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On the fractional stochastic differential problems

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Abstract

In this work, we give a short and important introduction to the fractional and stochastic calculus, then we present some different examples for the fractional and stochastic differential problems. Finally, we prove some qualitative properties (i.e., existence and uniqueness and asymptotic separation between two solutions) of the solutions of FSDEs, then we give an example to illustrate our main results.

Keywords: fractional calculus, stochastic calculus, fractional differential problems, stochastic differential problems, Banach fixed point theorem.

Résumé

Dans ce travail, nous donnons une introduction courte et importante au calcul fractionnaire et stochastique, puis nous présentons quelques exemples différents pour les problèmes différentiels fractionnaires et stochastiques. Enfin, nous prouvons quelques propriétés qualitatives (c'est-à-dire l'existence, l'unicité et la séparation asymptotique entre deux solutions) des solutions des EDSF, puis nous donnons un exemple pour illustrer nos résultats principaux.

Mots-clés: calcul fractionnaire, calcul stochastique, problèmes différentiels fractionnaires, problèmes différentiels stochastiques, théorème du point fixe de Banach.

ملخص

في هذا العمل، عرضنا مقدمة قصيرة ومهمة في الحساب الكسري والعشوائي، ثم طرحنا بعض الأمثلة المختلفة لمشاكل التفاضل. الكسري والعشوائي. وأخيرًا، أثبتنا بعض الخصائص النوعية (أي الوجود والوحدانية والفصل التراكمي بين حلين مختلفين) لحلول معادلات التفاضل الكسري والعشوائي، ثم قمنا بطرح مثالاً لتوضيح نتائجنا الرئيسية.

الكلمات المفتاحية: الحساب الكسري، الحساب الكسري، الحساب العشوائي، مشاكل التفاضل الكسري، مشاكل التفاضل العشوائي، نظرية النقطة الثابتة لباناخ.

Dedication

To my dear parents,

For your unconditional love, unwavering support, and endless sacrifices. This thesis is the result of your upbringing, encouragement, and inspiration.

To my family,

To my sisters,

You have been a true support. Your efforts are invaluable.

To my friends,

For your encouragement, smiles, and moments of relaxation that brightened this study period. Your advice and friendship have been invaluable to me.

This dedication is for you, as a testament to our unbreakable bond.

To all of you, this humble dedication.

Souheyla Rounek Morghad

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Souheyla Rounek Morghad

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NOTATIONS

- $\Gamma(z)$: the Gamma function.
- $\beta(z,w)$: the Beta function.
- E_α : the Mittag-Leffler function with a single parameter.
- $E_{\alpha,\beta}$: the Mittag-Leffler function with two parameters.
- ${}^cD_a^\alpha$: the fractional derivative in the sense of Grunwald-Letnikov.
- $I_a^\alpha f$: left-sided fractional integral of Riemann-Liouville of f of order α.
- $I_b^\alpha f$: right-sided fractional integral of Riemann-Liouville of f of order α.
- D_a^α : the fractional derivative in the sense of Riemann-Liouville of order α.
- ${}^cD_a^\alpha$: the fractional derivative in the sense of Caputo of order α.
- $v.a$: random variable.
- f_X : probability density.
- F^X : natural filtration.
- $\mathbb{N}$: set of natural numbers.
- $p.s.$: almost surely.
- $\mathbb{N}^*$: set of non-zero natural numbers.
- $\mathbb{R}$: set of real numbers.
- $\mathbb{R}_+$: set of positive real numbers $[0,+\infty[$.
- $\mathbb{C}$: set of complex numbers.
- Ω : sample space of a random experiment.
- $\mathcal{F}$: σ-algebra.
- $\mathcal{B}(\mathbb{R}^n)$: Borel σ-algebra of $\mathbb{R}^n$.
- $\mathbb{P}$: probability measure.
- $(\Omega,\mathcal{F},\mathbb{P})$: probability space.

- X : random variable.
- F_X : cumulative distribution function of the random variable X.
- $\mathbb{E}(X)$: expected value of the random variable X.
- $\sigma(X)$: σ-algebra generated by the random variable X.
- I : identity operator.
- $M.B$: Brownian motion.

INTRODUCTION

It is generally accepted that the concept of fractional calculus stems from a question posed in 1695 by the Marquis de L'Hôpital (1661-1704) to Gottfried Wilhelm Leibniz (1646-1716), who sought to understand the meaning of Leibniz's currently popular notation, $\frac{d^n y}{dx^n}$, for the nth derivative when n is a real number. L'Hôpital wondered what would happen if n took on a fractional value, such as $\frac{1}{2}$ or even $\frac{1}{12}$.

In his response on September 30, 1695, Leibniz wrote to L'Hôpital: "... It is an apparent paradox from which, one day, useful consequences will be drawn..."

Subsequently, references to fractional derivatives were made by mathematicians such as Euler in 1730, Lagrange in 1772, Laplace in 1812, Lacroix in 1819, Fourier in 1822, Liouville in 1832, Riemann in 1847, Greer in 1859, Holmgren in 1865, Griinwald in 1867, Letnikov in 1868, Sonin in 1869, Laurent in 1884, Nekrassov in 1888, Krug in 1890, and Weyl in 1917.

Fractional differential equations are equations involving fractional derivatives, i.e., derivatives of non-integer order, and they are often used to model physical, biological, economic, and social phenomena that exhibit nonlinear and complex behaviors.

The introduction of the stochastic component adds an additional dimension to these equations as it accounts for uncertainty and random fluctuations present in many real systems. Stochastic fractional differential equations are therefore used to model phenomena where dynamic behavior is influenced by both deterministic and random factors.

In this thesis, in the first chapter we give a short introduction and history of the notion of stochastic fractional differential equation of the Caputo type. This is achieved through a progression with step-by-step concepts.

Chapter 01: We introduced basic concepts that help us understand and solve a stochastic fractional differential equation. We discussed specific functions like the Gamma function, Beta function, and some properties of Laplace transforms. Additionally, we explored the fixed-point theory (Banach), we addressed fractional calculus which includes definitions of fractional derivatives such as Caputo, Riemann-Liouville, and fractional differential equations (FDEs) with examples.

Chapter 02: We began with basic definitions of stochastic calculus, which includes general reminders on probability, expectation, conditional expectation, etc. We also covered Brownian motion and stochastic integrals, along with some examples of stochastic equations.

Chapter 03: In this section, we give the equivalent integral equation (the solution) of a fractional stochastic differential problem of Caputo type, the existence and uniqueness of the solution is proved by the Banach fixed point theorem. Asymptotic separation between solutions of Caputo fractional stochastic differential equations are showed. Then we give an example on FSDE of Caputo type to illustrate our results.

CHAPTER 1

FRACTIONAL CALCULUS AND FRACTIONAL DIFFERENTIAL EQUATIONS

The concept of fractional calculus constitutes a discipline within analysis focusing on the study of integration and differentiation operations of non-integer order. Several definitions of fractional derivatives exist, yet regrettably, they are not all equivalent. In this chapter, the most commonly used definitions are presented, including those of Riemann-Liouville, Liouville, Caputo, and Grunwald-Letnikov.

Fractional calculus

1.1 fractional Derivation in the Grunwald-letnikov ov sense

this definition is based on the calculation of derivatives using finite differences [14].

let $f : \mathbb{R} \to \mathbb{R}^n$. for $h > 0$, denote the τ_h left translation operator :

$$\tau_h f(t) = f(t-h) \tag{1.1}$$

thus, we have

$$f'(t) = \lim_{h \to 0} \frac{1}{h}(f(t) - f(t-h)) = \lim_{h \to 0} \frac{1}{h}\left(id - \tau_h\right) f(t)$$

By denoting $\tau_h^2 = \tau_h \circ \tau_h$, we have : $\tau_h^2 f(t) = f(t-2h)$.

Regarding the second derivative

$$\begin{aligned} f''(t) &= \lim_{h \to 0} \left(\frac{1}{h}\left(id - \tau_h\right)\right)^2 f(t) \\ &= \lim_{h \to 0} \frac{1}{h^2}\left(id - 2\tau_h - \tau_h^2\right) f(t) \\ &= \lim_{h \to 0} \frac{1}{h^2}(f(t) - 2f(t-h) + f(t-2h)) \end{aligned}$$

More generally,the derivative n^{ime} of f is given by

$$\begin{aligned} f^{(n)}(t) &= \lim_{h\to 0}\frac{1}{h^n}\left(id-\tau_h\right)^n f(t) \\ &= \lim_{h\to 0}\frac{1}{h^n}\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)^{id^{n-k}}\left(-\tau_h\right)^k f(t) \\ &= \lim_{h\to 0}\frac{1}{h^n}\sum_{k=0}^{n}(-1)^k\left(\begin{array}{c} n \\ k \end{array}\right) f(t-kh) \end{aligned}$$

where

$$\left(\begin{array}{c} n \\ k \end{array}\right)=\frac{n!}{k!(n-k)!}=\frac{n(n-1)\ldots(n-k+1)}{k!}$$

It is possible to extender to $k>n$, by setting $\left(\begin{array}{c} n \\ k \end{array}\right)=0$. the formula (2.1) then becomes:

$$f^{(n)}(t)=\lim_{h\to 0}\frac{1}{h^n}\sum_{k=0}^{\infty}(-1)^k\left(\begin{array}{c} n \\ k \end{array}\right) f(t-kh)$$

the generalization of this formula using the Gamma function, for α non integer (with $0\leq n-1<\alpha<n$) by setting for $\alpha\in\mathbb{R}^+/\mathbb{N}$ et $k\in\mathbb{N}$, Note that $\left(\begin{array}{c} a \\ k \end{array}\right)=0$ even if $k>\alpha$

$${}^{G}D_a^{\mathrm{a}}f(t)=\lim_{h\to 0}\frac{1}{h^\alpha}\sum_{k=0}^{\infty}(-1)^k\left(\begin{array}{c} \alpha \\ k \end{array}\right) f(t-kh)$$

According to proposition 1.6, we have

$$(-1)^k\left(\begin{array}{c} \alpha \\ k \end{array}\right)=\frac{\Gamma(k-\alpha)}{\Gamma(k+1)\Gamma(-\alpha)}$$

this gives us:

$${}^{G}D_a^{\alpha}f(t)=\lim_{h\to 0}\frac{1}{h^\alpha}\sum_{k=0}^{\infty}\frac{\Gamma(k-\alpha)}{\Gamma(k+1)\Gamma(-\alpha)}f(t-kh)$$

and

$$\begin{aligned} {}_{G}D_a^{-\alpha}f(t) &= \lim_{h\to 0}\frac{1}{h^{-\alpha}}\sum_{k=0}^{\infty}\frac{\Gamma(k+\alpha)}{\Gamma(k+1)\Gamma(\alpha)}f(t-kh) \\ &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}f(\tau)d\tau \end{aligned}$$

if f is of class C^n then using integration by parts, we obtain :

$${}_{G}D_a^{-\alpha}f(t)=\sum_{k=0}^{n-1}\frac{f^{(k)}(a)(t-a)^{k+\alpha}}{\Gamma(k+\alpha+1)}+\frac{1}{\Gamma(n+\alpha)}\int_a^t (t-\tau)^{n+\alpha-1}f^{(n)}(\tau)d\tau$$

and also :

$${}^{G}D_a^{\alpha}f(t)=\sum_{k=0}^{n-1}\frac{f^{(k)}(\alpha)(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)}+\frac{1}{\Gamma(n-\alpha)}\int_a^t (t-\tau)^{n-\alpha-1}f^{(n)}(\tau)d\tau$$

Example 1.1.1
the derivative $f(t) = (t-a)^p$ in the sense of Grunwald-Letnikov. let p be non integer and $0 \leq n-1 < \alpha < n$ with $p > n-1$ then we have : $f^{(k)}(a) = 0$ for $k = 0, 1, \ldots, n-1$ and $f^{(n)}(\tau) = \frac{\Gamma(p+1)}{\Gamma(p-n+1)}(\tau - a)^{p-n}$ thus:

$$^{G}D_{a}^{\alpha}(t-a)^p = \frac{\Gamma(p+1)}{\Gamma(n-\alpha)\Gamma(p-n+1)} \int_a^t (t-\tau)^{n-n-1}(\tau-a)^{p-n}d\tau$$

Taking $\tau = a + s(t-a)$ we have :

$$\begin{aligned} {}_{G}D_{a}^{\alpha}(t-a)^p &= \frac{\Gamma(p+1)}{\Gamma(n-\alpha)\Gamma(p-n+1)}(t-a)^{p-\alpha} \int_0^1 (1-s)^{n-a+1} s^{p-n} ds \\ &= \frac{\Gamma(p+1)\beta(n-\alpha, p-n+1)}{\Gamma(n-\alpha)\Gamma(p-n+1)}(t-a)^{p-\alpha} \\ &= \frac{\Gamma(p+1)\Gamma(n-\alpha)\Gamma(p-n+1)}{\Gamma(n-\alpha)\Gamma(p-n+1)\Gamma(p-\alpha+1)}(t-a)^{p-a} \\ &= \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}(t-a)^{p-n} \end{aligned}$$

Remark 1.1.1
the derivative of a constant function in the sense of Grunwald-letnikov is neither zero nor constant.
if $f(t) = C$ and α is non integer we have : $f^{(k)}(t) = 0$ for $k = 1, 2, \ldots, n$

$$\begin{aligned} ^{G}D_{a}^{a}f(t) &= \frac{C}{\Gamma(1-\alpha)}(t-a)^{-a} + \underbrace{\sum_{k=1}^{n-1} \frac{f^{(k)}(a)(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)}}_{0} \\ &\quad + \underbrace{\frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-a-1} f^{n}(\tau) d\tau}_{0} \\ &= \frac{C}{\Gamma(1-\alpha)}(t-a)^{-a} \end{aligned}$$

1.1.1 Composition with derivatives of integer order

Proposition 1.1.1 *[28]*

for m a positive integer and α non integer with :

$$\frac{d^m}{dt^m}\left({}^{G}D_{\mathrm{a}}^{a}f(t)\right) = {}^{G}D_{a}^{m+a}f(t) \tag{1.2}$$

And

$$^{G}D_{a}^{\alpha}\left(\frac{d^m}{dt^m}(f(t))\right) = {}^{G}D_{a}^{m+\alpha}f(t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(a)(t-a)^{k-\alpha-m}}{\Gamma(k-\alpha-m+1)} \tag{1.3}$$

Proof
For m positive integer and α non integer with $(n-1 < \alpha < n)$ we have :

$$\begin{aligned} \tfrac{d^m}{dt^m}\left({}^{G}D_{a}^{\alpha}f(t)\right) = \textstyle\sum_{k=0}^{m-1} \tfrac{f^{(k)}(a)(t-a)^{k-(a+m)}}{\Gamma(k-(\alpha+m)+1)} + \tfrac{1}{\Gamma(n+m-(p+m))} \\ \times \textstyle\int_a^t (t-\tau)^{\mathrm{n}+m-(p+m)-1} f^{n+m}\tau d\tau \end{aligned}$$

then :

$$\frac{d^m}{dt^m}\left({}^G D_n^a f(t)\right) = {}^G D_a^{m+a} f(t)$$

but :

$$\begin{aligned}
{}^G D_{\mathrm{a}}\left(\frac{d^m}{dt^m} f(t)\right) &= \sum_{k=0}^{m-1} \frac{f^{(m+k)}(a)(t-a)^{(k-a)}}{\Gamma(k-\alpha+1)} + \frac{1}{\Gamma(n-p))} \int_a^t (t-\tau)^{n-\alpha-1} f^{n+m}(\tau) d\tau \\
&= \sum_{k=0}^{m+m-1} \frac{f^{(k)}(a)(t-a)^{k-(\alpha+m)}}{\Gamma(k-(\alpha+m)+1)} \\
&\quad + \frac{1}{\Gamma(n+m-(p+m))} \int_a^t (t-\tau)^{m+m-(p+m)-1} f^{n+m}(\tau) d\tau \\
&\quad - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(t-a)^{k-(a+m)}}{\Gamma(k-(\alpha+m)+1)} \\
&= {}^G D_a^{m+a} f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(t-a)^{k-\alpha-m}}{\Gamma(k-\alpha-m+1)}
\end{aligned}$$

Remark 1.1.2

It is deduced that fractional differentiation and conventional differentiation commute only if: $f^{(k)}(a) = 0$ *for all* $k = 0, 1, 2, \ldots, m-1$.

1.1.2 composition with fractional derivatives

Proposition 1.1.2

1. *if* $\alpha' < 0$ *and* $\alpha \in \mathbb{R}$ *then :*

$$ {}^G D_a^{\mathrm{a}}\left({}^G D_{\mathrm{a}}^{a'}(f(t))\right) = {}^G D_a^{\mathrm{a}+\mathrm{a}'} f(t) \tag{1.4}$$

2. *if* $0 \le m-1 < \alpha' < m$ *and* $\alpha < 0$ *then :*

$$ {}^G D_a^{\mathrm{a}}\left({}^C D_{\mathrm{a}}^{a'}(f(t))\right) = {}^G D_{\mathrm{a}}^{\mathrm{a}+\mathrm{a}'} f(t) \tag{1.5}$$

only if $f^{(k)}(a) = 0$ *for all* $k = 0, 1, 2, \ldots, m-2$

3. *Si* $0 \le m-1 < \alpha' < m$ *and* $0 \le n-1 < \alpha < n$ *then :*

$$\begin{aligned} {}^G D_a^a\left({}^G D_a^a(f(t))\right) &= {}^G D_a^a\left({}^G D_a^a(f(t))\right) \\ &= {}^G D_a^{a+a'} f(t) \end{aligned} \tag{1.6}$$

only if $f^{(k)}(a) = 0$ *for all* $k = 0, 1, 2, \ldots, r-2$ *with* $r = \max(m, n)$

Proof

1. if $\alpha < 0$ and $\alpha < 0$ then :

$$
\begin{aligned}
{}^G D_a^\alpha \left({}^G D_a^{a'}(f(t))\right) &= \frac{1}{\Gamma(-\alpha)} \int_a^t (t-\tau)^{-a-1} \left({}^G D_a^{a'}(f(\tau))\right) d\tau \\
&= \frac{1}{\Gamma(-\alpha)\Gamma(-\alpha')} \int_a^t (t-\tau)^{-\alpha-1} d\tau \int_a^t (\tau-s)^{-a'-1} f(s) ds \\
&= \frac{1}{\Gamma(-\alpha)\Gamma(-\alpha')} \int_a^t f(s) ds \int_a^t (\tau-s)^{-a'-1} (t-\tau)^{-a-1} d\tau \\
&= \frac{1}{\Gamma(-(\alpha+\alpha'))} \int_a^t (t-s)^{-a-a'-1} f(s) ds \\
&= {}^G D_a^{\mathrm{n}+\alpha'} f(t)
\end{aligned}
$$

if $\alpha' < 0$ and $0 \leq n-1 < \alpha < n$ we have $\alpha = n + (\alpha - n)$with $(\alpha - n) < 0$ then :

$$
\begin{aligned}
{}^G D_a^a \left({}^G D_a^{a'}(f(t))\right) &= \frac{d^n}{dt^n} \left\{{}^C D_a^{\mathrm{a-n}} \left({}^G D_a^{\mathrm{a}'}(f(t))\right)\right\} \\
&= \frac{d^n}{dt^n} \left({}^G D_a^{a'+a-n}(f(t))\right) \\
&= {}^G D_a^{\mathrm{a}+a'} f(t)
\end{aligned}
$$

2. for $0 \leq m-1 < \alpha' < m$ and $\alpha < 0$ we have :

$$
{}^G D_a^{a'} f(t) = \sum_{k=0}^{m-1} \frac{f^{(k)}(a)(t-a)^{k-a'}}{\Gamma(k-\alpha'+1)} + \frac{1}{\Gamma(m-\alpha')} \int_a^t (t-\tau)^{m-a'-1} f(m)(t) d\tau
$$

and $(t-a)^{k-a}$ they have non integrable singularities then ${}^G D_a^a \left({}^G D_a^a(f(t))\right)$ only exists $f_{(k)}(a) = 0$ for all $k = 0, 1, 2, \ldots, m-2$ this case we have :

$$
{}^G D_a^{a'} f(t) = \frac{f^{(m-1)}(a)(t-a)^{m-1-\alpha'}}{\Gamma(m-\alpha')} + {}^G D_a^{\alpha'-m} f^m(t)
$$

then :

$$
\begin{aligned}
{}^G D_a^\alpha \left({}^G D_a^{a'}(f(t))\right) &= \frac{f^{(m-1)}(a)(t-a)^{m-1-\alpha'-\alpha}}{\Gamma(m-\alpha'-\alpha)} + {}^G D_a^{\mathrm{a}+a'-m} f^m(t) \\
&= \frac{f^{(m-1)}(\alpha)(t-a)^{m-1-(a'+a)}}{\Gamma(m-\alpha'-\alpha)} \\
&+ \frac{1}{\Gamma(m-(\alpha'+\alpha))} \frac{1}{\Gamma(m-\alpha')} \int_a^t (t-\tau)^{m-(a+a')-1} f(m)(t) d\tau \\
&= {}^G D_a^{\alpha+a'} f(t)
\end{aligned}
$$

3. for $0 \leq m-1 < \alpha' < m$ and $0 \leq n-1 < \alpha < n$ we have :

$$
{}^G D_n^\alpha \left({}^G D_n^{\alpha'}(f(t))\right) = \frac{d^{\mathrm{n}}}{dt^n} \left\{{}^G D_\sim^{\mathrm{a-n}} \left({}^G D_n^{\mathrm{a}}(f(t))\right)\right\}
$$

Si $f^{(k)}(a) = 0$ for all $k = 0, 1, 2, \ldots, m-2$ then :

$$
{}^G D_{\mathrm{a}}^{a-n} \left({}^G D_{\mathrm{a}}^{a'}(f(t))\right) = {}^G D_a^{\mathrm{a}+\mathrm{a}'-m} f(t)
$$

therefore :

$$
\begin{aligned}
{}^G D_a^a \left({}^G D_{\mathrm{a}}'(f(t))\right) &= \frac{d^n G}{dt^2} D_a^{a+a'-n} f(t) \\
&= {}^G D_a^{a+a'} f(t)
\end{aligned}
$$

1.1.3 the Laplace transform of fractional derivative in the sense of Grunwald-Letnikov

let f be a function that has the Laplace transform $F(s)$. for $0 \leq \alpha < 1$ we have :

$$^{G}D_{0}^{\alpha} f(t) = \frac{f(0)t^{-\alpha}}{\Gamma(n-\alpha)} + \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} (t-\tau)^{-a} f'(\tau) d\tau \tag{1.7}$$

then :

$$\begin{aligned} L\left[{}^{G}D_{0}^{\alpha} f(t)\right](s) &= \frac{f(0)}{s^{1-a}} + \frac{1}{s^{1-\alpha}}[sF(s) - f(0)] \\ &= s^{\alpha} F(s) \end{aligned} \tag{1.8}$$

for $\alpha \geq 1$ there does not exist a Laplace transform in the classical sense, but in the sense of distributions, we also have:

$$L\left[{}^{G}D_{0}^{\infty} f(t)\right](s) = s^{a} F(s) \tag{1.9}$$

1.2 Riemann-Liouville Fractional Integral

This section introduces the elementary definitions and some properties of the Riemann-Liouville fractional integral.

Let f be a real, continuous, and integrable function on the interval $[a, b]$. We consider the integral

$$\begin{aligned} I^{1} f(t) &= \int_{a}^{t} f(\tau) d\tau \\ I^{2} f(t) &= \int_{a}^{t} I^{1} f(u) du \\ &= \int_{a}^{t} \left(\int_{a}^{u} f(s) ds \right) du \\ &= \int_{a}^{t} \left(\int_{s}^{t} du \right) f(s) ds \\ &= \int_{a}^{t} (t-s) f(s) ds \end{aligned}$$

By repeatedly applying this process n times, we obtain, according to Cauchy's formula:

$$I^{n} f(t) = \frac{1}{(n-1)!} \int_{a}^{t} (t-s)^{n-1} f(s) ds \tag{1.10}$$

And, using the generalization of the factorial function via the Gamma function: $\Gamma(n) = (n-1)!$. Riemann realized that the right-hand side could make sense even when n takes on non-integer values. He defined the fractional integral as follows:

Let $f \in C[a, b], \alpha \in \mathbb{R}_{+}$. The Riemann-Liouville fractional integral of f of order α, denoted by $I_{\mathrm{a}+}^{\mathrm{a}} f$, is defined by:

Definition 1.2.1 *[27]*
let $f \in C[a, b], \alpha \in \mathbb{R}_{+}$, we it the Riemann-Liouville fractional (left-sided) integral off of order α,denoted by $I_{\mathrm{a}+}^{\mathrm{a}} f$ the function defined by :

$$I_{\alpha+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} (t-s)^{\alpha-1} f(s) ds \tag{1.11}$$

the right-sided Riemann-Liouville fractional integral of the function f of order α, denoted by $I_{\mathrm{b}}^{\mathrm{a}}f$ the function defined by :

$$I_{b-}^{a} f(t)=\frac{1}{\Gamma(\alpha)} \int_{t}^{b}(t-s)^{\mathrm{a}-1} f(s) ds \tag{1.12}$$

Remark 1.2.1

For the remainder of this work, we will exclusively utilize the left-sided integral and employ the notation α.

throughout what follows, we will only use the left-sided integral and denote it as I_{a}^{a}.

Example 1.2.1

let $f(t)=t^{\beta}$ with $\beta>-1$ we have :

$$\begin{aligned} I_{\alpha}^{\alpha} f(t) &= I_{\alpha}^{\alpha} t^{\beta} \\ &= \frac{1}{\Gamma(\alpha)} \int_{a}^{t} s^{a}(t-s)^{\alpha-1} ds \end{aligned}$$

by setting $s=tu,(2.3)$ becomes

$$I_{\alpha}^{\alpha} f(t)=\frac{1}{\Gamma(\alpha)} \int_{0}^{1}(tu)^{\beta}(1-tu)^{\alpha-1} t du$$

Using the definition of the Beta function from (1.2) and Proposition 1.7, we obtain:

$$\begin{aligned} I_{a}^{\alpha} f(t) &= \frac{t^{\beta+\alpha}}{\Gamma(\alpha)} \int_{0}^{1} u^{\beta}(1-u)^{\mathrm{a}-1} t du \\ &= \frac{t^{\beta+a}}{\Gamma(\alpha)} \beta(\beta+1, \alpha) \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \end{aligned}$$

Proposition 1.2.1 *[12]*
Let f be an integrable and bounded function, and let α and α' be two strictly positive real numbers. Then

$$I_{a}^{\dot{\alpha}}\left[I_{a}^{\alpha'} f(t)\right]=I_{a}^{\alpha+a'} f(t) \tag{1.13}$$

Proof

$$\begin{aligned} I_{a}^{\alpha}\left[I_{a}^{\alpha'} f(t)\right] &= \frac{1}{\Gamma(\alpha)} \int_{0}^{t-\alpha} s^{\alpha'-1} I_{a}^{\alpha} f(t-s) ds \\ &= \frac{1}{\Gamma(\alpha) \Gamma\left(\alpha'\right)} \int_{0}^{t-a} s^{\alpha'-1} ds \int_{a}^{t-s}(t-s-u)^{\alpha-1} f(u) du \\ &= \frac{1}{\Gamma(\alpha) \Gamma\left(\alpha'\right)} \int_{a}^{t} f(u) du \int_{a}^{t-\alpha} t^{\alpha'-1}(t-u-s)^{\alpha-1} ds \end{aligned}$$

let $\quad s=v(t-u)$
Then $\quad ds=(t-u) dv$

Hence, it follows that:

$$
\begin{aligned}
& =\frac{1}{\Gamma(\alpha)\Gamma\left(\alpha^{\prime}\right)} \int_a^t f(u) d u \int_0^1(v(t-u))^{\alpha^{\prime}-1}(t-u-v(t-u))^{\alpha-1}(t-u) d v \\
& =\frac{1}{\Gamma(\alpha)\Gamma\left(\alpha^{\prime}\right)} \int_a^t(t-u)^{\alpha+\alpha^{\prime}-1} f(u) d u \int_0^1 v^{\alpha^{\prime}-1}(1-v)^{\alpha-1} d v \\
& =\frac{1}{\Gamma(\alpha)\Gamma\left(\alpha^{\prime}\right)} \int_a^t(t-u)^{\alpha+\alpha^{\prime}-1} f(u) d u \beta\left(\alpha^{\prime}, \alpha\right) \\
& =\frac{1}{\Gamma\left(\alpha+\alpha^{\prime}\right)} \int_a^t(t-u)^{\alpha+\alpha^{\prime}-1} f(u) d u \\
& =I_a^{\left(\alpha+\alpha^{\prime}\right)} f(t)
\end{aligned}
$$

Proposition 1.2.2
let f, g be two continuous and integrable function over $[a, b], I_a^{\infty}$ is linear, that is:

$$
\forall \gamma, \lambda \in \mathbb{R} \quad \alpha>0 \quad \text{on } a \quad I_a^\alpha[\lambda f(t)+\gamma g(t)]=\lambda I_a^a f(t)+\gamma I_a^a g(t) \tag{1.14}
$$

Proof

$$
\begin{aligned}
I_a^\alpha[\lambda f(t)+\gamma g(t)] & =\frac{1}{\Gamma(\alpha)} \int_a^t(t-s)^{\alpha-1}[\lambda f(s)+\gamma g(s)] d s \\
& =\lambda \frac{1}{\Gamma(\alpha)} \int_a^t(t-s)^{\alpha-1} f(s) d s+\gamma \frac{1}{\Gamma(\alpha)} \int_a^t(t-s)^{\alpha-1} g(s) d s \\
& =\lambda I_a^\alpha f(t)+\gamma I_a^\alpha g(t)
\end{aligned}
$$

Proposition 1.2.3 *[27]*
the Laplace transform of the Riemann-liouville fractional integral for $a=0$ of a function f,that has the Laplace transform $F(s)$ in the half-plane $\operatorname{Re}(s)>0$, is given by:

$$
L\left(I^\alpha f\right)(s)=s^{-\alpha} F(s) \tag{1.15}
$$

1.3 fractional derivative in the sense of Riemann-Liouville

Definition 1.3.1 *[16]*
let f be an integrable function over $[a, b]$ then the fractional derivative of order α (with $n-1<\alpha<n, n \in \mathbb{N}^*$) in the sense of Riemann-Liouville $D_a^{\mathrm{a}} f$ is defined by :

$$
D_a^\alpha f(t)=\frac{1}{\Gamma(n-\alpha)} \frac{d^n}{d t^n} \int_a^t \frac{f(s)}{(t-s)^{\alpha-n+1}} d s \tag{1.16}
$$

this fractional order derivative can also be written as follows :

$$
D_a^\alpha f(t)=\frac{d^m}{d t^n}\left\{I_a^{n-\alpha} f(t)\right\} \tag{1.17}
$$

Example 1.3.1

The Riemann-Liouville fractional derivative of $f(t)=(t-a)^p$. Let α be a non-integer with $0 \leq n-1<\alpha<n$ and $p>-1$, then we have:

$$
\begin{aligned}
D_a^\alpha f(t) & =D_{a(\alpha}^\alpha(t-a)^p \\
& =\frac{1}{\Gamma(n-\alpha)} \frac{d t}{d t^n} \int_a^t \frac{(\tau-a)^p}{(t-\tau)^{\alpha-n+1}} d \tau
\end{aligned}
$$

By changing the variable $\tau = a + s(t-a)$ we have :

$$\begin{aligned} D_a^{\alpha}(t-a)^p &= \frac{1}{\Gamma(n-\alpha)}\frac{dt^n}{dt^n}(t-a)^{\mathrm{n}+p-\alpha}\int_0^1 (1-s)^{\alpha-n+1}s^p ds \\ &= \frac{\Gamma(n+p-\alpha+1)\beta(n-\alpha,p+1)}{\Gamma(n-\alpha)\Gamma(p-\alpha+1)}(t-a)^{p-\alpha} \\ &= \frac{\Gamma(n+p-\alpha+1)\beta(n-\alpha,p+1)\Gamma(p+1)}{\Gamma(n-\alpha)\Gamma(p-\alpha+1)\Gamma(n+p-\alpha+1)}(t-a)^{p-\alpha} \\ &= \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}(t-a)^{p-\alpha} \end{aligned}$$

Proposition 1.3.1 *[21]*
if $\alpha = n \in \mathbb{N}$ we have :

$$D_a^0 f(t) = f(t), D_a^2 f(t) = f^{(2)}, \ldots, D_a^n f(t) = f^{(n)}(t) \tag{1.18}$$

Remark 1.3.1 *[26]*

The non-integer order derivative of a constant function in the Riemann-Liouville sense is neither zero nor constant, However, we have:

$$D_a^{\mathrm{a}} C = \frac{C}{\Gamma(1-\alpha)}(t-a)^{-a} \tag{1.19}$$

On note $\frac{d^m}{d^m}$ by D^n.

1.3.1 Composition with the fractional integral

Proposition 1.3.2
let $\alpha > 0$ et $n = [\alpha]+1$ then for every integer $m \in \mathbb{N}^*$ we have:

$$D_a^{\alpha} f(t) = D_a^m \left(I^{m-\alpha} f(t), \quad m > \alpha \tag{1.20}$$

Proof
as $m \geq n$, we have :

$$\begin{aligned} D_a^m I^{m-\alpha} f(t) &= D^n D^{m-n} I^{m-n} I^{n-a} f(t) \\ &= D^n I^{n-a} f(t) \\ &= D_a^a f(t) \end{aligned}$$

Lemma 1.3.1
let $\alpha > 0$ and $f \in L^1[a,b]$, then the equality:

$$D_a^{\alpha} I_a^{\alpha} f(t) = f(t) \tag{1.21}$$

is true almost every $t \in [a,b]$

Proof
Using the definition, we have :

$$\begin{aligned} D_a^{\alpha} I_a^{\alpha} f(t) &= D^n I_a^{n-\alpha} I_a^{\alpha} f(t) \\ &= D^n I_a^n f(t) \\ &= f(t) \end{aligned}$$

Theorem 1.3.1
let $\alpha, \beta > 0$ and $n-1 \leq \alpha < n, m-1 \leq \alpha < m$ such that $(n, m \in \mathbb{N})$ then :

1. if $\alpha > \beta > 0$, then for $f \in L^1[a,b]$ the equality :

$$D_a^{\beta}\left(I^{\alpha} f\right)(t) = I^{\alpha-\beta} f(t) \tag{1.22}$$

is valid almost everywhere on $[a,b]$.

2. if there exist a function $\varphi \in L^1[a,b]$ tel such that $f = I_a^{\alpha}\varphi$ then :

$$I_a^{\alpha} D_a^{\alpha} f(t) = f(t) \tag{1.23}$$

for almost every $x \in [a,b]$.

3. Si $\beta \geq \alpha > 0$ and the fractional derivative $D_{\alpha}^{s-a} f$ exist, then :

$$D_a^{\beta}\left(I_a^{\alpha} f\right)(t) = D_a^{\beta-\alpha} f(t) \tag{1.24}$$

Proof
Using definition 2.2 and proposition 2.3 we obtain:

1. for $\alpha > \beta > 0$, then for all $n \geq m$, we have :

$$\begin{aligned} D_a^{3}\left(I_a^{\alpha}\right) f(t) &= D^n I^{n-\beta}\left(I^{\alpha}\right) f(t) \\ &= D^n\left(I_a^{n+\alpha-\beta}\right) f(t) \\ &= D^n\left(I_a^n\left(I_a^{\alpha-\beta}\right) f(t)\right. \\ &= I_a^{\alpha-\beta} f(t) \end{aligned}$$

almost for every $t \in [a,b]$

2. by relation 2.5, we obtain :

$$\begin{aligned} I_a^{\alpha} D_a^{\alpha} f(t) &= I_a^{\alpha}\left(D_a^{\alpha} I_a^{\alpha}\varphi(t)\right) \\ &= I_a^{\alpha}\varphi(t) \\ &= f(t) \end{aligned}$$

3. on a :

$$\begin{aligned} D_a^{\beta}\left(I_a^{\alpha}\right) f(t) &= D_a^m I_a^{m-\beta} I_a^{\alpha} f(t) \\ &= D^m I^{m-(\beta-\alpha)} f(t) \\ &= D_a^{\beta-\alpha} f(t) \end{aligned}$$

Exist for $i-1 \leq \beta-\alpha < i$ et $i \leq m$

1.3.2 Composition with integer order derivatives

Theorem 1.3.2
let $\alpha, \beta > 0$ and $n-1 \leq \alpha < n, m-1 \leq \alpha < m$ such that $(n, m \in \mathbb{N})$ then: for $\alpha > 0$, $k \in \mathbb{N}^*$. if the fractional derivatives $D_a^{\alpha} f$ and $D_a^{k+a} f$ exist, then :

$$D^k\left(D_a^{\alpha} f(t)\right) = D_a^{k+a} f(t) \tag{1.25}$$

Proof
On a :

$$\begin{aligned} D^k\left[D_a^{\alpha} f(t)\right] &= D^k D^{\mathrm{n}} I_a^{n-\alpha} f(t) \\ &= D^{k+n} I_a^{k+n-a+k-k} f(t) \\ &= D^{k+n} I_a^{k+n-(a+k)} f(t) \\ &= D_a^{k+a} f(x) \end{aligned}$$

Hence the result.

Proposition 1.3.3 *[26]*
for $\alpha > 0, n \in \mathbb{N}^*$. if the fractional derivative $D_a^{n+\alpha} f$ and $1 \leq k \leq n-1$ exist, then

$$D_a^\alpha \left(D^n f(t)\right) = \left(D_a^{n+\alpha} f(t)\right) - \sum_{k=1}^{n-1} \frac{f^{(k)}(a)(t-a)^{k-\alpha-n}}{\Gamma(k-\alpha-n+1)} \tag{1.26}$$

Remark 1.3.2
Fractional differentiation and conventional differentiation commute only if: $f^{(k)}(a) = 0$ for all $k = 0, 1, 2, \ldots, n-1$.

1.3.3 composition with fractional derivatives

Proposition 1.3.4 *[26]*
for all $n-1 \leq \alpha < n$ and $m-1 \leq \beta < m$ we have :

$$D_a^\alpha \left(D_a^3 f(t)\right) = D_a^{\alpha+\beta} f(t) - \sum_{k=1}^{m} \left[D_a^{j-k} f(t)\right]_{t=a} \frac{(t-a)^{-\alpha-k}}{\Gamma(-\alpha-k+1)} \tag{1.27}$$

Proposition 1.3.5 *[26]*
for all $n-1 \leq \alpha < n$ and $m-1 \leq \beta < m$ we have:

$$D_a^\beta \left(D_a^\alpha f(t)\right) = D_a^{\alpha+\beta} f(t) - \sum_{k=1}^{n} \left[D_a^{\alpha k} f(t)\right]_{t-a} \frac{(t-a)^{-\beta-k}}{\Gamma(-\beta-k+1)} \tag{1.28}$$

assume that if $\alpha = \beta$ and $\left[D_a^{j-k} f(t)\right]_{t=a}$ for all $k = 1, 2, \ldots, m$ and $\left[D_a^{\mathrm{a}-k} f(t)\right]_{tma}$ for all $k = 1, 2, \ldots, n$

Fractional Derivative in the sense of Caputo

1.4 Fractional Derivative in the Caputo sense

In mathematical modeling, the use of Riemann-Liouville fractional derivatives involves initial conditions containing the limit values of fractional derivatives at the lower bound $t = a$. Mr. Caputo proposed a solution to this issue. This section provides a definition and some properties of the Caputo fractional derivative. This section provides a definition and some properties of the Caputo fractional derivative.

Definition 1.4.1 *[16]*
for any α , a strictly positive real number,the caputo fractional derivative ${}^cD_a^\alpha f$ of order α on $[a, b]$, is defined as :

$$\begin{aligned} {}^cD_a^\alpha f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^n(s)}{(t-s)^{\alpha-n+1}} ds \\ &= I_a^{n-\alpha} f^{(n)}(t) \end{aligned} \tag{1.29}$$

Example 1.4.1
the caputo derivative of $f(t) = (t-a)^p$. let α be non integer $0 \leq n-1 < \alpha < n$ and $p > -1$ then we have :

$$\begin{aligned} {}^cD_a^\alpha f(t) &= {}^cD_a^\alpha (t-a)^p \\ &= \frac{\Gamma(p+1)}{\Gamma(n-\alpha)\Gamma(p-n+1)} \int_a^t (\tau-a)^{p-n} (t-\tau)^{\mathrm{n}-\alpha-1} d\tau \end{aligned}$$

Taking $\tau = a + s(t-a)$ *we get :*

$$
\begin{aligned}
{}^{c}D_a^\alpha (t-a)^p &= \frac{\Gamma(p+1)}{\Gamma(n-\alpha)\Gamma(p-n+1)}(t-a)^{p-\alpha}\int_0^1 (1-s)^{n-\alpha-1}s^{p-n}ds \\
&= \frac{\Gamma(p+1)\beta(n-\alpha, p-n+1)}{\Gamma(n-\alpha)\Gamma(p-n+1)}(t-a)^{p-\alpha} \\
&- \frac{\Gamma(p+1)\Gamma(n-\alpha)\Gamma(p-n+1)}{\Gamma(n-\alpha)\Gamma(p-\alpha+1)\Gamma(p-\alpha+1)}(t \quad a)^{p-\alpha} \\
&= \frac{\Gamma(p+1)}{\Gamma(p\alpha+1)}(t-a)^{p-\alpha}
\end{aligned}
$$

1.4.1 composition with the fractional integral

Theorem 1.4.1 ([26],[27])
let $\alpha > 0$*, and* f *is a continuous function on* $[a, +\infty)$ *in* $\mathbb{R}$ *we have :*

$$I_a^\alpha \left({}^{c}D_a^\alpha f(t)\right) = f(t) - \sum_{k=1}^{n-1} \frac{f^{(k)}(a)(t-a)^k}{k!} \tag{1.30}$$

Theorem 1.4.2 *[23]*
let $\alpha > 0$ *and* f *be a continuous function on* $[a, +\infty)$ *in* $\mathbb{R}$ *we have :*

$${}^{c}D_a^\alpha \left(I_a^\alpha f\right)(t) = f(t) \tag{1.31}$$

Remark 1.4.1
the Caputo derivative operator can be considered as a left-inverse of the fractional integration operator, but it does not constitute a right-inverse.

Remark 1.4.2
the conclusion of theorem 2.14 indicates that differentiating a function f *in the Caputo is equivalent to a fractional derivative of the remainder in the Taylor expansion of* f*.*

Theorem 1.4.3 *[21]*
Si $\alpha = n \in \mathbb{N}$ *we have :*

$${}^{c}D_a^\alpha f(t) = f^{(n)}(t) \tag{1.32}$$

that is to say :

$${}^{c}D_a^0 f(t) = f(t), {}^{c}D_a^2 f(t) = f^{(2)}, \ldots, {}^{c}D_a^n f(t) = f^{(n)}(t) \tag{1.33}$$

1.5 Relationship between the Riemann-Liouville Fractional Derivative and the Caputo Fractional Derivative

The following theorem establishes the connection between the Caputo fractional and Riemann-Liouville fractional derivatives.

Theorem 1.5.1 ([24],[27])

let $\alpha \geq 0$ *(with* $m-1 \leq \alpha < n$ *and* $m \in \mathbb{N}^*$*) if* f *has* $m-1$ *derivatives at and if* ${}^{\circ}D_a^\alpha f$ *and* $D_a^a f$ *exist, then : for almost every* $t \in [a, +\infty)$ *:*

$${}^{c}D_a^\alpha f(t) = D_a^\alpha f(t) - \sum_{j=1}^{m-1} \frac{(t-a)^{j-a}}{\Gamma(-\alpha+1+j)} f^{(j)}(a) \tag{1.34}$$

Proof
we have :

$$I_a^\alpha f(t) = \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} f(a) + \frac{1}{\Gamma(\alpha+1)} \int_a^t (t-s)^\alpha f'(s)ds$$
$$= \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} f(a) + I_a^{\alpha+1} f'(t)$$
$$I_a^\alpha f(t) = \sum_{j=1}^{m-1} \frac{(t-a)^{\alpha+j}}{\Gamma(\alpha+1+j)} f^{(j)}(a) + I_a^{\alpha+n} f^n(t)$$

Setting $n = m$ and $\alpha = m - \alpha$ we find :

$$I_a^{m-\alpha} f(t) = I_a^{2m-\alpha} f^{(m)}(t) = \sum_{j=1}^{m-1} \frac{(t-a)^{m-\alpha+j}}{\Gamma(m-\alpha+1+j)} f^{(j)}(a)$$

then

$$\frac{d^m}{dt^m}\left[I_a^{m-\alpha} f(t)\right] = \frac{d^m}{dt^m}\left[I_a^{2m-\alpha} f(t)^{(m)} + \sum_{j=1}^{m-1} \frac{(t-a)^{m-\alpha+j}}{\Gamma(m-\alpha+1+j)} f^{(j)}(a)\right]$$
$$= \frac{d^m}{dt^m}\left[I_a^{2m-\alpha} f^{(m)}(t)\right] + \sum_{j=1}^{m-1} \frac{(t-a)^{j-\alpha}}{\Gamma(-\alpha+1+j)} f^{(j)}(a)$$
$$= I_a^{m-\alpha} f^{(m)}(t) + \sum_{j=1}^{m-1} \frac{(t-a)^{j-\alpha}}{\Gamma(-\alpha+1+j)} f^{(j)}(a)$$

Therefore

$$D_a^\alpha f(t) = {}^c D_a^\alpha f(t) + \sum_{j=1}^{m-1} \frac{(t-a)^{j-\alpha}}{\Gamma(-\alpha+1+j)} f^{(j)}(a)$$

Corrolaire 1.5.1 *[29]*
for $\alpha > 0$, we deduce that if $f^{(k)}(a) = 0$for $k = 0, 1, 2, \ldots, n-1, (n = [\alpha] + 1)$ then we will have

$$D_a^\alpha f(t) = {}^c D_a^\alpha f(t) \tag{1.35}$$

1.6 General properties of fractional derivatives

1.6.1 Linearity

Proposition 1.6.1 ([9],[25])

let f, g be two continuous functions on $[a, b]$, Fractional differentiation is a linear operation,i.e., for any: $\forall \gamma, \lambda \in \mathbb{R}, \alpha > 0$, we have

$$D^\alpha[\lambda f(t) + \gamma g(t)] = \lambda D^\alpha f(t) + \gamma D^\alpha g(t) \tag{1.36}$$

Where D^α denotes any sense of fractional derivative.

Example 1.6.1 *[25]*
- the linearity of fractional derivative in the sense of Grunwald-Letnikov:

let $\alpha, \beta \in \mathbb{C}$ we have :

$$
\begin{aligned}
{}^{G}D_{a}^{\alpha}[\lambda f(t) + \gamma g(t)] =& \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{n} (-1)^{k} \left(\begin{array}{c} \alpha \\ k \end{array} \right) [\lambda f(t-kh) + \gamma g(t-kh)] \\
=& \lambda \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{n} (-1)^{k} \left(\begin{array}{c} \alpha \\ k \end{array} \right) f(t-kh) \\
&+ \gamma \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{n} (-1)^{k} \left(\begin{array}{c} \alpha \\ k \end{array} \right) g(t-kh) \\
=& \lambda {}^{G}D_{a}^{\alpha} f(t) + \gamma {}^{G}D_{a}^{\alpha} g(t)
\end{aligned}
$$

- the linearity of fractional derivative in the sense of Riemann-Liouville:
Let $\alpha, \beta \in \mathbb{C}$ we have :

$$
\begin{aligned}
D_{a}^{\alpha}[\lambda f(t) + \gamma g(t)] =& \frac{1}{\Gamma(n-\alpha)} \frac{d^{n}}{dt^{n}} \int_{a}^{t} \frac{[\lambda f(s) + \gamma g(s)]}{(t-s)^{\alpha-n+1}} \\
=& \frac{\lambda}{\Gamma(n-\alpha)} \frac{d^{n}}{dt^{n}} \int_{a}^{t} \frac{f(s)}{(t-s)^{\alpha-n+1}} \\
&+ \frac{\gamma}{\Gamma(n-\alpha)} \frac{d^{n}}{dt^{n}} \int_{a}^{t} \frac{g(s)}{(t-s)^{\alpha-n+1}} \\
=& \lambda D_{a}^{\alpha} f(t) + \gamma D_{a}^{\alpha} g(t)
\end{aligned}
$$

1.6.2 Leibniz Rule

for an integer n we have

$$
\frac{d^{n}}{dt^{n}}(f(t)g(t)) = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) f^{k}(t) g^{n-1}(t) \tag{1.37}
$$

the generalization of this formula gives us

$$
D^{\alpha}(f(t)g(t)) = \sum_{k=0}^{n} \left(\begin{array}{c} \alpha \\ k \end{array} \right) f^{k}(t) D^{\alpha-k} g(t) + R_{n}^{\alpha}(t) \tag{1.38}
$$

where $n \geq \alpha + 1$ and

$$
R_{\mathrm{n}}^{\alpha}(t) = \frac{1}{n!\Gamma(-\alpha)} \int_{a}^{t} (t-\tau)^{-\alpha-1} g(\tau) d\tau \int_{\tau}^{t} f^{(n+1)}(\tau-\xi)^{\mathrm{n}}(\xi) d\xi \tag{1.39}
$$

with $\lim_{\mathrm{n} \to \infty} R_{\mathrm{n}}^{\alpha}(t) = 0$
if f and g are continuous on $[a, t]$ include all their derivatives ,the formula becomes :

$$
D^{\alpha}(f(t)g(t)) = \sum_{k=0}^{n} \left(\begin{array}{c} \alpha \\ k \end{array} \right) f^{k}(t) D^{\alpha-k} g(t) + R_{n}^{\alpha}(t) \tag{1.40}
$$

D^{α} is the fractional derivative in the sense of Grunwald-Letnikov and in the sense of Riemann-Liouville.

Definition 1.6.1
let $\alpha > 0$, $\alpha \notin \mathbb{N}, n = [\alpha] + 1$ and $f : A \sqsubset \mathbb{R}^{2} \to \mathbb{R}$, then:

$$
\mathcal{D}^{\alpha} y(t) = f(t, y(t)), \tag{1.41}
$$

is called a Riemann-Liouville fractional differential equation.
Similarly,

$$
{}^{C}\mathcal{D}^{\alpha} y(t) = f(t, y(t)), \tag{1.42}
$$

is called a Caputo fractional differential equation.

1.7 Riemann-Liouville fractional differential equation

Starting with the homogeneous Riemann-Liouville type equation.

Lemma 1.7.1

let $\alpha > 0$. If we assume that $u \in C(0,1) \cap L(0,1)$, then the Riemann-Liouville fractional differential equation is:

$$\mathcal{D}_{0^+}^{\alpha} u(t) = 0, \quad 0 < t < 1, \tag{1.43}$$

admits a unique solution

$$u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + ... + C_n t^{\alpha-n}.$$

where $C_m \in \mathbb{R}$, with $m = 1, 2, ..., n$.

Proof
Let $\alpha > 0$. According to Remark (22222????), we have:

$$\mathcal{D}_{0^+}^{\alpha} t^{\alpha-m} = 0, \qquad \textit{with} \quad m = 1, 2, ..., n.$$

Then, the fractional differential equation (3.5) admits a particular solution, such as

$$u(t) = C_m t^{\alpha-m}, \quad \textit{with} \quad m = 1, 2, ..., n. \tag{1.44}$$

where $C_m \in \mathbb{R}$.
Thus, the general solution of (3.5), given as a sum of particular solutions (3.6), i.e.

$$u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + ... + C_n t^{\alpha-n},$$

where $C_m \in \mathbb{R}$, *with* $m = 1, 2, ..., n$.

Lemma 1.7.2
Suppose that

$$u \in C(0,1) \cap L(0,1), \quad \textit{and} \quad D_{0^+}^{\alpha} u \in C(0,1) \cap L(0,1).$$

Then:

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + ... + C_n t^{\alpha-n}, \tag{1.45}$$

where $C_m \in \mathbb{R}$, with $m = 1, 2, ..., n$.

Proof
Let $\alpha > 0$. For all $u \in C(0,1) \cap L(0,1)$(Proposition 2.2.3), we have:

$$\begin{aligned}
\mathcal{I}_{0^+}^{\alpha} \mathcal{D}_{0^+}^{\alpha} u(t) &= u(t) - \sum_{k=1}^{n} \frac{(\mathcal{I}_{0^+}^{n-\alpha} u^{n-k})(0)}{\Gamma(\alpha - k + 1)} t^{\alpha-k} \\
&= u(t) - \left[\frac{(\mathcal{I}_{0^+}^{n-\alpha} u^{(n-1)})(0)}{\Gamma(\alpha)} t^{\alpha-1} + \frac{(\mathcal{I}_{0^+}^{n-\alpha} u^{(n-2)})(0)}{\Gamma(\alpha-1)} t^{\alpha-2} + ... + \frac{(\mathcal{I}_{0^+}^{n-\alpha} u)(0)}{\Gamma(\alpha-n+1)} t^{\alpha-n} \right]
\end{aligned}$$

We define $C_m = -\dfrac{(\mathcal{I}_{0^+}^{n-\alpha} u^{(n-m)})(0)}{\Gamma(\alpha - m + 1)} \in \mathbb{R}$, for each $m = 1, 2, ..., n$, we find the equality (1.45).

Lemma 1.7.3
Let $1 < \alpha \leq 2$, *and* $y \in C([0,1])$.
Then the unique solution to the boundary value problem

$$\begin{cases} \mathcal{D}^{\alpha}_{0+}u(t) + y(t) = 0, & 0 < t < 1 \\ u(0) = u(1) = 0 \end{cases} \tag{1.46}$$

is given by:

$$u(t) = \int_0^1 G(t,s)y(s)ds,$$

such as:

$$G(t,s) = \begin{cases} \dfrac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & \text{if} \quad 0 \leq s \leq t \leq 1 \\ \dfrac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}, & \text{if} \quad 0 \leq s \leq t \leq 1 \end{cases} \tag{1.47}$$

Proof
Applying $\mathcal{I}^{\alpha}_{0+}$, to equation 1.46, we obtain:

$$\mathcal{I}^{\alpha}_{0+}[\mathcal{D}^{\alpha}_{0+}u(t) + y(t)] = 0 \leftrightarrow \mathcal{I}^{\alpha}_{0+}\mathcal{D}^{\alpha}_{0+}u(t) + \mathcal{I}^{\alpha}_{0+}y(t) = 0.$$

According to Lemma 1.7.2, for $1 < \alpha \leq 2$ $(n = [\alpha] + 1 = 2)$, we have:

$$\mathcal{I}^{\alpha}_{0+}\mathcal{D}^{\alpha}_{0+}u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2}, \; C_1, C_2 \in \mathbb{R}$$

Thus,

$$u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \mathcal{I}^{\alpha}_{0+}y(t) = 0$$

which implies

$$u(t) = -\mathcal{I}^{\alpha}_{0+}y(t) - C_1 t^{\alpha-1} - C_2 t^{\alpha-2},$$

Therefore, the general solution of equation 1.46 is given by:

$$u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds - C_1 t^{\alpha-1} - C_2 t^{\alpha-2}. \tag{1.48}$$

The boundary conditions imply that:

$$\begin{cases} u(0) = 0 \Rightarrow 0 = -0 - 0 - \lim_{t\to 0} C_2 t^{\alpha-2} & \Rightarrow C_2 = 0, \\ u(1) = 0 \Rightarrow 0 = -\dfrac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}y(s)ds - C_1 & \Rightarrow C_1 = -\dfrac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}y(s)ds. \end{cases}$$

The integro-differential equation 1.48 is equivalent to:

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}y(s)ds \\ &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^t (1-s)^{\alpha-1}y(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_t^1 (1-s)^{\alpha-1}y(s)ds \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t [t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}]y(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_t^1 (1-s)^{\alpha-1}y(s)ds \\ &= \int_0^t \frac{[t(1-s)]^{\alpha-1} - (t-s)^{(\alpha-1)}}{\Gamma(\alpha)}y(s)ds + \int_t^1 \frac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}y(s)ds \\ &= \int_0^1 G(t,s)y(s)ds. \end{aligned}$$

The proof is complete.

1.8 Caputo fractional differential equation

Starting with the homogeneous Caputo-type equation.

Lemma 1.8.1
Let $\alpha > 0$. If we assume that $u \in C(0,1) \cap L(0,1)$, then the Caputo-type fractional differential equation is:

$$ {}^{C}\mathcal{D}^{\alpha}_{0+}u(t) = 0, \tag{1.49}$$

admits a unique solution

$$u(t) = C_0 + C_1 t + C_2 t^2 + \cdots + C_{n-1}t^{n-1}.$$

where $C_m \in \mathbb{R}$, with $m = 0, 1, 2, \ldots, n-1$.

Proof
let $\alpha > 0$. According to Remark 2.2.4, we have:

$$ {}^{C}\mathcal{D}^{\alpha}_{0+}t^m = 0, \quad \text{for} \quad m = 0, 1, 2, \ldots, n-1.$$

So the fractional differential equation 1.49 admits a particular solution, such as

$$u(t) = C_m t^m, \quad \text{for} \quad m = 0, 1, 2, \ldots, n-1. \tag{1.50}$$

where $C_m \in \mathbb{R}$.

The general solution of 1.49, given as a sum of particular solutions 1.50, i.e.,

$$u(t) = C_0 + C_1 t + C_2 t^2 + \cdots + C_{n-1}t^{n-1}.$$

Lemma 1.8.2
Assume that $u \in C^n([0,1])$. then:

$$\mathcal{I}^{\alpha}_{0+} + {}^{C}\mathcal{D}^{\alpha}_{0+}u(t) = u(t) + C_0 + C_1 t + C_2 t^2 + \ldots + C_{n-1}t^{n-1}. \tag{1.51}$$

where $C_m \in \mathbb{R}$, with $m = 0, 1, 2, \ldots, n-1$.

Proof
Let $\alpha > 0$. for all $u \in C^n([0,1])$ (Proposition 2.2.4) we have

$$\begin{aligned}\mathcal{I}^{\alpha}_{0+}\, {}^{C}\mathcal{D}^{\alpha}_{0+}u(t) &= u(t) - \sum_{k=0}^{n-1} \frac{u^k(0)}{k!}t^k \\ &= u(t) - \left[u(0) + u'(0)t + \frac{u''(0)}{2}t^2 + \ldots + \frac{u^{(n-1)}(0)}{(n-1)!}t^{n-1}\right]\end{aligned}$$

We pose $C_m = -\dfrac{u^{(m)}(0)}{m!} \in \mathbb{R}$, for each $m = 0, 1, 2, \ldots, n-1$,We easily find the equality 1.51

Lemma 1.8.3
let $1 < \alpha \le 2$, and $y \in C([0,1])$.
Then the unique solution to the boundary value problem is:

$$\begin{cases} {}^{C}\mathcal{D}^{\alpha}_{0+}u(t) = y(t), \quad 0 < t < 1 \\ u(0) + u'(0) = 0, \ u(1) + u'(1) = 0 \end{cases}, \tag{1.52}$$

is given by:

$$u(t) = \int_0^1 G(t,s)y(s)ds,$$

such as:

$$G(t,s) = \begin{cases} \frac{(1-t)(1-s)^{\alpha-1} + (t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-t)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} & \text{if} \quad 0 \le s \le t \le 1 \\ \frac{(1-t)(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-t)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} & \text{if} \quad 0 \le t \le s \le 1 \end{cases} \tag{1.53}$$

Proof
Applying $\mathcal{I}_{0+}^{\alpha}$, to equation 1.52 we obtain:

$$\mathcal{I}_{0+}^{\alpha}\left[{}^{C}\mathcal{D}_{0+}^{\alpha}u(t)-y(t)\right]=0\Leftrightarrow\mathcal{I}_{0+}^{\alpha}\,{}^{C}\mathcal{D}_{0+}^{\alpha}u(t)-\mathcal{I}_{0+}^{\alpha}y(t)=0.$$

According to Lemma 1.8.2, for $1<\alpha\leq 2$ ($n=[\alpha]+1=2$),we have:

$$\mathcal{I}_{0+}^{\alpha}\,{}^{C}\mathcal{D}_{0+}^{\alpha}u(t)=u(t)+C_0+C_1t,\ C_0,C_1,C_2\in\mathbb{R},$$

thus,

$$u(t)+C_0+C_1t-\mathcal{I}_{0+}^{\alpha}y(t)=0,$$

which implies

$$u(t)=\mathcal{I}_{0+}^{\alpha}y(t)-C_0-C_1t,$$

Therefore, the general solution of equation 1.52is given by:

$$u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)ds-C_0-C_1t. \tag{1.54}$$

The boundary conditions imply that:

$$\begin{cases} u(0)+u'(0)=0 & \Rightarrow C_0+C_1=0\\ u(1)+u'(1)=0 & \Rightarrow C_0+2C_1=(\mathcal{I}_{0+}^{\alpha}y)(1)+(\mathcal{I}_{0+}^{\alpha}y)'(1)\end{cases}$$

thus

$$\begin{cases} C_0 &=-\left(\mathcal{I}_{0+}^{\alpha}y)(1)-(\mathcal{I}_{0+}^{\alpha}y\right)'(1)\\ &=-\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^1(1-s)^{\alpha-1}y(s)ds-\dfrac{1}{\Gamma(\alpha-1)}\displaystyle\int_0^1(1-s)^{\alpha-2}y(s)ds\\ C_1 &=(\mathcal{I}_{0+}^{\alpha}y)(1)-(\mathcal{I}_{0+}^{\alpha}y)'(1)\\ &=\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^1(1-s)^{\alpha-1}y(s)ds+\dfrac{1}{\Gamma(\alpha-1)}\displaystyle\int_0^1(1-s)^{\alpha-2}y(s)ds\end{cases}$$

The integro-differential equation 1.52 is equivalent to:

$$\begin{aligned} u(t)=&\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)ds+\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}y(s)ds\\ &+\frac{1}{\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}y(s)ds-\frac{t}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}y(s)ds\\ &-\frac{t}{\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}y(s)ds\\ =&\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)ds+\frac{(1-t)}{\Gamma(\alpha)}\int_0^t(1-s)^{\alpha-1}y(s)ds\\ &+\frac{(1-t)}{\Gamma(\alpha)}\int_t^1(1-s)^{\alpha-1}y(s)ds+\frac{(1-t)}{\Gamma(\alpha-1)}\int_0^t(1-s)^{\alpha-2}y(s)ds\\ &+\frac{(1-t)}{\Gamma(\alpha-1)}\int_t^1(1-s)^{\alpha-2}y(s)ds\\ =&\int_0^t\left[\frac{(t-s)^{\alpha-1}+(1-t)(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(1-t)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right]y(s)ds\\ &+\int_t^1\left[\frac{(1-t)(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(1-t)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right]y(s)ds\\ =&\int_0^1 G(t,s)y(s)ds.\end{aligned}$$

The proof is complete.

1.9 Existence and uniqueness of the solution

This section constitutes a preliminary part in which fundamental concepts and results of the theory of functional analysis are recalled (Banach contraction principle, equicontinuity, Schauder's theorem, Arzela-Ascoli theorem,...). Subsequently, the question of existence and uniqueness of the solution for the boundary value problem of fractional order differential equation will be addressed.

1.9.1 Some fixed point theorems

Definition 1.9.1 *Let (E, d) be a complete metric space and $F : E \to E$ be a continuous function.*

1. *We say that $s \in E$ is a fixed point of F if $f(u) = u$.*
2. *We say that F is a contraction mapping if it is Lipschitz with Lipschitz constant $0 < L < 1$,i.e., if there exists $0 < L < 1$, such that*

$$\forall u, v \in E,\ d(F(u), F(v)) \leq Ld(u, v), \qquad 0 < L < 1.$$

Theorem 1.9.1 *(Arzelà–Ascoli)*
Let A be a subset of $C(J; E)$; A is relatively compact in $C(J; E)$ if and only if the following conditions are satisfied:

1. *The set A is bounded, i.e., there exists a constant $k > 0$ such that:*

$$||f|| \leq k \ for\ every\ x \in J\ and\ f \in A.$$

2. *The set A is equicontinuous, i.e., for every $\epsilon > 0$, there exists $\delta > 0$ such that*

$$|t_1 - t_2| < \delta \Rightarrow ||f(t_1) - f(t_2) \leq \epsilon \ for\ every\ t_1, t_2 \in J\ and\ f \in A.$$

3. *for every $x \in J$ the set $\{f(x), f \in A\} \subset E$is relatively compact.*

Theorem 1.9.2 *(Banach)*
Let X be a Banach space, and let $F : X \to X$. be a contraction operator. Then F admits a unique fixed point.
i.e., $\exists! u \in X$ such that $Fu = u$ The second fixed point theorem that we will state is the Schauder Fixed Point Theorem.

1.10 Cauchy problem fractional order differential equation

The existence and uniqueness of the solution to a Cauchy problem for fractional-order differential equations (using the Caputo derivative) will be studied, where the problem is given in the following form:

$$\begin{cases} {}^{C}\mathcal{D}^{\alpha}y(t) = f(t, y(t)) & t \in [o, T],\ 0 < \alpha < 1 \\ y(0) = y_0,\ y_0 \in \mathbb{R} \end{cases} \tag{1.55}$$

tell that $f : [0, T] \in \mathbb{R} \to \mathbb{R}$ is a continuous function.

Lemma 1.10.1
Let $0 < \alpha < 1$ and let $h : [0, T] \to \mathbb{R}$ be a continuous function. A function y is a solution to the Cauchy problem

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) dx. \tag{1.56}$$

Proof
We apply operator $\mathcal{I}^\alpha$ to equation 1.55 and we find

$$\begin{aligned} \mathcal{I}^{\alpha C}\mathcal{D}^{\alpha C}\mathcal{D}^\alpha y = \mathcal{I}^\alpha f(t) &\Rightarrow y(t) + c_0 = \mathcal{I}^\alpha h(t) \\ &\Rightarrow y(t) = \mathcal{I}^\alpha h(t) - c_0 \end{aligned}$$

The initial condition gives

$$y(0) = (\mathcal{I}^\alpha h)(0) - c_0 = -c_0 \Rightarrow c_0 = -y_0.$$

Thus,

$$\begin{aligned} y(t) &= \mathcal{I}^\alpha h(t) - (y_0) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) dx + y_0. \end{aligned}$$

in return

$$\begin{aligned} y(t) &= y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) dx \\ &= \mathcal{I}^\alpha h(t) + y_0. \end{aligned}$$

we apply ${}^C\mathcal{D}^\alpha$ to the integral equation 1.56.

$$\begin{aligned} {}^C\mathcal{D}^\alpha y(t) &= {}^C\mathcal{D}^\alpha(\mathcal{I}^\alpha h)(t) + {}^C\mathcal{D}^\alpha(y_0) \\ &= h(t). \end{aligned}$$

Thus, it remains to verify that $y(0) = y_0$,

$$\begin{aligned} y(0) &= \mathcal{I}^\alpha h(0) + y_0 = 0 + y_0 \\ &= y_0. \end{aligned}$$

Then y is a solution to the problem 1.56.

Theorem 1.10.1
Let $0 < \alpha < 1$ and $f : [0;T] \times \mathbb{R} \to \mathbb{R}$ and satisfies the following Lipschitz condition:

$$|f(t,y) - f(t,z)| \leq k|y-z|, \ \forall t \in [0,T], \ \textit{and } y, z \in \mathbb{R}.$$

$$\frac{kT^\alpha}{\Gamma(\alpha+1)} < 1,$$

There exists a unique solution to the Cauchy problem 1.55.

Proof
We use the Banach fixed point theorem 1.9.2.

We transform problem 1.55 into a fixed point problem (Lemma 1.10.1), considering the operator

$$\begin{aligned} F : C([o,T],\mathbb{R}) &\to C([o,T],\mathbb{R}) \\ y \to F(y)(t) &= y_0 + \frac{1}{\Gamma} \int_0^t (t-s)^{\alpha-1} f(s,y(s)) dx. \end{aligned}$$

where $C([o,T],\mathbb{R})$ is the Banach space of continuous functions y defined on $[o,T]$ in $\mathbb{R}$, equipped with the norm

$$||y|| = \sup_{t\in[o,T]} |y(t)|.$$

It is clear that the fixed points of the operator F are the solutions to problem 1.55. F is well defined, indeed: if $y(t) \in C([o,T],\mathbb{R})$, Then $Fy(t) \in C([o,T],\mathbb{R})$.
To show that F has a fixed point, it suffices to demonstrate that F is a contraction; indeed, if $y_1, y_2 \in C([o,T],\mathbb{R}), t \in [o,T]$ By using the Lipschitz condition, we obtain:

$$\begin{aligned}
|Fy_1 - Fy_2| &= \left| \frac{1}{\Gamma(\alpha)} \int_0^t (f(s,y_1(s))) - (f(s,y_2(s)))(t-s)^{\alpha-1} ds \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (|f(s,y_1(s))) - (f(s,y_2(s)))|(t-s)^{\alpha-1} ds \\
&\leq \frac{k}{\Gamma(\alpha)} \int_0^t |y_1(s) - y_2(s)|(t-s)^{\alpha-1} ds \\
&\leq \frac{k}{\Gamma(\alpha)} ||y_1 - y_2|| \int_0^t (t-s)^{\alpha-1} ds \\
&\leq \frac{kT^\alpha}{\Gamma(\alpha+1)} ||y_1 - y_2||
\end{aligned}$$

It states that due to the property $\frac{kT^\alpha}{\Gamma(\alpha+1)} < 1$, F is a contraction, and according to Banach's Fixed Point Theorem, F has a unique fixed point, which is the solution to problem (3.17).

CHAPTER 2

STOCHASTIC CALCULUS AND STOCHASTIC DIFFERENTIAL EQUATIONS

2.1 probability Basics

2.1.1 probability space

Definition 2.1.1
A sigma-algebra (or σ-algebra) the probability space Ω is defined as a family $\mathcal{F}$ of subsets of Ω (called events) satisfying the following properties :

1. *the empty set $\varnothing$ belongs to$\mathcal{F}$.*
2. *if an event A is in $\mathcal{F}$, then its complement A^c is also in $\mathcal{F}$.*
3. *if $(A_n)_{n=1}^{\infty}$ is a sequence of events belonging to $\mathcal{F}$, then the union of all these events, $\bigcup_{n=1}^{\infty} A_n$, is also in $\mathcal{F}$.*

Definition 2.1.2
the probability measure on the probability space $(\Omega, \mathcal{F})$ is defined as a function $\mathbb{P}$ de $\mathcal{F}$to the interval $[0, 1]$, satisfying the following conditions :

1. *the probability of the certain event, $\mathbb{P}(\Omega)$, is equal to 1.*
2. *for any sequence of events A_n belonging to $\mathcal{F}$ and pairwise disjoint, the probability of the union of these events, $\mathbb{P}\left(\bigcup_{n=0}^{\infty} A_n\right)$, is equal to the infinite sum of individual probabilities, $\sum_{n=0}^{\infty} \mathbb{P}(A_n)$.*

Definition 2.1.3
A probability space is defined as a triplet $(\Omega, \mathcal{F}, \mathbb{P})$ where : - Ω is a set, - $\mathcal{F}$ is a sigma-algebra (or σ-tribe) on Ω, - $\mathbb{P}$ is a probability measure defined on $(\Omega, \mathcal{F})$.

2.1.2 Random variable

Definition 2.1.4
let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space. A random variable on $(\Omega, \mathcal{F}, \mathrm{P})$, is any function $X : \Omega \to \mathbb{R}$ such that :

$$\{\omega \in \Omega : X(\omega) \in B\} = \{X \in B\} \in \mathcal{F}, \forall B \in \mathbb{B}(\mathbb{R}) \tag{2.1}$$

2.1.3 Expectation of a Random variable

Definition 2.1.5 *(cumulative distribution function)*
the cumulative distribution function of a random variable X defined on $(\Omega, \mathcal{F}, \mathrm{P})$ is the function $F_{\boldsymbol{X}}(x)$ defined on $\mathbb{R}$ by :

$$F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \leq x\}) \tag{2.2}$$

Definition 2.1.6
If the cumulative distribution function $F_X(x)$ is differentiable, the derivative of this function, denoted $f_X(x)$, is called the probability density function of the random variable X. :

$$\frac{\partial F_X(x)}{\partial x} = f_X(x) \tag{2.3}$$

Definition 2.1.7
the mathematical expectation or mean, denoted $\mathbb{E}(X)$,is defined as follows:

1. ***Discrete case**, when the random variable X takes discrete values (i.e., integers) in a given interval, whether bounded or unbounded.*

$$\mathbb{E}(X) = \sum_{K=1}^{\infty} x_K \mathbb{P}\left(X = x_K\right) \tag{2.4}$$

2. ***continuous case** Si X is a real-valued random variable (absolutely continuous)*

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} x f_X(x) dx \tag{2.5}$$

Definition 2.1.8
let X and Y defined:

$$\begin{aligned} \operatorname{Var}(X, Y) &= \mathbb{E}\left((X - \mathbb{E}(X))^2\right) \\ &= \mathbb{E}\left(X^2\right) - \mathbb{E}(X)^2 \geq 0 \end{aligned} \tag{2.6}$$

$$\begin{aligned} \operatorname{Cov}(X, Y) &= \mathbb{E}((X - \mathbb{E}(X))(Y - \mathbb{E}(Y))) \\ &= \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) \end{aligned} \tag{2.7}$$

conditional Expectation

1. **conditioning with respect to an event** $B \in \mathcal{F}$:

   let $A \in \mathcal{F}$:

$$\mathbb{P}(A/B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \tag{2.8}$$

   let X be an integrable random variable defined $\mathbb{E}(|X|) < \infty$) :

$$\mathbb{E}(X/B) = \frac{\mathbb{P}\left(X 1_B\right)}{\mathbb{P}(B)} \text{ si } \mathbb{P}(B) \neq 0 \tag{2.9}$$

2. **conditioning for a random variable (taking values in the countable set):**
   let X be an integrable random variable:

$$\mathbb{E}(X/Y) = \psi(Y) \tag{2.10}$$

   where

$$\psi(y) = \mathbb{E}(X/Y = y), y \in D \tag{2.11}$$

3. **Conditioning with respect to a sigma-algebra $\mathcal{F}_1$**
   let X be an integrable random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{F}_1$ be a sub-sigma-algebra of $\mathcal{F}$:

   Definition 2.1.9
   the conditional expectation of X with respect to $\mathcal{F}_1$. denoted $\mathbb{E}(X/\mathcal{F}_1)$ is any random variable Z such that $\mathbb{E}(|z|) < \infty$ that satisfies:
   i) Z is a random variable $\mathcal{F}_1$-measurable.
   ii) $\mathbb{E}(XU) = \mathbb{E}(ZU)$,for all bounded $\forall U$ measurable random variables $\mathcal{F}_1$.

Proposition 2.1.1
let X and Y be to integrable random variables and $\mathcal{F}_1 \subset \mathcal{F}$, then:

1. $\mathbb{E}(aX + Y/\mathcal{F}_1) = a\mathbb{E}(X/\mathcal{F}_1) + \mathbb{E}(Y/\mathcal{F}_1)$.
2. *If $X \leq Y$ then $\mathbb{E}(X/\mathcal{F}_1) \leq \mathbb{E}(Y/\mathcal{F}_1)$.*
3. $\mathbb{E}(\mathbb{E}(X/\mathcal{F}_1)) = \mathbb{E}(X)$ *(taking $A = \Omega$ in the definition).*
4. *If X is independent of $\mathcal{F}_1$ then $\mathbb{E}(X/\mathcal{F}_1) = \mathbb{E}(X)$, meaning that in the absence of any information about X,the best estimate of X is its expectation.*
5. *If X is $\mathcal{F}_1$ measurable, then $\mathbb{E}(X/\mathcal{F}_1) = X$. this expresses the fact that $\mathcal{F}_1$ already contains all the information about X.*
6. *If X is $\mathcal{F}_1-$ measurable and $\mathbb{E}(|XY|) < +\infty$, then $\mathbb{E}(XY/\mathcal{F}_1) = X\mathbb{E}(Y/\mathcal{F}_1)$.*
7. *If $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}$, then $\mathbb{E}(\mathbb{E}(X/\mathcal{F}_2)/\mathcal{F}_1) = \mathbb{E}(X/\mathcal{F}_1)$.*

2.1.4 Convergence of sequences of random variables

let $(X_n)_{n=1}^{\infty}$ be a sequence of random variables and X another random variable, all defined on $(\Omega, \mathcal{F}, \mathbb{P})$. there are several ways to the sequence (X_n) to X.

- **Convergence in probability**:

$$X_n \xrightarrow[n\to\infty]{\overset{P}{\to}} X \quad \text{si} \quad : \forall \epsilon > 0, \lim_{n\to\infty} \mathbb{P}(\omega \in \Omega : X_n(\omega) - X(\omega) > \epsilon) = 0 \tag{2.12}$$

- **Almost sure convergence**:

$$X_n \xrightarrow[n\to\infty]{\longrightarrow} X \text{ p.s si } : \mathbb{P}\left(\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right) = 1 \tag{2.13}$$

- **Convergence in mean** (or convergence in L^1):

$$\lim_{n\to\infty} \mathbb{E}\left(|X_n - X|^1\right) = 0 \tag{2.14}$$

- **Quadratic convergence** (or convergence in L^2):

$$\lim_{n\to\infty} \mathbb{E}\left(|X_n - X|^2\right) = 0 \tag{2.15}$$

2.2 Filtration and Stochastic Processes

2.2.1 Filtration

Definition 2.2.1
A filtration the context of a probability space $(\Omega, B, \mathbb{P})$*, is defined as an increasing sequence* $(\mathcal{F}_n)_{n\geq 0}$ *sub-sigma algebras of* B*, i.e.,* $\mathcal{F}_t$ *is contained in* $\mathcal{F}_s$ *for all* $t \leq s$*.*

Definition 2.2.2 *[19]*
Given a measurable space $(\Omega, \mathcal{F})$ *,a real-valued random variable* X *is said to be a measurable function from* $(\Omega, \mathcal{F})$ *to* $\mathbb{R}$ *if :*

$$X^{-1}(B) \in \mathcal{F} \quad \forall B \in \mathbb{B}(\mathbb{R}) \tag{2.16}$$

Definition 2.2.3 *[18]*
the sigma algebra generated by a family of random variables $(X_t, t \in [0, T])$ *is the smallest sigma algebra containing the sets* $X_t^{-1}(B)$ *for all* $t \in [0, T]$ *and* $B \in \mathbb{B}(\mathbb{R})$*. It is denoted as* $\sigma(X_t, t \leq T)$

Definition 2.2.4
let $(\mathcal{F}_t)_{t\geq 0}$ *is said to be right continuous if :*

$$\mathcal{F}_t = \bigcap_{e>0} \mathcal{F}_{t+e} \forall t \geq 0 \tag{2.17}$$

It is left-continuous if :

$$\mathcal{F}_t = \sigma\left(\bigcup_{0<s<t} \mathcal{F}_s\right) \forall t > 0 \tag{2.18}$$

the same sequence of filtration is termed complete with respect to a probability measure $\mathbb{P}$ *when* $\mathcal{F}_0$ *includes all subsets of* $\mathcal{F}$ *with probability measure zero according to* $\mathbb{P}$*.*

Definition 2.2.5
A filtrated probability space,denoted as $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathbb{P})$*,is the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *equipped with the compatible filtration* $\{\mathcal{F}_t, t \geq 0\}$*.*

Definition 2.2.6
A filtration $(\mathcal{F}t)\, t \geq 0$ *is said to satisfy the usual conditions if it is both right-continuous and complete.*

2.2.2 stochastic process

In this section, we explore some fundamental concepts related to stochastic processes and begin by defining them.

Definition 2.2.7
let T *be anon-empty subset of* $\mathbb{R}$*. A stochastic process* $(X_t)_{t\in T}$ *in* $\mathbb{R}^n$ *is a family of random variables taking values in* $\mathbb{R}^n$ *indexed by* T*. for fixed* $\omega \in \Omega$ $t \longmapsto X_t(\omega)$ *is called trajectory.*

Definition 2.2.8 *(natural filtration)[18]*
the natural filtration of a stochastic process $X = \{X_t, t \geq 0\}$*, denote by* F^X*, is the increasing family of generated sigma-algebras generatedby*$\{X(s), 0 \leq s \leq t\}$*.* $t \geq 0$ *that is :*

$$F^X = \left\{F_t^X = \sigma(\{X(s), 0 \leq s \leq t\}), t \geq 0\right\} \tag{2.19}$$

Definition 2.2.9
A process $X = (X)_{t\geq 0}$ *is measurable if the mapping :*

$$X : \mathbb{R} \times \Omega \to \mathbb{R}^n$$
$$(t, \omega) \mapsto X_t(\omega)$$

is measurable with respect $\mathbb{B}(\mathbb{R}^+) \otimes \mathcal{F}$ *and* $\mathbb{B}(\mathbb{R}^n)$

Definition 2.2.10
A process $(X_t)_{t\in T}$ *is said to be continuous if for almost every* $w \in \Omega, t \to X_t(w)$ *is continuous (i.e., the trajectories are continuous).*

2.3 Brownian motion

2.3.1 Gaussian vector

In all that follows, $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a complete probability space.

Definition 2.3.1 *[10]*
A random variable X *defined on* $(\Omega, \mathcal{F}, \mathbb{P})$ *is said to be a Gaussian or normal random variable with parameters* $(m, \sigma^2), (m \in \mathbb{R}, \sigma \in \mathbb{R}^*_+)$ *if its density function* f_X *is given by:*

$$f_X = \frac{1}{\sigma\sqrt{2\pi}} e^{\left(-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right)}$$

In this case, its law $\mathbb{P}_X$ *is given by*

$$\forall A \in \mathbb{B}(\mathbb{R}) \quad \mathbb{P}_X(A) = \int_{\mathcal{F}} f_X(x)dx$$

And it is noted

$$X \sim \mathcal{N}\left(m, \sigma^2\right)$$

If $m = 0$*, the vector* X *is said to be centered.*

Remark 2.3.1
When the standard deviation σ *is zero, the random variable* X *is constant, meaning that* X *is almost surely equal to the mean* m*, i.e.,* $\mathbb{P}$.

Proposition 2.3.1 *[20]*
A random variable X *following the normal distribution* $\mathcal{N}(m, \sigma^2)$ *has:*

- *Expected value:* $\mathbb{E}[X] = m$.
- *Variance:* $\text{Var}(X) = \sigma^2$.
- $\text{Cov}(X_s, X_t) = \min(s, t) \quad \forall 0 \leq s, t < T$.

Definition 2.3.2
$X = (X_1, X_2, \ldots, X_n)$ *is a Gaussian random vector if all linear combinations of its components are Gaussian, that is, for any choice of coefficients* $a_1, \ldots, a_n \in \mathbb{R}$*, the random variable* $\sum_{i=1}^n a_i X_i$ *is Gaussian.*

Definition 2.3.3
A process $X = (X_t)_{t\in T}$ *is a Gaussian process if all its finite-dimensional distributions are Gaussian, i.e., for all* $n \geq 1$ *and for any choice of times* $t_1 < t_2 < \ldots < t_n \in T$*, the vector* $(X_{t_1}, \ldots, X_{t_n})$ *is Gaussian.*

Proposition 2.3.2 *[22]*
If the random vector (X_1, X_2) *is Gaussian, then the random variables* X_1 *and* X_2 *are independent if and only if their covariance* $Cov(X_1, X_2)$ *is zero.*

Proposition 2.3.3 *[22]*
Any vector of independent Gaussian random variables is a Gaussian vector.

2.3.2 Brownian motion

Brownian motion derives its name from the chaotic trajectories first observed by Robert Brown in 1827, when he witnessed the irregular movement of pollen particles suspended in a liquid. This movement, resulting from random collisions between the pollen particles and the molecules of the liquid, leads to the dispersion or diffusion of pollen in the liquid.

Definition 2.3.4 *(Standard Brownian Motion)*
A standard Brownian motion in dimension d over a time interval $T = [0, T]$ or over the set of positive real numbers $\mathbb{R}^+$ is a continuous process with values in $\mathbb{R}^d$, denoted by $(W_t)_{t\in T} = \left(W_t^1, \ldots, W_t^d\right)_{t\in T}$, which satisfies the following properties:
- $W_0 = 0$ almost surely. - For all $0 \leq s < t$ in T, the increment $W_t - W_s$ is independent of the information up to time s, $\sigma(W_u, u \leq s)$. - For all $0 \leq s < t$ in T, the increment $W_t - W_s$ follows a centered normal distribution, with a variance-covariance matrix $(t-s)I_d$, where I_d is the identity matrix of size d.

Definition 2.3.5 *(Brownian motion with respect to a filtration)*
A vectorial Brownian motion in dimension d over a time interval $T = [0, T]$ or over the set of positive real numbers $\mathbb{R}^+$ with respect to a filtration $\mathcal{F} = (\mathcal{F}_t)_{t\in T}$ is a continuous process $\mathcal{F}$-adapted taking values in $\mathbb{R}^d$, denoted by $(W_t)_{t\in T} = \left(W_t^1, \ldots, W_t^d\right)_{t\in T}$, which satisfies the following properties:
$W_0 = 0$ almost surely. For all $0 \leq s < t$ in T, the increment $W_t - W_s$ is independent of $\mathcal{F}_s$. For all $0 \leq s < t$ in T, the increment $W_t - W_s$ follows a centered normal distribution, with a variance-covariance matrix of $(t-s)I_d$, where I_d is the identity matrix of size d.

Remark 2.3.2
Un mouvement brownien standard est un mouvement brownien par rapport à sa propre filtration naturelle.

Remark 2.3.3 *[17]*
From this definition, it follows that for $t \geq s \geq 0$,

$$W_t - W_s \sim W_{t-s} \sim \mathcal{N}(0, t-s)$$

which means:

$$\mathbb{E}(W_t - W_s) = 0 \text{ et } \mathbb{E}\left((W_t - W_s)^2\right) = t - s$$

Proposition 2.3.4 *[10]*
Let $W = (W_t)_{t\geq 0}$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then:

1. *Symmetry: The process $(-W) = (-W_t)_{t\geq 0}$ is also a Brownian motion.*
2. *Scale Change: For all $\lambda > 0$, the process $W^\lambda = \left(W_t^\lambda\right)_{t\geq 0}$ defined by $W_t^\lambda = \left(\frac{1}{\lambda}\right) W\lambda^2 t$ is a Brownian motion.*
3. *Simple Markov Property: For all $s \geq 0$, if $\mathcal{F}_s := \sigma(W_u, u \leq s)$ and $W_t^{(s)} = Wt + s - W_s$, then the process $W^{(s)} = (W_t^s)_{t\geq 0}$ is a Brownian motion independent of $\mathcal{F}_s$.*

Total, Quadratic, and Bounded Variation

Definition 2.3.6 *[30]*
The infinitesimal variation of order p of a process X_t defined on the interval $[0, T]$ associated with a subdivision $\Pi_n = (t_1^n, \ldots, t_n^n)$ is defined by:

$$V_T^p(\Pi) = \sum_{i=1}^{n} \left| X_{t_i^n} - X_{t_{i-1}^n} \right|^p$$

If $V_T^p(\Pi)$ has a limit in a certain sense (almost sure convergence, L^p convergence) when

$$\pi_n = \|\Pi_n\|_\infty = \max_{i \leq n} \left| t_{i+1}^n - t_i^n \right| \to 0$$

The limit does not depend on the chosen subdivision, and we then call it the order p variation of X_t on $[0, T]$. In particular:

- *if $p = 1$, the limit is called the total variation of X_t on $[0, T]$.*
- *if $p = 2$, the limit is called the quadratic variation of X_t on $[0, T]$ and is denoted by $< X >_T$.*

Definition 2.3.7 *[30]*
A process X_t is a process with bounded variation over $[0, T]$ if it has bounded variation trajectory by trajectory, meaning that

$$\sup_{\pi_n} \sum_{i=1}^{n} \left| X_{t_i} - X_{t_{i-1}} \right| < \infty \quad a.s$$

Remark 2.3.4
If the total variation of a process exists almost surely, then it is defined as:

$$V_T^p = \sup_{\pi \in P} \sum_{i=1}^{n} \left| X_{t_i} - X_{t_{i-1}} \right| \qquad a.s$$

where P is the set of possible subdivisions of the interval $[0, T]$.

Proposition 2.3.5 *[30]*
The quadratic variation over the interval $[0, T]$ of the Brownian motion exists in $L^2(\Omega)$ and is equal to T. Furthermore, if the subdivision Π_n satisfies $\sum_{n=1}^{\infty} \pi_n < \infty$, then there is almost sure convergence, and thus:

$$< W >_T = T$$

Lemma 2.3.1
For any $\gamma, t > 0$ and $\alpha \in [\frac{1}{2}; 1]$, the following inequality is true:

$$\frac{\gamma}{\Gamma(2\alpha - 1)} \int_0^t (t-s)^{2\alpha-2} E_{2\alpha-1}\left(\gamma s^{2\alpha-1}\right) ds \leq E_{2\alpha-1}\left(\gamma t^{2\alpha-1}\right)$$

Lemma 2.3.2
For $u, v \in H^2$ and $0 < c < 1$:

$$\|u\|^2 \leq \frac{1}{1-c}\|u-v\|^2 + \frac{1}{c}\|v\|^2$$

2.4 Stochastic Integral

Let's start by defining the integral for elementary processes. Then, we extend the definition to adapted processes having a second-order moment, using a result on complete spaces. Finally, we look at the Itô formula, as well as the integral with respect to an Itô process.

2.4.1 Definition

Definition 2.4.1 *(Wiener Integral)[31]*
The Wiener integral is simply an integral of the type

$$\int_0^t X_s \, dW_s$$

where X_t processes are defined for $t \in [0,T]$ on C. With C being the set of functions $X : [0,T] \times \Omega \to \mathbb{R}$, continuous and stochastic, and $a), b), c)$ are satisfied, where

(a) X is $\mathcal{B}([0,t]) \times \mathcal{F}$-measurable

(b) $X(t,\cdot) : \Omega \to \mathbb{R}$ is $\mathcal{F}_t$-measurable

(c) $X(t,\cdot) \in L^2(\Omega)$ and $\mathbb{E}\left[\int_0^T |X(t,\cdot)|^2 dt\right] < \infty$, i.e., $X \in L^2([0,T] \times \Omega)$

2.4.2 propriete of stochastic integral

Additivity in Time For $0 \le s \le t$,

$$\int_0^t \xi_t \, dB = \int_0^s \xi_t \, dB_s + \int_s^t \xi_t \, B_s.$$

Itô's Isometry For an adapted process ξ in L^2,

$$\mathbb{E}\left[\left(\int_0^t \xi \, dB_s\right)^2\right] = \mathbb{E}\left[\int_0^t \xi^2 \, dt\right],$$

where W is a standard Wiener process.

Martingale Property : If (X_t) is an adapted process such that $\left(\int_0^t X_s^2 \, ds\right)$ is finite for all t, then the integral $\left(\int_0^t X_s \, dB_s\right)$ is a martingale.

Adaptedness and Measurability The stochastic integral $\int \xi \, dB$ is adapted and measurable with respect to the filtration with which B is adapted.

2.4.3 Itô's Formulas

A new class of processes will be introduced, in relation to which a stochastic integral will be defined: this is the family of Itô processes. This class allows the establishment of several practical formulas that form the basis of stochastic differential and integral calculus. We begin with the first formulation of Itô's formula.[30]

2.4.4 Itô process

Definition 2.4.2 *An Ito process or stochastic integral is a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ adopted to $\mathcal{F}_t$ which can be written in the form*

$$X_t = X_0 + \int_0^t U_s \, ds + \int_0^t V_s \, dB_s, \tag{2}$$

where $U, V \in \mathcal{L}_2$. As a shorthand notation, we will write (2) as

$$dX_t = U_t \, dt + V_t \, dB_t.$$

Thus B_t^2 is an Ito process:

$$B_t^2 = \int_0^t ds + 2\int_0^t B_s\,dB_s \quad or \quad d(B_t^2) = dt + 2B_t\,dB_t.$$

Note the difference from the usual differentiation: $d(x^2) = 2x\,dx$. The additional term dt arises because Brownian motion B is not differentiable and instead has quadratic variation.

1st Itô Formula:

Let X_t be a Itô process, and let $f \in C^2$, then:

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)d\langle X, X\rangle_s$$

2nd Itô Formula

Let (X_t) be a Itô process, and $f \in C^2$ be a function of two variables, then for all $t \geq 0$, then:

$$f(t, X_t) = f(0, X_0) + \int_0^t \frac{\partial f}{\partial t}(s, X_s)\,ds + \int_0^t \frac{\partial f}{\partial x}(s, X_s)\,dX_s + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_s)\,d\langle X\rangle_s$$

3rd Itô Formula:

Let X_t and Y_t be Itô processes, then for all $t \geq 0$, then:

$$X_tY_t = X_0Y_0 + \int_0^t Y_s dX_s + \int_0^t X_s dY_s + \int_0^t d\langle X, Y\rangle_s$$

2.5 Stochastic differential equations (SDE)

2.5.1 definition

Definition 2.5.1 *A stochastic differential equation on $\mathbb{R}^d$ with drift coefficient b and diffusion coefficient σ is given in the form:*

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t & \forall t \in [0, T] \\ X_0 = \xi \end{cases}$$

where

- *T is a strictly positive real number.*
- *$b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$ are two Borel functions.*
- *$(W_t)_{t\geq 0}$ is a standard Brownian motion on $\mathbb{R}^d$ defined on a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.*
- *ξ is an arbitrary random variable independent of the Brownian motion, belonging to $\mathbb{R}^n$.*

We will specify the notions of existence and uniqueness of solutions to a stochastic differential equation (SDE).

2.5.2 Existence and Uniqueness of Solutions (SDEs)

Theorem 2.5.1 *[10]*
Suppose that the coefficients b and σ satisfy the following two conditions: Assume there exists a positive constant K such that for all $t \geq 0$, $X, Y \in \mathbb{R}^n$:
- H1 Lipschitz Condition:

$$|b(t,X) - b(t,Y)| + |\sigma(t,X) - \sigma(t,Y)| \leq K|X - Y|$$

- H2 Linear Growth Condition:

$$|b(t,X)| \leq K(1 + |X|), \quad |\sigma(t,X)| \leq K(1 + |X|)$$

Then the SDE admits, for any square-integrable initial condition ξ ($\mathbb{E}(|\xi|^2) < \infty$), a strong solution $(X_t)_{t\in[0,T]}$, almost surely continuous. This solution is unique and furthermore satisfies:

$$\mathbb{E}\left(\sup_{0\leq t\leq T} |X_t|^2\right) < \infty$$

Proof.: See [10]

Example 2.5.1
Let's consider the following SDE:

$$dX_t = a(b - X_t)dt + \sigma dW(t)$$

We need to verify:

1. $|b(x,t) - b(y,t)| + |a(x,t) - a(y,t)| \leq K|x - y|, \forall t \geq 0$
2. $|b(x,t)|^2 + |a(x,t)|^2 \leq K^2\left(1 + x^2\right), \forall t \geq 0$
3. $E\left[X_0^2\right] < \infty$

We have:

$$\begin{aligned}|b(x,t) - b(y,t)| + |a(x,t) - a(y,t)| &= |a(b-x) - a(b-y)| + |\sigma - \sigma| \\ &= |a||x-y|\end{aligned}$$

since:

$$|x| \leq \left\{\begin{array}{ll} 1 & \text{if } |x| \leq 1 \\ x^2 & \text{if } |x| \geq 1 \end{array}\right. \leq \begin{cases} 1 + x^2 & \text{if } |x| \leq 1 \\ 1 + x^2 & \text{if } |x| \geq 1 \end{cases}$$

thus

$$\begin{aligned}|b(x,t)|^2 + |a(x,t)|^2 &= |a(b-x)|^2 + |\sigma|^2 \\ &= a^2(b-x)^2 + \sigma^2 \\ &= a^2\left(b^2 - 2bx + x^2\right) + \sigma^2 \\ &\leq a^2\left(b^2 + 2|b||x| + x^2\right) + \sigma^2 \\ &\leq a^2\left(b^2 + 2|b|\left(1 + x^2\right) + x^2\right) + \sigma^2 \\ &= a^2\left(b^2 + 2|b|\right) + \sigma^2 + (2|b| + 1)x^2 \\ &\leq \max\left(a^2\left(b^2 + 2|b|\right) + \sigma^2(2|b| + 1)\right)\left(1 + x^2\right)\end{aligned}$$

So let's set $K = \max\left(|a|, \sqrt{a^2\left(b^2 + 2|b|\right) + \sigma^2}, \sqrt{2|b| + 1}\right)$.
Since the initial condition wasn't specified, we only need to choose X_0 to be square-integrable to fulfill condition (3).

Ornstein-Uhlenbeck Process as a Solution to the Langevin Equation

The Langevin equation $\frac{d}{dt}V = -\gamma V + L(t)$ in the Itô formalism can be written as:

$$dV_t = -\gamma V_t dt + \sigma dB_t$$
$$V(0) = v_0$$

which has a solution according to theorem
Here, dB_t replaces a mathematically ill-defined random force $L(t)$. So we have:

$$dV_t = -\gamma V_t dt + L(t)$$

For each trajectory $V_t(\omega)$, we would use the method of variation of constants. This method is compatible with our formalism. By setting

$$C_t = V_t e^{\gamma t}$$

we have, applying Itô's formula to $f(t,x) = e^{\gamma t} x$:

$$dC_t = \gamma C_t dt + e^{\gamma t}\left(-\gamma V_t dt + \sigma dB_t\right) = e^{\gamma t}\sigma dB_t$$

and thus

$$V_t = e^{-\gamma t} v_0 + \int_0^t e^{-\gamma(t-s)} \sigma dB_s$$

Application to Finance: Geometric Brownian Motion and the Black-Scholes Model

In this model, the price of a stock is governed by the SDE

$$dS_t = S_t(\mu dt + \sigma dB_t)$$

$$S_0 = s_0$$

We set:

$$Y_t = \log(S_t)$$

As we have no guarantee that S_t does not vanish, we will perform a formal calculation without justification. We apply Itô's formula with the function $f(t,x) = \log x$. We get

$$d\log\left(S_t\right) = \left(\mu dt + \sigma dB_t\right) - \sigma^2 dt = \left(\mu - \sigma^2/2\right) dt + \sigma dB_t$$

By integrating both sides, we obtain

$$Y_t = \log\left(s_0\right) + \left(\mu - \sigma^2/2\right) t + \sigma B_t \quad \text{or} \quad S_t = s_0 e^{\left(\mu - \sigma^2/2\right)t + \sigma B_t}$$

CHAPTER 3

ASYMPTOTIC SEPARATION BETWEEN SOLUTIONS OF CAPUTO FRACTIONAL STOCHASTIC DIFFERENTIAL EQUATIONS

3.1 Introduction

Consider a Caputo fractional stochastic differential equation (for short Caputo FSDE) of order $\alpha \in \left[\frac{1}{2}, 1\right]$ of the following form:

$$^{c}D^{\alpha}_{0+}X(t) = b(t, X(t)) + \sigma(t, X(t))\frac{dW(t)}{dt}, \tag{3.1}$$

where $b, \sigma : [0, \infty)\mathbb{R}^d \to \mathbb{R}^d$ are measurable and $(W_t)_t \in [o, \infty]$ is a standard scalar Brownian motion on an underlying complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := \{\mathcal{F}_t\}_{t\in[0,\infty)}, \mathbb{P})$. For each $t \in [0, \infty)$, let $\mathcal{X}_t := \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P})$ denote the space of all $\mathcal{F}_t$ -measurable, mean square integrable functions $f = (f_1, ..., f_d)^T : \Omega \to \mathbb{R}^d$ with

$$||f||_{ms} := \sqrt{\sum_{i=1}^{d} \mathbb{E}(|f_i|^2)} = \sqrt{\mathbb{E}||f||^2},$$

where $\mathbb{R}^d$ is endowed with the standard Euclidean norm.A process $X : [0, \infty) \to \mathbb{L}(\Omega, \mathcal{F}, \mathbb{P})$ is said to be $\mathbb{F}-$adapted if $X(t) \in \mathcal{X}_t$ for all $t \geq 0$.For each $\eta \in \mathcal{X}_0$, a $\mathbb{F}$-adapted process X is called a solution of (3.1) with the initial condition $X(0) = \eta$ if the following equality holds for $t \in [0, \infty]$

$$X(t) = \eta + \frac{1}{\Gamma(\alpha)}\left(\int_0^t (t-\tau)^{\alpha-1} b(\tau, X(\tau))d\tau + \int_0^t (t-\tau)^{\alpha-1}\sigma(\tau, X(\tau))dW_\tau\right) \tag{3.2}$$

where $\Gamma(\alpha) := \int_0^\infty \tau^{\alpha-1}\exp(-\tau)d\tau$ is the Gamma function. In the remain- ing of the article, we assume that the coefficients b and σ satisfy the following standard conditions:

(H1) There exists $L > 0$ such that for all $x, y \in \mathbb{R}^d$, $t \in [o, \infty)$

$$||b(t, x) - b(t, y)|| + ||\sigma(t, x) - \sigma(t, y)|| \leq L||x - y||.$$

(H2) $\sigma(\cdot, 0)$ is essentially bounded,i.e.

$$||\sigma(\cdot, 0)||_\infty := ess \sup_{\tau\in[0,\infty)} ||\sigma(\tau, 0)|| < \infty$$

and $b(\cdot, 0)$ is $\mathbb{L}^2$ integrable, i.e.

$$\int_0^\infty ||b(\tau, 0)||^2 d\tau < \infty$$

Our first result in this article is to show the global existence and uniqueness solutions of (3.1) when $(H1)$ and $(H2)$ hold. Furthermore, we also show the continuity dependence of solutions on the initial values.

We need this lemma: Here, the weight function is the Mittag–Leffler function $E_{2\alpha-1}(\cdot)$ defined as:

$$E_{2\alpha-1}(t) := \sum_{k=0}^{\infty} \frac{t^k}{\Gamma((2\alpha-1)k+1)} \quad \text{for all } t \in \mathbb{R}.$$

For more details on the Mittag–Leffler functions, we refer the reader to the book [?, d2] The following result is a technical lemma which is used later to estimate the operator T_η.

Lemma 3.1.1 *For any $\alpha > \frac{1}{2}$ and $\gamma > 0$, the following inequality holds:*

$$\frac{\gamma}{\Gamma(2\alpha-1)} \int_0^t (t-\tau)^{2\alpha-2} E_{2\alpha-1}(\gamma t^{2\alpha-1}) d\tau \leq E_{2\alpha-1}(\gamma t^{2\alpha-1}).$$

Proof. Let $\gamma > 0$ be arbitrary. Consider the corresponding linear Caputo fractional differential equation of the following form:

$$^{c}D_{0^+}^{2\alpha-1} x(t) = \gamma x(t). \tag{3.3}$$

The Mittag–Leffler function $E_{2\alpha-1}(\gamma t^{2\alpha-1})$ is a solution of 3.3, see, e.g., [?, d4] Hence, the following equality holds:

$$E_{2\alpha-1}(\gamma t^{2\alpha-1}) = 1 + \frac{\gamma}{\Gamma(2\alpha-1)} \int_0^t (t-\tau)^{2\alpha-2} E_{2\alpha-1}(\gamma t^{2\alpha-1}) d\tau,$$

which completes the proof.

Theorem 3.1.1 *[3]*
(Global existence and uniqueness and Continuity dependence on the initial values of solutions of Caputo FSDE). Suppose that $(H1)$ and $(H2)$ hold. Then

(i) for any $\eta \in \mathcal{X}_0$, the initial value problem 3.1 with the initial condition $X(0) = \eta$ has a unique global solution on the whole interval $[0, \infty)$ denoted by $\varphi(\cdot, \eta)$;

(ii) on any bounded time interval $[0, T]$, where $T > 0$, the solution $\varphi(\cdot, \eta)$ depends continuously on η, i.e.

$$\lim_{\zeta \to \eta} \sup_{t \in [0,T]} ||\varphi(t, \zeta) - \varphi(t, \eta)||_{ms} = 0$$

Our next result is to establish a lower bound on the asymptotic separation between two distinct solutions of (3.1)

Theorem 3.1.2 *[1]*
Let $\eta, \zeta \in \mathcal{X}_0$ such that $\eta \neq \zeta$. Then, for any $\epsilon > 0$

$$\limsup_{t \to \infty} t^{\frac{2\alpha}{1-\alpha}+\epsilon} ||\varphi(t, \eta) - \varphi(t, \zeta)||_{ms} = \infty$$

Finally, we give an application of the main results concerning the mean square Lyapunov exponent of non-trivial solutions to a bounded bilinear Caputo FSDE. To formulate this result, we consider the following equation:

$$^{c}D_{0^+}^{\alpha} x(t) = A(t)x(t)) + B(t)x(t)\frac{dW(t)}{dt}, \tag{3.4}$$

where $A, B : [0, \infty) \to \mathbb{R}^{d\times d}$ are measurable and essentially bounded, i.e.,

$$\operatorname*{ess\,sup}_{t\in[0,\infty)]} ||A(t)||, \quad \operatorname*{ess\,sup}_{t\in[0,\infty)]} ||B(t)|| < \infty$$

By virtue of theorem 3.1.1, for each $\eta \in \mathcal{X}_0 \setminus \{0\}$, there exists a unique solution of 3.4, denoted by $\Phi(\cdot, \eta)$, satisfying the initial condition $X(0) = \eta$. The mean square Lyapunov exponent of $\Phi(\cdot, \eta)$ is defined by

$$\lambda_{ms}(\Phi(\cdot, \eta)) := \limsup_{t\to\infty} \frac{1}{t} \log ||\Phi(t, \eta)||_{ms} \tag{3.5}$$

see, e.g., [11]. In the following corollary, we show the non-negativity of the mean square Lyapunov exponent of an arbitrary non-trivial solution.

Corrolaire 3.1.1 *(Non-negativity of mean square Lyapunov exponent for solutions of linear Caputo fsde). The mean square Lyapunov exponent of a nontrivial solution of 3.4 is always non-negative, i.e.,*

$$\lambda_{ms}(\Phi(\cdot, \eta)) \geq 0 \quad for \ \ all \ \ \eta \in \mathcal{X}_0 \setminus \{0\}$$

Proof
Let $\eta \in \mathcal{X}_0 \setminus \{0\}$ be arbitrary. Using theorem 3.1.2, we obtain

$$\limsup_{t\to\infty} t^{\frac{2\alpha}{1-\alpha}+\epsilon} ||\phi(t, \eta)||_{ms} = \infty$$

where $\epsilon > 0$ is arbitrary. Hence, there exists $T > 0$ such that

$$||\phi(t, \eta)||_{ms} \geq t^{-\left(\frac{2\alpha}{1-\alpha}+\epsilon\right)} \ for \ \ all \ \ t \geq T,$$

which together with 3.5 implies that

$$\lambda_{ms}(\Phi(\cdot, \eta)) \geq \limsup_{t\to\infty} \frac{1}{t} \log \left(t^{-\left(\frac{2\alpha}{1-\alpha}+\epsilon\right)}\right) = 0.$$

3.2 Existence and uniqueness results

3.2.1 Existence, uniqueness, and continuity dependence on the initial values of solutions

Our aim in this subsection is to prove the result on global existence, uniqueness, and continuity dependence on the initial values of solutions to the equation 3.1. In fact, in order to prove Theorem 3.1.1(i) it is equivalent to show the existence and uniqueness solutions on an arbitrary interval $[0, T]$, where $T > 0$ is arbitrary. In what follows, we choose and fix a $T > 0$ arbitrarily. Let $\mathbb{H}^2([0, T])$ be the space of all the processes X which are measurable, $\mathbb{F}_T$-adapted, where $\mathbb{F}_T := \{\mathcal{F}\}_{t\in[0,T]}$, and satisfies that

$$||X||_{\mathbb{H}^2} := \sup_{0\leq t\leq T} ||X(t)||_{ms} < \infty$$

Obviously, $(\mathbb{H}^2([0, T]), ||.||_{\mathbb{H}^2})$ is a Banach space. For any $\eta \in \mathcal{X}_0$, we define an operator $T_\eta : \mathbb{H}^2([0, T]) \to \mathbb{H}^2([0, T])$ by

$$T_\eta\xi(t) := \eta + \frac{1}{\Gamma(\alpha)} \left(\int_0^t (t-\tau)^{\alpha-1} b(\tau, \xi(\tau) d\tau + \int_0^t (t-\tau)^{\alpha-1} \sigma(\tau, \xi(\tau)) dW_\tau \right). \tag{3.6}$$

The following lemma is devoted to showing that this operator is well-defined.

Lemma 3.2.1 *For any $\eta \in \mathcal{X}_0$, the operator T_η is well-defined.*

Proof
Let $\xi \in \mathbb{H}^2([0,T])$ be arbitrary. From the definition of $T_\eta \xi$ as in 3.6 and the inequality $\|x+y+z\|^2 \le 3(\|x\|^2 + \|y\|^2 + \|z\|^2)$ for all $x, y, z \in \mathbb{R}^d$, we have for all $t \in [0,T]$

$$\begin{aligned} \|T_\eta \xi(t)\|_{ms}^2 \le 3\|\eta\|_{ms}^2 + \frac{3}{\Gamma(\alpha)^2}\mathbb{E}\left(\left\|\int_0^t (t-\tau)^{\alpha-1} b(\tau,\xi(\tau))d\tau\right\|^2\right) \\ + \frac{3}{\Gamma(\alpha)^2}\mathbb{E}\left(\left\|\int_0^t (t-\tau)^{\alpha-1-\sigma}(\tau,\xi(\tau))dW_\tau\right\|^2\right). \end{aligned} \tag{3.7}$$

By the Hölder inequality, we obtain

$$\begin{aligned} \mathbb{E}\left(\left\|\int_0^t (t-\tau)^{\alpha-1} b(\tau,\xi(\tau))d\tau\right\|^2\right) &\le \int_0^t (t-\tau)^{2\alpha-2} d\tau \mathbb{E}\left(\int_0^t \|b(\tau,\xi(\tau))\|^2 d\tau\right) \\ &= \frac{t^{2\alpha-1}}{2\alpha-1}\mathbb{E}\left(\int_0^t \|b(\tau,\xi(\tau))\|^2 d\tau\right). \end{aligned} \tag{3.8}$$

From (H1), we derive

$$\|b(\tau,\xi(\tau))\|^2 \le 2(\|b(\tau,\xi(\tau)) - b(\tau,0)\|^2 + \|b(\tau,0)\|^2)$$

$$\le 2L^2\|\xi(\tau)\|^2 + 2\|b(\tau,0)\|^2.$$

Therefore,

$$\begin{aligned} \mathbb{E}\left(\int_0^t \|b(\tau,\xi(\tau))\|^2 d\tau\right) &\le 2L^2\mathbb{E}\left(\int_0^t \|\xi(\tau)\|^2 d\tau\right) + 2\int_0^t \|b(\tau,0)\|^2 d\tau. \\ &\le 2L^2 T \sup_{t\in[0,T]} \mathbb{E}(\|\xi(t)\|^2) + 2\int_0^T \|b(\tau,0)\| \end{aligned}$$

which together with 3.8 implies that

$$\mathbb{E}\left(\left\|\int_0^t (t-\tau)^{\alpha-1} b(\tau,\xi(\tau))d\tau\right\|^2\right) \le \frac{2L^2T^{2\alpha}}{2\alpha-1}\|\xi\|_{\mathbb{H}^2}^2 + \frac{2T^{2\alpha-1}}{2\alpha-1}\int_0^T \|b(\tau,0)\|^2 d\tau. \tag{3.9}$$

Now, using Itô's isometry (see e.g., [?, p. 87]), we obtain

$$\begin{aligned} \mathbb{E}\left(\left\|\int_0^t (t-\tau)^{\alpha-1}\sigma(\tau,\xi(\tau))dW_\tau\right\|^2\right) &= \sum_{1\le i\le d} \mathbb{E}\left(\int_0^t (t-\tau)^{\alpha-1}\sigma_i(\tau,\xi(\tau))dW\tau\right)^2 \\ &= \sum_{1\le i\le d} \mathbb{E}\left(\int_0^t (t-\tau)^{2\alpha-2}|\sigma_i(\tau,\xi(\tau))|^2 d\tau\right). \\ &= \mathbb{E}\left(\int_0^t (t-\tau)^{2\alpha-2}\|\sigma(\tau,\xi(\tau))\|^2 d\tau\right). \end{aligned}$$

From (H1), we also have

$$\|\sigma(\tau,\xi(\tau))\|^2 \le 2L^2\|\xi(\tau)\|^2 + 2\|\sigma(\tau,0)\|^2 \le 2L^2\|\xi(\tau)\|^2 + 2\|\sigma(\cdot,0)\|_\infty^2.$$

Therefore, for all $t \in [0,T]$ we have

$$\begin{aligned} &\mathbb{E}\left(\left\|\int_0^t (t-\tau)^{\alpha-1}\sigma(\tau,\xi(\tau))dW_\tau\right\|^2\right) \\ &\le 2L^2\mathbb{E}\left(\int_0^t (t-\tau)^{2\alpha-2}\|\xi(\tau)\|^2 d\tau\right) + 2\|\sigma(\cdot,0)\|_\infty^2 \int_0^t (t-\tau)^{2\alpha-2} d\tau \\ &\le 2L^2\frac{T^{2\alpha-1}}{2\alpha-1}\|\xi(t)\|_{\mathbb{H}^2}^2 + \frac{2T^{2\alpha-1}}{2\alpha-1}\int_0^T \|\sigma(\cdot,0)\|_\infty^2. \end{aligned}$$

This together with 3.7 and 3.9 implies that $\|T_\eta \xi\|_{\mathbb{H}^2} < \infty$. Hence, the map T_η is well-defined. To prove existence and uniqueness of solutions, we will show that the operator T_η defined as above is contractive under a suitable temporally weighted norm (cf. [?], Remark 2.1) for the same method to prove the existence and uniqueness of solutions of stochastic differential equations). Here, the weight function is the Mittag–Leffler function $E_{2\alpha-1}(\cdot)$ defined as:

$$E_{2\alpha-1}(t) := \sum_{k=0}^{\infty} \frac{t^k}{\Gamma((2\alpha-1)k+1)} \quad \text{for all } t \in \mathbb{R}.$$

For more details on the Mittag–Leffler functions, we refer the reader to the book [1, p. 16]. The following result is a technical lemma which is used later to estimate the operator T_η.

Lemma 3.2.2 *For any $\alpha > \frac{1}{2}$ and $\gamma > 0$, the following inequality holds:*

$$\frac{\gamma}{\Gamma(2\alpha-1)} \int_0^t (t-\tau)^{2\alpha-2} E_{2\alpha-1}(\gamma t^{2\alpha-1}) d\tau \leq E_{2\alpha-1}(\gamma t^{2\alpha-1}).$$

Proof of theorem 3.1.1. Let $T > 0$ be arbitrary. Choose and fix a positive constant γ such that

$$\gamma > \frac{3L^2(T+1)\Gamma(2\alpha-1)}{\Gamma(\alpha)^2}. \tag{3.10}$$

On the space $\mathbb{H}^2([0,T])$, we define a weighted norm $\|\cdot\|_\gamma$ as below

$$\|X\|_\gamma := \sup_{t\in[0,T]} \sqrt{\frac{\mathbb{E}(\|X(t)\|^2)}{E_{2\alpha-1}(\gamma t^{2\alpha-1})}} \quad \text{for all } X \in \mathbb{H}^2([0,T]). \tag{3.11}$$

Obviously, two norms $\|\cdot\|_{\mathbb{H}^2}$ and $\|\cdot\|_\gamma$ are equivalent. Thus, $(\mathbb{H}^2([0,T]), \|\cdot\|_\gamma)$ is also a Banach space.

(i) Choose and fix $\eta \in \mathcal{X}_0$. By virtue of Lemma 3.2.1, the operator T_η is well-defined. We will prove that the map T_η is contractive with respect to the norm $\|\cdot\|_\gamma$.

For this purpose, let $\xi, \hat{\xi} \in \mathbb{H}^2([0,T])$ be arbitrary. From 3.6 and the inequality $\|x+y\|^2 \leq 2(\|x\|^2 + \|y\|^2)$ for all $x, y \in \mathbb{R}^d$, we derive the following inequalities for all $t \in [0,T]$:

$$\begin{aligned}\mathbb{E}\left(\|T_\eta\xi(t) - T_\eta\hat{\xi}(t)\|^2\right) \leq \frac{2}{\Gamma(\alpha)^2}\mathbb{E}\left(\left\|\int_0^t (t-\tau)^{\alpha-1}(b(\tau,\xi(\tau)) - b(\tau,\hat{\xi}(\tau)))d\tau\right\|^2\right)\\ + \frac{2}{\Gamma(\alpha)^2}\mathbb{E}\left(\left\|\int_0^t (t-\tau)^{\alpha-1}(\sigma(\tau,\xi(\tau)) - \sigma(\tau,\hat{\xi}(\tau)))dW_\tau\right\|^2\right).\end{aligned}$$

Using the Hölder inequality and (H1), we obtain

$$\mathbb{E}\left(\left\|\int_0^t (t-\tau)^{\alpha-1}(b(\tau,\xi(\tau)) - b(\tau,\hat{\xi}(\tau)))d\tau\right\|^2\right) \leq L^2 t \int_0^t (t-\tau)^{2\alpha-2}\mathbb{E}(\|\xi(\tau) - \hat{\xi}(\tau)\|^2)d\tau.$$

On the other hand, by Itô's isometry and (H1), we have

$$\begin{aligned}&\mathbb{E}\left(\left\|\int_0^t (t-\tau)^{\alpha-1}(\sigma(\tau,\xi(\tau)) - \sigma(\tau,\hat{\xi}(\tau)))dW_\tau\right\|^2\right)\\ &= \mathbb{E}\left(\int_0^t (t-\tau)^{2\alpha-2}\|\sigma(\tau,\xi(\tau)) - \sigma(\tau,\hat{\xi}(\tau))\|^2 d\tau\right)\\ &\leq L^2 \int_0^t (t-\tau)^{2\alpha-2}\mathbb{E}(\|\xi(\tau) - \hat{\xi}(\tau)\|^2)d\tau.\end{aligned}$$

Thus, for all $t \in [0,T]$ we have

$$\mathbb{E}\left(\|T_\eta\xi(t) - T_\eta\hat{\xi}(t)\|^2\right) \le \frac{2L^2(t+1)}{\Gamma(\alpha)^2}\int_0^t (t-\tau)^{2\alpha-2}\mathbb{E}(\|\xi(\tau)-\hat{\xi}(\tau)\|^2)d\tau.$$

which together with the definition of $\|\cdot\|_\gamma$ as in (3.11) implies that

$$\frac{\mathbb{E}\left(||T_\eta\xi(t) - T_\eta\hat{\xi}(t)||^2\right)}{E_{2\alpha-1}(\gamma t^{2\alpha-1})} \le \frac{2L^2(t+1)}{\Gamma(\alpha)^2}\frac{\int_0^t (t-\tau)^{2\alpha-2}E_{2\alpha-1}(\gamma\tau^{2\alpha-1})\,d\tau}{E_{2\alpha-1}(\gamma t^{2\alpha-1})}\|\xi-\hat{\xi}\|_\gamma^2.$$

In light of Lemma 3.2.2, we have for all $t \in [0,T]$

$$\frac{\mathbb{E}\left(||T_\eta\xi(t) - T_\eta\hat{\xi}(t)||^2\right)}{E_{2\alpha-1}(\gamma t^{2\alpha-1})} \le \frac{2\Gamma(2\alpha-1)L^2(T+1)}{\Gamma(\alpha)^2\gamma^2}\|\xi-\hat{\xi}\|_\gamma^2.$$

Consequently,

$$\|T_\eta\xi - T_\eta\hat{\xi}\|_\gamma \le \kappa\|\xi-\hat{\xi}\|_\gamma, \quad \text{where} \quad \kappa := \sqrt{\frac{2\Gamma(2\alpha-1)L^2(T+1)}{\Gamma(\alpha)^2\gamma^2}}.$$

By (3.10), we have $\kappa < 1$ and therefore the operator T_η is a contractive map on $H^2([0,T]), \|\cdot\|_\gamma$. Using the Banach fixed point theorem, there exists a unique fixed point of this map in $H^2([0,T])$. This fixed point is also the unique solution of (3.1) with the initial condition $X(0) = \eta$. The proof of this part is complete.

(ii) Choose and fix $T > 0$ and $\eta, \zeta \in \mathcal{X}_0$. Since $\varphi(\cdot,\eta)$ and $\varphi(\cdot,\zeta)$ are solutions of (3.1) it follows that

$$\begin{aligned}\varphi(t,\eta) - \varphi(t,\zeta) = \eta - \zeta &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}(b(\tau,\varphi(\tau,\eta)) - b(\tau,\varphi(\tau,\zeta)))\,d\tau \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}(\sigma(\tau,\varphi(\tau,\eta)) - \sigma(\tau,\varphi(\tau,\zeta)))\,dW_\tau.\end{aligned}$$

Hence, using the inequality $\|x+y+z\|^2 \le 3(\|x\|^2+\|y\|^2+\|z\|^2)$ for all $x,y,z \in \mathbb{R}^d$, (H1), the Hölder inequality and Itô's isometry (see Part (i)), we obtain

$$\begin{aligned}\mathbb{E}\left(\|\varphi(t,\eta)-\varphi(t,\zeta)\|^2\right) \le &\frac{3L^2(t+1)}{\Gamma(\alpha)^2}\int_0^t (t-\tau)^{2\alpha-2}\mathbb{E}(\|\varphi(\tau,\eta)-\varphi(\tau,\zeta)\|^2)\,d\tau \\ &+ 3\mathbb{E}(\|\eta-\zeta\|^2).\end{aligned}$$

By definition of $\|\cdot\|_\gamma$, we have

$$\begin{aligned}\mathbb{E}\left(\|\varphi(t,\eta)-\varphi(t,\zeta)\|^2\right)\frac{E_{2\alpha-1}(\gamma t^{2\alpha-1})}{\Gamma(\alpha)^2\gamma^2} \le &\frac{3L^2(t+1)}{\Gamma(\alpha)^2}\int_0^t (t-\tau)^{2\alpha-2}E_{2\alpha-1}(\gamma\tau^{2\alpha-1})\,d\tau \\ &\times \|\varphi(\cdot,\eta)-\varphi(\cdot,\zeta)\|_\gamma^2 + 3\mathbb{E}(\|\eta-\zeta\|^2).\end{aligned}$$

By virtue of Lemma 3.2.2, we have

$$\|\varphi(\cdot,\eta)-\varphi(\cdot,\zeta)\|_\gamma^2 \le \frac{3L^2(T+1)\Gamma(2\alpha-1)}{\gamma\Gamma(\alpha)^2}\|\varphi(\cdot,\eta)-\varphi(\cdot,\zeta)\|_\gamma^2 + 3\|\eta-\zeta\|_{\mathrm{ms}}^2.$$

Thus, by (3.10) we have

$$\|\varphi(\cdot,\eta)-\varphi(\cdot,\zeta)\|_\gamma^2 \le \frac{3L^2(T+1)\Gamma(2\alpha-1)}{\gamma\Gamma(\alpha)^2}\|\varphi(\cdot,\eta)-\varphi(\cdot,\zeta)\|_\gamma^2 + 3\|\eta-\zeta\|_{\mathrm{ms}}^2.$$

Hence,

$$\lim_{\eta\to\zeta}\sup_{t\in[0,T]}\|\varphi(t,\eta)-\varphi(t,\zeta)\|_{\text{ms}}=0.$$

The proof is complete.
We conclude this section with a discussion on the gap in the proof of global existence of solutions for Caputo fractional stochastic differential equation in 3.3.

Remark 3.2.1
For $\alpha\in\left[\frac{1}{2},1\right]$, we consider a Caputo fractional stochastic differential equation on a Banach space X of the following form

$$^{c}D^{\alpha}_{0+}x(t)=b(t,x(t))+\sigma(t,x(t))dW_t,$$

where $t\in[0,T]$, $b,\sigma:[0,T]\times L^2(\Omega;X)\to L^2(\Omega;X)$ are measurable functions satisfying the following conditions:

(i) there exists a constant $L>0$ such that for all $t\in[0,T]$ and $x,y\in L^2$

$$\mathbb{E}(\|b(t,x)-b(t,y)\|^2)+\mathbb{E}(\|\sigma(t,x)-\sigma(t,y)\|^2)\le L\mathbb{E}(\|x-y\|^2);$$

(ii) the functions b,σ are bounded, i.e. for some $x_0\in L^2(\Omega;X)$ and $b>0$, there exists a constant $M>0$ such that

$$\mathbb{E}(\|b(t,x)\|^2)\le M^2,\quad \mathbb{E}(\|\sigma(t,x)\|^2)\le M^2$$

for all $(t,x)\in R_0:=\{(t,x):0\le t\le T,\mathbb{E}(\|x-x_0\|^2)\le b^2\}$.

3.2.2 A lower bound on the asymptotic separation of two distinct solutions

Proof of Theorem 3.1.2. Suppose a contradiction, i.e. there exists a positive constant $\lambda>\frac{2\alpha}{1-\alpha}$ such that

$$\limsup_{t\to\infty}t^{\lambda}\|\varphi(t,\eta)-\varphi(t,\zeta)\|_{\text{ms}}<\infty, \tag{3.12}$$

for some $\eta,\zeta\in\mathcal{X}_0,\eta\neq\zeta$. Then, there exist constants $T>0$ and $K>0$ such that

$$\|\varphi(t,\eta)-\varphi(t,\zeta)\|^2_{\text{ms}}\le Kt^{-2\lambda}\quad\text{for all } t\ge T. \tag{3.13}$$

From 3.2 and the inequality $\|x+y+z\|^2\le 3(\|x\|^2+\|y\|^2+\|z\|^2)$ for all $x,y,z\in\mathbb{R}^d$, we have

$$\begin{aligned}\|\eta-\zeta\|^2\le 3\|\varphi(t,\eta)-\varphi(t,\zeta)\|^2&+\frac{3}{\Gamma(\alpha)^2}\left\|\int_0^t(t-\tau)^{\alpha-1}(\sigma(\tau,\varphi(\tau,\eta))-\sigma(\tau,\varphi(\tau,\zeta)))\,dW_\tau\right\|^2\\&+\frac{3}{\Gamma(\alpha)^2}\left\|\int_0^t(t-\tau)^{\alpha-1}(b(\tau,\varphi(\tau,\eta))-b(\tau,\varphi(\tau,\zeta)))\,d\tau\right\|^2.\end{aligned}$$

Taking the expectation of both sides and using the Itô's isometry, (H1), we obtain

$$\begin{aligned}\|\eta-\zeta\|^2_{\text{ms}}\le 3\mathbb{E}(\|\varphi(t,\eta)-\varphi(t,\zeta)\|^2)&+\frac{3L^2}{\Gamma(\alpha)^2}\mathbb{E}\left(\int_0^t(t-\tau)^{\alpha-1}\|\varphi(\tau,\eta)-\varphi(\tau,\zeta)\|\,d\tau\right)^2\\&+\frac{3L^2}{\Gamma(\alpha)^2}\int_0^t(t-\tau)^{2\alpha-2}\|\varphi(\tau,\eta)-\varphi(\tau,\zeta)\|^2_{\text{ms}}\,d\tau.\end{aligned}$$

From (3.13), we derive that $\lim_{t\to\infty}\mathbb{E}(\|\varphi(t,\eta)-\varphi(t,\zeta)\|^2)=0$. Hence, to derive a contradiction and therefore to complete the proof it is sufficient to show that

$$\lim_{t\to\infty}I_1(t)=0,\quad\text{where } I_1(t):=\mathbb{E}\left(\int_0^t(t-\tau)^{\alpha-1}\|\varphi(\tau,\eta)-\varphi(\tau,\zeta)\|\,d\tau\right)^2 \tag{3.14}$$

and

$$\lim_{t\to\infty} I_2(t) = 0, \quad \text{where } I_2(t) := \int_0^t (t-\tau)^{2\alpha-2}\|\varphi(\tau,\eta)-\varphi(\tau,\zeta)\|_{\text{ms}}^2\,d\tau. \tag{3.15}$$

To prove (3.14), choose and fix $\delta \in \left(\frac{\alpha}{\lambda}, \frac{1-\alpha}{2}\right)$. Note that the existence of such a δ comes from the fact that $\frac{\alpha}{\lambda} < \frac{1-\alpha}{2}$ (equivalently, $\lambda > \frac{2\alpha}{1-\alpha}$). For $t > \max\{T^{1/\delta}, 1\}$, we have

$$\begin{aligned} I_1(t) &\le 2\mathbb{E}\left(\int_{t^\delta}^t (t-\tau)^{\alpha-1}\|\varphi(\tau,\eta)-\varphi(\tau,\zeta)\|\,d\tau\right)^2 \\ &\quad + 2\mathbb{E}\left(\int_0^{t^\delta} (t-\tau)^{\alpha-1}\|\varphi(\tau,\eta)-\varphi(\tau,\zeta)\|\,d\tau\right)^2. \end{aligned}$$

Using the Hölder inequality, we obtain

$$\begin{aligned} I_1(t) &\le 2\int_0^{t^\delta} (t-\tau)^{2\alpha-2}\,d\tau \int_0^{t^\delta} \|\varphi(\tau,\eta)-\varphi(\tau,\zeta)\|_{\text{ms}}^2\,d\tau \\ &\quad + 2\int_{t^\delta}^t (t-\tau)^{2\alpha-2}\,d\tau \int_{t^\delta}^t \|\varphi(\tau,\eta)-\varphi(\tau,\zeta)\|_{\text{ms}}^2\,d\tau. \end{aligned}$$

Since

$$\int_0^{t^\delta} (t-\tau)^{2\alpha-2}\,d\tau \le \frac{t^\delta}{(t-t^\delta)^{2\alpha-2}} \cdot \int_0^{t^\delta} (t-\tau)^{2\alpha-2}\,d\tau \le \frac{(t-t^\delta)^{2\alpha-1}}{2\alpha-1}$$

it follows together with (3.13) that

$$\begin{aligned} I_1(t) &\le \frac{2t^{2\delta}\sup_{\tau\ge 0}\|\varphi(\tau,\eta)-\varphi(\tau,\zeta)\|_{m_s}^2}{(t-t^\delta)^{2-2\alpha}} + \frac{2K(t-t^\delta)^{2\alpha-1}}{2\alpha-1}\int_{t^\delta}^t \tau^{-2\lambda}\,d\tau \\ &\le \frac{2t^{2\delta}\sup_{\tau\ge 0}\|\varphi(\tau,\eta)-\varphi(\tau,\zeta)\|_{m_s}^2}{(t-t^\delta)^{2-2\alpha}} + \frac{2K(t-t^\delta)^{2\alpha}}{(2\alpha-1)t^{2\delta\lambda-2\alpha}}. \end{aligned}$$

By definition of δ, we have $2\delta < 2-2\alpha$ and $2\alpha < 2\delta\lambda$. Hence, letting $t\to\infty$ in the preceding inequality yields that $\lim_{t\to\infty} I_1(t) = 0$ and thus (15) is proved. Concerning the assertion (16), let $t \ge T$ be arbitrary. By (3.13), we have

$$\begin{aligned} I_2(t) &\le \int_0^T (t-\tau)^{2\alpha-2}\|\varphi(\tau,\eta)-\varphi(\tau,\zeta)\|_{m_s}^2\,d\tau + K\int_T^t (t-\tau)^{2\alpha-2}\tau^{-2\lambda}\,d\tau \\ &\le \frac{T}{(t-T)^{2-2\alpha}}\sup_{\tau\ge 0}\|\varphi(\tau,\eta)-\varphi(\tau,\zeta)\|_{m_s}^2 + K\int_T^t (t-\tau)^{2\alpha-2}\tau^{-2\lambda}\,d\tau. \end{aligned}$$

Therefore,

$$\limsup_{t\to\infty} I_2(t) \le K\limsup_{t\to\infty}\int_T^t (t-\tau)^{2\alpha-2}\tau^{-2\lambda}\,d\tau.$$

Note that for $t \ge 2T$ we have

$$\begin{aligned} \int_T^t (t-\tau)^{2\alpha-2}\tau^{-2\lambda}\,d\tau &= \int_T^{t/2} (t-\tau)^{2\alpha-2}\tau^{-2\lambda}\,d\tau + \int_{t/2}^t (t-\tau)^{2\alpha-2}\tau^{-2\lambda}\,d\tau \\ &\le \frac{2^{2-2\alpha}}{t^{2-2\alpha}}\int_T^{t/2} \tau^{-2\lambda}\,d\tau + \left(\frac{t}{2}\right)^{-2\lambda}\int_{t/2}^t (t-\tau)^{2\alpha-2}\,d\tau \\ &\le \frac{2^{2-2\alpha}T^{-2\lambda+1}}{(2\alpha-1)t^{2-2\alpha}} + \frac{1}{2\alpha-1}\left(\frac{t}{2}\right)^{2\alpha-1-2\lambda}, \end{aligned}$$

which together with (17) and the fact that $\alpha \in \left(\frac{1}{2}, 1\right), \lambda > \frac{2\alpha}{1-\alpha} - \frac{1}{2}$, implies that $\lim_{t\to\infty} I_2(t) = 0$.

The proof is complete.

3.2.3 Example

Consider the following fractional Caputo stochastic differential equation:

$$
\begin{aligned}
{}^{C}D^{\frac{4}{5}}_{0+}X(t) &= \begin{bmatrix} \sin X_1 \\ X_2 + 3 \end{bmatrix} + \begin{bmatrix} \cos X_1 \\ \tan X_2 \end{bmatrix} \frac{\mathrm{d}W(t)}{\mathrm{d}t}, \quad t \in (0,1], \\
X(0) &= \begin{bmatrix} 4 \\ 5 \end{bmatrix},
\end{aligned}
\tag{3.16}
$$

where: $\alpha \in (\frac{1}{2}; 1)$ and Where

- $X(t) = \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix}$
- $W(t)$ is a Brownian motion
- $b(t, X(t)) = \begin{bmatrix} \sin X_1 \\ X_2 + 3 \end{bmatrix}, \sigma(t, X(t)) = \begin{bmatrix} \cos X_1 \\ \tan X_2 \end{bmatrix}$. are measurable functions

Then,
by the above lemma and theorems, the problem 3.16 has a unique solution.

CONCLUSION

In this thesis, we have studied several fractional Caputo differential equations. Subsequently, we extended them by adding the stochastic term, transforming them into fractional Caputo stochastic differential equations. We proved the existence and uniqueness of solutions by the Banach fixed point theorem.

As future perspectives, these equations can be further developed and explored or using the numerical methods.

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