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On the study of Bifurcations in Marguerre-von Kármán equations

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Dedication

To the one who supported me throughout my life, ***my dear mother***

To the one who was the reason for my existence and taught me freedom, ***my dear father***

To all my brothers, sisters and friends who helped me

To all my honorable teachers who did not spare me anything, especially my supervisor ***Pr. Abderrezak Ghezal***

I dedecated this humble work.

tourkia bendob.

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Abstract

Many mechanical phenomena are mathematically modeled as partial differential equations. Among these equations are the Marguerre-von Kármán equations, which constitute a mathematical model for the buckling of Marguerre-von Kármán shallow shells.

This work aims to study the bifurcation of solutions to the Marguerre-von Kármán equations, which are a system of semilinear elliptic equations of the fourth order. More precisely, we reduce the Marguerre-von Kármán equations to a single cubic operator equation; its second member depends on the function that defines the middle surface of the shallow shell and the applied forces to it. Next, we prove the general existence theorem of the reduced equation using the main theorem for pseudomonotone operators. We then study the bifurcation of the solution in the reduced equation, with the second member being small, in the neighborhood of the simple characteristic value of the linearized problem. Finally, using the mixed finite element scheme of Hermann-Miyoshi, the existence and convergence of the approximate branch of non-singular solutions were studied, and error estimates were obtained.

Key words

Marguerre-von Kármán equations, Bifurcation theory, Nonlinear elasticity, Buckling of shallow shells, Perturbation methods, Finite element methods.

Résumé

De nombreux phénomènes mécaniques sont modélisés mathématiquement sous forme d' équations aux dérivées partielles. Parmi ces équations, les équations de Marguerre-von Kármán, qui constituent un modèle mathématique pour le flambement des coques peu profondes de Marguerre-von Kármán.

Ce travail vise à étudier la bifurcation des solutions des équations de Marguerre-von Kármán, qui sont un système des équations elliptiques quasi linéaires du quatrième ordre. Plus précisément, nous réduisons les équations de Marguerre-von Kármán à une seule équation avec un opérateur cubique, son deuxième membre dépend de la fonction qui définit la surface moyenne de la coque peu profonde et des forces qui lui sont appliquées. Ensuite, nous prouvons le théorème d'existence général pour l'équation réduite, en utilisant le théorème principal sur les opérateurs pseudomonotone. Nous étudions ensuite la bifurcation des solutions dans l'équation réduite, dont le deuxième membre est petit, au voisinage de la valeur caractéristique simple du problème linéarisé. Enfin, en utilisant le schéma d'éléments finis mixtes de Hermann-Miyoshi, l'existence et la convergence de la branche approchée des solutions non singulier ont été étudiées avec des estimations d'erreurs obtenus.

Mots clés :

Équations de Marguerre-von Kármán, théorie de bifurcation, élasticité non linéaire, flambage des coques peu profondes, méthodes de perturbation, méthode des éléments finis.

ملخص

تنمذج العديد من الظواهر الميكانيكية رياضيا الى معادلات تفاضلية جزئية، من بين هذه المعادلات، معادلات مارغر- فون كارمان، والتي تشكل نموذجا رياضيا لإنبعاج أصداف مارغر- فون كارمان الضحلة.

يهدف هذا العمل الى دراسة تشعب حلول معادلات مارغر- فون كارمان التي هي عبارة عن جملة من المعادلات الناقصية شبه خطية من الدرجة الرابعة. بشكل أكثر دقة فإننا نختصر معادلات مارغر- فون كارمان الى معادلة واحدة ذات مؤثر تكعيبي، يعتمد طرفها الثاني على الدالة التي تحدد السطح الأوسط للقشرة الضحلة والقوى المطبقة عليه. بعد ذلك نثبت نظرية الوجود العام للمعادلة المختصرة، وذلك باستعمال النظرية الأساسية لمؤثر الرتابة الزائفة، حيث يكون طرفها الثاني صغير وهذا في جوار قيمة موضوعية بسيطة للمسألة الخطية. وأخيرا باستخدام مخطط العناصر المنتهية لهيرمان- ميوشي، تمت دراسة وجود وتقارب الفرع التقريبي للحلول غير الشاذة مع الحصول على تقديرات للخطأ.

كلمات مفتاحية:

معادلات مارغر- فون كارمان، نظرية التشعب، المرونة غير الخطية، التواء القشرة الضحلة، طريقة التشويش، طريقة العناصر المنتهية.

Contents

Notations and Definitions

- ω: connected bounded open subset of $\mathbb{R}^2$ with a Lipschitz-continuous boundary
- $\gamma = \partial\omega$: boundary of $\omega, 0 \in \gamma$
- $\overline{\omega}$: closure of a set ω.
- $d\gamma$: the length element along γ
- $\gamma(y)$: the arc joining 0 to the point $y \in \gamma$
- ∂_ν: the outer normal operator along the boundary γ
- ν_α : The unit outer normal vector
- $\epsilon > 0$: designates a parameter approaches zero
- $p' > 1$: the conjugate exponent of p i.e $\dfrac{1}{p} + \dfrac{1}{p'} = 1$
- C_a: constant depends of a.
- $f(.), f(.,.)$: function of one variable, two variable ...
- $f(.,b)$: partial mapping $x \to f(x,b)$.
- $f|_\Omega$: restriction of a function f to the set Ω.
- $\partial_\alpha = \dfrac{\partial}{\partial x_\alpha}, \partial_{\alpha\beta} = \dfrac{\partial^2}{\partial x_\alpha \partial x_\beta}$: partial derivative
- $\Delta = \partial_{\alpha\alpha}$: the Laplace operator
- $\Delta^2 = \Delta\Delta = \partial_{\alpha\alpha}\partial_{\beta\beta}$: the biharmonic operator
- I: Identity operator
- B: bilinear operator
- C: cubic operator
- A^{-1}: the inverse operator of A
- $D(A)$: the domain of A
- $\rho(A)$: the resolvent set of the operator A
- $\ker A = \{x \in D(A); Ax = 0\}$.
- $card\,\Omega$: number of elements of a set Ω
- DA: first Fréchet derivative of an operator A .
- $D^m A$: m^{th} Fréchet derivative of an operator A .
- $\mathscr{C}^m(\Omega)$: space of function m times continuously differentiable on Ω.
- $\mathscr{L}(X,Y)$: space of all continuous linear mappings from normed vector space X onto a normed vector space Y.
- $\mathscr{L}(X) = \mathscr{L}(X,X)$
- $X' = \mathscr{L}(X,\mathbb{R})$: dual space of X
- $\langle .,. \rangle$: duality pairing between a space and its dual.
- $supp f$: support of $f = \{x \in X; f(x) \neq 0\}$
- $\mathscr{D}(\Omega) = \{f \in \mathscr{C}^\infty(\Omega); supp\ f$ is compact $\}$
- $L^p(\Omega) = \left\{f; \int_\Omega |f|^p d\Omega < \infty\right\}$: the usual space of measurable whose p th power is Lebesgue integrable

- $W^{m,p}(\Omega)=\left\{f\in L^p(\Omega);\partial^\alpha f\in L^p(\Omega)\text{for all}\,|\alpha|\leq m\right\},1\leq p\leq\infty$: usual Sobolev space
- $H^m(\Omega)=W^{m,2}(\Omega)$
- $W^{-m,p'}(\Omega)=(W_0^{m,p}(\Omega))'$: dual of $W^{m,p}(\Omega)$
- $(.,.)_\Delta$: scalar product in H_0^2
- $\|.\|_{s,p}$: the norm in $W^{s,p}(.)$, $s\in\mathbb{R},p\geq 1$
- $|.|_{s,p}$: semi norm in $W^{s,p}(.),s\in\mathbb{N}$
- $\|.\|_s=\|.\|_{s,2}$ and $|.|_s=|.|_{s,2}$: the norm and the semi norm in L^p
- $X\oplus Y=\{x+y;x\in X,y\in Y\}$ where $X\cap Y=\{0\}$.
- $\hookrightarrow$: continuous embedding.
- $\rightarrow$: strong convergence .
- $\rightharpoonup$: weak convergence.

Introduction

The Bifurcation theory is of mechanical origin, dating back to the mid eighteenth century, work by Euler and Bernoulli on equilibrium of elastic beams. The mathematical treatment of this theory starts with Poincaré at the late of the nineteenth century on celestial mechanics. The literature on this theory is quite large, and bifurcation theory is used to solve a wade variety of problems related to bifurcation in science and engineering. In particular, applications are made for variety of PDEs, see, e.g., Rabinowitz [64], Chow and Hale [19], Bruter et al. [15], Kielhöfer [50].

Scientific and technological advances gave rise to many models of mathematics. Model of structural mechanics in this. These structures have been mostly used in plates, and shells; in particular, a shallow shell where the initial shell circumference is assumed to be small. Introduces mathematical formulation of these problems to a system of partial differentiation and boundary conditions with complex geometric shapes such as those in many shells.

In the first half of the last century, Marguerre [57], von Kármán and Tsien [69] presented a system of fourth-order semilinear elliptic equations, a two-dimensional equation for nonlinearly elastic shells a simple with plate subjected to boundary conditions [70]. In terms of both stability and energy, the von Kármán and Marguerre-von Kármán equations have attracted considerable mathematical and engineering interest. They play an important role in applied mathematics, with applications in many areas of complex engineering and have applications to many fields of solid mechanics. More detailed examples are the numerical models associated with these equations, can be found in, e.g., Ciarlet and Rabier [29], Ciarlet and Paumier [28], Ciarlet [23], Gratie [39], Ciarlet and Gratie [30] and Chueshov and Lasiecka [20]. In this direction, we refer to the works of Ghezal [4, 17, 33, 35] for generalized Marguerre-von Kármán equations. In addition, we refer to [58] and the references therein for linearly elastic shallow shells.

A very interesting phenomenon associated with the Marguerre-von Kármán and von Kármán equations is the appearance of the buckling. This situation is reflected in the multiplicity of solutions to the boundary value problems associated with these equations, which were mathematically treated as bifurcation problems besides Ciarlet and Rabier [29], many authors have studied bifurcation in von Kármán equations, e.g., Antman [2], Brilla [10, 11]

Berger [5, 7], Chien and Chen [18], Chow and Hale [19], Vanderbauwhede [68], Holder and Schaffer [43], Janczewska [45, 46]. Topological methods are among the most important methods for studying bifurcation problems. For results in this area, we refer to Goeleven et al. [38], Gratie [40, 41], Janczewska [44] and Kesaven [49]. Indeed, in recent years, the nonlinear governing equations concerning the transverse deflections of the shell and Airy stress function are able to characterize large shell deformations that are of primary interest in industrial applications. Notably, in the seminal work, Ciarlet and Rabier [29] have studied perturbed bifurcation in the von Kármán equations for plates the details are summarized in [23]. We therefore propose to extend this study further Marguerre-von Kármán detailed Geometry of Shallow Shells.

For numerical analysis, numerous works have been devoted to von Kármán equations. Finite element approaches have been successfully used, have taken on a variety of different types, see, e.g., [12, 26, 47, 55, 56, 63] for conforming, nonconforming and hybrid FEMs. Especially, we refer to [13, 14, 48, 59, 67] about mixed FEMs. More recently, we quote the work of Carstensen and Nataraj [16] for adaptive Morley FEMs. The FEMs for Marguerre-von Kármán equations have been analyzed in [36, 34].

In fact, the approximation of branch of nonsingular solutions of von Kármán equations for the buckling of a nonlinearly thin elastic plates has been studied using the mixed finite element scheme of Heltan-Hermann-Johnson by Brezzi et al. [13]. Next, Reinhart [67] used the general results of Brezzi et al. [13] to approximate the branch of nonsingular solutions of von Kármán equations using the mixed finite element scheme of Hermann-Miyoshi. Limited studies were done about bifurcation in Marguerre-von Kármán equations, see Rao [66], Léger and Miara [53] and Ghezal [34, 37].

The objective of this thesis is to study the bifurcating the trivial solution of the Marguerre-von Kármán equations in the neighborhood of simple characteristic value of the linearized problem. For this reason, we extend the results of Ciarlet [29] from plate to more general geometry shallow shell, i.e., from von Kármán to Marguerre-von Kármán equations under certain condition for the function θ. the second main results of this thesis is to study the existence and the convergence of the approximate branch of non singular solutions, by applying the technique of Brezzi et al. [13].

Now, we present the organization of this thesis:
In the first chapter, we have recalled the most important theories, lemmas, and propositions, found mostly in Haim Bresis [9] and Ciarlet [22], as well as the characteristic value, the compact operator, and its most important characteristics, which we need to study later in the coming chapters.

In the second chapter, we present the classical Marguerre-von Kármán equations and how they were transformed into equations in a simpler and less complex form. Then we converted the letter into a cubic operator equation with one unknown, and for that, we used special operators found at the beginning of this chapter. After that, the existence and regularity of the solution are proved by the main theorem of minimization problems and the embedding Sobolev space.

In the third chapter, we have developed a general introduction to the bifurcation theory from bifurcation points and some conditions of their existence, such as the implicit function theorem and the Krasnoselsk'ii theorem, whose form applies to the bifurcation problem in Marguerre-von Kármán equations, and based on it, we set the first condition $(H1)$ in this thesis for study. Then we present the bifurcation problem in Marguerre-von Kármán equations, and in the same way as before, we reduce it to the cubic operator equation of the bifurcation

in Marguerre-von Kármán. Then we study the general existence result for the cubic operator equation by applying the main theorem on pseudomonotone operators. From the condition $(H1)$ and Krasnoselsk's theorem, we defined the form of bifurcation points. After that, we studied the bifurcation for a homogeneous problem using the Kikuchi method. These results were found without proof in [32].
The first main result of this thesis is determining the solutions of the perturbed bifurcation in Marguerre-von Kármán equations by extending the study of Ciarlet and Rabier [29] for perturbed bifurcation in von Kármán equations for plates to the more general geometry of Marguerre-von Kármán shallow shells.

In the fourth chapter, we present the biharmonic problem and the mixed formulation and the most important results related to the regularity of its solution and we define the discrete problem which converge to the continuous problem. based on mixed formulation of biharmonic problem we introduce the mixed formulation of bifurcation in Marguerre-von Kármán and their continuous operator equation, by utilizing the mixed finite element scheme of Hermann-Miyoshi we present the discrete problem and its discrete operator equation. The last main results of this thesis is the study of existence of a branch of solutions for the discrete operator equation which approximates the branch of nonsingular solutions of the continuous operator equation, for this, we based on the methods and technics of Reinhart ([67]) and Brezzi et al. [13].

Chapter 1

Preliminaries

In this chapter of the thesis, we have recalled the most important Theories, Lemmas and Propositions found mostly in [9], which we need to study later in the coming chapters

1.1 Spectrum and the resolvent

Definition 1.1.1. *[9] Let X, Y be two Banach spaces and let $A : X \to Y$ is a bounded linear mapping ($A \in \mathscr{L}(X, Y)$), then the resolvent set $\rho(A)$ is defined by:*

$$\rho(A) = \{\lambda \in \mathbb{C}; (A - \lambda I) \textit{is bijective from X onto Y}\}.$$

We recall that $(A - \lambda I)$ is bijective means that $(A - \lambda I)$ has a bounded inverse ($(A - \lambda I)^{-1} \in \mathscr{L}(Y, X)$)

Definition 1.1.2. *[9] the spectrum of A denoted by $\sigma(A)$, is the complement of the resolvent set, i.e $\sigma(A) = \mathbb{C} - \{\rho(A)\}$ This definition equivalent to*

$$\sigma(A) = \mathbb{C} - \rho(A) = \{\lambda \in \mathbb{C}; (A - \lambda I) \textit{ has't a bounded inverse}\}.$$

Following [61] the spectrum $\sigma(A)$ is decomposed into three disjoint sets $P_\sigma(A), C_\sigma(A)$ and $R_\sigma(A)$ with the following properties:

•$P_\sigma(A)$ is the set of complex or real numbers λ for which $(A - \lambda I)$ does not have an inverse, $P_\sigma(A)$ is called the point spectrum of A.

•$C_\sigma(A)$ is the set of complex or real numbers λ for which $(A - \lambda I)$ has a discontinuous inverse with domain dense in X; $C_\sigma(A)$ is called the continuous spectrum of A.

•$R_\sigma(A)$ is the totality of complex or real numbers λ for which $(A - \lambda I)$ has an inverse whose domain is not dense in X; $R_\sigma(A)$ is called the residual spectrum of A.

Definition 1.1.3. *[61] A point $\lambda \in \mathbb{C}$ is called an eigenvalue of the operator A if there is a nonzero $\eta \in X$ such that:*

$$A(\eta) = \lambda\eta,$$

that is, the space $\ker(A - \lambda I)$ is at least one dimensional. Any $\eta \in \ker(A - \lambda I)$ is called an eigenvector.

• If λ is an eigenvalue of the operator A, then $\frac{1}{\lambda}$ is called a characteristic value of the operator A

Proposition 1.1.1. *[61] Let A be a linear bounded operator, then $\lambda_0 \in P_\sigma(A)$ if and only if λ_0 is an eigenvalue of A.*

1.2 Spectrum of compact and self adjoint operator

1.2.1 Compact operator

Definition 1.2.1. *[22] Let X and Y be two Banach spaces. A linear operator $T : X \to Y$ is said to be compact if the image under A of any bounded subset of X is a relatively compact subset of Y; in other words, whenever B is bounded in X, then $\overline{A(B)}$ is compact in Y.*

Following H. Bresis [9] we denote by $\mathcal{K}(X, Y)$ the set of all compact operators from X into Y.

Proposition 1.2.1. *[9] Let X, Y, and Z be three Banach spaces. Let $T \in \mathscr{L}(X, Y)$ and $S \in \mathcal{K}(Y, Z)$ [resp. $T \in \mathcal{K}(X, Y)$ and $S \in \mathscr{L}(Y, Z)$]. Then*

$$SoT \in \mathcal{K}(X, Z).$$

Theorem 1.2.1. *[9] Let $T \in \mathcal{K}(X)$,i.e, (T be a compact operator) with dim $X = \infty$, then we have:*
(a) $\quad 0 \in \sigma(T)$,
(b) $\quad \sigma(T) \backslash \{0\} = P_\sigma(T) \backslash \{0\}$,
(c) *one of the following cases holds:*

$$\sigma(T) = 0,$$
$$\sigma(T) \backslash 0 \textit{ is a finite set },$$
$$\sigma(T) \backslash 0 \textit{ is a sequence converging to 0 }.$$

1.2.2 Sequentially compact operator

Definition 1.2.2. *[72] Let X and Y be Banach spaces, a bounded linear operator $T : X \to Y$ is called sequentially compact if*

$$x_n \rightharpoonup x, \text{ then } T(x_n) \to T(x).$$

Remark 1.2.1. *[7] A compact linear operator is sequentially compact. If X is reflexive Banach spaces then a sequentially compact operator is compact.*

Definition 1.2.3. *[49] Let $T : \bar{\Omega} \to X$ be a compact mapping. We set*

$$A = I - T$$

A is called a compact perturbation of the identity.

1.2.3 Self adjoint operator

Theorem 1.2.2. *[22] Let $(X, (.,.)_X)$ and $(Y, (.,.)_Y)$ be two complex Hilbert spaces and let the operator $A \in \mathscr{L}(X, Y)$ (bounded and linear) be given.*
$a)$ There exist a unique operator $A^* \in \mathscr{L}(Y, X)$, called the adjoint of A, that satisfies

$$(Ax, y)_Y = (x, A^* y)_X \quad \text{for all } x \in X, y \in Y$$

The mapping $A \in \mathscr{L}(Y, X) \to A^* \in \mathscr{L}(Y, X)$ defined in this fashion is semilinear. Besides,

$$\|A\|_{\mathscr{L}(X,Y)} = \|A^*\|_{\mathscr{L}(Y,X)}.$$

b) The following relations hold:

$$(ImA)^{\perp} = \ker A^* \quad and \quad (ImA^*)^{\perp} = \ker A,$$
$$Y = \ker A^* \oplus \overline{ImA} \quad and \quad X = \ker A \oplus \overline{ImA^*}.$$

Definition 1.2.4. *[22] Let $(X, (.,.)_X)$ be an inner-product space. A linear operator $A : X \to X$is self-adjoint if it coincides with its adjoint A^*, i.e., if it satisfies*

$$(Ax, y)_X = (x, Ay)_X \quad for\ all\ x, y \in X$$

.

Theorem 1.2.3. *[22] Let $(X, (.,.)_X)$ be an inner-product space, and let $A : X \to X$ be a self-adjoint linear operator.*
(a) For any $x \in X$, the scalar $(Ax, x)_X$ is real.
(b) Let λ be any eigenvalue of A. Then λ is real. Moreover, $\lambda \geq 0$ if A is nonnegative-definite, and $\lambda > 0$ if A is positive-definite.
(c) Eigenvectors corresponding to distinct eigenvalues are orthogonal.
(d) If $A \in \mathscr{L}(X)$, the operator norm of A, viz., $\|A\| := \sup\limits_{x\neq 0} \dfrac{\|Ax\|}{\|x\|}$, is also given by

$$\|A\| = \sup_{x\neq 0} \frac{|\ (Ax, x)\ |}{\|x\|^2}$$

1.2.4 Spectral Theory for compact self-adjoint operators

Theorem 1.2.4. *[9] Let X be a separable Hilbert space and let A be a compact self-adjoint operator. Then there exists a Hilbert bases composed of eigenvectors of A.*

Theorem 1.2.5. *[22] Let $(X, (.,.)_X)$ be an infinite- dimensional inner-product space and let $A : X \to X$ be a compact and self-adjoint linear operator with an infinite-dimensional range. Then:*
$a)$ There exist an infinite sequence (λ_n) of eigenvalues of A and an infinite sequence (P_n) of corresponding eigenvectors that satisfy

$$|\ \lambda_1\ | = \|A\|,\ |\ \lambda_1\ | \geq |\ \lambda_2\ | \geq ... \geq |\ \lambda_n\ | \geq ...,\ \lambda_n \neq 0\, \textit{for all n} \geq 1,\ \lim_{n\to\infty} \lambda_n = 0$$

$$Ap_n = \lambda_n p_n \ \textit{ for all n } \geq 1 \ \textit{ and } (p_k, p_l) = \delta_{kl}\, \textit{for all k,l} \geq 1.$$

$$|\ \lambda_1\ | = \frac{|\ (Ap_1, p_1)\ |}{\|p_1\|^2} = \sup_{x\neq 0} \frac{|\ (Ax, x)\ |}{\|x\|^2}$$

$$|\ \lambda_n\ | = \frac{|\ (Ap_n, p_n)\ |}{\|p_n\|^2} = \sup_{\substack{x\neq 0 \\ (x,p_k)=0,\ 1\leq k\leq n-1}} \frac{|\ (Ax, x)\ |}{\|x\|^2} \textit{for all n} \geq 2.$$

$b)$ For any vector $x \in X$,

$$Ax = \sum_{n=1}^{\infty} \lambda_n (x, p_n) p_n$$

$c)$ *Let λ be any nonzero eigenvalue of A. then, there exists $n \geq 1$ such that $\lambda_n = \lambda$. Besides, the set $I(\lambda) := \{n \geq 1; \lambda_n = \lambda\}$ is finite, and*

$$\{p \in X; Ap = \lambda p\} = Span(p_n)_{n \in I(\lambda)}$$

$d)$ *the kernel of A is also given by*

$$\ker A = \{Span(p_n)\}^{\perp}$$

1.3 Orthogonality and Orthogonal Projection Operator

1.3.1 Orthogonality

Definition 1.3.1. *[22] Let $(X, (.,.)_X)$ be a real or complex inner-product space. two vectors $x \in X$ and $y \in X$ are said to be orthogonal if*

$$(x, y)_X = 0,$$

and the orthogonal complement of any nonempty subset Z of X is the subset of X defined as

$$Z^{\perp} = \{x \in X; (x, z) = 0 \, \textit{for all } z \in Z\}.$$

Theorem 1.3.1. *[22] Let Z be a nonempty subset of an inner-product space X. Then the set $Z^{\perp}$ is a closed subspace of X. Besides, $(\overline{Z})^{\perp} = Z^{\perp}$, and $Z \cap Z^{\perp} = \{0\}$ if $0 \in Z$ and $Z \cap Z^{\perp} = \phi$ if $0 \notin Z$.*

1.3.2 Direct sum Theorem and Orthogonal Projection Operator

Theorem 1.3.2. *[22] Let X be a real or complex Hilbert space and let M be a closed subspace of X. Then the space X is the direct sum*

$$X = M \oplus M^{\perp},$$

i.e., any element $x \in X$ can be written as

$$x = m + m^{\perp} \textit{ with } m \in M \textit{ and } m^{\perp} \in M^{\perp},$$

and such a decomposition is unique. In fact,

$$m = \mathscr{P}x \quad \textit{and} \quad m^{\perp} = \mathscr{P}^{\perp}x,$$

where $\mathscr{P} : X \to M$ denotes the projection operator from X onto M, and

$$\mathscr{P}^{\perp} = I - \mathscr{P}$$

is the projection operator from X onto $M^{\perp}$.

Lemma 1.3.1. *[22]*

$i)$ *Any orthogonal projection map $\mathscr{P}$ on a closed subspace M of Hilbert space X is self adjoint with $\mathscr{P}^2 = \mathscr{P}$, $\|\mathscr{P}\| = 1$, and $\|(1 - \mathscr{P})x\| = d(x, M)$. Conversely, any self-adjoint operator Q with $Q^2 = Q$ is the orthogonal projection of X onto $Q(X)$.*

$ii)$ *An orthogonal projection operator $\mathscr{P}$ of X onto M is compact if and only if $dim M < \infty$.*

1.4 Differentiability

Let $\mathscr{C}(X, Y)$ denote the set of continuous mappings between the topological spaces X and Y.

Definition 1.4.1. *[7] $f \in \mathscr{C}(X, Y)$ is Fréchet differentiable at x_0 if there is a linear operator $A \in L(X, Y)$ such that in a neighborhood U of x_0*

$$\|f(x) - f(x_0) - A(x - x_0)h\| = o(\|x - x_0\|). \tag{1.4.1}$$

In this case we write $A = f'(x_0)$, and $f'(x_0)$ is called the Fréchet derivative of f at x_0. If the mapping $x \to f'(x)$ of $X \to L(X, Y)$ is continuous at x_0, f is called $\mathscr{C}^1$ at x_0.

If we put $h = x - x_0$, where $x_0 + h \in U$, (1.4.1) become

$$\|f(x_0 + h) - f(x_0) - f'(x_0)h\| = o(\|h\|)$$

Definition 1.4.2. *[7] $f \in \mathscr{C}(X, Y)$ is Gateaux differentiable at x_0 if there an operator $Df(x_0, h) \in \mathscr{C}(X \times X, Y)$ such that*

$$\lim_{t\to 0} \|f(x_0 + th) - f(x_0) - tDf(x_0, h)\| = 0$$

for $(x_0 + th) \in U$, a neighborhood of x_0. Furthermore, $Df(x_0, h)$ is called the Gateaux derivative of f at x_0, and we write

$$\frac{d}{dt} f(x_0 + th)|_{t=0} = Df(x_0, h).$$

Lemma 1.4.1. *[7] If f is Gateaux differentiable at $x_0 + th$, $(0 \le t \le 1)$*

$$f(x_0 + h) - f(x_0) = \int_0^1 Df(x_0 + th, h)dt.$$

The following theorem gives the relationship between Gateaux and Fréchet differentiability

Theorem 1.4.1. *[7] If $f \in \mathscr{C}(X, Y)$ is Fréchet differentiable at x_0, it is Gateaux differentiable at x_0. Conversely, if the Gateaux derivative of f at x_0, $Df(x_0, h)$, is linear in h, i.e., $Df(x, .) \in L(X, Y)$ and is continuous in x as a map from $X \to L(X, Y)$, then f is Fréchet differentiable at x_0. In either case we have the formula $f'(x_0)y = Df(x_0, y)$.*

Lemma 1.4.2. *[7] The Fréchet and Gateaux derivatives are unique.*

1.5 Definition and some Properties of L^p Space

Definition 1.5.1. *[9] Let $(\Omega, \mathcal{M}, \mu)$ denote a measure space, We denote by $L^1(\Omega, \mu)$, or simply $L^1(\Omega)$ (or just L^1),the space of integrable functions from Ω into $\mathbb{R}$. and*

$$\|f\|_{L^1} = \|f\|_1 = \int_\Omega |f| d\mu = \int |f|$$

Definition 1.5.2. *[9] Let $p \in \mathbb{R}$ with $1 < p < \infty$; We set*

$$L^p(\Omega) = \left\{f : \Omega \to \mathbb{R}; f \textit{ is measurable and } |f|^p \in L^1(\Omega)\right\}.$$

with

$$\|f\|_{L^p} = \|f\|_p = \left[\int_\Omega |f(x)|^p d\mu\right]^{1/p}$$

for $p = \infty$ we have the following definition

Definition 1.5.3. *[9]*

$$L^\infty(\Omega) = \left\{ \begin{array}{ll} f : \Omega \to \mathbb{R}; & f \text{ is measurable and there is a constant } C \\ & \text{such that } |f(x)| \leqslant C \text{ on } \Omega \end{array} \right\}$$

with

$$\|f\|_{L^\infty} = \|f\|_\infty = \inf \{C; |f(x)| \leqslant C \text{ on } \Omega\}.$$

We recall that p' is the conjugate exponent of p if $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, for any $p, p' \geq 1$

Young Inequality

Lemma 1.5.1. *[9] For all $a, b \geq 0$ and for all $p, p' \geq 1$, were p' is the conjugate exponent of p we have*

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}$$

Hölders Inequality

Theorem 1.5.1. *[22] Assume that $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$ with $1 \leqslant p \leqslant \infty$ and p' is the conjugate exponent of p. Then $fg \in L^1(\Omega)$ and*

$$\int_\Omega | fg | \leqslant \|f\|_p \|g\|_{p'}$$

Reflexivity and dual space of L^p

Theorem 1.5.2. *[9] L^p is reflexive for any p, $1 < p < \infty$ and separable for $1 \leq p < \infty$ and the dual space of L^p is $L^{p'}$, i.e, $(L^p)' = L^{p'}$*

1.6 Sobolev Spaces, Definition and some Properties

Let $\Omega \subset \mathbb{R}^n$ be an open set and let $p \in \mathbb{R}$ with $1 \leq p \leq \infty$.

Definition 1.6.1. *[9] The Sobolev space $W^{1,p}(\Omega)$ is defined by*

$$W^{1,p}(\Omega) = \left\{ \begin{array}{ll} u \in L^p(\Omega); & \exists g_1, g_2 ... g_N \in L^p(\Omega) \text{ such that} \\ & \int_\Omega u \dfrac{\partial \varphi}{\partial x_i} = - \int_\Omega g_i \varphi \quad \forall \varphi \in C_c^\infty(\Omega) \; \forall i = 1, 2, ... N \end{array} \right\}$$

We set

$$H^1(\Omega) = W^{1,2}(\Omega)$$

The space $W^{1,p}(\Omega)$ is equipped with the norm

$$\|u\|_{1,2} = \|u\|_p + \sum_{i=1}^{N} \|\frac{\partial u}{\partial x_i}\|_p.$$

or sometimes with the equivalent norm

$$\|u\|_{1,2} = \left(\|u\|_p^p + \sum_{i=1}^{N} \| \frac{\partial u}{\partial x_i} \|_p^p \right)^{1/p} \qquad if 1 \leq p < \infty$$

Definition 1.6.2. *[9] Let $m \geq 2$ be an integer and let p be a real number with $1 \leq p \leq \infty$. We define by induction*

$$W^{m,p}(\Omega) = \left\{ u \in W^{m-1,p}(\Omega); \frac{\partial u}{\partial x_i} \in W^{m-1,p}(\Omega) \quad \forall i = 1, 2, ...N \qquad \right\}$$

Theorem 1.6.1. *[22]The space $W^{m,p}(\Omega)$ satisfies:*

1. *The space $W^{m,p}(\Omega)$ is Banach space,*
2. *$W^{m,p}(\Omega)$is separable if $1 \leq p < \infty$,*
3. *$W^{m,p}(\Omega)$ is reflexive if $1 < p < \infty$*
4. *$H^m(\Omega) = W^{m,2}(\Omega)$ is Hilbert space.*

Corollary 1.6.1. *[9] Let $\Omega \subset \mathbb{R}^N$ be an open set of class C^1 with $\Gamma = \partial\Omega$ bounded or else that $\Omega = \mathbb{R}^N_+$ Let $1 \leq p \leq \infty$. We have*

$$\begin{aligned} &W^{1,p}(\Omega) \hookrightarrow L^{p*}(\Omega), \quad where\ \frac{1}{p*} = \frac{1}{p} - \frac{1}{N}, \quad if\ p < N \\ &W^{1,p}(\Omega) \hookrightarrow L^{q}(\Omega) \qquad \forall q \in [p, +\infty) \qquad if\ p = N \\ &W^{1,p}(\Omega) \hookrightarrow L^{\infty}(\Omega) \qquad\qquad\qquad if\ p > N \end{aligned}$$

and all these injections are continuous. Moreover, if $p > N$ we have, for all $u \in W^{1,p}(\Omega)$,

$$| u(x) - u(y) | \leq C\|u\|_{W^{1,p}} | x - y |^{\alpha} \qquad x, y \in \Omega.$$

with $\alpha = 1 - \frac{N}{p}$ and C depends only on Ω, p, and N. In particular, $W^{1,p}(\Omega) \subset C(\overline{\Omega})$

Theorem 1.6.2 ([9] Rellich-Kondrachov)**.** *Suppose that Ω is bounded and of class $\mathscr{C}^1$. Then we have the following compact injections:*

$$\begin{aligned} &W^{1,p}(\Omega) \hookrightarrow L^{q}(\Omega), \quad \forall q \in [1, p*), \quad where\ \frac{1}{p*} = \frac{1}{p} - \frac{1}{N}, \qquad if\ p < N \\ &W^{1,p}(\Omega) \hookrightarrow L^{q}(\Omega) \qquad \forall q \in [p, +\infty) \qquad\qquad if\ p = N \\ &W^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega}) \qquad\qquad\qquad\qquad if\ p > N \end{aligned}$$

In particular, $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ with compact injection for all p (and all N).
In general case as in [29] we have the following compact injection

$$\begin{aligned} &W^{m,p}(\Omega) \hookrightarrow L^{q}(\Omega), \quad \forall 1 \leq q < p*, \quad where\ \frac{1}{p*} = \frac{1}{p} - \frac{m}{N}, \qquad if\ mp < N \\ &W^{m,p}(\Omega) \hookrightarrow L^{q}(\Omega) \qquad \forall q \in [p, +\infty[\qquad\qquad if\ mp = N \\ &W^{m,p}(\Omega) \hookrightarrow C(\overline{\Omega}) \qquad\qquad\qquad\qquad if\ mp > N \end{aligned}$$

and the injection

$$W^{m,p}(\Omega) \hookrightarrow L^{p*}(\Omega), \quad \text{where } \frac{1}{p*} = \frac{1}{p} - \frac{m}{N}, \qquad if\ mp < N$$

is continuous

Lemma 1.6.1. *[9] The space $W^{m,p}(\Omega)$ equipped with the norm*

$$\|u\|_{m,p} = \sum_{0 \leq |\alpha| \leq m} \|D^{\alpha} u\|_p$$

is a Banach space.
where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$ is the standard multi-index, with α_i an integer,

Remark 1.6.1. *[9] if Ω is smooth enough with $\gamma = \partial\Omega$ bounded, then the norm on $W^{m,p}(\Omega)$ is equivalent to the norm*

$$\|u\|_p + \sum_{|\alpha| = m} \|D^{\alpha} u\|_p$$

Lemma 1.6.2. *[9] The space $H^m(\Omega) = W^{m,2}(\Omega)$ equipped with the scalar product*

$$(u, v)_{H^m} = \sum_{0 \leq |\alpha| \leq m} (D^{\alpha} u, D^{\alpha} v)_{L^2}$$

is a Hilbert space.

Theorem 1.6.3. *[22]*
(a) Let Ω be an open subset of $\mathbb{R}^n$. Then

$$|v|_{2,2} = \|\Delta v\|_{0,2} \quad \text{for all } v \in H_0^2(\Omega).$$

(b) If Ω is of finite width, the semi-norm $v \to \|\Delta v\|_{0,2}$ becomes a norm over the space $H_0^2(\Omega)$, equivalent to the norm $\|v\|_{2,2}$.

1.7 Minimization Problem

Definition 1.7.1. *[49] Let $J : \Omega \subset X \to \mathbb{R}$ be a functional defined on a subset Ω of a Banach space X. We say that J is coercive if $J(x_n) \to +\infty$ whenever we have $x_n \in \Omega$ such that $\|x_n\| \to +\infty$*

Definition 1.7.2. *[49] Let X be a Banach space and let $\Omega \subset X$. We say that $J : \Omega \to \mathbb{R}$ is weakly lower semi-continuous (weakly l.s.c.) if $J^{-1}((-\infty, C])$ is weakly closed for all $C \in \mathbb{R}$.*
We say that J is weakly sequentially l.s.c. if whenever a sequence x_n in Ω converge weakly to $x \in \Omega$ we have

$$J(x) \leq \lim_{n \to \infty} \inf J(x_n)$$

Remark 1.7.1. *[49] Obviously, a weakly l.s.c. functional in X is weakly sequentially l.s.c.*

Theorem 1.7.1. *[7] Suppose $J(u)$ is a bounded functional defined on a (sequentially) weakly closed and nonempty subset M of a reflexive Banach X. Then if $J(u)$ is coercive on M (in the sense that $J(u) \to \infty$ whenever $\|u\| \to \infty$ with $u \in M$), and in addition $J(u)$ is weak lower semi-continuous on M, then $C = \inf J(u)$ over M is finite and attained at a point $u_0 \in M$.*

In particular, if $M = X$ and $J(u)$ is $\mathscr{C}^1$, then $J'(u_0) = 0$ so that $C = J(u_0)$ is a critical value of $J(u)$, and any element in $J^{-1}(C)$ is a critical point of $J(u)$.

Chapter 2

Classical Marguerre-von Kármán equations

Before presenting the Marguerre-von Kármán equations, we will need to determine some definitions and operators and their properties.

2.1 The bracket $[\xi, \eta]$

Definition 2.1.1. *[23] for any $\xi, \eta \in H^2(\omega)$ we define the brackets $[\xi, \eta]$ as follows:*

$$[\xi, \eta] = \xi_{xx}\eta_{yy} + \xi_{yy}\eta_{xx} - 2\xi_{xy}\eta_{xy}. \tag{2.1.1}$$

It is clearly that the bracket $[\xi, \eta]$ has the following property:

Lemma 2.1.1. *[29] the brackets $[.,.]$ is bilinear and symmetric form.*

Corollary 2.1.1. *[31] for $\xi, \eta \in H^2(\omega)$ we have:*

$$[\xi, \eta] = \frac{1}{2}\Delta(\xi\Delta\eta) - S_1(\xi S_1\eta) - 2S_2(\xi S_2\eta).$$

Where for $\eta \in H^2(\omega)$

$$S_1(\eta) = \frac{1}{2}\left(\frac{\partial^2\eta}{\partial x^2} - \frac{\partial^2\eta}{\partial y^2}\right), \quad S_2(\eta) = \frac{\partial^2\eta}{\partial x\partial y}.$$

Lemma 2.1.2. *[31] for any $\xi \in \mathscr{C}^2(\omega)$ and for $\eta \in \mathscr{C}^4(\omega)$, we have*

$$\begin{aligned}
[\xi, \eta] &= (\xi\eta_{yy})_{xx} + (\xi\eta_{xx})_{yy} - 2\,(\xi\eta_{xy})_{xy} \\
[\xi, \eta] &= (\xi_x\eta_{yy})_x + (\xi_y\eta_{xx})_y - (\xi_x\eta_{xy})_y - (\xi_y\eta_{xy})_x
\end{aligned}$$

Lemma 2.1.3. *[29] The following mapping*

$$T : (\zeta, \xi, \eta) \in (H^2(\omega))^3 \to \int_\omega [\zeta, \xi]\eta d\omega, \tag{2.1.2}$$

is trilinear continuous form; furthermore, if at least one of the three arguments ζ, ξ and η is in $H_0^2(\omega)$, then the mapping T will be a symmetric, so in this case we get

$$\left|\int_\omega [\zeta, \xi]\eta d\omega\right| \le C \mid \zeta \mid_{2,2}\mid \xi \mid_{1,4}\mid \eta \mid_{1,4} \tag{2.1.3}$$

Proof. [29] From the definition of bracket $[.,.]$ we get for $\zeta, \xi \in H^2(\omega), [\zeta, \xi] \in L^1(\omega)$ and from the embedding $H^2(\omega) \hookrightarrow \mathscr{C}^0(\overline{\omega})$ we conclude that there exists a constant C such that

$$\left| \int_\omega [\zeta, \xi] \eta d\omega \right| \leq C \mid [\zeta, \xi] \mid_{0,1} \mid \eta \mid_{0,\infty} \leq C \mid \zeta \mid_{2,2} \mid \xi \mid_{2,2} \|\eta\|_{2,2}$$

Hence the mapping T is continuous.

Let the functions ζ, ξ and η be in $\mathscr{C}^\infty(\overline{\omega})$, then we have

$$\begin{aligned}\int_\omega [\zeta, \xi] \eta d\omega &= \int_\omega \eta \left(\partial_{11}\zeta \partial_{22}\xi + \partial_{22}\zeta \partial_{11}\xi - 2\partial_{12}\zeta \partial_{12}\xi\right) d\omega \\ &= \int_\omega \left(\eta \partial_{11}\zeta \partial_{22}\xi - \eta \partial_{12}\zeta \partial_{12}\xi\right) d\omega \\ &\quad + \int_\omega \left(\eta \partial_{22}\zeta \partial_{11}\xi - \eta \partial_{12}\zeta \partial_{12}\xi\right) d\omega\end{aligned}$$

we set:

$$\begin{aligned}I_1 &= \int_\omega \left(\eta \partial_{11}\zeta \partial_{22}\xi - \eta \partial_{12}\zeta \partial_{12}\xi\right) d\omega \\ I_2 &= \int_\omega \left(\eta \partial_{22}\zeta \partial_{11}\xi - \eta \partial_{12}\zeta \partial_{12}\xi\right) d\omega\end{aligned}$$

with simple calculation we get:

$$\begin{aligned}I_1 &= \int_\omega \partial_2 \left(\eta \partial_{11}\zeta \partial_2\xi - \eta \partial_{12}\zeta \partial_1\xi\right) d\omega - \int_\omega \partial_2\xi \partial_2 \left(\eta \partial_{11}\zeta\right) d\omega + \int_\omega \partial_1\xi \partial_2 \left(\eta \partial_{12}\zeta\right) d\omega \\ I_2 &= \int_\omega \partial_1 \left(\eta \partial_{22}\zeta \partial_1\xi - \eta \partial_{12}\zeta \partial_2\xi\right) d\omega - \int_\omega \partial_1\xi \partial_1 \left(\eta \partial_{22}\zeta\right) d\omega + \int_\omega \partial_2\xi \partial_1 \left(\eta \partial_{12}\zeta\right) d\omega\end{aligned}$$

If at least one of the functions η, ζ and ξ is in $\mathscr{D}(\omega)$, then

$$\int_\omega \partial_2 \left(\eta \partial_{11}\zeta \partial_2\xi - \eta \partial_{12}\zeta \partial_1\xi\right) d\omega = 0$$

and

$$\int_\omega \partial_1 \left(\eta \partial_{22}\zeta \partial_1\xi - \eta \partial_{12}\zeta \partial_2\xi\right) d\omega = 0$$

then

$$\begin{aligned}I_1 &= -\int_\omega \partial_2\xi \partial_2 \left(\eta \partial_{11}\zeta\right) d\omega + \int_\omega \partial_1\xi \partial_2 \left(\eta \partial_{12}\zeta\right) d\omega \\ I_2 &= -\int_\omega \partial_1\xi \partial_1 \left(\eta \partial_{22}\zeta\right) d\omega + \int_\omega \partial_2\xi \partial_1 \left(\eta \partial_{12}\zeta\right) d\omega\end{aligned}$$

so,

$$I_1 + I_2 = \int_\omega \partial_1\xi \partial_2 \left(\eta \partial_{12}\zeta\right) d\omega + \int_\omega \partial_2\xi \partial_1 \left(\eta \partial_{12}\zeta\right) d\omega - \int_\omega \partial_2\xi \partial_2 \left(\eta \partial_{11}\zeta\right) + \partial_1\xi \partial_1 \left(\eta \partial_{22}\zeta\right) d\omega$$

hence

$$\begin{aligned}\int_\omega [\zeta,\xi]\eta d\omega =& I_1 + I_2 \\ &= \int_\omega \partial_{12}\zeta\left(\partial_1\xi\partial_2\eta + \partial_2\xi\partial_1\eta\right)d\omega \quad - \int_\omega \left(\partial_{11}\zeta\partial_2\xi\partial_2\eta + \partial_{22}\zeta\partial_1\xi\partial_1\eta\right)d\omega\end{aligned} \tag{2.1.4}$$

We have $\overline{\mathscr{C}^\infty(\overline{\omega})} = H^2(\omega)$ and $\overline{\mathscr{D}(\omega)} = H_0^2(\omega)$, then the relation (2.1.4) remains true if the functions ζ, ξ , and η belong to $H^2(\omega)$, and one of them belong to $H_0^2(\omega)$. We have both sides of (2.1.4) are continuous trilinear forms on $H^2(\omega)^3$, with respect to $\|.\|_{2.2}$, (we recall that $H^2(\omega) \hookrightarrow W^{1,4}(\omega)$); hence the inequality (2.1.3) holds. The mapping T becomes symmetric in this case: The left-hand side of (2.1.4) is unaltered if ζ and ξ are exchanged and likewise, the right-hand side is unaltered if ξ and η are exchanged.

□

Corollary 2.1.2 ([23], [29])**.** *Let $\eta \in H_0^2(\omega)$ satisfies:*

$$[\eta,\eta] = 0.$$

Then, $\eta = 0$.

Proof. [23], [29] If $[\eta,\eta] = 0$, for $\eta \in H_0^2(\omega)$ then for any $\chi \in H^2(\omega)$

$$\int_\omega [\eta,\eta]\chi d\omega = 0$$

by the symmetry of T established in (2.1.2), we get

$$\int_\omega [\eta,\eta]\chi d\omega = \int_\omega [\eta,\chi]\eta d\omega = 0$$

choosing $\chi(x_1,x_2) = \frac{1}{2}(x_1^2 + x_2^2)$, we get:

$$[\eta,\chi] = \Delta\eta$$

So,

$$\int_\omega [\eta,\chi]\eta d\omega = \int_\omega \eta\Delta\eta d\omega = |\eta|_{1,\omega}^2 = 0$$

Thus, $\eta = 0$ □

2.2 The Bilinear operator "B"

Definition 2.2.1. *[23], [29] for each pair $(\xi,\eta) \in H^2(\omega) \times H^2(\omega)$, we define the operator B as follows*

$$B : H^2(\omega) \times H^2(\omega) \to H_0^2(\omega),$$

the unique solution of the problem

$$\begin{cases} \Delta^2 B(\xi,\eta) = [\xi,\eta] & \text{in } \omega \\ B(\xi,\eta) \in H_0^2(\omega) \end{cases}$$

Furthermore, we have:

$$(\xi,\eta) \in (H^2(\omega))^2 \Rightarrow [\xi,\eta] \in L^1(\omega) \hookrightarrow H^{-2}(\omega)$$

Lemma 2.2.1. *[23], [29] The operator $B : (H^2(\omega))^2 \longrightarrow (H_0^2(\omega))^2$ is bilinear, symmetric and continuous.*

Proof. [29] The bilinearity and the symmetry of B is immediately consequence of bilinearity and symmetry of the brackets $[., .]$.
For the continuity we have the inverse biharmonic operator:

$$(\Delta^2)^{-1} : H^{-2}(\omega) \to H_0^2(\omega)$$

is continuous, from the definition of B, and the continuous embedding

$$L^1(\omega) \hookrightarrow H^{-2}(\omega) \tag{2.2.1}$$

we get:

$$\Delta^2 B(\xi, \eta) = [\xi, \eta]$$

equivalent to

$$B(\xi, \eta) = (\Delta^2)^{-1}[\xi, \eta]$$

and

$$\|B(\xi, \eta)\|_{2,2} = \|(\Delta^2)^{-1}[\xi, \eta]\|_{2,2} \leq C\|[\xi, \eta]\|_{-2,2}$$

the continuous embedding (2.2.1) gives

$$\|B(\xi, \eta)\|_{2,2} \leq C\|[\xi, \eta]\|_{-2,2} \leq C|[\xi, \eta]|_{0,1}$$

finally, the definition of brackets $[., .]$ gives:

$$\|B(\xi, \eta)\|_{2,2} \leq C|[\xi, \eta]|_{0,1} \leq C|\xi|_{2,2} \mid \eta \mid_{2,2}$$

therefore, the continuity of the operator B is proven. □

Lemma 2.2.2. *[23, 29] Let*

$$(\xi, \eta)_\Delta = \int_\omega \Delta\xi \Delta\eta \, d\omega$$

be an inner product over the space $H_0^2(\omega)$,
and let

$$|\xi|_\Delta = \|\Delta\xi\|_{0,2} = |\xi|_{2,2}$$

is a norm over the space $H_0^2(\omega)$, which precisely corresponds to the inner product $(., .)_\Delta$. Then

$$(B(\xi, \eta), \chi)_\Delta = (B(\xi, \chi), \eta)_\Delta \quad \forall (\xi, \eta, \chi) \in H^2(\omega) \times H_0^2(\omega) \times H_0^2(\omega),$$

and for all $(\xi, \eta, \chi) \in H^2(\omega) \times H^2(\omega) \times H_0^2(\omega)$

$$(B(\xi, \eta), \chi)_\Delta \leq C \, |\xi|_{1,4} \, |\eta|_{1,4} \, |\chi|_\Delta \, .$$

Hence

$$|B(\xi, \eta)|_\Delta \leq C \, |\xi|_{1,4} \, |\eta|_{1,4} \, . \tag{2.2.2}$$

Proof. [23, 29]

$$(B(\xi,\eta),\chi)_\Delta = \int_\omega \Delta B(\xi,\eta)\Delta\chi d\omega = \int_\omega \Delta^2 B(\xi,\eta)\chi d\omega = \int_\omega [\xi,\eta]\chi d\omega$$

by the symmetry of T (2.1.2), we fined

$$\int_\omega [\xi,\eta]\chi d\omega = \int_\omega [\xi,\chi]\eta d\omega = \int_\omega \Delta^2 B(\xi,\chi)\eta d\omega = \int_\omega \Delta B(\xi,\chi)\Delta\eta d\omega = (B(\xi,\chi),\eta)_\Delta$$

Since, $(B(\xi,\eta),\chi)_\Delta = \displaystyle\int_\omega [\xi,\eta]\chi \, d\omega$, and from (2.1.3) we get

$$(B(\xi,\eta),\chi)_\Delta \le \left| \int_\omega [\xi,\eta]\chi d\omega \right| \le C\,|\xi|_{1,4}\,|\eta|_{1,4}\,|\chi|_\Delta$$

furthermore

$$|B(\xi,\eta)|_\Delta = \sup_{\substack{\chi\neq 0\\ \chi\in H_0^2(\omega)}} \frac{(B(\xi,\eta),\chi)_\Delta}{|\chi|_\Delta} \le C\,|\xi|_{1,4}\,|\eta|_{1,4}\,.$$

□

Lemma 2.2.3. *[29] The operator $B : (H^2(\omega))^2 \longrightarrow (H_0^2(\omega))^2$ is sequentially compact, hence a fortiori continuous in the sense that (as usual, strong and weak convergences are noted $\rightarrow$ and $\rightharpoonup$ respectively)*

$$(\xi_n,\eta_n) \rightharpoonup (\xi,\eta) \ in \ (H^2(\omega))^2 \Rightarrow B(\xi_n,\eta_n) \rightarrow B(\xi,\eta) \ in \ H_0^2(\omega)$$

Proof. [29] Let $(\xi_n,\eta_n) \rightharpoonup (\xi,\eta) \ in \ H^2(\omega)\times H^2(\omega)$;
using the bilinearity of B, we may write

$$B(\xi_n,\eta_n) - B(\xi,\eta) = B(\xi_n-\xi,\eta) + B(\xi,\eta_n-\eta) + B(\xi_n-\xi,\eta_n-\eta)$$

and thus, by the inequality (2.2.2),

$$\begin{aligned} |B(\xi_n,\eta_n) - B(\xi,\eta)|_\Delta =& |B(\xi_n-\xi,\eta) + B(\xi,\eta_n-\eta) + B(\xi_n-\xi,\eta_n-\eta)|_\Delta \\ \le& C(|\xi_n-\xi|_{1,4}\,|\eta|_{1,4} + |\xi|_{1,4}\,|\eta_n-\eta|_{1,4} \\ &+ |\xi_n-\xi|_{1,4}\,|\eta_n-\eta|_{1,4}) \end{aligned}$$

The compact embedding $H^2(\omega) \hookrightarrow W^{1,4}(\omega)$ then shows that $B(\xi_n,\eta_n) \rightarrow B(\xi,\eta)$ in $H_0^2(\omega)$; hence, the operator B is sequentially compact. □

2.3 The linear operator "L"

Definition 2.3.1. *For fixed $\eta_0 \in H^2(\omega)$, we define the operator $L : H_0^2(\omega) \rightarrow H_0^2(\omega)$ as follows:*

$$L(\xi) = B(\eta_0,\xi).$$

From the properties of the operator B we conclude

Proposition 2.3.1. *The operator $L : H_0^2(\omega) \rightarrow H_0^2(\omega)$ is linear continuous, sequentially compact and self adjoint operator*

Proof. The linearity of L is immediately consequence of bilinearity of the operator B.
On the other hand by the continuity of B we get

$$\|L(\xi)\|_{2,2} = \|B(\eta_0, \xi)\|_{2,2} \leq \|B\|\|\eta_0\|_{2,2}\|\xi\|_{2,2} \leq C\|\xi\|_{2,2} \tag{2.3.1}$$

hence L is continuous operator
Also for all $\xi, \eta \in H_0^2(\omega)$ and from the Lemma(2.2.2) we get

$$(L(\xi), \eta)_\Delta = (B(\eta_0, \xi), \eta)_\Delta = (B(\eta_0, \eta), \xi)_\Delta = (L(\eta), \xi)_\Delta = (\xi, L(\eta))_\Delta \tag{2.3.2}$$

so, L is self-adjoint.
Finally, we will prove that the operator L is sequentially compact
By the same way as in proof of the Lemma (2.2.3) and for $\xi_n \rightharpoonup \xi \; in \; H_0^2(\omega)$, we find

$$\begin{aligned}\|L(\xi_n) - L(\xi)\|_{2,2} = \|L(\xi_n - \xi)\|_{2,2} &= \|B(\eta_0, \xi_n - \xi)\|_{2,2} \\ &\leq \|B\|\|\eta_0\|_{1,4}\|\xi_n - \xi)\|_{1,4} \\ &\leq C\|\xi_n - \xi)\|_{1,4}\end{aligned}$$

So, from the compact embedding $H^2(\omega) \hookrightarrow W^{1,4}(\omega)$ then shows that $L(\xi_n) \to L(\xi)$ in $H_0^2(\omega)$; hence the operator L is sequentially compact.

□

As a result of Theorem (1.2.5) we conclude:

Corollary 2.3.1. *the spectrum of the operator L consist only by the set of real eigenvalue.*

2.4 The cubic operator "C"

Definition 2.4.1. *[23] For $\eta \in H_0^2(\omega)$, we consider the operator C defined by:*

$$\begin{aligned}C : \eta \in H_0^2(\omega) &\longrightarrow H_0^2(\omega) \\ \eta &\longrightarrow C(\eta) = B(B(\eta, \eta), \eta).\end{aligned}$$

From the properties of the operator B we get:

Proposition 2.4.1. *[29] The operator $C : H_0^2(\omega) \longrightarrow H_0^2(\omega)$ satisfies the following properties:*
a) The operator C is a cubic operator in that

$$C(\alpha\eta) = \alpha^3 C(\eta) \text{ for all } \alpha \in \mathbb{R}, \text{ and } \eta \in H_0^2(\omega)$$

b) The operator $C : H_0^2(\omega) \to H_0^2(\omega)$ is continuous, and sequentially compact operator.

Proof. [29] a) Using the bilinearity of B we get:

$$\begin{aligned}C(\alpha\eta) &= B(B(\alpha\eta, \alpha\eta), \alpha\eta) \\ &= B(\alpha^2 B(\eta, \eta), \alpha\eta) \\ &= \alpha^2 B(B(\eta, \eta), \alpha\eta) \\ &= \alpha^3 \left(B(B(\eta, \eta), \eta)\right) = \alpha^3 C(\eta)\end{aligned}$$

b) It is clear that the operator C is continuous because it is a composition of continuous operators B
For the sequentially compact of B we have
Let $\xi_n \rightharpoonup \xi \in H_0^2(\omega)$, from the Lemma(2.2.3), we conclude that

$$B(\xi_n, \xi_n) \to B(\xi, \xi) \tag{2.4.1}$$

applying the operator B again, we find

$$B(B(\xi_n, \xi_n), \xi_n) \to B(B(\xi, \xi), \xi) \tag{2.4.2}$$

hence

$$C(\xi_n) \to C(\xi) \tag{2.4.3}$$

□

Proposition 2.4.2. *[29] a) for any $\eta \in H_0^2(\omega)$,*

$$(C\eta, \eta)_\Delta \geq 0,$$

$$(C\eta, \eta)_\Delta = 0 \Leftrightarrow \eta = 0.$$

and

$$\|C(\eta)\| \leq \|B\|^2 \|\eta\|^3$$

b) for any $(\xi, \eta) \in H_0^2(\omega)^2$ we have

$$(C(\xi) - C(\eta), \xi - \eta)_\Delta \leq \|B\|^2 \max\left\{\|\xi\|^2, \|\eta\|^2\right\} \|\xi - \eta\|^2, \tag{2.4.4}$$

and

$$\|C(\xi) - C(\eta)\| \leq 3\|B\|^2 \max\left\{\|\xi\|^2, \|\eta\|^2\right\} \|\xi - \eta\|. \tag{2.4.5}$$

Proof. [29]
a) from the Lemma (2.2.2) we get:

$$\begin{aligned}(C\eta, \eta)_\Delta &= (B(B(\eta, \eta), \eta), \eta)_\Delta \\ &= (B(\eta, B(\eta, \eta)), \eta)_\Delta \\ &= (B(\eta, \eta), B(\eta, \eta))_\Delta = \|B(\eta, \eta)\|_\Delta^2 \geq 0\end{aligned}$$

the bilinearity and the continuity of B show that

$$(C\eta, \eta)_\Delta = 0 \Leftrightarrow B(\eta, \eta) = 0 \Leftrightarrow \eta = 0.$$

$$\|C(\xi)\| = \|B(B(\xi, \xi), \eta)\| \leq \|B\|\|B(\xi, \xi)\|\|B\| \leq \|B\|^2 \|\xi\|^3$$

b) for any $h \in H_0^2(\omega)$ we have

$$(C(\xi) - C(\eta), \xi - \eta)_\Delta = \int_0^1 (DC(\xi + t(\xi - \eta)).(\xi - \eta), h)_\Delta \, dt \tag{2.4.6}$$

for any $\xi_0 \in H_0^2(\omega)$ and for any $h \in H_0^2(\omega)$ we have

$$\begin{aligned} C(\xi_0+h) =& B(B(\xi_0+h,\xi_0+h),\xi_0+h) = B(B(\xi_0,\xi_0)+B(h,h)+2B(\xi_0,h),\xi_0+h) \\ =& B(B(\xi_0,\xi_0),\xi_0)+B(B(\xi_0,\xi_0),h)+B(B(h,h),\xi_0)+B(B(h,h),h) \\ &+2B(B(\xi_0,h),\xi_0)+2B(B(\xi_0,h),h) \end{aligned} \tag{2.4.7}$$

then

$$\begin{aligned} C(\xi_0+h)-C(\xi_0) =& B(B(\xi_0,\xi_0),h)+B(B(h,h),\xi_0)+B(B(h,h),h) \\ &+2B(B(\xi_0,h),\xi_0)+2B(B(\xi_0,h),h) \end{aligned} \tag{2.4.8}$$

therefore

$$DC(\xi_0).h = B(B(\xi_0,\xi_0),h)+2B(B(\xi_0,h),\xi_0)$$

on the other hand and from the Lemma (2.2.2) we have:

$$\begin{aligned} (DC(\xi_0).h,-h)_\Delta &= -(B(B(\xi_0,\xi_0),h),h)_\Delta - 2(B(B(\xi_0,h),\xi_0),h)_\Delta \\ &= -\left(B(\xi_0,\xi_0),B(h,h)\right)_\Delta - 2\left\|B(\xi_0,h)\right\|^2 \\ &\leq -\left(B(\xi_0,\xi_0),B(h,h)\right)_\Delta \end{aligned} \tag{2.4.9}$$

Substitution h by $\xi-\eta$ and ξ_0 by $\xi+th$ in the relation (2.4.9) we get

$$(DC(\xi+t(\xi-\eta)).(\xi-\eta),\eta-\xi)_\Delta \leq -\left(B(\xi+t(\xi-\eta),\xi+t(\xi-\eta)),B(\xi-\eta,\xi-\eta)\right)_\Delta \tag{2.4.10}$$

and from (2.4.6) we get

$$\begin{aligned} (C(\xi)-C(\eta),\eta-\xi)_\Delta &= -\int_0^1 \left(B(\xi+t(\xi-\eta),\xi+t(\xi-\eta)),B(\xi-\eta,\xi-\eta)\right)_\Delta \, dt \\ &\leq \int_0^1 \left\|B(\xi+t(\xi-\eta),\xi+t(\xi-\eta))\right\| \, dt \left\|B(\xi-\eta,\xi-\eta)\right\| \\ &\leq \left\|B\right\|^2 \left\|\xi-\eta\right\|^2 \int_0^1 \left\|\xi+t(\xi-\eta)\right\|^2 dt \end{aligned}$$

as a result of convexity of the square of the norm in the last inequality we conclude the relation (2.4.4)

For any $(\xi,\eta,\phi) \in (H_0^2)^3$ we have

$$\|C(\xi)-C(\eta)\| = \sup_{\|\phi\|\leq 1} (C(\xi)-C(\eta),\phi)_\Delta$$

and from bilinearity of the operator B and the Lemma (2.2.2) we get

$$\begin{aligned} (C(\xi)-C(\eta),\phi)_\Delta &= \left(B(\xi-\eta,B(\xi,\xi)+B(\xi,\eta)+B(\eta,\eta)),\phi\right)_\Delta \\ &\leq \left\|B\right\|^2 \left\|\xi-\eta\right\| \left[\left\|\xi\right\|^2+\left\|\xi\right\|\left\|\eta\right\|+\left\|\eta\right\|^2\right] \left\|\phi\right\| \\ &\leq \left\|B\right\|^2 \left\|\xi-\eta\right\| \left[3\max\left\{\left\|\xi\right\|^2,\left\|\eta\right\|^2\right\}\right] \left\|\phi\right\| \end{aligned}$$

hence

$$\|C(\xi)-C(\eta)\| = \sup_{\|\phi\|\leq 1} (C(\xi)-C(\eta),\phi)_\Delta \leq 3\left\|B\right\|^2 \left\|\xi-\eta\right\| \left[\max\left\{\left\|\xi\right\|^2,\left\|\eta\right\|^2\right\}\right]$$

□

2.5 Classical Marguerre-von Kármán equations

Let ω be a bounded and simply-connected open subset of $\mathbb{R}^2$ with a Lipschitz-continuous boundary γ, ω being locally on a single side of γ, we assume $0 \in \gamma$ and we denote by $\gamma(y)$ the arc joining 0 to the point $y \in \gamma$. The unit outer normal vector and the outer normal operator along the boundary γ are denoted respectively by (ν_α) and ∂_ν. As shown in [28], the classical Marguerre-von Kármán equations are written as

$$\begin{cases} \frac{2E}{3(1-\upsilon^2)}\Delta^2\zeta_3 = 2[\varphi, \zeta_3+\theta] + p_3 & \text{in } \omega, \\ \Delta^2\varphi = -\frac{E}{2}[\zeta_3, \zeta_3 + 2\theta] & \text{in } \omega, \\ \zeta_3 = \partial_\nu\zeta_3 = 0 & \text{on } \gamma, \\ \varphi = \varphi_0 \text{ and } \partial_\nu\varphi = \varphi_1 & \text{on } \gamma, \end{cases}$$

where

$$\varphi_0(y) = -y_1\int_{\gamma(y)} h_2 d\gamma + y_2\int_{\gamma(y)} h_1 d\gamma + \int_{\gamma(y)}(x_1h_2 - x_2h_1)d\gamma,\ y \in \gamma,$$
$$\varphi_1(y) = -\nu_1\int_{\gamma(y)} h_2 d\gamma + \nu_2\int_{\gamma(y)} h_1 d\gamma,\ y \in \gamma,$$
$$[\eta, \xi] = \partial_{11}\eta\partial_{22}\xi + \partial_{22}\eta\partial_{11}\xi - 2\partial_{12}\eta\partial_{12}\xi.$$

θ be a known function that defines the middle surface of the shell, p_3 is also a known function that defines resultant of the vertical forces acting on the shell. The functions φ_0 and φ_1 are known functions of the appropriately density $(h_\alpha) : \gamma \to \mathbb{R}^2$ of the resultant of the horizontal forces acting on the lateral face of the shell. The constant E represent the Young modulus, and the constant υ represent the Poisson coefficient of the elastic material constituting of the shell. The unknown $\zeta_3 : \bar{\omega} \to \mathbb{R}$ is the vertical component of the displacement field of the middle surface of the shell and the unknown $\varphi : \bar{\omega} \to \mathbb{R}$ is the Airy function.

Now, we will write the classical Marguerre-von Kármán equations in a simpler form, using the following relations:

$$\begin{cases} \zeta_3 = (\frac{2}{3(1-\upsilon^2)})^{\frac{1}{2}}\xi, & \varphi = \frac{E}{3(1-\upsilon^2)}\tilde{\varphi}, \\ \\ \theta = (\frac{2}{3(1-\upsilon^2)})^{\frac{1}{2}}\tilde{\theta}, & p_3 = (\frac{2}{3(1-\upsilon^2)})^{\frac{3}{2}}Ef. \end{cases}$$

So, the classical Marguerre-von Kármán equations become

$$\begin{cases} \mu_1\Delta^2\xi = \mu_1[\tilde{\varphi}, \xi + \tilde{\theta}] + \mu_1 f \text{ in } \omega, \\ \mu_2\Delta^2\tilde{\varphi} = -\mu_2[\xi, \xi + 2\tilde{\theta}] \text{ in } \omega, \\ \xi = \partial_\nu\xi = 0 \text{ on } \gamma, \\ \tilde{\varphi} = \tilde{\varphi}_0 \text{ and } \partial_\nu\tilde{\varphi} = \tilde{\varphi}_1 \text{ on } \gamma, \end{cases} \tag{2.5.1}$$

Where

$$\mu_1 = \Big(\frac{2}{3(1-\upsilon^2)}\Big)^{\frac{3}{2}}E, \quad \text{and} \quad \mu_2 = \frac{E}{3(1-\upsilon^2)} \tag{2.5.2}$$

and

$$\tilde{\varphi}_0 = \frac{1}{\mu_2}\varphi_0 \quad \text{and} \quad \tilde{\varphi}_1 = \frac{1}{\mu_2}\varphi_1 \tag{2.5.3}$$

Since μ_1 and μ_2 different from zero, we infer that the unknowns $(\xi, \tilde{\varphi})$ satisfy the Marguerre-von Kármán equations

$$\begin{cases} \Delta^2\xi = [\tilde{\varphi}, \xi + \tilde{\theta}] + f & \text{in } \omega, \\ \Delta^2\tilde{\varphi} = -[\xi, \xi + 2\tilde{\theta}] & \text{in } \omega, \\ \xi = \partial_\nu \xi = 0 & \text{on } \gamma, \\ \tilde{\varphi} = \tilde{\varphi}_0 \text{ and } \partial_\nu \tilde{\varphi} = \tilde{\varphi}_1 & \text{on } \gamma \end{cases} \tag{2.5.4}$$

with

$$(\tilde{\varphi}_0, \tilde{\varphi}_1) \in H^{3/2}(\gamma) \times H^{1/2}(\gamma) \text{ and } f \in H^{-2}(\omega). \tag{2.5.5}$$

Then by the classical elliptic theory, there exists a unique function θ_0 be a solution of the boundary value problem

$$\begin{cases} \Delta^2\eta = 0 \text{ in } \omega, \\ \eta = \tilde{\varphi}_0 \text{ and } \partial_\nu \eta = \tilde{\varphi}_1 \text{ on } \gamma. \end{cases} \tag{2.5.6}$$

and there exists a unique function Fsuch that

$$\begin{cases} \Delta^2 F = f & in\ \omega \\ F \in H_0^2(\omega) \end{cases} \tag{2.5.7}$$

We set

$$\tilde{\varphi} = \Phi + \theta_0.$$

Therefore

$$\Phi = \partial_\nu \Phi = 0 \text{ on } \gamma.$$

So, from the definition of the function θ_0 we find:

$$\Delta^2\tilde{\varphi} = \Delta^2(\Phi + \theta_0) = \Delta^2\Phi + \Delta^2\theta_0 = \Delta^2\Phi$$

Thus, the Marguerre-von Kármán equations becomes:

$$\begin{cases} \Delta^2\xi = [\Phi + \tilde{\theta}_0, \xi + \tilde{\theta}] + \Delta^2 F & \text{in } \omega, \\ \Delta^2\Phi = -[\xi, \xi + 2\tilde{\theta}] & \text{in } \omega, \\ \xi = \partial_\nu \xi = 0 & \text{on } \gamma, \\ \Phi = \partial_\nu \Phi = 0 & \text{on } \gamma. \end{cases} \tag{2.5.8}$$

2.6 Cubic operator equation for Marguerre-von Kármán equations

We transform the Marguerre-von Kármán equations which represent in the system (2.5.8) onto single cubic operator equation

Theorem 2.6.1. *[66] Assume that $\tilde{\theta} \in H_0^2(\omega)$. Then the unknowns (ξ, Φ) satisfies the Marguerre-von Kármán equations (2.5.8) if and only if the function $\tilde{\xi}$ satisfies the cubic operator equation*

$$\tilde{\xi} - L_2(\tilde{\xi}) - L_1(\tilde{\xi}) + C(\tilde{\xi}) - \tilde{\mathfrak{F}} = 0 \tag{2.6.1}$$

where $\tilde{\xi} = \xi + \tilde{\theta} \in H_0^2(\omega)$. And the airy function Φ is given by:

$$\Phi = B(\tilde{\theta}, \tilde{\theta}) - B(\tilde{\xi}, \tilde{\xi})$$

Proof. [66] With a new unknown $\tilde{\xi}$, the system (2.5.8) becomes

$$\begin{cases} \Delta^2(\tilde{\xi} - \tilde{\theta}) = [\Phi + \tilde{\theta}_0, \tilde{\xi}] + \Delta^2 F \text{ in } \omega, \\ \Delta^2\Phi = -[\tilde{\xi} - \tilde{\theta}, \tilde{\xi} + \tilde{\theta}] \text{ in } \omega, \\ \tilde{\xi} = \partial_\nu \tilde{\xi} = 0 \text{ on } \gamma, \\ \Phi = \partial_\nu \Phi = 0 \text{ on } \gamma. \end{cases} \tag{2.6.2}$$

Using the bilinearity of Δ^2 and the bracket $[.,.]$, the first and second equation of (2.6.2) become

$$\Delta^2(\tilde{\xi}) = [\Phi, \tilde{\xi}] + [\tilde{\theta}_0, \tilde{\xi}] + \Delta^2(F + \tilde{\theta}) \text{ in } \omega, \tag{2.6.3}$$

$$\Delta^2\Phi = [\tilde{\theta}, \tilde{\theta}] - [\tilde{\xi}, \tilde{\xi}] \text{ in } \omega. \tag{2.6.4}$$

the definition of the bracket $[.,.]$ and the definition of the operator B gives

$$\Delta^2(\tilde{\xi}) = \Delta^2 B(\Phi, \tilde{\xi}) + \Delta^2 B(\tilde{\theta}_0, \tilde{\xi}) + \Delta^2(F + \tilde{\theta}) \tag{2.6.5}$$

$$\Delta^2\Phi = \Delta^2 B(\tilde{\theta}, \tilde{\theta}) - \Delta^2 B(\tilde{\xi}, \tilde{\xi}) \tag{2.6.6}$$

which implies that:

$$\Delta^2(\tilde{\xi}) = \Delta^2 \left(B(\Phi, \tilde{\xi}) + B(\tilde{\theta}_0, \tilde{\xi}) + (F + \tilde{\theta}) \right) \tag{2.6.7}$$

$$\Delta^2\Phi = \Delta^2 \left(B(\tilde{\theta}, \tilde{\theta}) - B(\tilde{\xi}, \tilde{\xi}) \right) \tag{2.6.8}$$

Since $\Phi, \tilde{\xi}, B(\eta, \eta)$ and $F + \tilde{\theta}$ are in $H_0^2(\omega)$ and since the bi-Laplacian Δ^2 operator is an isomorphism between the spaces $H_0^2(\omega)$ and $H^{-2}(\omega)$, we deduce that

$$\tilde{\xi} = B(\Phi, \tilde{\xi}) + B(\tilde{\theta}_0, \tilde{\xi}) + (F + \tilde{\theta}) \tag{2.6.9}$$

$$\Phi = B(\tilde{\theta}, \tilde{\theta}) - B(\tilde{\xi}, \tilde{\xi}) \tag{2.6.10}$$

Substituting equation (2.6.10) into equation (2.6.9)

$$\begin{aligned} \tilde{\xi} =& B\left(B(\tilde{\theta}, \tilde{\theta}) - B(\tilde{\xi}, \tilde{\xi}), \tilde{\xi} \right) + B(\tilde{\theta}_0, \tilde{\xi}) + (F + \tilde{\theta}) \\ =& B\left(B(\tilde{\theta}, \tilde{\theta}), \tilde{\xi} \right) - B\left(B(\tilde{\xi}, \tilde{\xi}), \tilde{\xi} \right) + B(\tilde{\theta}_0, \tilde{\xi}) + (F + \tilde{\theta}) \end{aligned} \tag{2.6.11}$$

Then

$$\tilde{\xi} - L_2(\tilde{\xi}) - L_1(\tilde{\xi}) + C(\tilde{\xi}) - \tilde{\mathfrak{F}} = 0 \tag{2.6.12}$$

Where

$$\begin{aligned} L_1(\tilde{\xi}) =& B\left(B(\tilde{\theta}, \tilde{\theta}), \tilde{\xi} \right) \\ L_2(\tilde{\xi}) =& B(\tilde{\theta}_0, \tilde{\xi}) \\ C(\tilde{\xi}) =& B\left(B(\tilde{\xi}, \tilde{\xi}), \tilde{\xi} \right) \\ \tilde{\mathfrak{F}} =& F + \tilde{\theta} \end{aligned} \tag{2.6.13}$$

□

2.7 Existence and regularity of solution

2.7.1 Existence of solution

Let we consider the following functional J defined on $H_0^2(\omega)$ by

$$J(\eta) = \frac{1}{2}\left(\|\eta\|_\Delta^2 - (L_1(\eta),\eta)_\Delta - (L_2(\eta),\eta)_\Delta\right) + \frac{1}{4}\left(C(\eta),\eta\right)_\Delta - (\tilde{\mathfrak{F}},\eta)_\Delta \tag{2.7.1}$$

Theorem 2.7.1. *[66] Assume that $\tilde{\theta} \in H_0^2(\omega)$. Then any solution of equation (2.6.1) is a critical point of the functional J which defined in* (2.7.1)

Proof. [66] Since the bilinear operator B is continuous from the space $(H^2(\omega))^2$ into the space $H_0^2(\omega)$. We get the operators L_1, L_2 and C are continuous, furthermore the functional J is of class $\mathscr{C}^\infty(\omega)$, and since $\tilde{\mathfrak{F}} = F + \tilde{\theta} \in H_0^2(\omega)$, then for all $h \in H_0^2(\omega)$ we have

$$\begin{aligned} J(\eta+h) =& \frac{1}{2}\left\{\|\eta+h\|_\Delta^2 - (L_1(\eta+h),\eta+h)_\Delta - (L_2(\eta+h),\eta+h)_\Delta\right\} \\ &+ \frac{1}{4}\left(C(\eta+h),\eta+h\right)_\Delta - (\tilde{\mathfrak{F}},\eta+h)_\Delta \end{aligned} \tag{2.7.2}$$

Using the bilinearity of scalar product and the linearity of operators L_1 and L_2 we get:

$$\begin{aligned} \|\eta+h\|_\Delta^2 =& \|\eta\|_\Delta^2 + \|h\|_\Delta^2 + 2\left(\eta,,h\right)_\Delta \\ (L_1(\eta+h),\eta+h)_\Delta =& (L_1(\eta),\eta)_\Delta + (L_1(h),h)_\Delta + 2\left(L_1(\eta),h\right)_\Delta \\ (L_2(\eta+h),\eta+h)_\Delta =& (L_2(\eta),\eta)_\Delta + (L_2(h),h)_\Delta + 2\left(L_2(\eta),h\right)_\Delta \\ (\tilde{\mathfrak{F}},\eta+h)_\Delta =& (\tilde{\mathfrak{F}},\eta)_\Delta + (\tilde{\mathfrak{F}},h)_\Delta \end{aligned} \tag{2.7.3}$$

And

$$\begin{aligned} (C(\eta+h),\eta+h)_\Delta &= \|B(\eta+h,\eta+h)\|_\Delta^2 = \left(B(\eta+h,\eta+h),B(\eta+h,\eta+h)\right)_\Delta \\ &= \left(B(\eta,\eta)+B(h,h)+2B(\eta,h),B(\eta,\eta)+B(h,h)+2B(\eta,h)\right)_\Delta \\ &= \left(B(\eta,\eta),B(\eta,\eta)\right)_\Delta + \left(B(h,h),B(h,h)\right)_\Delta + 4\left(B(\eta,h),B(\eta,h)\right)_\Delta \\ &\quad + 2\left(B(\eta,\eta),B(h,h)\right)_\Delta + 4\left(B(h,h),B(\eta,h)\right)_\Delta + 4\left(B(\eta,\eta),B(\eta,h)\right)_\Delta \\ &= \|B(\eta,\eta)\|_\Delta^2 + \|B(h,h)\|_\Delta^2 + 4\|B(\eta,h)\|_\Delta^2 + 2\left(B(\eta,\eta),B(h,h)\right)_\Delta \\ &\quad + 4\left(B(h,h),B(\eta,h)\right)_\Delta + 4\left(B(\eta,\eta),B(\eta,h)\right)_\Delta \end{aligned} \tag{2.7.4}$$

We have η, h and $B(.,.)$ are in $H_0^2(\omega)$, then from the Lemma (2.2.2) we conclude:

$$\begin{aligned} \left(B(h,h),B(\eta,h)\right)_\Delta &= \left(\eta,B(B(h,h),h)\right)_\Delta = \left(C(h),\eta\right)_\Delta \\ \left(B(\eta,\eta),B(\eta,h)\right)_\Delta &= \left(h,B(B(\eta,\eta),\eta)\right)_\Delta = \left(C(\eta),h\right)_\Delta \end{aligned} \tag{2.7.5}$$

So, the equation (2.7.4) becomes

$$\begin{aligned} (C(\eta+h),\eta+h)_\Delta =& \|B(\eta,\eta)\|_\Delta^2 + \|B(h,h)\|_\Delta^2 + 4\|B(\eta,h)\|_\Delta^2 \\ &+ 2\left(B(\eta,\eta),B(h,h)\right)_\Delta + 4\left(C(h),\eta\right)_\Delta + 4\left(C(\eta),h\right)_\Delta \end{aligned} \tag{2.7.6}$$

Substitution of (2.7.3) and (2.7.6) into (2.7.2) gives

$$\begin{aligned} J(\eta+h) =& \frac{1}{2}\left\|\eta\right\|_{\Delta}^{2}+\frac{1}{2}\left\|h\right\|_{\Delta}^{2}+(\eta,,h)_{\Delta} \\ &-\frac{1}{2}\left(L_1(\eta),\eta\right)_{\Delta}-\frac{1}{2}\left(L_1(h),h\right)_{\Delta}-\left(L_1(\eta),h\right)_{\Delta} \\ &-\frac{1}{2}\left(L_2(\eta),\eta\right)_{\Delta}-\frac{1}{2}\left(L_2(h),h\right)_{\Delta}-\left(L_2(\eta),h\right)_{\Delta} \\ &+\frac{1}{4}\left\|B(\eta,\eta)\right\|_{\Delta}^{2}+\frac{1}{4}\left\|B(h,h)\right\|_{\Delta}^{2}+\left\|B(\eta,h)\right\|_{\Delta}^{2} \\ &+\frac{1}{2}\left(B(\eta,\eta),B(h,h)\right)_{\Delta}+\left(C(h),\eta\right)_{\Delta}+\left(C(\eta),h\right)_{\Delta} \\ &-(\tilde{\mathfrak{F}},\eta)_{\Delta}-(\tilde{\mathfrak{F}},h)_{\Delta} \end{aligned} \tag{2.7.7}$$

So that

$$\begin{aligned} J(\eta+h)-J(\eta) =& \frac{1}{2}\left\|h\right\|_{\Delta}^{2}+(\eta,,h)_{\Delta}-\frac{1}{2}\left(L_1(h),h\right)_{\Delta}-\left(L_1(\eta),h\right)_{\Delta} \\ &-\frac{1}{2}\left(L_2(h),h\right)_{\Delta}-\left(L_2(\eta),h\right)_{\Delta}+\frac{1}{4}\left\|B(h,h)\right\|_{\Delta}^{2} \\ &+\frac{1}{2}\left(B(\eta,\eta),B(h,h)\right)_{\Delta}+\left(C(h),\eta\right)_{\Delta} \\ &+\left(C(\eta),h\right)_{\Delta}+\left\|B(\eta,h)\right\|_{\Delta}^{2}-(\tilde{\mathfrak{F}},h)_{\Delta}. \end{aligned} \tag{2.7.8}$$

Hence

$$\begin{aligned} J(\eta+h)-J(\eta) &= (\eta,h)_{\Delta}-\left(L_1(\eta),h\right)_{\Delta}-\left(L_2(\eta),h\right)_{\Delta}+\left(C(\eta),h\right)_{\Delta}-(\tilde{\mathfrak{F}},h)_{\Delta}+o(h^2) \\ &= \Big(\eta-L_1(\eta)-L_2(\eta)+C(\eta)-\tilde{\mathfrak{F}},h\Big)_{\Delta}+o(h^2) \end{aligned} \tag{2.7.9}$$

then yields

$$J'(\eta)=\eta-L_1(\eta)-L_2(\eta)+C(\eta)-\tilde{\mathfrak{F}} \tag{2.7.10}$$

□

Lemma 2.7.1. *[66] The functional J coercive on the space $H_0^2(\omega)$.*

In order to prove the coerciveness of functional J, we need the following Lemma.

Lemma 2.7.2. *[3] If one of the arguments ζ,ξ,η belongs to $H_0^2(\omega)$, then*

$$\left(B(\zeta,\xi),\eta\right)_{\Delta}=\left(B(\eta,\xi),\zeta\right)_{\Delta}. \tag{2.7.11}$$

This Lemma is found in the Lemma (2.2.2) in the case when $(\zeta,\xi,\eta)\in H^2(\omega)\times H_0^2(\omega)\times H_0^2(\omega)$, by the same argument and since the mapping T defined in (2.1.2) is symmetric if one of three arguments ζ,ξ,η belongs to $H_0^2(\omega)$, then we conclude that the Lemma (2.7.2) is verified.

Proof the Lemma 2.7.1. Let η_n be a sequence in $H_0^2(\omega)$, such that $\|\eta_n\|\to\infty$.
We set $\tilde{J}(\eta)=2J(\eta)$, then we get:

$$\begin{aligned} \tilde{J}(\eta_n) &= \left\|\eta_n\right\|_{\Delta}^{2}-\left(L_1(\eta_n),\eta_n\right)_{\Delta}-\left(L_2(\eta_n),\eta_n\right)_{\Delta}+\frac{1}{2}\left(C(\eta_n),\eta_n\right)_{\Delta}-2(\tilde{\mathfrak{F}},\eta_n)_{\Delta} \\ &= \left\|\eta_n\right\|_{\Delta}^{2}-(B(B(\tilde{\theta},\tilde{\theta}),\eta_n),\eta_n)_{\Delta}-(B(\tilde{\theta}_0,\eta_n),\eta_n)_{\Delta}+\frac{1}{2}(B(B(\eta_n,\eta_n),\eta_n),\eta_n)_{\Delta}-2(\tilde{\mathfrak{F}},\eta_n)_{\Delta} \end{aligned} \tag{2.7.12}$$

From the Lemma (2.7.2), we show that

$$\tilde{J}(\eta_n) = \|\eta_n\|_\Delta^2 - (B(\eta_n,\eta_n), B(\tilde{\theta},\tilde{\theta}))_\Delta - (B(\eta_n,\eta_n),\tilde{\theta}_0)_\Delta + \frac{1}{2}\|B(\eta_n,\eta_n)\|_\Delta^2 - 2(\tilde{\mathfrak{F}},\eta_n)_\Delta \tag{2.7.13}$$

Consequently for any $\epsilon > 0$, by the Cauchy-Schwarz and Young inequality

$$\begin{aligned} \tilde{J}(\eta_n) \geq \; & \|\eta_n\|_\Delta^2 - \frac{\epsilon}{2}\|B(\eta_n,\eta_n)\|_\Delta^2 - \frac{1}{2\epsilon}\|B(\tilde{\theta},\tilde{\theta})\|_\Delta^2 - \frac{\epsilon'}{2}\|B(\eta_n,\eta_n)\|_\Delta^2 - \frac{1}{2\epsilon'}\|\tilde{\theta}_0\|_\Delta^2 \\ & + \frac{1}{2}\|B(\eta_n,\eta_n)\|_\Delta^2 - 2\|\tilde{\mathfrak{F}}\|_\Delta\|\eta_n\|_\Delta \\ \geq \; & \|\eta_n\|_\Delta\left(\|\eta_n\|_\Delta - 2\|\tilde{\mathfrak{F}}\|_\Delta\right) + \left(\frac{1}{2} - \frac{\epsilon'+\epsilon}{2}\right)\|B(\eta_n,\eta_n)\|_\Delta^2 - \frac{1}{2\epsilon}\|B(\tilde{\theta},\tilde{\theta})\|_\Delta^2 - \frac{1}{2\epsilon'}\|\tilde{\theta}_0\|_\Delta^2, \end{aligned}$$

choosing $\epsilon + \epsilon' = 1$, we find that

$$\tilde{J}(\eta_n) \geq \|\eta_n\|_\Delta\left(\|\eta_n\|_\Delta - 2\|\tilde{\mathfrak{F}}\|_\Delta\right) - \frac{1}{2\epsilon}\|B(\tilde{\theta},\tilde{\theta})\|_\Delta^2 - \frac{1}{2(1-\epsilon)}\|\tilde{\theta}_0\|_\Delta^2,$$

then

$$\tilde{J}(\eta_n) \geq \|\eta_n\|_\Delta\left(\|\eta_n\|_\Delta - 2\|\tilde{\mathfrak{F}}\|_\Delta - \frac{1}{2\epsilon}\|B(\tilde{\theta},\tilde{\theta})\|_\Delta^2\|\eta_n\|_\Delta^{-1} - \frac{1}{2(1-\epsilon)}\|\tilde{\theta}_0\|_\Delta^2\|\eta_n\|_\Delta^{-1}\right),$$

thus

$$\lim_{\|\eta_n\|\to\infty} \tilde{J}(\eta_n) = \infty. \tag{2.7.14}$$

□

Remark 2.7.1. *B.Rao in [66], proved the coerciveness of the functional J by contradiction.*

Theorem 2.7.2. *[66] Assume that $\tilde{\theta} \in H_0^2(\omega)$. Then there exists at least one $\tilde{\xi} \in H_0^2(\omega)$ such that*

$$J(\tilde{\xi}) = \inf_{\eta\in H_0^2(\omega)} J(\eta) \tag{2.7.15}$$

Proof. [66] we see that the terms appearing in (2.7.1) such as $(L_1(\eta),\eta)_\Delta = \left(B(B(\tilde{\theta},\tilde{\theta}),\eta\right)_\Delta$, $(L_2(\eta),\eta)_\Delta = \left(B(\tilde{\theta}_0,\eta),\eta\right)_\Delta$ and $(C(\eta),\eta)_\Delta = \|B(\eta,\eta)\|_{2,2}^2$ are weakly continuous in $H_0^2(\omega)$, while the term $\|\eta\|_\Delta^2$ being strongly continuous and convex in $H_0^2(\omega)$, is weakly lower semi-continuous. We have thus proved the weak sequential lower semi-continuity of the functional J, and since the functional J is coercive, the existence of a function $\tilde{\xi} \in H_0^2(\omega)$ satisfying (2.7.15) is a consequence of the Theorem (1.7.1).

□

2.7.2 Regularity of solution

Proposition 2.7.1. *Assume that*

$$\tilde{\theta}_0 \in H^4(\omega) \text{ and } \tilde{\mathfrak{F}} = (\Delta^2)^{-1}f + \tilde{\theta} \in \left(H_0^2(\omega) \cap H^4(\omega)\right). \tag{2.7.16}$$

Then all solutions of Marguerre-von Kármán equations (2.5.8) *have the regularity*

$$(\tilde{\xi},\Phi) \in \left(H_0^2(\omega) \times H^4(\omega)\right)^2 \tag{2.7.17}$$

Proof. We use the same argument as in Ciarlet [29] and Rao [66] we have

$$\begin{cases} \Delta^2(\tilde{\xi}) = [\Phi, \tilde{\xi}] + [\tilde{\theta}_0, \tilde{\xi}] + \Delta^2(F + \tilde{\theta}) \text{ in } \omega, \\ \tilde{\xi} \in H_0^2(\omega) \end{cases} \tag{2.7.18}$$

From the condition (2.7.16) we get:

$$\begin{cases} \Delta^2(\tilde{\xi}) = [\Phi, \tilde{\xi}] + [\tilde{\theta}_0, \tilde{\xi}] + \Delta^2(F + \tilde{\theta}) \text{ in } L^1(\omega), \\ \tilde{\xi} \in H_0^2(\omega) \end{cases} \tag{2.7.19}$$

From a classical result of Peetre, we have:

$$L^1(\omega) \subset H^{-1-\epsilon}(\omega) \quad \forall \epsilon > 0$$

that is

$$\begin{cases} \Delta^2(\tilde{\xi}) \text{ in } H^{-1-\epsilon}(\omega), \\ \tilde{\xi} \in H_0^2(\omega) \end{cases} \tag{2.7.20}$$

$\Delta^2\tilde{\xi}$ in $H^{-1-\epsilon}(\omega)$ implies that $\tilde{\xi}$ in $H^{3-\epsilon}(\omega)$. So

$$\tilde{\xi} \in H_0^2(\omega) \cap H^{3-\epsilon}(\omega). \tag{2.7.21}$$

Clearly that from (2.7.21), the second derivatives of $\tilde{\xi}$ belong to $H^{1-\epsilon}(\omega)$ which is contained in $L^{\frac{2}{\epsilon}}(\omega)$ then

$$[\tilde{\xi}, \tilde{\xi}] \in L^q(\omega)$$

where

$$\frac{1}{q} = \frac{1}{\frac{2}{\epsilon}} + \frac{1}{\frac{2}{\epsilon}} = \epsilon$$

that is

$$[\tilde{\xi}, \tilde{\xi}] \in L^{\frac{1}{\epsilon}}(\omega) \tag{2.7.22}$$

So, a suitable choice of ϵ $(0 < \epsilon \leq 1)$ we get

$$[\tilde{\xi}, \tilde{\xi}] \in L^q(\omega) \quad \forall q \geq 1 \tag{2.7.23}$$

On the other hand, since $\tilde{\theta} \in (H_0^2(\omega) \cap H^4(\omega)) \subset (H_0^2(\omega) \cap H^3(\omega))$, then by the same arguments we get

$$[\tilde{\theta}, \tilde{\theta}] \in L^q(\omega) \quad \forall q \geq 1 \tag{2.7.24}$$

It follows from (2.6.4),(2.7.23) and (2.7.24) that

$$\begin{cases} \Delta^2\Phi = -[\tilde{\xi}, \tilde{\xi}] + [\tilde{\theta}, \tilde{\theta}] \text{ in } L^q(\omega), \; q \geq 1 \\ \Phi \in H_0^2(\omega) \end{cases} \tag{2.7.25}$$

wich implies that

$$\Phi \in H_0^2(\omega) \cap W^{4,q}(\omega) \quad \forall q \geq 1 \tag{2.7.26}$$

In particular for $q = 2$ we conclude that

$$\Phi \in H_0^2(\omega) \cap H^4(\omega) \tag{2.7.27}$$

So,

$$[\tilde{\xi}, \Phi] \in L^2(\omega) \tag{2.7.28}$$

then

$$\begin{cases} \Delta^2(\tilde{\xi}) = [\Phi, \tilde{\xi}] + [\tilde{\theta}_0, \tilde{\xi}] + \Delta^2(F + \tilde{\theta}) \text{ in } L^2(\omega), \\ \tilde{\xi} \in H_0^2(\omega) \end{cases} \tag{2.7.29}$$

which imply that

$$\tilde{\xi} \in H_0^2(\omega) \cap H^4(\omega) \tag{2.7.30}$$

□

Remark 2.7.2. *Rao [66] proved this regularity of solutions to these equations in the case when* $\tilde{\theta} \in H^2(\omega)$

Chapter 3

Bifurcation Problem

3.1 Introduction to the Bifurcation theory

Let X, Y and Z be three Banach spaces. We consider the following continuous mapping

$$\begin{aligned} \mathscr{F} : X \times Y &\to Z \\ (\lambda, x) &\to \mathscr{F}(\lambda, x) \end{aligned} \tag{3.1.1}$$

Assume that

$$\mathscr{F}(\lambda, 0) = 0 \text{ for all } \lambda \in X \tag{3.1.2}$$

3.1.1 Bifurcation point

Definition 3.1.1. *[49] A point $(\lambda_0, 0) \in X \times Y$ is said to be a bifurcation point if every neighborhood of this point in $X \times Y$ contains a solution $(\lambda, x), x \neq 0$ of the equation*

$$\mathscr{F}(\mu, x) = 0 \tag{3.1.3}$$

Remark 3.1.1. *[49] (3.1.3) Note that, essentially, only small neighborhoods count. The definition ensures the existence of a sequence $\{(\lambda_n, x_n)\}$ of nontrivial solutions such that $\lambda_n \to \lambda$ and $x_n \to x$ as $n \to \infty$.*
It does not guarantee the existence of a continuous branch of solutions $(\lambda, x(\lambda))$ with $x(\lambda) \to 0$ as $\lambda \to \lambda_0$.

The following theorems, give some necessary conditions for the existence of bifurcation point.

Theorem 3.1.1 (Implicit Function Theorem [7])**.** *Let X, Y, and Z be Banach spaces. Suppose $\mathscr{F}(x, y)$ is a continuous mapping of a neighborhood U of $(x_0, y_0) \in X \times Y$ into Z, $\mathscr{F}(x_0, y_0) = 0$, and $\partial_y \mathscr{F}(x_0, y_0)$ exists is continuous in X and is a linear homeomorphism of Y onto Z. Then there is a unique continuous mapping g defined in a neighborhood U, of x_0, $g : U \to Y$, such that*

$$g(x_0) = y_0 \textit{ and } \mathscr{F}(x, g(x)) = 0 \textit{ for } x \in U.$$

Corollary 3.1.1. *[7] If, in addition to the hypothesis of the implicit function theorem (3.1.1) $\partial_x \mathscr{F}(x, y)$ exists and is continuous for (x, y) near (x_0, y_0), then the function $g(x)$ is continuously differentiable for $x \in U$, and*

$$g'(x) = -\left[\partial_y \mathscr{F}(x, g(x))\right]^{-1} \partial_x \mathscr{F}(x, g(x)).$$

The following Proposition is a consequence of the implicit function Theorem

Proposition 3.1.1. *[49] Let $\mathscr{F} : X \times Y \to Z$ be differentiable. If $(\lambda_0, x_0) \in X \times Y$ is a bifurcation point, then $\partial_x \mathscr{F}(\lambda_0, x_0) : X \to Z$ is not an isomorphism.*

Remark 3.1.2. *The condition given in the precedent Proposition is only necessary not sufficient. For that we have the following counter-example.*

Example 1. *[49] Let $X = Z = \mathbb{R}^2$ and $Y = \mathbb{R}$. Let*

$$\mathscr{F}(\lambda, x) = (1-\lambda)\begin{pmatrix} z_1 \\ \\ z_2 \end{pmatrix} + \begin{pmatrix} z_2^3 \\ \\ -z_1^3 \end{pmatrix}.$$

With $x = (z_1, z_2)$
We have $\partial_x \mathscr{F}(\lambda, 0) = (1-\lambda)I$ which fails to be an isomorphism, only for $\lambda = 1$. However, $(1, 0)$ is not a bifurcation point. Indeed it is immediate to see that if (λ, x) is a solution to (3.1.3), then

$$\mathscr{F}(\lambda, x) = 0$$

equivalent to

$$\begin{cases} z_1 - \lambda z_1 + z_2^3 = 0 \\ z_2 - \lambda z_2 - z_1^3 = 0 \end{cases}$$

which equivalent to

$$\begin{cases} z_1 z_2 - \lambda z_1 z_2 + z_2^4 = 0 \\ -z_2 z_1 + \lambda z_2 z_1 + z_1^4 = 0 \end{cases}$$

Thus

$$z_2^4 + z_1^4 = 0$$

Hence

$$z_1 = z_2 = 0$$

Therefore, the trivial solutions are the only solutions to this equation and thus $(1, 0)$ is not a bifurcation point.

Theorem 3.1.2. *[49] Let $\mathscr{F} : \mathbb{R} \times X \to Y$ be a mapping of class $\mathscr{C}^m$ for some $m \geq 2$. Assume that $\mathscr{F}(\lambda_0, 0) = 0$. Assume further that*

a) $\partial_\lambda \mathscr{F}(\lambda_0, 0) = 0$

b) $\ker(\partial_\lambda \mathscr{F}(\lambda_0, 0))$ is 1-dimensional and spanned by $x_0 \in X$

c) Rang $(\partial_\lambda \mathscr{F}(\lambda_0, 0)) = Y_1$ which has one codimension.

d) With the obvious identifications,

$$\partial_{\lambda\lambda} \mathscr{F}(\lambda_0, 0) \in Y_1 \text{ and } \partial_{\lambda x} \mathscr{F}(\lambda_0, 0) x_0 \notin Y_1.$$

Then, $(\lambda_0, 0)$ is a bifurcation point and the set of solutions to $\mathscr{F}(\lambda, x) = 0$ near $(\lambda_0, 0)$ consists of two $\mathscr{C}^{m-2}$ curves Γ_1 and Γ_2 cutting only at $(\lambda_0, 0)$. Moreover, if $m > 2$, Γ_1 is tangent to the λ-axis at $(\lambda_0, 0)$ and can be parametrized by λ; i.e

$$\Gamma_1 = \{(\lambda, x(\lambda))\,;\, |\lambda| \leq \varepsilon\}$$

Γ_2 can be parametrized as

$$\Gamma_2 = \{(sx_0 + x_2(s), \lambda(s))\,;\, |s| \leq \varepsilon\}$$

with $x_2(0) = 0,\ x_2'(0) = 0$ and $\lambda(0) = \lambda_0$.

Remark 3.1.3. *[49] If $\mathscr{F}(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$, then Γ_1 is the λ- axis itself*

3.1.2 Krasnoselsk'ii theorem

The theorem of Krasnoselsk'ii is interested in determining the bifurcation points for the equations of the following form:

$$\mathscr{F}(\lambda, \eta) = \eta - \lambda L\eta + g(\lambda, \eta). \tag{3.1.4}$$

Where $L : X \to X$ be a compact bounded linear operator on the Banach space X.
We assume that there is a known branch of solutions to the equation:

$$\mathscr{F}(\lambda, \eta) = \eta - \lambda L\eta + g(\lambda, \eta) = 0. \tag{3.1.5}$$

Before the Krasnoselsk'ii theorem we have the following theorem:

Theorem 3.1.3. *[29] Assume that the mapping g satisfies the following assumptions:*

$$g(\lambda, 0) = 0. \tag{3.1.6}$$

and

$$\|g(\lambda, 0)\|_X = o(\|u\|_X) \tag{3.1.7}$$

uniformly with respect to in the bounded interval of $\mathbb{R}$.
Then, $(\lambda_0, 0)$ is a bifurcation point of the trivial solution, only if λ_0 is a characteristic value of L.

The reciprocal of the precedent theorem is false in general case. For that Krasnoselsk'ii gave a necessary and sufficient conditions in the following theorem

Theorem 3.1.4 (Krasnoselsk'ii[49]). *Assume that the mapping g is compact, furthermore satisfies the equations (3.1.6) and (3.1.7). Then $(\lambda_0, 0)$ is a bifurcation point if and only if λ_0 is a characteristic value of odd algebraic multiplicity of L.*

The proof It is based on topological degree methods.

3.1.3 Structure of solutions

After determining the bifurcation points, now in this paragraph we will present some important results that describe to us the structure of the set of solution of a nonlinear equation near a bifurcation point.

Theorem 3.1.5 (Morse Lemma [49]). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a mapping of class $\mathscr{C}^m$, for some $m \geq 2$. Assume that $f(0) = 0$, and $f'(0) = 0$ and that $f''(0)$ is a non-singular matrix. Then, in a neighborhood of the origin, there exists a change of coordinates $x \to y(x)$ which is a mapping of class $\mathscr{C}^{m-2}$ such that $y(0) = 0, y'(0) = I$ and*

$$f(x) = \frac{1}{2}(f''(0)y(x), y(x)).$$

Corollary 3.1.2. *[49] Let $n = 2$ and let f be as in the preceding theorem. If $f''(0)$ is an indefinite matrix then the set of solutions to the equation $f(x) = 0$ near the origin is a pair of curves which intersect only at the origin. If $m > 2$, these curves are $\mathscr{C}^1$ and they cut transversely.*

3.2 Bifurcation Problem in Marguerre-von Kármán equations

We recall that the classical Marguerre-von Kármán equations are represented as follows:

$$\begin{cases} \frac{2E}{3(1-v^2)}\Delta^2\zeta_3 = 2[\varphi, \zeta_3 + \theta] + p_3 & \text{in } \omega, \\ \Delta^2\varphi = -\frac{E}{2}[\zeta_3, \zeta_3 + 2\theta] & \text{in } \omega, \\ \zeta_3 = \partial_\nu\zeta_3 = 0 & \text{on } \gamma, \\ \varphi = \varphi_0 \text{ and } \partial_\nu\varphi = \varphi_1 & \text{on } \gamma, \end{cases}$$

Where the unknown $\zeta_3 : \bar{\omega} \to \mathbb{R}$ is the vertical component of the displacement field of the middle surface of the shell and the unknown $\varphi : \bar{\omega} \to \mathbb{R}$ is the Airy function.
Details about these equations are in section 2.5.

We consider here, the buckling of a nonlinearly thin elastic shallow shell under the compressive forces of von Kármán's type applied on its lateral face, such that, before deformation this forces is collinear to the normal of γ, and $\lambda > 0$ is a parameter measuring the magnitude of this forces, denotes the intensity of the lateral compression.

In this case, the Airy function be given by $\varphi + \lambda\theta_0$, where the function $\lambda\theta_0$ is the unique solution of the boundary value problem

$$\begin{cases} \Delta^2\eta = 0 & \text{in } \omega, \\ \eta = \varphi_0 \text{ and } \partial_\nu\eta = \varphi_1 & \text{on } \gamma. \end{cases}$$

The classical Marguerre-von Kármán equations becomes

$$\begin{cases} \frac{2E}{3(1-v^2)}\Delta^2\zeta_3 = 2[\varphi, \zeta_3 + \theta] + 2\lambda[\theta_0, \zeta_3 + \theta] + p_3 & \text{in } \omega, \\ \Delta^2\varphi = -\frac{E}{2}[\zeta_3, \zeta_3] & \text{in } \omega, \\ \zeta_3 = \partial_\nu\zeta_3 = 0 & \text{on } \gamma, \\ \varphi = \partial_\nu\varphi = 0 & \text{on } \gamma. \end{cases}$$

Next, we write the classical Marguerre-von Kármán equations in a simpler form, using the following relations:

$$\begin{cases} \zeta_3 = (\frac{2}{3(1-v^2)})^{\frac{1}{2}}\xi, \quad \varphi = \frac{E}{3(1-v^2)}\Phi, \quad \theta = (\frac{2}{3(1-v^2)})^{\frac{1}{2}}\tilde{\theta}, \\ \\ \theta_0 = \frac{E}{3(1-v^2)}\tilde{\theta}_0, \quad p_3 = (\frac{2}{3(1-v^2)})^{\frac{3}{2}}f. \end{cases}$$

Then, we find that the unknowns (ξ, Φ, λ) satisfy the bifurcation Marguerre-von Kármán equations

$$\begin{cases} \Delta^2 \xi = [\Phi, \xi + \tilde{\theta}] + \lambda[\tilde{\theta}_0, \xi + \tilde{\theta}] + f & \text{in } \omega, \\ \Delta^2 \Phi = -[\xi, \xi + 2\tilde{\theta}] & \text{in } \omega, \\ \xi = \partial_\nu \xi = 0 & \text{on } \gamma, \\ \Phi = \partial_\nu \Phi = 0 & \text{on } \gamma. \end{cases} \tag{3.2.1}$$

3.3 Cubic operator equation

We transform the Bifurcation Problem in Marguerre-von Kármán equations which represent in the system (3.2.1) onto single cubic operator equation

Lemma 3.3.1. *[3] Assume that $\tilde{\theta} \in H_0^2(\omega)$. Then the unknowns (λ, ξ, Φ) satisfies the Bifurcation Problem in Marguerre-von Kármán equations (3.2.1) if and only if the function $\tilde{\xi}$ satisfies the cubic operator equation*

$$\tilde{\xi} - \lambda L_2(\tilde{\xi}) - L_1(\tilde{\xi}) + C(\tilde{\xi}) = \tilde{\mathfrak{F}} \tag{3.3.1}$$

where $\tilde{\xi} = \xi + \tilde{\theta} \in H_0^2(\omega)$. And the airy function Φ is given by:

$$\Phi = \tilde{\chi} - B(\tilde{\xi}, \tilde{\xi}), \quad where\ \tilde{\chi} = B(\tilde{\theta}, \tilde{\theta}). \tag{3.3.2}$$

Proof. We follow the same method as is the proof of Theorem (2.6.1) □

Now, we define the variational problem of the Marguerre-von Kármán equations.
Let $\vartheta \in H_0^2(\Omega)$ be a test function, multiplying the first equation in (3.2.1) by $\vartheta \in H_0^2(\Omega)$, then by integrating we get

$$\int_\omega (\Delta^2 \xi) \vartheta d\omega - \int_\omega \big([\Phi, \xi + \tilde{\theta}] + \lambda[\tilde{\theta}_0, \xi + \tilde{\theta}] + f\big) \vartheta d\omega = 0. \tag{3.3.3}$$

Since $\tilde{\xi} = \xi + \tilde{\theta}$, and by using the two relations (3.3.2) and (2.6.13), we fined

$$\begin{aligned} &\int_\omega (\Delta^2 \xi) \vartheta d\omega - \int_\omega [B(\tilde{\theta}, \tilde{\theta}), \tilde{\xi}] \vartheta d\omega + \int_\omega [B(\tilde{\xi}, \tilde{\xi}), \tilde{\xi}] \vartheta d\omega \\ &- \lambda \int_\omega [\tilde{\theta}_0, \tilde{\xi}] \vartheta d\omega - \int_\omega \big(\Delta^2 (\tilde{\mathfrak{F}} - \tilde{\theta})\big) \vartheta d\omega = 0. \end{aligned} \tag{3.3.4}$$

Using the definition of operator B and its relationship to brackets $[.,.]$ we obtain

$$\begin{aligned} &\int_\omega \big(\Delta^2 (\xi + \tilde{\theta})\big) \vartheta d\omega - \int_\omega \Big(\Delta^2 B\big(B(\tilde{\theta}, \tilde{\theta}), \tilde{\xi}\big)\Big) \vartheta d\omega + \int_\omega \Big(\Delta^2 B\big(B(\tilde{\xi}, \tilde{\xi}), \tilde{\xi}\big)\Big) \vartheta d\omega \\ &- \lambda \int_\omega \big(\Delta^2 B(\tilde{\theta}_0, \tilde{\xi})\big) \vartheta d\omega - \int_\omega \big(\Delta^2 \tilde{\mathfrak{F}}\big) \vartheta d\omega = 0. \end{aligned} \tag{3.3.5}$$

Since $\tilde{\xi} = \xi + \tilde{\theta}$, and by integration by parts we get

$$\begin{aligned} &\int_\omega \Delta \tilde{\xi} \Delta \vartheta d\Omega - \int_\omega \Delta B\big(B(\tilde{\theta}, \tilde{\theta}), \tilde{\xi}\big) \Delta \vartheta d\omega + \int_\omega \Delta B\big(B(\tilde{\xi}, \tilde{\xi}), \tilde{\xi}\big) \Delta \vartheta d\omega \\ &- \lambda \int_\omega \Delta B(\tilde{\theta}_0, \tilde{\xi}) \Delta \vartheta d\omega - \int_\omega \Delta \tilde{\mathfrak{F}} \Delta \vartheta d\omega = 0. \end{aligned} \tag{3.3.6}$$

From (2.6.13), we obtain

$$\int_\omega \Delta\tilde{\xi}\Delta\vartheta d\omega - \int_\omega \Delta L_1(\tilde{\xi})\Delta\vartheta d\omega + \int_\omega \Delta C(\tilde{\xi})\Delta\vartheta d\omega \\ - \lambda \int_\omega \Delta L_2(\tilde{\xi})\Delta\vartheta d\omega - \int_\omega \Delta\tilde{\mathfrak{F}}\Delta\vartheta d\omega = 0. \tag{3.3.7}$$

Then we conclude that the operator equation (3.3.1) is equivalent to the following variational problem

$$\begin{cases} \text{Find } (\lambda, \tilde{\xi}) \in \mathbb{R} \times H_0^2(\Omega) \text{ such that,} \\ \left(\tilde{\xi} - \lambda L_2(\tilde{\xi}) + C(\tilde{\xi}) - L_1(\tilde{\xi}) - \tilde{\mathfrak{F}}, \vartheta\right)_\Delta = 0 \text{ for all } \vartheta \in H_0^2(\Omega), \end{cases} \tag{3.3.8}$$

The cubic operator equation (3.3.1) generalizes an operator equation originally introduced by Berger and Fife [6] and also Berger [5] . Details about this may be found in [30].

3.4 General existence result

3.4.1 Pseudomonotone operator

Definition 3.4.1. *[72] Let $\mathscr{A} : X \to X'$ be an operator from the real reflexive Banach space X into its dual space, $\mathscr{A}$ is called pseudomonotone if and only if for $u_n \rightharpoonup u$ in X as $n \to \infty$ and*

$$\lim_{n\to\infty} \sup \langle \mathscr{A}u_n, u_n - u\rangle \leq 0$$

implies

$$\langle \mathscr{A}u, u - w\rangle \leq \lim_{n\to\infty} \inf \langle \mathscr{A}u_n, u_n - w\rangle \quad \textit{for all } w \in X.$$

Proposition 3.4.1. *[72] Let $\mathscr{A}, \mathscr{B} : X \to X'$ be two operators defined on the real reflexive Banach space X. Then:*

- *a) If $\mathscr{A}$ is monotone and hemicontinuous, then $\mathscr{A}$ is pseudomonotone.*
- *b) If $\mathscr{A}$ is strongly continuous, then $\mathscr{A}$ is pseudomonotone.*
- *c) If $\mathscr{A}$ is continuous and dim $X < \infty$, then $\mathscr{A}$ is pseudomonotone.*
- *d) Additivity. If $\mathscr{A}$ and $\mathscr{B}$ are pseudomonotone, then $\mathscr{A} + \mathscr{B}$ is pseudomonotone.*
- *e) If $\mathscr{A}$ is monotone and hemicontinuous and $\mathscr{B}$ is strongly continuous, then $\mathscr{A} + \mathscr{B}$ is pseudomonotone.*
- *f) If $\mathscr{A}$ is pseudomonotone and $\mathscr{B}$ is monotone and hemicontinuous, then $\mathscr{A} + \mathscr{B}$ is pseudomonotone.*
- *g) If $\mathscr{A}$ is pseudomonotone and $\mathscr{B}$ is strongly continuous, then $\mathscr{A} + \mathscr{B}$ is pseudomonotone.*

The Main Theorem on Pseudomonotone Operators

We consider the operator equation

$$\mathscr{A}u = b, \quad u \in X, \tag{3.4.1}$$

along with the Galerkin method

$$\langle \mathscr{A}u_n - b, w_k \rangle = 0, \quad u_n \in X_n, \quad k = 1, ..., n, \tag{3.4.2}$$

where $X_n = span\,\{w_1, ..., w_n\}$

Theorem 3.4.1 (Brezis (1968) [72]). *Assume that:*
(i) the operator $\mathscr{A} : X \to X'$ is pseudomonotone, bounded, and coercive on the real, separable, and reflexive Banach space X with dim $X = \infty$.
(ii) Let $\{w_1, w_2...\}$ be a basis in X.
Then the following hold:
$a)$ Existence. For each $b \in X'$, the original equation (3.4.1) has a solution.
$b)$ Galerkin method. For fixed $b \in X'$ and for each $n \in \mathbb{N}$, the Galerkin equation (3.4.2) has a solution u_n. There exists a subsequence $(u_{n'})$ which converges weakly to a solution of the original equation (3.4.1). If the operator $\mathscr{A}$ satisfies $(S)_+$, then $(u_{n'})$ converges strongly to a solution of equation (3.4.1). If equation (3.4.1) has a unique solution u, then the total sequence (u_n) converges to u

3.4.2 Existence result

In order to prove the existence result of Marguerre-von Kármán equations, we will use the main Theorem on Pseudomonotone operators, so we will need to recall the most important properties of the operators L_1, L_2 and C as follows:
From the Proposition (2.4.1) .(b) the operator C is strongly continuous, also from the Proposition (2.3.1) we have the two operators L_1, and L_2 are strongly continuous. Hence the three operators L_1, L_2 and C are strongly continuous.

Let we consider the operator $\mathscr{A} : H_0^2(\omega) \to H_0^2(\omega)$ be defined by

$$\mathscr{A} = I - \lambda L_2 + C - L_1. \tag{3.4.3}$$

We have

Lemma 3.4.1. *[3] $\mathscr{A}$ is bounded and pseudomonotone operator.*

Proof. [3] Since the operators L_1, L_2 and C are strongly continuous, then the operator $\mathscr{A}$ is a strongly continuous perturbation of the identity. According to Proposition (3.4.1).(b) , $\mathscr{A}$ is pseudomonotone and bounded. □

Lemma 3.4.2. *[3] The operator $\mathscr{A}$ is coercive.*

Proof. [3] We have

$$\begin{aligned}(\mathscr{A}\tilde{\xi}, \tilde{\xi})_\Delta &= (\tilde{\xi} - \lambda L_2\tilde{\xi} + C\tilde{\xi} - L_1\tilde{\xi}, \tilde{\xi})_\Delta \\ &= \|\tilde{\xi}\|_\Delta^2 - \lambda(B(\tilde{\theta}_0, \tilde{\xi}), \tilde{\xi})_\Delta + (B(B(\tilde{\xi}, \tilde{\xi}), \tilde{\xi}), \tilde{\xi})_\Delta - (B(B(\tilde{\theta}, \tilde{\theta}), \tilde{\xi}), \tilde{\xi})_\Delta.\end{aligned}$$

From the Lemma (2.7.2), we show that

$$\begin{aligned}(\mathscr{A}\tilde{\xi},\tilde{\xi})_\Delta &= \|\tilde{\xi}\|_\Delta^2 - \lambda(B(\tilde{\xi},\tilde{\xi}),\tilde{\theta}_0)_\Delta + (B(\tilde{\xi},\tilde{\xi}),B(\tilde{\xi},\tilde{\xi}))_\Delta - (B(\tilde{\xi},\tilde{\xi}),B(\tilde{\theta},\tilde{\theta}))_\Delta \\ &= \|\tilde{\xi}\|_\Delta^2 - \lambda(B(\tilde{\xi},\tilde{\xi}),\tilde{\theta}_0)_\Delta + \|B(\tilde{\xi},\tilde{\xi})\|_\Delta^2 - (B(\tilde{\xi},\tilde{\xi}),B(\tilde{\theta},\tilde{\theta}))_\Delta.\end{aligned}$$

Consequently for any $\epsilon > 0$, by the Cauchy-Schwarz and Young inequality

$$\begin{aligned}(\mathscr{A}\tilde{\xi},\tilde{\xi})_\Delta \geq \;& \|\tilde{\xi}\|_\Delta^2 - \frac{\lambda\epsilon}{2}\|B(\tilde{\xi},\tilde{\xi})\|_\Delta^2 - \frac{\lambda}{2\epsilon}\|\tilde{\theta}_0\|_\Delta^2 + \|B(\tilde{\xi},\tilde{\xi})\|_\Delta^2 - \frac{1}{2}\|B(\tilde{\xi},\tilde{\xi})\|_\Delta^2 \\ & - \frac{1}{2}\|B(\tilde{\theta},\tilde{\theta})\|_\Delta^2 \\ \geq \;& \|\tilde{\xi}\|_\Delta^2 + \left(\frac{1}{2} - \frac{\lambda\epsilon}{2}\right)\|B(\tilde{\xi},\tilde{\xi})\|_\Delta^2 - \frac{\lambda}{2\epsilon}\|\tilde{\theta}_0\|_\Delta^2 - \frac{1}{2}\|B(\tilde{\theta},\tilde{\theta})\|_\Delta^2,\end{aligned}$$

choosing $\lambda\epsilon = 1$, we find that

$$(\mathscr{A}\tilde{\xi},\tilde{\xi})_\Delta \geq \|\tilde{\xi}\|_\Delta^2 - \frac{\lambda^2}{2}\|\tilde{\theta}_0\|_\Delta^2 - \frac{1}{2}\|B(\tilde{\theta},\tilde{\theta})\|_\Delta^2,$$

then

$$\frac{(\mathscr{A}\tilde{\xi},\tilde{\xi})_\Delta}{\|\tilde{\xi}\|_\Delta} \geq \|\tilde{\xi}\|_\Delta - \frac{\lambda^2}{2}\|\tilde{\theta}_0\|_\Delta^2\|\tilde{\xi}\|_\Delta^{-1} - \frac{1}{2}\|B(\tilde{\theta},\tilde{\theta})\|_\Delta^2\|\tilde{\xi}\|_\Delta^{-1}. \tag{3.4.4}$$

hence

$$\lim_{\|\tilde{\xi}\|\to\infty} \frac{(\mathscr{A}\tilde{\xi},\tilde{\xi})_\Delta}{\|\tilde{\xi}\|_\Delta} = \infty. \tag{3.4.5}$$

Since

$$\|\mathscr{A}(\tilde{\xi})\|_\Delta \geq \frac{(\mathscr{A}\tilde{\xi},\tilde{\xi})_\Delta}{\|\tilde{\xi}\|_\Delta}. \tag{3.4.6}$$

we conclude that

$$\lim_{\|\tilde{\xi}\|\to\infty} \|\mathscr{A}(\tilde{\xi})\|_\Delta = \infty. \tag{3.4.7}$$

□

Theorem 3.4.2. *[3] (Existence) The operator equation (3.3.1), has a solution for every fixed* $\lambda > 0$ *and every* $\tilde{\mathfrak{F}} \in H_0^2(\omega)$.

Proof. [3] From the Lemmas (3.4.2) and (3.4.1), the operator $\mathscr{A}$ is coercive, pseudomonotone and bounded. Hence, the main theorem on pseudomonotone operators (Theorem (3.4.1)) is due to Brezis (1968), it shows that, The operator equation $\mathscr{A}\tilde{\xi} = \tilde{\mathfrak{F}}$, has a solution.

For more details about the generalized problern for the von Kármán plate equations, see e.g., Theorem 65.A in [72]. □

3.5 Bifurcation Point for Marguerre-von Kármán equations

We consider the following operator equation corresponding to the homogeneous bifurcation problem in Marguerre-von Kármán equations which represent in (3.3.1):

$$\tilde{\xi} - \lambda L_2(\tilde{\xi}) - L_1(\tilde{\xi}) + C(\tilde{\xi}) = 0 \tag{3.5.1}$$

From the previous paragraph, we found that the equation (3.5.1) has a solution. Since $C(0) = 0$ and the operators L_1, L_2 are linear we get for any $\lambda_0 \in \mathbb{R}$ we have $(\lambda_0, 0) \in \mathbb{R} \times H_0^2(\omega)$ be a solution of the equation (3.5.1).
In the following Proposition we determine a Bifurcation Point of the trivial solution under certain conditions

Proposition 3.5.1. *[3] Assume that*

$$\|C\tilde{\xi} - L_1\tilde{\xi}\|_\Delta = \circ(\|\tilde{\xi}\|_\Delta), \tag{H1}$$

in a neighborhood of λ_0. Then $(\lambda_0, 0)$ is a bifurcation point of the trivial branch of solutions of (3.5.1) only if λ_0 is a characteristic value of L_2.

Proof. For the proof we use the same arguments as in Theorem 2.4-1 in [29].
We have $(\lambda_0, 0)$ is a Bifurcation Point of the trivial solution that is there exist $\tilde{\xi} \neq 0$ solution of (3.5.1).
Assume that λ_0 is not a characteristic value of the operator L_2, since L_2 is compact so from the Theorem (1.2.1) we get:

$$I - \lambda_0 L_2 \in Isom(H_0^2(\omega)) \tag{3.5.2}$$

and there exist a positive consent M independent of λ_0 such that

$$\|(I - \lambda_0 L_2)^{-1}\|_{\mathcal{L}(H_0^2)} \leq M \tag{3.5.3}$$

From (3.5.1) we find

$$\tilde{\xi} - \lambda L_2(\tilde{\xi}) = L_1(\tilde{\xi}) - C(\tilde{\xi})$$

$$(I - \lambda L_2)\tilde{\xi} = L_1(\tilde{\xi}) - C(\tilde{\xi})$$

So, from (3.5.2) and (3.5.3), and he assumption $(H1)$ we get

$$\tilde{\xi} = (I - \lambda L_2)^{-1}(L_1(\tilde{\xi}) - C(\tilde{\xi})) \tag{3.5.4}$$

and

$$\|\tilde{\xi}\|_\Delta \leq M\|C\tilde{\xi} - L_1\tilde{\xi}\|_\Delta = \circ(\|\tilde{\xi}\|_\Delta) \tag{3.5.5}$$

which is absurd with $\tilde{\xi} \neq 0$. Thus λ_0 is a characteristic value of the operator L_2. □

since the operators C and L_1 are compact, the operator $C - L_1$ is compact. Under the assumption $(H1)$, the Krasnosel'skii Theorem (Theorem (3.1.4)) proves the following Proposition which is a partial converse of Proposition (3.5.1)

Proposition 3.5.2. *[3] $(\lambda_0, 0)$ is a bifurcation point of the equation (3.5.1) if and only if λ_0 is a characteristic value of odd algebraic multiplicity of L.*

Since the operator L_2 is linear, compact, self-adjoint and positive definite we conclude that from the Theorem (1.2.5) L_2 has an infinite number of distinct characteristic values $\lambda_i > 0$, each of finite multiplicity, such that

$$0 < \lambda_1 < \lambda_2 < \ldots < \lambda_i < \ldots \to \infty$$

Where

$$\lambda_1 = \inf_{\substack{\eta \in H_0^2(\omega) \\ \eta \neq 0}} \frac{\|\eta\|_\Delta}{(L_2\eta, \eta)_\Delta}$$

3.6 Kikuchi method for homogeneous Marguerre-von Kármán equations

In this section we will present the results of Ghezal which found in [34] in more detail.
We assume that $\tilde{\theta} \in H_0^2$ and $\tilde{\mathfrak{F}} = F + \tilde{\theta} = 0$ $(i.e., (\Delta^2)^{-1}\tilde{f} = -\tilde{\theta})$ So, the operator equation of bifurcation problem in Marguerre-von Kármán equations which represent in (3.3.1) becomes:

$$\tilde{\xi} - \lambda L_2(\tilde{\xi}) - L_1(\tilde{\xi}) + C(\tilde{\xi}) = 0 \tag{3.6.1}$$

we have from the Proposition $(3.5.2)$, $(\lambda_0, 0) \in \mathbb{R} \times H_0^2(\omega)$ is a bifurcation point of Marguerre-von Kármán equations (3.6.1)
We want here to apply the Kikuchi method to find the solutions of equation (3.6.1) in the neighborhood of $(\lambda_0, 0)$.
The principle of Kikuchi method is based on finding the solution $(\lambda, \tilde{\xi})$ of (3.6.1), with λ in the neighborhood of λ_0 and $\tilde{\xi}$ of the form

$$\tilde{\xi} = \epsilon\phi_0 + \mathcal{V}, \tag{3.6.2}$$

with $\left\|\tilde{\xi}\right\|_\Delta$ is small , $\epsilon > 0$ is a parameter approaching to zero, ϕ_0 be a normalized eigenfunction of the operator L_2 corresponding to the simple eigenvalue λ_0, in the sense that

$$\phi_0 = \lambda_0 L_2 \phi_0, \text{ and } (L_2\phi_0, \phi_0)_\Delta = 1$$

Thus, finding the solution $\tilde{\xi} \in H_0^2(\omega)$ turns into the finding $v \in \{\phi_0\}^\perp$

Theorem 3.6.1. *[34] Let $\epsilon > 0$ and $(\lambda, \tilde{\xi}) \in H_0^2(\omega) \times \mathbb{R}$, with*

$$\tilde{\xi} = \epsilon\phi_0 + \mathcal{V}; \mathcal{V} \in \{\phi_0\}^\perp \tag{3.6.3}$$

Then $(\lambda, \tilde{\xi})$ be solution of the operator equation (3.6.1) if and only if

$$\mathcal{V} = ST_\epsilon(\tilde{\xi}), \tag{3.6.4}$$

and

$$\lambda = \lambda_0 + \frac{1}{\epsilon}(C(\tilde{\xi}) - L_1(\tilde{\xi}), \phi_0)_\Delta \tag{3.6.5}$$

Where

$$T_\epsilon\eta = \frac{1}{\epsilon}(C(\eta) - L_1(\eta), \phi_0)_\Delta L_2(\eta) - C(\eta) + L_1(\eta) \tag{3.6.6}$$

and the mapping $S : H_0^2 \to \{\phi_0\}^\perp$ be defined by

$$(I - \lambda_0 L_2)S\eta = P_0\eta; \tag{3.6.7}$$

such that P_0 is the orthogonal projection in $H_0^2(\omega)$ onto $\{\phi_0\}^\perp$.

Proof. Let $\tilde{\xi}$ of the form $\tilde{\xi} = \epsilon\phi_0 + \mathcal{V}$ be a solution of the equation (3.6.1), then

$$\epsilon\phi_0 + \mathcal{V} - \lambda L_2(\epsilon\phi_0 + \mathcal{V}) - L_1(\epsilon\phi_0 + \mathcal{V}) + C(\epsilon\phi_0 + \mathcal{V}) = 0 \tag{3.6.8}$$

the scalar product of the equation (3.6.8) with ϕ_0 gives

$$\begin{aligned} &(\epsilon\phi_0 + \mathcal{V} - \lambda L_2(\epsilon\phi_0 + \mathcal{V}) - L_1(\epsilon\phi_0 + \mathcal{V}) + C(\epsilon\phi_0 + \mathcal{V}), \phi_0)_\Delta \\ =&(\epsilon\phi_0 + \mathcal{V}, \phi_0)_\Delta - \lambda\left(L_2(\epsilon\phi_0 + \mathcal{V}), \phi_0\right)_\Delta - \left(L_1(\epsilon\phi_0 + \mathcal{V}), \phi_0\right)_\Delta \\ &+ \left(C(\epsilon\phi_0 + \mathcal{V}), \phi_0\right)_\Delta = 0 \end{aligned} \tag{3.6.9}$$

we have

$$\begin{aligned} (\epsilon\phi_0 + \mathcal{V}, \phi_0)_\Delta &= \epsilon\left(\phi_0, \phi_0\right)_\Delta + (\mathcal{V}, \phi_0)_\Delta \\ &= \epsilon\|\phi_0\|^2_\Delta = \epsilon\lambda_0 \end{aligned} \tag{3.6.10}$$

and

$$\begin{aligned} -\lambda\left(L_2(\epsilon\phi_0 + \mathcal{V}), \phi_0\right)_\Delta &= -\lambda\left(L_2(\epsilon\phi_0), \phi_0\right)_\Delta - \lambda\left(L_2(v), \phi_0\right)_\Delta \\ &= -\lambda\epsilon\left(L_2(\phi_0), \phi_0\right)_\Delta - \lambda\left(\mathcal{V}, L_2(\phi_0)\right)_\Delta \\ &= -\lambda\epsilon - \lambda\left(\mathcal{V}, \frac{\phi_0}{\lambda_0}\right)_\Delta = -\lambda\epsilon - \frac{\lambda}{\lambda_0}\left(\mathcal{V}, \phi_0\right)_\Delta \\ &= -\lambda\epsilon \end{aligned} \tag{3.6.11}$$

then the equation (3.6.9) become

$$\epsilon\lambda_0 - \lambda\epsilon - \left(L_1(\epsilon\phi_0 + \mathcal{V}), \phi_0\right)_\Delta + \left(C(\epsilon\phi_0 + \mathcal{V}), \phi_0\right)_\Delta = 0 \tag{3.6.12}$$

which equivalent to

$$\lambda_0 - \lambda = \frac{1}{\epsilon}\left(\left(L_1(\epsilon\phi_0 + \mathcal{V}), \phi_0\right)_\Delta - \left(C(\epsilon\phi_0 + \mathcal{V}), \phi_0\right)_\Delta\right) \tag{3.6.13}$$

So

$$\begin{aligned} \lambda &= \lambda_0 + \frac{1}{\epsilon}\left(C(\epsilon\phi_0 + \mathcal{V}) - L_1(\epsilon\phi_0 + \mathcal{V}), \phi_0\right)_\Delta \\ &= \lambda_0 + \frac{1}{\epsilon}\left(C(\tilde{\xi}) - L_1(\tilde{\xi}), \phi_0\right)_\Delta . \end{aligned} \tag{3.6.14}$$

The projection orthogonal of the equation (3.6.8) in $H^2_0(\omega)$ onto $\{\phi_0\}^\perp$ gives

$$v - \lambda L_2(v) - P_0 L_1(\epsilon\phi_0 + \mathcal{V}) + P_0 C(\epsilon\phi_0 + \mathcal{V}) = 0 \tag{3.6.15}$$

which equivalent to

$$\begin{aligned} & v - \lambda_0 L_2(v) + \lambda_0 L_2(v) - \lambda L_2(v) - P_0 L_1(\epsilon\phi_0 + \mathcal{V}) + P_0 C(\epsilon\phi_0 + \mathcal{V}) = 0 \\ \Leftrightarrow & \left(I - \lambda_0 L_2\right) v = \left(\lambda - \lambda_0\right) L_2(v) + P_0 L_1(\epsilon\phi_0 + \mathcal{V}) - P_0 C(\epsilon\phi_0 + \mathcal{V}) \end{aligned} \tag{3.6.16}$$

by applying the operator S which defined in (3.6.7) to both sides of the equation (3.6.16) we find

$$S\left(I - \lambda_0 L_2\right) v = \left(\lambda - \lambda_0\right) S L_2(v) + S P_0 L_1(\epsilon\phi_0 + \mathcal{V}) - S P_0 C(\epsilon\phi_0 + \mathcal{V}) \tag{3.6.17}$$

the definition of the operator S and since $P_0^2 = P_0$ (see the Lemma (1.3.1)) we get

$$S P_0 L_1(\epsilon\phi_0 + \mathcal{V}) = S L_1(\epsilon\phi_0 + \mathcal{V})$$

and

$$S P_0 C(\epsilon\phi_0 + \mathcal{V}) = S C(\epsilon\phi_0 + \mathcal{V})$$

hence, the relation (3.6.17) implies that

$$\begin{aligned} v &= (\lambda - \lambda_0)\, SL_2(v) + SL_1(\epsilon\phi_0 + \mathcal{V}) - SC(\epsilon\phi_0 + \mathcal{V}) \\ &= S\left[(\lambda - \lambda_0)L_2(v) + L_1(\epsilon\phi_0 + \mathcal{V}) - C(\epsilon\phi_0 + \mathcal{V})\right] \end{aligned} \tag{3.6.18}$$

Substitute the value of $\lambda - \lambda_0$ from the equation (3.6.14) into (3.6.18) we get

$$\begin{aligned} v &= S\left[\frac{1}{\epsilon}\left(C(\epsilon\phi_0 + \mathcal{V}) - L_1(\epsilon\phi_0 + \mathcal{V}), \phi_0\right)_\Delta L_2(v) + L_1(\epsilon\phi_0 + \mathcal{V}) - C(\epsilon\phi_0 + \mathcal{V})\right] \\ &= S\left[\frac{1}{\epsilon}\left(C(\tilde{\xi}) - L_1(\tilde{\xi}), \phi_0\right)_\Delta L_2(v) + L_1(\tilde{\xi}) - C(\tilde{\xi})\right] \end{aligned} \tag{3.6.19}$$

we have $SL_2(\phi_0) = \lambda_0^{-1}S\phi_0 = 0$, then

$$SL_2(v) = SL_2(v) + SL_2(\phi_0) = SL_2(\epsilon\phi_0 + \mathcal{V}) \tag{3.6.20}$$

thus

$$\begin{aligned} v &= S\left[\frac{1}{\epsilon}\left(C(\tilde{\xi}) - L_1(\tilde{\xi}), \phi_0\right)_\Delta L_2(\tilde{\xi}) + L_1(\tilde{\xi}) - C(\tilde{\xi})\right] \\ &= ST_\epsilon(\tilde{\xi}). \end{aligned} \tag{3.6.21}$$

□

The existence results is based on the following Lemma

Lemma 3.6.1. *[34] We let the operator $\Lambda_\epsilon : H_0^2(\omega) \to \mathbb{R}$ be defined by*

$$\Lambda_\epsilon\eta = \lambda_0 + \frac{1}{\epsilon}(C(\eta) - L_1(\eta), \phi_0)_\Delta \tag{3.6.22}$$

such that

$$T_\epsilon\eta = (\Lambda_\epsilon\eta - \lambda_0)L_2(\eta) - C(\eta) + L_1(\eta),\ \forall\eta \in H_0^2(\omega) \tag{3.6.23}$$

and we assume here that:

$$\|\tilde{\theta}\|_\Delta < c\epsilon \tag{3.6.24}$$

Then, there exists a constant c independent of ϵ, such that

1. $$\|S\eta\|_\Delta \le c\|\eta\|_\Delta, \forall\eta \in H_0^2(\omega) \tag{3.6.25}$$

2. $$\|L_1\zeta_1 - L_1\zeta_2\|_\Delta \le c\epsilon^2\|\zeta_1 - \zeta_2\|_\Delta, \forall\zeta_i \in U_\epsilon \tag{3.6.26}$$

3. $$\|C(\zeta_1) - C(\zeta_2)\|_\Delta \le c\epsilon^2\|\zeta_1 - \zeta_2\|_\Delta, \forall\zeta_i \in U_\epsilon \tag{3.6.27}$$

4. $$|\Lambda_\epsilon\zeta - \lambda_0| \le c\epsilon^2, \forall\zeta \in U_\epsilon \tag{3.6.28}$$

5. $$|\Lambda_\epsilon\zeta_1 - \Lambda_\epsilon\zeta_2| \le c\epsilon\|\zeta_1 - \zeta_2\|_\Delta, \forall\zeta_i \in U_\epsilon \tag{3.6.29}$$

6.

$$\|T_\epsilon \zeta\|_\Delta \leq c\epsilon^3, \forall \zeta \in U_\epsilon \tag{3.6.30}$$

7.

$$\|T_\epsilon \zeta_1 - T_\epsilon \zeta_2\|_\Delta \leq c\epsilon^2 \|\zeta_1 - \zeta_2\|_\Delta, \forall \zeta_i \in U_\epsilon \tag{3.6.31}$$

Where

$$U_\epsilon = \{\epsilon\phi_0 + \mathcal{V}; \mathcal{V} \in V_\epsilon\}; \tag{3.6.32}$$

$$V_\epsilon = \{\mathcal{V} \in H_0^2(\omega); \|v\|_\Delta \leq c\epsilon\}. \tag{3.6.33}$$

Proof. 1. the first relation (3.6.25) is immediate from the continuity of the operator S.

2. we have $L_1(\zeta) = B(B(\tilde{\theta}, \tilde{\theta}), \zeta)$ then from the condition (3.6.24) we get

$$\begin{aligned}\|L_1\zeta_1 - L_1\zeta_2\|_\Delta = \|L_1(\zeta_1 - \zeta_2)\|_\Delta &= \|B(B(\tilde{\theta}, \tilde{\theta}), \zeta_1 - \zeta_2)\|_\Delta \\ &\leq \|B\|_\Delta^2 \|\tilde{\theta}\|_\Delta^2 \|\zeta_1 - \zeta_2)\|_\Delta \\ &\leq c\epsilon^2 \|\zeta_1 - \zeta_2\|_\Delta, \forall \zeta_i \in U_\epsilon\end{aligned} \tag{3.6.34}$$

3. using the relation (2.4.5) we get

$$\begin{aligned}\|C(\zeta_1) - C(\zeta_2)\|_\Delta &\leq 3\|B\|_\Delta^2 \|\zeta_1 - \zeta_2)\|_\Delta \left[\max\left\{\|\zeta_1\|_\Delta^2, \|\zeta_2)\|_\Delta^2\right\}\right] \\ &\leq 3\|B\|_\Delta^2 \|\zeta_1 - \zeta_2)\|_\Delta \,(c\epsilon)^2 \\ &\leq c\epsilon^2 \|\zeta_1 - \zeta_2\|_\Delta, \forall \zeta_i \in U_\epsilon\end{aligned} \tag{3.6.35}$$

4. using definition of operators L_1, C and the condition (3.6.24), we get

$$\begin{aligned}|\Lambda_\epsilon \zeta - \lambda_0| &= |\frac{1}{\epsilon}(C(\zeta) - L_1(\zeta), \phi_0)_\Delta| \\ &\leq \frac{1}{\epsilon}\left[\|C(\zeta) - L_1(\zeta)\|_\Delta \|\phi_0\|_\Delta\right] \\ &\leq \frac{1}{\epsilon}\left[\|C(\zeta)\|_\Delta + \|L_1(\zeta)\|_\Delta\right] \|\phi_0\|_\Delta \\ &\leq \frac{1}{\epsilon}\left[\|B\|_\Delta^2 \|(\zeta)\|_\Delta^3 + \|B\|_\Delta^2 \|\tilde{\theta}\|_\Delta^2 \|(\zeta)\|_\Delta\right] \|\phi_0\|_\Delta \\ &\leq \frac{1}{\epsilon}\left[\|B\|_\Delta^2 (c\epsilon)^3 + \|B\|_\Delta^2 (c\epsilon)^3\right] \|\phi_0\|_\Delta \\ &\leq c\epsilon^2, \forall \zeta \in U_\epsilon\end{aligned} \tag{3.6.36}$$

5. using definition of Λ_ϵ and the two relations (3.6.26) and (3.6.27) we get

$$\begin{aligned}|\Lambda_\epsilon \zeta_1 - \Lambda_\epsilon \zeta_2| &= \left|\frac{1}{\epsilon}(C(\zeta_1) - L_1(\zeta_1), \phi_0)_\Delta - \frac{1}{\epsilon}(C(\zeta_2) - L_1(\zeta_2), \phi_0)_\Delta\right| \\ &= \frac{1}{\epsilon}\left|(C(\zeta_1) - C(\zeta_2) + L_1(\zeta_2) - L_1(\zeta_1), \phi_0)_\Delta\right| \\ &\leq \frac{1}{\epsilon}\left(\|C(\zeta_1) - C(\zeta_2)\|_\Delta + \|L_1(\zeta_2) - L_1(\zeta_1)\|_\Delta\right) \|\phi_0\|_\Delta \\ &\leq \frac{1}{\epsilon}\left(c\epsilon^2 \|\zeta_1 - \zeta_2\|_\Delta + c\epsilon^2 \|\zeta_1 - \zeta_2\|_\Delta\right) \|\phi_0\|_\Delta \\ &\leq c\epsilon \|\zeta_1 - \zeta_2\|_\Delta, \forall \zeta_i \in U_\epsilon\end{aligned} \tag{3.6.37}$$

6. By using the definition of $T_\epsilon\zeta, C$ and L_2, and the relations (3.6.28), (3.6.24) we fined

$$\begin{aligned}\|T_\epsilon\zeta\|_\Delta &= (\varLambda_\epsilon\eta-\lambda_0)L_2(\eta)-C(\eta)+L_1(\eta)\\ &\leq |\varLambda_\epsilon\zeta-\lambda_0|\,\|L_2(\zeta)\|_\Delta+\|C(\zeta)\|_\Delta+\|L_1(\eta)\|_\Delta\\ &\leq c\epsilon^2\|B\|_\Delta\|\tilde{\theta}_0\|_\Delta\|\zeta\|_\Delta+\|B\|_\Delta^2\|\zeta\|_\Delta^3+\|B\|_\Delta^2\|\tilde{\theta}\|_\Delta^2\|\zeta\|_\Delta\\ &\leq c\epsilon^3, \forall\zeta\in U_\epsilon\end{aligned} \tag{3.6.38}$$

7. Finally, the definition of $T_\epsilon\zeta$ and the relations (3.6.26),(3.6.27),(3.6.28) and (3.6.29) we get

$$\begin{aligned}\|T_\epsilon\zeta_1-T_\epsilon\zeta_2\|_\Delta =&\|(\varLambda_\epsilon\zeta_1-\lambda_0)L_2(\zeta_1)-C(\zeta_1)+L_1(\zeta_1)-(\varLambda_\epsilon\zeta_2-\lambda_0)L_2(\zeta_2)+C(\zeta_2)\\ &-L_1(\zeta_2)\|_\Delta\\ =&\|(\varLambda_\epsilon\zeta_1-\varLambda_\epsilon\zeta_2)L_2(\zeta_1)+(\varLambda_\epsilon\zeta_2-\lambda_0)L_2(\zeta_1-\zeta_2)-(C(\zeta_1)-C(\zeta_2))\\ &+(L_1(\zeta_1)-L_1(\zeta_2))\|_\Delta\\ \leq&\|(\varLambda_\epsilon\zeta_1-\varLambda_\epsilon\zeta_2)\|_\Delta\|L_2(\zeta_1)\|_\Delta+\|(\varLambda_\epsilon\zeta_2-\lambda_0)\|_\Delta\|L_2(\zeta_1-\zeta_2)\|_\Delta\\ &+\|(C(\zeta_1)-C(\zeta_2))\|_\Delta+\|(L_1(\zeta_1)-L_1(\zeta_2))\|_\Delta\\ \leq& c\epsilon\|\zeta_1-\zeta_2\|_\Delta\|L_2\|\|\zeta_1\|_\Delta+c\epsilon^2\|L_2\|\|\zeta_1-\zeta_2\|_\Delta\\ &+c\epsilon^2\|\zeta_1-\zeta_2\|_\Delta+c\epsilon^2\|\zeta_1-\zeta_2\|_\Delta\\ \leq& c\epsilon^2\|\zeta_1-\zeta_2\|_\Delta\end{aligned} \tag{3.6.39}$$

□

Theorem 3.6.2. *[34] For ϵ is small enough, then there exists a unique solution $(\lambda,\tilde{\xi})$ to the operator equation (3.6.1), with $\tilde{\xi}$ is of the form $\tilde{\xi}=\epsilon\phi_0+\mathcal{V}$; $\mathcal{V}\in\{\phi_0\}^\perp\cap V_\epsilon$, such that*

$$\tilde{\xi}\neq 0, \|\tilde{\xi}\|_\Delta=O(\epsilon), |\lambda-\lambda_0|=O(\epsilon^2). \tag{3.6.40}$$

Proof. The proof is immediately from the fixed point theorem and the Lemma (3.6.1). □

3.7 Perturbed Bifurcation Problem in Marguerre-von Kármán equations

3.7.1 Setting of the problem

In this section, we present the results of the article [3].
We then consider the following operator equation, corresponding to perturbed bifurcation problem in Marguerre-von Kármán equations

$$\tilde{\xi}-\lambda L_2\tilde{\xi}+C\tilde{\xi}-L_1\tilde{\xi}=\delta\tilde{\mathfrak{F}},\ (\lambda,\tilde{\xi})\in\mathbb{R}\times H_0^2(\omega), \tag{$\mathscr{E}_\delta$}$$

where $|\delta|$ is sufficiently small.

Proposition 3.7.1. *(Necessary bifurcation condition [3]) If $(\lambda_0,\tilde{\xi}_0)$ is a bifurcation point of ($\mathscr{E}_\delta$), then*

$$I-\lambda_0L_2+C'\tilde{\xi}_0-L_1\notin Isom\left(H_0^2(\omega)\right). \tag{3.7.1}$$

This follows immediately from the implicit function Theorem (Theorem (3.1.1)).

According to Proposition (3.5.2), we deduce that $(\lambda_i, 0)$ is a bifurcation point for equation $(\mathscr{E}_0)$ if and only if λ_i is a simple characteristic value of the operator L_2.

Next, we want to study the set of solutions of the operator equation $(\mathscr{E}_\delta)$, in the neighborhood $(\lambda_i, 0) \in \mathbb{R} \times H_0^2(\omega)$, that satisfy the following property

$$I - \lambda L_2 + C'\tilde{\xi}_0 - L_1 \notin Isom\left(H_0^2(\omega)\right). \tag{3.7.2}$$

where $\lambda_i > 0$ is the i^{th} simple characteristic value of the operator L_2, with normalized eigenfunction ϕ_i (i.e., $\|\phi_i\|_\Delta = 1$).

Let

$$\mathcal{K}_i = \ker(I - \lambda_i L_2), \tag{3.7.3}$$

the eigen-space generated by the fixed normalized eigenfunction ϕ_i.

We consider the following equation in the neighborhood of $(\lambda_i, 0, 0)$ satisfies

$$\tilde{\xi} - \lambda L_2\tilde{\xi} + C\tilde{\xi} - L_1\tilde{\xi} - \delta\tilde{\mathfrak{F}} = 0,\ (\lambda, \tilde{\xi}, \delta) \in \mathbb{R} \times H_0^2(\omega) \times \mathbb{R}. \tag{$\mathscr{E}$}$$

The equation $(\mathscr{E}_\delta)$ is different than the equation $(\mathscr{E})$, such that δ is fixed parameter in the first equation, but it is unknown in the second equation . In the sense that $(\lambda, \tilde{\xi}, \delta) \in \mathbb{R} \times H_0^2(\omega) \times \mathbb{R}$ (resp. the couple $(\lambda, \tilde{\xi}) \in \mathbb{R} \times H_0^2(\omega)$) is $\tilde{\xi}$-singular solution of the equation $(\mathscr{E})$ (resp. $\mathscr{E}_\delta$) if $(\lambda, \tilde{\xi}, \delta)$ (resp. $(\lambda, \tilde{\xi})$) is solution of the equation $(\mathscr{E})$ (resp. $\mathscr{E}_\delta$), and the condition $(3.7.2)$ is realized.

3.7.2 Reduction of the problem

We clearly have the two operators L_1 and $L_2 \in \mathscr{L}\left(H_0^2(\omega)\right)$ are compact, and since the cubic operator C is strongly continuous, then from the results of Krasnosel'skii in [52] we find that the operator $C'\tilde{\xi} \in \mathscr{L}\left(H_0^2(\omega)\right)$ is compact. Then we conclude that the necessary bifurcation condition $(3.7.2)$ is equivalent to the following compact perturbation of the identity

$$I - \lambda L_2 + C'\tilde{\xi} - L_1 \in \mathscr{L}\left(H_0^2(\omega)\right), \tag{3.7.4}$$

is not injective.

In all the following we will use the same approach in the von-Kármán equation, and we keep the same formula with adding the new operator represented in L_1.

Let $\mathcal{U}_i$ subset of $\mathbb{R}^2$ be an open neighborhoods to $(\lambda_i, 0)$, and let $\tilde{\mathcal{U}_i}$ subset of $\mathbb{R} \times H_0^2(\omega)$ be an open neighborhoods to $(\lambda_i, 0)$

that will be defined later, I_3 an open neighborhood of 0, and let $\mathcal{V}_i$ be a mapping define from $\mathcal{U}_i \times I_3$ such that $\mathcal{V}_i(\lambda_i, 0, 0) = 0$.

We will define the following mapping of class $\mathscr{C}^\infty$:

$$\mathcal{R}_i : \mathcal{U}_i \times I_3 \longrightarrow \mathbb{R},$$

as follows

$$\begin{aligned}\mathcal{R}_i(\lambda, \epsilon, \delta) = \left(1 - \frac{\lambda}{\lambda_i}\right)\epsilon + \left(C\left(\epsilon\phi_i + \mathcal{V}_i(\lambda, \epsilon, \delta)\right), \phi_i\right)_\Delta \\ - \left(L_1\left(\epsilon\phi_i + \mathcal{V}_i(\lambda, \epsilon, \delta)\right), \phi_i\right)_\Delta - \delta(\tilde{\mathfrak{F}}, \phi_i)_\Delta.\end{aligned} \tag{3.7.5}$$

We consider the equation (E) which consist to determine the triples $(\lambda, \epsilon, \delta) \in \mathcal{U}_i \times I_3$, such that:

$$\mathcal{R}_i(\lambda, \epsilon, \delta) = 0. \tag{E}$$

For fixed $\delta \in I_3$, the equation (E_δ) corresponds to determine the couples $(\lambda, \epsilon) \in \mathcal{U}_i$ such that:

$$\mathcal{R}_i(\lambda, \epsilon, \delta) = 0. \tag{E_δ}$$

We note that the triple $(\lambda, \epsilon, \delta) \in \mathcal{U}_i \times I_3$ (resp. the couple $(\lambda, \epsilon) \in \mathcal{U}_i$) is ϵ-singular solution of the equation (E) (resp. E_δ) if $(\lambda, \epsilon, \delta)$ (resp. (λ, ϵ)) is solution of the equation (E) (resp. E_δ), and the following condition

$$\partial_\epsilon \mathcal{R}_i(\lambda, \epsilon, \delta) = 0, \tag{3.7.6}$$

is realized.

Let the mapping $\mathscr{P}_i \in \mathscr{C}^\infty$ be defined as

$$\begin{aligned} \mathscr{P}_i : \mathcal{U}_i \times I_3 &\longrightarrow \tilde{\mathcal{U}_i} \times I_3, \\ (\lambda, \epsilon, \delta) &\longrightarrow (\lambda, \epsilon\phi_i + \mathcal{V}_i(\lambda, \epsilon, \delta), \delta). \end{aligned}$$

For fixed $\delta \in I_3$, let $\mathscr{P}_{i,\delta} \in \mathscr{C}^\infty$ the mapping be defined as

$$\begin{aligned} \mathscr{P}_{i,\delta} : \mathcal{U}_i &\longrightarrow \tilde{\mathcal{U}_i}, \\ (\lambda, \epsilon) &\longrightarrow (\lambda, \epsilon\phi_i + \mathcal{V}_i(\lambda, \epsilon, \delta)). \end{aligned}$$

Theorem 3.7.1. *[3] The $\epsilon-$singular solution of the equation (E) in $\mathcal{U}_i \times I_3$ (resp. (E_δ) in $\mathcal{U}_i$, for fixed $\delta \in I_3$) and $\tilde{\xi}$-singular solution of the equation $(\mathscr{E})$ in $\tilde{\mathcal{U}_i} \times I_3$ (resp. $(\mathscr{E}_\delta)$ in $\tilde{\mathcal{U}_i}$, for fixed $\delta \in I_3$) are bijective by the mapping $\mathscr{P}_i$ (resp. $\mathscr{P}_{i,\delta}$).*

The proof of this Theorem is based on the following two Lemmas

Lemma 3.7.1. *[3] There exist a neighborhood $\tilde{\mathcal{U}_i}$ of $(\lambda_i, 0) \in \mathbb{R} \times H_0^2(\omega)$ and a mapping $\Theta_i \in \mathscr{C}^\infty$:*

$$\Theta_i : \tilde{\mathcal{U}_i} \longrightarrow \mathcal{K}_i^\perp, \tag{3.7.7}$$

with

$$\Theta_i(\lambda_i, 0) = [I - \lambda_i L_2 - P_i L_1]^{-1} P_i L_1 \phi_i,$$

where P_i denotes the orthogonal projection operator onto $\mathcal{K}_i^\perp$, such that the two following conditions are equivalent

$$(\lambda, \tilde{\xi}) \in \tilde{\mathcal{U}_i},\ I - \lambda L_2 + C'\tilde{\xi} - L_1 \notin Isom\left(H_0^2(\omega)\right), \tag{3.7.8}$$

$$1 - \frac{\lambda}{\lambda_i} + (C'\tilde{\xi}.\left(\phi_i + \Theta_i(\lambda, \tilde{\xi})\right), \phi_i)_\Delta - (L_1\left(\phi_i + \Theta_i(\lambda, \tilde{\xi})\right), \phi_i)_\Delta = 0. \tag{3.7.9}$$

where $C'(\xi).(\eta)$ denotes the evaluation of the Fréchet derivative of C at ξ on η. Moreover, the $\ker(I - \lambda L_2 + C'\tilde{\xi} - L_1)$ has an 1-dimensional and generated by the vector $\phi_i + \Theta_i(\lambda, \tilde{\xi})$.

Proof. [3] Let $h \in H_0^2(\omega) \setminus \{0\}$ such that:

$$h - \lambda L_2 h + C'\tilde{\xi}.h - L_1 h = 0. \tag{3.7.10}$$

Using the decomposition of the Hilbert space $H_0^2(\omega)$ in the direct sum of spaces $\mathcal{K}_i$ and $\mathcal{K}_i^\perp$; $H_0^2(\omega) = \mathcal{K}_i \oplus \mathcal{K}_i^\perp$, we have

$$h = \alpha\phi_i + \eta;\ \alpha \in \mathbb{R},\ \eta \in \mathcal{K}_i^\perp. \tag{3.7.11}$$

So that

$$
\begin{aligned}
(h-\lambda L_2 h + C'\tilde{\xi}.h - L_1 h, \phi_i)_\Delta &= (1-\frac{\lambda}{\lambda_i})\alpha + \alpha(C'\tilde{\xi}.\phi_i,\phi_i)_\Delta + (C'\tilde{\xi}.\eta,\phi_i)_\Delta \\
&\quad - \alpha(L_1\phi_i,\phi_i)_\Delta - (L_1\eta,\phi_i)_\Delta \\
&= 0.
\end{aligned} \tag{3.7.12}
$$

Let P_i denote the orthogonal projection operator onto $\mathcal{K}_i^\perp$.

From (3.7.10) and the operator P_i, we find that

$$
(I-\lambda L_2)\eta + P_i C'\tilde{\xi}.\eta + \alpha P_i C'\tilde{\xi}.\phi_i - P_i L_1\eta - \alpha P_i L_1 \phi_i = 0. \tag{3.7.13}
$$

Since the operator L_2 is compact and self-adjoint, by using the spectral theory (Theorem (1.2.1).(b)), we conclude that the restriction of the operator $I-\lambda_i L_2$ on the space $\mathcal{K}_i^\perp$ is an isomorphism in $\mathcal{K}_i^\perp$, and since the set $Isom(\mathcal{K}_i^\perp)$ is open, then by the continuity, there exist a neighborhood $\tilde{\mathcal{U}}_i$ of $(\lambda_i, 0)$ such that for any $(\lambda,\tilde{\xi})\in\tilde{\mathcal{U}}_i$, we have

$$
I-\lambda L_2 + P_i C'\tilde{\xi} - P_i L_1 \in Isom(\mathcal{K}_i^\perp). \tag{3.7.14}
$$

Taking into account (3.7.13), for any $(\lambda,\tilde{\xi})\in\tilde{\mathcal{U}}_i$, we find that

$$
\eta = \alpha\varTheta_i(\lambda,\tilde{\xi}) \in \mathscr{C}^\infty, \tag{3.7.15}
$$

where

$$
\varTheta_i(\lambda,\tilde{\xi}) = -[I-\lambda L_2 + P_i C'\tilde{\xi} - P_i L_1]^{-1}[P_i C'\tilde{\xi} - P_i L_1]\phi_i. \tag{3.7.16}
$$

Since $C'(0)=0$ implies

$$
\varTheta_i(\lambda_i,0) = [I-\lambda_i L_2 - P_i L_1]^{-1} P_i L_1 \phi_i.
$$

Substitution of (3.7.15) into (3.7.12) gives

$$
(1-\frac{\lambda}{\lambda_i})\alpha + \alpha(C'\tilde{\xi}.\phi_i,\phi_i)_\Delta + \alpha(C'\tilde{\xi}.\varTheta_i,\phi_i)_\Delta - \alpha(L_1\phi_i,\phi_i)_\Delta - \alpha(L_1\varTheta_i,\phi_i)_\Delta = 0. \tag{3.7.17}
$$

Inserting (3.7.15) into (3.7.11) gives

$$
h = \alpha(\phi_i + \varTheta_i). \tag{3.7.18}
$$

Since $h\in H_0^2(\omega)\setminus\{0\}$ implies that $\alpha\neq 0$. Hence we infer from (3.7.17) that

$$
1-\frac{\lambda}{\lambda_i} + (C'\tilde{\xi}.\phi_i,\phi_i)_\Delta + (C'\tilde{\xi}.\varTheta_i,\phi_i)_\Delta - (L_1\phi_i,\phi_i)_\Delta - (L_1\varTheta_i,\phi_i)_\Delta = 0, \tag{3.7.19}
$$

this immediately implies the equivalence between the conditions (3.7.8) and (3.7.9).

Then we conclude that

$$
\ker(I-\lambda L_2 + C'\tilde{\xi} - L_1)
$$

has an 1-dimensional and generated by the vector $\phi_i + \varTheta_i(\lambda,\tilde{\xi})$. □

Lemma 3.7.2. *[3] There are an open interval $I_{1,i}$ containing λ_i, two open intervals I_2, I_3 containing 0, a connected open neighborhood V_i of 0 in the $\mathcal{K}_i^\perp$ space, and a unique mapping $\mathcal{V}_i \in \mathscr{C}^\infty$:*

$$\mathcal{V}_i : I_{1,i} \times I_2 \times I_3 \longrightarrow V_i \subset \mathcal{K}_i^\perp,$$

with $\mathcal{V}_i(\lambda_i, 0, 0) = 0$, such that by canonically identifying the open product $I_2 \times V_i$ to an open neighborhood of $0 \in H_0^2(\omega)$, the two following conditions are equivalent

$$(\lambda, \tilde{\xi}, \delta) \in I_{1,i} \times (I_2 \times V_i) \times I_3 \text{ solution of the equation } (\mathscr{E}), \tag{3.7.20}$$

$$(\lambda, \epsilon, \delta) \in I_{1,i} \times I_2 \times I_3 \text{ solution of the equation } (E). \tag{3.7.21}$$

Moreover

$$\partial_\lambda \mathcal{V}_i(\lambda_i, 0, 0) = 0 \in \mathcal{K}_i^\perp, \tag{3.7.22}$$

$$\partial_\delta \mathcal{V}_i(\lambda_i, 0, 0) = (I - \lambda_i L_2 - P_i L_1)^{-1} P_i \tilde{\mathfrak{F}} \in \mathcal{K}_i^\perp, \tag{3.7.23}$$

$$\partial_\epsilon \mathcal{V}_i(\lambda, \epsilon, \delta) = \Theta_i \left(\lambda, \epsilon\phi_i + \mathcal{V}_i(\lambda, \epsilon, \delta)\right) \in \mathcal{K}_i^\perp, \tag{3.7.24}$$

for any $(\lambda, \epsilon, \delta) \in I_{i,1} \times I_2 \times I_3$, and in particular

$$\partial_\epsilon \mathcal{V}_i(\lambda_i, 0, 0) = \left[I - \lambda_i L_2 - P_i L_1\right]^{-1} P_i L_1 \phi_i \in \mathcal{K}_i^\perp. \tag{3.7.25}$$

Finally, the mapping $\mathscr{P}_i \in \mathscr{C}^\infty$:

$$\begin{aligned} \mathscr{P}_i : I_{1,i} \times I_2 \times I_3 &\longrightarrow I_{1,i} \times (I_2 \times V_i) \times I_3, \\ (\lambda, \epsilon, \delta) &\longrightarrow \left(\lambda, \epsilon\phi_i + \mathcal{V}_i(\lambda, \epsilon, \delta), \delta\right), \end{aligned} \tag{3.7.26}$$

is an immersion at $(\lambda_i, 0, 0)$, injective in $I_{1,i} \times I_2 \times I_3$ and realized a bijection between the triples $(\lambda, \epsilon, \delta) \in I_{1,i} \times I_2 \times I_3$ which is solutions of the equation (E) and the triples $(\lambda, \tilde{\xi}, \delta) \in I_{1,i} \times (I_2 \times V_i) \times I_3$ which is solutions of the equation $(\mathscr{E})$.

Proof. [3] Let the triple $(\lambda, \tilde{\xi}, \delta) \in \mathbb{R} \times H_0^2(\omega) \times \mathbb{R}$ be a solution of the equation $(\mathscr{E})$. Using the decomposition of the space $H_0^2(\omega)$ in the direct sum of spaces $\mathcal{K}_i$ and $\mathcal{K}_i^\perp$, we obtain

$$\tilde{\xi} = \epsilon\phi_i + \mathcal{V}; \quad \epsilon \in \mathbb{R},\ \mathcal{V} \in \mathcal{K}_i^\perp.$$

The projection of the equation $(\mathscr{E})$ on the spaces $\mathcal{K}_i$ and $\mathcal{K}_i^\perp$ by P_i, gives the following equivalent equations

$$(1 - \frac{\lambda}{\lambda_i})\epsilon + (C(\epsilon\phi_i + \mathcal{V}), \phi_i)_\Delta - (L_1(\epsilon\phi_i + \mathcal{V}), \phi_i)_\Delta - \delta(\tilde{\mathfrak{F}}, \phi_i)_\Delta = 0, \tag{3.7.27}$$

$$(I - \lambda L_2)\mathcal{V} + P_i C(\epsilon\phi_i + \mathcal{V}) - P_i L_1(\epsilon\phi_i + \mathcal{V}) - \delta P_i \tilde{\mathfrak{F}} = 0. \tag{3.7.28}$$

Since $I - \lambda L_2 + P_i C' \tilde{\xi} - P_i L_1 \in Isom(\mathcal{K}_i^\perp)$, we can apply the implicit function theorem in the neighborhood of $(\lambda_i, 0, 0) \in \mathbb{R} \times \mathcal{K}_i^\perp \times \mathbb{R}$. We conclude that there exist a connected open neighborhood V_i of 0 in $\mathcal{K}_i^\perp$, an open neighborhood $I_{1,i}$ containing λ_i, two open intervals I_2, I_3 containing 0 and a unique mapping $\mathcal{V}_i$ of class $\mathscr{C}^\infty$:

$$\mathcal{V}_i : I_{1,i} \times I_2 \times I_3 \to V_i,$$

satisfies $\mathcal{V}_i(\lambda_i, 0, 0) = 0$ (since $\lambda = \lambda_i, \epsilon = 0, \delta = 0, \mathcal{V} = 0$ is solution of the equation (3.7.28)), such that in the open $I_{1,i} \times I_2 \times V_i \times I_3$ the equation (3.7.28) is equivalent to the conditions

$$(\lambda, \epsilon, \delta) \in I_{1,i} \times I_2 \times I_3, \tag{3.7.29}$$

$$\mathcal{V} = \mathcal{V}_i(\lambda, \epsilon, \delta) \in V_i. \tag{3.7.30}$$

Substitution of (3.7.30) into (3.7.27) gives the equivalence between the conditions (3.7.20) and (3.7.21). Since we have the possibility of taking the open $I_{1,i} \times (I_2 \times V_i)$ is contained in the open $\tilde{\mathcal{U}}_i$. Then if we replace v in the equation (3.7.28) with the mapping $\mathcal{V}_i$, we obtain

$$(I - \lambda L_2)\, \mathcal{V}_i(\lambda, \epsilon, \delta) + P_i C\left(\epsilon\phi_i + \mathcal{V}_i(\lambda, \epsilon, \delta)\right) - P_i L_1\left(\epsilon\phi_i + \mathcal{V}_i(\lambda, \epsilon, \delta)\right) - \delta P_i \tilde{\mathfrak{F}} = 0. \tag{3.7.31}$$

Using the partial derivatives of (3.7.31) with respect to the variables λ, δ and ϵ respectively, we obtain

$$\left[I - \lambda L_2 + P_i C'\left(\epsilon\phi_i + \mathcal{V}_i(\lambda, \epsilon, \delta)\right) - P_i L_1\right] \partial_\lambda \mathcal{V}_i(\lambda, \epsilon, \delta) = L_2\left(\mathcal{V}_i(\lambda, \epsilon, \delta)\right). \tag{3.7.32}$$

Since $\mathcal{V}_i(\lambda_i, 0, 0) = 0$ implies (3.7.22).

$$\partial_\delta \mathcal{V}_i(\lambda, \epsilon, \delta) = \left[I - \lambda L_2 + P_i C'\left(\epsilon\phi_i + \mathcal{V}_i(\lambda, \epsilon, \delta)\right) - P_i L_1\right]^{-1} P_i \tilde{\mathfrak{F}}.$$

Since $\mathcal{V}_i(\lambda_i, 0, 0) = 0$ and $C'(0) = 0$ imply (3.7.23).

$$\begin{aligned} \partial_\epsilon \mathcal{V}_i(\lambda, \epsilon, \delta) = &- \left[I - \lambda L_2 + P_i C'\left(\epsilon\phi_i + \mathcal{V}_i(\lambda, \epsilon, \delta)\right) - P_i L_1\right]^{-1} \left[P_i C'(\epsilon\phi_i\right. \\ &\left. + \mathcal{V}_i(\lambda, \epsilon, \delta)) - P_i L_1\right]\phi_i \\ = &\ \varTheta_i\left(\lambda, \epsilon\phi_i + \mathcal{V}_i(\lambda, \epsilon, \delta)\right). \end{aligned}$$

Since $\mathcal{V}_i(\lambda_i, 0, 0) = 0$ and $C'(0) = 0$ immediately imply (3.7.25).

Now we will prove the properties of the mapping $\mathscr{P}_i$.

Assuming that

$$\mathscr{P}_i(\lambda, \epsilon, \delta) = \mathscr{P}_i(\lambda', \epsilon', \delta'),$$

for $(\lambda, \epsilon, \delta)$, $(\lambda', \epsilon', \delta') \in I_{1,i} \times I_2 \times I_3$.

Then the definition of the mapping $\mathscr{P}_i$ implies that

$$\begin{cases} \lambda = \lambda', \\ \delta = \delta', \\ \epsilon\phi_i + \mathcal{V}_i(\lambda, \epsilon, \delta) = \epsilon'\phi_i + \mathcal{V}_i(\lambda', \epsilon', \delta'). \end{cases}$$

Since $\mathcal{V}_i \in \mathcal{K}_i^{\perp}$, then the projection of the two terms of the last equation on $\mathcal{K}_i$ gives $\epsilon = \epsilon'$, which proves the injectivity of $\mathscr{P}_i$.

Moreover the equivalence between the (3.7.20) and (3.7.21) shows that the image of the mapping $\mathscr{P}_i$ constitute the solution of the equation $(\mathscr{E})$ in the open $I_{1,i} \times (I_2 \times V_i) \times I_3$. Therefore the injectivity of $\mathscr{P}_i$ yields the bijectivity of $\mathscr{P}_i$.

From the explicit form of the derivative of $\mathscr{P}_i$ at $(\lambda_i, 0, 0)$, we obtain

$$\begin{aligned} \mathscr{P}_i'(\lambda_i, 0, 0).(x, \zeta, y) &= \mathscr{P}_i(\lambda_i + x, \zeta, y) - \mathscr{P}_i(\lambda_i, 0, 0) \\ &= (x, \zeta\phi_i + \partial_\lambda \mathcal{V}_i(\lambda_i, 0, 0)x + \partial_\epsilon \mathcal{V}_i(\lambda_i, 0, 0)\zeta \\ &\quad + \partial_\delta \mathcal{V}_i(\lambda_i, 0, 0)y, y). \end{aligned} \tag{3.7.33}$$

Use of (3.7.22),(3.7.23) and (3.7.25) in (3.7.33) gives

$$\begin{aligned}\mathscr{P}_i'(\lambda_i,0,0).(x,\zeta,y) = (x, \zeta(\phi_i + \left[I - \lambda L_2 - P_i L_1\right]^{-1} P_i L_1 \phi_i) \\ + y[I - \lambda_i L_2 - L_1]^{-1} P_i \tilde{\mathfrak{F}}, y).\end{aligned}$$

So that $\mathscr{P}_i'(\lambda_i,0,0)$ is injective, which immediately implies that the application $\mathscr{P}_i$ is immersion at $(\lambda_i,0,0)$. □

The Lemma (3.7.1) remains valid if we restricted the open $\tilde{\mathcal{U}_i}$ arbitrary, and for this reason, we can assume that

$$\tilde{\mathcal{U}_i} = I_{1,i} \times (I_2 \times V_i), \tag{3.7.34}$$

and we pose

$$\mathcal{U}_i = I_{1,i} \times I_2. \tag{3.7.35}$$

Similar arguments as in the Lemma (3.7.2) show that for fixed $\delta \in I_3$ the mapping $\mathscr{P}_{i,\delta}$ is a bijection between the solution of the equation (E_δ) in $\mathcal{U}_i$ and the solution of the equation $(\mathscr{E}_\delta)$ in $\tilde{\mathcal{U}_i}$.

Proof of the Theorem 3.7.1. [3]

Let $(\lambda,\epsilon,\delta) \in \mathcal{U}_i \times I_3$ be the $\epsilon-$singular solution of the equation (E).

Using (3.7.5) and (3.7.6), we obtain

$$\begin{aligned}\partial_\epsilon \mathcal{R}_i(\lambda,\epsilon,\delta) &= 1 - \frac{\lambda}{\lambda_i} + \left(C'\left(\epsilon\phi_i + \mathcal{V}_i(\lambda,\epsilon,\delta)\right).\left(\phi_i + \partial_\epsilon \mathcal{V}_i(\lambda,\epsilon,\delta)\right), \phi_i\right)_\Delta \\ &- \left(L_1'\left(\epsilon\phi_i + \mathcal{V}_i(\lambda,\epsilon,\delta)\right).\left(\phi_i + \partial_\epsilon \mathcal{V}_i(\lambda,\epsilon,\delta)\right), \phi_i\right)_\Delta \\ &= 0,\end{aligned} \tag{3.7.36}$$

Taking into account (3.7.24) and since L_1 is a linear operator, we have

$$\begin{aligned}\partial_\epsilon \mathcal{R}_i(\lambda,\epsilon,\delta) &= 1 - \frac{\lambda}{\lambda_i} + (C'(\epsilon\phi_i + \mathcal{V}_i(\lambda,\epsilon,\delta)).(\phi_i + \varTheta_i(\lambda, \epsilon\phi_i + \mathcal{V}_i(\lambda,\epsilon,\delta))), \phi_i)_\Delta \\ &- \left(L_1\left(\phi_i + \varTheta_i\left(\lambda, \epsilon\phi_i + \mathcal{V}_i(\lambda,\epsilon,\delta)\right)\right), \phi_i\right)_\Delta \\ &= 0.\end{aligned} \tag{3.7.37}$$

Accordingly to the Lemma (3.7.1) we deduce that the solution

$$\mathscr{P}_i(\lambda,\epsilon,\delta) = \left(\lambda, \epsilon\phi_i + \mathcal{V}_i(\lambda,\epsilon,\delta), \delta\right),$$

of the equation $(\mathscr{E})$ is $\tilde{\xi}-$singular.

Reciprocally, if $(\lambda,\tilde{\xi},\delta) \in \tilde{\mathcal{U}_i} \times I_3$ is solution of the equation $(\mathscr{E})$. Accordingly to the Lemma (3.7.2), we have

$$(\lambda,\tilde{\xi},\delta) = \mathscr{P}_i(\lambda,\epsilon,\delta),$$

with $\epsilon = (\tilde{\xi},\phi_i)_\Delta$.

Moreover if $(\lambda,\tilde{\xi},\delta)$ is $\tilde{\xi}-$singular solution, applying Lemma (3.7.1), it follows from (3.7.9) that (3.7.36) holds, which yields that $(\lambda,\epsilon,\delta) \in \mathcal{U}_i \times I_3$ is $\epsilon-$singular solution of the equation (E).

Of course, we immediately obtain that $\epsilon-$singular solutions of equation (E_δ) in $\mathcal{U}_i$ and $\tilde{\xi}$-singular solutions of equation $(\mathscr{E}_\delta)$ in $\tilde{\mathcal{U}_i}$ are bijective by the mapping $\mathscr{P}_{i,\delta}$. □

3.7.3 Study of $\epsilon-$singular solutions of equations (E) and (E_δ)

According the definition of the mapping $\mathcal{R}_i$ and the equation (E), also from the equivalence between (3.7.8) and (3.7.9), we conclude that, the $\epsilon-$ singular solutions of the equation (E) are the set of triples $(\lambda, \epsilon, \delta) \in \mathcal{U}_i \times I_3$ which satisfies:

$$\begin{cases} \mathcal{R}_i(\lambda, \epsilon, \delta) = 0, \\ \partial_\epsilon \mathcal{R}_i(\lambda, \epsilon, \delta) = 0. \end{cases} \tag{3.7.38}$$

We will introduce the mapping $\mathcal{S}_i$ of class $\mathscr{C}^\infty$

$$\mathcal{S}_i : \mathcal{U}_i \times I_3 \longrightarrow \mathbb{R}^2,$$

defined by

$$\mathcal{S}_i = (\mathcal{R}_i(\lambda, \epsilon, \delta), \partial_\epsilon \mathcal{R}_i(\lambda, \epsilon, \delta)),$$

so that the system (3.7.38) can be written as follows

$$\mathcal{S}_i(\lambda, \epsilon, \delta) = 0. \tag{3.7.39}$$

We recall that, we need to study the non trivial solutions bifurcating from the trivial solution at neighborhood of λ_i.

Clearly, $\mathcal{R}_i(\lambda_i, 0, 0) = 0$. But $\partial_\epsilon \mathcal{R}_i(\lambda_i, 0, 0) = 0$ implies that, the operator L_1 satisfies the following necessary condition

$$L_1 \left(\phi_i + \Theta_i(\lambda_i, 0)\right) \in \mathcal{K}_i^\perp \quad (i.e., \left(L_1 \left(\phi_i + \Theta_i(\lambda_i, 0)\right), \phi_i\right)_\Delta = 0). \tag{$C1$}$$

Lemma 3.7.3. *[3] Assume that the following conditions*

$$\left(L_1 \left(\partial_\delta \mathcal{V}_i(\lambda_i, 0, 0)\right), \phi_i\right)_\Delta + (\tilde{\mathfrak{F}}, \phi_i)_\Delta \neq 0, \tag{$C2$}$$

and

$$\frac{1}{\lambda_i} + \left(L_1 \left(\partial_\lambda \Theta_i(\lambda_i, 0)\right), \phi_i\right)_\Delta \neq 0, \tag{$C3$}$$

and the condition ($C1$) are realized. Then the set of solutions of the system (3.7.38) in the open $\mathcal{U}_i \times I_3$ is a curve of class $\mathscr{C}^\infty$ parametrized by ϵ.

Proof. [3]

We prove that the mapping $\mathcal{S}_i$ is submersion in $(\lambda_i, 0, 0) \in \mathcal{U}_i \times I_3$.

The Jacobian matrix of $\mathcal{S}_i$ at $(\lambda_i, 0, 0)$ is as follows

$$Jac\, \mathcal{S}_i(\lambda_i, 0, 0) = \begin{pmatrix} \partial_\lambda \mathcal{R}_i(\lambda_i, 0, 0) & \partial_\epsilon \mathcal{R}_i(\lambda_i, 0, 0) & \partial_\delta \mathcal{R}_i(\lambda_i, 0, 0) \\ \partial^2_{\lambda\epsilon} \mathcal{R}_i(\lambda_i, 0, 0) & \partial^2_{\epsilon\epsilon} \mathcal{R}_i(\lambda_i, 0, 0) & \partial^2_{\delta\epsilon} \mathcal{R}_i(\lambda_i, 0, 0) \end{pmatrix}.$$

We have

$$\begin{aligned} \partial_\lambda \mathcal{R}_i(\lambda, \epsilon, \delta) = -\frac{\epsilon}{\lambda_i} &+ \left(C'\left(\epsilon\phi_i + \mathcal{V}_i(\lambda, \epsilon, \delta)\right).\partial_\lambda \mathcal{V}_i(\lambda, \epsilon, \delta), \phi_i\right)_\Delta \\ &- \left(L_1 \left(\partial_\lambda \mathcal{V}_i(\lambda, \epsilon, \delta)\right), \phi_i\right)_\Delta, \end{aligned} \tag{3.7.40}$$

since $\partial_\lambda \mathcal{V}_i(\lambda_i, 0, 0) = 0$ it follows that

$$\partial_\lambda \mathcal{R}_i(\lambda_i, 0, 0) = 0. \tag{3.7.41}$$

From (3.7.37) and the condition $(C1)$, and since $C'(0)=0$, we obtain

$$\partial_\epsilon \mathcal{R}_i(\lambda_i,0,0)=0. \tag{3.7.42}$$

$$\begin{aligned}\partial_\delta \mathcal{R}_i(\lambda,\epsilon,\delta)&=\left(C'\left(\epsilon\phi_i+\mathcal{V}_i(\lambda,\epsilon,\delta)\right).\partial_\delta \mathcal{V}_i(\lambda,\epsilon,\delta),\phi_i\right)_\Delta\\&\quad-\left(L_1\left(\partial_\delta \mathcal{V}_i(\lambda,\epsilon,\delta)\right),\phi_i\right)_\Delta-(\tilde{\mathfrak{F}},\phi_i)_\Delta,\end{aligned}$$

since $C'(0)=0$ it follows that

$$\partial_\delta \mathcal{R}_i(\lambda_i,0,0)=-\left(L_1\left(\partial_\delta \mathcal{V}_i(\lambda_i,0,0)\right),\phi_i\right)_\Delta-(\tilde{\mathfrak{F}},\phi_i)_\Delta. \tag{3.7.43}$$

Also, we have

$$\begin{aligned}\partial^2_{\lambda\epsilon}\mathcal{R}_i(\lambda,\epsilon,\delta)=-\frac{1}{\lambda_i}&+(\partial_\lambda \mathcal{V}_i(\lambda,\epsilon,\delta)C''(\epsilon\phi_i+\mathcal{V}_i(\lambda,\epsilon,\delta)).(\phi_i+\Theta_i(\lambda,\epsilon\phi_i+\mathcal{V}_i(\lambda,\epsilon,\delta)))\\&+C'(\epsilon\phi_i+\mathcal{V}_i(\lambda,\epsilon,\delta)).\partial_\lambda\Theta_i(\lambda,\epsilon\phi_i+\mathcal{V}_i(\lambda,\epsilon,\delta)),\phi_i)_\Delta\\&-(L_1(\partial_\lambda\Theta_i\left(\lambda,\epsilon\phi_i+\mathcal{V}_i(\lambda,\epsilon,\delta)\right)),\phi_i)_\Delta,\end{aligned}$$

since $C'(0)=0$ and $C''(0)=0$ and also $\partial_\lambda \mathcal{V}_i(\lambda_i,0,0)=0$, it follows that

$$\partial^2_{\lambda\epsilon}\mathcal{R}_i(\lambda_i,0,0)=-\frac{1}{\lambda_i}-\left(L_1\left(\partial_\lambda\Theta_i(\lambda_i,0)\right),\phi_i\right)_\Delta. \tag{3.7.44}$$

$$\begin{aligned}\partial^2_{\epsilon\epsilon}\mathcal{R}_i(\lambda,\epsilon,\delta)&=((\phi_i+\partial_\epsilon \mathcal{V}_i(\lambda,\epsilon,\delta))C''(\epsilon\phi_i+\mathcal{V}_i(\lambda,\epsilon,\delta)).(\phi_i+\Theta_i(\lambda,\epsilon\phi_i+\mathcal{V}_i(\lambda,\epsilon,\delta)))\\&\quad+C'(\epsilon\phi_i+\mathcal{V}_i(\lambda,\epsilon,\delta)).\partial_\epsilon\Theta_i(\lambda,\epsilon\phi_i+\mathcal{V}_i(\lambda,\epsilon,\delta)),\phi_i)_\Delta\\&\quad-(L_1(\partial_\epsilon\Theta_i(\lambda,\epsilon\phi_i+\mathcal{V}_i(\lambda,\epsilon,\delta))),\phi_i)_\Delta,\end{aligned} \tag{3.7.45}$$

since $C'(0)=0$ and $C''(0)=0$ it follows that

$$\partial^2_{\epsilon\epsilon}\mathcal{R}_i(\lambda_i,0,0)=-\left(L_1\left(\partial_\epsilon\Theta_i(\lambda_i,0)\right),\phi_i\right)_\Delta. \tag{3.7.46}$$

$$\begin{aligned}\partial^2_{\delta\epsilon}\mathcal{R}_i(\lambda,\epsilon,\delta)&=(\partial_\delta \mathcal{V}_i(\lambda,\epsilon,\delta)C''(\epsilon\phi_i+\mathcal{V}_i(\lambda,\epsilon,\delta)).(\phi_i+\Theta_i(\lambda,\epsilon\phi_i+\mathcal{V}_i(\lambda,\epsilon,\delta)))\\&\quad+C'(\epsilon\phi_i+\mathcal{V}_i(\lambda,\epsilon,\delta)).\partial_\delta\Theta_i(\lambda,\epsilon\phi_i+\mathcal{V}_i(\lambda,\epsilon,\delta)),\phi_i)_\Delta\\&\quad-(L_1(\partial_\delta\Theta_i\left(\lambda,\epsilon\phi_i+\mathcal{V}_i(\lambda,\epsilon,\delta)\right)),\phi_i)_\Delta,\end{aligned}$$

since $C'(0)=0$ and $C''(0)=0$ it follows that

$$\partial^2_{\delta\epsilon}\mathcal{R}_i(\lambda_i,0,0)=-\left(L_1\left(\partial_\delta\Theta_i(\lambda_i,0)\right),\phi_i\right)_\Delta. \tag{3.7.47}$$

It remains for us to determine the partial derivatives of $\Theta_i(\lambda,\epsilon\phi_i+\mathcal{V}_i(\lambda,\epsilon,\delta))$ with respect to the variables λ, ϵ and δ in order to complete the computation the Jacobian matrix of $\mathcal{S}_i$ in $(\lambda_i,0,0)$.

From (3.7.16) we obtain

$$\begin{aligned}&\left[I-\lambda L_2-P_iL_1+P_iC'\left(\epsilon\phi_i+\mathcal{V}_i(\lambda,\epsilon,\delta)\right)\right]\Theta_i\left(\lambda,\epsilon\phi_i+\mathcal{V}_i(\lambda,\epsilon,\delta)\right)\\&+P_iC'\left(\epsilon\phi_i+\mathcal{V}_i(\lambda,\epsilon,\delta)\right).\phi_i-P_iL_1\phi_i=0.\end{aligned} \tag{3.7.48}$$

The partial derivatives of (3.7.48) with respect to the variables λ, ϵ and δ respectively, and by using the relations $C'(0)=0$ and $C''(0)=0$, we get

$$\partial_\lambda\Theta_i(\lambda_i,0)=\left[I-\lambda_iL_2-P_iL_1\right]^{-1}L_2\left(\Theta_i(\lambda_i,0)\right), \tag{3.7.49}$$

$$[I - \lambda_i L_2 - P_i L_1]\, \partial_\epsilon \varTheta_i(\lambda_i, 0) = 0, \tag{3.7.50}$$

then yields

$$\partial_\epsilon \varTheta_i(\lambda_i, 0) = 0, \tag{3.7.51}$$

thus

$$\partial_{\epsilon\epsilon} \mathcal{R}_i(\lambda_i, 0, 0) = 0. \tag{3.7.52}$$

Next, we have

$$[I - \lambda_i L_2 - P_i L_1]\, \partial_\delta \varTheta_i(\lambda_i, 0) = 0, \tag{3.7.53}$$

then yields

$$\partial_\delta \varTheta_i(\lambda_i, 0) = 0, \tag{3.7.54}$$

thus

$$\partial_{\delta\epsilon} \mathcal{R}_i(\lambda_i, 0, 0) = 0. \tag{3.7.55}$$

We deduce From the previous computations, we conclude that the $Jac\ \mathcal{S}_i(\lambda_i, 0, 0)$ is as follows

$$\begin{pmatrix} 0 & 0 & -(L_1(\partial_\delta \mathcal{V}_i(\lambda_i, 0, 0)), \phi_i)_\Delta - (\tilde{\mathfrak{F}}, \phi_i)_\Delta \\ \\ -\dfrac{1}{\lambda_i} - (L_1(\partial_\lambda \varTheta_i(\lambda_i, 0)), \phi_i)_\Delta & 0 & 0 \end{pmatrix}.$$

The previous two conditions ($C2$) and ($C3$) prove that the rank of $Jac\ \mathcal{S}_i(\lambda_i, 0, 0)$ is 2, hence $\mathcal{S}_i$ is submersion at $(\lambda_i, 0, 0)$. Also, in the neighborhood of $(\lambda_i, 0, 0) \in \mathcal{U}_i \times I_3$, and From the implicit function theorem we conclude that, the ensemble of solution to the equation (3.7.39) is a submanifolds of class $\mathscr{C}^\infty$ of one dimensional. This curve parametrized by ϵ in the neighborhood of 0, which implies that in the neighborhood of 0 there exist two functions $\tilde{\lambda}_i(\epsilon)$ and $\tilde{\delta}_i(\epsilon)$ of class $\mathscr{C}^\infty$, such that the equation (3.7.39) is equivalent to the following relations

$$\lambda = \tilde{\lambda}_i(\epsilon), \quad \delta = \tilde{\delta}_i(\epsilon) \text{ with } \tilde{\lambda}_i(0) = \lambda_i, \text{ and } \tilde{\delta}_i(0) = 0. \tag{3.7.56}$$

□

Lemma 3.7.4. *[3] Assume that the conditions ($C1$), ($C2$) and the following condition*

$$(\tilde{\mathfrak{F}}, \phi_i)_\Delta \neq 0, \tag{$C4$}$$

are realized. Then we have

$$\begin{cases} \tilde{\lambda}_i'(0) = 0, \\ \tilde{\delta}_i'(0) = 0, \\ \tilde{\lambda}_i''(0) = 6\lambda_i (C\left(\phi_i + \varTheta_i(\lambda_i, 0)\right), \phi_i)_\Delta, \\ \tilde{\delta}_i''(0) = 0, \\ \tilde{\delta}_i^{(3)}(0) = -12 \dfrac{(C\left(\phi_i + \varTheta_i(\lambda_i, 0)\right), \phi_i)_\Delta}{(\tilde{\mathfrak{F}}, \phi_i)_\Delta}. \end{cases} \tag{3.7.57}$$

Proof. [3] Substituting λ and δ by $\tilde{\lambda}_i(\epsilon)$ and $\tilde{\delta}_i(\epsilon)$ respectively in the second equation of (3.7.38) we get

$$\begin{aligned} &\frac{1}{\lambda_i} . \left(\tilde{\lambda}_i(\epsilon) - \lambda_i \{ 1 - (L_1(\phi_i + \varTheta_i(\tilde{\lambda}_i(\epsilon), \epsilon\phi_i + \mathcal{V}_i(\tilde{\lambda}_i(\epsilon), \epsilon, \tilde{\delta}_i(\epsilon)))), \phi_i)_\Delta \} \right) \\ &= (C'(\epsilon\phi_i + \mathcal{V}_i(\tilde{\lambda}_i(\epsilon), \epsilon, \tilde{\delta}_i(\epsilon))).\{\phi_i + \varTheta_i(\tilde{\lambda}_i(\epsilon), \epsilon\phi_i + \mathcal{V}_i(\tilde{\lambda}_i(\epsilon), \epsilon, \tilde{\delta}_i(\epsilon)))\}, \phi_i)_\Delta, \end{aligned} \tag{3.7.58}$$

for all $\epsilon \in I_2$.

Dividing (3.7.58) by $\epsilon^2 \neq 0$ and since the mapping $C' : H_0^2(\omega) \longrightarrow \mathscr{L}(H_0^2(\omega), \mathbb{R})$ is homogeneous of degree two we fined

$$\begin{aligned}&\frac{1}{\epsilon^2\lambda_i}.\left(\tilde{\lambda}_i(\epsilon) - \lambda_i\{1 - (L_1(\phi_i + \mathit{\Theta}_i(\tilde{\lambda}_i(\epsilon), \epsilon\phi_i + \mathcal{V}_i(\tilde{\lambda}_i(\epsilon), \epsilon, \tilde{\delta}_i(\epsilon)))), \phi_i)_\Delta\}\right)\\ &= (C'(\phi_i + \frac{\mathcal{V}_i(\tilde{\lambda}_i(\epsilon), \epsilon, \tilde{\delta}_i(\epsilon))}{\epsilon}).(\phi_i + \mathit{\Theta}_i(\tilde{\lambda}_i(\epsilon), \epsilon\phi_i + \mathcal{V}_i(\tilde{\lambda}_i(\epsilon), \epsilon, \tilde{\delta}_i(\epsilon)))), \phi_i)_\Delta,\end{aligned} \tag{3.7.59}$$

We have

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\mathcal{V}_i\left(\tilde{\lambda}_i(\epsilon), \epsilon, \tilde{\delta}_i(\epsilon)\right) = \tilde{\lambda}_i'(0)\partial_\lambda\mathcal{V}_i(\lambda_i, 0, 0) + \partial_\epsilon\mathcal{V}_i(\lambda_i, 0, 0) + \tilde{\delta}_i'(0)\partial_\delta\mathcal{V}_i(\lambda_i, 0, 0). \tag{3.7.60}$$

Using the condition $(C1)$ and the relations (3.7.22), (3.7.23) and (3.7.25), we obtain

$$\begin{aligned}\lim_{\epsilon\to 0}\frac{\tilde{\lambda}_i(\epsilon) - \lambda_i}{\epsilon^2\lambda_i} =&(C'(\phi_i + \mathit{\Theta}_i(\lambda_i, 0) + \tilde{\delta}_i'(0)(I - \lambda_i L_2 - P_i L_1)^{-1}P_i\tilde{\mathfrak{F}})\\ &.\left(\phi_i + \mathit{\Theta}_i(\lambda_i, 0)\right), \phi_i)_\Delta.\end{aligned} \tag{3.7.61}$$

Using the Taylor expansion of $\tilde{\lambda}_i(\epsilon)$ at 0, we conclude that

$$\tilde{\lambda}_i'(0) = 0, \tag{3.7.62}$$

and

$$\begin{aligned}\tilde{\lambda}_i''(0) = 2\lambda_i(&C'(\phi_i + \mathit{\Theta}_i(\lambda_i, 0) + \tilde{\delta}_i'(0)(I - \lambda_i L_2 - P_i L_1)^{-1}P_i\tilde{\mathfrak{F}})\\ &.\left(\phi_i + \mathit{\Theta}_i(\lambda_i, 0)\right), \phi_i)_\Delta.\end{aligned} \tag{3.7.63}$$

Similarly, replacing λ and δ by $\tilde{\lambda}_i(\epsilon)$ and $\tilde{\delta}_i(\epsilon)$ respectively in the first equation of (3.7.38) we find that

$$\begin{aligned}\tilde{\delta}_i(\epsilon)(\tilde{\mathfrak{F}}, \phi_i)_\Delta = &-\frac{1}{\lambda_i}.\left(\epsilon\tilde{\lambda}_i(\epsilon) - \lambda_i\{\epsilon - (L_1(\epsilon\phi_i + \mathcal{V}_i(\tilde{\lambda}_i(\epsilon), \epsilon, \tilde{\delta}_i(\epsilon))), \phi_i)_\Delta\}\right)\\ &+ \left(C\left(\epsilon\phi_i + \mathcal{V}_i(\tilde{\lambda}_i(\epsilon), \epsilon, \tilde{\delta}_i(\epsilon))\right), \phi_i\right)_\Delta.\end{aligned} \tag{3.7.64}$$

Dividing (3.7.64) by $\epsilon^3 \neq 0$ and since the operator C is homogeneous of degree 3 we obtain

$$\begin{aligned}\frac{\tilde{\delta}_i(\epsilon)}{\epsilon^3}(\tilde{\mathfrak{F}}, \phi_i)_\Delta = &-\frac{1}{\epsilon^2\lambda_i}.\left(\tilde{\lambda}_i(\epsilon) - \lambda_i\{1 - (L_1(\phi_i + \frac{\mathcal{V}_i(\tilde{\lambda}_i(\epsilon), \epsilon, \tilde{\delta}_i(\epsilon))}{\epsilon}), \phi_i)_\Delta\}\right)\\ &+ \left(C(\phi_i + \frac{\mathcal{V}_i(\tilde{\lambda}_i(\epsilon), \epsilon, \tilde{\delta}_i(\epsilon))}{\epsilon}), \phi_i\right)_\Delta,\end{aligned} \tag{3.7.65}$$

Then the condition $(C4)$ implies that the limit of $\dfrac{\tilde{\delta}_i(\epsilon)}{\epsilon^3}$ does exist.

Passing to the limit in (3.7.65), using Taylor expansion at 0 of $\tilde{\delta}_i(\epsilon)$ and taking into account (3.7.60), $(C1)$ and $(C2)$ show that

$$\tilde{\delta}_i'(0) = -\frac{\left(L_1\left(\phi_i + \mathit{\Theta}_i(\lambda_i, 0)\right), \phi_i\right)_\Delta}{(\tilde{\mathfrak{F}}, \phi_i)_\Delta + (L_1((I - \lambda_i L_2 - P_i L_1)^{-1}P_i\tilde{\mathfrak{F}}), \phi_i)_\Delta} = 0, \tag{3.7.66}$$

and

$$\tilde{\delta}_i''(0) = 0. \tag{3.7.67}$$

Use of $\tilde{\delta}_i'(0) = 0$ in (3.7.63) gives

$$\tilde{\lambda}_i''(0) = 2\lambda_i (C'\left(\phi_i + \Theta_i(\lambda_i, 0)\right).\left(\phi_i + \Theta_i(\lambda_i, 0)\right), \phi_i)_\Delta. \tag{3.7.68}$$

Taking into account the following relation (C is homogeneous operator of degree 3)

$$(C'\left(\phi_i + \Theta_i(\lambda_i, 0)\right).\left(\phi_i + \Theta_i(\lambda_i, 0)\right), \phi_i)_\Delta = 3(C\left(\phi_i + \Theta_i(\lambda_i, 0)\right), \phi_i)_\Delta, \tag{3.7.69}$$

this gives the third relation in (3.7.57).

We have

$$\lim_{\epsilon \to 0} \frac{\tilde{\delta}_i(\epsilon)}{\epsilon^3} = \lim_{\epsilon \to 0} \frac{\tilde{\delta}_i'(0)}{\epsilon^2} + \frac{1}{6}\tilde{\delta}_i^{(3)}(0), \tag{3.7.70}$$

hence

$$\tilde{\delta}_i^{(3)}(0) = \frac{6}{(\tilde{\mathfrak{F}}, \phi_i)_\Delta}\left[(C\left(\phi_i + \Theta_i(\lambda_i, 0)\right), \phi_i)_\Delta - \lim_{\epsilon \to 0}\frac{\tilde{\lambda}_i(\epsilon) - \lambda_i}{\epsilon^2 \lambda_i}\right]. \tag{3.7.71}$$

It follows from (3.7.61) that

$$\begin{aligned}\tilde{\delta}_i^{(3)}(0) = \frac{6}{(\tilde{\mathfrak{F}}, \phi_i)_\Delta}\big[&(C\left(\phi_i + \Theta_i(\lambda_i, 0)\right), \phi_i)_\Delta - (C'\left(\phi_i + \Theta_i(\lambda_i, 0)\right)\\ &.\left(\phi_i + \Theta_i(\lambda_i, 0)\right), \phi_i)_\Delta\big].\end{aligned} \tag{3.7.72}$$

Use of (3.7.69) in (3.7.72) gives the last relation in (3.7.57).

□

Proposition 3.7.2. *[3] Assume that the following condition*

$$(C\left(\phi_i + \Theta_i(\lambda_i, 0)\right), \phi_i)_\Delta \neq 0, \tag{C5}$$

and the conditions ($C1$),($C2$),($C3$) and($C4$) are realized. Then, the equation (E_δ) has a unique $\epsilon-$singular solution in $\mathcal{U}_i$ for fixed $\delta \in I_3$, denoted $(\lambda_i^*(\delta), \epsilon_i^*(\delta))$. Moreover, we have

$$\lambda_i^*(0) = \lambda_i, \ \epsilon_i^*(0) = 0. \tag{3.7.73}$$

For $\delta \in I_3 \backslash \{0\}$, we have

$$sign\left(\epsilon_i^*(\delta)\right) = -sign\left(\delta(\tilde{\mathfrak{F}}, \phi_i)_\Delta \times (C\left(\phi_i + \Theta_i(\lambda_i, 0)\right), \phi_i)_\Delta\right), \tag{3.7.74}$$

$$sign\left(\lambda_i^*(\delta) - \lambda_i\right) = sign\left((C\left(\phi_i + \Theta_i(\lambda_i, 0)\right), \phi_i)_\Delta\right). \tag{3.7.75}$$

Proof. [3] The condition ($C5$) implies that the mapping $\tilde{\delta}_i$ satisfies

$$\tilde{\delta}_i^{(3)}(0) \neq 0. \tag{3.7.76}$$

Taylor expansion at 0 of $\tilde{\delta}_i(\epsilon)$ gives

$$\tilde{\delta}_i(\epsilon) = \frac{1}{6}\tilde{\delta}_i^{(3)}(0)\epsilon^3\left(1 + R(\epsilon)\right), \tag{3.7.77}$$

where $R(\epsilon) : I_2 \longrightarrow \mathbb{R}$ is a $\mathscr{C}^\infty$ function satisfies

$$R(\epsilon) = 0(\epsilon), \tag{3.7.78}$$

in the neighborhood of $\epsilon = 0$.

The characteristic of mapping $\tilde{\delta}_i(\epsilon)$ in Lemma (3.7.4) shows that, the finding an $\epsilon-$singular solution of the equation (E_δ) in $\mathcal{U}_i$ for fixed $\delta \in I_3$ is equivalent to solving the following equation

$$\tilde{\delta}_i(\epsilon) = \delta. \tag{3.7.79}$$

From (3.7.77) and the last relation in (3.7.57), we obtain

$$\epsilon^3 \left(1 + R(\epsilon)\right) = -\frac{\delta(\tilde{\mathfrak{F}}, \phi_i)_\Delta}{2(C\left(\phi_i + \Theta_i(\lambda_i, 0)\right), \phi_i)_\Delta}. \tag{3.7.80}$$

Let

$$g(\epsilon) = \epsilon^3 \left(1 + R(\epsilon)\right), \tag{3.7.81}$$

is a function of class $\mathscr{C}^\infty$ and strictly increasing function in the neighborhood of $\epsilon = 0$. For $\epsilon \in I_2$ and $\delta \in I_3$ are small enough, the equation (3.7.80) has a unique solution, denoted $\epsilon_i^*(\delta)$ which satisfies (3.7.74).

Also, the characteristic of mapping $\tilde{\lambda}_i(.)$ in Lemma (3.7.4) shows that, the $\epsilon-$singular solution of the equation (E_δ) is the following couple

$$\left(\lambda_i^*(\delta), \epsilon_i^*(\delta)\right),$$

where $\lambda_i^*(\delta) = \tilde{\lambda}_i\left(\epsilon_i^*(\delta)\right)$.

Taylor expansion of $\tilde{\lambda}_i(\epsilon)$ in the neighborhood of $\epsilon = 0$ gives

$$\tilde{\lambda}_i(\epsilon) = \lambda_i + 3\lambda_i(C\left(\phi_i + \Theta_i(\lambda_i, 0)\right), \phi_i)_\Delta \epsilon^2 + o(\epsilon^3). \tag{3.7.82}$$

We can choose the interval I_2 a small enough such that

$$\tilde{\lambda}_i(\epsilon) - \lambda_i = 3\lambda_i(C\left(\phi_i + \Theta_i(\lambda_i, 0)\right), \phi_i)_\Delta \epsilon^2. \tag{3.7.83}$$

Since $\lambda_i > 0$ (see Theorem (3.4.2)), we obtain

$$sign\left(\tilde{\lambda}_i(\epsilon) - \lambda_i\right) = sign\left((C\left(\phi_i + \Theta_i(\lambda_i, 0)\right), \phi_i)_\Delta\right), \tag{3.7.84}$$

for all $\epsilon \in I_2$. Then, we obtain (3.7.75).

$\square$

We give the following definition due to Ciarlet and Rabier [29].

Definition 3.7.1. *[3] Let $\delta \in I_3$ and let $(\lambda_i^*(\delta), \epsilon_i^*(\delta)) \in \mathcal{U}_i$ be the $\epsilon-$singular solution of the equation (E_δ). The point $(\lambda_i^*(\delta), \epsilon_i^*(\delta))$ is called a nondegenerate turning point of the equation (E_δ) if*

$$\partial_\lambda \mathcal{R}_i\left(\lambda_i^*(\delta), \epsilon_i^*(\delta), \delta\right) \neq 0, \tag{3.7.85}$$

$$\partial^2_{\epsilon\epsilon} \mathcal{R}_i\left(\lambda_i^*(\delta), \epsilon_i^*(\delta), \delta\right) \neq 0. \tag{3.7.86}$$

Proposition 3.7.3. *[3] Assume that the conditions ($C1$),($C2$),($C3$),($C4$) and ($C5$) are realized. Then the $\epsilon-$singular solution $(\lambda_i^*(\delta), \epsilon_i^*(\delta))$ of the equation (E_δ) is nondegenerate turning point of this equation for all $\delta \in I_3 \backslash \{0\}$. Furthermore*

$$\begin{aligned} & sign\left(\partial_\lambda \mathcal{R}_i\left(\lambda_i^*(\delta), \epsilon_i^*(\delta), \delta\right)\right) \\ & \quad = sign\left(\delta(C\left(\phi_i + \Theta_i(\lambda_i, 0)\right), \phi_i)_\Delta \times \left((L_1\left(\partial_\delta \mathcal{V}_i(\lambda_i, 0, 0)\right), \phi_i)_\Delta + (\tilde{\mathfrak{F}}, \phi_i)_\Delta\right)\right), \end{aligned} \tag{3.7.87}$$

$$\begin{aligned} & sign\left(\partial_{\epsilon\epsilon} \mathcal{R}_i\left(\lambda_i^*(\delta), \epsilon_i^*(\delta), \delta\right)\right) = \\ & \qquad -sign\left(\delta(\tilde{\mathfrak{F}}, \phi_i)_\Delta \times \left(\frac{1}{\lambda_i} + (L_1\left(\partial_\lambda \Theta_i(\lambda_i, 0)\right), \phi_i)_\Delta\right)\right). \end{aligned} \tag{3.7.88}$$

Proof. From the relations (3.7.57), the Taylor expansion of the functions $\tilde{\lambda}_i'(\epsilon)$ and $\tilde{\delta}'(\epsilon)$ in the neighborhood of $\epsilon = 0$ gives

$$\tilde{\lambda}_i'(\epsilon) = \tilde{\lambda}_i''(0)\epsilon + o(\epsilon^2), \tag{3.7.89}$$

$$\tilde{\delta}'(\epsilon) = \frac{1}{2}\tilde{\delta}^{(3)}(0)\epsilon^2 + o(\epsilon^3). \tag{3.7.90}$$

So, we can assume that the interval I_2 is small enough such that for any $\epsilon \in I_2 \backslash \{0\}$, we have

$$\tilde{\lambda}'(\epsilon) \neq 0, \quad \tilde{\delta}'(\epsilon) \neq 0, \tag{3.7.91}$$

and

$$sign\left(\tilde{\lambda}_i'(\epsilon)\right) = sign\left(\tilde{\lambda}_i''(0)\epsilon\right), \tag{3.7.92}$$

$$sign\left(\tilde{\delta}_i'(\epsilon)\right) = sign\left(\tilde{\delta}_i^{(3)}(0)\right). \tag{3.7.93}$$

The functions $\tilde{\lambda}_i(\epsilon)$ and $\tilde{\delta}_i(\epsilon)$ satisfying

$$\mathcal{R}_i\left(\tilde{\lambda}_i(\epsilon), \epsilon, \tilde{\delta}_i(\epsilon)\right) = 0, \tag{3.7.94}$$

$$\partial_\epsilon \mathcal{R}_i\left(\tilde{\lambda}_i(\epsilon), \epsilon, \tilde{\delta}_i(\epsilon)\right) = 0, \tag{3.7.95}$$

Using the derivative of (3.7.94) and taking into account the relation (3.7.95), we obtain

$$\partial_\lambda \mathcal{R}_i\left(\tilde{\lambda}_i(\epsilon), \epsilon, \tilde{\delta}_i(\epsilon)\right) = -\partial_\delta \mathcal{R}_i\left(\tilde{\lambda}_i(\epsilon), \epsilon, \tilde{\delta}_i(\epsilon)\right) \frac{\tilde{\delta}_i'(\epsilon)}{\tilde{\lambda}_i'(\epsilon)}, \tag{3.7.96}$$

for $\epsilon \in I_2 \backslash \{0\}$.

From (3.7.43) and ($C2$) we have

$$\partial_\delta \mathcal{R}_i(\lambda_i, 0, 0) = -(L_1\left(\partial_\delta \mathcal{V}_i(\lambda_i, 0, 0)\right), \phi_i)_\Delta - (\tilde{\mathfrak{F}}, \phi_i)_\Delta \neq 0. \tag{3.7.97}$$

We next deduce from the continuity with the assumption that the interval I_2 is small enough such that

$$sign\left(-\partial_\delta \mathcal{R}_i\left(\tilde{\lambda}_i(\epsilon), \epsilon, \tilde{\delta}_i(\epsilon)\right)\right) = sign\left((L_1\left(\partial_\delta \mathcal{V}_i(\lambda_i, 0, 0)\right), \phi_i)_\Delta + (\tilde{\mathfrak{F}}, \phi_i)_\Delta\right), \tag{3.7.98}$$

for $\epsilon \in I_2$.

From (3.7.92), (3.7.93) and (3.7.96) we obtain

$$\begin{aligned} &sign\left(\partial_\lambda \mathcal{R}_i\left(\tilde{\lambda}_i(\epsilon),\epsilon,\tilde{\delta}_i(\epsilon)\right)\right)\\ &\qquad = sign\left(\left(L_1\left(\partial_\delta \mathcal{V}_i(\lambda_i,0,0)\right),\phi_i\right)_\Delta + (\tilde{\mathfrak{F}},\phi_i)_\Delta\right)\times sign\left(\frac{\tilde{\delta}_i^{(3)}(0)\epsilon}{\tilde{\lambda}_i''(0)}\right), \end{aligned} \tag{3.7.99}$$

for $\epsilon\in I_2\backslash\{0\}$.

From (3.7.57) and since $\lambda_i>0$ we have

$$\begin{aligned} &sign\left(\partial_\lambda \mathcal{R}_i\left(\tilde{\lambda}_i(\epsilon),\epsilon,\tilde{\delta}_i(\epsilon)\right)\right)\\ &\qquad = -sign\left(\frac{(L_1(\partial_\delta \mathcal{V}_i(\lambda_i,0,0)),\phi_i)_\Delta + (\tilde{\mathfrak{F}},\phi_i)_\Delta}{(\tilde{\mathfrak{F}},\phi_i)_\Delta}\right)\times sign(\epsilon). \end{aligned} \tag{3.7.100}$$

For a fixed $\delta\in I_3\backslash\{0\}$, if $(\lambda_i^*(\delta),\epsilon_i^*(\delta))$ is the $\epsilon-$singular solution of the equation (E_δ) and taking into account (3.7.74), $(C2)$, $(C4)$ and $(C5)$, the last relation (3.7.100) becomes

$$\begin{aligned} &sign\left(\partial_\lambda \mathcal{R}_i\left(\lambda_i^*(\delta),\epsilon_i^*(\delta),\delta\right)\right)\\ &= -sign\left(\frac{\left(L_1\left(\partial_\delta \mathcal{V}_i(\lambda_i,0,0)\right),\phi_i\right)_\Delta + (\tilde{\mathfrak{F}},\phi_i)_\Delta}{(\tilde{\mathfrak{F}},\phi_i)_\Delta}\right)\times sign(\epsilon_i^*(\delta))\\ &= sign\left(\frac{\left(L_1\left(\partial_\delta \mathcal{V}_i(\lambda_i,0,0)\right),\phi_i\right)_\Delta + (\tilde{\mathfrak{F}},\phi_i)_\Delta}{(\tilde{\mathfrak{F}},\phi_i)_\Delta}\right)\times sign\left(\delta(\tilde{\mathfrak{F}},\phi_i)_\Delta\times\left(\left(C\left(\phi_i+\Theta_i(\lambda_i,0)\right),\phi_i\right)_\Delta\right)\right.\\ &\quad = sign\left(\delta(C\left(\phi_i+\Theta_i(\lambda_i,0)\right),\phi_i)_\Delta\times\left(\left(L_1\left(\partial_\delta \mathcal{V}_i(\lambda_i,0,0)\right),\phi_i\right)_\Delta + (\tilde{\mathfrak{F}},\phi_i)_\Delta\right)\right)\\ &\quad \neq 0. \end{aligned} \tag{3.7.101}$$

Now using the derivative of (3.7.95) and dividing it by $\epsilon\neq 0$, it follows that

$$\begin{aligned} &\frac{1}{\epsilon}\partial^2_{\epsilon\epsilon}\mathcal{R}_i\left(\tilde{\lambda}_i(\epsilon),\epsilon,\tilde{\delta}_i(\epsilon)\right)\\ &\qquad = -\partial^2_{\lambda\epsilon}\mathcal{R}_i\left(\tilde{\lambda}_i(\epsilon),\epsilon,\tilde{\delta}_i(\epsilon)\right)\frac{\tilde{\lambda}_i'(\epsilon)}{\epsilon} - \partial^2_{\delta\epsilon}\mathcal{R}_i\left(\tilde{\lambda}_i(\epsilon),\epsilon,\tilde{\delta}_i(\epsilon)\right)\times\frac{\tilde{\delta}_i'(\epsilon)}{\epsilon}. \end{aligned} \tag{3.7.102}$$

Passing to the limit in (3.7.102), we obtain

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\partial^2_{\epsilon\epsilon}\mathcal{R}_i\left(\tilde{\lambda}_i(\epsilon),\epsilon,\tilde{\delta}_i(\epsilon)\right) = -\partial^2_{\lambda\epsilon}\mathcal{R}_i\left(\lambda_i,0,0\right)\tilde{\lambda}_i''(0) - \partial^2_{\delta\epsilon}\mathcal{R}_i\left(\lambda_i,0,0\right)\tilde{\delta}_i''(0). \tag{3.7.103}$$

Using (3.7.44) and (3.7.57), we obtain

$$\begin{aligned} &\lim_{\epsilon\to 0}\frac{1}{\epsilon}\partial^2_{\epsilon\epsilon}\mathcal{R}_i\left(\tilde{\lambda}_i(\epsilon),\epsilon,\tilde{\delta}_i(\epsilon)\right)\\ &\qquad = 6\lambda_i\left(\frac{1}{\lambda_i} + \left(L_1\left(\partial_\lambda\Theta_i(\lambda_i,0)\right),\phi_i\right)_\Delta\right)\times\left(C\left(\phi_i+\Theta_i(\lambda_i,0)\right),\phi_i\right)_\Delta. \end{aligned} \tag{3.7.104}$$

Since $\lambda_i>0$, we obtain

$$\begin{aligned} &sign\left(\partial^2_{\epsilon\epsilon}\mathcal{R}_i\left(\tilde{\lambda}_i(\epsilon),\epsilon,\tilde{\delta}_i(\epsilon)\right)\right)\\ &\quad = sign\left(\left(C\left(\phi_i+\Theta_i(\lambda_i,0)\right),\phi_i\right)_\Delta\times\left(\frac{1}{\lambda_i} + \left(L_1\left(\partial_\lambda\Theta_i(\lambda_i,0)\right),\phi_i\right)_\Delta\right)\right)\times sign(\epsilon). \end{aligned} \tag{3.7.105}$$

For a fixed $\delta \in I_3 \backslash \{0\}$, if $(\lambda_i^*(\delta), \epsilon_i^*(\delta))$ is the $\epsilon-$singular solution of the equation (E_δ) and taking into account (3.7.74), $(C3)$, $(C4)$ and $(C5)$, the last relation (3.7.105) becomes

$$\begin{aligned} & sign\left(\partial_{\epsilon\epsilon}\mathcal{R}_i\left(\lambda_i^*(\delta), \epsilon_i^*(\delta), \delta\right)\right) = \\ & \qquad - sign\left(\left(C\left(\phi_i + \varTheta_i(\lambda_i, 0)\right), \phi_i\right)_\Delta \times \left(\frac{1}{\lambda_i} + \left(L_1\left(\partial_\lambda \varTheta_i(\lambda_i, 0)\right), \phi_i\right)_\Delta\right)\right) \\ & \qquad \times sign\left(\delta(\tilde{\mathfrak{F}}, \phi_i)_\Delta \times \left(\left(C\left(\phi_i + \varTheta_i(\lambda_i, 0)\right), \phi_i\right)_\Delta\right)\right. \\ & \qquad = -sign\left(\delta(\tilde{\mathfrak{F}}, \phi_i)_\Delta \times \left(\frac{1}{\lambda_i} + \left(L_1\left(\partial_\lambda \varTheta_i(\lambda_i, 0)\right), \phi_i\right)_\Delta\right)\right) \\ & \qquad \neq 0. \end{aligned} \tag{3.7.106}$$

□

3.7.4 Solution set of the equation (E_δ)

In this section we determine the solutions of the equation (E_δ) in the case $\delta \neq 0$.

Lemma 3.7.5. *[3] Assume that the following assumptions*

$$\left(C\left(\phi_i + \varTheta_i(\lambda_i, 0)\right), \phi_i\right)_\Delta > 0, \tag{H2}$$

$$\begin{aligned} & sign\left(\left(L_1\left(\partial_\delta \mathcal{V}_i(\lambda_i, 0, 0)\right), \phi_i\right)_\Delta + (\tilde{\mathfrak{F}}, \phi_i)_\Delta\right) \\ & \qquad \times sign\left((\tilde{\mathfrak{F}}, \phi_i)_\Delta \times \left(\frac{1}{\lambda_i} + \left(L_1\left(\partial_\lambda \varTheta_i(\lambda_i, 0)\right), \phi_i\right)_\Delta\right)\right) > 0, \end{aligned} \tag{H3}$$

and the condition $(C1)$ are realized. Then in the neighborhood of the $\epsilon-$singular solution $(\lambda_i^*(\delta), \epsilon_i^*(\delta))$, the solution set of the equation (E_δ) contains one curve of class $\mathscr{C}^\infty$.

Proof. [3] Since $\epsilon-$singular solution $(\lambda_i^*(\delta), \epsilon_i^*(\delta))$ is nondegenerate turning point of the equation (E_δ), we obtain

$$\mathcal{R}_i\left(\lambda_i^*(\delta), \epsilon_i^*(\delta), \delta\right) = 0, \text{ and } \partial_\lambda \mathcal{R}_i\left(\lambda_i^*(\delta), \epsilon_i^*(\delta), \delta\right) \neq 0.$$

Then by the implicit function theorem we conclude that in the neighborhood of $(\lambda_i^*(\delta), \epsilon_i^*(\delta))$ in $\mathcal{U}_i$, the set of solutions of equation (E_δ) is given by a function $\lambda_{i,\delta}(\epsilon)$ of class $\mathscr{C}^\infty$ satisfying

$$\lambda = \lambda_{i,\delta}(\epsilon) \text{ with } \lambda_{i,\delta}\left(\epsilon_i^*(\delta)\right) = \lambda_i^*(\delta).$$

So that

$$\mathcal{R}_i\left(\lambda_{i,\delta}(\epsilon), \epsilon, \delta\right) = 0. \tag{3.7.107}$$

Using the derivative of (3.7.107) with respect to the variable ϵ, we get

$$\partial_\lambda \mathcal{R}_i\left(\lambda_{i,\delta}(\epsilon), \epsilon, \delta\right) \lambda_{i,\delta}'(\epsilon) + \partial_\epsilon \mathcal{R}_i\left(\lambda_{i,\delta}(\epsilon), \epsilon, \delta\right) = 0, \tag{3.7.108}$$

Taking $\epsilon = \epsilon_i^*(\delta)$ in (3.7.108), we obtain

$$\lambda_{i,\delta}'\left(\epsilon_i^*(\delta)\right) = 0. \tag{3.7.109}$$

Again, using the derivative of (3.7.108) with respect to the variable ϵ and taking into account the relation (3.7.109), we get

$$\lambda''_{i,\delta}\left(\epsilon_i^*(\delta)\right) = -\frac{\partial_{\epsilon\epsilon}\mathcal{R}_i\left(\lambda_i^*(\delta), \epsilon_i^*(\delta), \delta\right)}{\partial_\lambda \mathcal{R}_i\left(\lambda_i^*(\delta), \epsilon_i^*(\delta), \delta\right)}. \tag{3.7.110}$$

From the relations (3.7.87) and (3.7.88) and the conditions (H2) and (H3), we obtain

$$\lambda''_{i,\delta}(\epsilon) > 0, \tag{3.7.111}$$

in the neighborhood of $\epsilon_i^*(\delta)$.

Finally, use of (3.7.109) and (3.7.111), which means that the function $\lambda_{i,\delta}(.)$ is strictly convex, and the strict minimum is attained at $\epsilon_i^*(\delta)$.

□

The previous lemma does not represent all solutions of the equation (E_δ) in $\mathcal{U}_i$ but only in the neighborhood of turning point $(\lambda_i^*(\delta), \epsilon_i^*(\delta))$. To determine the other solutions we need the following analysis.

We assume that the interval I_2 is bounded and the mapping $\mathcal{R}_i$ is defined on $I_{1,i} \times \bar{I}_2 \times I_3$.

From (3.7.42) and (3.7.52) we have

$$\partial_\epsilon \mathcal{R}_i(\lambda_i, 0, 0) = \partial^2_{\epsilon\epsilon}\mathcal{R}_i(\lambda_i, 0, 0) = 0.$$

From (3.7.45) we obtain

$$\begin{aligned}
\partial^3_{\epsilon\epsilon\epsilon}\mathcal{R}_i(\lambda, \epsilon, \delta) = {} & ((\partial_{\epsilon\epsilon}\mathcal{V}_i(\lambda, \epsilon, \delta) C''(\epsilon\phi_i + \mathcal{V}_i(\lambda, \epsilon, \delta)) + (\phi_i + \partial_\epsilon \mathcal{V}_i(\lambda, \epsilon, \delta))^2 \\
& \times C'''(\epsilon\phi_i + \mathcal{V}_i(\lambda, \epsilon, \delta))).(\phi_i + \Theta_i(\lambda, \epsilon\phi_i + \mathcal{V}_i(\lambda, \epsilon, \delta))) \\
& + (\phi_i + \partial_\epsilon \mathcal{V}_i(\lambda, \epsilon, \delta))C''(\epsilon\phi_i + \mathcal{V}_i(\lambda, \epsilon, \delta)).\partial_\epsilon\Theta_i(\lambda, \epsilon\phi_i + \mathcal{V}_i(\lambda, \epsilon, \delta)) \\
& + (\phi_i + \partial_\epsilon \mathcal{V}_i(\lambda, \epsilon, \delta))C''(\epsilon\phi_i + \mathcal{V}_i(\lambda, \epsilon, \delta)).\partial_\epsilon\Theta_i(\lambda, \epsilon\phi_i + \mathcal{V}_i(\lambda, \epsilon, \delta)) \\
& + C'(\epsilon\phi_i + \mathcal{V}_i(\lambda, \epsilon, \delta)).\partial_{\epsilon\epsilon}\Theta_i(\lambda, \epsilon\phi_i + \mathcal{V}_i(\lambda, \epsilon, \delta)), \phi_i)_\Delta \\
& - (L_1(\partial_{\epsilon\epsilon}\Theta_i\left(\lambda, \epsilon\phi_i + \mathcal{V}_i(\lambda, \epsilon, \delta))\right), \phi_i)_\Delta.
\end{aligned}$$

Since $C'(0) = 0 \in \mathscr{L}_1\left(H_0^2(\omega)\right)$, $C''(0) = 0 \in \mathscr{L}_2\left(H_0^2(\omega)\right)$ and C is homogeneous operator of degree 3, using (3.7.24) we have

$$\partial_{\epsilon\epsilon\epsilon}\mathcal{R}_i(\lambda_i, 0, 0) = 6(C\left(\phi_i + \Theta_i(\lambda_i, 0)\right), \phi_i)_\Delta - (L_1\left(\partial_{\epsilon\epsilon}\Theta_i(\lambda_i, 0)\right), \phi_i)_\Delta, \tag{3.7.112}$$

where

$$\begin{aligned}
\partial_{\epsilon\epsilon}\Theta_i(\lambda_i, 0) = {} & \left[I - \lambda_i L_2 - P_i L_1\right]^{-1}\left((\phi_i + \Theta_i(\lambda_i, 0))^2\right. \\
& \times \left(P_i C'''(0).\Theta_i(\lambda_i, 0) + P_i C'''(0).\phi_i)\right),
\end{aligned}$$

it follows from the relations (3.7.48) and (3.7.51).

If we assume that

$$\partial^3_{\epsilon\epsilon\epsilon}\mathcal{R}_i(\lambda_i, 0, 0) \neq 0, \tag{$H4$}$$

and the interval I_2 is small enough, the Taylor expansion of the function $\mathcal{R}_i(\lambda_i, ., 0)$, shows that the following equation

$$\mathcal{R}_i(\lambda_i, \epsilon, 0) = 0,$$

has only the trivial solution $\epsilon = 0$ in the compact interval $\bar{I}_2$.

We will need the following definition due to Ciarlet and Rabier [29].

Definition 3.7.2. *[3] Let $\partial I_2 = \bar{I}_2 \backslash I_2$ is a reduced set to two points. The neighborhood $\mathcal{U}_i \times I_3$ is called an adapted neighborhood if the equation (E_δ) has no solution in $I_{1,i} \times \partial I_2$, for fixed $\delta \in I_3$, where $\mathcal{U}_i = I_{1,i} \times I_2$.*

For $\delta \in I_3$ and $\lambda \in I_{1,i}$, we define

$$\Upsilon(\lambda, \delta) = \{\epsilon \in I_2;\ \mathcal{R}_i(\lambda, \epsilon, \delta) = 0\}. \tag{3.7.113}$$

Accordingly to the local inversion theorem, we deduce that $\Upsilon(\lambda, \delta)$ is a finite set, for any $\delta \in I_3$ and any $\lambda \in I_{1,i}$ such that $\lambda \neq \lambda_i^*(\delta)$. Therefore we can consider

$$card\ \Upsilon(\lambda, \delta) \in \mathbb{N}.$$

For any $\delta \in I_3$, we define the following open subintervals

$$I_{1,i}^-(\delta) = \{\lambda \in I_{1,i}; \lambda < \lambda_i^*(\delta)\},$$

$$I_{1,i}^+(\delta) = \{\lambda \in I_{1,i}; \lambda > \lambda_i^*(\delta)\},$$

so that

$$I_{1,i} = I_{1,i}^-(\delta) \cup \{\lambda_i^*(\delta)\} \cup I_{1,i}^+(\delta).$$

The same arguments as in [29] show that the following Proposition.

Proposition 3.7.4. *[3] Assume that the neighborhood $\mathcal{U}_i \times I_3$ is adapted and the conditions $(C1)$,(H2),(H3) and $(H4)$ are realised. Then*

$$card\ \Upsilon(\lambda, \delta) = 1, \tag{3.7.114}$$

for $\delta \in I_3$ and $\lambda \in I_{1,i}^-(\delta)$,

$$card\ \Upsilon(\lambda, \delta) = 3, \tag{3.7.115}$$

for $\delta \in I_3$ and $\lambda \in I_{1,i}^+(\delta)$,

$$card\ \Upsilon\left(\lambda_i^*(\delta), \delta\right) = 2, \tag{3.7.116}$$

$\delta \in I_3 \backslash \{0\}$.

3.8 Solution set of the bifurcation Marguerre-von Kármán equations

3.8.1 Unperturbed case

Here we are interested in the solution set of the following equation

$$\tilde{\xi} - \lambda L_2 \tilde{\xi} + C\tilde{\xi} - L_1 \tilde{\xi} = 0,\ (\lambda, \tilde{\xi}) \in \mathbb{R} \times H_0^2(\omega). \tag{$\mathscr{E}_0$}$$

Clearly, Proposition (3.7.3) is incorrect for $\delta = 0$, because

$$\partial_\lambda \mathcal{R}_i\left(\lambda_i, 0, 0\right) = \partial_{\epsilon\epsilon}^2 \mathcal{R}_i\left(\lambda_i, 0, 0\right) = 0.$$

Before giving the solution set of $(\mathscr{E}_0)$, we will need to give the solution set of (E_0).

Lemma 3.8.1. *[3] Assume that the conditions $(C1)$ and $(C3)$ are realized. Then, in a neighborhood of $(\lambda_i, 0)$, the solution set of the equation (E_0) is a pair of $\mathscr{C}^\infty$ curves which intersect only at $(\lambda_i, 0)$ and they cut transversally, one of which is the $\lambda-$axis.*

Proof. [3] From the Jacobian matrix of $\mathcal{S}_i$ in $(\lambda_i, 0, 0)$, the Hessienn matrix of the mapping $\mathcal{R}_i(.,.,0)$ at $(\lambda_i, 0)$ may be written as

$$\begin{pmatrix} \partial^2_{\lambda\lambda}\mathcal{R}_i(\lambda_i, 0, 0) & \partial^2_{\lambda\epsilon}\mathcal{R}_i(\lambda_i, 0, 0) \\ \\ \partial^2_{\epsilon\lambda}\mathcal{R}_i(\lambda_i, 0, 0) & \partial^2_{\epsilon\epsilon}\mathcal{R}_i(\lambda_i, 0, 0) \end{pmatrix}.$$

From the partial derivative of (3.7.40) with respect to λ, we obtain

$$\begin{aligned} \partial_{\lambda\lambda}\mathcal{R}_i(\lambda, \epsilon, \delta) = (\partial_\lambda \mathcal{V}_i(\lambda, \epsilon, \delta) C''\left(\epsilon\phi_i + \mathcal{V}_i(\lambda, \epsilon, \delta)\right).\partial_\lambda \mathcal{V}_i(\lambda, \epsilon, \delta) + C'(\epsilon\phi_i \\ + \mathcal{V}_i(\lambda, \epsilon, \delta)).\partial_{\lambda\lambda}\mathcal{V}_i(\lambda, \epsilon, \delta), \phi_i)_\Delta - \left(L_1\left(\partial_{\lambda\lambda}\mathcal{V}_i(\lambda, \epsilon, \delta)\right), \phi_i\right)_\Delta. \end{aligned}$$

On the other hand, using the partial derivative of (3.7.32) with respect to the λ, we get

$$\begin{aligned} &\left[I - \lambda L_2 + P_i C'\left(\epsilon\phi_i + \mathcal{V}_i(\lambda, \epsilon, \delta)\right) - P_i L_1\right] \partial_{\lambda\lambda}\mathcal{V}_i(\lambda, \epsilon, \delta) + \partial_\lambda \mathcal{V}_i(\lambda, \epsilon, \delta) \\ &\times P_i C''\left(\epsilon\phi_i + \mathcal{V}_i(\lambda, \epsilon, \delta)\right).\partial_\lambda \mathcal{V}_i(\lambda, \epsilon, \delta) = L_2\left(\partial_\lambda \mathcal{V}_i(\lambda, \epsilon, \delta)\right). \end{aligned}$$

It follows from (3.7.22) that $\partial_{\lambda\lambda}\mathcal{V}_i(\lambda_i, 0, 0) = 0$, which implies that

$$\partial_{\lambda\lambda}\mathcal{R}_i(\lambda_i, 0, 0) = 0.$$

From the Jacobian matrix of $\mathcal{S}_i$ in $(\lambda_i, 0, 0)$, we get

$$\partial_{\epsilon\lambda}\mathcal{R}_i(\lambda_i, 0, 0) = \partial_{\lambda\epsilon}\mathcal{R}_i(\lambda_i, 0, 0) = -\frac{1}{\lambda_i} - \left(L_1\left(\partial_\lambda \Theta_i(\lambda_i, 0)\right), \phi_i\right)_\Delta,$$

and

$$\partial_{\epsilon\epsilon}\mathcal{R}_i(\lambda_i, 0, 0) = 0.$$

So that, the Hessainn matrix of the mapping $\mathcal{R}_i(.,.,0)$ in $(\lambda_i, 0)$ can be written as

$$\begin{pmatrix} 0 & -\frac{1}{\lambda_i} - \left(L_1\left(\partial_\lambda \Theta_i(\lambda_i, 0)\right), \phi_i\right)_\Delta \\ \\ -\frac{1}{\lambda_i} - \left(L_1\left(\partial_\lambda \Theta_i(\lambda_i, 0)\right), \phi_i\right)_\Delta & 0 \end{pmatrix}.$$

The condition $(C3)$, implies that

$$Hess_{(\lambda_i, 0)}\left(\mathcal{R}_i(.,.,0)\right) = -\left(\frac{1}{\lambda_i} + \left(L_1\left(\partial_\lambda \Theta_i(\lambda_i, 0)\right), \phi_i\right)_\Delta\right)^2 < 0.$$

Accordingly to the Morse lemma we deduce that the solution set of the equation (E_0) in the neighborhood of $(\lambda_i, 0)$ consists of two $\mathscr{C}^\infty$ curves cutting each other transversally. This in turn implies that $(\lambda_i, 0) \in \mathcal{U}_i$ is a bifurcation point.

Since $\mathcal{V}_i$ is uniquely determined as in Lemma (3.7.2) and $v = \mathcal{V}_i(\lambda, 0, 0) = 0$ satisfies the equation (3.7.28), we conclude that the trivial branch $(\lambda, 0) \in \mathcal{U}_i$ is one of the two branches of solutions of (E_0).

□

Theorem 3.8.1. *[3] Assume that the conditions ($C1$), ($C3$) and the assumption (H2) are realized. Then, the equation (E_0) has a unique trivial solution for $\lambda \leq \lambda_i$ and has three distinct solutions for $\lambda > \lambda_i$.*

Proof. [3] The equation (E_0) can be written as

$$\left(1-\frac{\lambda}{\lambda_i}\right)\epsilon-(L_1(\epsilon\phi_i+\mathcal{V}_i(\lambda,\epsilon,0)),\phi_i)_\Delta+\left(C\left(\epsilon\phi_i+\mathcal{V}_i(\lambda,\epsilon,0)\right),\phi_i\right)_\Delta=0. \tag{3.8.1}$$

Since the operator C is homogeneous of degree 3, for $\epsilon \neq 0$, the equation (3.8.1) reads as follows

$$\left(1-\frac{\lambda}{\lambda_i}\right)-(L_1(\phi_i+\frac{\mathcal{V}_i(\lambda,\epsilon,0)}{\epsilon}),\phi_i)_\Delta+\epsilon^2(C(\phi_i+\frac{\mathcal{V}_i(\lambda,\epsilon,0)}{\epsilon}),\phi_i)_\Delta=0. \tag{3.8.2}$$

Taking into account

$$\lim_{\epsilon\to 0}\frac{\mathcal{V}_i(\lambda,\epsilon,0)}{\epsilon}=\partial_\epsilon\mathcal{V}_i(\lambda,0,0),$$

and since $\partial_\epsilon\mathcal{V}_i(\lambda,0,0)=\mathit{\Theta}_i(\lambda,0)$, by continuity of the mapping $\partial_\epsilon\mathcal{V}_i$, we can choose the neighborhood $\mathcal{U}_i$ a small enough such that

$$\left(1-\frac{\lambda}{\lambda_i}\right)-\left(L_1\left(\phi_i+\mathit{\Theta}_i(\lambda,0)\right),\phi_i\right)_\Delta+\epsilon^2\left(C\left(\phi_i+\mathit{\Theta}_i(\lambda,0)\right),\phi_i\right)_\Delta=0. \tag{3.8.3}$$

From the conditions ($C1$) and (H2), we deduce that, there are no nontrivial solutions for $\lambda \leq \lambda_i$, while there are two nontrivial solutions given by the parabola bifurcating from the trivial branch at $(\lambda_i, 0)$, symmetric with respect to the $\lambda-$axis. □

We now give the set solution of the equation ($\mathscr{E}_0$) by using the bijective mapping $\mathscr{P}_{i,0}$.

Theorem 3.8.2. *[3] Assume that the conditions ($C1$), ($C3$) and the assumption (H2) are realized. Then, there exists a neighborhood $\tilde{\mathcal{U}}_i$ of $(\lambda_i, 0)$ in $\mathbb{R}\times H_0^2(\omega)$ such that, the equation ($\mathscr{E}_0$) has a unique trivial solution for $\lambda \leq \lambda_i$ and has three distinct solutions for $\lambda > \lambda_i$.*

3.8.2 Perturbed case

Now we are interested in the solution set of the following equation

$$\tilde{\xi}-\lambda L_2\tilde{\xi}+C\tilde{\xi}-L_1\tilde{\xi}-\delta\tilde{\mathfrak{F}}=0, \quad (\lambda,\tilde{\xi})\in\mathbb{R}\times H_0^2(\omega),\ \delta\neq 0 \tag{$\mathscr{E}_\delta$}$$

According to Proposition (3.7.4), we immediately establish the following main results about the set solution of the equation ($\mathscr{E}_\delta$), by using the bijective mapping $\mathscr{P}_{i,\delta}$.

Theorem 3.8.3. *[3] We assume that the conditions ($C1$) and the assumptions (H2),(H3),($H4$) are realized. Then, there exists $\lambda_i^*(\delta) > \lambda_i$ and a neighborhood $\tilde{\mathcal{U}}_i$ of $(\lambda_i, 0)$ in $\mathbb{R}\times H_0^2(\omega)$ such that, for any $\delta \in I_3\backslash\{0\}$, all the solutions of the ($\mathscr{E}_\delta$) in $\tilde{\mathcal{U}}_i$ lie on two distinct continuous curves, one of them having a nondegenerate turning point $\left(\lambda_i^*(\delta),\tilde{\xi}_i^*(\delta)\right)$. Furthermore, the solution is unique if $\lambda < \lambda_i^*(\delta)$, two distinct solutions if $\lambda = \lambda_i^*(\delta)$, one of which is $\tilde{\xi}_i^*(\delta)$, and three distinct solutions if $\lambda > \lambda_i^*(\delta)$.*

To complete this study, it remains to give the following diagram represent the solution set of the bifurcation in Marguerre-von Kármán equations. (3.1).

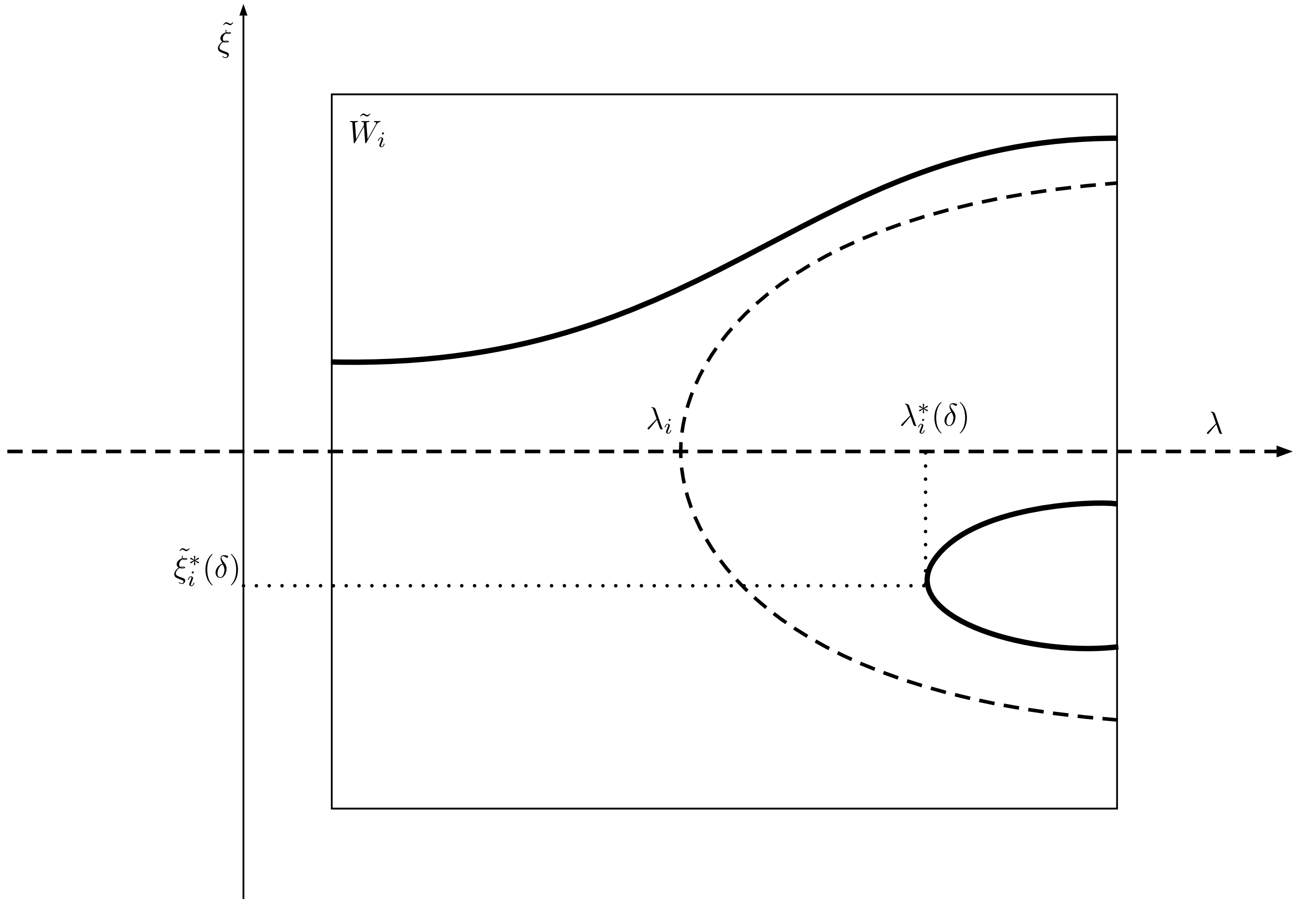

Figure 3.1: Bifurcation diagram for Marguerre–von Kármán equations. Unperturbed bifurcation diagram represented with a dashed line; and perturbed bifurcation diagram represented with a solid line, under the assumption that $sign\big(\delta(\tilde{\mathfrak{F}}, \phi_i)_\Delta\big) > 0$.

Chapter 4

Mixed finite element approximation

4.1 The biharmonic problem (The model problem)

Let ω be a bounded and open subset of $\mathbb{R}^2$ with a Lipschitz-continuous boundary γ. We consider the following Dirichlet problem for the biharmonic operator:
Find ξ in $H_0^2(\omega)$ satisfying:

$$\begin{aligned} \Delta^2\xi &= f \qquad \text{in } \omega, \\ \xi &= \partial_\nu \xi = 0 \quad \text{on } \gamma. \end{aligned} \tag{4.1.1}$$

We know that the operator Δ^2 is an isomorphism from $H^{-2}(\omega)$ to $H_0^2(\omega)$, So the biharmonic problem (4.1.1) has a unique solution for any $f \in H^{-2}(\omega)$.
The corresponding variational formulation of the model problem $(4.1.1)$, which is classical when dealing with elasticity problems in linear plate theory, consists in

$$\begin{cases} \text{Find } \xi \in \mathrm{H}_0^2(\omega) \text{ such that ,} \\ a_\nu(\xi,\eta) = (f,\eta), \quad \forall \eta \in \mathrm{H}_0^2(\omega). \end{cases} \tag{4.1.2}$$

Where $a_\nu(.,.)$ is bilinear form defined by

$$\begin{aligned} a_\nu(\xi,\eta) &= \int_\omega (\Delta\xi\Delta\eta - (1-\nu)[\xi,\eta])dx \\ &= \int_\omega \left\{\Delta\xi\Delta\eta - (1-\nu)\left(\partial_{11}\xi\partial_{22}\eta + \partial_{22}\xi\partial_{11}\eta - 2\partial_{12}\xi\partial_{12}\eta\right)\right\} dx, \end{aligned} \tag{4.1.3}$$

with $0 < \nu < 1$ is the Poisson ratio of the plate.
(f,η) is linear form defined by:

$$(f,\eta) = \int_\omega f\eta dx.$$

4.1.1 Mixed Formulation of the model Problem

Let $w \in \mathrm{H}_0^2(\omega)$, we define $(\sigma_w) = ((\sigma_w)_{i,j}, 1 \le i,j \le 2)$ by

$$(\sigma_w)_{i,j} = \frac{\partial^2 w}{\partial x_i \partial x_j},$$

So that

$$[\xi,\eta]=[[\sigma_\xi,\sigma_\eta]]=(\sigma_\xi)_{11}(\sigma_\eta)_{22}+(\sigma_\xi)_{22}(\sigma_\eta)_{11}-2(\sigma_\xi)_{12}(\sigma_\eta)_{12},$$

with this notation, we get

Proposition 4.1.1. *[67] Let $0<\nu<1$. Then the problem (4.1.2) is equivalent to the mixed problem:*

$$\begin{aligned}&\textit{Find } (\sigma,\xi)\in\Sigma\times\mathscr{V} \textit{ satisfying}:\\ &\forall\tau\in\Sigma,\ A_\nu(\sigma,\tau)+B_\nu(\tau,\xi)=0, && (4.1.4)\\ &\forall v\in\mathscr{N},\ -B_\nu(\sigma,v)=\int_\omega fv\,dx. && (4.1.5)\end{aligned}$$

Where:

$$\begin{aligned}&\sigma=\sigma_\xi\\ &\Sigma=\{\sigma=(\sigma_{ij}),1\le i,j\le 2,\sigma_{ij}\in L^2(\omega),\sigma_{12}=\sigma_{21}\}=(L^2(\omega))^4_s,\\ &\mathscr{V}=\mathrm{H}^2_0(\omega),\\ &A_\nu(\sigma,\tau)=\int_\omega\{(\sigma_{11}+\sigma_{22})(\tau_{11}+\tau_{22})-(1-\nu)[[\sigma,\tau]]\}dx\\ &B_\nu(\sigma,v)=-\int_\omega\{(\sigma_{11}+\sigma_{22}).\Delta v-(1-\nu)(\sigma_{11}\partial_{22}v+\sigma_{22}\partial_{11}v-2\sigma_{12}\partial_{12}v)\}dx.\\ &\textit{With } \partial_{ij}v=\frac{\partial^2 v}{\partial x_i\partial x_j}.\end{aligned}$$

For the regularity of solution, we have the following Proposition:

Proposition 4.1.2. *[67] Let ω be a domain with a Lipschitz continuous boundary γ. For f in $L^2(\omega)$, the solution of (4.1.4)-(4.1.5) satisfies*

$$\begin{aligned}&\xi\in\mathrm{H}^3(\omega)\cap\mathrm{H}^2_0(\omega),\\ &\sigma_{ij}\in\mathrm{H}^1(\omega),\ \forall i,j\quad 1\le i,j\le 2\end{aligned}$$

Proof. see [51] ☐

As a result of this regularity of solution, where an integration by parts is used, we have the following Proposition which gives a new Mixed formulation of the problem (4.1.2):

Proposition 4.1.3 ([67]). *Under the assumptions of Proposition (4.1.2), the unique solution of (4.1.4)-(4.1.5) is also the unique solution of:*
Find $(\sigma,\xi)\in\tilde\Sigma\times\tilde{\mathscr{V}}$ such that:

$$\begin{aligned}&\forall\tau\in\tilde\Sigma,\ A_\nu(\sigma,\tau)+\tilde B_\nu(\tau,\xi)=0,\\ &\forall v\in\tilde{\mathscr{V}},\ -\tilde B_\nu(\sigma,v)=\int_\omega fv\,dx.\end{aligned}\tag{4.1.6}$$

Where:

$$\begin{aligned}&\tilde\Sigma=(\mathrm{H}^1(\omega))^4_s=\{\sigma=(\sigma_{ij}),1\le i,j\le 2,\sigma_{ij}\in\mathrm{H}^1(\omega),\sigma_{12}=\sigma_{21}\},\\ &\tilde{\mathscr{V}}=\mathrm{H}^1_0(\omega),\\ &\tilde B_\nu(\sigma,v)=\int_\omega\{\partial_1\sigma_{11}\partial_1 v+\nu(\partial_1\sigma_{22}\partial_1 v+\partial_2\sigma_{11}\partial_2 v)+\partial_2\sigma_{22}\partial_2 v\\ &\qquad+(1-\nu)(\partial_1\sigma_{12}\partial_2 v+\partial_2\sigma_{12}\partial_1 v)\}dx.\end{aligned}$$

4.1.2 Mixed approximation

Assuming that ω is a convex polygonal domain. For each value of a real parameter $h > 0$ which tends to zero, we let $(\mathscr{T}_h)$ be a regular triangulations family of $\overline{\omega}$, in the sense of Reinhart [67], i.e., every triangle $K \in \mathscr{T}_h$ contains a circle with radius $C_0 h$ and is contained in a circle with radius $C_0^{-1} h$.

For each integer $k \geq 1$, we shall denote by $\mathbb{P}_k$ the space of all polynomials of degree $\leq k$. Let us introduce the two finite-dimensional spaces

$$\mathscr{X}_h^k = \left\{ v_h \in \mathscr{C}^0(\overline{\omega}) \,; \forall K \in \mathscr{T}_h,\ v_h|_K \in \mathbb{P}_k \right\}, \tag{4.1.7}$$

$$\mathscr{X}_h^{0(k)} = \mathscr{X}_h^k \cap \mathrm{H}_0^1 = \left\{ v_h \in \mathscr{X}_h^k;\ v_h = 0 \ on\ \gamma \right\}. \tag{4.1.8}$$

We set

$$\mathscr{V}_h = \mathscr{X}_h^{0(k)}, \quad \Sigma_h = (\mathscr{X}_h^k)_s^4, \tag{4.1.9}$$

The Herrmann-Miyoshi scheme is defined by:
Find $(\sigma_h, \xi_h) \in \Sigma_h \times \mathscr{V}_h$, such that

$$\begin{aligned} &\forall \tau_h \in \Sigma_h, \qquad A_\nu(\tau_h, \sigma_h) + \tilde{B}_\nu(\tau_h, \xi_h) = 0, \\ &\forall v_h \in \mathscr{V}_h, \qquad -\tilde{B}_\nu(\sigma_h, v_h) = \int_\omega f v_h dx. \end{aligned} \tag{4.1.10}$$

Since any solution corresponding homogeneous problem is necessarily equal to zero, we conclude that these finite dimensional problems have a unique solution.

4.1.3 Convergence of the Approximate Solutions for the Model Problem

In this paragraph, we will recall the most important results of the approximations to the Model Problem, through which we reach of the convergence of the operator T_h, which will be define later

In the following, we consider (σ, ξ), (σ_h, ξ_h) are the unique solution of the problems (4.1.5) and (4.1.10) respectively, then we have the following estimates

L^2 Error Estimates

If the unique solution (σ, ξ) of the model problem (4.1.5) satisfies $\xi \in H^4(\omega)$, the following error estimates hold:

Lemma 4.1.1 (estimate of $\|\sigma - \sigma_h\|_{0,2}$).

$$\|\sigma - \sigma_h\|_{0,2} \leq C h^{1/2} |\log h| \|\xi\|_{4,2} \tag{4.1.11}$$

Proof. see [67] □

Lemma 4.1.2 (estimate of $\|\xi - \xi_h\|_{0,2}$).

$$\|\xi - \xi_h\|_{0,2} \leq C h |\log h|^2 \, \|\xi\|_{4,2} \tag{4.1.12}$$

Proof. see [67] □

As a result of the two Lemmas (4.1.1) and (4.1.2) we find

Theorem 4.1.1.

$$\|\xi - \xi_h\|_{0,2} + h|\log h|\|\sigma - \sigma_h\|_{0,2} \leq Ch|\log h|^2\|\xi\|_{4,2} \tag{4.1.13}$$

Proof. see [67] ☐

Corollary 4.1.1. *The error between the exact solution ξ and the approximate solution ξ_h may be estimated in the H^1 norm by:*

$$\|\xi - \xi_h\|_{1,2} \leq Ch^{\frac{3}{4}}|\log h|^{\frac{3}{2}}\|\xi\|_{4,2}. \tag{4.1.14}$$

Proof. see [67] ☐

L^∞ Error Estimates

Theorem 4.1.2. *If $\xi \in H^4(\omega)$, the error estimate in the L^∞ norm is:*

$$\|\xi - \xi_h\|_{0,\infty,} \leq Ch|\log h|^3\|\xi\|_{4,2}. \tag{4.1.15}$$

Proof. see [67] ☐

Remark 4.1.1. *In the case of piecewise quadratic approximation (k = 2), Rannacher in ([65]) prove the following estimate*

Theorem 4.1.3. *[65] if $\xi \in H^3(\omega) \cap H_0^2(\omega)$ and if $0 < \nu < 1$ we have*

$$\|\xi - \xi_h\|_{1,2} + h\|\sigma - \sigma_h\|_{0,2} \leq Ch^2\|\xi\|_{3,2} \tag{4.1.16}$$

if, in addition $\xi \in H^4(\omega) \cap W^{3,\infty}(\omega)$ then

$$\|\xi - \xi_h\|_{1,\infty} + h\|\sigma - \sigma_h\|_{0,\infty} \leq Ch^2|\log h|^{\frac{3}{2}}\left\{\|\xi\|_{3,2} + \|\xi\|_{4,2}\right\} \tag{4.1.17}$$

4.2 Mixed formulation of Marguerre-von Kármán equations

For simplicity in notation, we write ψ^1 and ψ^2 instead of $\tilde{\xi}$ and Φ, respectively. So the bifurcation problem in Marguerre-von Kármán equations becomes

$$\begin{cases} \Delta^2\psi^1 = [\psi^2, \psi^1] + \lambda[\tilde{\theta}_0, \psi^1] + \tilde{f} \text{ in } \omega, \\ \Delta^2\psi^2 = -[\psi^1, \psi^1] + [\tilde{\theta}, \tilde{\theta}] \text{ in } \omega, \\ \psi^1 = \partial_\nu\psi^1 = 0 \text{ on } \gamma, \\ \psi^2 = \partial_\nu\psi^2 = 0 \text{ on } \gamma. \end{cases} \tag{4.2.1}$$

4.2.1 Weak formulation

As is well known, any smooth solution of the boundary value problem (4.2.1) also satisfies the following variational problem

$$\begin{cases} \text{Find } (\lambda, \psi^1, \psi^2) \in \mathbb{R} \times H_0^2(\omega) \times H_0^2(\omega) \text{ such that,} \\ \displaystyle\int_\omega \left\{ \Delta^2 \psi^1 - [\psi^2, \psi^1] - \lambda[\theta_0, \psi^1] - \tilde{\mathfrak{F}} \right\} \eta^1 d\omega = 0, \ \forall \eta^1 \in H_0^2(\omega), \\ \displaystyle\int_\omega \left\{ \Delta^2 \psi^2 + [\psi^1, \psi^1] - [\theta, \theta] \right\} \eta^2 d\omega = 0, \ \forall \eta^2 \in H_0^2(\omega). \end{cases} \tag{4.2.2}$$

Therefore, the previous variational problem take the particularly short forms

$$\begin{cases} \text{Find } (\lambda, \psi^1, \psi^2) \in \mathbb{R} \times H_0^2(\omega) \times H_0^2(\omega) \text{ such that,} \\ a_\nu(\psi^1, \eta^1) + b(\psi^1, \psi^2, \eta^1) + L(\eta^1) = 0, \ \forall \eta^1 \in H_0^2(\omega), \\ a_\nu(\psi^2, \eta^2) + b_\theta(\psi^1, \eta^2) = 0, \ \forall \eta^2 \in H_0^2(\omega), \end{cases} \tag{4.2.3}$$

where

$$\begin{aligned} a_\nu(\zeta, \eta) &= \int_\omega (\Delta\zeta \Delta\eta - (1-\nu)[\zeta, \eta]) d\omega, \\ b(\zeta, \varphi, \eta) &= -\int_\omega ([\varphi, \zeta] + \lambda[\theta_0, \zeta]) \eta d\omega, \\ b_\theta(\zeta, \eta) &= \int_\omega ([\zeta, \zeta] - [\theta, \theta]) \eta d\omega, \\ L(\eta) &= -\int_\omega \tilde{\mathfrak{F}} \eta d\omega. \end{aligned}$$

4.2.2 Mixed formulation

Let $\eta \in H^2(\omega)$, we define $(\sigma_\eta) = ((\sigma_\eta)_{\alpha\beta}, 1 \leq \alpha, \beta \leq 2)$ by

$$(\sigma_\eta)_{\alpha\beta} = \frac{\partial^2 \eta}{\partial y_\alpha \partial y_\beta}.$$

Use analogous arguments as in Proposition 4.1.2 due to Reinhart [67] for the biharmonic equation, we show

Proposition 4.2.1. *Let $0 < \nu < 1$. Then the problem (4.2.3) is equivalent to the following mixed problem*

$$\begin{cases} \textit{Find } ((\sigma^1, \psi^1), (\sigma^2, \psi^2)) \in (\Sigma \times \mathscr{V})^2 \textit{ such that}, \\ A_\nu(\sigma^1, \tau) + B_\nu(\tau, \psi^1) = 0, \ \forall \tau \in \Sigma, \\ B_\nu(\sigma^1, \eta^1) = -\int_\omega \left\{ [[\sigma^2, \sigma^1]] + \lambda[[\sigma_{\theta_0}, \sigma^1]] + \tilde{\mathfrak{F}} \right\} \eta^1 d\omega, \ \forall \eta^1 \in \mathscr{V}, \\ A_\nu(\sigma^2, \tau) + B_\nu(\tau, \psi^2) = 0, \ \forall \tau \in \Sigma, \\ B_\nu(\sigma^2, \eta^2) = -\int_\omega \left\{ -[[\sigma^1, \sigma^1]] + [[\sigma_\theta, \sigma_\theta]] \right\} \eta^2 d\omega, \ \forall \eta^2 \in \mathscr{V}, \end{cases} \tag{4.2.4}$$

where

$$\begin{aligned}
&\Sigma = (L^2(\omega))^4_s = \left\{\sigma = (\sigma_{\alpha\beta}), 1 \leq \alpha, \beta \leq 2, \sigma_{\alpha\beta} \in L^2(\omega), \sigma_{12} = \sigma_{21}\right\},\\
&\mathscr{V} = H^2_0(\omega),\\
&A_\nu(\sigma, \tau) = \int_\omega \{(\sigma_{11} + \sigma_{22})(\tau_{11} + \tau_{22}) - (1 - \nu)[[\sigma, \tau]]\}\, d\omega,\\
&B_\nu(\sigma, \eta) = \int_\omega \{(\sigma_{11} + \sigma_{22})\Delta\eta - (1 - \nu)(\sigma_{11}\partial_{22}\eta + \sigma_{22}\partial_{11}\eta - 2\sigma_{12}\partial_{12}\eta)\}\, d\omega,\\
&[[\sigma, \tau]] = \sigma_{11}\tau_{22} + \sigma_{22}\tau_{11} - 2\sigma_{12}\tau_{12}.
\end{aligned}$$

Next, we obtain

Proposition 4.2.2. *Assume that the following conditions*

$$\theta \in H^2_0(\omega) \cap H^3(\omega), (\varphi_0, \varphi_1) \in H^{\frac{5}{2}}(\gamma) \times H^{\frac{3}{2}}(\gamma) \text{ and } \tilde{f} \in L^2(\omega), \tag{4.2.5}$$

are realized. Then the mixed formulation of Marguerre-von Kármán equations (4.2.4) *may be equivalently written as*

$$\begin{cases}
\text{Find } ((\sigma^1, \psi^1), (\sigma^2, \psi^2)) \in \left(\tilde{\Sigma} \times \tilde{\mathscr{V}}\right)^2 \text{ such that,}\\
A_\nu(\sigma^1, \tau) + \tilde{B}_\nu(\tau, \psi^1) = 0,\ \forall \tau \in \tilde{\Sigma},\\
\tilde{B}_\nu(\sigma^1, \eta^1) = -\int_\omega \left\{[[\sigma^2, \sigma^1]] + \lambda[[\sigma_{\theta_0}, \sigma^1]] + \tilde{\mathfrak{F}}\right\} \eta^1 d\omega,\ \forall \eta^1 \in \tilde{\mathscr{V}},\\
A_\nu(\sigma^2, \tau) + \tilde{B}_\nu(\tau, \psi^2) = 0,\ \forall \tau \in \tilde{\Sigma},\\
\tilde{B}_\nu(\sigma^2, \eta^2) = -\int_\omega \{-[[\sigma^1, \sigma^1]] + [[\sigma_\theta, \sigma_\theta]]\}\, \eta^2 d\omega,\ \forall \eta^2 \in \tilde{\mathscr{V}},
\end{cases} \tag{4.2.6}$$

where

$$\begin{aligned}
&\tilde{\Sigma} = (H^1(\omega))^4_s = \left\{\sigma = (\sigma_{\alpha\beta}), 1 \leq \alpha, \beta \leq 2, \sigma_{\alpha\beta} \in H^1(\omega), \sigma_{12} = \sigma_{21}\right\},\\
&\tilde{\mathscr{V}} = H^1_0(\omega),\\
&\tilde{B}_\nu(\sigma, \eta) = \int_\omega \{\partial_1\sigma_{11}\partial_1\eta + \nu(\partial_1\sigma_{22}\partial_1\eta + \partial_2\sigma_{11}\partial_2\eta) + \partial_2\sigma_{22}\partial_2\eta\\
&\qquad + (1 - \nu)(\partial_1\sigma_{12}\partial_2\eta + \partial_2\sigma_{12}\partial_1\eta)\}d\omega.
\end{aligned}$$

Proof. The conditions (4.2.5) imply that $(\sigma_\theta)_{\alpha\beta} \in H^1(\omega)$ and $(\sigma_{\theta_0})_{\alpha\beta} \in H^1(\omega)$. On the other hand, since $H^1 \hookrightarrow L^p(\omega), \forall\ 1 < p < \infty$, hence

$$[[\sigma^2, \sigma^1]] + \lambda[[\sigma_{\theta_0}, \sigma^1]] + \tilde{\mathfrak{F}} \ \text{ and } \ -[[\sigma^1, \sigma^1]] + [[\sigma_\theta, \sigma_\theta]]$$

are in $L^2(\omega)$. Accordingly to Proposition (4.1.3), which is related to the regularity of the solution to the mixed formulation for the biharmonic equation due to Kondrat'ev [51]. Hence we obtain the required equivalent with $\sigma^1 = \sigma_{\psi^1}$ and $\sigma^2 = \sigma_{\psi^2}$. □

4.2.3 Continuous operator equation

Now, in this paragraph, will be reduce the mixed formulation ((4.2.6)) to continuous operator equation form.

For a given $\varepsilon > 0$, let $p = \dfrac{2}{1 - \varepsilon}$ and let q denote the conjugate exponent (i.e, $\dfrac{1}{p} + \dfrac{1}{q} = 1$).

We define the following two spaces

$$\mathscr{Y} = (W_0^{1,p}(\omega))^2 \times ((L^2(\omega))_s^4)^2, \tag{4.2.7}$$

$$\mathscr{W} = (W^{-1,q}(\omega))^2. \tag{4.2.8}$$

Let us introduce the following mapping

$$G_\theta : (\lambda, \boldsymbol{X} = ((\psi^1, \psi^2), (\sigma^1, \sigma^2))) \in \mathbb{R} \times \mathscr{Y} \to G_\theta(\lambda, \boldsymbol{X}) \in \mathscr{W}, \tag{4.2.9}$$

defined by

$$G_\theta(\lambda, \boldsymbol{X}) = \Big(-\lambda[[\sigma_{\theta_0}, \sigma^1]] - [[\sigma^1, \sigma^2]] - \tilde{f},\ [[\sigma^1, \sigma^1]] - [[\sigma_\theta, \sigma_\theta]]\Big). \tag{4.2.10}$$

The special notation G_θ, indicates that G_θ also depends on θ.

Also, we consider the following linear continuous operator

$$T : \boldsymbol{g} = (g^1, g^2) \in (H^{-2}(\omega))^2 \longrightarrow T\boldsymbol{g} = ((\eta^1, \eta^2), (\tau^1, \tau^2)) \in (H_0^2(\omega))^2 \times (L^2(\omega)_s^4)^2, \tag{4.2.11}$$

defined by

$$\left.\begin{cases} \Delta^2 \eta^\kappa = g^\kappa \text{ in } \omega, \\ \eta^\kappa = \partial_\nu \eta^\kappa = 0 \text{ on } \gamma, \\ \tau_{\alpha\beta}^\kappa = \dfrac{\partial^2 \eta^\kappa}{\partial y_\alpha \partial y_\beta}, \quad 1 \leq \alpha, \beta \leq 2, \end{cases}\right\} \kappa = 1, 2. \tag{4.2.12}$$

We have by the Sobolev embedding Theorem, $H_0^2(\omega) \hookrightarrow W_0^{1,p}(\omega)$ with completely continuous embedding, so $(W_0^{1,p}(\omega))' \subset (H_0^2(\omega))'$ that is $W^{-1,q}(\omega) \subset H^{-2}(\omega)$.Then we get:

Lemma 4.2.1. *[67]*

$$T \in \mathscr{L}(\mathscr{W}; \mathscr{Y}) \ \textit{ is compact operator }. \tag{4.2.13}$$

Finally, the mixed formulation (4.2.6) is reduced to a continuous operator equation, as follows

$$\begin{cases} \text{Find } \ (\lambda, \boldsymbol{X}) \in \mathbb{R} \times \mathscr{Y} \ \text{ such that,} \\ F(\lambda, \boldsymbol{X}) \equiv \boldsymbol{X} + TG_\theta(\lambda, \boldsymbol{X}) = 0. \end{cases} \tag{4.2.14}$$

4.3 Discrete problem

In this paragraph, we use the same finite-dimensional spaces that we defined in (4.1.2) for discrete problem of biharmonic operator

$$\mathscr{X}_h^{(k)} = \left\{ \eta_h \in \mathscr{C}^0(\bar{\omega}); \forall K \in \mathscr{T}_h, \eta_h|_K \in \mathbb{P}_k \right\}, \tag{4.3.1}$$

$$\mathscr{X}_h^{0(k)} = \mathscr{X}_h^{(k)} \cap H_0^1(\omega) = \Big\{ \eta_h \in \mathscr{X}_h^{(k)}; \forall I \text{ node of } \mathscr{T}_h, I \in \gamma, \eta_h(I) = 0 \Big\}. \tag{4.3.2}$$

Where, $(\mathscr{T}_h)$ be a regular triangulations family of $\overline{\omega}$, every triangle $K \in \mathscr{T}_h$ contains a circle with radius $C_0 h$ and is contained in a circle with radius $C_0^{-1} h$, $h > 0$ which tends to zero.
$\mathbb{P}_k$ the space of all polynomials of degree less than or equal to k.

We set

$$\mathscr{V}_h = \mathscr{X}_h^{0(k)}, \quad \Sigma_h = (\mathscr{X}_h^{(k)})_s^4, \tag{4.3.3}$$

and

$$\mathscr{Y}_h = \mathscr{V}_h^2 \times \Sigma_h^2. \tag{4.3.4}$$

Therefore, the mixed Herrmann-Miyoshi scheme of (4.2.6) is defined as

$$\begin{cases} \text{Find } ((\sigma_h^1, \psi_h^1), (\sigma_h^2, \psi_h^2)) \in (\Sigma_h \times \mathscr{V}_h)^2 \text{ such that,} \\ A_\nu(\sigma_h^1, \tau_h) + \tilde{B}_\nu(\tau_h, \psi_h^1) = 0, \ \forall \tau_h \in \Sigma_h, \\ \tilde{B}_\nu(\sigma_h^1, \eta_h^1) = -\int_\omega \left\{ [[\sigma_h^2, \sigma_h^1]] + \lambda[[\sigma_{\theta_0}, \sigma_h^1]] + \tilde{\mathfrak{F}} \right\} \eta_h^1 d\omega, \ \forall \eta_h^1 \in \mathscr{V}_h, \\ A_\nu(\sigma_h^2, \tau_h) + \tilde{B}_\nu(\tau_h, \psi_h^2) = 0, \ \forall \tau_h \in \Sigma_h, \\ \tilde{B}_\nu(\sigma_h^2, \eta_h^2) = -\int_\omega \left\{ -[[\sigma_h^1, \sigma_h^1]] + [[\sigma_\theta, \sigma_\theta]] \right\} \eta_h^2 d\omega, \ \forall \eta_h^2 \in \mathscr{V}_h. \end{cases} \tag{4.3.5}$$

We define the following linear operator

$$T_h : \boldsymbol{g} = (g^1, g^2) \in (H^{-1}(\omega))^2 \longrightarrow T_h \boldsymbol{g} = ((\eta_h^1, \eta_h^2), (\tau_h^1, \tau_h^2)) \in \mathscr{Y}_h, \tag{4.3.6}$$

where $T_h \boldsymbol{g}$ is the unique solution of

$$\left. \begin{cases} A_\nu(\varsigma_h, \tau_h^\kappa) + \tilde{B}_\nu(\varsigma_h, \eta_h^\kappa) = 0, \forall \varsigma_h \in \Sigma_h, \\ \tilde{B}_\nu(\tau_h^\kappa, \zeta_h) = - < g^\kappa, \zeta_h >, \forall \zeta_h \in \mathscr{V}_h, \end{cases} \right\} \kappa = 1, 2, \tag{4.3.7}$$

where $< ., . >$ is the duality pairing between $H^{-1}(\omega)$ and $H_0^1(\omega)$.

We infer from the definitions of the operators T_h and G_θ that the scheme (4.3.5) is equivalent to the following discrete operator equation

$$\begin{cases} \text{Find } (\lambda, \boldsymbol{X}_h = ((\psi_h^1, \psi_h^2), (\sigma_h^1, \sigma_h^2))) \in \mathbb{R} \times \mathscr{Y}_h \text{ such that,} \\ F_h(\lambda, \boldsymbol{X}_h) \equiv \boldsymbol{X}_h + T_h G_\theta(\lambda, \boldsymbol{X}_h) = 0. \end{cases} \tag{4.3.8}$$

4.4 Approximation of a branch of nonsingular solutions: General case

In this section, we will remember the main Theorems that proven by F. Brezzi et al in [13], which relate to the existence and the convergence of the approximate problem of the following form:

$$F(\lambda, u) = u + TG(\lambda, u). \tag{4.4.1}$$

Where $G : \Lambda \times V \to W$ is mapping of class $\mathscr{C}^i$, with Λ be a compact interval of $\mathbb{R}$, V and W be two Banach spaces.
The operator $T : W \to V$ be a linear continuous mapping (i.e $T \in \mathscr{L}(W; V)$).

In all the sequel, We assume that:
(i) for all $(\lambda, u) \in \Lambda \times V$, the operator $TD_uG(\lambda, u) \in \mathscr{L}(V; V)$ is compact
(ii) there exists a branch $\{(\lambda, u(\lambda)); \lambda \in \Lambda\}$ of nonsingular solutions of the equation

$$F(\lambda, u) = 0. \tag{4.4.2}$$

The nonsingular solutions of the equation (4.4.2), in the sense of Brezzi et al. [13], means that:

$$\begin{cases} \lambda \to \boldsymbol{u}(\lambda) \text{ is a continuous function from } \Lambda \text{ into } V, \\ F(\lambda, \boldsymbol{u}(\lambda)) = 0, \\ D_{\boldsymbol{u}} F(\lambda, \boldsymbol{u}(\lambda)) \text{ is an isomorphism of } V, \end{cases} \tag{4.4.3}$$

So, the implicit function theorem show that the mapping $\lambda \in \Lambda \to u(\lambda)$ is a $\mathscr{C}^1$ function from Λ into V .

Let us introduce three Banach spaces $\tilde{V}, \tilde{W}$ and Z such that

$$Z \hookrightarrow V \hookrightarrow \tilde{V}, \ \tilde{W} \hookrightarrow W \text{ with continuous embeddings,} \tag{4.4.4}$$

We assume that, for $u \in Z, DG(\lambda, u)$ may be extended as an operator of $\mathscr{L}(\mathbb{R} \times \tilde{V}; \tilde{W})$ with the following properties:

(i) for $(\lambda, u) \in \Lambda \times Z$, the operator

$$TD_u G(\lambda, u) \in \mathscr{L}(\tilde{V}; \tilde{V}) \text{is compact} \tag{4.4.5}$$

(ii) the mapping $(\lambda, u) \in \Lambda \times Z \to DG(\lambda, u) \in \mathscr{L}(\mathbb{R} \times \tilde{V}; \tilde{W})$ is Lipschitz continuous on the bounded subsets of $\Lambda \times Z$, i.e. there exists a function $L : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ monotonically increasing with respect to each variable such that for all $\lambda, \lambda* \in \Lambda$, and all $u, u* \in Z$

$$\begin{aligned} &\|DG(\lambda^*, u^*) - DG(\lambda, u)\|_{\mathscr{L}(\mathbb{R}\times\tilde{V};\tilde{W})} \\ &\qquad \le L\left(|\lambda^*| + \|u^*\|_Z, |\lambda| + \|u\|_Z\right)\left(|\lambda^* - \lambda| + \|u^* - u\|_Z\right) \end{aligned} \tag{4.4.6}$$

Moreover, we suppose that for any $\lambda \in \Lambda, u(\lambda)$ belongs to Z and the function

$$\lambda \to \boldsymbol{u}(\lambda) \text{ is a continuous function from } \Lambda \text{ into } Z. \tag{4.4.7}$$

Now, we define the approximate problem of (4.4.1)

Let V_h be a finite-dimensional subspace of the space Z, where $h > 0$ is a real positive parameter which will tend to zero and an operator $T_h \in \mathscr{L}(\tilde{W}, V_h)$. We set:

$$F_h(\lambda, u_h) = u_h + T_h G(\lambda, u_h). \quad \lambda \in \Lambda, u_h \in V_h \tag{4.4.8}$$

So, the approximate problem consists in finding a pair $(\lambda, u_h) \in \Lambda \times V_h$ solution of the equation

$$F_h(\lambda, u_h) = 0 \tag{4.4.9}$$

The following Theorems show the existence of a branch $(\lambda, u_h(\lambda)); \lambda \in \Lambda$ of solutions of the equation (4.4.9) which approximates the branch of nonsingular solutions of (4.4.2) under specific assumption:

We assume that the following inverse inequality holds

$$\|v_h\|_Z \le Ch^{-r}\|v_h\|_{\tilde{V}} \tag{4.4.10}$$

for some $r > 0$ and for all $v_h \in V_h$, with $C > 0$ be a constant independent of h and λ.

Moreover, we suppose that there exists a function $\Pi_h u : \lambda \to \Pi_h u(\lambda) \in V_h$ such that

$$\|\Pi_h u(\lambda^*) - \Pi_h u(\lambda)\|_{\tilde{V}} \le C\,|\lambda^* - \lambda| \quad \text{for all } \lambda, \lambda^* \in \Lambda \tag{4.4.11}$$

and

$$\lim_{h\to 0} \sup_{\lambda\in\Lambda} \|u(\lambda) - \Pi_h u(\lambda)\|_Z = 0 \tag{4.4.12}$$

Theorem 4.4.1. *Assume the hypotheses (4.4.5), (4.4.6), (4.4.7), (4.4.10), (4.4.11) and (4.4.12). Assume in addition that:*

$$\lim_{h\to 0} \|T - T_h\|_{\mathscr{L}(\tilde{W};\tilde{V})} = 0 \tag{4.4.13}$$

Then, under the condition

$$\lim_{h\to 0} \sup_{\lambda\in\Lambda} h^{-r}\|F_h(\lambda, \Pi_h u(\lambda))\|_{\tilde{V}} = 0, \tag{4.4.14}$$

and for $h < h_0$ small enough, there exists a constant $b > 0$ independent of h and a unique $\mathscr{C}^1$ mapping $\lambda \in \Lambda \to u_h(\lambda) \in V_h$ such that for all $\lambda \in \Lambda$

$$\begin{gathered} F_h(\lambda, u_h(\lambda)) = 0, \\ \|u_h(\lambda) - \Pi_h u(\lambda)\|_{\tilde{V}} \leq b h^r \end{gathered} \tag{4.4.15}$$

Moreover, we have for some constant $K_0 > 0$ independent of h and λ:

$$\|u_h(\lambda) - u(\lambda)\|_{\tilde{V}} \leq K_0 \left\{ \|u(\lambda) - \Pi_h u(\lambda)\|_{\tilde{V}} + \|(T - T_h) G(\lambda, u(\lambda))\|_{\tilde{V}} \right\}. \tag{4.4.16}$$

Proof. see [13] □

Remark 4.4.1. *[13] We can replace the condition (4.4.14) in Theorem [4.4.1] with the following relation*

$$\begin{gathered} \lim_{h\to 0} \sup_{\lambda\in\Lambda} h^{-r}\|u(\lambda) - \Pi_h u(\lambda)\|_{\tilde{V}} = 0, \\ \lim_{h\to 0} \sup_{\lambda\in\Lambda} h^{-r}\|(T_h - T) G(\lambda, u(\lambda))\|_{\tilde{V}} = 0. \end{gathered} \tag{4.4.17}$$

Corollary 4.4.1. *[13] As a result of Theorem [4.4.1], we obtain*

$$\lim_{h\to 0} \sup_{\lambda\in\Lambda} \|u_h(\lambda) - u(\lambda)\|_{\tilde{V}} = 0 \tag{4.4.18}$$

The next Theorem proves the approximate of the first derivative $u'(\lambda)$ of the function $u(\lambda)$, for that, let we introduce the following mappings:

$$\begin{gathered} F^{(1)} : \Lambda \times V^2 \to V, \\ G^{(1)} : \Lambda \times V^2 \to W, \\ F_h^{(1)} : \Lambda \times V_h^2 \to V_h. \end{gathered} \tag{4.4.19}$$

where

$$f^{(1)}\left(x, y, y^{(1)}\right) = D_x f(x, y) + D_y f(x, y) . y^{(1)}$$

We introduce a function $\Pi_h u' : \lambda \in \Lambda \to \Pi_h u'(\lambda) \in V_h$ and we assume that

$$\lim_{h\to 0} \sup_{\lambda\in\Lambda} h^{-r}\|u'(\lambda) - \Pi_h u'(\lambda)\|_{\tilde{V}} = 0, \tag{4.4.20}$$

Then we get:

Theorem 4.4.2. *Assume that the assumption (4.4.20) and the hypotheses of Theorem (4.4.1)are realized. Then, we have for some constant K_1 independent of h and λ*

$$\begin{aligned} \|u_h'(\lambda) - u'(\lambda)\|_{\tilde{V}} \leq K_1 (h^{-r} [&\|u(\lambda) - \Pi_h u(\lambda)\|_{\tilde{V}} + \|(T - T_h) G(\lambda, u(\lambda))\|_{\tilde{V}}] \\ &+ \|u(\lambda) - \Pi_h u(\lambda)\|_Z + \|u'(\lambda) - \Pi_h u'(\lambda)\|_{\tilde{V}} \\ &+ \|(T - T_h) G^1(\lambda, u(\lambda), u'(\lambda))\|_{\tilde{V}}) \end{aligned} \tag{4.4.21}$$

Proof. see [13] □

Corollary 4.4.2. *[13] Under the additional hypothesis (4.4.17) and using (4.4.12) and (4.4.20), and as a consequence of the Theorem [4.4.2] we obtain*

$$\lim_{h\to 0}\sup_{\lambda\in\Lambda}\|u_h(\lambda)-u(\lambda)\|_{\tilde{V}}=0 \tag{4.4.22}$$

Finally, it remains for us to prove the approximation of the m^{th} derivative $u^{(m)}(\lambda)$ of $u(\lambda)$; for this purpose, we suppose that the mapping:

$$\begin{aligned} G :&\Lambda\times V\to W\\ &(\lambda,u)\to G(\lambda,u)\end{aligned}$$

is a $\mathscr{C}^m$ mapping, for $m\geq 2$. So, the function $u:\lambda\in\Lambda\to u(\lambda)$ is of class $\mathscr{C}^m$, and we suppose that, for $u\in Z$, the l^{th} derivative $D^lG(\lambda,u)\in\mathscr{L}_l(\mathbb{R}\times V;W), 2\leq l\leq m$, may be extended as a l-linear continuous operator of $(\mathbb{R}\times Z)^{l-1}\times(\mathbb{R}\times\tilde{V})$ into $\tilde{W}$, such that for all $\lambda,\lambda^*\in\Lambda$ and all $u,u^*\in Z$

$$\begin{aligned}&\|D^lG(\lambda^*,u^*)-D^lG(\lambda,u)\|_{\mathscr{L}_l(\mathbb{R}\times Z,\ldots\mathbb{R}\times Z,\mathbb{R}\times\tilde{V};W)}\\ &\leq L(\mid\lambda^*\mid+\|u^*\|_Z,\mid\lambda\mid+\|u\|_Z)\left(\mid\lambda^*-\lambda\mid+\|u^*-u\|_Z\right),\quad 2\leq l\leq m.\end{aligned} \tag{4.4.23}$$

Moreover, we assume that

$$\text{the function }\lambda\to u(\lambda)\text{ is of class }\mathscr{C}^{m-1} from\ \Lambda\ into\ Z \tag{4.4.24}$$

As in (4.4.19) we introduce the following mappings, which define by induction

$$\begin{aligned} F^{(l)}&:\Lambda\times V^{l+1}\to V,\\ G^{(l)}&:\Lambda\times V^{l+1}\to W,\\ F_h^{(l)}&:\Lambda\times V_h^{l+1}\to V_h,\quad 2\leq l\leq m\end{aligned}$$

where

$$\begin{aligned} f^{(l+1)}\left(x,y,y^{(1)},...,y^{(l+1)}\right)&=D_xf^{(l)}\left(x,y,y^{(1)},...,y^{(l)}\right)\\ &+D_yf^{(l)}\left(x,y,y^{(1)},...,y^{(l)}\right).y^{(1)}+\sum_{i=1}^{l}D_{y^{(i)}}f^{(l)}\left(x,y,y^{(1)},...,y^{(l)}\right).y^{(i+1)}\end{aligned}$$

We introduce a function $\Pi_hu^{(l)}:\lambda\in\Lambda\to\Pi_hu^{(l)}(\lambda)\in V_h,\ 2\leq l\leq m$ and we suppose that

$$\begin{aligned}&\lim_{h\to 0}\sup_{\lambda\in\Lambda}\|u^{(l)}(\lambda)-\Pi_hu^{(l)}(\lambda)\|_Z=0\quad 1\leq l\leq m-1\\ &\lim_{h\to 0}\sup_{\lambda\in\Lambda}\|u^{(l)}(\lambda)-\Pi_hu^{(l)}(\lambda)\|_{\tilde{V}}=0\end{aligned} \tag{4.4.25}$$

So, we get

Theorem 4.4.3. *Assume that the hypotheses (4.4.23), (4.4.24), and (4.4.25) hold. In addition, we assume that all the hypotheses of Theorem (4.4.2) are realized. Then, $\lambda\to u_h(\lambda)$ is a function of class $\mathscr{C}^m$ from Λ into V_h and we have for some constant $K_m>0$ independent of λ and h*

$$\begin{aligned}\|u_h^{(m)}(\lambda) - u^{(m)}(\lambda)\|_{\tilde{V}} \leq{} & K_m(h^{-mr}\,(\|u(\lambda) - \Pi_h u(\lambda)\|_{\tilde{V}} + \|(T - T_h)G(\lambda, u(\lambda))\|_{\tilde{V}}) \\ & + h^{(1-m)r}\|u(\lambda) - \Pi_h u(\lambda)\|_Z \\ & + \sum_{i=1}^{m} h^{(1-m)r}(\|u_h^{(l)}(\lambda) - \Pi_h u^{(l)}(\lambda)\|_{\tilde{V}} + \|(T - T_h)G^l(\lambda, u(\lambda), ..., u^l(\lambda))\|_{\tilde{V}}))\end{aligned} \tag{4.4.26}$$

Proof. see [13] □

4.5 Approximation of a branch of nonsingular solutions for Marguerre-von Kármán equations

In the third chapter of this thesis, we concluded that there are two branches of solutions to the bifurcation problem in Marguerre-von Kármán equations, and this for λ in the neighborhood of the characteristic value λ_0 of the linearized problem and for small second side (i.e $\|(\Delta^2)^{-1}f + \tilde{\theta}\|_{2,2}$ is small). one of these branches is a branch of singular solutions and the other is a branch of nonsingular solutions.

We rememeber that in the sense of Brezzi et al. [13] $\{(\lambda, \boldsymbol{X}(\lambda)); \lambda \in \Lambda\}$ is branch of nonsingular solutions of the operator equation (4.2.14), i.e.,

$$\begin{cases} \lambda \to \boldsymbol{X}(\lambda) \text{ is a continuous function from } \Lambda \text{ into } \mathscr{Y}, \\ F(\lambda, \boldsymbol{X}(\lambda)) = 0, \\ D_{\boldsymbol{X}}F(\lambda, \boldsymbol{X}(\lambda)) \text{ is an isomorphism of } \mathscr{Y}, \end{cases} \tag{4.5.1}$$

under the assumptions that

$$\Lambda \text{ is a compact of } \mathbb{R}. \tag{4.5.2}$$

Therefore, we want to prove the existence of a branch $\{(\lambda, X_h(\lambda)); \lambda \in \Lambda\}$ of solutions of the discrete operator equation (4.3.8) which approximates the branch of nonsingular solutions of the continuous operator equation (4.2.14). For that purpose we will check the conditions of the Theories (4.4.1), (4.4.2), (4.4.3).
Let we define the following spaces

$$\tilde{\mathscr{Y}} = (H_0^1(\omega))^2 \times (L^2(\omega)_s^4)^2, \quad \tilde{\mathscr{W}} = (H^{-1}(\omega))^2, \quad \mathfrak{Z} = (W_0^{1.p}(\omega))^2 \times (L^p(\omega)_s^4)^2, \tag{4.5.3}$$

satisfying

$$\mathfrak{Z} \hookrightarrow \mathscr{Y} \hookrightarrow \tilde{\mathscr{Y}},\ \tilde{\mathscr{W}} \hookrightarrow \mathscr{W} \text{ with continuous embeddings,} \tag{4.5.4}$$

where $p = \dfrac{2}{1-\varepsilon}$ with $\varepsilon > 0$, arbitrary small.

Throughout this Section, we always denote by $C > 0$ a generic constant independent of λ and h.

We assume that

$$\lambda \to \boldsymbol{X}(\lambda) \text{ is a continuous function from } \Lambda \text{ into } \mathfrak{Z}. \tag{4.5.5}$$

We gather in the next lemmas some auxiliary results, which are essential for the proof of Theorem 4.5.1.

We begin with the following approximation properties of the operator T_h. This result was proven by Reinhart in [67] based on the results of approximation properties of the biharmonic problem that where found in the previous paragraph.

Lemma 4.5.1. *[67] For all $\boldsymbol{g} \in \tilde{\mathscr{W}}$, we have the estimates*

$$\|(T - T_h)\boldsymbol{g}\|_{\tilde{\mathscr{Y}}} \le Ch^{1/2}|\log h|\|\boldsymbol{g}\|_{\tilde{\mathscr{W}}},\ (\textit{piecewise lineare element}). \tag{4.5.6}$$

$$\|(T - T_h)\boldsymbol{g}\|_{\tilde{\mathscr{Y}}} \le Ch\|\boldsymbol{g}\|_{\tilde{\mathscr{W}}},\ (\textit{piecewise quadratic element}), \tag{4.5.7}$$

where

$$\|\boldsymbol{g}\|_{\tilde{\mathscr{W}}} = (\|g^1\|^2_{-1,\omega} + \|g^2\|^2_{-1,\omega})^{1/2} \text{ with } \boldsymbol{g} = (g^1, g^2),$$

$$\|\boldsymbol{Y}\|_{\tilde{\mathscr{Y}}} = (\|\eta^1\|^2_{1,\omega} + \|\eta^2\|^2_{1,\omega} + \|\tau^1\|^2_{0,\omega} + \|\tau^2\|^2_{0,\omega})^{1/2} \text{ with } \boldsymbol{Y} = ((\eta^1, \eta^2), (\tau^1, \tau^2)) \in \mathscr{Y}.$$

Now, we will check some properties the mapping $DG_\theta(\lambda, \boldsymbol{X})$.
From (3.1.6) We have

$$G_\theta(\lambda, \boldsymbol{X}) = \Big(-\lambda[[\sigma_{\theta_0}, \sigma^1]] - [[\sigma^1, \sigma^2]] - \tilde{f},\ [[\sigma^1, \sigma^1]] - [[\sigma_\theta, \sigma_\theta]]\Big). \tag{4.5.8}$$

clearly that

$$D_\lambda G_\theta(\lambda, \boldsymbol{X}) = \left(-[[\sigma_{\theta_0}, \sigma^1]],\ 0\right),$$

and for all $\boldsymbol{X} = ((\psi^1, \psi^2), (\sigma^1, \sigma^2))$ and $\boldsymbol{Y} = ((\eta^1, \eta^2), (\tau^1, \tau^2))$, we have

$$\begin{aligned}
D_{\boldsymbol{X}} G_\theta(\lambda, \boldsymbol{X}) \cdot \boldsymbol{Y} &= G_\theta(\lambda, \boldsymbol{X} + \boldsymbol{Y}) - G_\theta(\lambda, \boldsymbol{X}) \\
&= \Big(-\lambda[[\sigma_{\theta_0}, \sigma^1 + \tau^1]] - [[\sigma^1 + \tau^1, \sigma^2 + \tau^2]] - \tilde{f},\ [[\sigma^1 + \tau^1, \sigma^1 + \tau^1]] - [[\sigma_\theta, \sigma_\theta]]\Big) \\
&\quad - \Big(-\lambda[[\sigma_{\theta_0}, \sigma^1]] - [[\sigma^1, \sigma^2]] - \tilde{f},\ [[\sigma^1, \sigma^1]] - [[\sigma_\theta, \sigma_\theta]]\Big) \\
&= \left(-[[\lambda\sigma_{\theta_0}, \tau^1]] - [[\tau^1, \sigma^2]] - [[\sigma^1, \tau^2]],\ [[\tau^1, \sigma^1]] + [[\sigma^1, \tau^1]]\right), \\
&= \left(-[[\lambda\sigma_{\theta_0} + \sigma^2, \tau^1]] - [[\sigma^1, \tau^2]],\ 2[[\sigma^1, \tau^1]]\right).
\end{aligned} \tag{4.5.9}$$

Consequently, we get the following two Lemmas.

Lemma 4.5.2. *For $(\lambda, \boldsymbol{X}) \in \Lambda \times \mathfrak{Z}$, we have*

$$TD_{\boldsymbol{X}} G_\theta(\lambda, \boldsymbol{X}) \in \mathscr{L}(\tilde{\mathscr{Y}}, \tilde{\mathscr{Y}}) \textit{ compact}. \tag{4.5.10}$$

Proof. From (4.5.9) we have:

$$D_{\boldsymbol{X}} G_\theta(\lambda, \boldsymbol{X}) \cdot \boldsymbol{Y} = \left(-[[\lambda\sigma_{\theta_0} + \sigma^2, \tau^1]] - [[\sigma^1, \tau^2]],\ 2[[\sigma^1, \tau^1]]\right),$$

The condition (4.2.5) imply that $\sigma_{\theta_0} \in H^1(\omega) \subset (L^p(\omega)^4_s)$, then If $\boldsymbol{X} \in \mathfrak{Z}$ and $\boldsymbol{Y} \in \tilde{\mathscr{Y}}$, we have

$$D_{\boldsymbol{X}} G_\theta(\lambda, \boldsymbol{X}) \cdot \boldsymbol{Y} \in L^s(\omega)^2, \text{ where } \frac{1}{s} = \frac{1}{p} + \frac{1}{2}, \quad (s > 1)$$

that is $D_{\boldsymbol{X}} G_\theta(\lambda, \boldsymbol{X}) \cdot \boldsymbol{Y} \in L^{\frac{2}{2-\epsilon}}(\omega)^2$, so by the Sobolev embedding theorem we get:

$$D_{\boldsymbol{X}} G_\theta(\lambda, \boldsymbol{X}) \cdot \boldsymbol{Y} \in H^{-1}(\omega)^2$$

. By the same argument for $(\sigma_{\theta_0}) \in (L^p(\omega)^4_s)$ we get $D_\lambda G_\theta(\lambda, \boldsymbol{X}) \in L^s(\omega)^2, (s = \frac{2}{2-\epsilon})$.

Hence, for $\boldsymbol{X} \in \mathfrak{Z},\ DG_\theta(\lambda, \boldsymbol{X}) \in \mathscr{L}(\mathbb{R} \times \tilde{\mathscr{Y}}, \tilde{\mathscr{W}})$ and

$$\|DG_\theta(\lambda, \boldsymbol{X})\|_{\mathscr{L}(\mathbb{R} \times \tilde{\mathscr{Y}}, \tilde{\mathscr{W}})} \leq C \, \|\boldsymbol{X}\|_{\mathfrak{Z}}$$

and since $T \in \mathscr{L}(\tilde{\mathscr{W}}, \tilde{\mathscr{Y}})$ is compact, then from the Proposition (1.2.1) we conclude that the operator $T D_{\boldsymbol{X}} G_\theta(\lambda, \boldsymbol{X}) \in \mathscr{L}(\tilde{\mathscr{Y}}, \tilde{\mathscr{Y}})$ is compact for $\boldsymbol{X} \in \mathfrak{Z}$. $\square$

On the other hand, we have the mappings $\boldsymbol{X} \to D_{\boldsymbol{X}} G_\theta(\lambda, \boldsymbol{X})$ and $\lambda \to D_\lambda G_\theta(\lambda, \boldsymbol{X})$ are linear continuous mappings, we conclude the following Lemma

Lemma 4.5.3. *The mapping*

$$(\lambda, \boldsymbol{X}) \in \mathbb{R} \times \mathfrak{Z} \to DG_\theta(\lambda, \boldsymbol{X}) \in \mathscr{L}(\mathbb{R} \times \tilde{\mathscr{Y}}, \tilde{\mathscr{W}}), \tag{4.5.11}$$

is Lipschitz continuous on the bonded subset of $\Lambda \times \mathfrak{Z}$, *that is there exist a constant* C *such that*

$$\begin{aligned} \|DG_\theta(\lambda, \boldsymbol{X}) - DG_\theta(\lambda^*, \boldsymbol{X}^*)\|_{\mathscr{L}(\mathbb{R} \times \tilde{\mathscr{Y}}, \tilde{\mathscr{W}})} &\leq C \left\{ |\lambda - \lambda^*| + \|\boldsymbol{X} - \boldsymbol{X}^*\|_{\mathfrak{Z}} \right\}, \\ &\quad \forall (\lambda, \lambda^*) \in \Lambda^2, \forall (\boldsymbol{X}, \boldsymbol{X}^*) \in \mathfrak{Z}^2. \end{aligned} \tag{4.5.12}$$

Moreover, since $D^2 G_\theta$ *does not depend on* λ *and* $\boldsymbol{X}$, *then this implies that*

$$D^m G_\theta \text{ is Lipschitz continuous for } m \geq 1. \tag{4.5.13}$$

Remark 4.5.1. *We have*

$$DG_\theta(\lambda, \boldsymbol{X}) = DG_0(\lambda, \boldsymbol{X}) = DG(\lambda, \boldsymbol{X}), \tag{4.5.14}$$

where $DG(\lambda, \boldsymbol{X})$ *is defined in [67], as follows*

$$G(\lambda, \boldsymbol{X}) = \left(-\lambda[[\sigma_{\theta_0}, \sigma^1]] - [[\sigma^1, \sigma^2]] - f,\ [[\sigma^1, \sigma^1]] \right). \tag{4.5.15}$$

Now, we consider a uniformly regular family of triangulations $\mathscr{T}_h$. Therefore, the following inverse inequality is valid

$$\|\eta_h\|_{k,p} \leq C h^{\frac{2}{p} - 1} \|\eta_h\|_{k,2},\ \forall \eta_h \in \mathscr{X}_h^{(k)}. \tag{4.5.16}$$

This implies

$$\|\boldsymbol{Y}_h\|_{\mathfrak{Z}} \leq C h^{-\varepsilon} \|\boldsymbol{Y}_h\|_{\tilde{\mathscr{Y}}},\ \forall \boldsymbol{Y}_h \in \mathscr{Y}_h. \tag{4.5.17}$$

Let π_h be the classical interpolation operator defined by

$$\pi_h :\ \boldsymbol{Y} = (\boldsymbol{\eta}, \boldsymbol{\tau}) \in (H^2(\omega) \cap H_0^1(\omega))^2 \times (H^1(\omega)^4_s)^2 \longrightarrow \pi_h \boldsymbol{Y} = (\boldsymbol{\eta}_h, \boldsymbol{\tau}_h) \in \mathscr{Y}_h, \tag{4.5.18}$$

where the function $\boldsymbol{\eta}_h \in \mathscr{V}_h^2$ interpolates $\boldsymbol{\eta}$ at the usual finite element nodes and $\boldsymbol{\tau}_h$ is the orthogonal projection in $(L^2(\omega)^4)^2$ of $\boldsymbol{\tau}$ onto the space Σ_h^2.

Finally, we will also need the following result.

Lemma 4.5.4. *[67] If* $\boldsymbol{\eta} \in (H^{k+1}(\omega))^2$ *and* $\boldsymbol{\tau} \in (H^k(\omega)^4_s)^2$*. Then*

$$\|\boldsymbol{Y} - \pi_h \boldsymbol{Y}\|_{\tilde{\mathscr{Y}}} + h^\varepsilon \|\boldsymbol{Y} - \pi_h \boldsymbol{Y}\|_3 \leq C h^k \{\|\boldsymbol{\eta}\|_{k+1,\omega} + \|\boldsymbol{\tau}\|_{k,\omega}\}. \tag{4.5.19}$$

We now give the main result of this chapter.

Theorem 4.5.1. *Assume that the hypothesis* (4.2.5) *and* (4.5.2) *hold, and that* $\{(\lambda, \boldsymbol{X}(\lambda)); \lambda \in \Lambda\}$ *is a branch of nonsingular solutions of the Marguerre-von Kármán equations* (4.2.14)*. Then, for all* $\varepsilon > 0$ *sufficiently small, there exists a constant* $b_\varepsilon > 0$ *independent of* h *and, for* $h \leq h_0$ *small enough, a unique branch* $\{(\lambda, \boldsymbol{X}_h(\lambda)); \lambda \in \Lambda\}$ *of solutions of the discrete operator equation* (4.3.8) *such that*

$$\sup_{\lambda \in \Lambda} \|\boldsymbol{X}(\lambda) - \boldsymbol{X}_h(\lambda)\|_{\tilde{\mathscr{Y}}} \leq b_\varepsilon h^\varepsilon. \tag{4.5.20}$$

Furthermore, $\lambda \to \boldsymbol{X}_h(\lambda)$ *is a function of class* $\mathscr{C}^\infty$ *from* Λ *into* $\mathscr{Y}_h$*. If, in addition,* $(\psi^1, \psi^2) \in (H^4(\omega))^2$*, we obtain the error estimates*

$$\|\boldsymbol{\psi}_h^{(m)}(\lambda) - \boldsymbol{\psi}^{(m)}(\lambda)\|_{(H_0^1(\omega))^2} + \|\boldsymbol{\sigma}_h^{(m)}(\lambda) - \boldsymbol{\sigma}^{(m)}(\lambda)\|_{(L^2(\omega)^4_s)^2} \leq C_\varepsilon L_k^m(h), \tag{4.5.21}$$

where the exponent (m) *means the derivative of order* m *with respect to* λ*,* $\boldsymbol{\psi} = (\psi^1, \psi^2)$*,* $\boldsymbol{\sigma} = (\sigma^1, \sigma^2)$*,* $C_\varepsilon > 0$ *is a constant independent of* h *and* λ*, and*

$$L_k^m(h) = \begin{cases} h^{\frac{1}{2} - m\varepsilon} \mid \log h \mid & (\textit{linear elements}), \\ h^{1 - m\varepsilon} & (\textit{quadratic elements}). \end{cases}$$

Proof. We want to apply Theorems (4.4.1), (4.4.2) and (4.4.3).
First, use (4.5.6) and (4.5.7) gives

$$\lim_{h \to 0} \|T - T_h\|_{\mathscr{L}(\tilde{\mathscr{W}}, \tilde{\mathscr{Y}})} = 0. \tag{4.5.22}$$

It follows from (4.5.13) that G_θ is a mapping of class $\mathscr{C}^\infty$, so that we have

$$\lambda \to \boldsymbol{X}(\lambda) \text{ is a function of class } \mathscr{C}^\infty \text{ from } \Lambda \text{ into } \mathscr{Y}. \tag{4.5.23}$$

Using the continuity of the mapping $\lambda \to \boldsymbol{X}(\lambda)$, we get

$$\|\pi_h \boldsymbol{X}(\lambda) - \pi_h \boldsymbol{X}(\lambda^*)\|_{\tilde{\mathscr{Y}}} \leq C \mid \lambda - \lambda^* \mid, \ \forall \lambda, \lambda^* \in \Lambda. \tag{4.5.24}$$

From (4.5.19) with $k = 1$, we obtain

$$\sup_{\lambda \in \Lambda} \Big\{ \|\boldsymbol{X}^{(m)}(\lambda) - \pi_h \boldsymbol{X}^{(m)}(\lambda)\|_{\tilde{\mathscr{Y}}} + h^\varepsilon \|\boldsymbol{X}^{(m)}(\lambda) - \pi_h \boldsymbol{X}^{(m)}(\lambda)\|_3 \Big\} \leq C h, \quad \forall \lambda \in \Lambda, m \geq 0, \tag{4.5.25}$$

where $\boldsymbol{X}^{(0)}$ stand for $\boldsymbol{X}$.
Then, the above relation implies that

$$\lim_{h \to 0} \sup_{\lambda \in \Lambda} \|\boldsymbol{X}^{(m)}(\lambda) - \pi_h \boldsymbol{X}^{(m)}(\lambda)\|_3 = 0, \tag{4.5.26}$$

$$\lim_{h \to 0} \sup_{\lambda \in \Lambda} \|\boldsymbol{X}^{(m)}(\lambda) - \pi_h \boldsymbol{X}^{(m)}(\lambda)\|_{\tilde{\mathscr{Y}}} = 0, \tag{4.5.27}$$

$$\lim_{h\to 0}\sup_{\lambda\in\Lambda} h^{-\varepsilon}\|\boldsymbol{X}^{(m)}(\lambda)-\pi_h\boldsymbol{X}^{(m)}(\lambda)\|_{\tilde{\mathscr{Y}}}=0. \tag{4.5.28}$$

Taking into account the Lemma 4.5.1, we obtain

$$\lim_{h\to 0}\sup_{\lambda\in\Lambda} h^{-\varepsilon}\|(T-T_h)G_\theta(\lambda,\boldsymbol{X}(\lambda))\|_{\tilde{\mathscr{Y}}}=0. \tag{4.5.29}$$

On the other hand, we have

$$\|F_h(\lambda,\pi_h\boldsymbol{X}(\lambda))\|_{\tilde{\mathscr{Y}}}\leq C\{\|\boldsymbol{X}(\lambda)-\pi_h\boldsymbol{X}(\lambda)\|_{\tilde{\mathscr{Y}}}+\|(T-T_h)G(\lambda,\boldsymbol{X}(\lambda))\|_{\tilde{\mathscr{Y}}}\}, \tag{4.5.30}$$

hence it follows from (4.5.28) and (4.5.26) that

$$\lim_{h\to 0}\sup_{\lambda\in\Lambda} h^{-\varepsilon}\|F_h(\lambda,\pi_h\boldsymbol{X}(\lambda))\|_{\tilde{\mathscr{Y}}}=0. \tag{4.5.31}$$

In conclusion, for (4.5.12), (4.5.5), (4.5.17), (4.5.24), (4.5.26), (4.5.22), (4.5.31), (4.5.27), (4.5.13) and (4.5.23), we check all the assumptions of Theorems (4.4.1), (4.4.2) and (4.4.3), with $r=\varepsilon$. Then, we show that there exists a constant $b_\varepsilon>0$ independent of h and, for $h\leq h_0(\varepsilon)$ small enough, a unique function of class $\mathscr{C}^\infty$: $\lambda\in\Lambda\to\boldsymbol{X}_h(\lambda)=(\boldsymbol{\psi}_h(\lambda),\boldsymbol{\sigma}_h(\lambda))\in\mathscr{Y}_h$, such that

$$F_h(\lambda,\boldsymbol{X}_h(\lambda))=0,\ \forall\lambda\in\Lambda. \tag{4.5.32}$$

Furthermore, (4.5.20) and (4.5.21) hold. □

Conclusion and perspectives

In this thesis, the problem of bifurcation in the Marguerre-von Kármán equations was treated, starting from the bifurcation points and moving to the branches of the bifurcated solutions from the trivial solution. We also treated the perturbed bifurcation problem in these equations; finally we arrive at the existence of two branches of the solution, one of which is a branch of non-singular solutions and the other is a branch of $\epsilon-$singular solutions, and we based our study on generalizing the results of Ciarlet [29] from the plate to shallow shells under some assumptions on the function θ, which defines the middle surface of the shallow shell. In the second work of this thesis, based on the methods and techniques are due to Reinhart [67] and Brezzi et al. [13], we studied the numerical approximation of the branch of nonsingular solutions of Marguerre-von Kármán equations for nonlinearly elastic shallow shells; Also Under some assumptions on the function θ.

If we restrict our attention to the important case where function $\theta \equiv 0$ in $\bar{\omega}$, the shallow shell becomes a plate. Then we recover the numerical approximation for the branch of nonsingular solutions of the von Kármán equations, which was done by Reinhart [67]. Also, we recover the bifurcation problem in von Kármán equations, which was done by Ciarlet [29].

In our future work, we shall study the numerical approximation of the branch of singular solutions with a turning point and a bifurcation point for the Marguerre-von Kármán equations. Then we study the numerical experiments that justify these theoretical results.

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