is a continuous function} \end{aligned} \tag{2.3}$$

$\forall T < \infty$ $\exists C_T < \infty$ such that for all $t \in [0,T]$, x^1, $x^2 \in \mathbb{R}^d$

$$\begin{aligned} \left\|\sigma(t,x^1) - \sigma(t,x^2)^2\right\| &\leq C_T \left|x^1 - x^2\right|, \\ \left\|b(t,x^1) - b(t,x^2)^2\right\| &\leq C_T \left|x^1 - x^2\right|. \end{aligned} \tag{2.4}$$

Since $t \mapsto \sigma(t,0)$ and $t \mapsto b(t,0)$ are continuous and hence bounded on$[0,T]$ for every $T < \infty$, using the Lipschitz conditions (2.4), we can conclude that for each $T < \infty$, $\exists K_T < \infty$ such that

$$\begin{aligned} \|\sigma(t,x)\| &\leq K_T(1+|x|), \\ \|b(t,x)\| &\leq K_T(1+|x|). \end{aligned} \tag{2.5}$$

We will need the following lemma, known as Gronwall's lemma, for proving unique ness of solution to (2.2) under the Lipschitz conditions.

Lemme 2.1 *Let $\beta(t)$ be a bounded measurable function on $[0,T]$ satisfying, for some $0 \leq a < \infty, 0 < b < \infty$,*

$$\beta(t) \leq a + b\int_0^t \beta(s)ds, 0 \leq t \leq T. \tag{2.6}$$

Then

$$\beta(t) \leq ae^{bt}. \tag{2.7}$$

Proof. Let

$$g(t) = e^{-bt}\int_0^t \beta(s)ds.$$

Then by definition, g is absolutely continuous and

$$\acute{g}(t) = e^{-bt}\beta(t) - be^{-b_t}\int_0^t \beta(s)ds. \text{ a.e.}$$

Where almost everywhere refers to the Lebesgue measure on $\mathbb{R}$. Using (2.6), it follows that

$$\acute{g}(t) \leq ae^{-b_t} \text{ a.e.}$$

Hence (using $g(0) = 0$ and that g is absolutely continuous) $g(t) \leq \frac{a}{b}(e^{-b_t} - 1)$ from which we get

$$\int_0^t \beta(s)ds \leq \frac{a}{b}(e^{-b_t} - 1).$$

The conclusion $\beta(t) \leq ae^{bt}$ follows immediately from (2.6). ■

Lemme 2.2 *Let $Y, Z \in \Bbbk_m$ and let $\zeta = \Lambda(Y)$ and $\eta = \Lambda(Z)$. Then for $0 \leq t \leq T$ one has*

$$\mathbb{E}\left[\sup_{0\leq s\leq t} |\zeta_s - \eta_s|^2\right] \leq 3\mathbb{E}\left[|Y_0 - Z_0|^2\right] + 3C_T^2(4+T)\int_0^t \mathbb{E}\left[|Y_s - Z_s|^2\right] ds.$$

Proof. *Let us note that*

$$\zeta_t - \eta_t = Y_0 - Z_0 + \int_0^t \left[\sigma(s, Y_s) - \sigma(s, Z_s)\right] dW_s + \int_0^t \left[b(s, Y_s) - b(s, Z_s)\right] ds \tag{2.8}$$

and hence this time using the Lipschitz condition (2.4) along with the growth inequality we now have

$$\mathbb{E}\left[\sup_{0\leq s\leq t} |\zeta_s - \eta_s|^2\right] \leq 3\mathbb{E}\left[|Y_0 - Z_0|^2\right] + 4\mathbb{E}\left[\int_0^t \|\sigma(s, Y_s) - \sigma(s, Z_s)\|^2 ds\right] + \mathbb{E}\left[(\int_0^t |b(s, Y_s) - b(s, Z_s)| ds\right.$$

$$\leq 3\mathbb{E}\left[|Y_0 - Z_0|^2\right] + 3C_T^2(4+T)\int_0^t \mathbb{E}\left[|Y_s - Z_s|^2\right] ds.$$

■

Corollaire 2.1 *Suppose $Y, Z \in \mathbb{k}_m$ be such that $Y_0 = Z_0$. Then for $0 \leq t \leq T$*

$$\mathbb{E}\left[\sup_{0\leq s\leq t} |\Lambda(Y)_s - \Lambda(Z)_s|^2\right] \leq 3C_T^2(4+T)\int_0^t \mathbb{E}\left[|Y_s - Z_s|^2\right] ds.$$

We are now in a position to prove the main result of this section.

Théoréme 2.1 *Suppose σ, b satisfy conditions (2.3) and (2.4) and X_0 is a $\mathcal{F}_0$ measurable $\mathbb{R}^m$-valued random variable with $\mathbb{E}\left[|X_0|^2\right]$. Then there exists a process X such that $\mathbb{E}\left[\int_0^T |X_s|^2 ds\right] < \infty$ and*

$$X_t = X_0 + \int_0^t \sigma(s, X_s) dW_s + \int_o^t b(s, X_s) ds. \tag{2.9}$$

Further if X is another process such that $\tilde{X}_t X_0$, $\mathbb{E}\left[\int_0^T \left|\tilde{X}_s\right|^2 ds\right] < \infty$ for all $T < \infty$ and

$$\tilde{X}_t = \tilde{X}_0 + \int_0^t \sigma(s, \tilde{X}_s) dW_s + \int_o^t b(s, \tilde{X}_s) ds$$

then $X = \tilde{X}$, i.e. $\mathbb{P}(X_t = \tilde{X}_t \ \forall t) = 1$.

Proof. Let us first prove uniqueness. Let X and $\tilde{X}$ be as in the statement of the theorem. Then, using Corollary (2.1) it follows that

$$u(t) = \mathbb{E}\left[\sup_{s\leq t} \left|X_s - \tilde{X}_s\right|^2\right]$$

satisfies for $0 \leq t \leq T$ (recalling $X_0 = \tilde{X}_0$)

$$u(t) = 3C_T^2(4+T)\int_0^t \mathbb{E}\left[\left|X_s - \tilde{X}_s\right|^2\right] ds.$$

Hence u is bounded and satisfies

$$u(t) = 3C_T^2(4+T)\int_0^t u(s)ds,\ \ 0 \le t \le T.$$

By (Gronwall's) Lemma (2.1), it follows that $u(t) = 0, 0 \le t \le T$ for every $T < \infty$.

Hence

$$X = \tilde{X}.$$

We will now construct a solution. Let $X_t^1 = X_0$ for all $t \ge 0$. Note that $X^1 \in \mathbb{k}_m$. Now define X^n inductively by

$$X^{n+1} = \Lambda(X^n).$$

Since $X_t^1 = X_0$ for all t $X_0^2 = X_0^1$,

$$X_t^2 - X_t^1 = \int_0^t \sigma(s, X_0)dW_s + \int_o^t b(s, X_0)ds$$

and hence

$$\mathbb{E}\left[\sup_{s\le t}\left|X_s^2 - X_s^1\right|^2\right] \le 2K_T^2(4+T)(1+\mathbb{E}\left[|X_0|^2\right])t. \tag{2.10}$$

Note that $X_0^n = X_0^1 = X_0$ for all $n \ge 1$ and hence from Lemma (2.2) it follows that for $n \ge 2$, for $0 \le t \le T$,

$$\mathbb{E}\left[\sup_{s\le t}\left|X_s^{n+1} - X_s^n\right|^2\right] \le 3C_T^2(4+T)\int_0^t \mathbb{E}\left[\left|X_s^n - X_s^{n-1}\right|^2\right]ds.$$

Thus defining for $n \ge 1$, $u_n = \mathbb{E}\left[\sup_{s\le t}|X_s^{n+1} - X_s^n|^2\right]$ we have for $n \ge 2$, for $0 \le t \le T$,

$$u_n(t) \le 3C_T^2(4+T)\int_0^t u_{n-1}(s)ds. \tag{2.11}$$

As seen in (2.10),

$$u_1(t) \leq 2K_T^2(4+T)(1+\mathbb{E}\left[|X_0|^2\right])t$$

and hence using (2.11), which is true for $n \geq 2$, we can deduce by induction on n that for a constant $\tilde{C}_T = 3(C_T^2 + K_T^2)(4+T)(1+\mathbb{E}\left[|X_0|^2\right])$

$$u_n(t) \leq \frac{(\tilde{C}_T)^n t^n}{n}, \; 0 \leq t \leq T.$$

Thus $\sum_n \sqrt{u_n(T)} < \infty$ for every $T < \infty$ which is same as

$$\sum_{n=1}^{\infty} \left\| \sup_{s \leq T} \left|X_s^{n+1} - X_s^n\right|^2 \right\|_2 < \infty \tag{2.12}$$

$\|Z\|_2$ denoting the $\mathbb{L}^2(\mathbb{P})$ norm here. The relation (2.12) implies

$$\left\| \sum_{n=1}^{\infty} \sup_{s \leq T} \left|X_s^{n+1} - X_s^n\right|^2 \right\|_2 < \infty \tag{2.13}$$

as well as

$$\begin{aligned} \sup_{k\geq 1} \left\| \left[\sup_{s \leq T} \left|X_s^{n+k} - X_s^n\right| \right] \right\|_2 &\leq \sup_{k \geq 1} \left\| \left[\sum_{j=1}^{\infty} \sup_{s \leq T} \left|X_s^{j+1} - X_s^j\right| \right] \right\|_2 \\ &\leq \left[\sum_{j=1}^{\infty} \left\| \sup_{s \leq T} \left|X_s^{j+1} - X_s^j\right| \right\|_2 \right] \to 0 \text{ as } n \text{ tends to } \infty. \end{aligned} \tag{2.14}$$

Let $N = \cup_{T=1}^{\infty} \left\{ \omega : \sum_{n=1}^{\infty} \sup_{s \leq T} |X_s^{n+1}(\omega) - X_s^n(\omega)| = \infty \right\}$. Then by (2.13), $\mathbb{P}(N) = 0$ and for $\omega \notin \mathbb{N}$, $X_s^n(\omega)$ converges uniformly on $[0,T]$ for every $T < \infty$. So let us define X as follows :

$$X_t(\omega) = \lim_{n \to \infty} X_s^n(\omega) \text{ if } \omega \in N^c$$

$$X_t(\omega) = 0 \text{ if } \omega \in N^c$$

By definition, X is a continuous adapted process (since by assumption $N \in \mathcal{F}_0$) and X^n converges to X uniformly in $[0,T]$ for every T almost surely. Using Fatou's lemma and (2.14) we get

$$\left\| \left[\sup_{s \leq T} |X_s - X_s^n| \right] \right\|_2 \leq \liminf \left\| \left[\sum_{j=n}^{n+k} \sup_{s \leq T} |X_s^{j+1} - X_s^j| \right] \right\|_2 \tag{2.15}$$

$$\leq \left[\sum_{j=n}^{\infty} \left\| \sup_{s \leq T} |X_s^{j+1} - X_s^j| \right\|_2 \right] \to 0 \text{ as } n \text{ tends to } \infty.$$

In particular, $X \in \mathbb{k}_m$. Since $\Lambda(X^n) = X^{n+1}$ by definition, (2.15) also implies that

$$\lim_{n\to\infty} \left\| \left[\sup_{s \leq T} |X_s - \Lambda(X^n)_s| \right] \right\|_2 = 0 \tag{2.16}$$

while (2.15) and Corollary (2.1) (remembering that $X_0^n = X_0$ for all n) imply that

$$\lim_{n\to\infty} \left\| \left[\sup_{s \leq T} |\Lambda(X)_s - \Lambda(X^n)_s| \right] \right\|_2 = 0. \tag{2.17}$$

From (2.16) and (2.17) it follows that $X = \Lambda(X)$ or that X is a solution to the **SDE (2.9).**

■

Chapitre 3

PSDERB with non-Lipschitz coefficients : Existence and uniqueness

Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous, while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $B\colon = (B(t))_{0\leq t\leq T}$ be a standard one-dimensional Brownian motion, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ Let $\mathcal{L}^2([a,b];\mathbb{R})$ denote the family of $\mathcal{F}_t$-measurable, $\mathbb{R}$-valued processes $f(t) = \{f(t,\omega)\}, t \in [a,b]$ such that $\int_0^t |f(t)|^2 \, dt < \infty$ a.s. Where the coefficient $\alpha \in [0,1]$, and β is a Borel-measurable functions from $[0,T] \to (0,1]$.

Here $(\beta(t)L_t^0; 0 < \beta(t) \leq 1)$ denotes the local time at 0 for the time t of the semi-martingale X. Its role is to push upward the process X in order to keep it above 0, that is, to have the condition $X \geq 0$ satisfied. One of the possible ways to define it is through the limit,

$$L_t^0(X) = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_0^t 1_{[-\varepsilon,\varepsilon]}(X_s) d\langle X\rangle_s, t \geq 0.$$

as it satisfies, for example, $L_t^0(X) = \int_0^t 1_{\{X_s=0\}} \, dL_s^0(X)$, see [[10], Proposition 3.2].

Now, we define the sequence of the Carathéodory approximate solutions $X^n\colon [-1,T] \to \mathbb{R}$.

For all $n \geq 1$, we define

$$\begin{cases} X^n = x_0 = 0, -1 \leq t \leq 0. \\ X^n = x_0 + \int\limits_0^t \sigma(s, X^n(s - \frac{1}{n}))dB(s) + \int\limits_0^t b(s, X^n(s - \frac{1}{n}))ds + \alpha \max\limits_{0 \leq s \leq t} X^n(s - \frac{1}{n}) + \beta(t) L_t^0(X^n), \\ t \in [0, T]. \\ X^n \geq 0, x_0 \geq 0 \text{ for all t} \geq 0. \end{cases} \tag{3.1}$$

According to Skorokhod's equation [[10], Chapter VI, Lemma 1.3], and PSDERB (1), we get

$$L_t^0(X^n) = \frac{1}{\beta(t)} \max \left[0, \max_{s \in [0,t]} \left(- \left(X(0) + \int\limits_0^s \sigma(u, X(u))dB(u) + \int\limits_0^s b(u, X(u))du + \alpha \max_{0 \leq u \leq s} X^n(u) \right) \right) \right].$$

We assume that the discretization of the local time $(L_t^0(X))$ is the local time of the Carathéodory approximation $(L_t^0(X^n))$ Therefore. the Carathéodory approximation, of $(L_t^0(X^n))$ is given by

$$L_t^0(X^n) \leq \frac{1}{\beta(t)} \max \left[0, \max_{s \in [0,t]} \left(- \left(\begin{array}{c} X(0) + \int\limits_0^t \sigma(u, X^n(u - \frac{1}{n}))dB(u) \\ + \int\limits_0^t b(u, X^n(u - \frac{1}{n}))du + \alpha \max\limits_{0 \leq u \leq s} X^n(u - \frac{1}{n}) \end{array} \right) \right) \right]. \tag{3.2}$$

Note that $X^n(t)$ can be calculated step by step on the intervals, $\left[0, \frac{1}{n}\right)$, $\left[\frac{1}{n}, \frac{2}{n}\right)$,...etc.

3.1 PSDERB with non-Lipschitz coefficients : Existence and uniqueness

Let us consider the following perturbed **SDE** with reflecting boundary, defined by

$$\begin{cases} \{X(t) = X(0) + \int\limits_0^t \sigma(s, X(s))dB(s) + \int\limits_0^t b(s, X(s))ds + \alpha \max\limits_{0\leq s\leq t} X(s) + \beta(t)L_t^0(X), X(t) \geq 0, \\ X(0) = 0, \forall t \geq 0. \end{cases} \tag{3.3}$$

where $\alpha \in [0,1]$, and $\beta : [0,T] \to (0,1]$, and $b, \sigma : [0,T] \times \mathbb{R} \to \mathbb{R}$, are a Borel-measurable functions, and the initial value $X(0) = x_0$ is independent of B and satisfies $\mathbb{E}|x_0|^2 < \infty$.

To obtain the main results, we give the following conditions.

Assumption For any $x, y \in \mathbb{R}$ and $t \in [0,T]$,there exists a function $\rho(.)$ such that,

$$|\sigma(t,x) - \sigma(t,y)| \vee |b(t,x) - b(t,y)| \leq \rho(|x-y|).$$

Where $\rho(u)$ is *a* concave non-decreasing continuous function such that $\rho(0) = 0$ and $\int_{0^+} \frac{u}{\rho^2(u)} du = \infty$.

Remarque 3.1 *Since $\rho(.)$ is concave and $\rho(0) = 0$ one can find a pair of positive constants a and ζ such that,*

$$\rho(u) \leq a + \zeta u \text{ for } u \geq 0.$$

Assumption For any $t \in [0,T]$ and $x, y \in \mathbb{R}$, there exists a positive constant C such that,$|\sigma(t,0)| \vee |b(t,0)| \leq C$.

Assumption The coefficients satisfy : $0 \leq \alpha < \frac{\beta}{1+\beta}$. where $\beta = \min\{\beta(t); t \in [0,T]\}$.

Remarque 3.2 *Assumptions 3.1 imply that, $\alpha \in \left[0, \frac{1}{2}\right)$ .Now, we state our main result.*

Théoréme 3.1 *Under Assumptions 3.1, 3.1, and 3.1, there exists a unique $\mathcal{F}_t$-adapted solution $\{X(t)\}_{t\geq 0}$ to 3.3. Moreover, for any $T > 0$,*

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{0\leq t\leq T} |X^n(t) - X(t)|^2\right] = 0.$$

To prove Theorem (3.1), we will need the following Lemma.

Lemme 3.1 *Under Assumptions 3.1, 3.1, and 3.1, there exists a constant $C>0$, which does not depend on n, such that*

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|X^n(t)|^2\right]\leq C.$$

Proof. For any $t\in[0,T]$ we have

$$\begin{aligned}
&|X^n(t)| \\
&=\left|\int_0^t \sigma(s,X^n(s-\frac{1}{n}))dB(s)+\int_0^t b(s,X^n(s-\frac{1}{n}))ds+\alpha\max_{0\leq s\leq t}X^n(s-\frac{1}{n})+\beta(t)L_t^0(X^n)\right| \\
&\leq\left|\int_0^t \sigma(s,X^n(s-\frac{1}{n}))dB(s)+\int_0^t b(s,X^n(s-\frac{1}{n}))ds+\alpha\max_{0\leq s\leq t}X^n(s-\frac{1}{n})\right|+\left|\beta(t)L_t^0(X^n)\right|.
\end{aligned} \tag{3.4}$$

Equation (1.1) implies that,

$$\begin{aligned}
&L_t^0(X^n) \\
&\leq\frac{1}{\beta(t)}\max_{0\leq u\leq s}\left|\int_0^t \sigma(u,X^n(u-\frac{1}{n}))dB(u)+\int_0^t b(u,X^n(u-\frac{1}{n}))du+\alpha\max_{0\leq u\leq s}X^n(u-\frac{1}{n})\right|.
\end{aligned} \tag{3.5}$$

By using (3.4) and (3.5) we obtain,

$$\begin{aligned}
|X^n(t)|&\leq 2\max_{0\leq u\leq s}\left|\int_0^t \sigma(u,X^n(u-\frac{1}{n}))dB(u)+\int_0^t b(u,X^n(u-\frac{1}{n}))du+\alpha\max_{0\leq u\leq s}X^n(u-\frac{1}{n})\right| \\
&\leq 2\max_{0\leq s\leq t}\left|\int_0^t \sigma(u,X^n(u-\frac{1}{n}))dB(u)+\int_0^t b(u,X^n(u-\frac{1}{n}))du+\alpha\max_{\frac{-1}{n}\leq u\leq 0}X^n(u)+\alpha\max_{0\leq u\leq s}X^n(u)\right| \\
&\leq 2\max_{0\leq s\leq t}\left|\int_0^t \sigma(u,X^n(u-\frac{1}{n}))dB(u)+\int_0^t b(u,X^n(u-\frac{1}{n}))du+\alpha\max_{0\leq u\leq s}X^n(u)\right|.
\end{aligned}$$

Therefore

$$(1-2\alpha)\max_{0\leq t\leq T}|X^n(t)| \leq 2\max_{0\leq s\leq t}\left|\int_0^t \sigma(u,X^n(u-\frac{1}{n}))dB(u)+\int_0^t b(u,X^n(u-\frac{1}{n}))du\right|. \quad (3.6)$$

By the Hölder inequality and the Burkholder –Davis–Gundy inequality, we get

$$\begin{aligned}&(1-2\alpha)^2\mathbb{E}\left[\sup_{0\leq t\leq t_1}|X^n(t)|^2\right]\\ &\leq 8(8\mathbb{E}\int_0^{t_1}\left|\sigma(u,X^n(u-\frac{1}{n}))-\sigma(u,0)\right|^2du+2T\mathbb{E}\int_0^{t_1}\left|b(u,X^n(u-\frac{1}{n}))-b(u,0)\right|^2du\\ &+8\mathbb{E}\int_0^{t_1}|\sigma(u,0)|^2du+2T\mathbb{E}\int_0^{t_1}|b(u,0)|^2du).\end{aligned}$$

By Assumptions 3.1 and 3.1

$$(1-2\alpha)^2\mathbb{E}\left[\sup_{0\leq t\leq t_1}|X^n(t)|^2\right]\leq 8(2(T+4)\mathbb{E}\int_0^{t_1}\rho^2(\left|X^n(u-\frac{1}{n})\right|)du+2(T+4)TC).$$

Then the Jensen inequality implies that,

$$(1-2\alpha)^2\mathbb{E}\left[\sup_{0\leq t\leq t_1}|X^n(t)|^2\right]\leq 8(2(T+4)\int_0^{t_1}\rho^2((\mathbb{E}\left|X^n(u-\frac{1}{n})\right|^2)^{\frac{1}{2}})du+2(T+4)Tc).$$

Let, $h(x)=\rho^2(x^{\frac{1}{2}})$; it follows that,

$$\mathbb{E}\left[\sup_{0\leq t\leq t_1}|X^n(t)|^2\right]\leq\frac{16(T+4)TC}{(1-2\alpha)^2}+\frac{16(T+4)}{(1-2\alpha)^2}\int_0^{t_1}h(\mathbb{E}\left|(X^n(u-\frac{1}{n}))\right|^2)du. \quad (3.7)$$

Since $\frac{\rho(x)}{x}$ and $\rho'_+(x)$ are non-negative, nonincreasing functions, we have that

$$h'_+(x)=x^{-\frac{1}{2}}\rho(x^{\frac{1}{2}})\rho'_+(x).$$

is a non-negative, nonincreasing function which implies that h is a non-negative, non-decreasing concave function. Note that $\rho(0) = 0$, then $h(0) = 0$, and there exists a pair of positive constants a and ζ such that,

$$h(u) \leq a + \zeta u \text{ for } u \geq 0.$$

We therefore have,

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leq t\leq t_1} |X^n(t)|^2\right] &\leq \frac{16(T+4)T(C+2)}{(1-2\alpha)^2} + \frac{16(T+4)\zeta}{(1-2\alpha)^2}\int_0^{t_1} \mathbb{E}\left|(X^n(u-\frac{1}{n}))\right|^2 du \\ &\leq \frac{16(T+4)T(C+a)}{(1-2\alpha)^2} + \frac{16(T+4)\zeta}{(1-2\alpha)^2}\int_0^{t_1} \mathbb{E}\sup_{0\leq u\leq t} |X^n(t)|^2 \, dt.\end{aligned}$$

Set,

$$\mathbb{E}\left[\sup_{0\leq t\leq t_1} |X^n(t)|^2\right] \leq \frac{16(T+4)T(C+a)}{(1-2\alpha)^2} e^{\frac{16(T+4)\zeta}{(1-2\alpha)^2}t}.$$

one can have

$$\mathbb{E}\left[\sup_{0\leq t\leq t_1} |X^n(t)|^2\right] \leq C$$

for any $n \geq 1$. This completes the proof. ■

Lemme 3.2 *Under Assumptions 3.1, 3.1, and 3.1, for any $0 \leq s \leq t \leq T$, there exists a constant $C > 0$, which does not depend on n, such that*

$$\mathbb{E}\left|X^n(t) - X^n(s)\right|^2 \leq C(t-s).$$

Proof. For all $n \geq 1$, and $0 \leq s \leq t \leq T$, we have

$$\begin{aligned} & |X^n(t) - X^n(s)| \\ & = \left| \begin{array}{l} \int_s^t \sigma(u, X^n(u - \frac{1}{n}))dB(u) + \int_s^t b(u, X^n(u - \frac{1}{n}))du + \alpha \max_{0 \leq u \leq t} X^n(u - \frac{1}{n}) + \beta(t) L_t^0(X^n) \\ -\alpha \max_{0 \leq u \leq s} X^n(u - \frac{1}{n}) + \beta(s) L_s^0(X^n) \end{array} \right| \end{aligned} \tag{3.8}$$

$$\begin{aligned} & \leq \left| \int_s^t \sigma(u, X^n(u - \frac{1}{n}))dB(u) + \int_s^t b(u, X^n(u - \frac{1}{n}))du \right| + \left| \beta(t) L_t^0(X^n) - \beta(s) L_s^0(X^n) \right| \\ & + \left| \alpha(\max_{0 \leq u \leq t} X^n(u - \frac{1}{n}) - \max_{0 \leq u \leq s} X^n(u - \frac{1}{n})) \right|. \end{aligned} \tag{3.9}$$

Applying the inequality $\max\{a, b\} = \frac{1}{2}[a + b + |a - b|]$ for $a, b \in \mathbb{R}$, and the inequality

$||a|-|b|| \leq |a-b|$, Equation (3) implies that

$$
\begin{aligned}
&\left|\beta(t)L_t^0(X^n)-\beta(s)L_s^0(X^n)\right| \\
&= \left| \begin{array}{l} \max\left[\begin{array}{c} 0, \max\limits_{0\leq v\leq t}(-\int\limits_0^v \sigma(u,X^n(u-\frac{1}{n}))dB(u) - \int\limits_0^v b(u,X^n(u-\frac{1}{n}))du \\ -\alpha \max\limits_{0\leq u\leq v} X^n(u-\frac{1}{n})) \end{array}\right] \\ -\max\left[\begin{array}{c} 0, \max\limits_{0\leq w\leq s}(-\int\limits_0^w \sigma(u,X^n(u-\frac{1}{n}))dB(u) - \int\limits_0^w b(u,X^n(u-\frac{1}{n}))du \\ -\alpha \max\limits_{0\leq u\leq w} X^n(u-\frac{1}{n})) \end{array}\right] \end{array}\right| \\
&= \left| \begin{array}{l} \frac{1}{2}(0+\max\limits_{0\leq v\leq t}(-\int\limits_0^v \sigma(u,X^n(u-\frac{1}{n}))dB(u) - \int\limits_0^v b(u,X^n(u-\frac{1}{n}))du - \alpha\max\limits_{0\leq u\leq v} X^n(u-\frac{1}{n})) \\ +\left|0-\max\limits_{0\leq v\leq t}(-\int\limits_0^v \sigma(u,X^n(u-\frac{1}{n}))dB(u) - \int\limits_0^v b(u,X^n(u-\frac{1}{n}))du - \alpha\max\limits_{0\leq u\leq v} X^n(u-\frac{1}{n})\right|) \\ -\frac{1}{2}(0+\max\limits_{0\leq w\leq s}(-\int\limits_0^w \sigma(u,X^n(u-\frac{1}{n}))dB(u) - \int\limits_0^w b(u,X^n(u-\frac{1}{n}))du - \alpha\max\limits_{0\leq u\leq w} X^n(u-\frac{1}{n})) \\ -\left|0-\max\limits_{0\leq w\leq s}(-\int\limits_0^w \sigma(u,X^n(u-\frac{1}{n}))dB(u) - \int\limits_0^w b(u,X^n(u-\frac{1}{n}))du - \alpha\max\limits_{0\leq u\leq w} X^n(u-\frac{1}{n})\right| \end{array}\right| \\
&= \left| \begin{array}{l} \frac{1}{2}(2\left|\max\limits_{0\leq v\leq t}(-\int\limits_0^v \sigma(u,X^n(u-\frac{1}{n}))dB(u) - \int\limits_0^v b(u,X^n(u-\frac{1}{n}))du - \alpha\max\limits_{0\leq u\leq v} X^n(u-\frac{1}{n})\right| \\ -\frac{1}{2}(2\left|\max\limits_{0\leq w\leq s}(-\int\limits_0^w \sigma(u,X^n(u-\frac{1}{n}))dB(u) - \int\limits_0^w b(u,X^n(u-\frac{1}{n}))du - \alpha\max\limits_{0\leq u\leq w} X^n(u-\frac{1}{n})\right| \end{array}\right| \\
&= \left| \begin{array}{l} \left|\max\limits_{0\leq v\leq t}(-\int\limits_0^v \sigma(u,X^n(u-\frac{1}{n}))dB(u) - \int\limits_0^v b(u,X^n(u-\frac{1}{n}))du - \alpha\max\limits_{0\leq u\leq v} X^n(u-\frac{1}{n})\right| \\ -\left|\max\limits_{0\leq w\leq s}(-\int\limits_0^w \sigma(u,X^n(u-\frac{1}{n}))dB(u) - \int\limits_0^w b(u,X^n(u-\frac{1}{n}))du - \alpha\max\limits_{0\leq u\leq w} X^n(u-\frac{1}{n})\right| \end{array}\right| \\
&\leq \left| \begin{array}{c} \max\limits_{0\leq v\leq t}(-\int\limits_0^v \sigma(u,X^n(u-\frac{1}{n}))dB(u) - \int\limits_0^v b(u,X^n(u-\frac{1}{n}))du \\ -\alpha\max\limits_{0\leq u\leq v} X^n(u-\frac{1}{n})) \\ -\max\limits_{0\leq u\leq v}(-\int\limits_0^w \sigma(u,X^n(u-\frac{1}{n}))dB(u) - \int\limits_0^w b(u,X^n(u-\frac{1}{n}))du \\ -\alpha\max\limits_{0\leq u\leq w} X^n(u-\frac{1}{n})) \end{array}\right|.
\end{aligned}
$$

Let,

$$Y^n(v) = -\int_0^v \sigma(u, X^n(u-\frac{1}{n}))dB(u) - \int_0^v b(u, X^n(u-\frac{1}{n}))du - \alpha \max_{0\le u\le v} X^n(u-\frac{1}{n}).$$

This implies,

$$\left|\max_{0\le v\le t} Y^n(v) - \max_{0\le v\le s} Y^n(u)\right| = \left|\left[\max_{0\le v\le s} Y^n(v), \max_{s\le v\le t} Y^n(v)\right] \max_{0\le v\le s} Y^n(v)\right|.$$

Applying the inequality $\max\{a,b\} = \frac{1}{2}\left[a+b+|a-b|\right]$ for $a, b \in \mathbb{R}$, we get

$$\max_{0\le v\le t} Y^n(v) = \frac{1}{2}\left[\max_{0\le v\le s} Y^n(v) + \max_{s\le v\le t} Y^n(v) + \left|\max_{0\le v\le s} Y^n(v) + \max_{s\le v\le t} Y^n(v)\right|\right].$$

If $\max_{s\le v\le t} Y^n(v) \le \max_{0\le v\le s} Y^n(v)$, then

$$\left|\max_{0\le v\le t} Y^n(v) - \max_{0\le v\le s} Y^n(v)\right| = 0.$$

If $\max_{0\le v\le s} Y^n(v) \le \max_{s\le v\le t} Y^n(v)$, then

$$\left|\max_{0\le v\le t} Y^n(v) - \max_{0\le v\le s} Y^n(v)\right| = \left|\max_{s\le v\le t} Y^n(v) - \max_{0\le v\le s} Y^n(v)\right|.$$

Since, $\max_{0\le v\le s} Y^n(v) \le \max_{s\le v\le t} Y^n(v)$ and $Y^n(s) \le \max_{0\le v\le s} Y^n(v)$, we get

$$\begin{aligned}\left|\max_{0\le v\le t} Y^n(v) - \max_{0\le u\le s} Y^n(v)\right| &\le \left|\max_{s\le v\le t} Y^n(v) - Y^n(v)\right| \\ &\le \max_{s\le v\le t} \left|Y^n(v) - Y^n(s)\right|.\end{aligned}$$

Therefore,

$$\left|\beta(t)L_t^0(X^n) - \beta(s)L_s^0(X^n)\right| \leq \max_{s\leq v\leq t}\left|\int_s^v \sigma(u, X^n(u-\frac{1}{n}))dB(u) - \int_s^v b(u, X^n(u-\frac{1}{n}))du\right| \tag{3.10}$$

$$+ \alpha \max_{s\leq v\leq t}\left|X^n(v-\frac{1}{n}) - X^n(s-\frac{1}{n})\right|.$$

Equations (3.8) and (3.10) imply that,

$$\begin{aligned}|X^n(t) - X^n(s)| &\leq 2\max_{s\leq v\leq t}\left|\int_s^v \sigma(u, X^n(u-\frac{1}{n}))dB(u) + \int_s^v b(u, X^n(u-\frac{1}{n}))du\right| \\ &+ 2\alpha \max_{s\leq v\leq t}\left|X^n(v-\frac{1}{n}) - X^n(s-\frac{1}{n})\right| \\ &\leq 2\max_{s\leq v\leq t}\left|\int_s^v \sigma(u, X^n(u-\frac{1}{n}))dB(u) + \int_s^v b(u, X^n(u-\frac{1}{n}))du\right| \\ &+ 2\alpha \max_{s\leq v\leq t}\left|X^n(v-\frac{1}{n}) - X^n(s-\frac{1}{n})\right|.\end{aligned}$$

We therefore have,

$$|X^n(t_1) - X^n(s_1)|$$

$$= \left|\begin{array}{l}\int_{s_1}^{t_1} \sigma(u, X^n(u-\frac{1}{n}))dB(u) + \int_{s_1}^{t} b(u, X^n(u-\frac{1}{n}))du + \alpha \max_{0\leq u\leq t_1} X^n(u-\frac{1}{n}) + \beta(t_1)L_{t_1}^0(X^n) \\ -\alpha \max_{0\leq u\leq s_1} X^n(u-\frac{1}{n}) + \beta(s_1)L_{s_1}^0(X^n)\end{array}\right| \tag{3.11}$$

$$\leq \left|\int_{s_1}^{t_1} \sigma(u, X^n(u-\frac{1}{n}))dB(u) + \int_{s_1}^{t} b(u, X^n(u-\frac{1}{n}))du + \alpha \max_{0\leq u\leq t_1} X^n(u-\frac{1}{n})\right| \tag{3.12}$$

$$+ \left|\beta(t_1)L_{t_1}^0(X^n)\right| + \left|\beta(s_1)L_{s_1}^0(X^n)\right| \tag{3.13}$$

Equation (1.1) implies that,

$$\left|\beta(t_1)L^0_{t_1}(X^n)\right| \leq \int_0^{t_1} \sigma(u, X^n(u-\frac{1}{n}))dB(u) + \int_0^{t_1} b(u, X^n(u-\frac{1}{n}))du + \alpha \max_{0\leq u\leq t_1} X^n(u-\frac{1}{n})$$

So,

$$\max_{s\leq s_1\leq t_1\leq t} |X^n(t_1) - X^n(s_1)| \leq 2 \max_{s\leq s_1\leq t_1\leq t} \left| \int_{s_1}^{t_1} \sigma(u, X^n(u-\frac{1}{n}))dB(u) + \int_{s_1}^{t_1} b(u, X^n(u-\frac{1}{n}))du \right|$$
$$+ 2\alpha \max_{s\leq s_1\leq t_1\leq t} |X^n(t_1) - X^n(s_1)|$$

$$\max_{s\leq s_1\leq t_1\leq t} |X^n(t_1) - X^n(s_1)| \leq \frac{2}{(1-2\alpha)} \max_{s\leq s_1\leq t_1\leq t} \left| \int_{s_1}^{t_1} \sigma(u, X^n(u-\frac{1}{n}))dB(u) + \int_{s_1}^{t_1} b(u, X^n(u-\frac{1}{n}))du \right|.$$

Hence,

$$\mathbb{E}\left|X^n(t) - X^n(s)\right|^2 \leq \frac{4}{(1-2\alpha)^2} \left(\mathbb{E}\left| \int_s^t \sigma(u, X^n(u-\frac{1}{n}))dB(u) \right|^2 + \left| \int_s^t b(u, X^n(u-\frac{1}{n}))du \right|^2 \right).$$

Then Lemma (3.1) yields,

$$\begin{aligned}
\mathbb{E}\left|X^n(t) - X^n(s)\right|^2 &\leq \frac{4}{(1-2\alpha)^2} \left((t-s)\mathbb{E}\left| \int_s^t b(u, X^n(u-\frac{1}{n})) \right|^2 du + 4\mathbb{E}\int_s^t \left|\sigma(u, X^n(u-\frac{1}{n}))\right|^2 du \right) \\
&\leq \frac{4(T+4)\tilde{L}}{(1-2\alpha)^2} \int_s^t (1 + \mathbb{E}\left|X^n(u-\frac{1}{n})\right|^2)du \\
&\leq \frac{4(T+4)\tilde{L}}{(1-2\alpha)^2}(1+C)(t-s).
\end{aligned}$$

Where $\tilde{L} = 2\max(a+C, \zeta)$ The proof is therefore complete. ■

Now, let us apply the above lemmas to prove theorem 3.1

Proof. of Theorem 3.1. First, we will show that the sequence $\{X^n(t)\}$ is a Cauchy sequence in $\mathbb{L}^2([0,T];\mathbb{R})$ For any $n \geq m \geq 1$ we get

$$\begin{aligned}
&|X^n(t) - X^m(t)| \\
&\leq \left| \int_0^t \left[\sigma(s, X^n(s-\frac{1}{n})) - \sigma(s, X^m(s-\frac{1}{m})) \right] dB(s) + \int_0^t \left[b(s, X^n(s-\frac{1}{n})) - b(s, X^m(s-\frac{1}{m})) \right] ds \right| \\
&+ \left| \sigma(\max_{0\leq s\leq t} X^n(s-\frac{1}{n}) - \max_{0\leq s\leq t} X^m(s-\frac{1}{m})) \right| + \left| \beta(t)(L_t^0(X^n) - L_t^0(X^m)) \right|.
\end{aligned}$$

Applying the inequality $\max\{a,b\} = \frac{1}{2}\left[a+b+|a-b|\right]$ for $a, b \in \mathbb{R}$ and the inequality$||a|-|b|| \leq |a-b|$, Equation (1.1) implies that

$$\begin{aligned}
&\left|\beta(t)(L_t^0(X^n) - L_t^0(X^m))\right| \\
&= \left| \begin{array}{l} \max\left[0, -\max\limits_{o\leq v\leq t}(\int_0^v \sigma(u, X^n(u-\frac{1}{n}))dB(u) - \int_0^v b(u, X^n(u-\frac{1}{n}))du - \alpha \max\limits_{o\leq u\leq v} X^n(u-\frac{1}{n})\right] \\ -\max\left[0, -\max\limits_{o\leq v\leq t}(\int_0^v \sigma(u, X^m(u-\frac{1}{m}))dB(u) - \int_0^v b(u, X^m(u-\frac{1}{m}))du - \alpha \max\limits_{o\leq u\leq v} X^m(u-\frac{1}{m})\right] \end{array} \right|
\end{aligned}$$

$$\begin{aligned}
&\left|\beta(t)(L_t^0(X^n) - L_t^0(X^m))\right| \\
&\leq \left| \begin{array}{l} \max\limits_{o\leq v\leq t}(-\int_0^v \sigma(u, X^n(u-\frac{1}{n}))dB(u) - \int_0^v b(u, X^n(u-\frac{1}{n}))du - \alpha \max\limits_{o\leq u\leq v} X^n(u-\frac{1}{n})) \\ -\max\limits_{o\leq v\leq t}(\int_0^v \sigma(u, X^m(u-\frac{1}{m}))dB(u) - \int_0^v b(u, X^m(u-\frac{1}{m}))du - \alpha \max\limits_{o\leq u\leq v} X^m(u-\frac{1}{m})) \end{array} \right|.
\end{aligned}$$

Let

$$Y^n(v) = -\int_0^v \sigma(u, X^n(u-\frac{1}{n}))dB(u) - \int_0^v b(u, X^n(u-\frac{1}{n}))du - \alpha \max_{o\leq u\leq v} X^n(u-\frac{1}{n}).$$

This implies,

$$\left|\max_{o\leq v\leq t} Y^n(v) - \max_{o\leq v\leq t} Y^m(v)\right| \leq \max_{o\leq v\leq t} \left|Y^n(v) - Y^m(v)\right|.$$

Therefore

$$\begin{aligned}
&\left|\beta(t)(L_t^0(X^n) - L_t^0(X^m))\right| \\
&\leq \max_{o\leq v\leq t}\left|\int_0^v \left[\sigma(s, X^n(s-\frac{1}{n})) - \sigma(s, X^m(s-\frac{1}{m}))\right] dB(s) + \int_0^v \left[b(s, X^n(s-\frac{1}{n})) - b(s, X^m(s-\frac{1}{m}))\right] ds\right| \\
&+ \alpha \max_{0\leq s\leq t}\left|X^n(s-\frac{1}{n}) - X^m(s-\frac{1}{m})\right|.
\end{aligned}$$

Then

$$\begin{aligned}
&\left|X^n(t) - X^m(t)\right| \\
&\leq 2\left|\int_0^t \left[\sigma(s, X^n(s-\frac{1}{n})) - \sigma(s, X^m(s-\frac{1}{m}))\right] dB(s)\right| \\
&+ 2\left|\int_0^t \left[b(s, X^n(s-\frac{1}{n})) - b(s, X^m(s-\frac{1}{m}))\right] ds\right| + 2\alpha\max_{0\leq s\leq t}\left|X^n(s-\frac{1}{n}) - X^m(s-\frac{1}{m})\right| \\
&= 2\left|\int_0^t \left[\sigma(s, X^n(s-\frac{1}{n})) - \sigma(s, X^m(s-\frac{1}{m}))\right] dB(s)\right| + 2\left|\int_0^t \left[b(s, X^n(s-\frac{1}{n})) - b(s, X^m(s-\frac{1}{m}))\right] ds\right| \\
&2\alpha\max_{0\leq s\leq t}\left|X^n(s-\frac{1}{n}) - X^m(s-\frac{1}{m}) + X^m(s-\frac{1}{n}) - X^m(s-\frac{1}{n})\right| \\
&\leq 2\left|\int_0^t \left[\sigma(s, X^n(s-\frac{1}{n})) - \sigma(s, X^m(s-\frac{1}{m}))\right] dB(s)\right| \\
&+ 2\left|\int_0^t \left[b(s, X^n(s-\frac{1}{n})) - b(s, X^m(s-\frac{1}{m}))\right] ds\right| \\
&+ 2\alpha\max_{0\leq s\leq t}\left|X^n(s-\frac{1}{n}) - X^m(s-\frac{1}{n})\right| + 2\alpha\max_{0\leq s\leq t}\left|X^m(s-\frac{1}{n}) - X^m(s-\frac{1}{m})\right|.
\end{aligned}$$

By using the inequality,

$$(a+b+c)^2 \leq 3(a^2+b^2+c^2).$$

For any $\mathcal{L} \in \mathbb{N}, a_i \in \mathbb{R}$ and $q \geq 0$,we obtain that

$$\begin{aligned}
&(1-2\alpha)^2\mathbb{E}\left[\sup_{0\leq t\leq t_1} |X^n(t)-X^m(t)|^2\right] \\
&\leq 12(\alpha^2\mathbb{E}\left[\sup_{0\leq t\leq t_1} \left|X^n(t-\frac{1}{n})-X^m(t-\frac{1}{m})\right|^2\right] \\
&+\mathbb{E}\sup_{0\leq t\leq t_1}\left|\int_0^t \left[\sigma(s,X^n(t-\frac{1}{n}))-\sigma(s,X^m(t-\frac{1}{m}))\right]dB(s)\right|^2 \\
&+\mathbb{E}\sup_{0\leq t\leq t_1}\left|\int_0^t \left[b(s,X^n(t-\frac{1}{n}))-b(s,X^m(t-\frac{1}{m}))\right]ds\right|^2). \qquad (3.14)
\end{aligned}$$

By the Hölder inequality and the Burkholder –Davis–Gundy inequality, we get,

$$\begin{aligned}
&(1-2\alpha)^2\mathbb{E}\left[\sup_{0\leq t\leq t_1} |X^n(t)-X^m(t)|^2\right] \\
&\leq (4\mathbb{E}\int_0^{t_1} \left|\sigma(s,X^n(t-\frac{1}{n}))-\sigma(s,X^m(t-\frac{1}{m}))\right|^2 dB(s) \\
&+T\mathbb{E}\left|\int_0^{t_1} \left[b(s,X^n(t-\frac{1}{n}))-b(s,X^m(t-\frac{1}{m}))\right]\right|^2 ds \\
&+\alpha^2\mathbb{E}\sup_{0\leq t\leq t_1}\left|X^m(t-\frac{1}{n})-X^m(t-\frac{1}{m})\right|^2).
\end{aligned}$$

By Assumption 3.1 and the Jensen inequality, we have

$$\begin{aligned}
&(1-2\alpha)^2\mathbb{E}\left[\sup_{0\le t\le t_1}|X^n(t)-X^m(t)|^2\right]\\
&\le 12((4+T)\mathbb{E}\int_0^{t_1}\rho^2(\left|X^n(s-\frac{1}{n}))-X^m(s-\frac{1}{m})\right|)ds\\
&+\alpha^2\mathbb{E}\sup_{0\le t\le t_1}\left|X^m(t-\frac{1}{n})-X^m(t-\frac{1}{m})\right|^2)\\
&\le 12((4+T)\int_0^{t_1}\rho^2((\mathbb{E}\left|X^n(s-\frac{1}{n}))-X^m(s-\frac{1}{m})\right|^2)^{\frac{1}{2}})ds\\
&+\alpha^2\mathbb{E}\sup_{0\le t\le t_1}\left|X^m(t-\frac{1}{n})-X^m(t-\frac{1}{m})\right|^2).
\end{aligned}$$

Similar to (3.7), one obtains

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\le t\le t_1}|X^n(t)-X^m(t)|^2\right]\le \frac{12}{(1-2\alpha)^2}((4+T)\int_0^{t_1}h(\mathbb{E}\left|X^n(s-\frac{1}{n}))-X^m(s-\frac{1}{m})\right|^2)ds\\
+\alpha^2\mathbb{E}\sup_{0\le t\le t_1}\left|X^m(t-\frac{1}{n})-X^m(t-\frac{1}{m})\right|^2).
\end{aligned}$$

Since $h(.)$ is concave, we have $h(a+b) \leq h(a)+h(b)$. Then Lemma 3.2 yields,

$$\begin{aligned}
&(1-2\alpha)^2\mathbb{E}\left[\sup_{0\leq t\leq t_1}|X^n(t)-X^m(t)|^2\right] \\
&\leq \frac{12}{(1-2\alpha)^2}(2(4+T)\int_0^{t_1} h(\mathbb{E}\left|X^n(s-\frac{1}{n}))-X^m(s-\frac{1}{m})\right|^2)ds \\
&+2(4+T)\int_0^{t_1} h(\mathbb{E}\left|X^m(t-\frac{1}{n})-X^m(t-\frac{1}{m})\right|^2)ds \\
&+\alpha^2\mathbb{E}\sup_{0\leq t\leq t_1}\left|X^m(t-\frac{1}{n})-X^m(t-\frac{1}{m})\right|^2) \\
&\leq \frac{24(4+T)}{(1-2\alpha)^2}(\int_0^{t_1} h(\mathbb{E}\left|X^n(s-\frac{1}{n}))-X^m(s-\frac{1}{m})\right|^2)ds \\
&+\int_0^{t_1} h(C(\frac{1}{m}-\frac{1}{n}))ds)+\frac{12\alpha^2}{(1-2\alpha)^2}C(\frac{1}{m}-\frac{1}{n}).
\end{aligned} \tag{3.15}$$

Where,

$$\begin{aligned}
&\int_0^{t_1} h(\mathbb{E}\left|X^n(s-\frac{1}{n}))-X^m(s-\frac{1}{m})\right|^2)ds \\
&\leq \int_0^{t_1} h(\mathbb{E}\sup_{0\leq\theta\leq s}\left|X^n(\theta-\frac{1}{n}))-X^m(\theta-\frac{1}{m})\right|^2)ds \\
&\leq \int_0^{t_1} h(\mathbb{E}\sup_{-\frac{1}{n}\leq v\leq s-\frac{1}{n}}|X^n(v)-X^m(v)|^2)ds+\mathbb{E}\sup_{-0\leq v\leq s-\frac{1}{n}}|X^n(v)-X^m(v)|^2)ds \\
&\leq h(0)T\int_0^{t_1} h(\mathbb{E}\sup_{-0\leq s\leq t}|X^n(s)-X^m(s)|^2)dt.
\end{aligned} \tag{3.16}$$

Inserting (3.15) into (3.16), we obtain that

$$(1-2\alpha)^2\mathbb{E}\left[\sup_{0\le t\le t_1}|X^n(t)-X^m(t)|^2\right]$$
$$\le \frac{24(4+T)}{(1-2\alpha)^2}\frac{24(4+T)}{(1-2\alpha)^2}\int_0^{t_1} h(\mathbb{E}\sup_{0\le s\le t}|X^n(s)-X^m(s)|^2)dt+12C(n,m). \tag{3.17}$$

Where,

$$C(n,m)=\frac{2(T+4)T}{(1-2\alpha)^2}Ch(\frac{1}{m}-\frac{1}{n})+\frac{\alpha^2}{(1-2\alpha)^2}C(\frac{1}{m}-\frac{1}{n}).$$

By Bahri inéqalities, we get

$$\mathbb{E}\left[\sup_{0\le t\le t_1}|X^n(t)-X^m(t)|^2\right]\le G^{-1}(G(12C(n,m))+\frac{24(4+T)T}{(1-2\alpha)^2}).$$

Where $G(t)=\int_0^{t_1}\frac{ds}{h(s)}$. Obviously, G is a strictly increasing function, then G has an inverse function which is strictly increasing, and $G^{-1}(\infty)=0$ Note that when $n,m\to\infty$ then $C(n,m)\to 0$. Recalling $\int_{0^+}\frac{ds}{h(s)}=\int_{0^+}\frac{s}{\rho^2(s)}ds=\infty$, we have

$$G(12C(n,m))\to-\infty.$$

And,

$$G^{-1}(G(12C(n,m))+\frac{24(4+T)T}{(1-2\alpha)^2}\to 0.$$

We therefore have,

$$\limsup_{n,m\to\infty}\mathbb{E}\left[\sup_{0\le t\le t_1}|X^n(t)-X^m(t)|^2\right]$$
$$\le\limsup_{n,m\to\infty}G^{-1}(G(12C(n,m))+\frac{24(4+T)T}{(1-2\alpha)^2})=0. \tag{3.18}$$

Which implies that $(X^n(t))_{n\geq 1}$ is a Cauchy sequence. Denote the limit by $X(t)$. Letting $m \to \infty$ in (3.18) yields,

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{0\leq t\leq T} |X^n(t) - X(t)|^2\right] = 0.$$

Similar to (3.17), (3.18), we can show that $X(t)$ is a unique solution of Equation (nder non-Lipschitz conditions. Then, the proof is completed.

To see the generality of our results, let us give a few examples of the func tion $\rho(.)$ let $\delta \in (0,1)$ be sufficiently small, define

$$\rho_1(u) := \begin{cases} 0, \text{ if } u = 0, \\ u\sqrt{\log(u^{-1})}, \text{ if } u \in (0,\delta], \\ \delta\sqrt{\log(\delta^{-1})}, \text{ if } u \in [\delta, +\infty], \end{cases}$$

and,

$$\rho_2(u) := \begin{cases} 0, \text{ if } u = 0, \\ u\sqrt{\log(u^{-1})}\log\log(u^{-1}), \text{ if } u \in (0,\delta], \\ \delta\sqrt{\log(\delta^{-1})}\log\log(\delta^{-1}), \text{ if } u \in [\delta, +\infty], \end{cases}$$

They are all concave non-decreasing functions satisfying, $\int_{0^+} \frac{u}{\rho_i^2(u)} du = \infty$, $i = 1, 2$. ■

Conclusion

In this work, we study the carathéodory approximate solution for a class of one-dimensional perturbed stochastic differential equations with reflecting boundary (PSDERB). We prove that PSDERB have a unique solution and show that the carathéodory approximate solution converges to the solution of PSDERB whose both drift and diffusion coefficients are non-Lipschitz.

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Annex : Some mathematical tools

Théoréme 3.2 *(**Burkholder–Davis–Gundy Inequality** [11]) Let M_t be a continuous local martingale with $M_o = 0$, a.s, and suppose $2 \leq p < \infty$. There exists a constant c_1 depending on p such that for any finite stopping time T,*

$$E(M_T^*)p \leq c_1 E(M)_T^{p/2}.$$

Proof. There is nothing to prove if the left-hand side is zero, so we may assume it is positive. First suppose M_T^* is bounded by a positive constant K. Note for $p \geq 2$ the function $x \to |x|^p$ is C^2. By Doob's inequalities and then Ito's formula (and the fact that $|Ms| \geq 0$), we have

$$\begin{aligned} E|M_T^*|^p &\leq c\mathbb{E}|M_T|^p \\ &= cE\int_0^T p|M_s|^{p-1} dM_s + \frac{1}{2}cE\int p(p-1)|M_s|^{p-1} d\langle M\rangle_s \\ &\leq cE\int_0^T (M_T^*)^{p-2} d\langle M\rangle_s \\ &= cE[(M_T^*)^{p-2}\langle M\rangle_T]. \end{aligned}$$

(Recall our convention about constants and the letter c.) Using Holder's inequality with exponents $p/(p-2)$ and $p/2$, we obtain

$$E(M_T^*)^p \leq c(E(M_T^*)^p)^{\frac{p}{2}}(E\langle M\rangle_T^{\frac{p}{2}})^{\frac{2}{p}}.$$

Dividing both sides by $(E(M_T^*)^p)^{\frac{p-2}{p}}$ and then taking both sides to the power $p/2$gives our result.

We then apply the above to $T \wedge U_k$, where $U_k = inf\{t : |M_t| \geq K\}, let K \to \infty$, and use Fatou's lemm. ■

Théoréme 3.3 ***(Hölder Inequality [4])*** *Let p and q be conjugate exponents. Then, if f and g are two measurable functions from E into $\mathbb{R}$,*

$$\int |fg|\, d\mu \leq \|f\|_p \|g\|_q$$

In particular, $fg \in \mathbb{L}^1(E, A, \mu)$ if $f \in \mathbb{L}^p(E, A, \mu)$ and $g \in \mathbb{L}^q(E, A, \mu)$.

Remark In the last assertion, we implicitly use the fact that, if f and g are defined up to a set of zero μ-measure, the product fg (as well as the sum $f + g$ is also well defined up to a set of zero μ-measure.

Proof. If $\|f\|_p = 0$, we have $f = 0$, μ a.e., which implies $|fg|\, d\mu = 0$, and the inequality is trivial. We can thus assume that $\|f\|_p > 0$ and $\|g\|_q > 0$. Without loss of generality, we can also assume that $f \in L^p(E, A, \mu)$ and $g \in \mathbb{L}^q(E, A, \mu)$ (otherwise $\|f\|_p \|g\|_q = \infty$ and there is nothing to prove).

The case $p = 1$ and $q = \infty$ is very easy, since we have $|fg|\, d\mu \leq \|g\|_\infty |f|\, \mu a.e.$,

which implies

$$\int |fg|\, d\mu \leq \|g\|_\infty \int \|g\|_\infty \|f\|$$

In what follows we therefore assume that $1 < p < \infty$ (and thus $1 < q < \infty$). Let $\alpha \in (0, 1)$. Then, for every

$$x \in \mathbb{R}_+ x^\alpha - \alpha x \leq 1 - \alpha.$$

Indeed, define $\Phi_{\alpha.}(x) = x^\alpha - \alpha x$ for $x \geq 0$. Then, for $x > 0$, we have $\Phi'_{\alpha.}(x) = \alpha(x^{\alpha-1} - 1)$, and thus$\Phi'(x) > 0$ if $x \in (0, 1)$ and $\Phi'(x) < 0$ if $x \in (1, \infty)$. Hence $\Phi_{\alpha.}$ attains its maximum at $x = 1$, which gives the desired inequality. By applyingthis inequality to $x = \frac{u}{v}$, where $u \geq 0$

and $v > 0$, we get

$$u^{\alpha}v^{1-\alpha} \leq \alpha u + (1-\alpha)v,$$

and this inequality still holds if $v = 0$. We then take $\alpha = \frac{1}{p}$(so that $1 - \alpha = \frac{1}{q}$) and

$$u = \frac{|f(x)|^p}{\|f\|_p^p}, v = \frac{|g(x)|^q}{\|g\|_q^q}$$

to arrive at

$$u = \frac{|f(x)g(x)|}{\|f\|_p \|g\|_q} \leq \frac{1}{p}\frac{|f(x)|^p}{\|f\|_p^p} + \frac{1}{q}\frac{|g(x)|^q}{\|g\|_q^q}.$$

By integrating the latter inequality with respect to $\mu(dx)$, we get

$$\frac{1}{\|f\|_p \|g\|_q}\int |fg|\, d\mu \ \leq \frac{1}{p} + \frac{1}{q} = 1$$

which completes the proof. ■

Théoréme 3.4 ***(Jensen's Inequality*** *[4]) Suppose that μ is a probability measure, and let $\varphi : \mathbb{R} \to \mathbb{R}_+$ be a convex function. Then, for every $f \in L^1(E, A, \mu)$*

$$\int \varphi \circ f d\mu \geq \varphi(\int f d\mu).$$

Remark The integral $\int \varphi \circ f \ d\mu$ is well defined as the integral of a nonnegative measurable function.

Proof. Set

$$\varepsilon_{\varphi} = \{(a, b) \in \mathbb{R}^2 : \forall x \in \mathbb{R}, \varphi(x) \geq ax + b\}.$$

Then elementary properties of convex functions show that

$$\forall x \in R, \varphi(x) = \underset{(a,b)\in\varepsilon_{\varphi}}{sup} \ (ax + b).$$

Since $\varphi \circ f \geq af + b$ for every $(a, b) \in \varepsilon_\varphi$, we get

$$\int \varphi \circ f d\mu \geq \sup_{(a,b)\in\varepsilon_\varphi} \int (af + b) d\mu = \sup_{(a,b)\in\varepsilon_\varphi} \int (a \int f d\mu + b) = \varphi(\int f d\mu).$$

■

Théoréme 3.5 *(**Doob's Maximal Inequality**) Let M be a martingale or a positive sub-martingale. Then, for $\lambda > 0$, $n \geq 1$ one has*

$$\mathbb{P}(\max_{0\leq k\leq n} |M_k| > \lambda) \leq \frac{1}{\lambda}\mathbb{E}\left[|M_n| \, 1_{\left\{ \max_{0\leq k\leq n} |M_k| > \lambda \right\}} \right]. \tag{3.19}$$

Further, for $1 < p < \infty$, there exists a universal constant C depending only on p such that

$$\mathbb{E}\left[(\max_{0\leq k\leq n} |M_k|)^p \right] \leq C\mathbb{E}\left[|M_n|^p \right].$$

Théoréme 3.6 *(**Gronwall Inequality**) Let A, $B \in \mathbb{V}^+$ (increasing processes with $A_0 \geq 0$, $B_0 \geq 0$) and a stopping time τ be such that $A_0 \geq 0$, $B_{\tau^-} \leq M$. Suppose that for all stopping times $\sigma \leq \tau$*

$$\mathbb{E}\left[A_{\sigma^-} \right] \leq a + \beta \mathbb{E}\left[\int_{[0,\sigma)} A^- dB \right].$$

For $\alpha > 0$ let $C(\alpha) = \sum_{j=0}^{[\alpha]} \alpha^j$. Then we have

$$\mathbb{E}\left[A_{\tau^-} \right] \leq 2aC(2\beta M).$$

Lemme 3.3 *Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous, non-decreasing function satisfying $h(0) = 0$, and $\int \frac{du}{h(u)} = +\infty$. Let $u(.)$ be a Borel measurable bounded non-negative func tion defined on $[0, T]$ satisfying,*

$$u(t) \leq u_0 + \int_0^t v(s)h(u(s))ds, t \in [0, T].$$

where $u_0 > 0$ *and* $v(.)$ *is a non-negative integrable function on* $[0, T]$*. Then we have*

$$u(t) \leq G(G(u_0) + \int_0^t v(s)ds).$$

where $\frac{G(t)=\int_{t_0}^{t} du}{h(u)}$ *is well defined for some* $t_0 > 0$ *and* G^{-1} *is the inverse function of* G*. In particular, if* $u_0 = 0$*, then* $u(t) = 0$ *for all* $t \in [0, T]$ *.*

Proposition 3.1

1. Let f be a nonnegative measurable function on E. There exists an increasing sequence $(f_n)_{n\in\mathbb{N}}$ of nonnegative simple functions such that $f_n \to f$ pointwise as $n \to \infty$. If f is also bounded, the sequence $(f_n)_{n\in\mathbb{N}}$ can be chosen to converge uniformly to f.

2. Let f and g be two nonnegative measurable functions on E and $a, b \in \mathbb{R}_+$. Then

$$\int (af + bg)d\mu = a\int f d\mu + b\int g d\mu.$$

3. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative measurable functions on E. Then

$$\int (\sum_{n\in\mathbb{N}} f_n)d\mu = \sum_{n\in\mathbb{N}} \int f_n d\mu.$$

Remarque 3.3 $\sum_{n\in\mathbb{N}} f_n$ *is an increasing limit of nonnegative measurable functions and thus also measurable.*

Théoréme 3.7 *Let* $P \in [1, \infty)$ *.*

1. If $(E, \mathcal{A}, \mu)$ is a measure space, the set of all integrable simple functions is dense in $\mathbb{L}^p(E, \mathcal{A}, \mu)$.

2. If (E, d) is a metric space, and μ is an outer regular measure on $(E, \mathcal{B}(E))$, then the set of all bounded Lipschitz functions that belong to $\mathbb{L}^p(E, \mathcal{B}(E), \mu)$ is dense in $\mathbb{L}^p(E, \mathcal{B}(E), \mu)$.

3. If (E, d) is a separable locally compact metric space, and μ is a Radon measure on E, then the set of all Lipschitz functions with compact support is dense in $\mathbb{L}^p(E, \mathcal{B}(E), \mu)$.

Théoréme 3.8 *Suppose M is an r.c.l.l.$(\mathcal{F}_.)$-martingale and σ and τ are $(\mathcal{F}_.)$ stopping times with $\sigma \leq \tau$. Suppose X is an r.c.l.l adapted process. Let*

$$N_t = X_{\sigma \Lambda t}(M_{\tau \Lambda t} - M_{\sigma \Lambda t}).$$

Then N is a $(\mathcal{F}_.)$-martingale if either (i) X is bounded or if (ii) $\mathbb{E}\left[X_\sigma^2\right] < \infty$ and M is square integrable.

Théoréme 3.9 *Let $\mathcal{F}$ be a σ-field on Ω, and let $\mathbb{Q}$ be a probability measure on $(\Omega, \mathcal{F})$. Suppose $\mathbb{G} \subseteq \mathbb{B}(\Omega, \mathcal{F})$ be an algebra such that $\sigma(\mathbb{G}) = \mathcal{F}$. Further, $\exists f^n \in \mathbb{G}$ such that $f^n < f^{n+1}$ and f^n converges to 1 pointwise. Then $\mathbb{G}$ is dense in $\mathbb{L}\left(\Omega, \mathcal{F}, \mathbb{Q}\right)$.*

Théoréme 3.10 *Let T be a positive constant and let $\Delta(t)$, $0 \leq t \leq T$, be an adapted stochastic process that satisfies $\mathbb{E}\int_0^T \Delta(t)^2 dt < \infty$. Then $\int_0^t \Delta(u) dW(u)$ defined by*

$$\int_0^t \Delta(u) dW(u) = \lim_{n \to \infty} \int_0^t \Delta_n(u) dW(u),$$

has the following properties.

(i) (Continuity) As a function of the upper limit of integmtion t, the paths of $I(t)$ are continuous.

(ii) (Adaptivity) For each t, $I(t)$ is $\mathcal{F}(t)$-measumble.

(iii) (Linearity) If $I(t) = \int_0^t \Delta(u) dW(u)$ and $J(t) = \int_0^t \Gamma(u) dW(u)$, then $I(t) \pm J(t) = \int_0^t \left(\Delta(u) \pm \Gamma(u)\right) dW(u)$; furthermore, for every constant $cI(t) = \int_0^t c\Delta(u) dW(u)$.

(iv) (Martingale) $J(t)$ is a martingale.

(v) (Itô isometry) $\mathbb{E}I^2(t) = \mathbb{E}\int\limits_0^t \Delta^2(u)du.$

(vi) (Quadratic variation) $[I, I](t) = \int\limits_0^t \Delta^2(u)du.$

Proposition 3.2 *Let H be a separable real Hilbert space. There exist a propability space (Ω, $\mathcal{F}$, $\mathbb{P}$) and a family $X(h)$, $h \in H$, of random variables on this space, such that*

i the map $h \to X(h)$ is linear ;

ii for each h, the r.v. $X(h)$ is gaussian centered and

$$\mathbb{E}\left[X(h)^2\right] = \|h\|_H^2 .$$

الملخص:

في هذه المذكرة نحن مهتمين بالمعادلات التفاضلية العشوائية العكسية بحاجز ذات البعد الواحد، حيث قمنا بدراسة الوجود و الوحدانية بناءا على التقريب الكاثيودوري، حيث أثبتنا أن الحل التقريبي الكاثيودوري لهذه المعادلة يتقارب إلى حل المعادلة الأم تحت شرط اللاليبشيتز.

الكلمات المفتاحية :

المعادلات التفاضلية العشوائية العكسية ذات البعد الواحد؛ التقريب الكاثيودوري؛ الشرط اللا ليبشيتز.

ABSTRACT:

In this memory, we are interested about the perturbed stochastic differential equations with reflecting boundary, we studied the existence and uniqueness of the solution to this equation based on Carathéodory approximate, where we proved that the Carathéodory approximate solution for this equation converges to the solution under non-Lipschitz condition.

Keywords:

Perturbed stochastic differential equations with reflecting boundary; Carathéodory approximate; non-Lipschitz.

RÉSUMÉ :

Dans ce mémoire, nous nous intéressons aux équations différentielles stochastiques perturbées avec des conditions de bord réfléchissantes. Nous avons étudié l'existence et l'unicité de la solution de cette équation en nous basant sur une approximation de Carathéodory. Nous avons démontré que la solution approchée de Carathéodory pour cette équation converge vers la solution dans le cas d'une condition non lipschitzienne.

Mot clés :

L'équations différentielles stochastiques perturbées avec un barriér réfléchies; approximatives de Carathéodory; non lipschitzienne.