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DOCTORATE IN SCIENCES

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Study of particular polynomials defined by recurrent linear sequences with applications

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Contents

Introduction

The Fibonacci sequence is a well-known integer sequence in which each number is the sum of the two preceding ones, starting from 0 and 1. The sequence begins: $0, 1, 1, 2, 3, 5, 8, 13, 21$, and so on. It possesses a wide range of applications across different domains, such as mathematics, natural sciences, and computer science.

Polynomial Fibonacci sequences extend the concept of Fibonacci numbers by using polynomials instead of integers. In these sequences, each term is obtained by adding the previous terms, just like in the usual Fibonacci sequence, but the addition is done using polynomial arithmetic. This means that instead of simply summing the last two terms, the terms are added using polynomial addition.

For example, the polynomial Fibonacci sequence might start with the polynomials: $0, 1, \rho, \rho^2 + 1, \rho^3 + 2\rho, \rho^4 + 3\rho^2 + 1, \rho^5 + 4\rho^3 + 3\rho......$, and so on. Each term is obtained by adding the previous terms using polynomial addition rules.

Polynomial Fibonacci sequences have connections to various areas of mathematics, including combinatorics, algebraic geometry, and number theory. They provide a way to explore more complex relationships between numbers and polynomials, leading to interesting patterns and properties

The tribonacci sequence takes the concept of the famous Fibonacci sequence and adds an intriguing twist. While the Fibonacci sequence is characterized by the sum of the last two terms to generate the next, the tribonacci sequence goes a step further. It begins with three initial terms, and from there, each subsequent term is the sum of the three preceding terms. The tribonacci sequence orginally studied in 1963 by M. Feinberg. For $n > 2$, this sequence exhibit a more intricate interplay of numbers, revealing patterns that have captivated mathematicians and scientists alike.

Now, let us explore the realm of polynomial tribonacci sequences. Imagine blending the captivating nature of the tribonacci sequence with the complexity of polynomials. In polynomial tribonacci sequences, the terms aren't just ordinary numbers; they take the form of polynomials. These polynomials are generated by adding the previous terms using polynomial addition rules, adding an extra layer of sophistication to the sequence.

For instance, consider a polynomial tribonacci sequence that starts with the following polynomials: For $n > 2$, $p_n(\rho) = \rho^2 p_{n-1}(\rho) + \rho p_{n-2}(\rho) + p_{n-3}(\rho)$,with initial conditions $p_0(\rho) = 0$, $p_1(\rho) = 1$, and $p_2(\rho) = \rho^2$, this sequence was introduced in [13]
The first few tribonacci polynomials are : $1, \rho, \rho^2, \rho^3 + \rho^2, \rho^4 + 2\rho^3 + \rho^2, \rho^5 + 3\rho^4 + 3\rho^3 + \rho^2,$
In[1],[11] [17] ,[22] The authors explored Binet's formula, various identities, and additional properties of the tribonacci numbers.
In [25] Ramirez and Sirvent introduced the tribonacci polynomial triangle
In [21], the authors define a generalized tribonacci sequence and they give some properties of this sequence using matrix methods.
Various studies are linked to the generalized tribonacci numbers and polynomials. In one such study, presented in [17], the authors introduced the generalized tribonacci numbers and established an explicit formula for these numbers, even extending it to cover cases with negative subscripts. In[18]the authors give a new generalization of the tribonacci like polynomial which are $P_n(\rho, \varrho, \varpi)$ and $L_n(r, s, w)$. So they investigated of Binet's formula, some identities and other properties of the tribonacci like polynomials
In 1892, Corrado Segre introduced bicomplex numbers, defined by four elements: 1, i, j, and ij. These elements are subject to the following properties:

$$i^2 = -1, j^2 = -1, ij = ji \tag{1}$$

However, the Irish mathematician William Rowan Hamilton introduced Quaternions as an extension of complex numbers. A quaternion q is defined in the form:

$$q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$$

Here, a, b, c, and d are real numbers, and i, j, and k are fundamental Quaternions units.
So In [20],[29] the author introduce the bicomplex generalized tribonacci quaternions. Furthermore,describe a Binet's formula, generating functions, and some proprieties also related

to a matrix representation.

In this thesis an attempt is made to develop a theory of generalizations of the tribonacci sequence and polynomials by generalizing initial values. and the general form of this sequence.

So in the first chapter we define generalized tribonacci polynomials $(p'_n(\rho))_{n\in\mathbb{N}}$, where $p'_n(\rho) = \rho^2 p'_{n-1}(\rho) + \rho p'_{n-2}(\rho) + p'_{n-3}(\rho)$, $n \geqslant 4$, with $p_1(\rho) = a$, $p_2(\rho) = b$ and $p_3(\rho) = c\rho^2$, by using the coefficients $P'(n,j)$ we drive an explicit formula of generalized tribonacci polynomials, Also, we establish some properties of the tribonacci polynomials. Similarly, we study the jacobsthal polynomials $(J_n(x))_{n\in\mathbb{N}}$, where $J_n(\rho) = J_{n-1}(\rho) + \rho J_{n-2}(\rho) + \rho^2 J_{n-3}(\rho)$, $n \geqslant 4$, with $J_1(\rho) = J_2(\rho) = 1$, $J_3(\rho) = \rho + 1$ and describe some properties.
The primary focus of this research involves the exploration of generalizations of tribonacci polynomials, aiming to derive a more comprehensive sequence along with its various identities. The wealth of results in the realm of generalization has sparked our interest and prompted a thorough investigation into this topic. The objective is to unveil broader patterns and properties within the context of tribonacci-like sequences, contributing to a deeper understanding of their mathematical structures.

CHAPTER 1. Preliminaries

1.1 Tribonacci numbers [17]

Definition 1.1 *[17] The tribonacci numbers, denoted as* $(P_n)_{n\in\mathbb{N}}$*, are a sequence of numbers defined by the following recurrence relation.*

$$\begin{cases} P_n = P_{n-1} + P_{n-2} + P_{n-3}, n \geq 4 \\ P_1 = P_2 = 1 \\ P_3 = 2 \end{cases} \tag{1.1}$$

The first tibonacci numbers are 1, 1, 2, 4,7, 13, 24, 44, 81....

1.2 Tribonacci-Lucas numbers [17]

Definition 1.2 *[17] The tribonacci Lucas numbers,denoted by* $(L_n)_{n\in\mathbb{N}}$*, form a sequence defined by the following recurrence relation.*

$$\begin{cases} L_n = L_{n-1} + L_{n-2} + L_{n-3}, n \geq 4 \\ L_0 = 3 \\ L_1 = 1 \\ L_2 = 3 \end{cases} \tag{1.2}$$

The first tibonacci-Lucas numbers are 3, 1, 3, 7,11, 21, 39, ...

1.3 Generalized tribonacci Sequence (GTS)[5]

Definition 1.3 *The generalized tribonacci sequence* $(P'_n)_{n\in\mathbb{N}}$ *, are defined by the following recurrence relation*

$$\begin{cases} P'_n = P'_{n-1} + P'_{n-2} + P'_{n-3}, n \geq 4 \\ P'_1 = d \\ P'_2 = e \\ P'_3 = f \end{cases} \tag{1.3}$$

where $d, e, f \in \mathbb{Z}$

The first tribonacci numbers are $d,\ e,\ f,\ d+e+f,\ d+2e+2f,\ 2d+3e+4f, \ldots.$

Theorem 1.1 *[5] Let* $(P'_n)_{n\in\boldsymbol{N}}$ *denote the nth term of the (GTS). Then we have*

$$\begin{cases} P'_n = dP_{n-3} + e(P_{n-4} + P_{n-3}) + fP_{n-2}, n \geq 4. \\ P'_1 = d \\ P'_2 = e \\ P'_3 = f \end{cases} \tag{1.4}$$

where $d, e, f \in \mathbb{Z}$

1.4 Generating function [21]

Theorem 1.2 *The tribonacci numbers* $(P_n)_{n\in\mathbb{N}}$ *can be expressed using Binet's formula as follows:*

$$P_n = \frac{\kappa^{n+1}}{(\kappa-\vartheta)(\kappa-\eta)} + \frac{\vartheta^{n+1}}{(\vartheta-\kappa)(\vartheta-\eta)} + \frac{\eta^{n+1}}{(\eta-\kappa)(\eta-\vartheta)} \tag{1.5}$$

where κ, ϑ *and* η *are the distinct the roots of the equation* $t^3 - t^2 - t - 1 = 0$

1.5 Tribonacci polynomials [17]

Definition 1.4 *The tribonacci polynomials, denoted by $(p_n(\rho))_{n\in\mathbb{N}}$, are a sequence of polynomials defined by the following recurrence relation:*

$$\begin{cases} p_n(\rho) = \rho^2 p_{n-1}(\rho) + \rho p_{n-2}(\rho) + p_{n-3}(\rho), \\ p_0(\rho) = 0 \\ p_1(\rho) = 1 \\ p_2(\rho) = \rho^2 \end{cases} \tag{1.6}$$

Notice that $p_n(1) = P_n$, the nth tribonacci number.

The first tribonacci polynomials are :

$p_1(\rho) = 1$

$p_2(\rho) = \rho^2$

$p_3(\rho) = \rho^4 + \rho$

$p_4(\rho) = \rho^6 + 2\rho^3 + 1$

$p_5(\rho) = \rho^8 + 3\rho^5 + 3\rho^2$

$p_6(\rho) = \rho^{10} + 4\rho^7 + 6\rho^4 + 2\rho$

1.6 Tribonacci-Lucas polynomials [17]

Definition 1.5 *The sequence of polynomials, represented by $(L_n(r))_{n\in\mathbb{N}}$, is known as the tribonacci polynomials and is defined through the following recurrence relation:*

$$\begin{cases} L_{n+3}(\rho) = \rho^2 L_{n+2}(\rho) + \rho L_{n+1}(\rho) + L_n(\rho) \\ L_0(\rho) = 3 \\ L_1(\rho) = \rho^2 \\ L_2(\rho) = \rho^4 + 2\rho. \end{cases}$$

The first tribonacci-Lucas polynomials are

$L_0(\rho) = 3$

$L_1(\rho) = \rho^2$

$L_2(\rho) = \rho^4 + 2\rho$
$L_3(\rho) = \rho^2 + 2\rho + 1$
$L_4(\rho) = \rho^6 + 3\rho^3 + 3$
$L_5(\rho) = \rho^8 + 4\rho^5 + 6\rho^2$

1.6.1 Tricobsthal polynomials [30]

Definition 1.6 *The tricobsthal polynomials are defined by:*

$$\begin{cases} \mathfrak{J}_n(\rho) = \mathfrak{J}_{n-1}(\rho) + \rho\mathfrak{J}_{n-2}(\rho) + \rho^2\mathfrak{J}_{n-3}(\rho), for\ n \geqslant 4 \\ \mathfrak{J}_1(\rho) = \mathfrak{J}_2(\rho) = 1 \\ \mathfrak{J}_3(\rho) = \rho + 1 \end{cases} \tag{1.7}$$

The initial conditions for the tribonacci polynomials and the tribonacci numbers satisfy the property $\mathfrak{J}_n(1) = p_n(1) = P_n$, where P_n represents the n-th tribonacci number. So The generalized tricobsthal polynomials can be defined as follows

Definition 1.7 *The generalized tricobsthal polynomials are defined by :*

$$\begin{cases} \mathfrak{J}'_n(\rho) = \mathfrak{J}'_{n-1}(\rho) + \rho\mathfrak{J}'_{n-2}(\rho) + \rho^2\mathfrak{J}'_{n-3}(\rho), for\ n \geqslant 4 \\ \mathfrak{J}'_1(\rho) = d \\ \mathfrak{J}'_2(\rho) = e \\ \mathfrak{J}'_3(\rho) = c_1\rho + c_0 \end{cases} \tag{1.8}$$

where parameters $c_1 \in \mathbb{Z}$ and $d, e, c_0 \in \mathbb{Z}_+$.

Theorem 1.3 *[30] The nth tricobsthal polynomials is given by*

$$\mathfrak{J}_n(\rho) = \frac{(\iota'_1)^{n+1}}{(\iota'_1 - \iota_2)(\iota'_1 - \iota'_3)} + \frac{(\iota'_2)^{n+1}}{(\iota'_2 - \iota'_1)(\iota'_2 - \iota'_3)} + \frac{(\iota'_3)^{n+1}}{(\iota'_3 - \iota'_1)(\iota'_3 - \iota'_2)} \tag{1.9}$$

where ι'_1, ι'_2 and ι'_3 are the distinct roots of the characteristic polynomials of (1.7)

$$t^3 - t^2 - \rho t - \rho^2 = 0 \tag{1.10}$$

Theorem 1.4 *[30] The Binet formula for generalized tricobsthal polynomials defined by* (1.8) *is*

$$\mathfrak{J}'_n(\rho) = A_1\iota'^{n-1}_1 + A_2\iota'^{n-1}_2 + A_3\iota'^{n-1}_3 \tag{1.11}$$

where n is positive integer, $r \neq 0$ and

$$A_1 = \frac{\mathfrak{J}_3'(\rho) - (\iota'_2 + \iota'_3)\mathfrak{J}_2'(\rho) + \iota'_2\iota'_3\mathfrak{J}_1'(\rho)}{(\iota'_1 - \iota'_2)(\iota'_1 - \iota'_3)}$$
$$A_2 = \frac{\mathfrak{J}_3'(\rho) - (\iota'_1 + \iota'_3)\mathfrak{J}_2'(\rho) + \iota'_1\iota'_3\mathfrak{J}_1'(\rho)}{(\iota'_2 - \iota'_1)(\iota'_2 - \iota'_3)}$$
$$A_3 = \frac{\mathfrak{J}_3'(\rho) - (\iota'_1 + \iota'_2)\mathfrak{J}_2'(\rho) + \iota'_1\iota'_2\mathfrak{J}_1'(\rho)}{(\iota'_3 - \iota'_1)(\iota'_3 - \iota'_2)}$$

where $\iota'_1, \iota'_2, \iota'_3$ are different solutions of characteristic polynomials $t^3 - t^2 - \rho t - \rho^2 = 0$ of (1.8)

When $d = e = c_1 = c_0 = 1$, then one gets the generating tricobsthal polynomials

1.7 Generating functions of the tribonacci polynomials [30]

Theorem 1.5 *The nth tribonacci Polynomials is given by*

$$p_n(\rho) = \frac{\iota_1^{n+1}}{(\iota_1 - \iota_2)(\iota_1 - \iota_3)} + \frac{\iota_2^{n+1}}{(\iota_2 - \iota_1)(\iota_2 - \iota_3)} + \frac{\iota_3^{n+1}}{(\iota_3 - \iota_1)(\iota_3 - \iota_2)} \tag{1.12}$$

where ι_1, ι_2 and ι_3 are the distinct roots of the characteristic polynomial of (1.6)

$$t^3 - \rho^2 t^2 - \rho t - 1 = 0 \tag{1.13}$$

1.8 Generalized tribonacci like polynomials and Generalized tribonacci-Lucas like polynomials

Definition 1.8 *the generalized tribonacci-like polynomials are defined by :*

$$\begin{cases} P_n(\rho, \varrho, \varpi) = \rho P_{n-1}(\rho, \varrho, \varpi) + \varrho P_{n-2}(\rho, \varrho, \varpi) + \varpi P_{n-3}(\rho, \varrho, \varpi), n \geq 3 \\ P_0(\rho, \varrho, \varpi) = 0 \\ P_1(\rho, \varrho, \varpi) = 1 \\ P_2(\rho, \varrho, \varpi) = \rho. \end{cases} \tag{1.14}$$

Definition 1.9 *The generalized tribonacci-like polynomials are defined by :*

$$\begin{cases} L_n(\rho,\varrho,\varpi) = \rho L_{n-1}(\rho,\varrho,\varpi) + \varrho L_{n-2}(\rho,\varrho,\varpi) + \varpi L_{n-3}(\rho,\varrho,\varpi), n \geq 3 \\ L_0(\rho,\varrho,\varpi) = 3 \\ L_1(\rho,\varrho,\varpi) = \rho \\ L_2(\rho,\varrho,\varpi) = \rho^2 + 2\varrho. \end{cases} \tag{1.15}$$

The first term of $P_n(\rho,\varrho,\varpi)$ and $L_n(\rho,\varrho,\varpi)$ are as follows

n	$P_n(\rho,\varrho,\varpi)$	$L_n(\rho,\varrho,\varpi)$
0	0	3
1	1	ρ
2	ρ	$\rho^2 + 2\varrho$
3	$\rho^2 + \varrho$	$\rho^3 + 3\rho\varrho + 3\varpi$
4	$\rho^3 + 2\rho\varrho + \varpi$	$\rho^4 + 4\rho^2\varrho + 4\rho\varpi + 2\varrho^2$
5	$\rho^4 + 3\rho^2\varrho + 2\rho\varpi + \varrho^2$	$\rho^5 + 5\rho^3\varrho + 5\rho\varrho^2 + 5\rho^2\varpi + 5\varrho\varpi$
6	$\rho^5 + 4\rho^3\varrho + 3\rho\varrho^2 + 3\rho^2\varpi + 2\varrho\varpi$	$\rho^6 + 6\rho^4\varrho + 9\rho^2\varrho^2 + 6\rho^3\varpi + 12\rho\varrho\varpi + 2\varrho^3 + 3\varpi^2$

Generating function

The generating function of $P_n(x,y,z)$ are

$$h(t) = \sum_{n=0}^{\infty} P_n(\rho,\varrho,\varpi)t^n = \frac{t}{1 - \rho t - \varrho t^2 - \varpi t^3} \tag{1.16}$$

and

$$k(t) = \sum_{n=0}^{\infty} L_n(\rho,\varrho,\varpi)t^n = \frac{3 - 2\rho t - \varrho t^2}{1 - \rho t - \varrho t^2 - \varpi t^3} \tag{1.17}$$

The characteristic equation of the recurrences 1.14 and 1.15 is

$$t^3 - \rho t^2 - \varrho t - \varpi = 0 \tag{1.18}$$

1.9 Binet's formula

Theorem 1.6 *For $n \geq 1$ the Binet's formula for the generalized tribonacci like polynomials*

$$P_n(\rho, \varrho, \varpi) = \frac{\kappa^{n+1}}{(\kappa - \vartheta)(\kappa - \eta)} + \frac{\vartheta^{n+1}}{(\vartheta - \kappa)(\vartheta - \eta)} + \frac{\eta^{n+1}}{(\eta - \kappa)(\eta - \vartheta)} \tag{1.19}$$

and

$$L_n(\rho, \varrho, \varpi) = \kappa^n + \vartheta^n + \eta^n \tag{1.20}$$

where κ and ϑ, η root of the equation

$$t^3 - \rho t^2 - \varrho t - \varpi = 0$$

Let Q be the matrix define by

$$Q = \begin{pmatrix} \rho & \varrho & \varpi \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

hence $det(Q) = \varpi$, and

$$Q^n = \begin{pmatrix} P_{n+1}(\rho, \varrho, \varpi) & \varrho P_n(\rho, \varrho, \varpi) + \varpi P_{n-1}(\rho, \varrho, \varpi) & \varpi P_n(\rho, \varrho, \varpi) \\ P_n(\rho, \varrho, \varpi) & \varrho P_{n-1}(\rho, \varrho, \varpi) + \varpi P_{n-2}(\rho, \varrho, \varpi) & \varpi P_{n-1}(\rho, \varrho, \varpi) \\ P_{n-1}(\rho, \varrho, \varpi) & \varrho P_{n-2}(\rho, \varrho, \varpi) + \varpi P_{n-3}(\rho, \varrho, \varpi) & \varpi P_{n-2}(\rho, \varrho, \varpi) \end{pmatrix}$$

The **hypergeometric function** They can be describe for $|s| < 1$ by the hypergeometric series:

$${}_2F_1(d, e; f; s) = \sum_{n=0}^{\infty} \frac{(d)_n (e)_n}{(f)_n} \frac{s^n}{n!}$$

The notation $(d)_n$ given by

$$(d)_n = \begin{cases} 1 & \text{si } n = 0 \\ d(d+1)\cdots(d+n-1) & \text{si } n > 0 \end{cases}$$

.

CHAPTER 2

. Generalized tribonacci sequence and Polynomials

The tribonacci polynomials are the natural extensions of tribonacci sequence, these polynomials are studied in one variable defined by recurrence relation
$p_n(\rho) = \rho^2 p_{n-1}(\rho) + \rho p_{n-2}(\rho) + p_{n-3}(\rho), n \geqslant 3$ where $p_0(\rho) = 0, p_1(\rho) = 1, p_2(\rho) = \rho^2$. with property $p_n(1) = P_n$ is n-th tibonacci number. Another interesting family of such kind polynomials is Jacobsthal polynomials or generalized Jacobsthal polynomials [30].

2.1 Some Properties of tribonacci sequence

Our main focus will be on presenting the tribonacci and tribonacci-Lucas sequence, by giving some important results of the tribonacci sequence.

Definition 2.1 *Let* $\{H_n\}_{n\geq 0}$ *be the sequence defined by*

$$\begin{cases} H_{n+2} = H_{n+1} + H_n + H_{n-1}, n \geqslant 3 \\ H_0 = 3 \\ H_1 = 0 \\ H_2 = 2. \end{cases} \tag{2.1}$$

Theorem 2.1 *For* $n \geq 0$*, we have*
(a) $H_n = L_n - P_n$
(b) $H_n = 3P_{n+1} - 3P_n - P_{n-1}$
(c) $9H_{n+2} - 2H_{n+1} + 35H_n = 41L_n$

Proof *Parts (a) and (b) follows easily by induction on n*

(c) To prove this we will use (a) :

$$
\begin{aligned}
41L_n &= 41\left(H_n + P_n\right) \\
&= \left(35H_n + 6H_n\right) + 41L_n \\
&= 35H_n + 6\left(3P_{n+1} - 3P_n - P_{n-1}\right) + 41P_n \\
&= 35H_n + 18P_{n+1} + 23P_n - 6P_{n-1} \\
&= 35H_n + 18P_{n+1} + 23P_n - 6\left(P_{n+2} - P_{n+1} - P_n\right) \\
&= 35H_n - 6P_{n+2} + 24P_{n+1} + 29P_n \\
&= 35H_n - 6P_{n+2} + 24P_{n+1} + 27\left(P_{n+3} - P_{n+2} - P_{n+1}\right) + 2P_n \\
&= 35H_n + 27P_{n+3} - 33P_{n+2} - 3P_{n+1} + 2P_n \\
&= 35H_n - 2\left(3P_{n+2} - 3P_{n+1} - P_n\right) + 9\left(3P_{n+3} - 3P_{n+2} - P_{n+1}\right) \\
&= 35H_n - 2H_{n+1} + 9H_{n+2}
\end{aligned}
$$

as required. ■

Corollary 2.1 *[34] for n ≥ 2 we have*

(a) $L_n = P_n + 2P_{n-1} + 3P_{n-2}$

(b) $\sum_{i=0}^{n} P_i = \frac{1}{2}\left(P_{n+2} + P_n - 1\right)$

(c) $\sum_{i=0}^{n} P_{2i} = \frac{1}{2}\left(P_{2n+1} + P_{2n} - 1\right)$

(d) $\sum_{i=0}^{n} P_{2i+1} = \frac{1}{2}\left(P_{2n+2} + P_{2n+1}\right)$

Proof *(a) By induction on n*

For (b), using the equation (1.1), we can get the following relations:

$$
\begin{aligned}
P_0 &= P_3 - P_2 - P_1 \\
P_1 &= P_4 - P_3 - P_2 \\
P_2 &= P_5 - P_4 - P_3 \\
&\cdot \\
&\vdots \\
P_{n-2} &= P_{n+1} - P_n - P_{n-1} \\
P_{n-1} &= P_{n+2} - P_{n+1} - P_n \\
P_n &= P_{n+3} - P_{n+2} - P_{n+1}
\end{aligned}
$$

If we add the equations side by side. we get

$$\sum_{i=0}^{n} P_i = P_{n+3} - P_2 - \sum_{i=0}^{n} P_{i+1} = P_{n+3} - P_2 - \sum_{i=1}^{n} P_i - P_{n+1} - P_0 + P_0$$

$$= P_{n+3} - P_{n+1} - P_2 + P_0 - \sum_{i=0}^{n} P_i = P_{n+2} + P_n - P_2 + P_0 - \sum_{i=1}^{n} P_i$$

and so

$$\sum_{i=0}^{n} P_i = \frac{1}{2}\left(P_{n+2} + P_n - 1\right)$$

(iii) the statement (c) and (d) follows similarly ■

Lemma 1 *Some identities for the tribonacci and tribonacci-Lucas numbers.*

$$1) L_{-m}P_{-m+s} - P_{-2m+s} = L_m P_s - P_{m+s}$$

$$2) L_{-m}L_{-m+s} - L_{-2m+s} = L_m L_s - L_{m+s}$$

Proof *To prove (1) et (2) using Binet's formulas for the tribonacci and tribonacci-Lucas numbers* ■

Proposition 2.2 *[34] For $n \geq 0$, we have*

$$\sum_{i=0}^{n} P_{2i+1} = \frac{1}{2}\left(P_{2n+2} + P_{2n+1}\right)$$

$$\sum_{i=0}^{n} L_{2i+1} = \frac{1}{2}\left(L_{2n+2} + L_{2n+1} - 2\right)$$

$$\sum_{i=0}^{n} P_{3i+1} = \frac{1}{2}\left(P_{3n+4} - 4P_{3n+1} + P_{3n-2} + 1\right)$$

$$\sum_{i=0}^{n} P_{2i+1} = \frac{1}{2}\left(P_{3n+5} - 4P_{3n+2} + P_{3n-1} - 1\right)$$

$$\sum_{i=0}^{n} L_{2i+1} = \frac{1}{2}\left(L_{3n+4} - 4L_{3n+1} + L_{3n-2} - 4\right)$$

$$\sum_{i=0}^{n} L_{2i+1} = \frac{1}{2}\left(L_{3n+5} - 4L_{3n+2} + L_{3n-1} - 2\right)$$

Theorem 2.2 *Let P_n be the nth tribonacci number , then for $n \geq 0, k \geq 2$ we have*

$$P_{n+k} = P_{k-1}P_n + \left(P_{k-2} + P_{k-1}\right)P_{n+1} + P_k P_{n+2}.$$

2.2 Generating functions of the generalized tribonacci sequence

Theorem 2.3 *Let* $(P'_n)_{N\in\mathbb{N}}$ *be the nth term of the generalized tribonacci sequence (GTS), then*

$$P'_n = \frac{\kappa^{n-3}((d+e)\vartheta+e+f\kappa^2)}{(\kappa-\vartheta)(\kappa-\eta)} + \frac{\vartheta^{n-3}((d+e)\vartheta+e+f\vartheta^2)}{(\vartheta-\kappa)(\vartheta-\eta)} + \frac{\eta^{n-3}((d+e)\eta+e+f\eta^2)}{(\eta-\kappa)(\eta-\vartheta)}$$

Proof *By combining the Theorem 1.1 and equation*(1.5)*, we get*

$$\begin{aligned} P'_n &= fP_{n-2} + e[P_{n-3}+P_{n-4}(x)] + dP_{n-3} \\ &= \frac{f\kappa^{n-1}+e[\kappa^{n-2}+\kappa^{n-3}]+d\kappa^{n-2}}{(\kappa-\vartheta)(\kappa-\eta)} + \frac{f\vartheta^{n-1}+e[\vartheta^{n-2}+\vartheta^{n-3}]+d\vartheta^{n-2}}{(\vartheta-\kappa)(\vartheta-\eta)} + \\ &\quad + \frac{f\eta^{n-1}+e[\eta^{n-2}+\eta^{n-3}]+d\eta^{n-2}}{(\eta-\kappa)(\eta-\vartheta)} \end{aligned}$$

Finally

$$P'_n = \frac{\kappa^{n-3}((d+e)\kappa+e+f\kappa^2)}{(\kappa-\vartheta)(\kappa-\eta)} + \frac{\vartheta^{n-3}((d+e)\vartheta+e+f\vartheta^2)}{(\vartheta-\kappa)(\vartheta-\eta)} + \frac{\eta^{n-3}((d+e)\eta+e+f\eta^2)}{(\eta-\kappa)(\eta-\vartheta)}$$

■

2.3 Identities on the tribonacci sums [11]

Let S_n *be the sums of tribonacci sequence define by*

$$S_n = \sum_{k=0}^{n} P_k$$

By adding the initial conditions to the recurrence (1.1) *and combining the first n recurrence relations, we can obtain a result.*

$$S_n = 1 + S_{n-1} + S_{n-2} + S_{n-3}$$

where $S_0 = S_1 = 0,\ S_2 = 1,$

Theorem 2.4 *For $n \geq 4$ and $k \geq 4$ we have*

(a) $S_n = 2S_{n-1} - S_{n-4}$

(b) $S_{n+k} = -S_{k-2}S_n - S_{k-3}S_{n+1} - S_{k-4}S_{n+2} + S_{k-1}S_{n+3}$

Proof *(a) we have*

$$S_n = 1 + S_{n-1} + S_{n-2} + S_{n-3}$$

and

$$S_{n-1} = 1 + S_{n-2} + S_{n-3} + S_{n-4}$$

So

$$S_n + S_{n-4} = 1 + S_{n-1} + S_{n-2} + S_{n-3} + S_{n-4}$$

Then

$$S_n = 2S_{n-1} - S_{n-4}$$

(b) By induction on k ■

2.4 Tribonacci polynomials

Theorem 2.5 *For $n \geq 2$, we have*

$$L_n(\rho) = \rho^2 P_n(\rho) + 2\rho P_{n-1}(\rho) + 3P_{n-2}(\rho),$$

Theorem 2.6 *For $n \geq 0$,we have*

$$p_n(\rho) = \frac{1}{\lambda}\left(\iota_1^{n+1}(\iota_2 - \iota_3) + \iota_2^{n+1}(\iota_3 - \iota_1) + \iota_3^{n+1}(\iota_1 - \iota_2)\right)$$

where $\lambda = (\iota_1 - \iota_2)(\iota_1 - \iota_3)(\iota_2 - \iota_3)$ and ι_1, ι_2 , ι_3 are the distinct roots of the characteristic polynomial of (1.6)

$$t^3 - \rho^2 t^2 - \rho t - 1 = 0$$

Proof *Observe that*

$$0 = |Q - tI_3| = \left| \begin{pmatrix} \rho^2 - t & \rho & 1 \\ 1 & -t & 0 \\ 0 & 1 & -t \end{pmatrix} \right| = -t^3 + \rho^2 t^2 + \rho t + 1$$

and ι_1, ι_2 *and* ι_3 *be the eigenvalues of* Q
Let A *define by*

$$A = \begin{pmatrix} \iota_1^2 & \iota_2^2 & \iota_3^2 \\ \iota_1 & \iota_2 & \iota_3 \\ 1 & 1 & 1 \end{pmatrix}$$

then

$$A^{-1} = \frac{1}{\lambda}\begin{pmatrix} \iota_2 - \iota_3 & \iota_3^2 - \iota_2^2 & \iota_2\iota_3(\iota_2 - \iota_3) \\ \iota_3 - \iota_1 & \iota_1^2 - \iota_3^2 & \iota_1\iota_3(\iota_3 - \iota_1) \\ \iota_1 - \iota_2 & \iota_2^2 - \iota_1^2 & \iota_1\iota_2(\iota_1 - \iota_2) \end{pmatrix}$$

where $\lambda = (\iota_1 - \iota_2)(\iota_1 - \iota_3)(\iota_2 - \iota_3)$ *Now, let*

$$B = \begin{pmatrix} \iota_1 & 0 & 0 \\ 0 & \iota_2 & 0 \\ 0 & 0 & \iota_3 \end{pmatrix}$$

Then $Q = ADA^{-1}$, *so we obtain*

$$\begin{aligned} Q^n &= (ABA^{-1})^n \\ &= AB^nA^{-1} \\ &= \frac{1}{\lambda}\begin{pmatrix} \iota_1^2 & \iota_2^2 & \iota_3^2 \\ \iota_1 & \iota_2 & \iota_3 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} \iota_1 & 0 & 0 \\ 0 & \iota_2 & 0 \\ 0 & 0 & \iota_3 \end{pmatrix}\begin{pmatrix} \iota_2 - \iota_3 & \iota_3^2 - \iota_2^2 & \iota_2\iota_3(\iota_2 - \iota_3) \\ \iota_3 - \iota_1 & \iota_1^2 - \iota_3^2 & \iota_1\iota_3(\iota_3 - \iota_1) \\ \iota_1 - \iota_2 & \iota_2^2 - \iota_1^2 & \iota_1\iota_2(\iota_1 - \iota_2) \end{pmatrix} \end{aligned}$$

Finally

$$p_n(\rho) = \frac{1}{\lambda}\left(\iota_1^{n+1}(\iota_2 - \iota_3) + \iota_2^{n+1}(\iota_3 - \iota_1) + \iota_3^{n+1}(\iota_1 - \iota_2)\right)$$

■

Remark 2.7 *For* $n \geq 3$ *we can extend the sequence* $(p_n(x))_{n\geq 0}$ *to the negative integers by*

$$\begin{cases} p_{-n}(\rho) = p_{-(n-3)}(\rho) - \rho^2 p_{-(n-2)}(\rho) - \rho p_{-(n-1)}(\rho) \\ p_0(\rho) = p_{-1}(\rho) = 0 \\ p_{-2}(\rho) = 1 \end{cases} \tag{2.2}$$

Proposition 2.3 *For $n \geq 3$, we have*

$$1)\quad p_{-n}(\rho) = (1+\rho^3)p_{-n+3}(\rho) - \rho p_{-n+4}(\rho)$$

$$2)\quad p_{-n}(\rho) = [p_{n-1}(\rho)]^2 - p_n(\rho)p_{n-2}(\rho)$$

Proof 1) *Using the* (2.2), *we have*

$$\begin{aligned} p_{-n}(\rho) &= p_{-(n-3)}(\rho) - \rho^2 p_{-(n-2)}(\rho) - \rho p_{-(n-1)}(\rho) \\ &= p_{-(n-3)}(\rho) - \rho^2 p_{-(n-2)}(\rho) - \rho[p_{-(n-4)}(\rho) - \rho^2 p_{-(n-3)}(\rho) - \rho p_{-(n-2)}(\rho)] \\ &= (1+\rho^3)p_{-n+3}(\rho) - \rho p_{-n+4}(\rho) \end{aligned}$$

2)

We have [17]

$$P = \begin{bmatrix} \rho^2 & \rho & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Indicating that $det(P) = 1$, and

$$Q^n = \begin{pmatrix} p_{n+1}(\rho) & \rho p_n(\rho) + p_{n-1}(\rho) & p_n(\rho) \\ p_n(\rho) & \rho p_{n-1}(\rho) + p_{n-2}(\rho) & p_{n-1}(\rho) \\ p_{n-1}(\rho) & \rho p_{n-2}(\rho) + p_{n-3}(\rho) & p_{n-2}(\rho) \end{pmatrix}$$

since $Q^{-n} = (Q^n)^{-1}$,it follows that

$$Q^{-n} = \begin{pmatrix} p_{-n+1}(\rho) & \rho p_{-n}(\rho) + p_{-n-1}(\rho) & p_{-n}(\rho) \\ p_{-n}(\rho) & \rho p_{-n-1}(\rho) + p_{-n-2}(\rho) & p_{-n-1}(\rho) \\ p_{-n-1}(\rho) & \rho p_{-n-2}(\rho) + p_{-n-3}(\rho) & p_{-n-2}(\rho) \end{pmatrix}$$

and

$$(Q^n)^{-1} = \begin{pmatrix} [p_{n-2}(\rho)]^2 - p_{n-1}(\rho)p_{n-3}(\rho) & p_n(\rho)p_{n-3}(\rho) - p_{n-1}(\rho)p_{n-2}(\rho) & [p_{n-1}(\rho)]^2 - p_n(\rho)p_{n-2}(\rho) \\ [p_{n-1}(\rho)]^2 - p_n(\rho)p_{n-2}(\rho) & p_{n+1}(\rho)p_{n-2}(\rho) - p_n(\rho)p_{n-1}(\rho) & [p_n(\rho)]^2 - p_{n+1}(\rho)p_{n-1}(\rho) \\ [p_n(\rho]^2 - p_{n+1}(\rho)p_{n-1}(\rho) & p_{n+2}(\rho)p_{n-1}(\rho) - p_{n+1}(\rho)p_n(\rho) & [p_{n+1}(\rho)]^2 - p_{n+2}(\rho)p_n(\rho) \end{pmatrix}$$

Then

$$p_{-n}(\rho) = [p_{n-1}(\rho)]^2 - p_n(\rho)p_{n-2}(\rho)$$

■

2.5 Properties

Definition 2.2 *The generalized tribonacci polynomials $p'_n(\rho)$ are defined by recurrence linear*

$$\begin{cases} p'_n(\rho) = \rho^2 p'_{n-1}(\rho) + \rho p'_{n-2}(\rho) + p'_{n-3}(\rho), n \geqslant 4 \\ p'_0(\rho) = d \\ p'_1(\rho) = e \\ p'_2(\rho) = fr^2. \end{cases} \tag{2.3}$$

Notice that $p'_n(1) = P'_n$.

The fisrt Ten generalized tribonacci polinomilas are:

$p'_0(\rho) = d$

$p'_1(\rho) = e$

$p'_2(\rho) = f\rho^2$

$p'_3(\rho) = f\rho^4 + e\rho + d$

$p'_4(\rho) = f\rho^6 + (e+f)\rho^3 + d\rho^2 + e$

$p'_5(\rho) = f\rho^8 + (e+2f)\rho^5 + d\rho^4 + (2e+f)\rho^2 + d\rho$

$p'_6(\rho) = f\rho^{10} + (e+3f)\rho^7 + d\rho^6 + (3e+3f)\rho^4 + 2d\rho^3 + 2e\rho + d$

$p'_7(\rho) = f\rho^{12} + (e+4f)\rho^9 + d\rho^8 + (4e+6f)\rho^6 + 3d\rho^5 + (5e+2f)\rho^3 + 3d\rho^2 + e$

$p'_8(\rho) = f\rho^{14} + (e+5f)\rho^{11} + d\rho^{10} + (5e+10f)\rho^8 + 4d\rho^7 + (9e+7f)\rho^5 + 6d\rho^4 + (5e+f)\rho^2 + 2d\rho$

$p'_9(r) = fr^{16} + (e+6f)r^{13} + dr^{12} + (6e+15f)r^{10} + 5dr^9 + (14e+16f)r^7 + 10dr^6 +$
$(13e+6f)r^4 + 7dr^3 + 3er + d$

Theorem 2.8 *For $n > 4$, we have*

$$p'_n(\rho) = fp_{n-2}(\rho) + e[\rho p_{n-3}(\rho) + p_{n-4}(\rho)] + dp_{n-3}(\rho) \tag{2.4}$$

Proof *Let Q be the matrix of ordre 3 defined by :*

$$Q = \begin{bmatrix} \rho^2 & \rho & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

such that $detQ = 1$,

So

$$\begin{pmatrix} p'_4(\rho) \\ p'_3(\rho) \\ p'_2(\rho) \end{pmatrix} = \begin{pmatrix} \rho^2 & \rho & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} p'_2(\rho) \\ p'_1(\rho) \\ p'_0(\rho) \end{pmatrix}.$$

Then

$$\begin{pmatrix} p'_{n+3}(\rho) \\ p'_{n+2}(\rho) \\ p'_{n+1}(\rho) \end{pmatrix} = \begin{pmatrix} \rho^2 & \rho & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} p'_2(\rho) \\ p'_1(\rho) \\ p'_0(\rho) \end{pmatrix}$$

It is well know that (see [17])

$$\begin{pmatrix} \rho^2 & 1 & 0 \\ \rho & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^n = \begin{pmatrix} p_{n+1}(\rho) & p_n(\rho) & p_{n-1}(\rho) \\ \rho p_n(\rho) + p_{n-1}(\rho) & \rho p_{n-1}(\rho) + p_{n-2}(\rho) & \rho p_{n-2}(\rho) + p_{n-3}(\rho) \\ p_n(r) & p_{n-1}(r) & p_{n-2}(r) \end{pmatrix}$$

which implies

$$\begin{pmatrix} p'_{n+3}(\rho) \\ p'_{n+2}(\rho) \\ p'_{n+1}(\rho) \end{pmatrix} = \begin{pmatrix} p_{n+1}(\rho) & \rho p_n(\rho) + p_{n-1}(\rho) & p_n(\rho) \\ P_n(\rho) & \rho p_{n-1}(\rho) + p_{n-2}(\rho) & p_{n-1}(\rho) \\ p_{n-1}(\rho) & \rho p_{n-2}(\rho) + p_{n-3}(\rho) & p_{n-2}(\rho) \end{pmatrix} \begin{pmatrix} p'_2(\rho) \\ p'_1(\rho) \\ p'_0(\rho) \end{pmatrix}.$$

Then

$$p'_{n+1}(\rho) = p'_2(\rho)p_{n-1}(\rho) + p'_1(\rho)[\rho p_{n-2}(\rho) + p_{n-3}(\rho)] + p'_0(\rho)p_{n-2}(\rho).$$

Finally,

$$p'_n(\rho) = f\rho^2 p_{n-2}(\rho) + e[\rho p_{n-3}(\rho) + p_{n-4}(\rho)] + dp_{n-3}(\rho).$$

■

Remark 2.9 *If $x = 1$, we have*

$$P'_n = fP_{n-2} + e[P_{n-3} + P_{n-4}] + dP_{n-3}$$

Proposition 2.4 *For every integer $n \geq 6$, we have*

$$p'_n(\rho) = p'_1(\rho) + p'_2(\rho) + p'_3(\rho) + (\rho^2 + \rho)\sum_{k=3}^{n-3} p'_k(\rho) + (\rho^2 - 1)(p'_{n-1}(\rho) + p'_{n-2}(\rho)) + \rho p'_{n-2}(\rho)$$

Proof *From the definition of* $(p_n(\rho))_{n\geq 1}$*, we have successively*

$$\begin{aligned} p'_n(\rho) &= \rho^2 p'_{n-1}(\rho) + \rho p'_{n-2}(\rho) + p'_{n-3}(\rho) \\ p'_{n-1}(\rho) &= \rho^2 p'_{n-2}(\rho) + \rho p'_{n-3}(\rho) + p'_{n-4}(\rho) \\ p'_{n-2}(\rho) &= \rho^2 p'_{n-3}(\rho) + \rho p'_{n-4}(\rho) + p'_{n-5}(\rho) \\ p'_{n-3}(\rho) &= \rho^2 p'_{n-4}(\rho) + \rho p'_{n-5}(\rho) + p'_{n-6}(\rho) \\ &\vdots \qquad \vdots \qquad \vdots \\ p'_4(\rho) &= \rho^2 p'_3(\rho) + \rho p'_2(\rho) + p'_1(\rho) \\ p'_3(\rho) &= \rho^2 p'_2(\rho) + \rho p'_1(\rho) + p'_0(\rho). \end{aligned}$$

summing and simplifying the right hand side, we obtain

$$p'_n(\rho) = p'_1(\rho)+p'_2(\rho)+p'_3(\rho)+(\rho^2+\rho)\sum_{k=3}^{n-3} p'_k(\rho)+(\rho^2-1)(p'_{n-1}(\rho)+p'_{n-2}(\rho))+\rho(p'_2(\rho)+p'_{n-2}(\rho))$$

■

2.6 Generating functions

Theorem 2.10 *The nth generalized tribonacci polynomials satisfies*

$$p'_n(\rho) = \frac{\kappa^{n-3}((d+e\rho)\kappa + e + f\kappa^2)}{(\kappa-\vartheta)(\kappa-\eta)} + \frac{\vartheta^{n-3}((d+e\rho)\vartheta + e + f\vartheta^2)}{(\vartheta-\kappa)(\vartheta-\iota_3)} + \frac{\eta^{n-3}((a+b\rho)\eta + e + f\eta^2)}{(\eta-\kappa)(\eta-\vartheta)}$$

where κ*,* ϑ *and* η *are the distinct roots of the equation* (1.13)

Proof *By combining (2.4) with the Eq* (1.12)*, we get*

$$\begin{aligned} p'_n(\rho) &= fp_{n-2}(\rho) + e[\rho p_{n-3}(\rho) + p_{n-4}(\rho)] + dp_{n-3}(\rho) \\ &= \frac{f\kappa^{n-1} + e[\rho\kappa^{n-2} + \kappa^{n-3}] + d\kappa^{n-2}}{(\kappa-\vartheta)(\kappa-\eta)} + \frac{f\vartheta^{n-1} + e[\rho\vartheta^{n-2} + \vartheta^{n-3}] + d\vartheta^{n-2}}{(\vartheta-\kappa)(\vartheta-\eta)} + \\ &\quad + \frac{f\eta^{n-1} + e[\rho\eta^{n-2} + \eta^{n-3}] + d\eta^{n-2}}{(\eta-\iota_1)(\eta-\vartheta)} \end{aligned}$$

Thus,

$$p'_n(\rho) = \frac{\kappa^{n-3}((d+ex)\kappa + e + f\kappa^2)}{(\kappa-\vartheta)(\kappa-\eta)} + \frac{\vartheta^{n-3}((d+e\rho)\vartheta + e + f\vartheta^2)}{(\vartheta-\kappa)(\vartheta-\eta)} + \frac{\eta^{n-3}((d+e\rho)\eta + e + f\eta^2)}{(\eta-\kappa)(\eta-\vartheta)}$$

■

Proposition 2.5 *For* $n \geqslant 4$, *we have*

$$\begin{aligned}\kappa^n &= p_{n-1}(\rho)\kappa^2 + [\rho p_{n-2}(\rho) + p_{n-3}(\rho)]\kappa + p_{n-2}(\rho)\\ \vartheta^n &= p_{n-1}(\rho)\vartheta^2 + [\rho p_{n-2}(\rho) + p_{n-3}(\rho)]\vartheta + p_{n-2}(\rho)\\ \eta^n &= p_{n-1}(\rho)\eta^2 + [\rho p_{n-2}(\rho) + p_{n-3}(\rho)]\eta + p_{n-2}(\rho)\end{aligned} \tag{2.5}$$

Proof *We proceed by Induction on* n, *the result it is clear for* $\mathrm{n} = 4$ *and* $\mathrm{n} = 5$ *by hypothesis. Assume that it is true for* i *such that* $0 \leq i \leq \mathbf{k}$, *then*

$$\kappa^k = p_{k-1}(\rho)\kappa^2 + [\rho p_{k-2}(\rho) + p_{k-3}(\rho)]\kappa + p_{k-2}(r)$$

So

$$\begin{aligned}\kappa^{k+1} &= \kappa\left[p_{k-1}(\rho)\kappa^2 + [\rho p_{k-2}(\rho) + p_{k-3}(\rho)]\kappa + p_{r-2}(\rho)\right]\\ &= p_{k-1}(\rho)\kappa^3 + [\rho p_{k-2}(\rho) + p_{k-3}(\rho)]\kappa^2 + p_{k-2}(\rho)\kappa\\ &= p_{k-1}(\rho)(\rho^2\kappa^2 + \rho\kappa + 1) + [\rho p_{k-2}(\rho) + p_{k-3}(\rho)]\kappa^2 + p_{k-2}(\rho)\kappa\\ &= [\rho^2 p_{k-1}(\rho) + \rho p_{k-2}(\rho) + p_{k-3}]\kappa^2 + [\rho p_{k-1}(\rho) + p_{k-2}(\rho)]\kappa + p_{k-1}(\rho).\end{aligned}$$

Finally

$$\kappa^{k+1} = p_k(\rho)\kappa^2 + [r p_{k-1}(\rho) + p_{k-2}(\rho)]\kappa + p_{k-1}(\rho)$$

Thus, the result is true $\forall n \in \mathbb{N}$.
Similarly, we find that :

$$\vartheta^n = p_{n-1}(\rho)\vartheta^2 + [\rho p_{n-2}(\rho) + p_{n-3}(\rho)]\vartheta + p_{n-2}(\rho)$$

and

$$\eta^n = p_{n-1}(\rho)\eta^2 + [\rho p_{n-2}(\rho) + p_{n-3}(\rho)]\eta + p_{n-2}(\rho)$$

■

Proposition 2.6 *For every integer* $n \geq 3$, *we have*

$$(\kappa^n + \vartheta^n + \eta^n) = p_{n+1}(r) + \rho p_{n-1}(\rho) + 2p_{n-2}(\rho) \tag{2.6}$$

Proof *By using Eq.* (1.12) *and the Eq.* (2.6) *and taking in to account that*

$$\begin{cases} 1 = \kappa\vartheta\eta \\ r = -(\vartheta\eta + \vartheta\kappa + \kappa\eta). \end{cases}$$

we get

$$\begin{aligned}
(R) &= \frac{\kappa^{n+2}+\rho\kappa^{n}+2\kappa^{n-1}}{(\kappa-\vartheta)(\kappa-\eta)}+\frac{\vartheta^{n+2}+\rho\vartheta^{n}+2\vartheta^{n-1}}{(\vartheta-\kappa)(\vartheta-\eta)}+\frac{\eta^{n+2}+\rho\eta^{n}+2\eta^{n-1}}{(\eta-\kappa)(\eta-\vartheta)}\\
&= \frac{\kappa^{n}\left(\kappa^{2}+r+\frac{2}{\kappa}\right)}{(\kappa-\vartheta)(\kappa-\eta)}+\frac{\vartheta^{n}\left(\vartheta^{2}+r+\frac{2}{\vartheta}\right)}{(\vartheta-\kappa)(\vartheta-\eta)}+\frac{\eta^{n}\left(\eta^{2}+r+\frac{2}{\eta}\right)}{(\eta-\kappa)(\eta-\vartheta)}\\
&= \frac{\kappa^{n}\left(\vartheta+\rho+2\vartheta\eta\right)}{(\kappa-\vartheta)(\kappa-\eta)}+\frac{\vartheta^{n}\left(\vartheta^{2}+\rho+2\kappa\eta\right)}{(\vartheta-\kappa)(\vartheta-\eta)}+\frac{\eta^{n}\left(\eta^{2}+\rho+2\kappa\vartheta\right)}{(\eta-\kappa)(\eta-\vartheta)}\\
&= \frac{\kappa^{n}\left(\kappa^{2}+\rho+2\vartheta\eta\right)}{(\kappa^{2}-(-\rho-\eta\vartheta)+\vartheta\eta)}+\frac{\vartheta^{n}\left(\vartheta^{2}+\rho+2\kappa\eta\right)}{(\vartheta^{2}-(-r-\eta\kappa)+\eta\kappa)}+\frac{\eta^{n}\left(\eta^{2}+\rho+2\kappa\vartheta\right)}{(\eta^{2}-(-r-\kappa\vartheta)+\kappa\vartheta)}\\
&= \kappa^{n}+\vartheta^{n}+\eta^{n}
\end{aligned}$$

■

Theorem 2.11 *For $n \geq 1$, we have*

$$p_{3n-1}(\rho)=\sum_{i=0}^{n}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}\rho^{i+j}p_{i+j-1}(\rho)$$

Proof *Let ι is a root of the equation (1.13) . Then*

$$\iota^{3}=\rho^{2}\iota^{2}+\rho\iota+1$$

this shows

$$\begin{aligned}
\left(\iota^{3}\right)^{n} &= \left(\iota^{3}-1+1\right)^{n}\\
&= \sum_{i=0}^{n}\binom{n}{i}\left(\iota^{3}-1\right)^{i}\\
&= \sum_{i=0}^{n}\binom{n}{i}\left(\rho^{2}\iota^{2}+\rho\iota\right)^{i}\\
&= \sum_{i=0}^{n}\binom{n}{i}(\rho\iota)^{i}\sum_{j=0}^{i}\binom{i}{j}(\rho\iota)^{j}\\
&= \sum_{i=0}^{n}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}(\rho\iota)^{i+j}\\
&= \sum_{i=0}^{n}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}(r)^{i+j}\iota^{i+j}
\end{aligned}$$

If we replace ι *to* κ, ϑ, η *, and using the equation (1.13). Then we obtain*

$$\begin{aligned} p_{3n-1}(\rho) &= \frac{\kappa^{3n}}{(\kappa-\vartheta)(\kappa-\eta)} + \frac{\vartheta^{3n}}{(\vartheta-\kappa)(\vartheta-\eta)} + \frac{\eta^{3n}}{(\eta-\kappa)(\eta-\vartheta)} \\ &= \sum_{i=0}^{n}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}\rho^{i+j}\left(\frac{\kappa^{i+j}}{(\kappa-\vartheta)(\kappa-\eta)} + \frac{\vartheta^{i+j}}{(\vartheta-\kappa)(\vartheta-\eta)} + \frac{\eta^{i+j}}{(\eta-\kappa)(\eta-\vartheta)}\right) \\ &= \sum_{i=0}^{n}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}(\rho)^{i+j}p_{i+j-1}(\rho) \end{aligned}$$

■

2.7 Generating Matrix of the tricobsthal polynomials

Let Q' *be the matrix defind by*

$$Q' = \begin{bmatrix} 1 & \rho & \rho^2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

We can proved that by induction on n

$$(Q')^n = \begin{pmatrix} 1 & \rho & \rho^2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} \mathfrak{J}_{n+1}(\rho) & \mathfrak{J}_{n+2}(\rho) - \mathfrak{J}_{n+1}(\rho) & r^2\mathfrak{J}_n(\rho) \\ \mathfrak{J}_n(\rho) & \mathfrak{J}_{n+1}(\rho) - \mathfrak{J}_n(\rho) & \rho^2\mathfrak{J}_{n-1}(\rho) \\ \mathfrak{J}_{n-1}(\rho) & \mathfrak{J}_n(\rho) - \mathfrak{J}_{n-1}(\rho) & \rho^2\mathfrak{J}_{n-2}(\rho) \end{pmatrix}$$

Theorem 2.12 *For* $n \geq 4$*, we have*

$$\mathfrak{J}'_n(\rho) = (c_1\rho + c_0)\mathfrak{J}_{n-2}(\rho) + e[\mathfrak{J}_{n-1}(\rho) - \mathfrak{J}_{n-2}(\rho)] + d\rho^2\mathfrak{J}_{n-3}(\rho) \tag{2.7}$$

where $c_1 \geq 0$ *and* $d, e, c_0 \in \mathbb{N}^*$ *.*

Proof *By using the generating matrix for the sequence* (1.8)*. Then*

$$\begin{pmatrix} \mathfrak{J}'_{n+3}(\rho) \\ \mathfrak{J}'_{n+2}(\rho) \\ \mathfrak{J}'_{n+1}(\rho) \end{pmatrix} = \begin{pmatrix} 1 & \rho & \rho^2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} \mathfrak{J}'_3(\rho) \\ \mathfrak{J}'_2(\rho) \\ \mathfrak{J}'_1(\rho) \end{pmatrix}$$

So

$$\begin{pmatrix} \mathfrak{J}'_{n+3}(\rho) \\ \mathfrak{J}'_{n+2}(\rho) \\ \mathfrak{J}'_{n+1}(\rho) \end{pmatrix} = \begin{pmatrix} \mathfrak{J}_{n+1}(\rho) & \mathfrak{J}_{n+2}(\rho) - \mathfrak{J}_{n+1}(\rho) & \rho^2\mathfrak{J}_n(\rho) \\ \mathfrak{J}_n(\rho) & \mathfrak{J}_{n+1}(\rho) - \mathfrak{J}_n(\rho) & \rho^2\mathfrak{J}_{n-1}(\rho) \\ \mathfrak{J}_{n-1}(\rho) & \mathfrak{J}_n(\rho) - \mathfrak{J}_{n-1}(\rho) & \rho^2\mathfrak{J}_{n-2}(\rho) \end{pmatrix} \begin{pmatrix} \mathfrak{J}'_3(\rho) \\ \mathfrak{J}'_2(\rho) \\ \mathfrak{J}'_1(\rho) \end{pmatrix}$$

Then

$$\mathfrak{J}'_{n+1}(\rho) = \mathfrak{J}'_3(\rho)\mathfrak{J}_{n-1}(\rho) + \mathfrak{J}'_2(\rho)[\mathfrak{J}_n(\rho) - \mathfrak{J}_{n-1}(\rho)] + \rho^2\mathfrak{J}'_1(\rho)\mathfrak{J}_{n-2}(\rho).$$

Finally

$$\mathfrak{J}'_n(\rho) = (c_1\rho + c_0)\mathfrak{J}_{n-2}(\rho) + e[\mathfrak{J}_{n-1}(\rho) - \mathfrak{J}_{n-2}(\rho)] + d\rho^2\mathfrak{J}_{n-3}(\rho)$$

■

Theorem 2.13 *For $n \geq 0$, we have*

$$\mathfrak{J}_n(x) = \frac{1}{\lambda}\left[({\iota'}_1)^{n+1}({\iota'}_2 - {\iota'}_3) + ({\iota'}_2)^{n+1}({\iota'}_3 - {\iota'}_1) + ({\iota'}_3)^{n+1}({\iota'}_1 - {\iota'}_2)\right]$$

where $\lambda' = ({\iota'}_1 - {\iota'}_2)({\iota'}_1 - {\iota'}_3)({\iota'}_2 - {\iota'}_3)$ and ${\iota'}_1$, ${\iota'}_2$, ${\iota'}_3$ are the distinct roots of the characteristic polynomial of 1.7

$$t^3 - t^2 - \rho t - \rho^2 = 0$$

Proposition 2.7 *For every $n \geq 3$, we have*

$$\left({\iota'}^n_1 + {\iota'}^n_2 + {\iota'}^n_3\right) = \mathfrak{J}_{n+1}(\rho) + \rho\mathfrak{J}_{n-1}(\rho) + 2r^2\mathfrak{J}_{n-2}(\rho) \tag{2.8}$$

where ${\iota'}_1$, ${\iota'}_2$ and ${\iota'}_3$ are the distinct roots of the equation (1.7)

Proof *Using Eq.* (1.9) *and Eq.* (2.8), *and taking in to account that*

$$\begin{cases} r^2 = {\iota'}_1{\iota'}_2{\iota'}_3 \\ \rho = -({\iota'}_2{\iota'}_3 + {\iota'}_2{\iota'}_1 + {\iota'}_1{\iota'}_3). \end{cases},$$

it follows that

$$
\begin{aligned}
(R) &= \frac{\iota'^{n+2}_1 + \rho\iota'^n_1 + 2\rho^2\iota'^{n-1}_1}{(\iota'_1-\iota'_2)(\iota'_1-\iota'_3)} + \frac{\iota'^{n+2}_2 + \rho\iota'^n_2 + 2\rho^2\iota'^{n-1}_2}{(\iota'_2-\iota'_1)(\iota'_2-\iota'_3)} + \frac{\iota'^{n+2}_3 + \rho\iota'^n_3 + 2\rho^2\iota'^{n-1}_3}{(\iota'_3-\iota'_1)(\iota'_3-\iota'_2)} \\
&= \frac{\iota'^n_1\left(\iota'^2_1+\rho+\dfrac{2\rho^2}{\iota'_1}\right)}{(\iota'_1-\iota'_2)(\iota'_1-\iota'_3)} + \frac{\iota'^n_2\left(\iota'^2_2+\rho+\dfrac{2\rho^2}{\iota'_2}\right)}{(\iota'_2-\iota'_1)(\iota'_2-\iota'_3)} + \frac{\iota'^n_3\left(\iota'^2_3+\rho+\dfrac{2\rho^2}{\iota'_3}\right)}{(\iota'_3-\iota'_1)(\iota'_3-\iota'_2)} \\
&= \frac{\iota'^n_1\left(\iota'^2_1+\rho+2\iota'_2\iota'_3\right)}{(\iota'^2_1-(-\rho-\iota'_3\iota'_2)+\iota'_2\iota'_3)} + \frac{\iota'^n_2\left(\iota'^2_2+\rho+2\iota'_1\iota'_3\right)}{(\iota'^2_2-(-\rho-\iota'_3\iota'_1)+\iota'_3\iota'_1)} + \\
&+ \frac{\iota'^n_3\left(\iota'^2_3+\rho+2\iota'_1\iota'_2\right)}{(\iota'^2_3-(-\rho-\iota'_1\iota'_2)+\iota'_1\iota'_2)} \\
&= \iota'^n_1 + \iota'^n_2 + \iota'^n_3
\end{aligned}
$$

■

Proposition 2.8 *For $n \geqslant 3$, we have*

$$
\begin{aligned}
\iota'^n_1 &= \mathfrak{J}_{n-1}(\rho)\iota'^2_1 + [\mathfrak{J}_n(\rho) - \mathfrak{J}_{n-1}(\rho)]\iota'_1 + \rho^2\mathfrak{J}_{n-2}(\rho) \\
\iota'^n_2 &= \mathfrak{J}_{n-1}(\rho)\iota'^2_2 + [\mathfrak{J}_n(\rho) - \mathfrak{J}_{n-1}(\rho)]\iota'_2 + \rho^2\mathfrak{J}_{n-2}(\rho) \\
\iota'^n_3 &= \mathfrak{J}_{n-1}(\rho)\iota'^2_3 + [\mathfrak{J}_n(\rho) - \mathfrak{J}_{n-1}(\rho)]\iota'_3 + \rho^2\mathfrak{J}_{n-2}(\rho)
\end{aligned}
$$

Proof *The result is straight forward by induction on n.* ■

Theorem 2.14 *For $n \geq 1$, we have*

$$\mathfrak{J}_{3n-1}(\rho) = \sum_{i=0}^{n}\sum_{k=0}^{i}\binom{n}{i}\binom{i}{k}(\rho)^{2n-i-k}\mathfrak{J}_{i+k-1}(\rho)$$

Proof *We proceed as for the Theorem (2.11) , we have*

$$\iota'^3 = \iota'^2 + \rho\iota' + \rho^2$$

where ι' is a root of the equation (1.7), which implies

$$\begin{aligned}
\left(\iota'^3\right)^n &= \left(\iota'^3 - \rho^2 + \rho^2\right)^n \\
&= \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} (\rho)^{2n-2i} \left(\iota'^3 - \rho^2\right)^i \\
&= \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} (\rho)^{2n-2i} \left(\iota'^2 + \rho\iota'\right)^i \\
&= \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} (\rho)^{2n-2i} \sum_{k=0}^{i} \begin{pmatrix} i \\ k \end{pmatrix} \iota'^{2k} (\rho\iota')^{i-k} \\
&= \sum_{i=0}^{n} \sum_{k=0}^{i} \begin{pmatrix} n \\ i \end{pmatrix} \begin{pmatrix} i \\ k \end{pmatrix} (\rho)^{2n-i-k} (\iota')^{i+k} \\
&= \sum_{i=0}^{n} \sum_{k=0}^{i} \begin{pmatrix} n \\ i \end{pmatrix} \begin{pmatrix} i \\ k \end{pmatrix} (\rho)^{2n-i-k} (\iota')^{i+k}
\end{aligned}$$

replacing ι with $\iota'_1, \iota'_2, \iota'_3$ and combining the letter with the equation (1.9), we get

$$\begin{aligned}
\mathfrak{J}_{3n-1}(\rho) &= \frac{(\iota'_1)^{3n}}{(\iota'_1 - \iota'_2)(\iota'_1 - \iota'_3)} + \frac{(\iota'_2)^{3n}}{(\iota'_2 - \iota'_1)(\iota'_2 - \iota'_3)} + \frac{(\iota'_3)^{3n}}{(\iota'_3 - \iota'_1)(\iota'_3 - \iota'_2)} \\
&= \sum_{i=0}^{n} \sum_{k=0}^{i} \begin{pmatrix} n \\ i \end{pmatrix} \begin{pmatrix} i \\ k \end{pmatrix} \rho^{2n-i-k} \left(\frac{(\iota'_1)^{i+k}}{(\iota'_1 - \iota'_2)(\iota'_1 - \iota'_3)} + \frac{(\iota'_2)^{i+k}}{(\iota'_2 - \iota'_1)(\iota'_2 - \iota'_3)} + \frac{(\iota'_3)^{i+k}}{(\iota'_3 - \iota'_1)(\iota'_3 - \iota'_2)}\right) \\
&= \sum_{i=0}^{n} \sum_{k=0}^{i} \begin{pmatrix} n \\ i \end{pmatrix} \begin{pmatrix} i \\ k \end{pmatrix} (\rho)^{2n-i-k} \mathfrak{J}_{i+k-1}(\rho)
\end{aligned}$$

■

CHAPTER

3. Generalized tribonacci Like Polynomials

One of the well-known generalizations of Fibonacci polynomials is the tribonacci polynomials. tribonacci polynomials are defined by a third-order homogeneous recurrence relation. These polynomials have been extensively studied (see [8], [17], [26]). In this Chapter, we define two new extension $(P_n(\rho,\varrho,\varpi))_{n\in\mathbb{N}}$ and $(L_n(\rho,\varrho,\varpi))_{n\in\mathbb{N}}$ of the tribonacci polynomials, which we called respectively the generalized tribonacci like polynomial and the generalized lucas-Tribonacci like polynomials has been introduced using the homogeneous recurrence relation defined by

$$\begin{cases} P_n(\rho,\varrho,\varpi)=\rho P_{n-1}(\rho,\varrho,\varpi)+\varrho P_{n-2}(\rho,\varrho,\varpi)+\varpi P_{n-3}(\rho,\varrho,\varpi), n\geq 3 \\ P_0(\rho,\varrho,\varpi)=0 \\ P_1(\rho,\varrho,\varpi)=1 \\ P_2(\rho,\varrho,\varpi)=r. \end{cases} \tag{3.1}$$

and

$$\begin{cases} L_n(\rho,\varrho,\varpi)=\rho L_{n-1}(\rho,\varrho,\varpi)+\varrho L_{n-2}(\rho,\varrho,\varpi)+\varpi L_{n-3}(\rho,\varrho,\varpi), n\geq 3 \\ L_0(\rho,\varrho,\varpi)=3 \\ L_1(\rho,\varrho,\varpi)=r \\ L_2(\rho,\varrho,\varpi)=r^2+2s. \end{cases} \tag{3.2}$$

.

Firstly, we establish some properties of thess two polynomials and yielding connexion between theme, also we find their generating functions by investigating some appropriate matrices. Moreover, we find explicit formulas for the partial derivatives of these polynomials. Next, we introduce tune new generating matrices,for those polynomials and theirs *summations, this leads in particular to explicit formulas for such sums. Finally, we explore exhibiting various properties related to them*

3.1 Properties

Theorem 3.1 *For $n \geqslant 2$*

1) $P_{3n-1}(\rho,\varrho,\varpi)=\sum_{i=0}^{n}\sum_{j=0}^{i}\begin{pmatrix} n \\ i \end{pmatrix}\begin{pmatrix} i \\ j \end{pmatrix}\rho^j\varrho^{i-j}\varpi^{n-i}P_{i+j-1}(\rho,\varrho,\varpi)$

2) $L_{3n}(\rho,\varrho,\varpi)=\sum_{i=0}^{n}\sum_{j=0}^{i}\begin{pmatrix} n \\ i \end{pmatrix}\begin{pmatrix} i \\ j \end{pmatrix}\rho^j\varrho^{i-j}\varpi^{n-i}L_{i+j}(\rho,\varrho,\varpi)$

Proof *Let ι is a root of the equation (1.18),*

$$\iota^3=\rho\iota^2+\varrho\iota+\varpi$$

which implies

$$\begin{aligned}\iota^{3n}&=(\iota^3-\varpi+\varpi)^n\\&=\sum_{i=0}^{n}\begin{pmatrix} n \\ i \end{pmatrix}w^{n-i}(\iota^3-\varpi)^i\\&=\sum_{i=0}^{n}\begin{pmatrix} n \\ i \end{pmatrix}w^{n-i}(\rho\iota^2+\varrho\iota)^i\\&=\sum_{i=0}^{n}\begin{pmatrix} n \\ i \end{pmatrix}\varpi^{n-i}\sum_{j=0}^{i}\begin{pmatrix} i \\ j \end{pmatrix}(\rho\iota^2)^j(\varrho\iota))^{i-j}\\&=\sum_{i=0}^{n}\begin{pmatrix} n \\ i \end{pmatrix}\sum_{j=0}^{i}\begin{pmatrix} i \\ j \end{pmatrix}\varpi^{n-i}\rho^j\varrho^{i-j}\iota^{i+j}\end{aligned}$$

If we replace ι with κ, ϑ and η , and combined them with the equation (1.19), we get

$$\begin{aligned}P_{3n-1}(\rho,\varrho,\varpi)&=\frac{\kappa^{3n}}{(\kappa-\vartheta)(\kappa-\eta)}+\frac{\vartheta^{3n}}{(\vartheta-\kappa)(\vartheta-\eta)}+\frac{\eta^{3n}}{(\eta-\kappa)(\eta-\vartheta)}\\&=\sum_{i=0}^{n}\sum_{k=0}^{i}\begin{pmatrix} n \\ i \end{pmatrix}\begin{pmatrix} i \\ j \end{pmatrix}\rho^j\varrho^{i-j}\varpi^{n-i}P_{i+j-1}(\rho,\varrho,\varpi)\end{aligned}$$

2) *by the same methods withe combine the equation (1.20)*

■

Theorem 3.2 *[18] For $n \geq 2$*

$$P_n(\rho,\varrho,\varpi)=\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}\sum_{j=0}^{i}\binom{i}{j}\binom{n-i-j-1}{i}\rho^{n-2i-j-1}\varrho^{i-j}\varpi^{j}.$$

and

$$L_n(\rho,\varrho,\varpi)=\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}\sum_{j=0}^{i}\frac{n}{n-i-j}\binom{i}{j}\binom{n-i-j}{i}\rho^{n-2i-j}\varrho^{i-j}\varpi^{j}.$$

Proof *1) By induction on n, the statement is true for. $n=1$ and $n=2$. If we consider it valid for $n=k-1$ and $n=k$ where $k\geq 3$.*

Then

$$\begin{aligned}P_{k+1}(\rho,\varrho,\varpi)=&\rho P_k(\rho,\varrho,\varpi)+\varrho P_{k-1}(\rho,\varrho,\varpi)+\varpi P_{k-2}(\rho,\varrho,\varpi)\\ =&\rho\sum_{i=0}^{\lfloor\frac{k-1}{2}\rfloor}\sum_{j=0}^{i}\binom{i}{j}\binom{k-i-j-1}{i}\rho^{k-2i-j-1}\varrho^{i-j}\varpi^{j}\\ &+\varrho\sum_{i=0}^{\lfloor\frac{k-2}{2}\rfloor}\sum_{j=0}^{i}\binom{i}{j}\binom{k-i-j-2}{i}\rho^{k-2i-j-2}\varrho^{i-j}\varpi^{j}\\ &+\varpi\sum_{i=0}^{\lfloor\frac{k-3}{2}\rfloor}\sum_{j=0}^{i}\binom{i}{j}\binom{k-i-j-3}{i}\rho^{k-2i-j-3}\varrho^{i-j}\varpi^{j}.\end{aligned}$$

choosing $k=2t$, we obtain

$$\begin{aligned}P_{k+1}(\rho,\varrho,\varpi)=&\sum_{i=0}^{t-1}\sum_{j=0}^{i}\binom{i}{j}\binom{2t-i-j-1}{i}\rho^{2t-2i-j}\varrho^{i-j}\varpi^{j}\\ &+\sum_{i=0}^{t-1}\sum_{j=0}^{i}\binom{i}{j}\binom{2t-i-j-2}{i}\rho^{2t-2i-j-2}\varrho^{i-j+1}\varpi^{j}\\ &+\sum_{i=0}^{t-2}\sum_{j=0}^{i}\binom{i}{j}\binom{2t-i-j-3}{i}\rho^{2t-2i-j-3}\varrho^{i-j}\varpi^{j+1}.\end{aligned}$$

By the Pascal's formula, So

$$\begin{aligned}P_{k+1}(\rho,\varrho,\varpi)=&\sum_{i=0}^{t}\sum_{j=0}^{i}\binom{i}{j}\binom{2t-i-j}{i}\rho^{2t-2i-j}\varrho^{i-j}\varpi^{j}\\ =&\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}\sum_{j=0}^{i}\binom{i}{j}\binom{k-i-j}{i}\rho^{k-2i-j}\varrho^{i-j}\varpi^{j}\end{aligned}$$

by $k = 2t$. If k is odd,The formula is valid.

(2) The proof similarly. ■

Proposition 3.1 *[7] If $n \geq 2$ we have*

1) $L_n(\rho,\varrho,\varpi) = \rho P_n(\rho,\varrho,\varpi) + 2\varrho P_{n-1}(\rho,\varrho,\varpi) + 3\varpi P_{n-2}(\rho,\varrho,\varpi)$

2) $P_n(\rho,\varrho,\varpi) = \dfrac{(-\rho - 9w\varrho\varpi^2)L_{n-1}(\rho,\varrho,\varpi) + (2\varpi\rho^2 - \rho\varrho^2 - 3\varrho\varpi)L_n(\rho,\varrho,\varpi) + (-6\rho\varpi + 2\rho^2)L_{n+1}(\rho,\varrho,\varpi}{-18\rho\varrho\varpi - 27\varpi^2 - 4\rho^3\varpi + \rho^2\varrho^2 + 4\varrho^3}$

Proof 1) *Using the generating matrix corresponding to the recurrence relation* (1.15), *we obtain:*

$$\begin{pmatrix} L_3(\rho,\varrho,\varpi) \\ L_2(\rho,\varrho,\varpi) \\ L_1(\rho,\varrho,\varpi) \end{pmatrix} = \begin{pmatrix} r & s & w \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} L_2(\rho,\varrho,\varpi) \\ L_1(\rho,\varrho,\varpi) \\ L_0(\rho,\varrho,\varpi) \end{pmatrix}.$$

as we all know

$$(Q)^n = \begin{pmatrix} \rho & \varrho & \varpi \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} P_{n+1}(\rho,\varrho,\varpi) & P_{n+2}(\rho,\varrho,\varpi) - \rho P_{n+1}(\rho,\varrho,\varpi) & \varpi P_n(\rho,\varrho,\varpi) \\ P_n(r)(\rho,\varrho,\varpi) & P_{n+1}(\rho,\varrho,\varpi) - \rho P_n(\rho,\varrho,\varpi) & \varpi P_{n-1}(\rho,\varrho,\varpi) \\ P_{n-1}(r)(\rho,\varrho,\varpi) & P_n(\rho,\varrho,\varpi) - \rho P_{n-1}(\rho,\varrho,\varpi) & \varpi P_{n-2}(\rho,\varrho,\varpi) \end{pmatrix}$$

This means

$$\begin{pmatrix} L_{n+2}(\rho,\varrho,\varpi) \\ L_{n+1}(\rho,\varrho,\varpi) \\ L_n(r,s,w) \end{pmatrix} = \begin{pmatrix} P_{n+1}(\rho,\varrho,\varpi) & P_{n+2}(\rho,\varrho,\varpi) - \rho P_{n+1}(\rho,\varrho,\varpi) & \varpi P_n(\rho,\varrho,\varpi) \\ P_n(\rho,\varrho,\varpi) & P_{n+1}(\rho,\varrho,\varpi) - \rho P_n(\rho,\varrho,\varpi) & \varpi P_{n-1}(\rho,\varrho,\varpi) \\ P_{n-1}(\rho,\varrho,\varpi) & P_n(\rho,\varrho,\varpi) - \rho P_{n-1}(\rho,\varrho,\varpi) & \varpi P_{n-2}(\rho,\varrho,\varpi) \end{pmatrix} \begin{pmatrix} L_2(\rho,\varrho,\varpi) \\ L_1(\rho,\varrho,\varpi) \\ L_0(\rho,\varrho,\varpi) \end{pmatrix}$$

So

$$\begin{aligned} L_n(\rho,\varrho,\varpi) &= (\rho^2 + 2\varrho)P_{n-1}(\rho,\varrho,\varpi) + \varpi[P_n(\rho,\varrho,\varpi) - \varpi P_{n-1}(\rho,\varrho,\varpi)] + 3\varpi P_{n-2}(\rho,\varrho,\varpi) \\ &= \rho P_n(\rho,\varrho,\varpi) + 2\varrho P_{n-1}(\rho,\varrho,\varpi) + 3wP_{n-2}(\rho,\varrho,\varpi). \end{aligned}$$

2) *From* (1) *We have*

$$L_n(\rho,\varrho,\varpi) = \rho P_n(\rho,\varrho,\varpi) + 2\varrho P_{n-1}(\rho,\varrho,\varpi) + 3\varpi P_{n-2}(\rho,\varrho,\varpi)$$

So we Obtain:

$$\begin{cases} L_{n-1}(\rho,\varrho,\varpi) &= 3P_n(\rho,\varrho,\varpi) - 2\rho P_{n-1}(\rho,\varrho,\varpi) - \varpi P_{n-2}(\rho,\varrho,\varpi) \\ L_n(\rho,\varrho,\varpi) &= \rho P_n(\rho,\varrho,\varpi) + 2\varrho P_{n-1}(\rho,\varrho,\varpi) + 3\varpi P_{n-2}(\rho,\varrho,\varpi) \\ L_{n+1}(\rho,\varrho,\varpi) &= (\rho^2 + 2\varrho)P_n(\rho,\varrho,\varpi) + (\rho\varrho + 3\varpi)P_{n-1}(\rho,\varrho,\varpi) + (\rho\varpi)P_{n-2}(\rho,\varrho,\varpi) \end{cases} \tag{3.3}$$

By Cramer, we obtain
$P_n(\rho,\varrho,\varpi)=\dfrac{|A|}{|B|}$, *where*

$$C=\begin{bmatrix} 3 & -2\rho & -\varrho \\ \rho & 2\varrho & 3\varpi \\ \rho^2+2\varrho & \rho\varrho+3\varpi & \rho\varpi \end{bmatrix} \text{ and, } E=\begin{bmatrix} L_{n-1}(\rho,\varrho,\varpi) & -2\rho & -\varrho \\ L_n(\rho,\varrho,\varpi) & 2\varrho & 3\varpi \\ L_{n+1}(\rho,\varrho,\varpi) & \rho\varrho+3\varpi & \rho\varpi \end{bmatrix}$$

$|C|=det(B)=-18\rho\varrho\varpi-27\varpi^2-4\rho^3\varpi+\rho^2\varrho^2+4\varrho^3$

$|E|=det(A)=(-\rho\varrho\varpi-9\varpi^2)L_{n-1}(\rho,\varrho,\varpi)+(2\varpi\rho^2-\rho\varrho^2-3\varrho\varpi)L_n(\rho,\varrho,\varpi)+(-6\rho\varpi+2\varrho^2)L_{n+1}(\rho,\varrho,\varpi$

■

Theorem 3.3 *[18] For $n\geq 2$, then*

$$\begin{vmatrix} P_{n+2}(\rho,\varrho,\varpi) & P_{n+1}(\rho,\varrho,\varpi) & P_n(\rho,\varrho,\varpi) \\ P_{n+1}(\rho,\varrho,\varpi) & P_n(\rho,\varrho,\varpi) & P_{n-1}(\rho,\varrho,\varpi) \\ P_n(\rho,\varrho,\varpi) & P_{n-1}(\rho,\varrho,\varpi) & P_{n-2}(\rho,\varrho,\varpi) \end{vmatrix}=-\varpi^{n-1}.$$

Proof *Let Q matrix*

$$Q=\begin{pmatrix} \rho & \varrho & \varpi \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $det(Q)=\varpi$, and

$$|Q^n|=\begin{vmatrix} P_{n+1}(\rho,\varrho,\varpi) & \varrho P_n(\rho,\varrho,\varpi)+\varpi P_{n-1}(\rho,\varrho,\varpi) & \varpi P_n(\rho,\varrho,\varpi) \\ P_n(\rho,\varrho,\varpi) & \varrho P_{n-1}(\rho,\varrho,\varpi)+\varpi P_{n-2}(\rho,\varrho,\varpi) & \varpi P_{n-1}(\rho,\varrho,\varpi) \\ P_{n-1}(\rho,\varrho,\varpi) & \varrho P_{n-2}(\rho,\varrho,\varpi)+\varpi P_{n-3}(\rho,\varrho,\varpi) & \varpi P_{n-2}(\rho,\varrho,\varpi) \end{vmatrix}$$

Using the determinants of the matrices Q and Q^n, we obtain

$$\begin{vmatrix} P_{n+1}(\rho,\varrho,\varpi) & P_n(\rho,\varrho,\varpi) & P_{n-1}(\rho,\varrho,\varpi) \\ \varrho P_n(\rho,\varrho,\varpi)+\varpi P_{n-1}(\rho,\varrho,\varpi) & \varrho P_{n-1}(\rho,\varrho,\varpi)+\varpi P_{n-2}(\rho,\varrho,\varpi) & \varrho P_{n-2}(\rho,\varrho,\varpi)+zP_{n-3}(\rho,\varrho,\varpi) \\ \varpi P_n(\rho,\varrho,\varpi) & \varpi P_{n-1}(\rho,\varrho,\varpi) & \varpi P_{n-2}(\rho,\varrho,\varpi) \end{vmatrix}=\varpi^n$$

Multiplying the first row of Q^n by r and adding it to the second row, followed by exchanging rows 1 and 2, results in

$$\begin{vmatrix} P_{n+2}(\rho,\varrho,\varpi) & P_{n+1}(\rho,\varrho,\varpi) & P_n(\rho,\varrho,\varpi) \\ P_{n+1}(\rho,\varrho,\varpi) & P_n(\rho,\varrho,\varpi) & P_{n-1}(\rho,\varrho,\varpi) \\ \varpi P_n(\rho,\varrho,\varpi) & \varpi P_{n-1}(\rho,\varrho,\varpi) & \varpi P_{n-2}(\rho,\varrho,\varpi) \end{vmatrix}=-\varpi^n$$

From the properties of determinants, we can drive :

$$\begin{vmatrix} P_{n+2}(\rho,\varrho,\varpi) & P_{n+1}(\rho,\varrho,\varpi) & P_n(\rho,\varrho,\varpi) \\ P_{n+1}(\rho,\varrho,\varpi) & P_n(\rho,\varrho,\varpi) & P_{n-1}(\rho,\varrho,\varpi) \\ P_n(\rho,\varrho,\varpi) & P_{n-1}(\rho,\varrho,\varpi) & P_{n-2}(\rho,\varrho,\varpi) \end{vmatrix} = -\varpi^{n-1}$$

■

3.2 Generating Function

Theorem 3.4 *for* $n \geq 0$

$$P_n(\rho,\varrho,\varpi) = \frac{1}{\lambda}\left(\kappa^{n+1}(\vartheta-\eta) + \vartheta^{n+1}(\kappa-\eta) + \eta^{n+1}(\kappa-\vartheta)\right)$$

where $\lambda = (\kappa-\vartheta)(\kappa-\eta)(\vartheta-\eta)$

Proof *The validity of this proof relies on the diagonalization of the generating matrix* Q.

$$Q = \begin{pmatrix} \rho & \varrho & \varpi \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

The characteristic equation is $-t^3 + \rho t^2 + \varrho t + \varpi = 0$
Let κ, ϑ *and* η *be the roots of the* $-t^3 + \rho t^2 + \varrho t + \varpi = 0$
Let A *is a matrix defined by*

$$A = \begin{pmatrix} \kappa^2 & \vartheta^2 & \eta^2 \\ \kappa & \vartheta & \eta \\ 1 & 1 & 1 \end{pmatrix} \text{ and, } A^{-1} = \frac{1}{\lambda}\begin{pmatrix} \vartheta-\eta & \eta^2-\vartheta^2 & \vartheta\eta(\vartheta-\eta) \\ \eta-\kappa & \kappa^2-\eta^2 & \kappa\eta(\eta-\kappa) \\ \kappa-\vartheta & \eta^2-\kappa^2 & \kappa\vartheta(\kappa-\vartheta) \end{pmatrix}$$

where $\lambda = (\kappa-\vartheta)(\kappa-\eta)(\vartheta-\eta)$ *Now, let*

$$B = \begin{pmatrix} \kappa & 0 & 0 \\ 0 & \vartheta & 0 \\ 0 & 0 & \eta \end{pmatrix}$$

i.e., B is the diagonal matrix

Then $Q = ABA^{-1}$, *we obtain*

$$\begin{aligned} Q^n &= (ABA^{-1})^n \\ &= AB^nA^{-1} \\ &= \frac{1}{\lambda}\begin{pmatrix} \kappa^2 & \vartheta^2 & \eta^2 \\ \kappa & \vartheta & \eta \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} \kappa^n & 0 & 0 \\ 0 & \vartheta^n & 0 \\ 0 & 0 & \eta^n \end{pmatrix}\begin{pmatrix} \vartheta-\eta & \eta^2-\vartheta^2 & \vartheta\eta(\vartheta-\eta) \\ \eta-\kappa & \kappa^2-\eta^2 & \kappa\eta(\eta-\kappa) \\ \kappa-\vartheta & \vartheta^2-\kappa^2 & \kappa\vartheta(\kappa-\vartheta) \end{pmatrix} \end{aligned}$$

After calculating the right side and matching with the matrix Q^n, *we obtain*

$$P_n(\rho,\varrho,\varpi) = \frac{1}{\lambda}\left(\kappa^{n+1}(\vartheta-\eta) + \vartheta^{n+1}(\kappa-\eta) + \eta^{n+1}(\kappa-\vartheta)\right)$$

■

Proposition 3.2 *For* $n \geq 2$, *we have*

1) $\displaystyle\sum_{n=0}^{\infty} P_n(\rho,\varrho,\varpi)t^n = te^{\rho t+\varrho t^2}\left({}_2F_1\left(n+1,1;1;\varpi t^3\right)\right)$

2) $\displaystyle\sum_{n=0}^{\infty} L_{k+n}(\rho,\varrho,\varpi)t^n = \frac{L_k(\rho,\varrho,\varpi) + [L_{k+1}(\rho,\varrho,\varpi) - \rho L_k(\rho,\varrho,\varpi)]t + \varpi L_{k-1}(\rho,\varrho,\varpi)t^2}{1-\rho t-\varrho t^2-\varpi t^3}$

3) $\displaystyle\sum_{n=0}^{\infty} P_{k+n}(\rho,\varrho,\varpi)t^n = \frac{P_k(\rho,\varrho,\varpi) + [P_{k+1}(\rho,\varrho,\varpi) - \rho P_k(\rho,\varrho,\varpi)]t + \varpi P_{k-1}(\rho,\varrho,\varpi)t^2}{1-\rho t-\varrho t^2-\varpi t^3}$

Proof 1) *From the generating function we can derive the following :*

$$\begin{aligned}
\sum_{n=0}^{\infty} P_n(\rho,\varrho,\varpi)t^n &= t\left(1-\rho t-\varrho t^2-\varpi t^3\right)^{-1}\\
&= t\sum_{n=0}^{\infty} t^n(\rho+\varrho t+\varpi t^2)^n\\
&= t\sum_{n=0}^{\infty} t^n\sum_{m=0}^{n}\left(\begin{array}{c} n\\ m\end{array}\right) w^m t^{2m}(\rho+\varrho t)^{n-m}\\
&= t\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(n+m)!}{n!m!}\varpi^m t^{n+3m}(\rho+\varrho t)^n\\
&= t\sum_{n=0}^{\infty}(\rho t+\varrho t^2)^n\sum_{m=0}^{\infty}\frac{(n+m)!}{n!m!}\varpi^m t^{3m}\\
&= t\sum_{n=0}^{\infty}\frac{(\rho t+\varrho t^2)^n}{n!}\sum_{m=0}^{\infty}\frac{(n+m)!}{m!}\varpi^m t^{3m}\\
&= te^{\rho t+\varrho t^2}\sum_{m=0}^{\infty}\frac{(n+1)_m(1)_m}{(1)_m}\frac{(\varpi t^3)^m}{m!}\\
&= te^{\rho t+\varrho t^2}\left({}_2F_1\left(n+1,1;1;\varpi t^3\right)\right)
\end{aligned}$$

2) if we combine to the equation (1.20)*, we obtain*

$$\begin{aligned}
\sum_{n=0}^{\infty} L_{m+n}(\rho,\varrho,\varpi)t^n &= \sum_{n=0}^{\infty}\left(\kappa^{m+n}+\vartheta^{m+n}+\eta^{n+m}\right)t^n\\
&= \kappa^m\sum_{n=0}^{\infty}\kappa^n t^n+\vartheta^m\sum_{n=0}^{\infty}\vartheta^n t^n+\eta^m\sum_{n=0}^{\infty}\eta^n t^n\\
&= \frac{\kappa^m}{1-\kappa t}+\frac{\vartheta^m}{1-\vartheta t}+\frac{\eta^m}{1-\eta t}\\
&= \frac{L_m(\rho,\varrho,\varpi)+[L_{m+1}(\rho,\varrho,\varpi)-\rho L_m(\rho,\varrho,\varpi)]t+\varpi L_{m-1}(\rho,\varrho,\varpi)t^2}{1-\rho t-\varrho t^2-\varpi t^3}
\end{aligned}$$

■

Theorem 3.5 *For any integer* $n \geq 3$

$$(\kappa^n+\vartheta^n+\eta^n) = P_{n+1}(\rho,\varrho,\varpi)+\varrho P_{n-1}(\rho,\varrho,\varpi)+2\varpi P_{n-2}(\rho,\varrho,\varpi) \tag{3.4}$$

Proof *we know that*

$$\begin{cases} \varpi = \kappa\vartheta\eta\\ \rho = -(\vartheta\eta+\vartheta\kappa+\kappa\eta). \end{cases}$$

and Combining the Eq. (1.19) *in the R.H.S. of Eq.* (3.4) *We obtain*

$$\begin{aligned}(RHS) &= \frac{\kappa^{n+2}+\varrho\kappa^{n}+2\varpi\kappa^{n-1}}{(\kappa-\vartheta)(\kappa-\eta)}+\frac{\vartheta^{n+2}+\varrho\vartheta^{n}+2\varpi\vartheta^{n-1}}{(\vartheta-\kappa)(\vartheta-\eta)}+\frac{\eta^{n+2}+\varrho\eta^{n}+2\varpi\eta^{n-1}}{(\eta-\kappa)(\eta-\vartheta)}\\ &= \frac{\kappa^{n}\left(\kappa^{2}+\varrho+2\vartheta\eta\right)}{(\kappa-\vartheta)(\kappa-\eta)}+\frac{\vartheta^{n}\left(\vartheta^{2}+\varrho+2\kappa\eta\right)}{(\vartheta-\kappa)(\vartheta-\eta)}+\frac{\eta^{n}\left(\eta^{2}+\varrho+2\kappa\vartheta\right)}{(\eta-\kappa)(\eta-\vartheta)}\\ &= \kappa^{n}+\vartheta^{n}+\eta^{n}\end{aligned}$$

■

Proposition 3.3 *For* $n\geq 3$*, we have*

$$\kappa^{n}=P_{n-1}(\rho,\varrho,\varpi)\kappa^{2}+\left[P_{n}(\rho,\varrho,\varpi)-\rho P_{n-1}(\rho,\varrho,\varpi)\right]\kappa+\varpi P_{n-2}(\rho,\varrho,\varpi)$$
$$\vartheta^{n}=P_{n-1}(\rho,\varrho,\varpi)\vartheta^{2}+\left[P_{n}(\rho,\varrho,\varpi)-\rho P_{n-1}(\rho,\varrho,\varpi)\right]\vartheta+wP_{n-2}(\rho,\varrho,\varpi)$$
$$\eta^{n}=P_{n-1}(\rho,\varrho,\varpi)\eta^{2}+\left[P_{n}(\rho,\varrho,\varpi)-\rho P_{n-1}(\rho,\varrho,\varpi)\right]\eta+\varpi P_{n-2}(\rho,\varrho,\varpi)$$

Proof *By indection on n* ■

The tribonacci like polynomials $(P_n(\rho,\varrho,\varpi))_{n\geq 0}$ *can be extended to negative subscripts by introducing the following definition:*

$$\begin{cases} P_{-n}(\rho,\varrho,\varpi)=\dfrac{P_{-(n-3)}(\rho,\varrho,\varpi)-\rho P_{-(n-2)}(\rho,\varrho,\varpi)-P_{-(n-1)}(\rho,\varrho,\varpi)}{\varpi}, n\geq 3\\ P_{0}(\rho,\varrho,\varpi)=P_{-1}(\rho,\varrho,\varpi)=0\\ P_{-2}(\rho,\varrho,\varpi)=\dfrac{1}{\varpi}\end{cases} \tag{3.5}$$

Proposition 3.4 *For* $n\geq 3$*, we have*

1) $P_{-n}(\rho,\varrho,\varpi)=[\frac{\varrho^{2}-\rho\varpi}{\varpi^{2}}]P_{-n+2}(\rho,\varrho,\varpi)+[\frac{\varpi+\rho\varrho}{\varpi^{2}}]P_{-n+3}(\rho,\varrho,\varpi)-[\frac{\varrho}{\varpi^{2}}]P_{-n+4}(\rho,\varrho,\varpi)$

2) $P_{-n}(\rho,\varrho,\varpi)=\frac{[P_{n-1}(\rho,\varrho,\varpi)]^{2}-P_{n}(\rho,\varrho,\varpi)P_{n-2}(\rho,\varrho,\varpi)}{\varpi^{n-1}}$

Proof 1) *From the recurrence relation* (3.5)*, we obtain*

$$\begin{aligned}P_{-n}(\rho,\varrho,\varpi) &= \frac{P_{-(n-3)}(\rho,\varrho,\varpi)-\rho P_{-(n-2)}(\rho,\varrho,\varpi)-\varrho P_{-(n-1)}(\rho,\varrho,\varpi)}{w}\\ &= \frac{P_{-(n-3)}(\rho,\varrho,\varpi)-\rho P_{-(n-2)}(\rho,\varrho,\varpi)}{\varpi}-\varrho\Bigg(\frac{P_{-(n-4)}(\rho,\varrho,\varpi)-\rho P_{-(n-3)}(\rho,\varrho,\varpi)-\varrho P_{-(n-2)}(\rho,}{\varpi^{2}}\\ &= [\frac{\varrho^{2}-\rho\varpi}{\varpi^{2}}]P_{-n+2}(\rho,\varrho,\varpi)+[\frac{\varpi+\rho\varrho}{\varpi^{2}}]P_{-n+3}(\rho,\varrho,\varpi)-[\frac{\varrho}{\varpi^{2}}]T_{-n+4}(\rho,\varrho,\varpi)\end{aligned}$$

2) Let Q matrix

$$Q = \begin{pmatrix} \rho & \varrho & \varpi \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that with $det(Q) = w$, *and*

$$Q^n = \begin{pmatrix} P_{n+1}(\rho,\varrho,\varpi) & \varrho P_n(\rho,\varrho,\varpi) + \varpi P_{n-1}(\rho,\varrho,\varpi) & \varpi P_n(\rho,\varrho,\varpi) \\ P_n(\rho,\varrho,\varpi) & \varrho P_{n-1}(\rho,\varrho,\varpi) + \varpi P_{n-2}(\rho,\varrho,\varpi) & \varpi P_{n-1}(\rho,\varrho,\varpi) \\ P_{n-1}(\rho,\varrho,\varpi) & \varrho P_{n-2}(\rho,\varrho,\varpi) + \varpi P_{n-3}(\rho,\varrho,\varpi) & \varpi P_{n-2}(\rho,\varrho,\varpi) \end{pmatrix}$$

since $Q^{-n} = (Q^n)^{-1}$, *than*

$$Q^{-n} = \begin{pmatrix} P_{-n+1}(\rho,\varrho,\varpi) & \varrho P_{-n}(\rho,\varrho,\varpi) + \varpi P_{-n-1}(\rho,\varrho,\varpi) & \varpi P_{-n}(\rho,\varrho,\varpi) \\ P_{-n}(\rho,\varrho,\varpi) & \varrho P_{-n-1}(\rho,\varrho,\varpi) + \varpi P_{-n-2}(\rho,\varrho,\varpi) & \varpi P_{-n-1}(\rho,\varrho,\varpi) \\ P_{-n-1}(\rho,\varrho,\varpi) & \varrho P_{-n-2}(\rho,\varrho,\varpi) + \varpi P_{-n-3}(\rho,\varrho,\varpi) & \varpi P_{-n-2}(\rho,\varrho,\varpi) \end{pmatrix}$$

and

$$Q^{-n} = \frac{1}{w^n} \begin{pmatrix} w^2\left[(P_{n-2}(Y))^2 - P_{n-1}(Y)P_{n-3}(Y)\right] & w^2\left[P_n(Y)P_{n-3}(Y) - P_{n-1}(Y)P_{n-2}(Y)\right] & w^2\left[(P_{n-1}(Y))^2 - P_n(Y)P_{n-2}(Y)\right] \\ w\left[(P_{n-1}(Y))^2 - P_n(Y)P_{n-2}(Y)\right] & w\left[P_{n+1}(Y)P_{n-2}(Y) - P_n(Y)P_{n-1}(Y)\right] & w\left[(P_n(Y))^2 - P_{n+1}(Y)P_{n-1}(Y)\right] \\ \left[(P_n(Y))^2 - P_{n+1}(Y)P_{n-1}(Y)\right] & \left[P_{n+2}(Y)P_{n-1}(Y) - P_{n+1}(Y)P_n(Y)\right] & \left[P_{n+1}(Y)\right]^2 - P_{n+2}(Y)P_n(Y) \end{pmatrix}$$

where $(Y) = (\rho,\varrho,\varpi)$

Finally

$$P_{-n}(\rho,\varrho,\varpi) = \frac{[P_{n-1}(\rho,\varrho,\varpi)]^2 - P_n(\rho,\varrho,\varpi)P_{n-2}(\rho,\varrho,\varpi)}{w^{n-1}}$$

■

3.3 Partial Derivative

Theorem 3.6 *[7] For $n \geq 4$, we have*

$$1)\frac{\partial P_n(\rho,\varrho,\varpi)}{\partial \rho} = \sum_{k=1}^{n-1} P_k(\rho,\varrho,\varpi)P_{n-k}(\rho,\varrho,\varpi)$$

$$2)\frac{\partial P_n(\rho,\varrho,\varpi)}{\partial \varrho} = \sum_{k=1}^{n-2} P_k(\rho,\varrho,\varpi)P_{n-k-1}(\rho,\varrho,\varpi)$$

$$3)\frac{\partial P_n(\rho,\varrho,\varpi)}{\partial \varpi} = \sum_{k=2}^{n-2} P_{k-1}(\rho,\varrho,\varpi)P_{n-k-1}(\rho,\varrho,\varpi)$$

Proof *Taking the derivative of Equation Eq. (1.16) with respect to ρ, we get*

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{\partial P_n(\rho,\varrho,\varpi)}{\partial \rho} t^n &= \left(\frac{t}{1-\rho t-\varrho t^2-\varpi t^3}\right)^2 \\ &= \left[\sum_{n=0}^{\infty} P_n(\rho,\varrho,\varpi)t^n\right]^2 \\ &= \sum_{n=0}^{\infty}\left[\sum_{i=0}^{n} P_k(\rho,\varrho,\varpi)P_{n-k}(\rho,\varrho,\varpi)\right]t^n\end{aligned}$$

Equating the coefficients of t^n, we get

$$\begin{aligned}\frac{\partial P_n(\rho,\varrho,\varpi)}{\partial \rho} &= \sum_{k=0}^{n} P_k(\rho,\varrho,\varpi)P_{n-k}(\rho,\varrho,\varpi) \\ \frac{\partial P_n(\rho,\varrho,\varpi)}{\partial \rho} &= \sum_{i=1}^{n-1} P_k(\rho,\varrho,\varpi)P_{n-k}(\rho,\varrho,\varpi)\end{aligned}$$

Since $P_0(\rho,\varrho,\varpi) = 0$

2) *Taking the derivative of Equation Eq. (1.16) with respect to ϱ, we get*

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{\partial P_n(\rho,\varrho,\varpi)}{\partial s} t^n &= \frac{t^3}{(1-\rho t-\varrho t^2-\varpi t^3)^2} \\ &= \left(\frac{t}{1-\rho t-\varrho t^2-\varpi t^3}\right)\left(\frac{t^2}{1-\rho t-\varrho t^2-\varpi t^3}\right) \\ &= \left[\sum_{n=0}^{\infty} P_n(\rho,\varrho,\varpi)t^n\right]\left[\sum_{n=0}^{\infty} P_{n-1}(\rho,\varrho,\varpi)t^n\right] \\ &= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} P_k(\rho,\varrho,\varpi)P_{n-k-1}(\rho,\varrho,\varpi)\right]t^n\end{aligned}$$

Since $P_0(\rho,\varrho,\varpi)=P_{-1}(\rho,\varrho,\varpi)=0$. Equating the coefficients of t^n, we get

$$\frac{\partial P_n(\rho,\varrho,\varpi)}{\partial s}=\sum_{k=1}^{n-2}P_k(\rho,\varrho,\varpi)P_{n-k-1}(\rho,\varrho,\varpi)$$

3) *Taking the derivative of Equation Eq. (1.16) with respect to ϖ, we get*

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\partial P_n(\rho,\varrho,\varpi)}{\partial w}t^n &= \frac{t^4}{\left(1-\rho t-\varrho t^2-\varpi t^3\right)^2}\\ &= \left[\sum_{n=0}^{\infty}P_{n-1}(\rho,\varrho,\varpi)t^n\right]^2\\ &= \sum_{n=0}^{\infty}\left[\sum_{k=2}^{n-2}P_{k-1}(\rho,\varrho,\varpi)P_{n-k-1}(\rho,\varrho,\varpi)\right]t^n\end{aligned}$$

■

Remark 3.7 *For $n\geq 4$, we have*

$$\frac{\partial P_n(\rho,\varrho,\varpi)}{\partial \rho}=\frac{\partial P_{n+1}(\rho,\varrho,\varpi)}{\partial \varrho}=\frac{\partial P_{n+2}(\rho,\varrho,\varpi)}{\partial \varpi}.$$

.

Theorem 3.8 *[18] Let $P_n(\rho,\varrho,\varpi)$ and $L_n(\rho,\varrho,\varpi)$ be n-th generalized tribonacci like polynomials. Then*

$$r\frac{\partial L_n(\rho,\varrho,\varpi)}{\partial \rho}+\varrho\frac{\partial L_n(\rho,\varrho,\varpi)}{\partial \varrho}+\varpi\frac{\partial L_n(\rho,\varrho,\varpi)}{\partial \varpi}=nP_{n+1}(\rho,\varrho,\varpi).$$

Proof *Using partial derivations of the $L_n(\rho,\varrho,\varpi)$, we obtain*

$$\begin{aligned}\frac{\partial L_n(\rho,\varrho,\varpi)}{\partial x} &= \frac{\partial}{\partial \rho}\left(\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}\sum_{j=0}^{i}\frac{n}{n-i-j}\binom{i}{j}\binom{n-i-j}{i}\rho^{n-2i-j}\varrho^{i-j}\varpi^j\right)\\ &= \sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}\sum_{j=0}^{i}\frac{n}{n-i-j}(n-2i-j)\binom{i}{j}\binom{n-i-j}{i}\rho^{n-2i-j-1}\varrho^{i-j}\varpi^j\\ &= n\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}\sum_{j=0}^{i}\binom{i}{j}\binom{n-i-j-1}{i}\rho^{n-2i-j-1}\varrho^{i-j}\varpi^j\\ &= nP_n(\rho,\varrho,\varpi).\end{aligned}$$

Similarly, we get

$$\frac{\partial L_n(\rho,\varrho,\varpi)}{\partial y} = nP_{n-1}(\rho,\varrho,\varpi)$$

■

Proposition 3.5 *For* $n \geq 1$ *. the partial derivative of* (∂L_n) *with respect to those variables, we obtain*

1) $\dfrac{\partial L_n(\rho,\varrho,\varpi)}{\partial \rho} = \dfrac{\partial P_{n+1}(\rho,\varrho,\varpi)}{\partial \rho} + \varrho\dfrac{\partial P_n(\rho,\varrho,\varpi)}{\partial \varrho} + 2\varpi\dfrac{\partial P_n(\rho,\varrho,\varpi)}{\partial \varpi}$

2) $\dfrac{\partial L_n(\rho,\varrho,\varpi)}{\partial \varrho} = 2\dfrac{\partial P_n(\rho,\varrho,\varpi)}{\partial \rho} - \rho\dfrac{\partial P_n(\rho,\varrho,\varpi)}{\partial \varrho} + \varpi\dfrac{\partial P_{n-1}(\rho,\varrho,\varpi)}{\partial \varpi}$

3) $\dfrac{\partial L_n(\rho,\varrho,\varpi)}{\partial \varpi} = 3\dfrac{\partial P_n(\rho,\varrho,\varpi)}{\partial \varrho} - 2\rho\dfrac{\partial P_n(\rho,\varrho,\varpi)}{\partial \varpi} - \varrho\dfrac{\partial P_{n-1}(\rho,\varrho,\varpi)}{\partial \varpi}$

Proof 1) *Differentiating Eq (1.17) with respect to r,*

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\partial L_n(\rho,\varrho,\varpi)}{\partial \rho}t^n &= \frac{t+\varrho t^3+2\varpi t^4}{\left(1-\rho t-\varrho t^2-\varpi t^3\right)^2}\\ &= \sum_{n=0}^{\infty}\left(\frac{\partial P_{n+1}(\rho,\varrho,\varpi)}{\partial \rho} + \varrho\frac{\partial P_n(\rho,\varrho,\varpi)}{\partial \varrho} + 2\varpi\frac{\partial P_n(\rho,\varrho,\varpi)}{\partial w}\right] t^n\end{aligned}$$

Equating the coefficients of t^n*, we get*

$$\frac{\partial L_n(\rho,\varrho,\varpi)}{\partial r} = \frac{\partial P_{n+1}(\rho,\varrho,\varpi)}{\partial \rho} + \varrho\frac{\partial P_n(\rho,\varrho,\varpi)}{\partial \varrho} + 2\varpi\frac{\partial P_n(\rho,\varrho,\varpi)}{\partial \varpi}$$

(2) *and* (3) *By using the same technic* ■

3.4 Sums of usual tribonacci like polynomials

Let $S_n(\rho,\varrho,\varpi)$ *denote the sum of usuel Tribonacci like polynomial which defined by*

$$S_n(\rho,\varrho,\varpi) = \sum_{k=0}^{n} P_n(\rho,\varrho,\varpi)$$

The first few term

$S_0(\rho, \varrho, \varpi) = 0$

$S_1(\rho, \varrho, \varpi) = 1$

$S_2(\rho, \varrho, \varpi) = 1 + \rho$

$S_3(\rho, \varrho, \varpi) = \rho^2 + \rho + \varrho + 1$

$S_4(\rho, \varrho, \varpi) = \rho^3 + \rho^2 + \rho + \varrho + 2\rho\varrho + \varpi + 1$

$S_5(\rho, \varrho, \varpi) = \rho^4 + \rho^3 + \rho^2 + \varrho^2 + 3\rho^2\varrho + \rho + \varrho + 2\rho\varrho + 2\rho\varpi + \varpi + 1$

$S_6(\rho, \varrho, \varpi) = \rho^5 + \rho^4 + \rho^3 + \rho^2 + \varrho^2 + 4\rho^3\varrho + 3\rho^2\varrho + 3\rho\varrho^2 + 3\rho^2\varpi + 2\rho\varrho + 2\rho\varpi + 2\varrho\varpi + \rho + \varrho + \varpi + 1$

Lemma 2 *For $n \geq 3$,*

$$S_n(\rho, \varrho, \varpi) = 1 + \rho S_{n-1}(\rho, \varrho, \varpi) + \varrho S_{n-2}(\rho, \varrho, \varpi) + \varpi S_{n-3}(\rho, \varrho, \varpi) \tag{3.6}$$

Proof *Induction on n.* ■.

Considering the matrix F, and H_n as shown:

$$F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \rho & \varrho & \varpi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, H_n = \begin{bmatrix} 1 & 0 & 0 & 0 \\ S_n(\rho, \varrho, \varpi) & P_{n+1}(\rho, \varrho, \varpi) & P_{n+2}(\rho, \varrho, \varpi) - \rho P_{n+1}(\rho, \varrho, \varpi) & \varpi P_n(\rho, \varrho, \varpi \\ S_{n-1}(\rho, \varrho, \varpi) & P_n(\rho, \varrho, \varpi) & P_{n+1}(\rho, \varrho, \varpi) - \rho P_n(\rho, \varrho, \varpi) & \varpi P_{n-1}(\rho, \varrho, \\ S_{n-2}(\rho, \varrho, \varpi) & P_{n-1}(\rho, \varrho, \varpi) & P_n(\rho, \varrho, \varpi) - \rho P_{n-1}(\rho, \varrho, \varpi) & \varpi P_{n-2}(\rho, \varrho, \end{bmatrix}$$

Corollary 3.6 *For $n \geq 2$, we have $F^n = H_n$*

Proof *Induction on n, with using lemma (2),* ■

Corollary 3.7 *For $n \geq 0$ and $m \geq 3$, we have*

$S_{n+m}(\rho, \varrho, \varpi) = S_n(\rho, \varrho, \varpi) + P_{n+1}(\rho, \varrho, \varpi)S_m(\rho, \varrho, \varpi) + [P_{n+2}(\rho, \varrho, \varpi) - \rho P_{n+1}(\rho, \varrho, \varpi)]S_{m-1}(\rho, \varrho, \varpi) + \varpi P_n(\rho,$

Proof *From the identity $F^{k+l} = F^k F^l$ and Corollary 3.6, the result is clear* ■

Theorem 3.9 *For $n \geq 2$, we have*

$$S_n(\rho, \varrho, \varpi) = \frac{P_{n+2}(\rho, \varrho, \varpi) + (1 - \rho)P_{n+1}(\rho, \varrho, \varpi) + \varpi P_n(\rho, \varrho, \varpi) - 1}{\rho + \varrho + \varpi - 1}$$

Proof *Using the diagonalization of the generating matrix F,*

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & \rho & \varrho & \varpi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

So the characteristic equation of the generating matrix F is $(1-t)(-t^3+\rho t^2+\varrho t+\varpi)$ Let κ, ϑ and η be the roots of the characteristic equation $-t^3+\rho t^2+\varrho t+\varpi=0$,
So the matrix M defined by

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \left(\dfrac{1}{1-\rho-\varrho-\varpi}\right) & \kappa^2 & \vartheta^2 & \eta^2 \\ \left(\dfrac{1}{1-\rho-\varrho-\varpi}\right) & \kappa & \vartheta & \eta \\ \left(\dfrac{1}{1-\rho-\varrho-\varpi}\right) & 1 & 1 & 1 \end{pmatrix} \text{ and } N = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \kappa & 0 & 0 \\ 0 & 0 & \vartheta & 0 \\ 0 & 0 & 0 & \eta \end{pmatrix}$$

So $AM = MN$ with. Then $A^nM = MN^n$. By Corollary 3.6 we write $H_nM = MN^n$. We get

$$S_n(\rho,\varrho,\varpi) + \frac{P_{n+1}(\rho,\varrho,\varpi)+P_{n+2}(\rho,\varrho,\varpi)-\rho P_{n+1}(\rho,\varrho,\varpi)+\varpi P_n(\rho,\varrho,\varpi)}{1-\rho-\varrho-\varpi} = \frac{1}{1-\rho-\varrho-\varpi}$$

Finally

$$S_n(\rho,\varrho,\varpi) = \frac{P_{n+2}(\rho,\varrho,\varpi)+(1-\rho)P_{n+1}(\rho,\varrho,\varpi)+\varpi P_n(\rho,\varrho,\varpi)-1}{\rho+\varrho+\varpi-1}$$

■

Generating matrix of the sums of usual tribonacci like polynomials

Corollary 3.8 *[7] The sequence $S_n(\rho,\varrho,\varpi)$ satisfies the following recursion, for $n > 4$*

$$\begin{cases} S_n(\rho,\varrho,\varpi) = (1+\rho)S_{n-1}(\rho,\varrho,\varpi) + (\varrho-\rho)S_{n-2}(\rho,\varrho,\varpi) + (\varpi-\varrho)S_{n-3}(\rho,\varrho,\varpi) - \varpi S_{n-4}(\rho,\varrho,\varpi) \\ S_0(\rho,\varrho,\varpi) = 0 \\ S_1(\rho,\varrho,\varpi) = 1 \\ S_2(\rho,\varrho,\varpi) = 1+\rho. \\ S_3(\rho,\varrho,\varpi) = \rho^2+\rho+\varrho+1. \end{cases}$$

Then the characteristic polynomials of the rucurrance relation (3.8) *given by*

$$t^4 - (1+\rho)t^3 - (\varrho - \rho)t^2 - (\varpi - \varrho)t + \varpi = 0 \tag{3.7}$$

Since $t^4 - (1+\rho)t^3 - (\varrho - \rho)t^2 - (\varpi - \varrho)t + \varpi = (t-1)(t^3 - \rho t^2 - \varrho t - \varpi)$.
Where $1, \kappa$, ϑ *and* η *the roots of the characteristic polynomials*

The Generating matrix of the sums of usual tribonacci like polynomials defined by

$$L = \begin{pmatrix} 1+\rho & \varrho - \rho & \varpi - \rho & -\varpi \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Proposition 3.9 *For* $n \geq 3$, *we have*

$$(L)^n = \begin{pmatrix} S_{n+1}(Y) & -S_{n+1}(Y) + \varrho S_n(Y) + \varpi S_{n-1}(Y) + 1 & -S_{n+2}(Y) + \rho S_{n+1}(Y) + \varpi S_n(Y) + 1 \\ S_n(Y) & -S_n(Y) + \varrho S_{n-1}(Y) + \varpi S_{n-2}(Y) + 1 & -S_{n+1}(Y) + \rho S_n(Y) + \varpi S_{n-1}(Y) + 1 \\ S_{n-1}(Y) & -S_{n-1}(Y) + \varrho S_{n-2}(Y) + \varpi S_{n-3}(Y) + 1 & -S_n(Y) + \rho S_{n-1}(Y) + \varpi S_{n-2}(Y) + 1 \\ S_{n-2}(Y) & -S_{n-2}(Y) + \varrho S_{n-3}(Y) + \varpi S_{n-4}(Y) + 1 & -S_{n-1}(Y) + \rho S_{n-2}(Y) + \varpi S_{n-3}(Y) + 1 \end{pmatrix}$$

where $S_n(Y) = S_n(\rho, \varrho, \varpi)$

Proof *It can be proved by indection on* n, ■

Theorem 3.10 *[7] forn* ≥ 3, *we have*

$$S_n(\rho, \varrho, \varpi) = \frac{\kappa^{n+2}}{(\kappa - 1)(\kappa - \vartheta)(\kappa - \eta)} + \frac{\vartheta^{n+2}}{(\vartheta - 1)(\vartheta - \kappa)(\vartheta - \eta)} + \frac{\eta^{n+2}}{(\eta - 1)(\eta - \kappa)(\eta - \vartheta)} \tag{3.8}$$

where $1, \kappa$, ϑ *and* η *be the roots of the characteristic equation*

$$t^4 - (1+\rho)t^3 - (\varrho - \rho)t^2 - (\varpi - \varrho)t + \varpi = 0$$

Proof

The characteristic equation of the generating matrix L *is*

$$0 = |L - tI_3| = \begin{vmatrix} 1+\rho-t & \varrho - \rho & \varpi - \rho & -\varpi \\ 1 & -t & 0 & 0 \\ 0 & 1 & -t & 0 \\ 0 & 0 & 1 & -t \end{vmatrix} = t^4 - (1+\rho)t^3 - (\varrho - \rho)t^2 - (\varpi - \varrho)t + \varpi$$

and $1, \kappa$, ϑ and η be the roots of this polynomials.
Let M is a matrix of eigenvalues, we obtain

$$M = \begin{pmatrix} \kappa^3 & \vartheta^3 & \eta^3 & 1 \\ \kappa^2 & \vartheta^2 & \eta^2 & 1 \\ \kappa & \vartheta & \eta & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$M^{-1} = \frac{1}{\lambda} \begin{pmatrix} [(\vartheta-1)(\eta-1)(\vartheta-\eta)] & -\begin{vmatrix} \vartheta^3 & \eta^3 & 1 \\ \vartheta & \eta & 1 \\ 1 & 1 & 1 \end{vmatrix} & \begin{vmatrix} \vartheta^3 & \eta^3 & 1 \\ \vartheta^2 & \eta^2 & 1 \\ 1 & 1 & 1 \end{vmatrix} & -\begin{vmatrix} \vartheta^3 & \eta^3 & 1 \\ \vartheta^2 & \eta^2 & 1 \\ \vartheta & \eta & 1 \end{vmatrix} \\ -[(\kappa-1)(\eta-1)(\kappa-\eta)] & \begin{vmatrix} \kappa^3 & \eta^3 & 1 \\ \kappa & \eta & 1 \\ 1 & 1 & 1 \end{vmatrix} & -\begin{vmatrix} \kappa^3 & \eta^3 & 1 \\ \kappa^2 & \eta^2 & 1 \\ 1 & 1 & 1 \end{vmatrix} & \begin{vmatrix} \kappa^3 & \eta^3 & 1 \\ \kappa^2 & \eta^2 & 1 \\ \kappa & \eta & 1 \end{vmatrix} \\ (\kappa-1)(\vartheta-1)(\kappa-\vartheta) & -\begin{vmatrix} \kappa^3 & \vartheta^3 & 1 \\ \kappa & \vartheta & 1 \\ 1 & 1 & 1 \end{vmatrix} & \begin{vmatrix} \kappa^3 & \vartheta^3 & 1 \\ \kappa^2 & \vartheta^2 & 1 \\ 1 & 1 & 1 \end{vmatrix} & -\begin{vmatrix} \kappa^3 & \vartheta^3 & 1 \\ \kappa^2 & \vartheta^2 & 1 \\ \kappa & \vartheta & 1 \end{vmatrix} \\ -[(\kappa-\vartheta)(\kappa-\eta)(\vartheta-\eta)] & \begin{vmatrix} \kappa^3 & \vartheta^3 & \eta^3 \\ \kappa & \vartheta & \eta \\ 1 & 1 & 1 \end{vmatrix} & -\begin{vmatrix} \kappa^3 & \vartheta^3 & \eta^3 \\ \kappa^2 & \vartheta^2 & \eta^2 \\ 1 & 1 & 1 \end{vmatrix} & \begin{vmatrix} \kappa^3 & \vartheta^3 & \eta^3 \\ \kappa^2 & \vartheta^2 & \eta^2 \\ \kappa & \vartheta & \eta \end{vmatrix} \end{pmatrix}$$

where $\lambda = (\kappa-1)(\eta-1)(\vartheta-1)(\kappa-\vartheta)(\kappa-\eta)(\vartheta-\mathfrak{C})$
And, let H is the diagonal matrix

$$D = \begin{pmatrix} \kappa & 0 & 0 & 0 \\ 0 & \vartheta & 0 & 0 \\ 0 & 0 & \eta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then $L = MHM^{-1}$, so we obtain $L^n = MH^nM^{-1}$. Then

$$S_n(\rho,\varrho,\varpi)=\frac{\kappa^{n+2}}{(\kappa-1)(\kappa-\vartheta)(\kappa-\eta)}+\frac{\vartheta^{n+2}}{(\vartheta-1)(\vartheta-\kappa)(\vartheta-\eta)}+\frac{\eta^{n+2}}{(\eta-1)(\eta-\kappa)(\eta-\vartheta)}$$

■

Proposition 3.10 *For $n \geq 3$, we have*

$$\begin{aligned}P_{n+k}(\rho,\varrho,\varpi)&=P_{k-1}(\rho,\varrho,\varpi)P_{n+2}(\rho,\varrho,\varpi)+\left[P_k(\rho,\varrho,\varpi)-\rho P_{k-1}(\rho,\varrho,\varpi)\right]P_{n+1}(\rho,\varrho,\varpi)+\\&+\varpi P_{k-2}(\rho,\varrho,\varpi)P_n(\rho,\varrho,\varpi)\\S_{n+k}(\rho,\varrho,\varpi)&=S_{k-2}(\rho,\varrho,\varpi)S_{n+3}(\rho,\varrho,\varpi)+\left[S_{k-1}(\rho,\varrho,\varpi)-(1+\rho)S_{k-2}(\rho,\varrho,\varpi)\right]S_{n+2}(\rho,\varrho,\varpi)+\\&+\left[(\varpi-\varrho)S_{k-3}(\rho,\varrho,\varpi)-\varpi S_{k-4}(\rho,\varrho,\varpi)\right]S_{n+1}(\rho,\varrho,\varpi)+[-\varpi S_{k-3}(\rho,\varrho,\varpi)]S_n(\rho,\varrho,\varpi)\end{aligned}$$

Proof *by indection on n* ■

Examples

1) $P_{n+4}(\rho,\varrho,\varpi)=(\rho^2+\varrho)P_{n+2}(\rho,\varrho,\varpi)+(\rho\varrho+\varpi)P_{n+1}(\rho,\varrho,\varpi)+(\rho\varpi)P_n(\rho,\varrho,\varpi)$

$P_{n+5}(\rho,\varrho,\varpi)=[r^3+2rs+w]T_{n+2}(\rho,\varrho,\varpi)+[s^2+r^2s+rw]T_{n+1}(\rho,\varrho,\varpi)+[\varpi\rho^2+\varpi\varrho]P_n(\rho,\varrho,\varpi)$

$\vdots \quad \vdots \quad \vdots$

2) $S_{n+4}(\rho,\varrho,\varpi)=(\rho+1)S_{n+3}(\rho,\varrho,\varpi)+(\varrho-\rho)S_{n+2}(\rho,\varrho,\varpi)+(\varpi-\varrho)S_{n+1}(\rho,\varrho,\varpi)+(-\varpi)S(n)(\rho,\varrho,\varpi)$

$$\begin{aligned}S_{n+5}(\rho,\varrho,\varpi)&=(\rho^2+\rho+\varrho+1)S_{n+3}(\rho,\varrho,\varpi)+(\varrho+\rho\varrho-\rho-\rho^2+\varpi-\varrho)S_{n+2}(\rho,\varrho,\varpi)+\\&+(-\varpi\rho-\varpi)S(n)(\rho,\varrho,\varpi)+\left([\varpi-\varrho][r^2+\rho+\varrho+1]-\varpi\right)S_{n+1}(\rho,\varrho,\varpi)\end{aligned}$$

$\vdots \quad \vdots \quad \vdots$

CHAPTER

4. Tribonacci and tribonacci-Lucas Quaternion like polynomials

Quaternions were introduced by the Irish mathematician William Rowan Hamilton in 1843. So in this chapter we are treated the tribonacci and tribonacci-Lucas Quaternion like polynomials and we proved some results that are simulate to those in chapter 3, and develop some formula for theirs sums Furthermore, we provide the explicit formula for the partial derivative of these polynomials.

4.1 Some properties

Definition 4.1 *For $n \geq 0$, the tribonacci and tribonacci-Lucas quaternion like polynomials are defined by*

$$\begin{cases} Q_{P,n}(\rho,\varrho,\varpi) = P_n(\rho,\varrho,\varpi) + P_{n+1}(\rho,\varrho,\varpi)\boldsymbol{i} + P_{n+2}(\rho,\varrho,\varpi)\boldsymbol{j} + P_{n+3}(\rho,\varrho,\varpi)\boldsymbol{k}, n \geq 0 \\ Q_{P,0}(\rho,\varrho,\varpi) = \boldsymbol{i} + \rho\boldsymbol{j} + (\rho^2+\varrho)\boldsymbol{k} \\ Q_{P,1}(\rho,\varrho,\varpi) = 1 + \rho\boldsymbol{i} + (\rho^2+\varrho)\boldsymbol{j} + (\rho^3+2\rho\varrho+\varpi)\boldsymbol{k} \\ Q_{P,2}(\rho,\varrho,\varpi) = \rho + (\rho^2+\varrho)\boldsymbol{i} + (\rho^3+2\rho\varrho+\varpi)\boldsymbol{j} + (\rho^4+3\rho^2\varrho+2\rho\varpi+\varrho^2)\boldsymbol{k} \end{cases} \tag{4.1}$$

and

$$\begin{cases} Q_{L,n}(\rho,\varrho,\varpi) = L_n(\rho,\varrho,\varpi) + L_{n+1}(\rho,\varrho,\varpi)\boldsymbol{i} + L_{n+2}(\rho,\varrho,\varpi)\boldsymbol{j} + L_{n+3}(\rho,\varrho,\varpi)\boldsymbol{k}, n \geq 0 \\ Q_{L,0}(\rho,\varrho,\varpi) = 3 + \rho\boldsymbol{i} + (\rho^2+\varrho)\boldsymbol{j} + (\rho^3+3\rho\varrho+3\varpi)\boldsymbol{k} \\ Q_{L,1}(\rho,\varrho,\varpi) = \rho + (\rho^2+\varrho)\boldsymbol{i} + (\rho^3+3\rho\varrho+3\varpi)\boldsymbol{j} + (\rho^4+4\rho^2\varrho+4\rho\varpi+2\varrho^2)\boldsymbol{k} \\ Q_{L,2}(\rho,\varrho,\varpi) = (\rho^2+\varrho) + (\rho^3+3\rho\varrho+3\varpi)\boldsymbol{i} + (\rho^4+4\rho^2\varrho+4\rho\varpi+2\varrho^2)\boldsymbol{j} + (\rho^5+5\rho^3\varrho+5\rho\varrho^2+5\rho^2\varpi+5\varrho\varpi \end{cases} \tag{4.2}$$

where i, j, k are quaternionic units which satisfy the multiplication rules (1)

Proposition 4.1 *For $n \geq 0$, the following identities hold*

$$\begin{aligned} (i)\ \ & Q_{P,n+3}(\rho,\varrho,\varpi) = \rho Q_{P,n+2}(\rho,\varrho,\varpi) + \varrho Q_{P,n+1}(\rho,\varrho,\varpi) + \varpi Q_{P,n}(\rho,\varrho,\varpi) \\ (ii).\ \ & Q_{L,n+3}(\rho,\varrho,\varpi) = \rho Q_{L,n+2}(\rho,\varrho,\varpi) + \varrho Q_{L,n+1}(\rho,\varrho,\varpi) + \varpi Q_{L,n}(\rho,\varrho,\varpi) \end{aligned} \tag{4.3}$$

Proof

i) From combining the equation (3.1) *with the definition* (4.1)*,we obtain*

$$\begin{aligned} \rho Q_{P,n+2}(\rho,\varrho,\varpi) + \varrho Q_{P,n+1}(\rho,\varrho,\varpi) + \varpi Q_{P,n}(\rho,\varrho,\varpi) &= \rho P_{n+2}(\rho,\varrho,\varpi) + \rho P_{n+3}(\rho,\varrho,\varpi)\boldsymbol{i} + \rho P_{n+4}(\rho,\varrho,\varpi)\boldsymbol{j} \\ &+ \rho P_{n+5}(\rho,\varrho,\varpi)\boldsymbol{k} + \varrho P_{n+1}(r,s,w) + \varrho P_{n+2}(\rho,\varrho,\varpi)\boldsymbol{i} + \\ &\varrho P_{n+3}(\rho,\varrho,\varpi)\boldsymbol{j} + \varrho P_{n+4}(\rho,\varrho,\varpi)\boldsymbol{k} + \varpi P_{n}(\rho,\varrho,\varpi) + \\ &+ \varpi P_{n+1}(\rho,\varrho,\varpi)\boldsymbol{i} + \varpi P_{n+2}(\rho,\varrho,\varpi)\boldsymbol{j} + \varpi P_{n+3}(\rho,\varrho,\varpi \\ &= P_{n+3}(\rho,\varrho,\varpi) + P_{n+4}(\rho,\varrho,\varpi)\boldsymbol{i} + P_{n+5}(\rho,\varrho,\varpi)\boldsymbol{j} + \\ &+ P_{n+6}(\rho,\varrho,\varpi)\boldsymbol{k} \\ &= Q_{P,n+3}(\rho,\varrho,\varpi) \end{aligned}$$

ii) Similarly, we obtain

$$Q_{L,n+3}(\rho,\varrho,\varpi) = \rho Q_{L,n+2}(\rho,\varrho,\varpi) + \varrho Q_{L,n+1}(\rho,\varrho,\varpi) + \varpi Q_{L,n}(\rho,\varrho,\varpi)$$

■

4.2 Generating functions

Theorem 4.1 *For $n \geq 1$, we have*

$$1)\ \ H_T(t) = \sum_{n=0}^{\infty} Q_{P,n}(\rho,\varrho,\varpi)t^n = \frac{t + \mathbf{i} + (\rho + \varrho t + \varpi t^2)\,\mathbf{j} + (\rho^2 + \varrho + \rho\varrho t + \varpi t + \rho\varpi t^2)\,\mathbf{k}}{1 - \rho t - \varrho t^2 - \varpi t^3}$$

$$2) K_L(t) = \sum_{n=0}^{\infty} Q_{L,n}(\rho,\varrho,\varpi)t^n = \frac{3 - 2\rho t - 2\varrho t^2 + (\rho + \varrho t + (\rho\varrho + 3\varpi)t^2)\,\mathbf{i} + ((\rho\varpi + \varrho^2)t^2 + (2\rho\varrho + 3\varpi)t + \rho^2 + \varrho}{1 - \rho t - \varrho t^2 - \varpi t^3}$$
$$+ \frac{+(\varpi\rho^2 + 2\varrho\varpi)t^2 + (\varrho\rho^2 + \rho\varpi + 2\varrho^2)t + \rho^3 + 3\rho\varrho + 3\varpi)\mathbf{k}}{1 - \rho t - \varrho t^2 - \varpi t^3}$$

Proof 1) *Let* $H_T(t)=\sum_{n=0}^{\infty}Q_{P,n}(\rho,\varrho,\varpi)t^n$ *and* $K_L(t)=\sum_{n=0}^{\infty}Q_{L,n}(\rho,\varrho,\varpi)t^n$. *Then we get the following equation*

$$\begin{aligned}(1-\rho t-\varrho t^2-\varpi t^3)\,H_T(t)&=Q_{P,0}(Y)+(Q_{P,1}(Y)-\rho Q_{P,0}(Y))\,t+(Q_{P,2}(Y)-\rho Q_{P,1}(Y)-\varrho Q_{P,0}(Y))\,t^2\\&+\textstyle\sum_{n=3}^{\infty}(Q_{P,n}(Y)-\rho Q_{P,n-1}(Y)-\varrho Q_{P,n-2}(Y)-\varpi Q_{P,n-3}(Y))\,t^n\end{aligned}$$

since, for each $n\geq 3$, *the coefficient of* t^n *is zero in the right-hand side of this equation, we obtain*

$$\begin{aligned}H_P(t)&=\frac{Q_{P,0}(\rho,\varrho,\varpi)+(Q_{P,1}(\rho,\varrho,\varpi)-\rho Q_{P,0}(\rho,\varrho,\varpi))\,t+(Q_{P,2}(\rho,\varrho,\varpi)-\rho Q_{P,1}(r,s,w)-\varrho Q_{P,0}(\rho,\varrho,\varpi))}{1-\rho t-\varrho t^2-\varpi t^3}\\&=\frac{t+\mathbf{i}+(\rho+\varrho t+\varpi t^2)\,\mathbf{j}+(\rho^2+\varrho+\rho\varrho t+\varpi t+\rho\varpi t^2)\,\mathbf{k}}{1-\rho t-\varrho t^2-\varpi t^3}\end{aligned}$$

2) *Similarly, we get*

$$\begin{aligned}K_L(t)&=\frac{3-2\rho t-2\varrho t^2+(\rho+\varrho t+(\rho\varrho+3\varpi)t^2)\,\mathbf{i}+((\rho\varpi+\varrho^2)t^2+(2\rho\varrho+3\varpi)t+\rho^2+\varrho)\,\mathbf{j}}{1-\rho t-\varrho t^2-\varpi t^3}\\&+\frac{+((\varpi\rho^2+2\varrho\varpi)t^2+(\varrho\rho^2+\rho\varpi+2\varrho^2)t+\rho^3+3\rho\varrho+3\varpi)\,\mathbf{k}}{1-\rho t-\varrho t^2-\varpi t^3}\end{aligned}$$

■

Proposition 4.2 *If* $n\geq 2$ *we have*

1) $Q_{P,n}(\rho,\varrho,\varpi)=Q_{P,1}(\rho,\varrho,\varpi)P_n(\rho,\varrho,\varpi)+(Q_{P,2}(\rho,\varrho,\varpi)-\rho Q_{P,1}(\rho,\varrho,\varpi))P_{n-1}(\rho,\varrho,\varpi)+$ $+\varpi Q_{P,0}(\rho,\varrho,\varpi)P_{n-2}(\rho,\varrho,\varpi)$

2) $Q_{L,n}(\rho,\varrho,\varpi)=Q_{L,1}(\rho,\varrho,\varpi)P_n(\rho,\varrho,\varpi)+(Q_{L,2}(\rho,\varrho,\varpi)-\rho Q_{L,1}(\rho,\varrho,\varpi))P_{n-1}(\rho,\varrho,\varpi)+$ $\varpi Q_{L,0}(\rho,\varrho,\varpi)P_{n-2}(\rho,\varrho,\varpi)$

Proof 1) *from the recurrence relation* (4.3). *We have*

$$\begin{pmatrix}Q_{P,3}(\rho,\varrho,\varpi)\\Q_{P,2}(\rho,\varrho,\varpi)\\Q_{P,1}(\rho,\varrho,\varpi)\end{pmatrix}=\begin{pmatrix}\rho&\varrho&\varpi\\1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}Q_{P,2}(\rho,\varrho,\varpi)\\Q_{P,1}(\rho,\varrho,\varpi)\\Q_{P,0}(\rho,\varrho,\varpi)\end{pmatrix}.$$

it is well known that

$$(Q)^n=\begin{pmatrix}\rho&\varrho&\varpi\\1&0&0\\0&1&0\end{pmatrix}^n=\begin{pmatrix}P_{n+1}(\rho,\varrho,\varpi)&P_{n+2}(\rho,\varrho,\varpi)-\rho P_{n+1}(\rho,\varrho,\varpi)&\varpi P_n(\rho,\varrho,\varpi)\\P_n(x)(\rho,\varrho,\varpi)&P_{n+1}(\rho,\varrho,\varpi)-\rho P_n(\rho,\varrho,\varpi)&\varpi P_{n-1}(\rho,\varrho,\varpi)\\P_{n-1}(\rho,\varrho,\varpi)&P_n(\rho,\varrho,\varpi)-\rho P_{n-1}(\rho,\varrho,\varpi)&\varpi P_{n-2}(\rho,\varrho,\varpi)\end{pmatrix}$$

which implies

$$\begin{pmatrix} Q_{P,n+2}(\rho,\varrho,\varpi) \\ Q_{P,n+1}(\rho,\varrho,\varpi) \\ Q_{P,n}(\rho,\varrho,\varpi) \end{pmatrix} = \begin{pmatrix} P_{n+1}(\rho,\varrho,\varpi) & P_{n+2}(\rho,\varrho,\varpi)-\rho P_{n+1}(\rho,\varrho,\varpi) & \varpi P_{n}(\rho,\varrho,\varpi) \\ P_{n}(\rho,\varrho,\varpi) & P_{n+1}(\rho,\varrho,\varpi)-\rho P_{n}(\rho,\varrho,\varpi) & \varpi P_{n-1}(\rho,\varrho,\varpi) \\ P_{n-1}(\rho,\varrho,\varpi) & P_{n}(\rho,\varrho,\varpi)-\rho P_{n-1}(\rho,\varrho,\varpi) & \varpi P_{n-2}(\rho,\varrho,\varpi) \end{pmatrix} \begin{pmatrix} Q_{P,2}(\rho,\varrho,\varpi \\ Q_{P,1}(\rho,\varrho,\varpi \\ Q_{P,0}(\rho,\varrho,\varpi \end{pmatrix}$$

Then

$$Q_{P,n}(\rho,\varrho,\varpi) = Q_{P,1}(\rho,\varrho,\varpi)P_n(\rho,\varrho,\varpi) + (Q_{P,2}(\rho,\varrho,\varpi) - \rho Q_{P,1}(\rho,\varrho,\varpi))P_{n-1}(\rho,\varrho,\varpi) + \\ + \varpi Q_{P,0}(\rho,\varrho,\varpi)P_{n-2}(\rho,\varrho,\varpi)$$

■

4.3 Binet's formulas

Corollary 4.3 *For $n \geq 0$, we have*

$$Q_{P,n}(\rho,\varrho,\varpi) = \frac{\underline{\kappa}\kappa^{n+1}}{(\kappa-\vartheta)(\kappa-\eta)} + \frac{\underline{\vartheta}\vartheta^{n+1}}{(\vartheta-\kappa)(\vartheta-\eta)} + \frac{\underline{\eta}\eta^{n+1}}{(\eta-\kappa)(\eta-\vartheta)} \tag{4.4}$$

and

$$Q_{L,n}(\rho,\varrho,\varpi) = \underline{\kappa}\kappa^n + \underline{\vartheta}\vartheta^n + \underline{\eta}\eta^n \tag{4.5}$$

where $\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}$ are quaternion units which satisfy the multiplication rules (1) and

$$\underline{\kappa} = 1 + \kappa\boldsymbol{i} + \kappa^2\boldsymbol{j} + \kappa\boldsymbol{k}$$
$$\underline{\vartheta} = 1 + \vartheta\boldsymbol{i} + \vartheta^2\boldsymbol{j} + \vartheta^3\boldsymbol{k}$$
$$\underline{\eta} = 1 + \eta\boldsymbol{i} + \eta^2\boldsymbol{j} + \eta^3\boldsymbol{k}$$

Proof *From the Binet's formula of the generalized tribonacci like polynomials, we have*

1) $$Q_{P,n}(\rho,\varrho,\varpi) = P_n(\rho,\varrho,\varpi) + P_{n+1}(\rho,\varrho,\varpi)\boldsymbol{i} + P_{n+2}(\rho,\varrho,\varpi)\boldsymbol{j} + P_{n+3}(\rho,\varrho,\varpi)\boldsymbol{k}$$
$$= \frac{\kappa^{n+1}+\boldsymbol{i}\kappa^{n+2}+\boldsymbol{j}\kappa^{n+3}+\boldsymbol{k}\kappa^{n+4}}{(\kappa-\vartheta)(\kappa-\eta)} + \frac{\vartheta^{n+1}+\boldsymbol{i}\vartheta^{n+2}+\boldsymbol{j}\vartheta^{n+3}+\boldsymbol{k}\vartheta^{n+4}}{(\vartheta-\kappa)(\vartheta-\eta)} + \\ + \frac{\eta^{n+1}+\boldsymbol{i}\eta^{n+2}+\boldsymbol{j}\eta^{n+3}+\boldsymbol{k}\eta^{n+4}}{(\eta-\kappa)(\eta-\vartheta)}$$
$$= \frac{\underline{\kappa}\kappa^{n+1}}{(\kappa-\vartheta)(\kappa-\eta)} + \frac{\underline{\vartheta}\vartheta^{n+1}}{(\vartheta-\kappa)(\vartheta-\eta)} + \frac{\underline{\eta}\eta^{n+1}}{(\eta-\kappa)(\eta-\vartheta)}$$

2) *Similarly, using the same technique* ■

4.4 Some proprieties of the sums

Theorem 4.2 *[16] For $n \geqslant 2$*

1) $Q_{P,3n-1}(\rho,\varrho,\varpi) = \sum_{k=0}^{n}\sum_{m=0}^{k}\begin{pmatrix} n \\ k \end{pmatrix}\begin{pmatrix} k \\ m \end{pmatrix}\rho^m\varrho^{k-m}\varpi^{n-k}Q_{P,k+m-1}(\rho,\varrho,\varpi)$

2) $Q_{L,3n}(\rho,\varrho,\varpi) = \sum_{k=0}^{n}\sum_{m=0}^{r}\begin{pmatrix} n \\ k \end{pmatrix}\begin{pmatrix} k \\ m \end{pmatrix}\rho^m\varrho^{k-m}\varpi^{n-k}Q_{L,k+m}(\rho,\varrho,\varpi)$

Proof *Let $\mathfrak{S}$ is root of the equation (1.18),*

$$\mathfrak{S}^3 = \rho\mathfrak{S}^2 + \varrho\mathfrak{S} + \varpi$$

which implies

$$\begin{aligned}
\mathfrak{S}^{3n} &= (\mathfrak{S}^3 - \varpi + \varpi)^n \\
&= \sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix} w^{n-k}(\mathfrak{S}^3 - w)^k \\
&= \sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix} \varpi^{n-k}(\rho\mathfrak{S}^2 + \varrho\mathfrak{S})^k \\
&= \sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix} \varpi^{n-k}\sum_{m=0}^{r}\begin{pmatrix} k \\ m \end{pmatrix}(\rho\mathfrak{S}^2)^m(\varrho\mathfrak{S}))^{k-m} \\
&= \sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}\sum_{m=0}^{k}\begin{pmatrix} k \\ m \end{pmatrix}\varpi^{n-k}\rho^m\varrho^{k-m}\mathfrak{S}^{k+m}
\end{aligned}$$

If we replace to κ, ϑ, η by $\mathfrak{S}$, and using the equation (4.4). *Then we obtain*

$$\begin{aligned}
Q_{T,3n-1}(\rho,\varrho,\varpi) &= \frac{\underline{\kappa}\alpha^{3n}}{(\kappa-\vartheta)(\kappa-\eta)} + \frac{\underline{\vartheta}\vartheta^{3n}}{(\vartheta-\kappa)(\vartheta-\eta)} + \frac{\underline{\eta}\eta^{3n}}{(\eta-\kappa)(\eta-\vartheta)} \\
&= \sum_{k=0}^{n}\sum_{m=0}^{k}\begin{pmatrix} n \\ k \end{pmatrix}\begin{pmatrix} k \\ m \end{pmatrix}\rho^m\varrho^{k-m}\varpi^{n-k}Q_{T,k+m-1}(\rho,\varrho,\varpi)
\end{aligned}$$

2) *By the same methods and using the equation (4.5)*

■

Proposition 4.4 *[16] For $n \geq 2$, we have*

1) $\sum_{n=0}^{\infty} Q_{L,n+m}(\rho,\varrho,\varpi)t^n = \frac{Q_{L,m}(\rho,\varrho,\varpi)+[Q_{L,m+1}(\rho,\varrho,\varpi)-\rho Q_{L,m}(\rho,\varrho,\varpi)]t+\varpi Q_{L,m-1}(\rho,\varrho,\varpi)t^2}{1-\rho t-\varrho t^2-\varpi t^3}$

2) $\sum_{n=0}^{\infty} Q_{T,n+m}(\rho,\varrho,\varpi)t^n = \frac{Q_{T,m}(\rho,\varrho,\varpi)+[Q_{T,m+1}(\rho,\varrho,\varpi)-\rho Q_{T,m}(\rho,\varrho,\varpi)]t+\varpi Q_{T,m-1}(\rho,\varrho,\varpi)t^2}{1-\rho t-\varrho t^2-\varpi t^3}$

Proof 1 *According to the equation* (4.4), *we get*

$$\begin{aligned}
\sum_{n=0}^{\infty} Q_{L,m+n}(\rho,\varrho,\varpi)t^n &= \sum_{n=0}^{\infty}\left(\underline{\kappa}\mathfrak{A}^{m+n}+\underline{\vartheta}\vartheta^{m+n}+\underline{\eta}\eta^{n+m}\right)t^n \\
&= \underline{\kappa}\kappa^m\sum_{n=0}^{\infty}\kappa^n t^n+\underline{\vartheta}\vartheta^m\sum_{n=0}^{\infty}\beta^n t^n+\underline{\eta}\gamma^m\sum_{n=0}^{\infty}\eta^n t^n \\
&= \frac{\underline{\kappa}\kappa^m}{1-\kappa t}+\frac{\underline{\vartheta}\vartheta^m}{1-\vartheta t}+\frac{\underline{\eta}\eta^m}{1-\eta t} \\
&= \frac{Q_{L,m}(\rho,\varrho,\varpi)+[Q_{L,m+1}(\rho,\varrho,\varpi)-\rho Q_{L,m}(\rho,\varrho,\varpi)]t+\varpi Q_{L,m-1}(\rho,\varrho,\varpi)t^2}{1-\rho t-\varrho t^2-\varpi t^3}
\end{aligned}$$

2) *Analogously, by using the same techniques* ∎

Proposition 4.5 *[16] For any integer $n \geq 3$*

$$Q_{L,n}(\rho,\varrho,\varpi)=Q_{P,n+1}(\rho,\varrho,\varpi)+\varrho Q_{P,n-1}(\rho,\varrho,\varpi)+2\varpi Q_{P,n-2}(\rho,\varrho,\varpi) \tag{4.6}$$

Proof *Combining the Corollary* (4.3) *by the Eq.* (4.6) *and taking in to account that*

$$\begin{cases}
\varpi=\kappa\vartheta\eta \\
\varrho=-(\vartheta\eta+\vartheta\kappa+\kappa\eta). \\
\rho=\kappa+\vartheta+\eta
\end{cases}$$

we get

$$\begin{aligned}
(R) &= \frac{\underline{\kappa}\kappa^{n+2}+\varrho\underline{\kappa}\kappa^n+2\varpi\underline{\kappa}\kappa^{n-1}}{(\kappa-\vartheta)(\kappa-\eta)}+\frac{\underline{\vartheta}\vartheta^{n+2}+s\underline{\vartheta}\vartheta^n+2\varpi\underline{\vartheta}\vartheta^{n-1}}{(\vartheta-\kappa)(\vartheta-\eta)}+\frac{\underline{\eta}\eta^{n+2}+s\underline{\eta}\eta^n+2w\underline{\eta}\eta^{n-1}}{(\eta-\kappa)(\eta-\vartheta)} \\
&= \frac{\underline{\kappa}\kappa^n\left(\kappa^2+\varrho+2\vartheta\eta\right)}{(\mathfrak{A}-\vartheta)(\kappa-\eta)}+\frac{\underline{\vartheta}\vartheta^n\left(\vartheta^2+\varrho+2\kappa\eta\right)}{(\vartheta-\kappa)(\vartheta-\eta)}+\frac{\underline{\eta}\eta^n\left(\eta^2+\varrho+2\kappa\vartheta\right)}{(\eta-\kappa)(\eta-\vartheta)} \\
&= \underline{\kappa}\kappa^n+\underline{\vartheta}\vartheta^n+\underline{\eta}\eta^n \\
&= Q_{L,n}(\rho,\varrho,\varpi)
\end{aligned}$$

∎

Proposition 4.6 *For $n \geq 3$, we have*

$$1)\ \underline{\kappa}\kappa^{n+2} = Q_{P,n+2}(\rho,\varrho,\varpi)\kappa^2 + \left[Q_{P,n+3}(\rho,\varrho,\varpi) - \rho Q_{P,n+2}(\rho,\varrho,\varpi)\right]\kappa + \eta Q_{P,n+1}(\rho,\varrho,\varpi)$$

$$2)\ \underline{\vartheta}\vartheta^{n+2} = Q_{P,n+2}(\rho,\varrho,\varpi)\vartheta^2 + \left[Q_{P,n+3}(\rho,\varrho,\varpi) - \rho Q_{P,n+2}(\rho,\varrho,\varpi)\right]\vartheta + \varpi Q_{P,n+1}(\rho,\varrho,\varpi)$$

$$3)\ \underline{\eta}\eta^{n+2} = Q_{P,n+2}(\rho,\varrho,\varpi)\eta^2 + \left[Q_{P,n+3}(\rho,\varrho,\varpi) - \rho Q_{P,n+2}(\rho,\varrho,\varpi)\right]\eta + \varpi Q_{P,n+1}(\rho,\varrho,\varpi)$$

Proof

$$\begin{aligned}
(RHS) &= \kappa\left[P_{n+2}(\rho,\varrho,\varpi) + P_{n+3}(\rho,\varrho,\varpi)\boldsymbol{i} + P_{n+4}(\rho,\varrho,\varpi)\boldsymbol{j} + P_{n+5}(\rho,\varrho,\varpi)\boldsymbol{k}\right] + \\
&+ \kappa[P_{n+3}(\rho,\varrho,\varpi) + P_{n+4}(\rho,\varrho,\varpi)\boldsymbol{i} + P_{n+5}(\rho,\varrho,\varpi)\boldsymbol{j} + P_{n+6}(\rho,\varrho,\varpi)\boldsymbol{k}] - \\
&- \rho\kappa\left[P_{n+2}(\rho,\varrho,\varpi) + P_{n+3}(\rho,\varrho,\varpi)\boldsymbol{i} + P_{n+4}(\rho,\varrho,\varpi)\boldsymbol{j} + P_{n+5}(\rho,\varrho,\varpi)\boldsymbol{k}\right] + \\
&+ \varpi\left[P_{n+1}(\rho,\varrho,\varpi) + P_{n+2}(\rho,\varrho,\varpi)\boldsymbol{i} + P_{n+3}(\rho,\varrho,\varpi)\boldsymbol{j} + P_{n+4}(\rho,\varrho,\varpi)\boldsymbol{k}\right] \\
&= \big(P_{n+2}(\rho,\varrho,\varpi)\kappa^2 + \left[P_{n+3}(\rho,\varrho,\varpi) - \rho P_{n+2}(\rho,\varrho,\varpi)\right]\kappa + \varpi P_{n+1}(\rho,\varrho,\varpi)\big) + \\
&+ \boldsymbol{i}\,\big(P_{n+3}(\rho,\varrho,\varpi)\rho^2 + \left[P_{n+4}(\rho,\varrho,\varpi) - \rho P_{n+3}(\rho,\varrho,\varpi)\right]\kappa + \rho P_{n+2}(\rho,\varrho,\varpi)\big) + \\
&+ \boldsymbol{j}\,\big(P_{n+4}(\rho,\varrho,\varpi)\kappa^2 + \left[P_{n+5}(\rho,\varrho,\varpi) - \rho P_{n+4}(\rho,\varrho,\varpi)\right]\kappa + \varpi P_{n+3}(\rho,\varrho,\varpi)\big) + \\
&+ \boldsymbol{k}\,\big(P_{n+5}(\rho,\varrho,\varpi)\kappa^2 + \left[P_{n+6}(\rho,\varrho,\varpi) - \rho P_{n+5}(\rho,\varrho,\varpi)\right]\kappa + \varpi P_{n+4}(\rho,\varrho,\varpi)\big) +
\end{aligned}$$

From the identities

$$\kappa^n = P_{n-1}(\rho,\varrho,\varpi)\kappa^2 + \left[P_n(\rho,\varrho,\varpi) - \rho P_{n-1}(\rho,\varrho,\varpi)\right]\kappa + \varpi P_{n-2}(\rho,\varrho,\varpi).$$

We obtain

$$\begin{aligned}
Q_{P,n+2}(\rho,\varrho,\varpi)\kappa^2 + \left[Q_{P,n+3}(\rho,\varrho,\varpi) - \rho Q_{P,n+2}(\rho,\varrho,\varpi)\right]\kappa + \varpi Q_{P,n+1}(\rho,\varrho,\varpi) &= \kappa^{n+2} + \boldsymbol{i}\kappa^{n+3} + \boldsymbol{j}\kappa^{n+4} + \boldsymbol{k}\kappa^{n} \\
&= \underline{\kappa}\kappa^{n+2}
\end{aligned}$$

where $\underline{\kappa} = 1 + \kappa\boldsymbol{i} + \kappa^2\boldsymbol{j} + \kappa^3\boldsymbol{k}$

2) *and* 3) *Analogously, we get*

$$\underline{\vartheta}\vartheta^{n+2} = Q_{P,n+2}(\rho,\varrho,\varpi)\vartheta^2 + \left[Q_{P,n+3}(\rho,\varrho,\varpi) - \rho Q_{P,n+2}(\rho,\varrho,\varpi)\right]\vartheta + \varpi Q_{P,n+1}(\rho,\varrho,\varpi)$$

and

$$\underline{\eta}\eta^{n+2} = Q_{P,n+2}(\rho,\varrho,\varpi)\eta^2 + \left[Q_{P,n+3}(\rho,\varrho,\varpi) - \rho Q_{P,n+2}(\rho,\varrho,\varpi)\right]\eta + \varpi Q_{P,n+1}(\rho,\varrho,\varpi)$$

■

Theorem 4.3 *For $n \geq 2$, we have*

$$1) \sum_{m=0}^{n} Q_{P,m}(\rho,\varrho,\varpi) = \frac{Q_{P,n+2}(\rho,\varrho,\varpi) + (1-\rho)[Q_{P,n+1}(\rho,\varrho,\varpi) - Q_{P,0}(\rho,\varrho,\varpi) - Q_{P,1}(\rho,\varrho,\varpi)] + \varpi Q_{P,n}(\rho,\varrho,\varpi) - Q_{P,}}{\rho+\varrho+\varpi-1}$$

$$2) \sum_{m=0}^{n} Q_{L,m}(\rho,\varrho,\varpi) = \frac{Q_{L,n+2}(\rho,\varrho,\varpi) + (1-\rho) Q_{L,n+1}(\rho,\varrho,\varpi) + \varpi Q_{P,n}(\rho,\varrho,\varpi) - (H+Q+R)}{\rho+\varrho+\varpi-1}$$

Where $H = \underline{\kappa}(\vartheta-1)(\eta-1),\ Q = \underline{\vartheta}(\kappa-1)(\eta-1),\ R = \underline{\eta}(\kappa-1)(\vartheta-1)$

Proof 1) *From the equation 1.8, we have successively*

$$\begin{aligned}
Q_{P,n}(\rho,\varrho,\varpi) &= \rho Q_{P,n-1}(\rho,\varrho,\varpi) + \varrho Q_{P,n-2}(\rho,\varrho,\varpi) + \varpi Q_{P,n-3}(\rho,\varrho,\varpi) \\
Q_{P,n-1}(\rho,\varrho,\varpi) &= \rho Q_{P,n-2}(\rho,\varrho,\varpi) + \varrho Q_{P,n-3}(\rho,\varrho,\varpi) + \varpi Q_{P,n-4}(\rho,\varrho,\varpi) \\
&\vdots \qquad \vdots \qquad \vdots \\
Q_{P,4}(\rho,\varrho,\varpi) &= \rho Q_{P,3}(\rho,\varrho,\varpi) + \varrho Q_{P,2}(\rho,\varrho,\varpi) + \varpi Q_{P,1}(\rho,\varrho,\varpi) \\
Q_{P,3}(\rho,\varrho,\varpi) &= \rho Q_{P,2}(\rho,\varrho,\varpi) + \varrho Q_{P,1}(\rho,\varrho,\varpi) + \varpi Q_{P,0}(\rho,\varrho,\varpi)
\end{aligned}$$

After making a side-by-side assumption, we obtain

$$\sum_{k=3}^{n} Q_{P,m}(\rho,\varrho,\varpi) = \rho \sum_{k=2}^{n-1} Q_{P,m}(\rho,\varrho,\varpi) + \varrho \sum_{k=1}^{n-2} Q_{P,m}(\rho,\varrho,\varpi) + \varpi \sum_{k=0}^{n-3} Q_{P,m}(\rho,\varrho,\varpi)$$

by simplifying the last equality

$$\begin{aligned}
\sum_{m=3}^{n} Q_{P,m}(\rho,\varrho,\varpi) &= \sum_{m=0}^{n} Q_{P,m}(\rho,\varrho,\varpi) - Q_{P,0}(\rho,\varrho,\varpi) - Q_{P,1}(\rho,\varrho,\varpi) - Q_{P,2}(\rho,\varrho,\varpi) \\
\rho \sum_{m=2}^{n-1} Q_{P,m}(\rho,\varrho,\varpi) &= \rho \sum_{m=0}^{n} Q_{P,m}(\rho,\varrho,\varpi) - \rho Q_{P,0}(\rho,\varrho,\varpi) - \rho Q_{P,1}(\rho,\varrho,\varpi) - \rho Q_{P,n}(\rho,\varrho,\varpi) \\
\varrho \sum_{m=1}^{n-2} Q_{P,m}(\rho,\varrho,\varpi) &= \varrho \sum_{m=0}^{n} Q_{P,m}(\rho,\varrho,\varpi) - \varrho Q_{P,0}(\rho,\varrho,\varpi) - \varrho Q_{P,n-1}(\rho,\varrho,\varpi) - \varrho Q_{P,n}(\rho,\varrho,\varpi) \\
w \sum_{m=0}^{n-3} Q_{P,m}(\rho,\varrho,\varpi) &= \varpi \sum_{m=0}^{n} Q_{P,m}(\rho,\varrho,\varpi) - \varpi Q_{P,n-2}(\rho,\varrho,\varpi) - \varpi Q_{P,n-1}(\rho,\varrho,\varpi) - \varpi Q_{P,n}(\rho,\varrho,\varpi)
\end{aligned}$$

Then

$$(1-\rho-\varrho-\varpi)\sum_{k=3}^{n}Q_{P,m}(\rho,\varrho,\varpi) = (1-\rho)[Q_{P,0}(\rho,\varrho,\varpi)+Q_{P,1}(\rho,\varrho,\varpi)] + Q_{P,2}(\rho,\varrho,\varpi) - \varrho Q_{P,0}(\rho,\varrho,\varpi)$$
$$- \varpi Q_{P,n}(\rho,\varrho,\varpi) + [-\rho Q_{P,n}(\rho,\varrho,\varpi) - \varrho Q_{P,n-1}(\rho,\varrho,\varpi) - \varpi Q_{P,n-2}(\rho,\varrho,\varpi)]$$
$$+ [-\varrho Q_{P,n}(\rho,\varrho,\varpi) - \varpi Q_{P,n-1}(\rho,\varrho,\varpi)]$$

Finally

$$\sum_{m=0}^{n}Q_{P,m}(\rho,\varrho,\varpi) = \frac{Q_{P,n+2}(\rho,\varrho,\varpi)+(1-\rho)[Q_{P,n+1}(\rho,\varrho,\varpi)-Q_{P,0}(\rho,\varrho,\varpi)-Q_{P,1}(\rho,\varrho,\varpi)]+\varpi Q_{P,n}(\rho,\varrho,\varpi)-Q_{P,2}(}{\rho+\varrho+\varpi-1}$$

2) *from the Benit formula (4.4),we have*

$$\sum_{m=0}^{n}Q_{L,m}(\rho,\varrho,\varpi) = \sum_{m=0}^{n}\left(\underline{\kappa}\kappa^m + \underline{\vartheta}\vartheta^m + \underline{\eta}\eta^m\right)$$
$$= \frac{\underline{\kappa}(\kappa^{n+1}-1)}{\kappa-1} + \frac{\underline{\vartheta}(\vartheta^{n+1}-1)}{\vartheta-1} + \frac{\underline{\eta}(\eta^{n+1}-1)}{\eta-1}$$
$$= \frac{Q_{L,n+2}(\rho,\varrho,\varpi)+(1-\rho)Q_{L,n+1}(\rho,\varrho,\varpi)+\varpi Q_{P,n}(\rho,\varrho,\varpi)-(H+Q+R)}{\rho+\varrho+\varpi-1}$$

Where $H=\underline{\kappa}(\vartheta-1)(\eta-1),\ Q=\underline{\vartheta}(\kappa-1)(\eta-1),\ R=\underline{\eta}(\kappa-1)(\vartheta-1)$

■

Proposition 4.7 *If* $2\varrho+2\rho\varpi+\rho^2-\varrho^2+\varpi^2-1=(\rho+\varrho+\varpi-1)(\rho-\varrho+\varpi+1)\neq 0$ *then*

$$\sum_{k=0}^{n}Q_{P,2k}(\rho,\varrho,\varpi) = \frac{(-\varrho+1)Q_{P,2n+2}(\rho,\varrho,\varpi)+(\varpi+\rho\varrho)Q_{P,2n+1}(\rho,\varrho,\varpi)+\left(\varpi^2+\rho\varpi\right)Q_{P,2n}(\rho,\varrho,\varpi)+}{(\rho+\varrho+\varpi-1)(\rho-\varrho+\varpi+1)}$$
$$+\frac{(-1+\varrho)Q_{P,2}(\rho,\varrho,\varpi)+(-\varpi-\rho\varrho)Q_{P,1}(\rho,\varrho,\varpi)+\left(-1+\rho^2-\varrho 2+\rho\varpi+2\varrho\right)Q_{P,0}(\rho,\varrho,\varpi}{(\rho+\varrho+\varpi-1)(\rho-\varrho+\varpi+1)}$$

and

$$\sum_{m=0}^{n}Q_{P,2m+1}(\rho,\varrho,\varpi) = \frac{(\rho+\varpi)Q_{P,2n+2}(\rho,\varrho,\varpi)+\left(\varrho-\varrho^2+\varpi^2+\rho\varpi\right)Q_{P,2n+1}(\rho,\varrho,\varpi)+(\varpi-\varrho\varpi)Q_{P,2n}(\rho,\varrho,\varpi)+}{(\rho+\varrho+\varpi-1)(\rho-\varrho+\varpi+1)}$$
$$+\frac{(-\rho-\varpi)Q_{P,2}(\rho,\varrho,\varpi)+\left(-1+\varrho+\rho^2+\rho\varpi\right)Q_{P,1}(\rho,\varrho,\varpi)+(-\varpi+\varrho\varpi)Q_{P,0}(\rho,\varrho,\varpi)}{(\rho-\varrho+\varpi+1)(\rho+\varrho+\varpi-1)}$$

Proof *(1) Using the recurrence relation*

$$Q_{P,n}(\rho,\varrho,\varpi)=\rho Q_{P,n-1}(\rho,\varrho,\varpi)+\varrho Q_{P,n-2}(\rho,\varrho,\varpi)+\varpi Q_{P,n-3}(\rho,\varrho,\varpi)$$

i.e.

$$rQ_{P,n-1}(\rho,\varrho,\varpi)=Q_{P,n}(\rho,\varrho,\varpi)-\varrho Q_{P,n-2}(\rho,\varrho,\varpi)-\varpi Q_{P,n-3}(\rho,\varrho,\varpi)$$

we obtain

$$\begin{aligned}
\rho Q_{P,3}(\rho,\varrho,\varpi)&=Q_{P,4}(\rho,\varrho,\varpi)-\varrho Q_{P,2}(\rho,\varrho,\varpi)-\varpi Q_{P,1}(\rho,\varrho,\varpi)\\
\rho Q_{P,5}(\rho,\varrho,\varpi)&=Q_{P,6}(\rho,\varrho,\varpi)-\varrho Q_{P,4}(\rho,\varrho,\varpi)-\varpi Q_{P,3}(\rho,\varrho,\varpi)\\
&\vdots\\
\rho Q_{P,2n-1}(\rho,\varrho,\varpi)&=Q_{P,2n}(\rho,\varrho,\varpi)-\varrho Q_{P,2n-2}(\rho,\varrho,\varpi)-\varpi Q_{P,2n-3}(\rho,\varrho,\varpi)\\
\rho Q_{P,2n+1}(\rho,\varrho,\varpi)&=Q_{P,2n+2}(\rho,\varrho,\varpi)-\varrho Q_{P,2n}(\rho,\varrho,\varpi)-\varpi Q_{P,2n-1}(\rho,\varrho,\varpi)
\end{aligned}$$

Now, if we add the above equations side by side, we get

$$\rho\left(-Q_{P,1}(\rho,\varrho,\varpi)+\sum_{k=0}^{n}Q_{P,2k+1}(\rho,\varrho,\varpi)\right)=\left(Q_{P,2n+2}(\rho,\varrho,\varpi)-Q_{P,2}(\rho,\varrho,\varpi)-Q_{P,0}(\rho,\varrho,\varpi)+\sum_{k=0}^{n}Q_{P,2k}(\right.$$

$$-\varrho\left(-Q_{P,0}(\rho,\varrho,\varpi)+\sum_{k=0}^{n}Q_{P,2k}(\rho,\varrho,\varpi)\right)-\varpi\left(-Q_{P,2n+1}(\rho,\varrho,\varpi)+\sum_{k=0}^{n}Q_{P,2k+1}(\rho,\varrho,\varpi)\right)\quad(4.7)$$

In the same way, by utilizing the recurring relationship (4.3)*, we get*

$$\begin{aligned}
\rho Q_{P,2}(\rho,\varrho,\varpi)&=Q_{P,3}(\rho,\varrho,\varpi)-\varrho Q_{P,1}(\rho,\varrho,\varpi)-\varpi Q_{P,0}(\rho,\varrho,\varpi)\\
\rho Q_{P,4}(\rho,\varrho,\varpi)&=Q_{P,5}(\rho,\varrho,\varpi)-\varrho Q_{P,3}(\rho,\varrho,\varpi)-\varpi Q_{P,2}(\rho,\varrho,\varpi)\\
\rho Q_{P,6}(\rho,\varrho,\varpi)&=Q_{P,7}(\rho,\varrho,\varpi)-\varrho Q_{P,5}(\rho,\varrho,\varpi)-\varpi Q_{P,4}(\rho,\varrho,\varpi)\\
&\vdots\\
\rho Q_{P,2n-2}(\rho,\varrho,\varpi)&=Q_{P,2n-1}(\rho,\varrho,\varpi)-\varrho Q_{P,2n-3}(\rho,\varrho,\varpi)-\varpi Q_{P,2n-4}(\rho,\varrho,\varpi)\\
\rho Q_{P,2n}(\rho,\varrho,\varpi)&=Q_{P,2n+1}(\rho,\varrho,\varpi)-\varrho Q_{P,2n-1}(\rho,\varrho,\varpi)-\varpi Q_{P,2n-2}(\rho,\varrho,\varpi)
\end{aligned}$$

Now, if we add the above equations, we obtain

$$\varrho\left(-Q_{P,0}(\rho,\varrho,\varpi)+\sum_{k=0}^{n}Q_{P,2k}(\rho,\varrho,\varpi)\right)=-Q_{P,1}(\rho,\varrho,\varpi)+\sum_{k=0}^{n}Q_{P,2k+1}(\rho,\varrho,\varpi)-\varrho\left(-Q_{P,2n+1}(\rho,\varrho,\varpi)+\sum_{k=0}^{n}Q_{P,2k+1}\right.$$

$$-\varpi\left(-Q_{P,2n}(\rho,\varrho,\varpi)+\sum_{k=0}^{n}Q_{P,2k}(\rho,\varrho,\varpi)\right) \qquad (4.8)$$

Then, solving the system (4.7) (4.8) , *the required results of (1) and (2) follow.* ■

4.5 Partial derivative

Theorem 4.4 *[16] For $n \geq 4$, we have*

1) $\dfrac{\partial Q_{P,n}(\rho,\varrho,\varpi)}{\partial\rho}=\boldsymbol{k}P_{n+2}(\rho,\varrho,\varpi)+\sum_{k=1}^{n+1}P_k(\rho,\varrho,\varpi)Q_{P,n-k}(\rho,\varrho,\varpi)$

2) $\dfrac{\partial Q_{P,n}(\rho,\varrho,\varpi)}{\partial\varrho}=\boldsymbol{k}P_{n+1}(\rho,\varrho,\varpi)+\sum_{k=1}^{n}P_k(\rho,\varrho,\varpi)Q_{P,n-k-1}(\rho,\varrho,\varpi)$

3) $\dfrac{\partial Q_{P,n}(\rho,\varrho,\varpi)}{\partial\varpi}=\boldsymbol{k}P_{n}(\rho,\varrho,\varpi)+\sum_{k=2}^{n}P_{k-1}(\rho,\varrho,\varpi)Q_{P,n-k-1}(\rho,\varrho,\varpi)$

Proof *combining the definition* (4.1), *withe the partial derivative of generalized tribonacci like polynomials, we obtain*

$$\begin{aligned}\frac{\partial Q_{P,n}(\rho,\varrho,\varpi)}{\partial\rho}&=\frac{\partial P_n(\rho,\varrho,\varpi)}{\partial\rho}+\frac{\partial P_{n+1}(\rho,\varrho,\varpi)}{\partial\rho}\boldsymbol{i}+\frac{\partial P_{n+2}(\rho,\varrho,\varpi)}{\partial\rho}\boldsymbol{j}+\frac{\partial P_{n+3}(\rho,\varrho,\varpi)}{\partial\rho}\boldsymbol{k}\\&=\sum_{k=1}^{n-1}P_k(\rho,\varrho,\varpi)P_{n-k}(\rho,\varrho,\varpi)+\boldsymbol{i}\sum_{k=1}^{n}P_k(\rho,\varrho,\varpi)P_{n+1-k}(\rho,\varrho,\varpi)+\boldsymbol{j}\sum_{k=1}^{n+1}P_k(\rho,\varrho,\varpi)P_{n+2-k}(\rho,\varrho,\\&\quad+\boldsymbol{k}\sum_{k=1}^{n+2}P_k(\rho,\varrho,\varpi)P_{n+3-k}(\rho,\varrho,\varpi)\\&=\boldsymbol{k}P_{n+2}(\rho,\varrho,\varpi)+\sum_{k=1}^{n+1}P_k(\rho,\varrho,\varpi)\left[P_{n-k}(\rho,\varrho,\varpi)+\boldsymbol{i}P_{n+1-k}(\rho,\varrho,\varpi)+\boldsymbol{j}P_{n+2-k}(\rho,\varrho,\varpi)+\boldsymbol{k}P_{n+3}\right.\\&=\boldsymbol{k}P_{n+2}(\rho,\varrho,\varpi)+\sum_{k=1}^{n+1}P_k(\rho,\varrho,\varpi)Q_{P,n-k}(\rho,\varrho,\varpi)\end{aligned}$$

The same technic, we get (2), *and* (3)

■

Remark 4.5 *For $n \geq 4$, the relation between partial derivative ($\partial Q_{T,n}$) with respect to those variables*

$$\frac{\partial Q_{P,n}(\rho,\varrho,\varpi)}{\partial\rho}=\frac{\partial Q_{P,n+1}(\rho,\varrho,\varpi)}{\partial\varrho}=\frac{\partial Q_{P,n+2}(\rho,\varrho,\varpi)}{\partial\varpi}.$$

.

Lemma 3 *If* $n \geq 2$ *we have*

$$Q_{L,n}(\rho,\varrho,\varpi) = \rho Q_{P,n}(\rho,\varrho,\varpi) + 2\rho Q_{P,n-1}(\rho,\varrho,\varpi) + 3\varpi Q_{P,n-2}(\rho,\varrho,\varpi)$$

Proof *By induction on n* ■

Proposition 4.8 *For* $n \geq 1$ *. the partial derivative of* (∂L_n) *with respect to those variables, given by*

$$\begin{aligned}
1)\frac{\partial Q_{L,n}(\rho,\varrho,\varpi)}{\partial \rho} &= \frac{\partial Q_{P,n+1}(\rho,\varrho,\varpi)}{\partial \rho} + \varrho\frac{\partial Q_{P,n}(\rho,\varrho,\varpi)}{\partial \varrho} + 2\varpi\frac{\partial Q_{P,n}(\rho,\varrho,\varpi)}{\partial \varpi}\\
2)\frac{\partial Q_{L,n}(\rho,\varrho,\varpi)}{\partial \varrho} &= 2\frac{\partial Q_{P,n}(\rho,\varrho,\varpi)}{\partial \rho} - \rho\frac{\partial Q_{P,n}(\rho,\varrho,\varpi)}{\partial \varrho} + \varpi\frac{\partial Q_{P,n-1}(\rho,\varrho,\varpi)}{\partial \varpi}\\
3)\frac{\partial Q_{L,n}(\rho,\varrho,\varpi)}{\partial \varpi} &= 3\frac{\partial Q_{P,n}(\rho,\varrho,\varpi)}{\partial \varrho} - 2\rho\frac{\partial Q_{P,n}(\rho,\varrho,\varpi)}{\partial \varpi} - \varrho\frac{\partial Q_{P,n-1}(\rho,\varrho,\varpi)}{\partial \varpi}
\end{aligned}$$

Let $S'_n(\rho,\varrho,\varpi)$ *denote the sum of tribonacci quaternions like polynomial which defined by*

$$S'_n(\rho,\varrho,\varpi) = \sum_{k=0}^{n} Q_{P,n}(\rho,\varrho,\varpi)$$

Theorem 4.6 *[16] For* $n \geq 2$*, we have*

$$S'_n(Y) = \frac{Q_{P,n+2}(Y) + (1-\rho)Q_{P,n+1}(Y) + \varpi Q_{P,n}(Y) - Q_{P,1}(Y) - \varpi Q_{P,-1}(Y) + (\rho-1)Q_{P,0}(Y)}{\rho+\varrho+\varpi-1}$$

where $(Y) = (\rho,\varrho,\varpi)$

Theorem 4.7 *forn* ≥ 3*, we have*

$$S'_n(x,y,z) = \frac{\underline{\kappa}\kappa^{n+2}}{(\kappa-1)(\kappa-\vartheta)(\kappa-\eta)} + \frac{\underline{\vartheta}\vartheta^{n+2}}{(\vartheta-1)(\vartheta-\kappa)(\vartheta-\eta)} + \frac{\underline{\eta}\eta^{n+2}}{(\eta-1)(\eta-\kappa)(\eta-\vartheta)} \quad (4.9)$$

Arabic Abstract

إرتأينا في هذه المذكرة إلى دراسة بعض الخواص المتعلقة بمتاتلية تريبوناتشي و كثير الحدود الخاص بها $(p_n(r))_{n\in\mathbb{N}}$

لتليها دراسة كثير الحدود بحدود أولية معممة كإستخراج عبارة الحد العام بإستعمال المصفوفات وكذلك المشتقات الجزئية و بعض المجاميع

ثم قمنا بتعميم الحدود الأولى ومعاملات كثير الحدود $(T_n(r, s, w))_{n\in\mathbb{N}}$ حيث تم إسقاط كل الخواص اللتي درسناها على هذا النوع من كثيرات الحدود وذلك من خلال تعميم كامل الدراسات و بالأخص المجاميع و المشتقات الجزئية و العلاقة بين $(T_n(r, s, w))_{n\in\mathbb{N}}$ و

$(L_n(r, s, w))_{n\in\mathbb{N}}$

وفي الأخير قمنا بإدراج ما تم استنتاجه وتوضيفه في كثيرات الحدود ذات البعد الرابع

كالمشتقات الجزئية و المجاميع تحت إسم $(QT_{,n}(r, s, w))_{n\in\mathbb{N}}$

Abstract

The primary goal of this research is to delve into the well-known tribonacci sequence, exploring its inherent identities. The aim is to extend and generalize these findings to encompass a broader class of sequences and their associated polynomials. Within this investigation, we develop a clear and explicit formula for the generalized tribonacci polynomials. Furthermore, we systematically establish and examine various properties inherent to these polynomials. Through this exploration, the research seeks to contribute to a deeper understanding of the generalized tribonacci polynomials and their properties.

Similarly, we investigate Jacobsthal polynomials $(J_n(\rho))_{n\in\mathbb{N}}$, *We also describe some of the properties of Jacobsthal polynomials.*
after that we are present an alternative formula for both generalized tribonacci polynomials $(P_n(\rho,\varrho,\varpi))_{n\in\mathbb{N}}$ *and generalized tribonacci-Lucas polynomials* $(Ln(x,y,z))_{n\in\mathbb{N}}$. *These formulas are derived through the application of combinatorial calculus, along with an examination of their summation. Additionally, we provide explicit expressions for the partial derivatives of these polynomials, denoted as* $\partial P_n(\rho,\varrho,\varpi)$ *and* $\partial L_n(\rho,\varrho,\varpi)$, *concerning one of their variables. Furthermore, we discuss various properties associated with these polynomials and their derivatives.*
Finally we introduce the generalized tribonacci and generalized tribonacci Quaternions like polynomial. We commence by establishing several fundamental identities related to these quaternions. Subsequently, we derive Binet's formula, generating functions, and a summation formula specifically tailored for this class of quaternions.

Résumé

Dans cette thèse, nous avons obtenu quelques résultats sur les suites bien connue de tribonacci et ses polynomials. De plus, nous établissons diverses propriétés de ces polynômes De manière similaire, nous examinons les polynômes de Jacobsthal $(J_n(x))_{n\in\mathbb{N}}$ *et décrivons certaines de leurs propriétés.*

Ensuite, nous présentons une formule alternative pour les polynômes généralisés de tribonacci $(P_n(\rho, \varrho, \varpi))\, n \in \mathbb{N}$ *ainsi que les polynômes généralisés tribonacci-Lucas* $(L_n(\rho, \varrho, \varpi))\, n \in \mathbb{N}$*. De plus, nous fournissons des expressions explicites pour les dérivées partielles de ces polynômes, notées* $\partial P_n(\rho, \varrho, \varpi)$ *et* $\partial L_n(\rho, \varrho, \varpi)$*, en ce qui concerne l'une de leurs variables. En outre, nous discutons de diverses propriétés associées à ces polynômes et à leurs dérivées.*

Enfin, nous introduisons les polynômes généralisés de tribonacci et les polynômes généralisés de tribonacci Quaternions. Nous commençons par établir plusieurs identités fondamentales liées à ces quaternions. Ensuite, nous dérivons la formule de Binet, les fonctions génératrices et une formule de sommation spécifiquement adaptée à cette classe de quaternions.

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