University of kasdi Merbah Ouargla
Faculty of Mathematics and matrical Science
Department Mathics

Thesis For the MASTER Degree in Modilisation and Numerical Analysis
presented by: **Ghali Chaima**

ON THE STUDY THE EXISTENCE AND STABILITY SOLUTION OF HYPERPOLIC PROBLEM WITH LOGARITMIC NONLINEARITY

In front of the jury composed of :

President:	Ziane Djeloul	MCA	University of Ouargla
Supervisor:	Otmani Sadok	MCB	University of Ouargla
Examiner:	Kaliche Keltoum	MCB	University of Ouargla
Examiner:	Lacheheb Ilyes	MCB	University of Ouargla

LMD Master 2023/2024

Acknowledgment

First, we thank God for givinig us the strength to achieve our goal and our accomplish our work.

First of all, i would like to thank my supervisor ***Otmani Sadok*** *for their patience, their attendance and their especially their wise advice, who help me to make this thesis.*

I thank as well, professor ***Ziane*** *who accepted to be the headmaster of the jury of this thesis .*

I also thank the members of the jury who had great honor to help me during their effective evaluation in this thesis.

In addition, i thank all the teachers of Mathematical Department for their valuable support and help.

To conclude, i thank all my relatives and friends whose their support and encouragement a big help to me.

Dedication

I dedicate this fruitful work of a long years of my hard work :
To my dear parents, my husband and my mather Naima who have done their best to complit my studies, who gave me their valuable advice, support and encouragement.

To the respect professor **otmani Sodok** for his important help and accomplishment during this work

I dedicate as well :

To the whole of my dear family . To all the teachers of Department od Mathematics. T o all my friends

Ghali
Chaima

Notation

Ω : bounded domain in $\mathbb{R}^2$

∇u : gradient of u.

Δu : Laplace of u.

D(Ω) : distribution space.

$C^k(\Omega)$: space of functions k-times continuously differentiable in Ω.

$L^p(\Omega)$: space of functions p-th power integrated on with measure of dx.

$\|f\|_p = \left(\int_\Omega |f|^p \right)^{\frac{1}{p}}$.

$W^{1,p} = \{u \in L^p(\Omega), \nabla u \in L^p(\Omega)\}$.

H: Hilbert space.

$H_0^1(\Omega) = W_0^{1,2}(\Omega)$.

<u>If X is a Banach space</u>

$L^P(0,T,X) = \Big\{ f : (0,T) \longrightarrow \text{X is measurable} : \int_0^T \|f(t)\|_X^p dt < \infty \Big\}$.

$L^\infty(0,T,X) = \Big\{ f : (0,T) \longrightarrow \text{X is measurable} : ess - \sup_{t\in[0,T]} \|f(t)\|_X^p < \infty \Big\}$.

$C^k([0,T];X)$:Space of functions k times continuously differentiable from $[0,T] \longrightarrow X$.

$D([0,T];X)$: space of functions continuously differentiable with compact support in $[0,T]$.

Contents

Introduction

In recent decades, viscoelastic wave equations with acoustic boundary conditions have garnered significant attention from many researchers. It is well-known that viscoelastic materials exhibit memory effects, where their mechanical response is influenced by the history of the materials themselves. Mathematically, these damping effects are modeled using integro-differential operators. Consequently, differential equations that incorporate memory effects have become a vibrant area of research in recent years. We can mention some works [1, 3, 4, 6, 8, 11, 12]. The logarithmic nonlinearity is of great interest in physics due to its natural occurrence in various fields such as inflation cosmology, supersymmetric field theories, quantum mechanics, and nuclear physics [14, 19]. In this thesis, we details the work published by Al-Gharabli et al in 2019 see[1], that is the existence and stability solution of viscoelastic wave equation:

$$u_{tt} + \Delta^2 u + u - \int_0^t g(t-s)\Delta^2 u(s)ds = ku\ln|u|, \text{ in } \Omega \times (0,\infty) \tag{1}$$

with the boundary condition:

$$u = \frac{\partial u}{\partial \nu} = 0, \text{ in } \partial\Omega \times (0,\infty) \tag{2}$$

and the initial condition:

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \text{ in } \Omega. \tag{3}$$

where Ω is a bounded domain in $\mathbb{R}^n, n \in \mathbb{N}$, with a smooth boundary $\partial\Omega = \Gamma$, ν is the unit outer normal to $\partial\Omega$ and k is a small positive real number. The kernel g is a positive nonincreasing function defined on $\mathbb{R}_+$. $(u_0, u_1,)$ are the initial data belonging to a suitable function space. In [17] Messoudi studied the following equation

$$u_{tt} + \Delta^2 u + |u_t|^{m-2}u_t = |u|^p - 2u, \qquad \text{in } \Omega \times (0,T) \tag{4}$$

with the boundary condition:

$$u = \frac{\partial u}{\partial \nu} = 0, \text{ in } \partial\Omega \times (0,\infty) \tag{5}$$

and the initial condition:

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \text{ in } \Omega. \tag{6}$$

established an existence result and showed that the solution continues to exist globally if $m \geq p$, and blows up in finite time if $m < p$ and the initial energy is negative. Santos and Junior in [20] studied the following system:

$$\begin{cases} u_{tt} + \Delta^2 u = 0, & \text{in } \Omega \times (0, \infty), \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \Gamma_0 \times (0, \infty), \\ -u + \int_0^t g_1(t-s)\beta_1 u(s)ds = 0, & \text{on } \Gamma_1 \times (0, \infty), \\ \frac{\partial u}{\partial \nu} + \int_0^t g_2(t-s)\beta_2 u(s)ds = 0, & \text{on } \Gamma_2 \times (0, \infty), \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & \text{in } \Omega, \end{cases} \tag{7}$$

where

$$\beta_1 u = \Delta u + (1-\mu) B_1 u \quad \text{and} \quad \beta_2 u = \frac{\partial \Delta u}{\partial \mu} + (1-\mu)\frac{\partial B_2 u}{\partial \eta}$$

with

$$B_1 u = 2\nu_1 \nu_2 u_{xy} - \nu_1^2 u_{yy} - \nu_2^2 u_{xx} \quad \text{and} \quad B_2 u = (\nu_1 - \nu_2)\, u_{xy} + \nu_1 \nu_2 \,(u_{yy} - u_{xx})\,.$$

Liu and Sun in [11] considered the equation

$$u_{tt} - \Delta u + \alpha(t) \int_0^t g(t-s)\Delta u(s)ds = 0, \qquad \text{in } \Omega \times (0, \infty)$$

with a homogeneous Dirichlet condition on a portion of the boundary and acoustic boundary conditions on the rest of the boundary. The authors established a general decay result, which depends on the behavior of both α and g, by using the perturbed energy functional technique.Cavalcanti et al. [7] considered the equation

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)ds - \gamma \Delta u_t = 0, \qquad \text{in } \Omega \times (0, \infty)$$

subject to Dirichlet boundary conditions. Taking $0 \leq \rho \leq \frac{2}{n-2}$ if $n \geq 3$ or $\rho > 0$ if $n = 1, 2$ and assuming that the kernel g decays exponentially, the authors obtained global existence of solutions in the case $\gamma \geq 0$. They also proved that the solution energy decays exponentially when $\gamma > 0$. In [9] Górka studied The existence of global solution for the wave equation :

$$u_{tt} - u_{xx} + u = \varepsilon u \log |u|^2, \qquad x \in \Omega, t \in [0, T),$$

with the boundary condition

$$u(x,t) = 0, \qquad \text{on } \partial\Omega \times (0, T)$$

and the initial condition

$$u(x, 0) = u_0, \quad u_t(x, 0) = u_1 \qquad \text{in } \Omega,$$

where $\Omega = [a, b]$ is a one-dimensional interval, $(u_0, u_1) \in H_0^1 \times L_2$ and $\varepsilon \in [0, 1]$.In [5], Cazenave and Haraux considered

$$u_{tt} - \Delta u = k u \ln |u|, \qquad \text{in } \mathbb{R}^3$$

and established the existence and uniqueness of the solution for the Cauchy problem.

The body of this thesis is organized as follows:

- In chapter 1, we present some notations and material needed for our work.
- In chapter 2, by using the Faedo-Galerkin method we establish the local existence of the solutions of the problem then by using the energy of the problem we showed the global existence.
- In chapter 3, we study the stability result.

Chapter 1

Preliminaries

In this chapter, we present the elementary symbols, definitions and provide many tools on the basic concepts of inequalities and spaces, we will use later.

1.1 Banach Spaces

We first review some basic facts from calculus in the most important class of linear spaces the "Banach spaces".

Definition 1.1.1 *A Banach space is a complete normed linear space* X*. Its dual space* $X^{'}$ *is the linear space of all continuous linear functional* $f : X \longrightarrow \mathbb{R}$.

Proposition 1.1.2 $X^{'}$ *equipped with the norm*

$$\|f\|_{X'} = \sup\left\{ |f(u)| : \|u\|_X \leq 1 \right\},$$

is also a Banach space.

Definition 1.1.3 *Let* X *be a Banach space, and let* $(u_n)_{n\in\mathbb{N}}$ *be a sequence in* X*. Then* u_n *converges strongly to* u *in* X *if and only if*

$$\lim_{n\longrightarrow\infty} \|u_n - u\|_X = 0,$$

and this is denoted by $u_n \longrightarrow u$*, or* $\lim\limits_{n\longrightarrow\infty} u_n = u$.

Definition 1.1.4 *A sequence* (u_n) *in* X *is weakly convergent to* u *if and only if*

$$\lim_{n\longrightarrow\infty} f(u_n) = f(u),$$

for every $f \in X^{'}$ *and this is denoted by* $u_n \rightharpoonup u$.

1.2 Functional spaces

1.2.1 The L^p spaces

Definition 1.2.1 *[2] Let Ω be a domain in $\mathbb{R}^n (n \in \mathbb{N})$,for $1 \leq p < \infty$, the Lebesgue space $L^p(\Omega)$ is defined by:*

$$L^p(\Omega) = u : \Omega \longrightarrow \mathbb{R}, u \text{ is measurable and} \int_\Omega |u(x)|^p dx < \infty,$$

with the norm

$$\|u\|_p = \Big(\int_\Omega |u(x)|^p dx \Big)^{\frac{1}{p}},$$

In addition ,we define $L^\infty(\Omega)$ by:
$L^\infty(\Omega) = u : \Omega \longrightarrow \mathbb{R}, u$ is measurable and $\exists\, c > 0$ such that$|u(x)| \leq c$ a.e on Ω, equipped with the norme

$$\|u\|_\infty = ess \sup_{x \in \Omega} |u(x)| = \inf\{c : |u(x)| \leq c, \ \ a.e \quad on\ \Omega\}.$$

1.2.2 Hilbert spaces

Definition 1.2.2 *An inner product on a complex linear space X is a map*

$$(.,.) : X \times X \longrightarrow \mathbb{C}.$$

Such that ,for all x,y,z $\in X$ and$\lambda, \mu \in \mathbb{C}$: $(a)(x, \lambda y + \mu z) = \lambda(x, y) + \mu(x, y)$(linear in the second argument):
1.$(x, x) \geq 0$(nonnegative);
2.(x,x)=0 if and only if x=0(positive definite).
we call alinear space with an inner product a pre-Hilbert space. If X is a linear space with an inner product(.,.)then we can define an norm in X by:

$$\|x\| = \sqrt{(x, x)}.$$

A Hilbert space is a complete inner product space.

Definition 1.2.3 *A Hilbert space is a complete inner product space.*

Remark 1.2.4 *The spaces $L^p([a, b])$ are Banach spaces but they are not Hilbert spaces when $p \neq 2$.*

1.3 $L^p(0, T, X)$ Spaces

Definition 1.3.1 *Let X be a Banach space,denote by $L^p(0, T, X)$ the space of measurable functions*

$$f :]0, T[\longrightarrow X$$
$$t \longrightarrow f(t),$$

such that

$$\int_0^T (\|f(T)\|_X^p)^{\frac{1}{p}} dt < \infty$$

If $p = \infty$

$$\|f\|_{L^\infty(0,T,X)} = \sup_{t\in]0,T[} ess\|f(t)\|_X.$$

Theorem 1.3.2 *The space* $L^p(0,T,X)$*is a Banach space.*

1.4 Sobolev space

Definition 1.4.1 *[13]*

For k$\in \mathbb{N}$ *and 1*$\leq p \leq \infty$ *. We define the sobolev space*

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega), D^\alpha u \in L^p(\Omega) \forall\alpha \in \mathbb{N}^n \text{ with } |\alpha| \leq k\},$$

equipped with the norm

$$\|u\|_{k,p} = \Big(\sum_{|\alpha|\leq k} \|D^\alpha u\|_p^p\Big)^{\frac{1}{p}}, 1 \leq p < \infty$$

$$\|u\|_{k,\infty} = \max_{|\alpha|\leq k} \|D^\alpha u\|_\infty,$$

where $D^\alpha u$ *is the* α*-th weak derivative of u which is defined as*

$$\int_\Omega u(x)D^\alpha\varphi(x) = -1^{|\alpha|}\int_\Omega v(x)\varphi(x), \forall\varphi \in C_c^\infty(\Omega),$$

|α*|=*$\alpha_1 + ... + \alpha_n$*,and*

$$v = D^\alpha u = \frac{\partial^{|\alpha|}u}{\partial_{x1}\alpha_1 ... \partial_{xn}\alpha_n}.$$

. The space $W^{k,2}(\Omega)$ *is denoted by* $H^k(\Omega)$*, which is a Hilbert space with respect to the inner product*

$$(u,v)_{H^k} = \int_\Omega \sum_{|\alpha|\leq k} D^\alpha u(x)D^\alpha v(x)dx, \forall u,v \in H^k(\Omega).$$

Definition 1.4.2 *[13]*
We denote by $W_0^{k,p}(\Omega)$ *the closure of* $C_c^\infty(\Omega)$ *in* $W^{k,p}(\Omega)$*.*

Proposition 1.4.3 *[18](Green's formula)*

Let Ω *be an open subset of* $\mathbb{R}^d$*, with a Lipschitz boundary. Then for all* $u, v \in H^1(\Omega)$*,*

$$\int_\Omega (\frac{\partial u}{\partial x_i}v + \frac{\partial v}{\partial x_i}u)dx = \int_{\partial\Omega} \gamma_0(u)\gamma_0(v)\eta_i ds, \qquad i = 1, ..., d$$

Where η_i *is the i-th component of the outward normal vector* η*.*

1.5 Some inequalities

Theorem 1.5.1 *(Cauchy-Schwarz inequality)*
Let $u \in L^2(\Omega)$ *and* $v \in L^2(\Omega)$*,then* $uv \in L^1(\Omega)$ *and*

$$\|uv\|_1 \leq \|u\|_2 \|v\|_2.$$

Theorem 1.5.2 *(Höldr's inequality)*
Let $1 \leq p \leq \infty$*,if* $u \in L^p(\Omega)$ *and* $v \in L^q(\Omega)$*, then* $u\ v \in L^1(\Omega)$ *and*

$$\|uv\|_1 \leq \|u\|_p \|v\|_q,$$

where $\frac{1}{p} + \frac{1}{q}$.

Theorem 1.5.3 *(Young's inequality)*
Let $1 \leq p \leq \infty$ *,and* $a, b > 0$*.Then for any* $\epsilon > 0$*,we have*

$$uv \leq \epsilon u^p + C_\epsilon v^q$$

where

$$C_\epsilon = \frac{1}{q(\epsilon p)^{\frac{q}{p}}}.$$

For $p = q = 2$*, we have*

$$uv \leq \epsilon u^2 + \frac{v^2}{4\epsilon}.$$

Theorem 1.5.4 *(Poincare inequality) Assume that* Ω *is a bounded connected open set of* $\mathbb{R}^d$ *and that* Σ *is a (non-empty)Lipschitz continuous subset of the boundray* $\partial\Omega$*.Then there exists a constant* $C_\Omega > 0$ *such that*

$$\int_\Omega v^2(x)dx \leq C_\Omega \int_\Omega |\nabla v(x)|^2 dx,$$

for each $v \in H^1_\Sigma(\Omega)$*.*

Theorem 1.5.5 *[2](Fubini)*
Assume that $F \in L^1(\Omega_1 \times \Omega_2)$*.Then for a,e* $x \in \Omega_1$*.*
$F(x, y) \in L^1_y(\Omega_2)$ *and* $\int_{\Omega_2} F(x, y)d\mu_2 \in L^1_x(\Omega_1)$*. Similarly,for a,e.*$y \in \Omega_2$*,*
$F(x, y) \in L^1_x(\Omega_1)$ *and* $\int_{\Omega_1} F(x, y)d\mu_1 \in L^1_y(\Omega_1)$*. Moreover,oneb has*

$$\int_{\Omega_1} d\mu_1 \int_{\Omega_2} F(x, y)d\mu_2 = \int_{\Omega_2} d\mu_2 \int_{\Omega_1} F(x, y)d\mu_1 = \int\int_{\Omega_1 \times \Omega_2} F(x, y)d\mu_1 d\mu_2.$$

Theorem 1.5.6 *[2] (Jensen's inequality.)*
Assume $|\Omega| < \infty$*. Let* $j : \mathbb{R} \longrightarrow (-\infty, +\infty)$ *be a convex l.s.c. function,* $j \not\equiv +\infty$*. Let* $f \in L^1(\Omega)$ *be such that* $f(x) \in D(j)$*a.e. and* $j(f) \in L^1(\Omega)$*.then*

$$j\Big(\frac{1}{|\Omega|} \int_\Omega f\Big) \leq \frac{1}{|\Omega|} \int_\Omega j(f)$$

Lemma 1.5.7 *(Sobolev-Poincare inequality)*
Let q be a number with

$$2 \leq q < \infty, (n = 1, 2), 2 \leq q \leq \frac{2n}{n-2}(n \geq 3),$$

then there exists aconstant $C_s = c_s(\Omega, q)$ *such that*

$$\|u\|_q \leq c\|\nabla u\|_2, \qquad \text{for } u \in H_0^1(\Omega).$$

Lemma 1.5.8 *(Logaritmic sobolev inequality)*
Let u be any function in $H_0^1(\Omega)$ *and a be any positive real numbre .Then*

$$\int_\Omega u^2 \ln |u| dx \leq \frac{1}{2}\|u\|_2^2 \ln \|u\|_2^2 + \frac{a^2}{2\pi}\|\nabla u\|_2^2 - (1 + \ln a)\|u\|_2^2. \tag{1.1}$$

Corollary 1.5.9 *Let u be any function in* $H_0^2(\Omega)$ *and a be any postive real numbre .Then*

$$\int_\Omega u^2 \ln |u| dx \leq \frac{1}{2}\|u\|_2^2 \ln \|u\|_2^2 + \frac{c_p a^2}{2\pi}\|\Delta u\|_2^2 - (1 + \ln a)\|u\|_2^2 \tag{1.2}$$

Lemma 1.5.10 *(Logaritmic Gronwall inequality).Let c>0,*$\gamma \in L^1(0, T; \mathbb{R}^+)$ *and assume that the function* $\omega : [0, T] \longrightarrow [1, \infty)$ *satisfies*

$$\omega(t) \leq c\Big(1 + \int_0^t \gamma(s)\omega(s) \ln \omega(s) ds\Big), \qquad 0 \leq t \leq T. \tag{1.3}$$

Then

$$\omega(t) \leq c\, exp\Big(c\int_0^t \gamma(s)ds\Big), \qquad 0 \leq t \leq T. \tag{1.4}$$

Lemma 1.5.11

$$\int_0^t g(t-s)(\Delta u(s), \Delta u_t(t))ds = -\frac{1}{2}g(t)\|\Delta u\|_2^2 + \frac{1}{2}(g' \circ \Delta u)(t) - \frac{1}{2}\frac{d}{dt}\Big\{(g \circ \Delta u)(t) - \Big(\int_0^t g(s)\Big)\|\Delta u\|_2^2\Big\}$$

where

$$(g \circ \Delta u)(t) = \int_0^t g(t-s)\|\Delta u(s), \Delta u_t(t)\|^2 ds.$$

We considre the following hypotheses:
(A1) $g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is aC^1- nonincreasing function satisfying

$$g(0) > 0 \qquad \text{and} \qquad 1 - \int_0^\infty g(s)ds = l > 0 \tag{1.5}$$

(A2) There exist a nonincreasing differentiable function $\xi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$,with $\xi(0) > 0$,and a constant $1 \leq p < \frac{3}{2}$ such that

$$g'(t) \leq -\xi(t)g^p(t), \qquad \forall t \in \mathbb{R}^+. \tag{1.6}$$

(A3) The constant k in (1,1)satisfies0 $< k < k_0$,wherek_0 is the positive real number satisfying

$$\sqrt{\frac{2\pi l}{k_0 c_p}} = e^{\frac{-3}{2}-\frac{1}{k_0}} \tag{1.7}$$

. and c_p is the smallest positive number satisfying

$$\|\nabla u\|_2^2 \leq c_p \|\Delta u\|_2^2, \qquad \forall u \in H_0^2(\Omega)$$

where $\|.\|_2 = \|.\|_{L^2(\Omega)}$

Remark 1.5.12 *The function* $f(s) = \sqrt{\frac{2\pi l}{s c_p}} - e^{\frac{-3}{2}-\frac{1}{s}}$ *is a continuous and decreasing function on* $(0, \infty)$, *with* $\lim_{s\longrightarrow 0^+} f(s) = \infty$ *and* $\lim_{s\longrightarrow 0^+} f(x) = -e^{-\frac{3}{2}}$. *Then, there exists a unique* $k_0 > 0$ *such that* $f(k_0) = 0$. *Moreover,*

$$e^{\frac{-3}{2}-\frac{1}{s}} < \sqrt{\frac{2\pi l}{s c_p}}, \qquad \forall s \in (0, k_0). \tag{1.8}$$

Chapter 2

Local and Global existence of solution

2.1 Local existence.

In this section. we state and prove the local existence result for problem (1)-(3).

Definition 2.1.1 *Let $T > 0$ a function*

$$u \in C([0,T], H_0^2(\Omega)) \cap C^1([0,T], L^2(\Omega)) \cap C^2([0,T], H^{-2}(\Omega))$$

is called a weak solution of ((1)-(3)) on [0,T] if, for any $\omega \in H_0^2(\Omega)$ and $t \in$ [0,T],

$$\begin{cases} \int_\Omega u_{tt}(x,t)\omega(x)dx + \int_\Omega \Delta u(x,t)\Delta\omega(x)dx + \int_\Omega u(x,t)\omega(x)dx \\ -\int_\Omega \Delta\omega(x)\int_0^t g(t-s)\Delta u(s)dsdx = \int_\Omega u(x,t)\omega(x)\ \ln|u(x,t)|^k dx \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x). \end{cases} \tag{2.1}$$

Lemma 2.1.2 *let u be a solution of ((1)-(3)) then the modified energy of this problem is defend by:*

$$\begin{aligned} E(t) =& \frac{1}{2}\Big(\|u_t\|_2^2 + \Big(1 - \int_0^t g(s)\Big)\|\Delta u\|_2^2 \frac{k+2}{2}\|u\|_2^2\Big) \\ &- \frac{1}{2}\int_\Omega |u|^2 \ln|u|^k dx + \frac{1}{2}(g \circ \Delta u)(t). \end{aligned} \tag{2.2}$$

the energy $E(t)$ is nonincreasing because,

$$E'(t) = \frac{1}{2}(g' \circ \Delta u)(t) - \frac{1}{2}g(t)\|\Delta u\|_2^2 \le \frac{1}{2}(g' \circ \Delta u)(t) \le 0. \tag{2.3}$$

Proof. Let us replace ω by u_t in (2.1) and integrate by parts to obtain:

$$\int_\Omega u_{tt}u_t dx + \int_\Omega \Delta u \Delta u_t dx + \int_\Omega u u_t dx - \int_\Omega \Delta u_t \int_0^t g(t-s)\Delta u(s)dsdx = \int_\Omega u u_t \ln|u|^k dx.$$

So

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_t^2dx+\frac{1}{2}\frac{d}{dt}\int_\Omega(\Delta u)^2dx+\frac{1}{2}\frac{d}{dt}\int_\Omega u^2dx-g(t)\|\Delta u\|_2^2+\frac{1}{2}(g'\circ\Delta u)(t)$$
$$-\frac{1}{2}\frac{d}{dt}\Big\{(g\circ\Delta u)(t)-\Big(\int_0^t g(s)\Big)\|\Delta u\|_2^2\Big\}=\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\ln|u|^kdx-\frac{k}{4}\frac{d}{dt}\int_\Omega u^2dx,$$

implies

$$\frac{1}{2}\frac{d}{dt}\|u_t\|_2^2+\frac{1}{2}\frac{d}{dt}\|\Delta u\|_2^2+\frac{1}{2}\frac{d}{dt}\|u\|_2^2-g(t)\|\Delta u\|_2^2+\frac{1}{2}(g'\circ\Delta u)(t)$$
$$-\frac{1}{2}\frac{d}{dt}\Big\{(g\circ\Delta u)(t)-\Big(\int_0^t g(s)\Big)\|\Delta u\|_2^2\Big\}=\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\ln|u|^kdx-\frac{k}{4}\frac{d}{dt}\|u\|_2^2.$$

Thus

$$\frac{1}{2}\frac{d}{dt}\Big\{\|u_t\|_2^2+\|u\|_2^2+(g\circ\Delta u)(t)+\Big(1-\int_0^t g(s)\Big)\|\Delta u\|_2^2-\int_\Omega u^2\ln|u|^kdx+\frac{k}{4}\|u\|_2^2\Big\}$$
$$=-\frac{1}{2}g(t)\|\Delta u\|_2^2+\frac{1}{2}(g'\circ\Delta u)(t),$$

then

$$E(t)=\frac{1}{2}\Big(\|u_t\|_2^2+\Big(1-\int_0^t g(s)\Big)\|\Delta u\|_2^2\frac{k+2}{2}\|u\|_2^2\Big)-\frac{1}{2}\int_\Omega|u|^2\ln|u|^kdx+\frac{1}{2}(g\circ\Delta u)(t). \tag{2.4}$$

■

Theorem 2.1.3 *Assume that (A1) and (A2) hold and let* $(u_0,u_1)\in H_0^2(\Omega)\times L^2(\Omega)$ *Then the problem* ((1)-(3)) *has a weak solution*

$$u\in C([0,T],H_0^2(\Omega))\cap C^1([0,T],L^2(\Omega))\cap C^2([0,T],H^{-2}(\Omega)). \tag{2.5}$$

Proof. . The demonstration is based on the Faedo-Galarkin method which consist of carrying out the following

let $\{\omega_j\}_{j=1}^\infty$ be orthogonal basis of the "separable" space $H_0^2(\Omega)$ which is orthonormal in $L^2(\Omega)$.

let $V_m=span\{\omega_1,\omega_2...\omega_m\}$ and let the projection of the initial on the finite dimensional subspace V_m be given by

$$u_0^m(x)=\sum_{j=1}^m a_j\omega_j(x),\qquad\text{and}\qquad u_1^m(x)=\sum_{j=1}^m b_j\omega_j(x).$$

where

$$u_0^m\to u_0\,in\,H_0^2(\Omega),\qquad and\qquad u_1^m\to u\,in\,L^2(\Omega). \tag{2.6}$$

Search for an apporoximate solution

$$\boxed{u^m(x)=\sum_{j=1}^{m} h_j^m \omega_j(x)}$$

of the approximate problem in V_m :

$$\begin{cases} \int_\Omega u_{tt}^m\omega_j+\Delta u^m\Delta\omega_j+u^m\omega_j-\int_0^t g(t-s)\Delta u^m(s)ds\Delta\omega_j dx \\ \quad =\int_\Omega \omega_j u^m\ln|u^m|^k dx \quad ,\forall\omega_j\in V_m, \\ \qquad u^m(0)=u_0^m=\sum_{j=1}^m(u_0,\omega_j)\omega_j, \\ \qquad u_t^m(0)=u_1^m=\sum_{j=1}^m(u_1,\omega_j)\omega_j. \end{cases} \tag{2.7}$$

This leads to a system of ODE_s unknown functions $h_j^m(t)$. Based on standard existence theory for ODE,one can obtain functions

$$h_j:[0,t_m)\rightarrow\mathbb{R}\quad \text{j=1,2,...,m},$$

which satisfy (2.7) in a maximal interval $[0,t_m),t_m\in(0,T]$.

Next, we show that t_m=T and that the local solution is uniformly bounded independent of m and t.

$$\frac{d}{dt}E^m(t)\leqslant\frac{1}{2}(g'\circ\Delta u^m)(t)\leqslant 0, \tag{2.8}$$

where

$$E^m(t)=\frac{1}{2}\Big(\|u_t^m\|_2^2+\Big(1-\int_0^t g(s)\Big)\|\Delta u^m\|_2^2\frac{k+2}{2}\|u^m\|_2^2\Big)-\frac{1}{2}\int_\Omega|u^m|^2\ln|u^m|^k dx+\frac{1}{2}(g\circ\Delta u^m)(t). \tag{2.9}$$

From(2.8) we get

$$\frac{d}{dt}E^m(t)\leqslant 0,$$

so

$$\int_0^t\frac{d}{dt}E^m(t)\leqslant 0,$$

then

$$E^m(t)-E^m(0)\leqslant 0,$$

thus

$$E^m(t)\leqslant E^m(0),$$

therefore

$$2E^m(t)\leq 2E^m(0):=C.$$

The last inequality with (1.5) and the Logarithmic Sobolev inequality (1.2), we get

$$\|u_t^m\|_2^2+(g\circ\Delta u^m)(t)+\Big(l-\frac{k\alpha^2c_p}{2\pi}\Big)\|\Delta u^m\|_2^2+\Big(\frac{k+2}{2}+k(1+\ln\alpha)\Big)\|u^m\|_2^2 \\ \leq C+\frac{k}{2}\|u^m\|_2^2\ln\|u^m\|_2^2, \tag{2.10}$$

where $C = 2E^m(0)$. Choosing

$$e^{\frac{3}{2}-\frac{1}{k}} < \alpha < \sqrt{\frac{2\pi l}{kc_p}}, \tag{2.11}$$

will make

$$l - \frac{k\alpha^2 c_p}{2\pi} > 0 \qquad and \qquad \frac{k+2}{2} + k(1+\ln\alpha) > 0.$$

this selection is possible thanks to (A3). So, we get

$$\|u_t^m\|_2^2 + (g \circ \Delta u^m)(t) + \|\Delta u^m\|_2^2 + \|u^m\|_2^2 \leqslant c\Big(1 + \|u^m\|_2^2 \ln \|u^m\|_2^2\Big). \tag{2.12}$$

Let us note that

$$u^m(.,t) = u^m(.,0) + \int_0^t \frac{\partial u^m}{\partial s}(.,s).$$

Then using Cauchy Schwarz's inequality and Fubini, we get

$$\|u^m(t)\|_2^2 \leq 2\|u^m(0)\|_2^2 + 2\|\int_0^t \frac{\partial u^m}{\partial s}(s)\|_2^2 \tag{2.13}$$

$$\leq 2\|u^m(0)\|_2^2 + 2\|\Big(\int_0^t 1ds\Big)^{\frac{1}{2}}\Big(\int_0^t (u_t^m)^2 ds\Big)^{\frac{1}{2}}\|_2^2 \tag{2.14}$$

$$\leq 2\|u^m(t)\|_2^2 + 2T\int_0^t \|u_t^m(s)\|_2^2 ds,$$

hence, inequality (2.12) gives

$$\|u^m\|_2^2 \leqslant 2\|u^m(0)\|_2^2 + 2cT\Big(1 + \int_0^t \|u^m\|_2^2 \ln \|u^m\|_2^2 ds\Big). \tag{2.15}$$

If we put $C_1 = max\{2cT, 2\|u^m(0)\|_2^2\}$ (3.11) leads to

$$\|u^m\|_2^2 \leqslant 2C_1\Big(1 + \int_0^t \|u^m\|_2^2 \ln \|u^m\|_2^2 ds\Big).$$

Without loss of generality,we take$C_1 \geqslant 1$,which gives

$$\|u^m\|_2^2 \leq 2C_1\Big(1 + \int_0^t \Big(C_1 + \|u^m\|_2^2\Big) \ln \Big(C_1 + \|u^m\|_2^2\Big) ds\Big).$$

Applying the Logarithmic Gronwall inequality to the last inequality, we obtain the following estimate :

$$\|u^m\|_2^2 \leqslant 2C_1 e^{2C_1 T} := C_2.$$

Where

$$\begin{cases} w(t) = \|u^m\|_2^2, \\ \gamma(t) = 1 \Longrightarrow \int_0^t 1ds = T, \\ c = 2C_1. \end{cases}$$

Hence from inequality (2.12) it follows that:

$$\|u_t^m\|_2^2 + (g\circ\Delta u^m)(t) + \|\Delta u^m\|_2^2 + \|u^m\|_2^2 \leqslant c\Big(1 + C_2\ln C_2\Big) := C_3. \tag{2.16}$$

where C_3 is a positive constant independent of m and t, and since

$$\begin{cases} \|u_t^m\|_2^2 \geqslant 0, \\ \|\Delta u^m\|_2^2 \geqslant 0, \\ \|u^m\|_2^2 \geqslant 0, \\ (g\circ\Delta u^m)(t) \geqslant 0. \end{cases} \tag{2.17}$$

Then(2.16) give us

$$\begin{cases} \|u_t^m\|_2^2 \leqslant C_3, \\ \|\Delta u^m\|_2^2 \leqslant C_3, \\ \|u^m\|_2^2 \leqslant C_3. \end{cases} \Longrightarrow \begin{cases} \sup_{t\in(0,tm)} \|u_t^m\|_2^2 \leqslant C_3, \\ \sup_{t\in(0,tm)} \|\Delta u^m\|_2^2 \leqslant C_3, \\ \sup_{t\in(0,tm)} \|u^m\|_2^2 \leqslant C_3. \end{cases}$$

So,

$$\sup_{t\in(0,tm)} \|u_t^m\|_2^2 + \sup_{t\in(0,tm)} \|\Delta u^m\|_2^2 + \sup_{t\in(0,tm)} \|u^m\|_2^2 \leqslant 3\, C_3. \tag{2.18}$$

And we have

$$\begin{cases} \sup_{t\in(0,tm)} \|u_t^m\|_2^2 \geqslant 0, \\ \sup_{t\in(0,tm)} \|\Delta u^m\|_2^2 \geqslant 0, \\ \sup_{t\in(0,tm)} \|u^m\|_2^2 \geqslant 0. \end{cases} \tag{2.19}$$

So,

$$\begin{cases} \sup_{t\in(0,tm)} \|u_t^m\|_2^2 \leqslant 3C_3 < \infty, \\ \sup_{t\in(0,tm)} \|u^m\|_2^2 \leqslant 3C_3 < \infty. \end{cases} \tag{2.20}$$

Then,

$$\begin{cases} u^m \text{is unformly bounded,} \\ u_t^m \text{is unformly bounded..} \end{cases}$$

there fore , we can ewtend t_m to T .Moreover, we obtain from (2.20)

$$\begin{cases} u^m \text{ is uniformly bounded in } L^\infty(0,T;H_0^2(\Omega)), \\ u_t^m \text{is uniformly bounded in} L^\infty(0,T;L^2(\Omega)). \end{cases} \tag{2.21}$$

Which implies that there exists a subsequence of (u^m) (still denoted by (u^m)), such that

$$\begin{cases} u^m \rightharpoonup u \text{ weakly * in } L^\infty(0,T;H_0^2(\Omega)), \\ u_t^m \rightharpoonup u_t \text{ weakly * in } L^\infty(0,T;L^2(\Omega)), \\ u^m \rightharpoonup u \text{ weakly in } L^2(0,T;H_0^2(\Omega)), \\ u_t^m \rightharpoonup u_t \text{ weakly in } L^2(0,T;L^2(\Omega)). \end{cases} \tag{2.22}$$

Making use of Aubin-Lions theorem ,we get$H_0^2(\Omega)$ the embedding compact in $L^2(\Omega)$ than

$$u^m \longrightarrow u \text{ strongly in } L^2(0,T;L^2(\Omega)).$$

And

$$u^m \longrightarrow u \text{ a.e in } \Omega\times(0,T).$$

Since the map s $\longrightarrow \ln|s|^k$ is continuous on $\mathbb{R}$, we have the convergence

$$u^m\ln|u^m|^k \longrightarrow u\ln|u|^k\text{a.e.in } \Omega\times(0,T).$$

Using the embedding of H_0^2 in $L^\infty(\Omega)$(sine $\Omega\subset\mathbb{R}^2$),it is clear that $u^m\ln|u^m|^k$ is bounded in $L^\infty(\Omega\times(0,T))$.Next, taking into account the Lebesgue bounded convergence theorem(Ω is bounded),we get

$$u^m\ln|u^m|^k \longrightarrow u\ln|u|^k\text{strongly in } L^2(0,T;L^2(\Omega)). \tag{2.23}$$

Now we integrate (2.7) over (0,t) for every $\omega_j\in V_m$ to obtain,

$$\begin{aligned}\int_0^t\int_\Omega u_{tt}^m\omega_j dxds+\int_0^t\int_\Omega \Delta u^m(s)\Delta\omega_j dxds+\int_0^t\int_\Omega u^m(s)\omega_j dxds\\ -\int_0^t\int_\Omega\int_0^\tau g(\tau-s)\Delta u^m(s)\Delta\omega_j dsdxd\tau\\ =\int_0^t\int_\Omega \omega_j u^m(s)\ln|u^m(s)|^k dxds,\end{aligned}$$

so

$$\begin{aligned}\int_\Omega\int_0^t u_{tt}^m\omega_j dsdx+\int_0^t\int_\Omega \Delta u^m(s)\Delta\omega_j dxds+\int_0^t\int_\Omega u^m(s)\omega_j dxds\\ -\int_0^t\int_\Omega\int_0^\tau g(\tau-s)\Delta u^m(s)ds\Delta\omega_j dxd\tau\\ =\int_0^t\int_\Omega \omega_j u^m(s)\ln|u^m(s)|^k dxds,\end{aligned}$$

then

$$\begin{aligned}\int_\Omega u_t^m\omega_j dx-\int_\Omega u_1^m\omega_j dx+\int_0^t\int_\Omega \Delta u^m(s)\Delta\omega_j dxds+\int_0^t\int_\Omega u^m(s)\omega_j dxds\\ -\int_0^t\int_\Omega\int_0^\tau g(\tau-s)\Delta u^m(s)ds\Delta\omega_j dxd\tau\\ =\int_0^t\int_\Omega \omega_j u^m(s)\ln|u^m(s)|^k dxds.\end{aligned} \tag{2.24}$$

Convergences (2.6) ,(2.22) and (2.23) are sufficient to pass to the limit in (2.24) , as $m\to+\infty$, and get,for any $\omega\in V_m$ and m$\geq$ 1,

$$\begin{aligned}\int_\Omega u_t\omega_j dx = \int_\Omega u_1\omega_j dx - \int_0^t\int_\Omega \Delta u(s)\Delta\omega_j dsdx - \int_0^t\int_\Omega u(s)\omega_j dxds \\ + \int_0^\tau\int_\Omega \Delta\omega_j(x)\int_0^t g(\tau-s)\Delta u(s)dsdxd\tau \\ + \int_0^t\int_\Omega \omega_j u(s)\ln|u(s)|^k dsdx,\end{aligned} \tag{2.25}$$

Which implies that (2.25) is valid for any $\omega \in H_0^2(\Omega)$. Using the fact that the terms in the right-hand side of (2.25) are absolutely continuous since they are functions of t defined by integrals over(0,t) ,hence it is differentiable for a.e.t ∈(0,T) and any $\omega \in H_0^2(\Omega)$,

$$\begin{aligned}\int_\Omega u_{tt}(x,t)\omega_j(x)dx + \int_\Omega \Delta u(x,t)\Delta\omega_j(x)dx + \int_\Omega u(x,t)\omega_j(x)dx \\ - \int_\Omega \Delta\omega_j(x)\int_0^t g(t-s)\Delta u(s)dsdx = \int_\Omega \omega_j(x)u(x,t))\ln|u(x,t)|^k dx.\end{aligned} \tag{2.26}$$

To handle the initial conditions, we note that

$$u^m \rightharpoonup u \quad \text{weakly in } L^2(0,T;H_0^2(\Omega)) \text{ and } u_t^m \rightharpoonup u_t \quad \text{weakly in} L^2(0,T;L^2(\Omega)). \tag{2.27}$$

Since$H_0^2(\Omega)$ the embedding compact in $L^2(\Omega)$ thus, using Lion's Lemma, we obtain

$$u^m \longrightarrow u \text{ in } C([0,T];L^2(\Omega)). \tag{2.28}$$

Therefore,$u^m(x,0)$ makes sense and

$$u^m(x,0) \longrightarrow u(x,0) \quad \text{in } L^2(\Omega).$$

Also, we have

$$u^m(x,0) = u_0^m(x) \longrightarrow u_0(x) \text{ in } H_0^2(\Omega).$$

since the limit unique so

$$u(x,0) = u_0(x).$$

Now, multiply (2.7) by $\phi \in C_0^\infty(0,T)$and integrate over $(0,T)$,we obtain , for any$\omega \in V_m$,

$$\begin{aligned}-\int_0^T\int_\Omega u_t^m\omega_j\phi' dxdt = -\int_0^T\int_\Omega \Delta u^m(t)\Delta\omega_j\phi\, dxdt - \int_0^t\int_\Omega u^m\,\omega_j\phi dxdt \\ + \int_0^T\int_\Omega\int_0^t g(t-s)\Delta u^m(s)ds\,\Delta\omega_j\phi\, dxdt + \int_0^T\int_\Omega \omega_j\phi u^m\ln|u^m|^k dxdt,\end{aligned} \tag{2.29}$$

As $m \longrightarrow \infty$, we have, for any $\omega \in H_0^2(\Omega)$ and any $\phi \in C_0^\infty(0,T)$,

$$\begin{aligned}-\int_0^T\int_\Omega u_t\omega_j\phi' dxdt = -\int_0^T\int_\Omega \Delta u(t)\Delta\omega_j\phi\, dxdt - \int_0^t\int_\Omega u\,\omega_j\phi dxdt \\ + \int_0^T\int_\Omega\int_0^t g(t-s)\Delta u(s)ds\,\Delta\omega_j\phi\, dxdt + \int_0^T\int_\Omega \omega_j\phi u\ln|u|^k dxdt,\end{aligned} \tag{2.30}$$

This means

$$u_{tt} \in L^2([0,T], H^{-2}(\Omega))$$

. Recalling that $u_t \in L^2([0,T], L^2(\Omega))$, we obtain

$$u_t \in C([0,T], H^{-2}(\Omega)).$$

So, $u_t^m(x,0)$ makes sense and

$$u_t^m(x,0) \longrightarrow u_t(x,0) \quad \text{in } \; H^{-2}(\Omega).$$

But

$$u_t^m(x,0) = u_1^m(x) \longrightarrow u_1(x) \text{ in } L^2(\Omega).$$

Hence

$$u_t(x,0) = u_1(x).$$

■

2.2 Global existence

In this section, we state and prove a global existence result under smallness conditions on the initial data (u_o, u_1). For this purpose,we introduce the following functions:

$$J(t) = \frac{1}{2}\Big(\Big(1 - \int_0^t g(s)ds\Big)\|\Delta u\|_2^2 + \|u\|_2^2 + (g \circ \Delta\, u)(t) - \int_\Omega u^2 \ln |u|^k dx\Big) + \frac{k}{4}\|u\|_2^2, \tag{2.31}$$

and

$$I(t) = \Big(1 - \int_0^t g(s)ds\Big)\|\Delta u\|_2^2 + \|u\|_2^2 + (g \circ \Delta\, u)(t) - 3\int_\Omega u^2 \ln |u|^k dx. \tag{2.32}$$

Lemma 2.2.1 *The following inequalities hold:*

$$-kd_0\sqrt{|\Omega| c_*^3}\|\Delta u\|_2^{\frac{3}{2}} \le \int_\Omega u^2 \ln |u|^k dx \le kc_*^3\|\Delta u\|_2^3, \qquad \forall u \in H_0^2, \tag{2.33}$$

where $d_0 = \sup_{0<s<1}\sqrt{s}|\ln s|, |\Omega|$ *is the lebegue of*Ω*and* c_**is the smallest embedding constant*

$$\Big(\int_\Omega |u|^3\Big)^{\frac{1}{3}} \le c_*\|\Delta u\|_2, \qquad \forall u \in H_0^2 \tag{2.34}$$

(c_* exists thanks to the embedding of $H_0^2(\Omega)$ in L^∞)

Proof. Let

$$\Omega_1 = \{x \in \Omega : |u(x)| > 1\} \qquad \text{and} \qquad \Omega_2 = \{x \in \Omega : |u(x)| \le 1\}$$

So,using (2.33), we have

$$\begin{aligned}
\int_\Omega u^2 \ln |u|^k dx &= \overbrace{\int_{\Omega_2} u^2 \ln |u|^k dx}^{(-)} + \int_{\Omega_1} u^2 \ln |u|^k dx \\
&\le \int_{\Omega_1} u^2 \ln |u|^k dx \\
&= k\int_{\Omega_1} u^2 \ln |u| dx \\
&\le k\int_{\Omega_1} |u|^3 dx \qquad \text{because } \ln |u| \le |u| \\
&\le k\int_\Omega |u|^3 dx \\
&\le kc_*^3\|\Delta u\|_2^3, \qquad \text{by(2.34)}.
\end{aligned}$$

Then

$$\int_\Omega u^2 \ln |u|^k dx \le kc_*^3\|\Delta u\|_2^3 \qquad \forall u \in H_0^2(\Omega).$$

On other hand,

$$
\begin{aligned}
-\int_\Omega u^2 \ln|u|^k dx &= \overbrace{-\int_{\Omega_2} u^2 \ln|u|^k dx}^{(+)} \overbrace{-\int_{\Omega_1} u^2 \ln|u|^k dx}^{(-)} \\
&\le \int_{\Omega_2} u^2 \ln|u|^k dx \\
&\le -k \int_{\Omega_2} u^2 \ln|u| dx \\
&\le k \int_{\Omega_2} u^2 |\ln|u|| dx \qquad \text{because } \ln|u| \le |\ln|u|| \\
&\le k \int_{\Omega_2} |u|^{\frac{3}{2}} |u|^{\frac{1}{2}} |\ln|u|| dx \\
&\le k \int_{\Omega_2} |u|^{\frac{3}{2}} \sup_{0\le|u|\le 1} |u|^{\frac{1}{2}} |\ln|u|| dx \\
&= k \int_{\Omega_2} |u|^{\frac{3}{2}} d_0 dx \\
&\le k d_0 \int_\Omega |u|^{\frac{3}{2}} dx \\
&\le k d_0 \|1\|_2 \|u_1^{\frac{3}{2}}\| \qquad \text{using Hölder inequality} \\
&= k d_0 \sqrt{|\Omega|} \Big(\int_\Omega \Big(|u|^{\frac{3}{2}} \Big)^2 \Big)^{\frac{1}{2}} \\
&= k d_0 \sqrt{|\Omega|} \Big(\int_\Omega |u|^3 \Big)^{\frac{1}{2}} \\
&\le k d_0 \sqrt{|\Omega|} c_*^{\frac{3}{2}} \|\Delta u\|_2^{\frac{3}{2}} \qquad \text{by(2.34)} \\
&\le k d_0 \sqrt{|\Omega| c_*^3} \|\Delta u\|_2^{\frac{3}{2}}.
\end{aligned}
$$

Then,

$$
-k d_0 \sqrt{|\Omega| c_*^3} \|\Delta u\|_2^{\frac{3}{2}} \le \int_\Omega u^2 \ln|u|^k dx.
$$

■

Lemma 2.2.2 *Assume that (A1)-(A2). Let* $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$*such that*

$$
I(0) > 0 \qquad \textit{and} \qquad \sqrt{54} k c_*^3 \Big(\frac{E(0)}{l} \Big)^{\frac{1}{2}} < l. \tag{2.35}
$$

Then

$$
I(t) > 0 \qquad \forall t \in [0, T). \tag{2.36}
$$

Proof. From (2.32)

$$
\int_\Omega u^2 \ln|u|^k dx = \frac{1}{3} \Big(1 - \int_0^t g(s) ds \Big) \|\Delta u\|_2^2 + \frac{1}{3} \|u\|_2^2 + \frac{1}{3} (g \circ \Delta\, u)(t) - \frac{1}{3} I(t). \tag{2.37}
$$

Substitute (2.37) in (2.31), we find

$$J(t)=\frac{1}{3}\left(\left(1-\int_0^t g(s)ds\right)\|\Delta u\|_2^2+\|u\|_2^2+(g\circ\Delta\, u)(t)\right)+\frac{k}{4}\|u\|_2^2+\frac{1}{6}I(t). \tag{2.38}$$

Since $I(0)>0$ and $I$is continuous on $[0,T]$, there exists $t_0\in(0,T]$such that $I(t)>0$, for all $t\in[0,t_0)$ .Let us denote by t_0 the biggest real number in $(0,T]$ such that $I>0$ on $[0,t_0)$. if $t_0=T$, than (2.36) is satisfied. We assume by contradiction that $t_0\in(0,T)$.Thus $I(t_0)=0$ and

$$\|\Delta u(t)\|_2^2\le\frac{3}{l}J(t)\le\frac{3}{l}E(t)\le\frac{3}{l}E(0)\qquad\forall t\in(0,t_0], \tag{2.39}$$

in deed by (A1) and (2.38) we have

$$J(t)=\frac{l}{3}\|\Delta u\|_2^2+\underbrace{\frac{1}{3}\|u\|_2^2+\frac{1}{3}(g\circ\Delta\, u)(t)+\frac{k}{4}\|u\|_2^2+\frac{1}{6}I(t),}_{(+)\text{ because I,k and g are positive}}\qquad\forall t\in(0,t_0]$$

$$\begin{aligned}&\frac{l}{3}\|\Delta u\|_2^2\le J(t),\\&E(t)=\frac{1}{2}\|u\|_2^2+J(t)\le E(0),\\&0\le E(t)-J(t)\le E(0),\\&J(t)\le E(t)\le E(0),\end{aligned}$$

then

$$\frac{l}{3}\|\Delta u\|_2^2\le J(t)\le E(t)\le E(0).$$

If $\|\Delta u(t_0)\|_2^2=0$, then (2.33) and (2.34) give
$|u(t_0)|\le 0\Longrightarrow u(t_0)=0$, $(g\circ\Delta\, u)(t_0)=0$ and $\int_\Omega u^2(t_0)\ln|u(t_0)|^k dx=0$, then

$$0=I(t_0)=(g\circ\Delta\, u)(t_0)=\int_0^{t_0}g(s)\|\Delta u(s)\|_2^2ds. \tag{2.40}$$

Consequently, if $g>0$ on$[0,t_0)$,we get $\|\Delta u(s)\|_2=0$ $\forall s\in[0,t_0)$.
Then

$$I(t)=0,\forall t\in[0,t_0),$$

which is not true since $I>0$ on $[0,t_0)$. If g is not positive on $[0,t_0)$, then let $t_1\in[0,t_0]$ the smallest real number such that $g(t^1)=0$. Because $g(0)$>0 and g is positive ,nonincreasing and continuous on $\mathbb{R}^+$(condition (A1)),then $t_1>0$and $g=0$on $[t_1,\infty)$ .Therefore, from (2.40), we deduce that

$$0=\int_0^{t_0}g(s)\|\Delta u(s)\|_2^2ds=\int_0^{t_1}g(s)\|\Delta u(s)\|_2^2ds,$$

then $\|\Delta u(s)\|_2=0$, for any $s\in[0,t_1)$, which implies that$I(t)=0$,for any $t\in[0,t_1)$ .As before , this is a contradiction with the fact that $I>0$,on $[0,t_0)$. Than we conclude that $\|\Delta u(t_0)\|>0$.On The other hand , we have

$$I(t_0)\ge l\|\Delta u(t_0)\|_2^2-3\int_\Omega u^2(t_0)\ln|u(t_0)|^k dx.$$

By using (2.39) and lemma (2.2.2) we have

$$I(t_0) \geq \Big[l - 3kc_*^3\Big(\frac{6E(0)}{l}\Big)^{\frac{1}{2}}\Big]\|\Delta u(t_0)\|_2^2.$$

By recalling (2.35), we arrive at $I(t_0) > 0$, which contradicts the assumption $I(t_0) = 0$. Hence,$t_0 = T$ and then

$$I(t) > 0, \qquad \forall t \in [0, T).$$

■

Theorem 2.2.3 *(Global existence) Under the assumption of lemma* (2.1.2) *the local solution of* (1)-(3) *is global solution .*

Proof. after the definition of E(t) we have

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + J(t).$$

and

$$E(t) \leq E(0).$$

So

$$\|u_t\|_2^2 + 2J(t) \leq 2E(0) := C.$$

such that C is independent of t, implies

$$\begin{cases} \|u_t\|_2^2 \leq C \\ 2J(t) \leq C \end{cases}$$

which established the proof of the theorem (2.2.3). ■

Chapter 3

Stability result

3.1 Some necessary lemmas

We start by establishing several lemmas needed for the proof of the our main result.

Lemma 3.1.1 *Assume that g satisfies (A1).Then,for $u \in H_0^2(\Omega)$,we have*

$$\int_\Omega \Big(\int_0^t g(t-s)(u(t)-u(s))ds\Big)^2 dx \le c(g \circ \Delta u(T)),$$

and

$$\int_\Omega \Big(\int_0^t g'(t-s)(u(t)-u(s))ds\Big)^2 dx \le c(g' \circ \Delta u(T)).$$

Proof. We can simplify

$$\int_\Omega \Big(\int_0^t g(t-s)(u(t)-u(s))ds\Big)^2 dx$$

to

$$= \int_\Omega \Big(\int_0^t \sqrt{g(t-s)}\sqrt{g(t-s)}(u(t)-u(s))ds\Big)^2 dx.$$

by applying Cauchy-Schwarz's inequalities

$$\begin{aligned}
&\le \int_\Omega \Big(\int_0^t \Big(\sqrt{g(t-s)}\Big)^2 ds\Big)^{\frac{1}{2}} \Big(\int_0^t \Big(\sqrt{g(t-s)}(u(t)-u(s))ds\Big)^2 dx\Big)^{\frac{1}{2}} \\
&\le \int_\Omega \Big(\int_0^t g(t-s)ds\Big)^{\frac{1}{2}} \Big(\int_0^t \Big(g(t-s)(u(t)-u(s))^2 ds\Big)^2 dx\Big)^{\frac{1}{2}} \qquad (3.1) \\
&\le (1-l)\Big(\int_0^t g(t-s)ds\Big) \int_\Omega (u(t)-u(s))^2 dx
\end{aligned}$$

by applying Poincare's inequalities

$$\begin{aligned}
&\le (1-l)c\Big(\int_0^t g(t-s)ds\Big) \int_\Omega (\Delta u(t)-\Delta u(s))^2 dx \\
&= (1-l)c(g \circ \Delta u)(t) \\
&\le c(g \circ \Delta u)(t).
\end{aligned}$$

■

Lemma 3.1.2 *Assume that g satisfies (A1) and (A2). Then*

$$\int_0^\infty \xi(t) g^{1-\sigma}(t)dt < \infty, \qquad \forall\, \sigma < 2-p. \tag{3.2}$$

Proof. Using (A1) and (A2),we easily see that , for any $\sigma < 2-p$,

$$\xi(t)g^{1-\sigma}(t) = \xi(t)g^{1-\sigma}(t)g^p(t)g^{-p}(t) \leq -g'(t)g^{1-\sigma-p}(t).$$

Integrate the lest inquality over $(0,\infty)$,we obtain

$$\int_0^\infty \xi(t)g^{1-\sigma}(t)dt \leq -\int_0^\infty g'(t)g^{1-\sigma-p}(t)dt = \Big[-\frac{2-p-\sigma(t)}{2-p-\sigma}\Big]_0^\infty < \infty.$$

■

Lemma 3.1.3 *Assume that (A1)-(A3) and (4,5)hold and u is a solution of ((1)-(3)). Then, for any $0<\sigma<1$ we have*

$$(g\circ\Delta u)(t) \leq c\Big[\Big(\int_0^\infty g^{1-\sigma}(t)dt\Big)E(0)\Big]^{\frac{p-1}{p-1+\sigma}}(g^p\circ\Delta u)^{\frac{\sigma}{p-1+\sigma}}(t).$$

By taking $\sigma = \dfrac{1}{2}$, we get

$$(g\circ\Delta u)(t) \leq c\Big(\int_0^t g^{\frac{1}{2}}(s)ds\Big)^{\frac{2p-2}{2p-1}}(g^p\circ\Delta u)^{\frac{1}{2p-1}}(t). \tag{3.3}$$

and for any $\epsilon_0 \in (0,1)$

$$(g\circ\Delta u)^{\frac{1}{1+\epsilon_0}}(t) \leq c^{\frac{1}{1+\epsilon_0}}\Big(\int_0^t g^{\frac{1}{2}}(s)ds\Big)^{\frac{2p-2}{(2p-1)(1+\epsilon_0)}}(g^p\circ\Delta u)^{\frac{1}{(2p-1(1+\epsilon_0))}}(t). \tag{3.4}$$

Corollary 3.1.4 *Assume that (A1)-(A2) and (2.35)hold any u is a solution of ((1)-(3)). Then*

$$\xi(t)(g\circ\Delta u)(t) \leq c(-E'(t))^{\frac{1}{(2p-1)}}, \tag{3.5}$$

and, for any$\epsilon_0 \in (0,1)$

$$\xi(t)(g\circ\Delta u)^{\frac{1}{1+\epsilon_0}}(t) \leq c_{\epsilon_0}(-E'(t))^{\frac{1}{(2p-1)(1+\epsilon_0)}}, \tag{3.6}$$

Proof. Multiply both sides of (3.3) by $\xi(t)$ and use (3.2) and (2.3) to obtain

$$\begin{aligned}
\xi(t)(g\circ\Delta u)(t) &\leq c\,\xi^{\frac{2p-2}{(2p-1)}}(t)\Big(\int_0^t g^{\frac{1}{2}}(s)ds\Big)^{\frac{2p-2}{2p-1}}\xi^{\frac{1}{2p-1}}(t)(g^p\circ\Delta u)^{\frac{1}{2p-1}}(t)\\
&\leq c\Big(\int_0^t \xi(s)g^{\frac{1}{2}}(s)ds\Big)^{\frac{2p-2}{2p-1}}(\xi g^p\circ\Delta u)^{\frac{1}{2p-1}}(t) \qquad (3.7)\\
&\leq c\Big(\int_0^t \xi(s)g^{\frac{1}{2}}(s)ds\Big)^{\frac{2p-2}{2p-1}}(g'\circ\Delta u)^{\frac{1}{2p-1}}(t)\\
&\leq c(-E'(t))^{\frac{1}{2p-1}}.
\end{aligned}$$

For the proof (3.6) using (3.5) and because ξ is nonincreasing we obtain

$$\xi(t)(g\circ\Delta u)^{\frac{1}{1+\epsilon_0}}(t) = \xi^{\frac{\epsilon_0}{1+\epsilon_0}}\xi(t)(g\circ\Delta u)^{\frac{1}{1+\epsilon_0}}(t) \leq c(E'(t))^{\frac{1}{(2p-1)(1+\epsilon_0)}}. \tag{3.8}$$

■

Lemma 3.1.5 *Assume that (A1)-(A3)and* (2.35) *hold. Then The functional*

$$\psi(t)=\int_\Omega uu_t dx,$$

satisfies, along the solutions of ((1)-(3)*)*

$$\psi'(t)\leq \|u_t\|_2^2-\frac{l}{2}\|\Delta u\|_2^2-\|u\|_2^2+\int_\Omega u^2\ln|u|^k dx+c(g\circ\Delta u)(t). \tag{3.9}$$

Proof. We have

$$\psi'(t)=\int_\Omega u_{tt}udx+\|u_t\|_2^2.$$

we multiply Eq ((1)-(3)) by u and we integrate over Ω we get,

$$\int_\Omega u_{tt}udx=\|\Delta u\|_2^2-\|u\|_2^2+\int_\Omega u\int_0^t g(t-s)\Delta^2u(s)dsdx)+\int_\Omega u^2\ln|u|^k dx,$$

by using green formula we find

$$\int_\Omega u_{tt}udx=\|\Delta u\|_2^2-\|u\|_2^2+\int_\Omega \Delta u\int_0^t g(t-s)\Delta u(s)dsdx)+\int_\Omega u^2\ln|u|^k dx.$$

Then

$$\psi'(t)=\|u_t\|_2^2-\|\Delta u\|_2^2-\|u\|_2^2+\int_\Omega \Delta u\int_0^t g(t-s)\Delta u(sdsdx)+\int_\Omega u^2\ln|u|^k dx. \tag{3.10}$$

we use Yong's inequality for any $\mu>0$,to obtain

$$\begin{aligned}&\int_\Omega \Delta u(t)\Big(\int_0^t g(t-s)\Delta u(s)ds\Big)dx\\ &\leq\frac{\mu}{2}\int_\Omega(\Delta u)^2(t)dx+\frac{1}{2\mu}\int_\Omega\Big(\int_0^t g(t-s)\Delta u(s)ds\Big)^2dx.\end{aligned}$$

we add then term $\Delta u(t)-\Delta u(t)$ we get:

$$\begin{aligned}&\int_\Omega \Delta u(t)\Big(\int_0^t g(t-s)\Delta u(s)ds\Big)dx\\ &\leq\frac{\mu}{2}\int_\Omega(\Delta u)^2(t)dx+\frac{1}{2\mu}\int_\Omega\Big(\int_0^t g(t-s)[(\Delta u(s)-\Delta u(t))+\Delta u(t)]ds\Big)^2dx\\ &=\frac{\mu}{2}\int_\Omega(\Delta u)^2(t)dx+\frac{1}{2\mu}\int_\Omega\Big(\int_0^t g(t-s)[\Delta u(s)-\Delta u(t)]ds\Big)^2+(\int_0^t g(s)\Delta u(t)ds\Big)^2dx\\ &=\frac{\mu}{2}\int_\Omega(\Delta u)^2(t)dx+\frac{1}{2\mu}\int_\Omega\Big(\int_0^t g(t-s)[\Delta u(s)-\Delta u(t)]ds\Big)^2dx+(1-l)\|\Delta u\|^2\\ &\leq(1-l+\frac{\mu}{2})\|\Delta u\|^2+\frac{1}{2\mu}(1-l)(g\circ\Delta u)(t).\end{aligned}$$

So

$$\int_\Omega \Delta u(t)\Big(\int_0^t g(t-s)\Delta u(s)ds\Big)dx \leq (1-l+\frac{\mu}{2})\|\Delta u\|_2^2 + \frac{1}{2\mu}(1-l)(g\circ\Delta u)(t). \quad (3.11)$$

By choosing $\mu = l$

$$\psi'(t) \leq \|u_t\|_2^2 - \|\Delta u\|_2^2 - \|u\|_2^2 + (1-\frac{l}{2})\|\Delta u\|_2^2 + c(g\circ\Delta u)(t) + \int_\Omega u^2 \ln|u|^k dx. \quad (3.12)$$

So we find (3.9). ■

Lemma 3.1.6 *Assume that (A1)-(A2) and* (2.35) *hold. Then the functional*

$$\chi(t) = -\int_\Omega u_t \int_0^t g(t-s)(u(t)-u(s))dsdx,$$

satisfies, along the solutions of ((1)-(3)) and for any $\epsilon_0 \in (0,1)$ and $\delta > 0$

$$\begin{aligned}\chi'(t) \leq {} & \delta\|\Delta u\|_2^2 + \frac{c}{\delta}(-g'\circ\Delta u)(t) + \frac{c}{\delta}(g\circ\Delta u)(t) \\ & + \Big(\delta - \int_0^t g(s)ds\Big)\|u_t\|_2^2 + c_{\epsilon_0,\delta}(g\circ\Delta u)^{\frac{1}{1+\epsilon_0}}(t). \quad (3.13)\end{aligned}$$

Proof. We have

$$\begin{aligned}\chi'(t) &= -\frac{d}{dt}\Big(\int_\Omega u_t \int_0^t g(t-s)(u(t)-u(s))dsdx\Big) \\ &= -\int_\Omega \frac{d}{dt}\Big(u_t \int_0^t g(t-s)(u(t)-u(s))dsdx\Big) \\ &= -\int_\Omega u_{tt}\Big(\int_0^t g(t-s)(u(t)-u(s))ds\Big)dx - \int_\Omega u_t \frac{d}{dt}\Big(\int_0^t g(t-s)(u(t)-u(s))ds\Big)dx,\end{aligned}$$

and we have

$$\begin{aligned}\int_\Omega u_t \frac{d}{dt}\Big(\int_0^t g(t-s)(u(t)-u(s))ds\Big)dx &= \int_\Omega u_t \int_0^t g'(t-s)(u(t)-u(s))dsdx \\ &\quad + \int_\Omega u_t \int_0^t g(t-s)u_t dsdx \\ &= \int_\Omega u_t \int_0^t g'(t-s)(u(t)-u(s))dsdx \\ &\quad + \Big(\int_0^t g(t-s)ds\Big)\int_\Omega u_t^2 dx.\end{aligned}$$

we put $y = t-s$ then $s = t-y \Rightarrow ds = -dy$ $s_1 = 0 \Rightarrow y_1 = t, s_2 = t \Rightarrow y_2 = 0$

$$\begin{aligned}\int_0^t g(t-s)ds &= \int_t^0 g(y)(-dy) \\ &= -\int_t^0 g(y)dy \\ &= \int_0^t g(y)dy \\ &= \int_0^t g(s)ds.\end{aligned}$$

So,

$$\int_\Omega u_t \frac{d}{dt}\Big(\int_0^t g(t-s)(u(t)-u(s))ds\Big)dx = \int_\Omega u_t \int_0^t g'(t-s)(u(t)-u(s))dsdx + \Big(\int_0^t g(s)ds\Big)\int_\Omega u_t^2 dx,$$

then

$$\begin{aligned}\chi'(t) =& -\int_\Omega u_{tt}\Big(\int_0^t g(t-s)(u(t)-u(s))ds\Big)dx - \int_\Omega u_t \int_0^t g'(t-s)(u(t)-u(s))dsdx\\ &-\Big(\int_0^t g(s)ds\Big)\int_\Omega u_t^2 dx.\end{aligned}$$

We multiply ((1)-(3)) by $\int_0^t g(t-s)(u(t)-(s))ds$ and we integrate over Ω then

$$\begin{aligned}&-\int_\Omega u_{tt}\Big(\int_0^t g(t-s)(u(t)-u(s))ds\Big)dx\\ &=\int_\Omega \Delta u\int_0^t g(t-s)(\Delta u(t)-\Delta u(s))dsdx\\ &+\int_\Omega u\int_0^t g(t-s)(u(t)-u(s))dsdx\\ &+\int_\Omega\int_0^t g(t-s)(\Delta u(t)-\Delta u(s))ds\int_0^t g(t-s)\Delta u(s)dsdx\\ &-\int_\Omega u\ln|u|^k\int_0^t g(t-s)(u(t)-u(s))dsdx,\end{aligned}$$

then

$$\begin{aligned}\chi'(t) =& \int_\Omega \Delta u\int_0^t g(t-s)(\Delta u(t)-\Delta u(s))dsdx + \int_\Omega u\int_0^t g(t-s)(u(t)-u(s))dsdx\\ &+\int_\Omega\int_0^t g(t-s)(\Delta u(t)-\Delta u(s))ds\int_0^t g(t-s)\Delta u(s)dsdx\\ &-\int_\Omega u\ln|u|^k\int_0^t g(t-s)(u(t)-u(s))dsdx\\ &-\int_\Omega u_t\int_0^t g'(t-s)(u(t)-u(s))dsdx\\ &-\Big(\int_0^t g(s)ds\Big)\int_\Omega u_t^2dx. \qquad (3.14)\end{aligned}$$

Similarly to (3.10), we estimate the eight-hand side terms of (3.14). So, by using Young's inequality, the first term gives, for any $\delta > 0$,

$$\begin{aligned}\int_\Omega \Delta u\int_0^t g(t-s)(\Delta u(t)-\Delta u(s))dsdx &\le \frac{\delta}{4}\|\Delta u\|_2^2 + \frac{1}{\delta}\int_\Omega\Big(\int_0^t g(t-s)(\Delta u(t)-\Delta u(s))ds\Big)^2dx\\ &\le \frac{\delta}{4}\|\Delta u\|_2^2 + \frac{c}{\delta}(g\circ\Delta u)(t). \qquad (3.15)\end{aligned}$$

Using Lemma (3.1.1) .Young,s and Poincaré's inequality, the second and fifth terms lead to

$$\begin{aligned}\int_\Omega u\int_0^t g(t-s)(\Delta u(t)-\Delta u(s))dsdx &\leq \frac{\delta}{4}\|u\|_2^2+\frac{1}{\delta}\int_\Omega(\int_0^t g(t-s)(\Delta u(t)-\Delta u(s))ds)^2dx\\ &\leq \frac{\delta}{4}\|\Delta u\|_2^2+\frac{c}{\delta}(g\circ\Delta u)(t), \qquad (3.16)\end{aligned}$$

and

$$-\int_\Omega u_t\int_0^t g'(t-s)(\Delta u(t)-\Delta u(s))dsdx \leq \delta\|u_t\|_2^2-\frac{c}{\delta}(g'\circ\Delta u)(t). \qquad (3.17)$$

Similarly, the third term can be estimated as follows

$$\int_\Omega\int_0^t g(t-s)(\Delta u(t)-\Delta u(s))ds\int_0^t g(t-s)\Delta u(s)dsdx \qquad (3.18)$$

$$\leq \frac{1}{\delta}\int_\Omega\Big(\int_0^t g(t-s)(\Delta u(t)-\Delta u(s))ds\Big)^2+\frac{\delta}{4}\int_\Omega\Big(\int_0^t g(t-s)\Delta u(s)ds\Big)^2dx \qquad (3.19)$$

$$\leq \frac{\delta}{4}\|\Delta u\|_2^2+c\Big(1+\frac{1}{\delta}\Big)(g\circ\Delta u)(t). \qquad (3.20)$$

Let $\epsilon_0\in(0,1)$ and $f(s)=s^{\epsilon_0}(|\ln s|-s)$. Notice that f is continuous on $(0,\infty)$ and its limit at 0 is 0, and its limit at ∞ is $-\infty$.Then f has a maximum d_{ϵ_0} on $[0,\infty)$, so the following inequality holds:

$$s|\ln s|\leq s^2+d_{\epsilon_0}s^{1-\epsilon_0}, \qquad \forall s>0. \qquad (3.21)$$

By applying this inequality (3.21) and we have $H_0^2(\Omega)$ is embedding in $L^\infty(\Omega)\Longrightarrow \int_\Omega u^2dx\leq c_1\int_\Omega|u|$
and using young's inequality s.t $q=\dfrac{2}{1+\epsilon_0}, q^*=\dfrac{2}{1-\epsilon_0}$

$$\begin{aligned}&\int_\Omega u\ln|u|^k\int_0^t g(t-s)(u(t)-u(s))dsdx\\ &=k\int_\Omega u\ln|u|\int_0^t g(t-s)(u(t)-u(s))dsdx\\ &\leq k\int_\Omega(u^2+d_{\epsilon_0}|u|^{1-\epsilon_0})\Big|\int_0^t g(t-s)(u(t)-u(s))dsdx\Big|\\ &=k\int_\Omega u^2\Big|\int_0^t g(t-s)(u(t)-u(s))dsdx\Big|+k\int_\Omega d_{\epsilon_0}|u|^{1-\epsilon_0}\Big|\int_0^t g(t-s)(u(t)-u(s))dsdx\Big|\\ &\leq c\int_\Omega|u|\Big|\int_0^t g(t-s)(u(t)-u(s))ds\Big|dx\\ &+\delta_1\int_\Omega u^2dx+c_{\epsilon_0,\delta_1}\int_\Omega\Big|\int_0^t g(t-s)(u(t)-u(s))ds\Big|^{\frac{2}{1+\epsilon_0}}dx\\ &\leq c\delta_1\|\Delta u\|_2^2+\frac{c}{\delta_1}\int_\Omega\Big|\int_0^t g(t-s)(u(t)-u(s))ds\Big|^2dx\\ &+c_{\epsilon_0,\delta_1}\int_\Omega\Big|\int_0^t g(t-s)(u(t)-u(s))ds\Big|^{\frac{2}{1+\epsilon_0}}dx,\end{aligned}$$

such that $c = kc_1$ and $\delta_1 = kd_{\epsilon_0}$ then,puting $\dfrac{\delta}{4} = c\delta_1$ and using Holder's inequality and Lemma (3.1.1),we find

$$\begin{aligned}&\int_\Omega u\ln|u|^k \int_0^t g(t-s)(u(t)-u(s))dsdx\\ &\leq \frac{\delta}{4}\|\Delta u\|_2^2 + \frac{c}{\delta}(g\circ\Delta u)(t) + c_{\epsilon_0,\delta_1}(g\circ\Delta u)^{\frac{1}{1+\epsilon_0}}(t).\end{aligned} \tag{3.22}$$

The above inequalities imply (3.13). ■

Lemma 3.1.7 *Assume that (A1)-(A3)and* (2.35) *hold and let* $\epsilon_0 \in (0,1)$*.Then , for k small enough, there exist two positive constant* $\varepsilon_1, \varepsilon_2$ *such that the functional*

$$L(t) = E(t) + \varepsilon_1\psi(t) + \varepsilon_2\chi(t),$$

satisfies

$$L \sim E \tag{3.23}$$

and ,for any $t_0 > 0$*, there exists a positive constant m such that*

$$L'(t) \leq -mE(t) + c(g\circ\Delta u)(t) + c_{\epsilon_0}(g\circ\Delta u)^{\frac{1}{1+\epsilon_0}}(t), \qquad \forall t \geq t_0. \tag{3.24}$$

Proof. For the proof of (3.23) we have

$$\begin{aligned}\psi(t) = \int_\Omega uu_t dx &\leq \int_\Omega u^2 dx + \int_\Omega u_t^2 dx\\ &\leq \int_\Omega \Delta u^2 dx + \int_\Omega u_t^2 dx\\ &\leq \|\Delta u\|_2^2 + \|u_t\|_2^2,\end{aligned}$$

and we have

$$\begin{aligned}\chi(t) = -\int_\Omega u_t \int_0^t g(t-s)(u(t)-u(s))dsdx &\leq -\int_\Omega u_t^2 dx - \int_\Omega (\int_0^t g(t-s)(u(t)-u(s))ds)^2 dx\\ &\leq -\|u_t\|^2 + c(g\circ\Delta u)(t).\end{aligned} \tag{3.25}$$

then we see that using similar calculations as before,

$$\begin{aligned}|L(t)-E(t)| &= |\varepsilon_1\psi(t) + \varepsilon_2\chi(t)|\\ &= |\varepsilon_1\int_\Omega u_t u dx - \varepsilon_2\int_\Omega u_t \int_0^t g(t-s)(u(t)-u(s))dsdx|\\ &\leq \varepsilon_1\Big(\|\Delta u\|_2^2 + \|u_t\|_2^2\Big) + \varepsilon_2\Big(\|u_t\|^2 + c(g\circ\Delta u)(t)\Big)\\ &\leq c(\varepsilon_1+\varepsilon_2)\Big(\|u_t\|_2^2 + \|\Delta u\|_2^2 + (g\circ\Delta u)(t)\Big),\end{aligned}$$

there for, from (2.36) and (2.38), we obtain

$$|L(t)-E(t)| \leq c(\varepsilon_1+\varepsilon_2)\Big(\frac{1}{2}\|u_t\|_2^2 + J(t)\Big) = c(\varepsilon_1+\varepsilon_2)E(t),$$

because

$$\frac{1}{2}\|u_t\|_2^2 + J(t) = E(t)$$

then

$$-c(\varepsilon_1+\varepsilon_2)E(t) \le L(t) - E(t) \le c(\varepsilon_1+\varepsilon_2)E(t).$$

So

$$(1-c(\varepsilon_1+\varepsilon_2))E(t) \le L(t) \le (1+c(\varepsilon_1+\varepsilon_2))E(t).$$

Hence, for $\varepsilon_1, \varepsilon_2 > 0$ satisfying

$$(1-c(\varepsilon_1+\varepsilon_2)) > 0, \tag{3.26}$$

the equivalence (3.23) holds. Now ,we prove inequality (3.24) .Since g is positive and $g(0) > 0$ then , for any $t_0 > 0$, we have

$$\int_0^t g(s)ds \ge \int_0^{t_0} g(s)ds = g_0 > 0, \qquad \forall t \ge t_0.$$

We have$L'(t) = E'(t) + \varepsilon_1\psi'(t) + \varepsilon_2\chi'(t)$

So when we add and subtract $mE(t)$ to obtain

$$L'(t) = -mE(t) + mE(t) + E'(t) + \varepsilon_1\psi'(t) + \varepsilon_2\chi'(t).$$

By using (2.3),(3.9) (3.13) and the definition of $E(t)$, then ,for $t \ge t_0$ and any $m > 0$, we have

$$\begin{aligned} L'(t) \le & -mE(t) - \Big(\varepsilon_2(g_0-\delta) - \varepsilon_1 - \frac{m}{2}\Big)\|u_t\|_2^2 \\ & - \Big(\frac{l}{2}\varepsilon_1 - \varepsilon_2\delta - \frac{m}{2}\Big)\|\Delta u\|_2^2 - \Big(\varepsilon_1 - \frac{(k+2)m}{4}\Big)\|u\|_2^2 \\ & + \Big(k\varepsilon_1 - k\frac{m}{2}\Big)\int_\Omega u^2 \ln|u|dx + \Big(c\varepsilon_1 + \varepsilon_2\frac{c}{\delta} + \frac{m}{2}\Big)(g\circ\Delta u)(t) \\ & + \Big(\frac{1}{2} - \frac{c\varepsilon_2}{\delta}\Big)(g'\circ\Delta u)(t) + \varepsilon_2 c_{\epsilon 0,\delta}(g\circ\Delta u)^{\frac{1}{1+\epsilon_0}}(t). \end{aligned} \tag{3.27}$$

Using the Logarithmic Sobolev inequality, for $0 < m < 2\varepsilon_1$, we get

$$\begin{aligned} L'(t) \le & -mE(t) - \Big(\varepsilon_2(g_0-\delta) - \varepsilon_1 - \frac{m}{2}\Big)\|u_t\|_2^2 \\ & - \Big(\frac{l}{2}\varepsilon_1 - \varepsilon_2\delta - \frac{m}{2} - k\big(\varepsilon_1 - \frac{m}{2}\big)\frac{c_p a^2}{2\pi}\Big)\|\Delta u\|_2^2 \\ & - \Big(\varepsilon_1 - \frac{(k+2)m}{4} + k\big(\varepsilon_1 - \frac{m}{2}\big)(1+\ln a) + k\big(\frac{m}{4} - \frac{\varepsilon_1}{2}\big)\ln\|u\|_2^2\Big)\|u\|_2^2 \\ & + \Big(c\varepsilon_1 + \varepsilon_2\frac{c}{\delta} + \frac{m}{2}\Big)(g\circ\Delta u)(t) \\ & + \Big(\frac{1}{2} - \frac{c\varepsilon_2}{\delta}\Big)(g'\circ\Delta u)(t) + \varepsilon_2 c_{\epsilon 0,\delta}(g\circ\Delta u)^{\frac{1}{1+\epsilon_0}}(t). \end{aligned} \tag{3.28}$$

At this point we choose δ so small that

$$g_0 - \delta > \frac{1}{2}g_0 \quad \text{and} \quad \delta < \frac{lg_0}{16}.$$

Whence δ is fixed, the choice of any twe positive constants ε_1 and ε_2 satisfying

$$\frac{g_0}{4}\varepsilon_2 < \varepsilon_1 < \frac{g_0}{2}\varepsilon_2, \tag{3.29}$$

will make

$$k_1 := \varepsilon_2(g_0 - \delta) - \varepsilon_1 > 0 \quad \text{and} \quad k_2 := \frac{l}{2}\varepsilon_1 - \varepsilon_2\delta > 0.$$

Then ,we choose ε_1 and ε_2 so small so that (3.26)and (3.29) remain valid and,further

$$\frac{1}{2} - \frac{c\varepsilon_2}{\delta} > 0.$$

$$\begin{aligned}
L'(t) \le & -mE(t) - \Big(k_1 - \frac{m}{2}\Big)\|u_t\|_2^2 \\
& - \Big(k_2 - \frac{m}{2} - k\big(\varepsilon_1 - \frac{m}{2}\big)\frac{c_p a^2}{2\pi}\Big)\|\Delta u\|_2^2 \\
& - \Big(\varepsilon_1 - \frac{(k+2)m}{4} + k\big(\varepsilon_1 - \frac{m}{2}\big)(1 + \ln a) + k\big(\frac{m}{4} - \frac{\varepsilon_1}{2}\big)\ln\|u\|_2^2\Big)\|u\|_2^2 \\
& + c(g \circ \Delta u)(t) + c_{\epsilon 0,\delta}(g \circ \Delta u)^{\frac{1}{1+\epsilon_0}}(t).
\end{aligned}$$

Then, using (2.11)and selecting m and k so small that

$$\alpha_1 := k_1 - \frac{m}{2} > 0, \quad \alpha_2 := k_2 - \frac{m}{2} - k\big(\varepsilon_1 - \frac{m}{2}\big)\frac{c_p a^2}{2\pi},$$

and

$$\alpha_3 := \varepsilon_1 - \frac{(k+2)m}{4} + k\big(\varepsilon_1 - \frac{m}{2}\big)(1 + \ln a) + k\big(\frac{m}{4} - \frac{\varepsilon_1}{2}\big)\ln\|u\|_2^2 > 0.$$

Therefore, we arrive at the desired result (3.24). ■

Remark 3.1.8 *Using*(1.5) ,(2.2),(2.31),(2.36), *and*(2.38), *we have*

$$E(t) = J(t) + \frac{1}{2}\|u_t(t)\|_2^2 \ge J(t) \ge \frac{l}{6}\|\Delta u(t)\|_2^2,$$

then ,using (2.3)

$$\|\Delta u(t)\|_2^2 \le \frac{6}{l}E(t) \le \frac{6}{l}E(0). \tag{3.30}$$

So from (2.3) *and using Young's inequality ,we get*

$$\begin{aligned}
|E'(t)| &= \frac{1}{2}g(t)\|\Delta u(t)\|_2^2 - \frac{1}{2}(g' \circ \Delta u)(t) \\
&\le \frac{1}{2}g(t)\|\Delta u(t)\|_2^2 - \int_0^t g'(t-s)(\|\Delta u(t)\|_2^2 + \|\Delta u(s)\|_2^2)ds \qquad (3.31) \\
&\le \frac{1}{2}g(t)\|\Delta u(t)\|_2^2 - \int_0^t 2g'(t-s)E(t)ds \qquad (3.32) \\
&\le \frac{1}{2}g(t)\|\Delta u(t)\|_2^2 - 2E(t)\int_0^t g'(t-s)ds \\
&\le \frac{1}{2}g(t)\|\Delta u(t)\|_2^2 - 2E(0)\int_0^t g'(t-s)ds \\
&\le \frac{6}{l}\Big(\frac{1}{2}g(t) + 2g(0) - 2g(t)\Big)E(0) \\
&\le cE(0).
\end{aligned}$$

3.2 Main result

Theorem 3.2.1 *Let* $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega), \epsilon \in (0, 2p-1)$ *and* $t_0 > 0$*.Assume that (A1)-(A3) and (4,5) hold. Then, for k small enough, there exists a positive constant K such that the solution of (*(1)-(3)*)satisfies*

$$E(t) \leq K\Big(1 + \int_0^t \xi^{2p-1+\epsilon}(s)ds\Big)^{\frac{-1}{2p-2+\epsilon_1}}, \qquad \forall t \geq t_0. \tag{3.33}$$

Moreover, if there exist $\epsilon_1 \in (0, 2p-1)$*and* $t_0 > 0$ *such that*

$$\int_{t_0}^{\infty} \Big(1 + \int_{t_0} t\xi^{2p-1+\epsilon_1}(s)ds\Big)^{\frac{-1}{2p-2+\epsilon_1}} dt < \infty, \tag{3.34}$$

then, for any $\epsilon \in (0, p)$ *and* $t_0 > 0$*,there exists a positive constant* K' *such that the solution of (*(1)-(3)*) satisfies*

$$E(t) \leq K'\Big(1 + \int_0^t \xi^{p+\epsilon}(s)ds\Big)^{\frac{-1}{p-1+\epsilon_1}}, \qquad \forall t \geq t_0. \tag{3.35}$$

Proof. We multiply (3.24) by $\xi(t)$ and use (3.5) , (3.6) and (3.31)to get, for any $t \geq t_0$,

$$\begin{aligned} \xi(t)L'(t) &\leq -m\xi(t)E(t) + c(-E'(t))^{\frac{1}{2p-1}} + c(E'(t))^{\frac{1}{(2p-1)(1+\epsilon_0)}} \\ &\leq -m\xi(t)E(t) + c(-E'(t))^{\frac{\epsilon_0}{(2p-1)(1+\epsilon_0)}}(-E'(t))^{\frac{1}{(2p-1)(1+\epsilon_0)}} \\ &+ c(-E'(t))^{\frac{1}{(2p-1)(1+\epsilon_0)}} \\ &\leq -m\xi(t)E(t) + c(-E'(t))^{\frac{1}{(2p-1)(1+\epsilon_0)}} \underbrace{\Big(1 + (-E'(t))^{\frac{\epsilon_0}{(2p-1)(1+\epsilon_0)}}\Big)}_{<\infty}, \qquad \forall t \geq t_0. \end{aligned} \tag{3.36}$$

$$\leq -m\xi(t)E(t) + c(-E'(t))^{\frac{1}{(2p-1)(1+\epsilon_0)}}. \qquad \forall t \geq t_0 \tag{3.37}$$

Multiply the lest inequality by $\xi(t)^{\gamma}(t)E^{\gamma}(t)$,where$\gamma = 2(p-1)(1+\epsilon_0) - 1$, and notice that $\xi' \leq 0$ to obtain

$$\xi^{\gamma+1}E^{\gamma}(t)L'(t) \leq -m\xi^{\gamma+1}(t)E^{\gamma+1}(t) + c(\xi E(t))^{\gamma}(-E'(t))^{\frac{1}{\gamma+1}}, \forall t \geq t_0.$$

Use of Young's inequality, with $q = \gamma + 1$ and $q^* = \dfrac{\gamma+1}{\gamma}$, gives for any $\varepsilon' > 0$,

$$\begin{aligned} \xi^{\gamma+1}(t)E^{\gamma}(t)L'(t) &\leq -m\xi^{\gamma+1}(t)E^{\gamma+1}(t) + c(\varepsilon'\xi^{\gamma+1}(t)E^{\gamma+1} - c_{\varepsilon'}E'(t)) \\ &= -(m - \varepsilon' c)\xi^{\gamma+1}(t)E^{\gamma+1}(t) - cE'(t), \qquad \forall t \geq t_0. \end{aligned}$$

We then choose $0 < \varepsilon' < \dfrac{m}{c}$ and recall that $\xi' \leq 0$and$E' \leq 0$,to get, for $c_1 = m - \varepsilon' c$,

$$\Big(\xi^{\gamma+1}E^{\gamma}L(t)\Big)' = \Big((\xi^{\gamma+1})'E^{\gamma}L(t)\Big) + \Big(\xi^{\gamma+1}(E^{\gamma})'L(t)\Big) + \Big(\xi^{\gamma+1}E^{\gamma}L'(t)\Big)$$

So,

$$\Big(\xi^{\gamma+1}E^{\gamma}L(t)\Big)' \ \leq\ \xi^{\gamma+1}(t)E^{\gamma+1}(t)L'(t) \ \leq\ -c_1\xi^{\gamma+1}(t)E^{\gamma+1} - cE'(t), \qquad \forall t \geq t_0.$$

which implies

$$(\xi^{\gamma+1}E^{\gamma}L + cE)'(t) \leq -c_1\xi^{\gamma+1}(t)E^{\gamma+1}(t) \qquad \forall t \geq t_0,$$

Let $F = \xi^{\gamma+1}E^{\gamma}L + cE$. Then $F \sim E$ (thanks to(3.23)) So

$$\begin{aligned} F'(t) &\leq -c_1\xi^{\gamma+1}(t)E^{\gamma+1}(t) \\ &\leq -c\xi^{\gamma+1}(t)F^{\gamma+1}(t) = -c\xi^{(2p-1)(1+\epsilon_0)}(t)F^{(2p-1)(1+\epsilon_0)}(t) \quad \forall t \geq t_0. \end{aligned}$$

Integrating over (t_0, t) and using the fact that $F \sim E$ we obtain (3.33) with $\epsilon = (2p-1)\epsilon_0$. To establish (3.35), we use the idea of Messaoudi and Al-Khulaif[16] let

$$\eta(t) = \int_0^t \|\Delta u(t) - \Delta u(t-s)\|_2^2 ds.$$

Using (3.30), we get

$$\begin{aligned} \eta(t) &\leq 2\int_0^t (\|\Delta u(t)\|_2^2 + \|\Delta u(t-s)\|_2^2)ds \quad \text{ by use inequality} (a-b)^2 \leq 2a^2 + 2b^2 \\ &\leq \frac{12}{l}\int_0^t (E(t) + E(t-s))ds \\ &\leq \frac{24}{l}\int_0^t E(s)ds \\ &< \frac{24}{l}\int_0^\infty E(s)ds < \infty. \quad \text{by use } (3.33), (3.34) \end{aligned}$$

This implies that

$$\sup_{t>0} \eta^{1-\frac{1}{p}}(t) < \infty. \tag{3.38}$$

Assume that $\eta(t) > 0$. Then because ξ is nonincreasing,we find

$$\xi(t)(g \circ \Delta u)(t) \leq \frac{\eta(t)}{\eta(t)}\int_0^t (\xi^p(s)g^p(s))^{\frac{1}{p}}\|\Delta u(t) - \Delta u(t-s)\|_2^2 ds.$$

Applying Jensen's inequality to get

$$\xi(t)(g \circ \Delta u)(t) \leq \eta(t)\Big(\frac{1}{\eta(t)}\int_0^t \xi^p(s)g^p(s)\|\Delta u(t) - \Delta(t-s)\|_2^2 ds\Big)^{\frac{1}{p}}.$$

The mention that $g'(t) \leq -\xi(t)g^p(t)$, (3.38), we obtain

$$\begin{aligned} \xi(t)(g \circ \Delta u)(t) &\leq \eta^{1-\frac{1}{p}}(t)\Big(\xi^{p-1}(0)\int_0^t \xi(s)g^p(s)\|\Delta u(t) - \Delta(t-s)\|_2^2 ds\Big)^{\frac{1}{p}} \\ &\leq \eta^{1-\frac{1}{p}}(t)\Big(\xi^{p-1}(0)\int_0^t -g'(t)\|\Delta u(t) - \Delta(t-s)\|_2^2 ds\Big)^{\frac{1}{p}} \\ &\leq \Big(\eta(t)\xi(0)\Big)^{1-\frac{1}{p}}\Big(\int_0^t -g'(t)\|\Delta u(t) - \Delta(t-s)\|_2^2 ds\Big)^{\frac{1}{p}} \\ &\leq c(-g' \circ \Delta u)^{\frac{1}{p}}(t), \end{aligned}$$

and then, according to (2.3),

$$\xi(t)(g\circ\Delta u)(t) \le c(-E'(t))^{\frac{1}{p}} \tag{3.39}$$

So since ξ is nonincreasing,

$$\begin{aligned}\xi(t)(g\circ\Delta u)^{\frac{1}{1+\epsilon_0}}(t) &= (\xi^{\epsilon_0}(t)\xi(t)(g\circ\Delta u)(t))^{\frac{1}{1+\epsilon_0}}\\ &\le (\xi^{\epsilon_0}(0)\xi(t)(g\circ\Delta u)(t))^{\frac{1}{1+\epsilon_0}}\\ &\le c\xi(t)(g\circ\Delta u)(t))^{\frac{1}{1+\epsilon_0}}\\ &\le c(-E'(t))^{\frac{1}{(2p-1)(1+\epsilon_0)}}. \quad \text{by}(3.6)\end{aligned} \tag{3.40}$$

If $\eta(t)=0$,then $s \longrightarrow \Delta u(s)$ is a constant function on $[0,t]$.Therefore

$$(g\circ\Delta u)(t) = 0,$$

and hence (3.39)and (3.40) hold.
Now, multiplying (3.24) by $\xi(t)$ and using (3.31),(3.31) ,(3.39) and (3.40) to find,for any $t \ge t_0$ (as for,(3.36))

$$\begin{aligned}\xi(t)L'(t) &\le -m\xi(t)E(t) + c(-E'(t))^{\frac{1}{p}} + c(E'(t))^{\frac{1}{p(1+\epsilon_0)}}\\ &\le -m\xi(t)E(t) + c(-E'(t))^{\frac{\epsilon_0}{p(1+\epsilon_0)}}(-E'(t))^{\frac{1}{p(1+\epsilon_0)}} + c(-E'(t))^{\frac{1}{p(1+\epsilon_0)}}\\ &\le -m\xi(t)E(t) + c(-E'(t))^{\frac{1}{p(1+\epsilon_0)}}, \qquad \forall t \in\ge t_0.\end{aligned} \tag{3.41}$$

Inequality (3.36) with $2p-1$ replaced by p is exactly (3.41). Then the proof of (3.35) can be completed as for the one of (3.33)(by taking $\gamma = p(1+\epsilon_0)-1$ and $\epsilon = p\epsilon_0$). This completes the proof of our main result. ■

Bibliography

[1] M. M. Al-Gharabli, A. Guesmia and S. A. Messoudi, Existence and a general decay result for a viscoelastic plat equation with a logarithmic nonlinearity, *CPAA*, **18**, 159-180, (2019)

[2] H. Brezis, Functional analysis, Sobolev spaces and partial differential equations, Springer (2010).

[3] S. Berrimi, S.A. Messaoudi: Exponential decay of solutions to a viscoelastic equation with nonlinear localized damping. Electron. J. Differ. Equ. 88, 1–10 (2004)

[4] Y. Boukhatem, B. Benabderrahmane: Existence and decay of solutions for a viscoelastic wave equation with acoustic boundary conditions. Nonlinear Anal. 97, 191–209 (2014)

[5] T. Cazenave and A. Haraux, Equations d'evolution avec non-linearite logarithmique, Ann. Fac. Sci. Toulouse Math., 2 (1980), 21-51.

[6] M.M. Cavalcanti, , Oquendo, H.P.: Frictional versus viscoelastic damping in a semilinear wave equations. SIAM J. Control Optim. 42(4), 1310–1324 (2003)

[7] M.M.Cavalcanti, , Domingos Cavalcanti, V.N., Ferreira, J.: Existence and uniform decay for a nonlinear viscoelastic equation with strong damping. Math. Methods Appl. Sci. 24, 1043–1053 (2001)

[8] H.F.Di, Y.D. Shang: Existence, nonexistence and decay estimate of global solutions for a viscoelastic wave equation with nonlinear boundary damping and internal source terms. Eur J. Pure Appl. Math. 10, 668–701 (2017)

[9] P. Górka, Logarithmic Klein–Gordon equation, Acta Phys. Polon. B 40 (1) (2009) 59–66.

[10] X. Han and M.wang, General dacay estimate of energy for the sccond order cvolution equations with memory,Act Appl.Math.,110(2010),194-207.

[11] W.J. Liu: General decay rate estimate for a viscoelastic equation with weakly nonlinear time independent dissipation and source terms. J. Math. Phys. 50, 1–17 (2009)

[12] W. Liu, Y. Sun: General decay of solutions for a weak viscoelastic equation with acoustic boundary conditions. Z. Angew. Math. Phys. 65, 125–134 (2014)

[13] I. Lacheheb, On the stability of solutions for some viscoelastic problems PDEs , thesis of doctorate,universte ouargla(2021).

[14] J. Lagnese: asymptotic energy estimates for Kirchoff plates subject to weak viscoelastic damping. International Series of Numerical Mathematics, vol. 91. Birhauser, Verlag Bassel (1989)

[15] J. Lions, Quelques Methodes de Resolution des Problemes aux Limites Non Lineaires,second Edition,Duanod,Paris,2002.

[16] S.Messaoudi and W.Al-Khulaifi,General and optimal decay for aquasilinear viscoelastic equation,Applied Mathematics Letters,66(2017),16-22.

[17] S. Messaoudi, Global existence and nonexistence in a system of Petrovsky, Journal of Mathematical Analysis and Applications, 265, 296-308 (2002)

[18] J. Necas. Les méthodes directes en théorie des équations elliptiques. Masson, Paris, 1967.

[19] J.M. Rivera, , Lapa, E.C., Barreto, R.: Decay rates for viscoelastic plates with memory. J. Elast. 44, 61–87 (1996)

[20] M. Santos and F. junior, A boundary condition with memory for Kirchoff plates equations, Appl. Math. Comput., 148, 475-496 (2004)

ملخص

في هذه المذكرة سنتطرق الى تقديم وشرح دراسة سابقة حول وجود واستقرار حلول معادلة الامواج التى تحتوي على حد اللزوجة والمرونة بأظافة الى حد لوغاريتم غير الخطي حيث في البداية سنقدم للقارى مجموعة من المفاهيم والمتراجحات والفضاءات التي سنحتاجها في باقي المذكرة, ثم سنلاحظ ان المسالة تقبل حلا محلي باستخدام طريقة **Galerkin** بعد ذلك نثبت ان ذلك الحل هو حل عام, وفي الاخير سنقدم بالتفصيل دراسة استقرار الحل باستخدام الطاقة الخاصة بالمسالة.

كلمات مفتاحية: فضاءات سوبولاف ,الوجود المحلي والعام, الاستقرار ,الطاقة.

Abstract

In this thesis, we will present and explain a previous study on the existence and stability of solutions to the wave equation that includes terms for viscosity and elasticity, along with a nonlinear logarithmic term. We will begin by introducing the reader to a set of concepts, inequalities, and spaces that will be needed throughout the rest of the paper. Then, we will show that the problem admits a local solution using the Galerkin method, after which we will prove that this solution is indeed a global solution. Finally, we will provide a detailed study of the solution's stability by using the energy associated with the problem.

Key words : Sobolev spaces, Local and Global existence, Stability, Energy.

Résumé

Dans cette mémoire, nous présenterons et expliquerons une étude antérieure sur l'existence et la stabilité des solutions de l'équation des ondes, qui inclut des termes de viscosité et d'élasticité, ainsi qu'un terme logarithmique non linéaire. Nous commencerons par introduire au lecteur un ensemble de concepts, d'inégalités et d'espaces nécessaires pour le reste du document. Ensuite, nous montrerons que le problème admet une solution locale en utilisant la méthode de Galerkin, après quoi nous prouverons que cette solution est effectivement une solution globale. Enfin, nous réaliserons une étude détaillée de la stabilité de la solution en utilisant l'énergie associée au problème.

Mots Clé : Espaces Sobolev , Local et Globale existence, Stabilité, Énergie.