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TRAD Houssam Eddine

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Décalage De Fréquence Des Photons Dans Un Trou Noir Chargé En Rotation

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Soutenu publiquement devant le jury composé de

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A Dissertation

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TRAD Houssam Eddine

In Fulfillment of the Requirements for the Degree of

DOCTOR OF SCIENCES IN THEORETICAL PHYSICS

THEME

FREQUENCY SHIFTS OF PHOTONS IN CHARGED ROTATING BALCK HOLE

Publicaly defended in front of the jury members:

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To my parents and family, the reason of my existence.

To my son, the source of my strength.

To my angel, the flame of my hope.

To those who believed in me, a heartfelt gratitude.

To those who believed in October the seventh, surely, Allah's victory is near.

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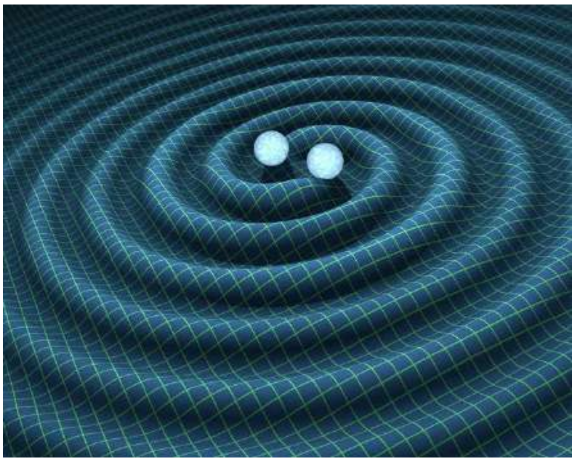
Contents

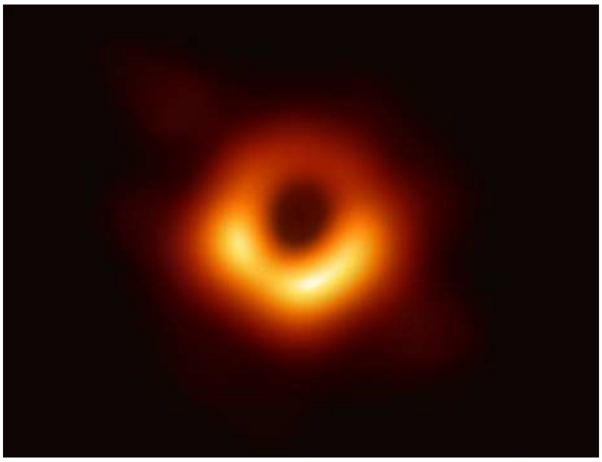
INTRODUCTION

Throughout history, the sky has been the only window to the outside world since the dawn of mankind. The bright celestial objects there occupied human imagination, passion, and curiosity. This was obvious from the very first models of the solar system found in cave markings. Humans always thought that the outside universe was a calm quiet place. This line of thinking dominated until the beginning of the 20th century, when the two pillars of modern physics, General Relativity and Quantum Mechanics have were founded [1][2]. Since then, humans understood that, while many regions in the universe are indeed calm and quiet, other corners are extremely violent and unpredicted. According to the current understanding, huge interstellar clouds of hydrogen and helium collapse under their own gravity. The tremendous gravity and temperature at the center smashes the gas nuclei together until the kinetic energy overcomes the coulomb barrier. At this point, nuclear fusion ignites at the heart and a main-sequence star is born. Every second, millions of tons of matter are transformed into energy, the equivalent of millions of nuclear bombs detonated simultaneously. At this stage, a delicate balance is established between gravity and the internal pressure. When the star exhausts its nuclear fuel, the internal pressure becomes weaker and the star will collapse under its own gravity to form a white dwarf star. If the star is massive enough, its fate will be the densest and smallest known stellar object in the universe: a neutron star [3]. Due to the conservation of angular momentum, neutron stars rotate at frequencies of several hundreds. However, if the star is way more massive, the collapse will produce the most violent and densest known object in the universe: a black hole [4]. These objects act like infinitely deep sinks in space. Anything that approaches them enough can never come back, not even light. As opposed to stars, considered as the cradle of visible light since the first human, black holes are the graves of light. It is therefore not surprising that such enigmatic destruction machines are a rich and fascinating topic, not only in physics but also in science-fiction, literature, and art.

Although predicted by General Relativity, the historical seeds of the concept of black holes goes back to the 18th century, when physicists John Michell and Pierre-Simon Laplace presented the idea of a "dark star", a massive celestial object with immense gravity that even light cannot escape [5][6]. Although their works were based on Newtonian mechanics and gravity, the black star radius they obtained was surprisingly the same as that of a neutral nonrotating black hole derived from General Relativity. The first breakthrough came in 1916, physicist Karl Schwarzschild found the first solution of Einstein equations that represent a black hole. In 1918, Hans Reissner and Gunnar Nordström found solution for the coupled Einstein-Maxwell equations, which represents a charged spherically symmetric black hole. Although these advancements and others gave them more interest, the very existence of black holes remained a debatable topic for decades. Many renowned physicists, such as Arthur Eddington, dismissed the idea that a star could collapse into such a compact object in real scenarios. They believed there must exist some physical mechanism that prevents such a collapse. However, this belief did not last long: in the 1930s, Robert Oppenheimer and his colleagues showed that a massive star would inevitably collapse into a black hole under its gravity at the end of their lives. In the 1960s, the discovery of quasars, tremendously bright objects at the center of galaxies, renewed the interest in black holes. With luminosities thousands of times greater than that of a whole galaxy like the Milky Way, it was believed that quasars are powered by accreting matter around supermassive black holes. In 1963, significant theoretical progress happened when Roy Kerr found the rotating neutral black hole solution to Einstein equations of General Relativity. Two years later, Erza Newman and his colleagues solved the Einstein-Maxwell equations for a charged rotating black hole, which represents the most general black hole solution. Furthermore, John Wheeler coined the term "black hole" to name those enigmatic objects [7]. The subject of black holes started witnessing widespread acceptance as real astrophysical objects rather than mere mathematical curiosity. The evidence of the existence of black holes reinforced by the discovery of many stellar-mass black holes in binary systems. Moreover, the existence of supermassive black holes at the heart of most galaxies was confirmed. The theoretical level witnessed the proposition of many models explaining the evolution of black holes and their possible role in the early universe. During the 1970s, a theory of black hole thermodynamics was developed by Jacob Bekenstein Brandon Carter, and James Bardeen. In 2015, another significant discovery was the detection of gravitational waves generated by two colliding black holes [8][9], billions of light years away from Earth, by the LIGO experiment. The last groundbreaking discoveries were the release of images of the supermassive black holes in the M87 and Milky galaxies, in 2019 and 2022 respectively, achieved by the EHT collaboration [10][11]. Today, the existence of black holes has been elevated from theoretical speculations into an established empirical fact, becoming fundamental astrophysical objects. On the other hand, the study of black holes is now a vital research area regarding their potential crucial role in the formation and evolution of galaxies, the early universe evolution, and most importantly, their combination between the effects of gravity and Quantum Mechanics.



(a) An artist's impression of gravitational waves from crashing black holes (2015).



(b) The first image ever of a black hole, by the EHT collaboration (2019).

Today, there are many pieces of evidence of the existence of black holes, though all of them are in fact indirect. This is due to the fact that black holes do not emit light in any wave length. Here we give some of those observations.

i) The detection of *ultraluminous X-ray sources* is a strong piece of evidence of the existence of black holes. When the gas of the accretion disk falls into the black hole due to friction, it heats up and radiates a tremendous amount of energy [12], mostly in the form of X-ray. One of these ultraluminous sources is quasars, the brightest known objects in the universe. Their extreme luminosity, thousands of times greater than that of average galaxies like ours, is believed to be powered by supermassive black holes. The reason behind this extreme brightness is that up to 40% of the rest mass of the infalling matter is converted from gravitational potential energy to radiation [13][14]. This is a very high conversion rate compared to only 0.7% of nuclear fusion at the heart of stars like the Sun.

ii) Another similar sources are *X-ray binaries*. These are systems composed of two bodies orbiting their common center of mass. The first body is a star while the second is either a black hole or neutron star. When matter falls from the companion star into the black hole, a tremendous amount of energy is radiated in the form of X-ray.

iii) *Trajectories of stars orbiting the center of galaxies* strongly suggest the existence of black holes. For many years, astronomers have been tracking the motion of stars orbiting the center of the Milky Way galaxy [15]. By fitting their trajectories to Keplerian orbits, they deduced that the gravitating body at the center was several millions more massive than our Sun while contained in a relatively small radius. The only known astrophysical object that has these characteristics is the black hole.

iv) *The detection of the gravitational waves* in 2015 on the 100th anniversary of General Relativity provided another highly accurate test of the theory. This significant achievement is expected to open the doors for a new era in astronomical observation. These gravitational waves were traveling ripples in the fabric of spacetime, produced by the merger of two stellar-mass black holes. The detected waves suggested that the separation between the two merging massive objects was only 350 km [8][9]. This indicated that the massive objects were extremely compact, making black holes the only known candidates. Therefore, the detection of gravitational waves is considered the most compelling piece of evidence of the existence of black holes.

v) *The first direct image of a black hole ever* was a groundbreaking astronomical achievement. In 2019, the Event Horizon Telescope collaboration [10] released the image of the supermassive black hole at the center of Messier 87 galaxy. Two years later, the same team released the first image of Sagittarius A*, the supermassive black hole at the center of the Milky Way galaxy. The images showed the black hole's accretion disks with details aligning with the theoretical predictions, hence providing compelling evidence of the existence of black holes.

The structure of black holes is governed by its metric, which gives the underlying spacetime a remarkable properties. Generally, black hole structure includes four elements [16][17][18]. The first and foremost element is the *event horizon*. It can be thought of as the no-return surface, beyond which nothing can escape, not even light. Outside the event horizon, the black hole acts as any massive celestial object, in the sense that a particle with enough high energy can in principle escape it. The second element is the *singularity*, a region of infinite curvature and density that represents the center of the black hole. The total mass of the black hole is concentrated there. The nature of singularities is an open question since the law of physics we know breaks down there and a quantum gravity theory is needed. *Accretion disk* is the third structure element, which features most black holes. It consists of gases and dust that are attracted and violently orbiting the black hole at very high speeds. The friction and radiation between its particles make it extremely hot and luminous. Some of the accretion disk matter would eventually spiral down, making the black hole even larger. The fourth element in the black hole's structure is the *ergosphere*. This region lies outside the event horizons but is still well under the influence of the black hole's gravitational field. The rotation of the black hole drags the surrounding fabric of spacetime with it, forcing any object in this region to rotate with it so that stationary observers cannot exist there.

Einstein field equations tell us that it is the distribution of energy and matter in spacetime that dictates its curvature. Consider a spherical symmetric distribution of energy and matter, a perfectly spherical planet or star, for instance. The external gravitational field in this case would depend on few parameters, such as the planet mass. This is in fact a very unrealistic ideal scenario. A real planet may have a very complex distribution of energy and matter due to the various landforms such as mountains, canyons and plains, each one representing a parameter that must be taken into account. This tremendous number of parameter would entail a very complex form curvature. Surprisingly, black holes do not exhibit this kind of complexity. Thanks to the *no hair theorem*, only three parameter are needed to describe any black hole solution for Einstein equations [17][18]. These are the mass of the black hole, its electric charge, and its angular momentum. Consequently, we can have four classes of black holes:

i) The *Schwarzschild* neutral and nonrotating black holes [19]. It is the spherically symmetric solution to Einstein equations in vacuum. It is believed that this idealized black hole is not likely to exist in real astrophysical situations. However, it can serve as a good approximation in many real scenarios, namely, when for neutral astrophysical bodies with very slow rotation. This black hole posses one event horizon concealing the central singularity and demarcating the region of no return from the rest of spacetime.

ii) The *Reissner-Nordström* charged nonrotating black holes [20][21]. It represents the spherically symmetric solution to the coupled Einstein and Maxwell equations in the presence of an electric charge [22]. There is a belief that such a black holes cannot exist because any initial charge in the black hole would be immediately neutralized by the opposite charges falling inwards from the accretion disk, leading to an uncharged Schwarzschild black hole. However, this topic is still far from being resolved [23]. The Reissner-Nordström solution can possess two event horizons concealing the central singularity, and has a more complex structure than the Schwarzschild one.

iii) The *Kerr* neutral rotating black holes [24]. It is the axially symmetric solution to Einstein equations in vacuum. In real astrophysics, this is the most probable scenario, taking into account the fact that most collapsing massive stars were rotating and that their angular momentum must be conserved. This solution have two event horizons concealing the central singularity, and an ergosphere, where stationary observers are not allowed. The spacetime structure is more complex and interesting compared to the previous solutions.

iv) The *Kerr-Newmann* charged rotating black holes [25]. This is the most general black hole solution for Einstein equations, as it is the solution to the coupled Einstein and Maxwell equations in the presence of an electric charge. As for the Reissner-Nordström case, the existence of an electric charge remains an open question. Nonetheless, the rich structure it exhibits is of significant importance for the understanding general black holes and the investigation of new solutions.

With their extreme gravitational field and enigmatic properties, black holes continue to pose significant challenges to physicists. After decades of research, our understanding of black holes still has shortcomings and unsolved problems. Physicists believe most of these problems boil down to the *absence of a quantum theory of gravity*, which represents the first and foremost unsolved black hole problem [26]. Another problem is the nature of *singularities*, the central region of a black with an infinite curvature and density. In this extremely gravitating small region, both the effect of gravity and Quantum Mechanics should be taken into account, that is why some physicists believe General Relativity as we know breaks down there. The existence of *naked singularities* poses another challenge. These are singularities not hidden by an event horizon from the rest of the spacetime. Their existence is forbidden by the debatable *cosmic censorship conjecture*. Another problem is the existence of *closed timelike curves* in some regions of Kerr black holes, the spinning ones. The existence of such curves is a serious pathology since it is in contradiction with causality. The mathematics of black holes leads to another problem which is *white holes*. Roughly speaking, these are the reverse of black holes, i.e., regions of spacetime with singularity and event horizon that cannot be entered from outside. Their existence and formation is an open question. One of the most debatable unsolved problems is *information paradox* [27]. According to quantum mechanics, quantum information (the mass, the spin) of every particle falling into the black hole must be preserved somehow inside, though it is beyond our reach. However, the combination of the effects of Quantum Mechanics with General Relativity leads to the *Hawking radiation*, which suggests that black holes gradually radiate and lose mass after a tremendously long time. Since the radiated photons carry no information, the black hole and the entire quantum information inside it could be permanently erased. Observation of distant supermassive black holes raised another difficult question about *the formation of supermassive black holes* [28]. Unlike stellar-mass black holes, which form after the collapse of stars, the formation of supermassive black holes remains an unsolved problem. Proposed mechanisms for their creation suggest that they formed in the early universe and grew through the accretion of matter and the merger with other black holes. However, over the last decades, observations showed the existence of quasars powered by supermassive black holes with high redshift. This means that they grew and formed in the early universe, way earlier than we thought they could.

As we mentioned earlier, general black holes are described only by three parameters: the mass M, the electric charge Q, and the rotational parameter $a = J/M$. The precise determination of those parameters is decisive for our understanding of black holes. Since most of the observational data we get from black holes is from their immediate surroundings (accretion disks, orbiting companions...), it is crucial to develop theories linking these data with the parameters of black holes. One important piece of data is the frequency red/blue shift z of photons emitted from particles orbiting the black hole. In this dissertation, we aim to develop a formula relating the observable red/blue shift z with the black hole parameters M, Q, and J. These parameters describe the Kerr-Newmann black hole, the most general case. The Dissertation is organized as follows. The first chapter is a review of General Relativity. We start from Newtonian spacetime and Special Relativity till the Einstein field equations. The second chapter is devoted to black holes. We derive the Schwarzschild and Reissner-Nordström metrics and explore the types, structures, and properties of the main black holes. In the third chapter, we investigate the relation between the Kerr-Newmann black hole parameters and the red/blue shifts of photons emitted by particles orbiting this black hole. We end the dissertation with a general conclusion.

Chapter 1

General Theory of Relativity

1.1 Special Theory of Relativity

The purpose of the present chapter is to develop the conceptual and mathematical key tools of General Relativity (GR). We can think of GR as a geometrical theory of gravity, where the gravitational force is just a manifestation of the curvature of the underlying *Spacetime*. Roughly speaking, spacetime is the background on which particles and fields evolve [16][22]. It is a four dimensional set of elements called *events*, labeled by four numbers, three spacial positions and time (t, x^1, x^2, x^3). The trajectory of a particle through spacetime consists of a set of events that we call the *world line*. Strictly speaking, however, a spacetime is a four dimensional mathematical structure called *differentiable manifold*, each point of which has four coordinates (x^0, x^1, x^2, x^3). In the present chapter, we will discuss the evolution of spacetime theories, from the Newtonian Mechanics, until the Einstein theory of General Relativity.

In order to get a flavor of upcoming details, let us take a glance the first and last steps. The first theory of gravity we encountered is Newtonian theory. It relies on two basic equations: the first one is Newton's law of gravity, the law giving the force $\mathbf{F}$ mutually exerted by tow bodies of masses M and m, separated by a distance vector $\mathbf{r} = r\,\mathbf{e_r}$,

$$\mathbf{F} = \frac{GMm}{r^2}\,\mathbf{e_r} \ , \tag{1.1}$$

where $\mathbf{e}(r)$ is a unite position vector. The second equation is Newton's second law of motion, which gives the acceleration of a body of mass m acted on by a force $\mathbf{F}$,

$$\mathbf{F} = m\mathbf{a} \ . \tag{1.2}$$

The first one describes how the gravitational field response to the existence of matter, while the second one describe the response of matter to those fields. In the framework of GR, as we shall see later, Newton's law of gravity is going to be replaced by the more complex, though mor precise, Einstein field equations,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} \ , \tag{1.3}$$

while the Newton's second law is will be replaced by the geodesic equation,

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0 \ . \tag{1.4}$$

We will discuss later the meaning of each equation.

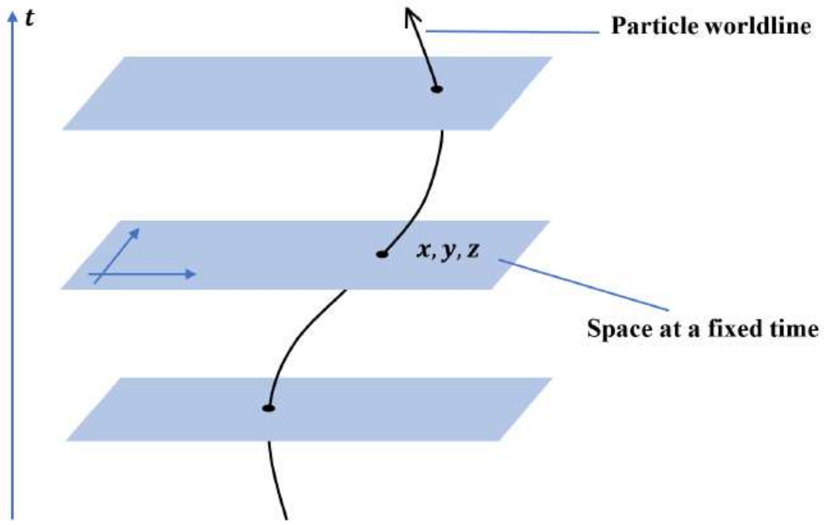


Fig. 1.1 – Newtonian slicing of spacetime into infinite three diemensional Euclidean spaces.

Newtonian gravity is an example of a classical dynamical system treated within the context of Newtonian spacetime. However, other classical dynamical systems, such as Maxwell's electrodynamics, operate within the context of Special Relativity spacetime rather than the Newtonian one

1.1.1 Newtonian Spacetime

The first spacetime encounter in physics is the Newtonian spacetime, the background space and time where Newtonian mechanics is applied. It could be seen as a space and time, separately, rather than a spacetime. Indeed, the Newtonian spacetime can be seen as a continuously parametrized infinite slices of three dimensional Euclidean space (Fig. 1.1). Mathematically, it can be represented by $R \times R^3$, where R^3 is a Euclidean manifold with a Euclidean metric $\eta_{\alpha\beta} = diag(1, 1, 1)$ [18][22]. Hence, some of the important consequences is the universal notion of simultaneity, i.e., if two events occur at the same time coordinate in one frame, they will be so in any other reference frame. In other words, if t =constant in one frame, we will get t' =constant in any other frame. Moreover, in the Newtonian spacetime, particles needs to always move forwards in time, whereas they are free to move forwards or backwards in space. Thus, there is no limit on the relative velocities of such particles.

Moreover, that fact that space and time are treated separately in Newtonian theory means that there is no metric then for the whole spacetime. We will see later how SR forced the idea of a single metric for the while spacetime, called Minkowski metric, which treat time and space on equal footing. Historically, Einstein approach to SR was kinematics, until Minkowski presented a geometrical approach to SR [29]. However, Einstein was initially dismissive of Minkowski's geometric interpretation, regarding it as "überflüssige Gelehrsamkeit", meaning superfluous learnedness [30].

1.1.2 Special Relativity

The second spacetime we shall deal with is the one of the Special Relativity (SR). In his original treatment [1], Einstein showed that SR relied on the following two postulate:

- *Principle of Relativity:* the laws of physics are invariant in all inertial frames of reference,

- *Constancy of the speed of light:* the speed of light is the same in all inertial frames of reference, regardless of the motion and direction of the light source.

An inertial frame of reference can be defined as an unaccelerated frame of reference. In the following, we will use the terminologies inertial frame of reference, inertial observer and inertial coordinate system interchangeably.

In fact, the principle of relativity dates back to Galileo, and then embodied in Newton's first law [31][32]. Galileo's law of inertia states that a body in a state of uniform motion remains in that state, unless acted upon by some external force. If two inertial frames R and R' are in uniform relative motion, and Galileo's law hold for R, then it will hold for R' too. Thus the laws of physics are the same in any inertial frame.

On the other hand, the second postulate, is Einstein's major contribution to SR, that led to many important results such as the invalidity of Galilean law of addition of velocities.

Lorentz Transformations

Another important result we can draw from the two postulates is the Lorentz transformations [2]. Suppose we have two inertial frames R and R', with coordinate systems (t, x, y, z) and (t', x', y', z'). If initially both frames axes coincide and $R'$is in relative uniform motion with speed v in the positive x-direction with respect to R, then the corresponding Lorentz transformation reads

$$\begin{aligned} ct' &= \gamma(ct - \beta x) \ , \\ x' &= \gamma(x - \beta ct) \ , \\ y' &= y \ , \\ z' &= z \ , \end{aligned} \tag{1.5}$$

where $\beta = v/c$ and $\gamma = (1 - \beta^2)^{-1/2}$and c is a conversion constant having the dimension of velocity. It turns out that this c is exactly the speed of propagation of electromagnetic waves in vacuum, the speed of light. Therefore, the importance of c does not come from being the speed of light, but rather from being the appropriate conversion constant that makes the Lorentz transformation working.

It's worth mentioning that the second postulate is not necessary for the derivation of Lorentz transformations. In fact, they can be obtained starting with the principle of relativity (first postulate) and the principle of inertia only [33].

Invariance of Spacetime Interval

Based on the two postulates, we can prove what is probably the most important result of SR, the invariance of the *spacetime interval*. In some inertial frame R, consider two arbitrary events for which the difference of coordinates is $(\Delta t, \Delta x^1, \Delta x^2, \Delta x^3)$. Then, the spacetime interval between these two arbitrary events is defined as [16][18][22]

$$\begin{aligned} \Delta s^2 &= -(c\Delta t)^2 + \left(\Delta x^1\right)^2 + \left(\Delta x^2\right)^2 + \left(\Delta x^3\right) \ , \\ &= \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu \ , \end{aligned} \tag{1.6}$$

where $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$ is the Minkowski metric and Einstein summation convention is applied over dummy indices. Greek indices are devoted for spacetime coordinates (from 0 to 3), while latin ones are devoted for spacial coordinates only (from 1 to 3). The important theorem we can prove starting from the two postulate of SR is that the interval between any two events is the same when calculated in any inertial frame, i.e., in any coordinate system

$$\Delta s'^2 = \Delta s^2 \ . \tag{1.7}$$

Since the interval is a property of the two events and not of the observer (coordinate system), we can use Δs^2 to classify the relation between any two events. If $\Delta s^2 > 0$, the events are said to be *spacelike separated.* If $\Delta s^2 < 0$, the events are said to be *timelike separated.* If $\Delta s^2 = 0$, the events are said to be *lightlike* or *null separated.* For some particular event A. the other events that are null separated from it constitute a cone called *light cone* of A.

For two infinitesimally separated events, the spacetime interval is (in natural units where $c = 1$)

$$\begin{aligned} ds^2 &= -dt^2 + \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 \\ &= \eta_{\mu\nu} dx^\mu dx^\nu \ . \end{aligned} \tag{1.8}$$

The Proper Time and Time Dilation

The *proper time* $\Delta\tau$ between two events A and B is defined as the square root of the negative interval between these two events

$$(\Delta\tau)^2 = -\Delta S^2 \ . \tag{1.9}$$

Consider a particle that is moving from events A to B particle with constant velocity, that is along a straight line in the spacetime diagram. This particle is at rest in its comoving frame R, i.e., $\Delta x^i = 0$ and the corresponding proper time in this frame is

$$(\Delta\tau)^2 = -\Delta t^2 \ . \tag{1.10}$$

Therefore, the proper time between two events is the time elapsed as seen by an observer moving on a straight line between the events. Note that, since they move on null curves, *proper time for massless particles is not well defined.*

In another frame R' that is moving with velocity v with respect to R, the proper time reads

$$\begin{aligned} (\Delta\tau) &= \left[(\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2\right]^{1/2} \\ &= (\Delta t')\left[1 - v^2\right]^{1/2} \ , \end{aligned} \tag{1.11}$$

which is nothing but the phenomenon of *time dilation* [2][34].

Four Velocity and Four Momentum

In the following, we will adopt the notation $\vec{A}$ or, in component notation A^μ for spacetime four vectors. By analogy with the interval, the scalar product between any 4-vectors $\vec{A}$ and $\vec{B}$ is defined as

$$\vec{A}.\vec{B} = A^\mu B_\mu = \eta_{\mu\nu} A^\mu B^\nu = -A^0B^0 + A^1B^1 + A^2B^2 + A^3B^3 \ . \tag{1.12}$$

By similarity, the scalar product is invariant in any inertial frame.

If a massive particle have the coordinates x^μ in some inertial frame, then its four velocity is defined by [22][32]

$$U^\mu = \frac{dx^\mu}{d\tau} \ . \tag{1.13}$$

Since particles must move at a velocity less than or equals the speed of light, i.e., they have $dx^i/dt \leqslant 1$. Using the definition above of four velocity and it norm, if follows that

$$\begin{aligned} U^\mu U_\mu = & -\left(\frac{dt}{d\tau}\right)^2 + \left(\frac{dx^i}{d\tau}\right)^2 = \left(\frac{dt}{d\tau}\right)^2 \left[-1 + \left(\frac{dx^i}{dt}\right)^2\right], \\ \Longrightarrow & \qquad U^\mu U_\mu < 0 \qquad \text{for massive particles} \end{aligned}$$

Moreover, a massive particle in general can have an acceleration, and hence, there is no inertial frame where it is at rest. However, at any moment , we can always define a *momentarily comoving reference frame* (MCRF), in which the particle is at rest at that moment only. In such a frame, we have $t = \tau, x^\mu =$constant, yielding $U^\mu = (1, 0, 0, 0)$ and $U^\mu U_\mu = -1$. Since the four velocity normalization is a scalar product, $U^\mu U_\mu = -1$, it holds in any inertial frame.

Similarly, we define the four momentum $p^\mu = (E, p^i)$, where E is the particle's energy. In any inertial frame, for a particle of mass m, the corresponding four momentum is

$$p^\mu = mU^\mu = m\frac{dx^\mu}{d\tau}, \tag{1.14}$$

and the normalization for four momentum is $p^\mu p_\mu = -m^2$. It turns out that this relation is valid for both types of particles

$$p^\mu p_\mu = -m^2 \qquad (\textit{for massive particles}), \tag{1.15}$$
$$p^\mu p_\mu = 0 \qquad (\textit{for massless particles}). \tag{1.16}$$

Note that, for massless particles, the four momentum p^μ is not related to any four velocity by a relation like (1.14), since the later is not well defined for this type of particles.

In the MCRF, where the particle is at rest, we have $p^\mu = (m, 0, 0, 0)$, while in an inertial frame moving with velocity v in the x direction with respect to the MCRF, we have [2][18]

$$p^\mu = (m\gamma, m\gamma v, 0, 0). \tag{1.17}$$

For small velocities $v \ll 1$, it is convenient to use the following expression as an approximation for the energy

$$E = m\gamma = m(1 - v^2)^{-1/2} \approx m + \frac{1}{2}mv^2, \tag{1.18}$$

This is the rest mass energy plus the Galilean kinetic energy.

Hence, in arbitrary frame we have

$$E^2 = m^2 + p^i p_i, \qquad \text{for massive particles.} \tag{1.19}$$

Consider an arbitrary observer who want to measure the energy of a particle. In the MCRF of this observer, its four velocity is $U^\mu_{obs} = (1, 0, 0, 0)$, while the particles four momentum is $p^\mu = (E, p^1, p^2, p^3)$, which give the following frame invariant expression

$$E = -p_\mu U^\mu_{obs}. \tag{1.20}$$

Why Are Photons Massless?

The reason why massive particles cannot move at speed c can be inferred from the momentum four vector expression. In any inertial frame,a particle of mass m have the four momentum vector $P^\mu = m(\gamma, v\gamma, 0, 0)$ and hence its energy is $E = m\gamma = m\left(1 - v^2\right)^{-1/2}$. If the particle were to

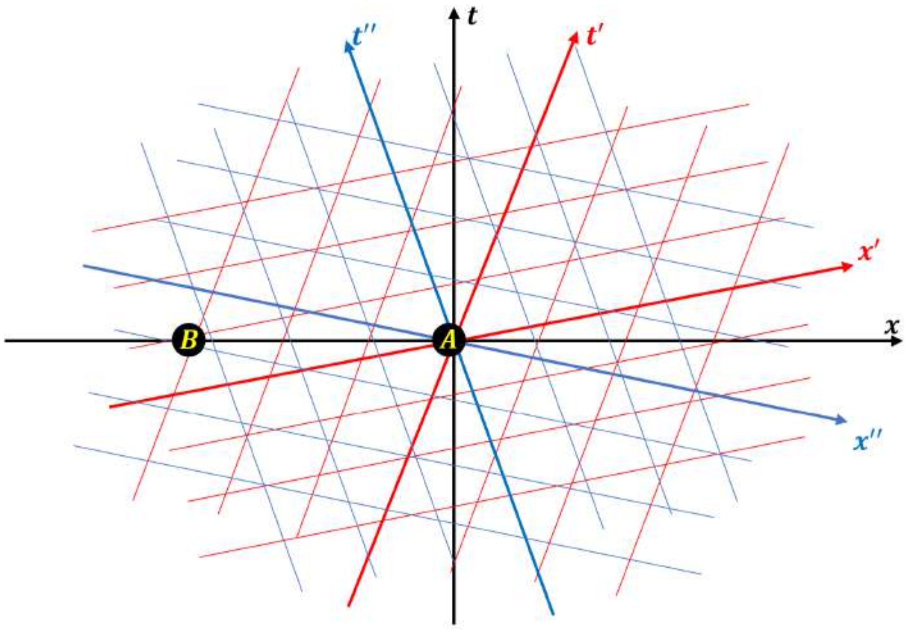


Fig. 1.2 – Two events A and B that are spacelike separated. Hence, there are frames in which they are simultaneous, and other frames in which they are not.

be accelerated to the speed of light, its energy would be infinite, which is physically senseless. Conversely, we conclude that any particle that is moving at c cannot have a mass. This means, for photons we have $p^\mu p_\mu = 0$, or equivalently [18][35]

$$E^2 = p^i p_i \ , \quad \text{for massless particles.} \tag{1.21}$$

Now, consider a photon of frequency ν that is moving in the x direction in a frame R. According to Quantum mechanics, the energy and momentum are hence given by $E = p^x = h\nu$. Suppose another frame R' is in relative motion with respect to frame R in the x direction. Using the Lorentz transformations (1.5), the energy $E' = h\nu'$ of this photon in R' is [2][18]

$$\begin{aligned} E' &= \gamma(E - \beta p^x) = \frac{(E - vp^x)}{(1 - v^2)^{1/2}} \\ h\nu' &= h\nu \frac{(1 - v)}{(1 - v^2)^{1/2}} = h\nu \left(\frac{1 - v}{1 + v}\right)^{1/2} \\ \nu' &= \nu \left(\frac{1 - v}{1 + v}\right)^{1/2} \ , \end{aligned} \tag{1.22}$$

which is nothing but the Doppler-shift formula.

Why Is c the Speed Limit?

An immediate consequence of the invariance of the spacetime interval is the following. Since the interval for a massive particle that is at rest ($\Delta x^i = 0, \Delta t \neq 0$) is negative in its comoving frame, any other observer will agree that massive particles move on timelike paths, i.e., a path constituted of timelike separated events. Since photons move on null paths, we conclude that no particle can move on a spacelike path. In other words, no particle can move faster than light.

Another reason why c is the maximum for particles is causality. Consider an inerial frame in which two events A and B are spacelike separated from each other. This means only only faster than light objects can travel between them, as portrayed in Fig. 1.2. Hence, there are frames where A and B are simultaneous, others where B precedes A, and others where A precedes B. Consequently, if an object were able to move faster than light in one reference frame from A to B, it would be traveling forward in time in some frames and backwards in time in others. This would lead to a violation of causality. This is one of the arguments why particles cannot move faster than light [36].

We can draw a similar conclusion from the Lorentz transformations. If the relative speed of the two frames exceeds the speed of light, the factor γ appearing in (1.5) would be imaginary, yielding an imaginary t' and x', which is physically meaningless [36].

In SR, space and time are intertwined and no longer treated separately. Unlike in Newtonian spacetime, where only rotations in space are allowed, SR allows for rotations of space and time into each other. This leads to the loss of the universal notion of simultaneity that exists the Newtonian case. If two events are simultaneous in one coordinate system, that is occurring at the same time coordinates, there exist other coordinate systems where these events are not simultaneous. In other words, the spacial slice of $t =$constant in one frame will be generally different from those of $t' =$constant in another.

1.2 Gravity and Special Relativity

SR remains a good theory, as long as gravity is not involved. Indeed, we will see in this section how the presence of gravity will impose limits on the validity of SR, and ultimately leads to the theory of General Relativity (GR). In SR, an inertial frame is one that can fill all of spacetime, in the sense that the whole space time can be covered by one single inertial frame whose coordinate points are always at rest with respect to the origin, and whose clocks run at the same rate everywhere. It turns out that this kind of construction is only possible in the absence of gravity. For this end, let's consider the following idealized thought experiment, originally suggested by Einstein [2][18][22][32].

1.2.1 The Gravitational Redshift

Suppose the following experiment (i) a particle of mass m is dropped and freely falls from rest from the top of a tower of height h on the surface of Earth, as portrayed in Fig. 1.3. Since its acceleration is $a = g$, the particle arrive at the ground with velocity $v^2 = v_0^2 + 2ah = 2gh$. Using the low velocities approximation for energy expression $E = m(1 - v^2)^{-1/2} \approx m + \frac{1}{2}mv^2$, the total energy at the ground is [2][32]

$$E_{bottom} = m + \frac{1}{2}mv^2 + 0[v^4] = m + mgh + 0[v^4]. \tag{1.23}$$

(ii)The experimenter on the ground has some photonic mass-energy converter that enables him to convert all of this energy into a single photon which has the same energy and travel upwards. (iii) Once the photon arrives at the top of the tower, its is again converted into a particle of rest mass $m_{top} = E_{top}$. This new mass is just the initial mass $m_{top} = m$, because, otherwise, any gain of mass would lead to a pertpetual motion, and any lose of mass would violate the energy

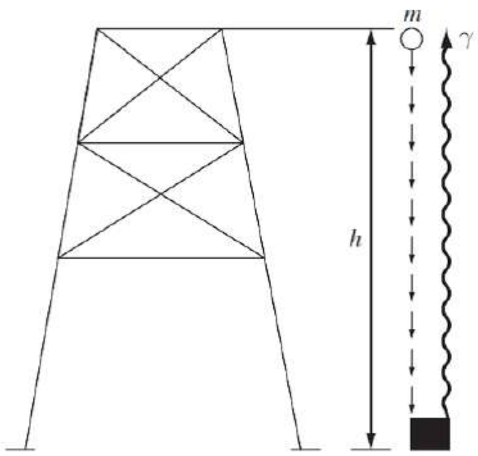


Fig. 1.3 – Einstein's thought experiment predicting the gravitational redshift.

conservation law. We thus end up with a redshift in the frequency of the photon given by

$$\begin{aligned} \frac{E_{top}}{E_{bottom}} &= \frac{m}{m + mgh + 0[v^4]} , \\ \frac{\nu_{top}(\text{lab frame})}{\nu_{bottom}(\text{lab frame})} &= 1 - gh + 0[v^4] . \end{aligned} \tag{1.24}$$

This expression predict a *gravitational redshift*: that a photon climbing in the Earth's gravitational field will lose some energy and get hence redshifted. Although this thought experiment is too idealized, predictions of the equation (1.24) has been verified to a remarkably high precision by Pound and Rebka [37], then by Pound and Snider [38]. Today, the most accurate verification of the gravitational redshift come from successful operating of the GPS navigation system that incorporates vital corrections to compensate for the redshift, without which it loses completely it's precision and becomes useless [39].

1.2.2 Are SR and Gravity Compatible?

The answer for this question can neither be yes, nor no. To see why, we will follow the argument made originally by Schild [40]. Consider the Pound-Rebka-Snider gravitational redshift from the point of view of a frame at rest on Earth, i.e., the laboratory frame, which is assumed to be inertial. The spacetime diagram appearing in Fig. 1.4 is the laboratory frame, where we can see that the top and bottom of the tower has vertical world lines since they are at rest relative to the laboratory. We will deal with the light as an electromagnetic wave, and look at the time Δt between two successive crests of the wave, which represent its period. As the light is emitted from the bottom to the top, the two successive crests of are represented by the wiggly lines. This lines has been drawn arbitrarily curved ,and not straight (null path), in order to account for any possibility that gravity may affect light in any unknown way. The fact that the gravitational field is not changing with time, its effect on both wave crests must be the same. Hence, based on the geometry of SR, the paths of the two crests are congruent and $\Delta t_{bot} = \Delta t_{top}$, or equivalently, in term of frequencies, $\nu_{bot} = \nu_{top}$. This is not in agreement with results of the Pound-Rebka-Snider experiment, meaning that our conclusion from Minkowski geometry (inertial frames) of SR for the laboratory frame are wrong. Thus, the frame at rest on Earth can not be inertial. However, it would be a hasty to say that this means inertial frames do not exist in the presence of gravity.

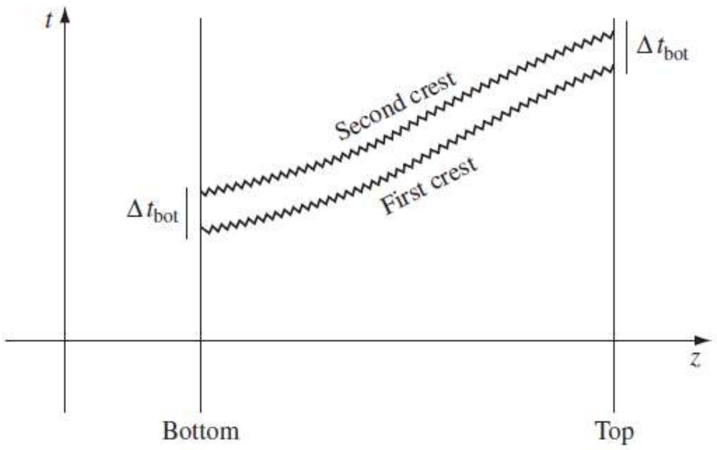


Fig. 1.4 – Spacetime diagram of the Pound-Rebka-Snider experiment in the laboratory frame.

All we have proved here is that, in the presence of gravity, the laboratory frame on Earth is not inertial. We shall see that we still can find inertial frames even in the presence of gravity.

1.2.3 Equivalence of Gravity and Acceleration

This is way that may help us in our search for inertial frames in the presence of gravity, the principle of inertia. This principle of inertia states that the particle persist in its state of rest or uniform motion unless acted upon by some forces. This principle is naturally a property of inertial frames. Thus, we can use the trajectory of the free particle, particles not acted upon by any force, to determine the desired inertial frames. For instance, in the case of the electromagnetic force, the inertial frames can be found by following the trajectories of the neutral particles. The same way can be used with all the other forces of nature, because their interaction with particles depend on their internal composition. But gravity is an exception. What makes gravity distinguished from all the other three forces is its *universality*: it affects all the particles in the same way, regardless of their internal composition. Consequently, there is no free particles (including photons as we have seen) in the presence of gravity that will mark the desired inertial frames [2][18][35].

However, even in the presence of gravity, there are frames in which particles do maintain a uniform velocity: the freely falling frames. If a frame is in free fall in the Earth's gravitational final, it will has the same acceleration as the freely falling particles do (at least, at low velocities, where Newtonian physics apply). Now, if we redefine gravity not to be a force, and free particles are only those acted on by gravity, we can draw the following important conclusion:in freely falling frames, free particles are moving at uniform velocity, and hence can be inertial frames.

The conclusion we have just made can be presented differently. Consider an accelerated rocked ship, in the empty space, far enough of any gravitational source. The unfixed objects relative to the rocket, which are free, appear to be accelerated towards the rear of the ship, with the same acceleration, regardless of their internal composition. Thus true inertial frames are those falling towards to the rear of the ship, at the same acceleration as free particles. However, the fixed object relative to the rocked appears to gain weight equal to the force required to keep it accelerating, that is proportional to their mass. By comparison, in the case of presence of gravity, the free particles are those in free fall, the inertial frames are the freely falling ones, while the force of weight is just the normal force, which we know is proportional to the mass of the object. Thus, we conclude that *the motion of freely falling particles in a uniform gravitational fields is the same as in uniformly accelerated frames, relative to inertial frames*. This is the equivalence principle between gravity and acceleration, a corner stone of Einstein theory of GR [2][22].

1.2.4 Inertial Freely Falling Frames

We can use the gravitational redshift experiment to prove that the equivalence principle, i.e., that the freely falling frames are inertial [32]. Let's consider this experiment from the point of view of a the frame that started its free fall at the moment photon was emitted from the bottom. When the photon reaches the receptor at the top, it has moved a distance h and hence has taken time $\Delta t = h$. Suppose the z-axis for both frames is directed up, so that, at the reception moment, the falling frame has acquired velocity $v = g\Delta t = gh$ in the negative z -direction relative to the laboratory frame. When the photon reach the receptor, for the falling fame, the photon's energy is $E' = h\nu'$, while for the laboratory frame, photon's energy and momentum are $E = h\nu$, $p_z = h\nu$ respectively. Using the Lorentz transformation between both frames, the resulting frequency can be obtained by [32]

$$\begin{aligned} E' &= \gamma(E - \beta p^x) = \frac{(E + vp^x)}{(1 - v^2)^{1/2}} , \\ h\nu' &= h\nu \frac{(1 + v)}{(1 - v^2)^{1/2}} \approx h\nu\,(1 + v) + 0[v^4] , \end{aligned}$$

$$\frac{\nu_{top}(\text{falling frame})}{\nu_{top}(\text{lab frame})} = 1 + gh + 0[v^4] . \tag{1.25}$$

Comparing (1.24) with (1.25), we get $\nu_{top}(\text{falling frame}) = \nu_{bottom}(\text{lab frame})$. But, the emission frequency is the same for both the falling and the laboratory frame because they were both initially at rest relative to each other. Hence, we have $\nu_{top}(\text{falling frame}) = \nu_{bottom}(\text{falling frame})$, meaning there is no redshift in the freely falling. This is a confirmation of the equivalence principle, in particular, that freely falling frames are inertial frames in the presence of gravity.

1.2.5 The Equivalence Principle

We have seen previously that frames accelerated uniformly relative to inertial frames are equivalent to uniform gravitational fields. The equivalence principle can be stated in an alternative equivalence way. The Newton law of the gravitational force $\mathbf{F} = m_g\ \mathbf{g}$, influencing a particle of gravitational mass m_g immersed in a gravitational field of intensity g. On the other hand, the Newton's second law $\mathbf{F} = m_i\ \mathbf{a}$, where $\mathbf{F}$ is the vectorial sum of forces acting on a body of inertial mass m_i and acceleration a. In the presence of the gravitational field only, we have $m_g\ \mathbf{g} = m_i\ \mathbf{a}$. Hence, the only way that gravity will affect all particles with the same acceleration, regardless of their internal composition is to set m_g/m_i =constant. Experiments have repeatedly shown that this ratio is one, hence the equivalence principle can be stated as the equality between the gravitational and inertial masses $m_g = m_i$ [34].

There are three important remarks concerning this equivalence principle. The first one concerns the spatial and temporal extensions of the freely falling frames. Although they are inertial, in the sense that free particles maintain uniform velocities relative to them, free falling frames can only fill all of spacetime if the gravitational field is uniform. In a more general gravitational fields, there is no single freely falling frame that can fill all of spacetime, as in the case of SR. The reason behind this restriction are the non-uniformities of the gravitational field, as we shall see. Consider for instance the Earth's gravitational field. A freely falling frame would not be inertial if it extends too horizontally, because then the direction of free fall will change. Moreover, due

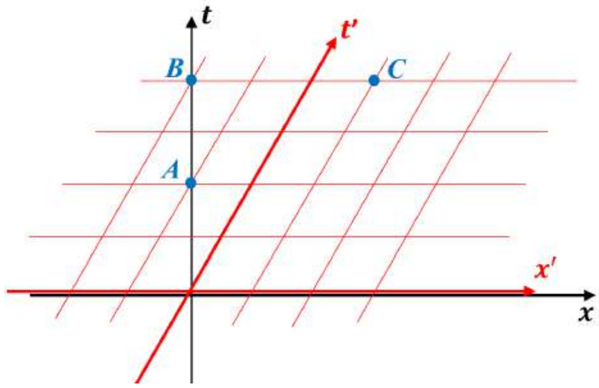


Fig. 1.5 – The Newtonian spacetime diagrame for two observers.

to the change of the strength of Earth's gravity with height, a freely falling frame can be inertial only in limited period of time. This would impose another restriction on the vertical extension of the inertial frame. This lead to the conclusion that, in nonuniform gravitational fields, freely falling frames are inertial only locally [2][16][18].

The second point is that there are an infinity of freely inertial frames at any point. Although they all accelerate towards Earth's center with the same rate, they have different velocities and different direction of spatial axes.

The third remark is that, the aforementioned versions of the equivalence principle refer to how bodies behave only under the influence of gravity. But what about the behavior of particles under an experiment other that falling? For this reason, it is called the *Weak Equivalence Principle*, or *WEP*. The more general version of the equivalence principle, called the *Einstein Equivalence Principle* (*EEP*), states that *the* law of physics have the same local form in a freely falling inertial frame as they do *in SR ; it is impossible to detect the existence of a gravitational field by means of local experiment* [22][35].

1.3 Three Spacetimes

Now, we will summarize the main properties of the three spacetimes we deal with in physics: the Newtonian spacetime of nonrelativistic physics, the flat and curved spacetimes of relativity.

1.3.1 Newtonian Spacetime

In Newtonian theory can be regarded as a theory of spacetime, where space and time are treated differently. We can think of it as an infinite number of three-dimensional Euclidean slices of space, each one is parameterized by a fixed value of time. Mathematically, this manifold is called $\mathbf{R}^3 \times \mathbf{R}$. Consider two arbitrary events A and B in the Newtonian spacetime, and two inertial frames R and R' in relative motion with uniform velocity, as portrayed in Fig. 1.5. For frames R and R', the time separation Δt_{AB} is invariant, no matter how much the spatial separation between A and B is. However, the two frames the spatial separation (distance) Δl_{AB} between events A and B can be invariant for R and R' only if the time separation between them is zero $\Delta t_{AB} = 0$, i.e., A and B are simultaneous.

Otherwise, frames R and R'will measure different spatial distances for events that are not happening at the same time. Therefore, in Newtonian theory, time is absolute, in the sense that all observers will agree on the time elapsed between any two events, while distance is not. We cannot hence define a metric on the whole spacetime, i.e., an invariant measure of spacetime distances that combine both space and time. We can only define an Euclidean metric $\eta_{\alpha\beta} = diag(1, 1, 1)$ on each three-dimensional space slice at a fixed time. This is a direct result for the structure $\mathbf{R}^3 \times \mathbf{R}$ [16][32].

1.3.2 SR Spacetime

In SR the situation is different. Einstein's second postulate of SR, the universality of the speed of light, imposed as we saw previously the invariance of the spacetime interval. Inertial frames will neither agree on time separation nor on spatial separation between any two events. They will only agree on the spacetime interval between them. The spacetime interval is the invariant measure of distances on the whole spacetime, that is, distances that unify both space and time measures. Mathematically then, SR spacetime has a simpler structure than Newton's. SR spacetime is a four-dimensional manifold $\mathbf{R}^4$, along with a Lorentzian metric $\eta_{\alpha\beta} = diag(-1, 1, 1, 1)$. Since the metric is constant throughout spacetime, in an inertial frame, coordinate points are always at rest with respect to the origin, and clocks run at the same rate everywhere [18][22][42].

1.3.3 GR Spacetime

Now, we will see how the notion of world lines is crucial in probing the spacetime geometry. As we know, SR spacetime can be described by one single global inertial frame, where free particles are the ones which are not acted upon by any force, except gravity. The world line of such free particles is a straight line. World lines of two free particles that started parallel, i.e., moving at the same velocity, will continue to be parallel indefinitely. Geometrically, this is called parallelism axiom, and is one property of Euclidean, or equivalently, flat geometry.

However, in the existence of gravity, the spacetime is more complicated. It accept only local inertial frames, the freely falling ones. Indeed, due to the presence of non-uniformities of the gravitational field, the freely falling frames cannot fill the whole spacetime, since it is not falling freely everywhere and all the time. Consider Earth's gravitational field, where two particles start a free fall from two close spatial points of the same height. In the first few seconds, their (three) velocities and world lines are parallel. However, after a while, they will get closer to each other as they approach the Earth, since they are actually falling towards the center of Earth. Their velocities and wold lines are no longer parallel. Thus, due to the presence of gravity, parallelism of world lines, and hence flat geometry, can only apply locally. Geometrically, the spaces that have this property are called curved manifolds, i.e., a differentiable manifold with a dynamic metric $g_{\alpha\beta}(x^i)$ that only locally can take its flat form $\eta_{\alpha\beta}$. The metric dependency on coordinates $g_{\alpha\beta}(x^i)$ generally means that the coordinate points are no longer at rest with respect to the origin, and clocks are not necessarily running at the same rate everywhere. Hence, due to the existence of the curvature, curved manifolds are flat only locally. This means that gravity is nothing but a spacetime curvature [2][22][32]. In this context, straight lines are generalized to geodesics, paths that maintain parallel, only locally.

An immediate consequence of the absence of a global inertial frame, is that it is physically meaningless to talk about relative velocities of far away objects. Since the inertial frame for each of these objects is falling at completely different velocity, relative velocity has no meaning. For this reason, in the context of GR, we can allow for relative velocities higher than the speed of light, without any violation of SR. This is exactly the case of relative velocities between far away galaxies, for example.

Another consequence is that inertial frames are no longer preferred, as is the case in SR. In GR, at each event, we can build a frame that is inertial (unaccelerated) only locally, which means that it is globally not inertial. In this sense, we say that GR allow for accelerated frames and hence all frames are equally acceptable.

1.4 Physics in Curved Spacetimes

As we have discussed earlier, the gravity is universal so that it influence all sorts of matter in the same way. This led Einstein to think of it as a property of the background spacetime rather than a conventional force. Therefore, the construction of a theory of General Relativity as a geometrical theory of gravity go through the following steps:

- The first step is the identification of spacetime as a manifold: SR imply that spacetime is a four-dimensional differentiable manifold with a metric (Riemannian manifold).
- The second step is to identify that SR spacetime is flat: at any event, the metric of spacetime can be put in the Lorentz form $\eta_{\alpha\beta}$ by an appropriate choice of coordinates.

Having said this, introducing gravity to the theory is through answering the following two questions:
(i) How the presence of a gravitational field influence the behavior of matter? In other words, how the presence of spacetime curvature influence the laws of physics? Practically, this is searching for the relativistic analog of Newton's second law in the presence of a gravitational field ϕ,

$$\vec{a} = \vec{\nabla}\phi \ . \tag{1.26}$$

(ii) How the presence of matter on the other hand creates the gravitational field? This is equivalent to searching for the relativistic version of Poisson's equation for the gravitational field ϕ created by mass density ρ,

$$\vec{\nabla}^2\phi = 4\pi G\rho \ , \tag{1.27}$$

G being Newton's constant of gravity. Now, we will answer these couple of questions.

- The third step is to find how gravity influence physical laws. The Einstein Equivalence Principle (EPP) implies that any local physical experiment, not involving gravity, will have the same results if performed in a freely falling inertial frame as if it were performed in the flat spacetime of SR. Mathematically, this means that the tensorial expressions of the laws of physics are the same in locally inertial frames as in SR. Practically, this means we locally have: [18][35]
  - The laws are written in a tensorial form,
  - Partial derivatives ∂_μ are replaced by covariant derivatives ∇_μ ,
  - SR flat metric $\eta_{\alpha\beta}$ is replaced by GR metric $g_{\alpha\beta}$,
  - The Riemann tensor $R^{\alpha}_{\beta\gamma\delta}$ cannot appear in any law.

  This steps are known as the *minimal-coupling-principle*.

As an example, we can look for the law of motion of free particles in the presence of gravity. In the absence of gravity, spacetime is flat and free particles move on straight world lines, i.e. with zero acceleration. Hence their equation of motion is

$$\frac{d^2x^i}{d\lambda^2} = \frac{dx^\alpha}{d\lambda}\partial_\alpha \frac{dx^\mu}{d\lambda} = 0 \tag{1.28}$$

, where λ the world line parameter. The generalization of the straight line equation in curved spacetime is the geodesic equation,

$$\frac{dx^\alpha}{d\lambda}\nabla_\alpha \frac{dx^\mu}{d\lambda} = \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0 \ , \tag{1.29}$$

where $\Gamma^\mu_{\alpha\beta}$ are the metric connections. In the Newtonian limit where [16][18][31]

▶ particles velocities are small compared to the speed of light, $dx^i/dt \ll 1$,

▶ the gravitational field is weak so that the metric is almost flat, $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$, with $|h|_{\alpha\beta} \ll 1$,

▶ the gravitational field is static, $\partial_0 g_{\alpha\beta} = 0$,

we can prove that (1.29) reduce to (1.26), as expected.

1.4.1 Einstein Field Equations

The fourth step now is to tackle the second question we asked previously: How matter create curvature? In other words, what is the relativistic supersede for Poisson's equation,

$$\vec{\nabla}^2\phi = 4\pi G\rho \ . \tag{1.30}$$

Einstein himself used an informal reasoning. In the right-hand side of this equation, the field source of the field is the mass density ρ. However, since mass and energy are interconvertible in relativity, a relativistic version of this equation should have all forms of energy and momentum as a source. The only way to represent this in a frame independent (covariant) way is the energy-momentum tensor $T^{\mu\nu}$. On the other hand, the left-hand side suggest that its generalization to relativity should be second order differential operator on the field, that is, on the metric $g_{\mu\nu}$. Thus, the desired equation would have the form, up to a proportionality constant k,

$$O[g]^{\mu\nu} = kT^{\mu\nu} \ , \tag{1.31}$$

where O should be a second order differential operator that produces a $\binom{2}{0}$ tensor, in order to get a tensorial equation. Plausible candidates for O that are both $\binom{2}{0}$ tensors and are constructed from $g_{\mu\nu}$, $\partial_\sigma g_{\mu\nu}$ and $\partial_\sigma\partial_\rho g_{\mu\nu}$ are the Ricci tensor $R^{\mu\nu}$, the Ricci scalar $R = R^\mu_{\ \mu}$. In fact, any tensor of the form

$$O^{\mu\nu} = R^{\mu\nu} + \alpha g^{\mu\nu} R + \Lambda g^{\mu\nu} \ , \tag{1.32}$$

will satisfy those requirements, provided α and Λ are constants. In fact there is another requirement that should be taken into account: the EEP demands that $T^{\mu\nu}$ be locally conserved, i.e., $\nabla_\mu T^{\mu\nu}$. This will fix the value of the first constant, $\alpha = 1/2$ [32]. For the sake of simplicity, we will drop the second constant Λ at the present, so that the field equations (1.31) becomes

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = kT^{\mu\nu} \ . \tag{1.33}$$

These are the Einstein field equations. In the Newtonian limit, described previously, these equations reproduce as expected Poisson's equation (1.27), if the we set $k = 8\pi G$. Using the geometrized unite system (where $c = G = 1$) and (1.32), the general form of Einstein's equations is [18][22][35]

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R + \Lambda g^{\mu\nu} = 8\pi T^{\mu\nu} \ . \tag{1.34}$$

Occasionally, we will need to write Einstein equations in another form. Taking $\Lambda = 0$ and contracting both sides of (1.34) give (in four dimensions)

$$R^{\mu\nu}g_{\mu\nu} - \frac{1}{2}g^{\mu\nu}Rg_{\mu\nu} = 8\pi T^{\mu\nu}g_{\mu\nu} \qquad \Longrightarrow \qquad -R = 8\pi T \ , \tag{1.35}$$

where $T = T^{\mu\nu}g_{\mu\nu}$. Inserting $R = 8\pi T$ in (1.33) yields the following useful form or Einstein equations

$$R^{\mu\nu} = 8\pi\left(T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T\right) \ . \tag{1.36}$$

In this case, the field equations in vacuum will take the following simple form

$$R^{\mu\nu} = 0 \ . \tag{1.37}$$

1.4.2 Derivation of Einstein Equations From the Least Action Principle

Now, we will derive the field equation in a more formal way, from the least action principle. In SR, the action of an arbitrary field φ^i is [22][31]

$$S = \int \mathcal{L}(\varphi^i, \partial_\mu\varphi^i)dx^4 \ , \tag{1.38}$$

where $\mathcal{L}$ is the Lagrangian density of the field. The transition to curved spacetimes is done by the generalization of derivative $\partial_\mu \longrightarrow \nabla_\mu$ and proper volume $dx^4 \longrightarrow \sqrt{-g}dx^4$, where g is the metric determinant. Thus, the action S above in GR becomes

$$S = \int \mathcal{L}(\varphi^i, \nabla_\mu\varphi^i)\sqrt{-g}dx^4 \ , \tag{1.39}$$

and $\mathcal{L}$ is a scalar rather than a density.

Since the dynamical variables of the field is $g_{\mu\nu}$, any nontrivial scalar that will made $\mathcal{L}$ must contain second derivatives of $g_{\mu\nu}$. That's because at any point, we can set $g_{\mu\nu}$ in its canonical form and its first derivatives to zero. It turns out that the only independent scalar constructed from the metric and its first and second derivatives is the Ricci scalar $R = R_{\mu\nu}g^{\mu\nu}$[16], hence the action S should be of the form

$$S_H = \gamma\int R\sqrt{-g}dx^4 \ = \gamma\int R_{\mu\nu}g^{\mu\nu}\sqrt{-g}dx^4 \tag{1.40}$$

where γ is a constant to be determined latter. This action is called the *Hilbert action* [35][31]. For simplicity, instead of plugging the Hilbert metric into Euler-Lagrange equations, as we usually do, we will directly vary the action S_H with respect to the dynamical variables $g_{\mu\nu}$. Moreover, we will use the fact that stationary point for the metric, where $\delta g_{\mu\nu} = 0$, are stationary point for its inverse $g^{\mu\nu}$, since we have

$$\delta g^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}\delta g_{\alpha\beta} \ . \tag{1.41}$$

Hence, varying equation (1.40) yields

$$\begin{aligned} \delta S_{HE} &= \gamma\int\left(\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} + \sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu} + R\delta\sqrt{-g}\right)dx^4 \\ &= \delta S_1 + \sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu} + \delta S_2. \end{aligned} \tag{1.42}$$

Let us start with the first term δS_1. Using the expression $R^{\rho}_{\mu\lambda\nu}$ in terms of the Christoffel symbols $\Gamma^{\rho}_{\nu\mu}$ (see appendices), we have

$$\begin{aligned} R^{\rho}_{\mu\lambda\nu} &= \partial_\lambda \Gamma^{\rho}_{\nu\mu} + \Gamma^{\rho}_{\lambda\sigma}\Gamma^{\sigma}_{\nu\mu} - \partial_\nu \Gamma^{\rho}_{\lambda\mu} - \Gamma^{\rho}_{\nu\sigma}\Gamma^{\sigma}_{\lambda\mu} \\ &\Rightarrow \delta R^{\rho}_{\mu\lambda\nu} = \partial_\lambda \delta\Gamma^{\rho}_{\nu\mu} + \delta\Gamma^{\rho}_{\lambda\sigma}\Gamma^{\sigma}_{\nu\mu} + \Gamma^{\rho}_{\lambda\sigma}\delta\Gamma^{\sigma}_{\nu\mu} - \partial_\nu \delta\Gamma^{\rho}_{\lambda\mu} - \delta\Gamma^{\rho}_{\nu\sigma}\Gamma^{\sigma}_{\lambda\mu} - \Gamma^{\rho}_{\nu\sigma}\delta\Gamma^{\sigma}_{\lambda\mu}. \end{aligned} \tag{1.43}$$

Since the variation of Christoffel symbols $\delta\Gamma^{\rho}_{\nu\mu}$ can be expressed as $\Gamma^{\rho}_{\nu\mu} \longrightarrow \Gamma^{\rho}_{\nu\mu} + \delta\Gamma^{\rho}_{\nu\mu}$, it is a tensor, and thus its covariant derivative is

$$\begin{aligned} \nabla_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) &= \partial_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) + \Gamma^{\rho}_{\lambda\sigma}\delta\Gamma^{\sigma}_{\nu\mu} - \Gamma^{\sigma}_{\lambda\nu}\delta\Gamma^{\rho}_{\sigma\mu} - \Gamma^{\sigma}_{\lambda\mu}\delta\Gamma^{\rho}_{\sigma\nu} \ . \\ \nabla_\nu(\delta\Gamma^{\rho}_{\lambda\mu}) &= \partial_\nu(\delta\Gamma^{\rho}_{\lambda\mu}) + \Gamma^{\rho}_{\nu\sigma}\delta\Gamma^{\sigma}_{\lambda\mu} - \Gamma^{\sigma}_{\nu\lambda}\delta\Gamma^{\rho}_{\sigma\mu} - \Gamma^{\sigma}_{\nu\mu}\delta\Gamma^{\rho}_{\sigma\lambda} \end{aligned} \tag{1.44}$$

This expression will allow us to rewrite (1.43) as

$$\begin{aligned} \delta R^{\rho}_{\mu\lambda\nu} &= \left[\partial_\lambda \delta\Gamma^{\rho}_{\nu\mu} + \Gamma^{\rho}_{\lambda\sigma}\delta\Gamma^{\sigma}_{\nu\mu} - \delta\Gamma^{\rho}_{\nu\sigma}\Gamma^{\sigma}_{\lambda\mu}\right] - \left[\partial_\nu \delta\Gamma^{\rho}_{\lambda\mu} + \Gamma^{\rho}_{\nu\sigma}\delta\Gamma^{\sigma}_{\lambda\mu} - \delta\Gamma^{\rho}_{\lambda\sigma}\Gamma^{\sigma}_{\nu\mu}\right] \\ &= \left[\nabla_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) + \Gamma^{\sigma}_{\lambda\nu}\delta\Gamma^{\rho}_{\sigma\mu}\right] - \left[\nabla_\nu(\delta\Gamma^{\rho}_{\lambda\mu}) + \Gamma^{\sigma}_{\nu\lambda}\delta\Gamma^{\rho}_{\sigma\mu}\right] \\ &= \nabla_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\rho}_{\lambda\mu}) \ . \end{aligned} \tag{1.45}$$

Hence, the term δS_1 becomes

$$\begin{aligned} \delta S_1 &= \gamma \int dx^4 \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} = \gamma \int dx^4 \sqrt{-g} g^{\mu\nu} \delta R^{\lambda}_{\mu\lambda\nu} \\ &= \gamma \int dx^4 \sqrt{-g} g^{\mu\nu} \left[\nabla_\lambda(\delta\Gamma^{\lambda}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\lambda}_{\lambda\mu})\right] \\ &= \gamma \int dx^4 \sqrt{-g} \nabla_\sigma \left[g^{\mu\nu}(\delta\Gamma^{\sigma}_{\nu\mu}) - g^{\mu\sigma}(\delta\Gamma^{\lambda}_{\lambda\mu})\right] \ , \end{aligned} \tag{1.46}$$

where we have used metric compatibility $\nabla_\sigma g^{\mu\nu} = 0$ and relabeled some dummy indices. This is an integration over the volume of the whole spacetime of a divergence term that can be transformed, using Stokes theorem, into a boundary term at infinity

$$\delta S_1 = \gamma \int_\Sigma dx^4 \sqrt{|g|} \nabla_\sigma [...] = \gamma \int_{\partial\Sigma} dx^3 \sqrt{|\varepsilon|} n_\sigma [...] \ , \tag{1.47}$$

where ε and n_σ are determinant and normal on the hypersurface $\partial\Sigma$. If we make variations vanish at infinity, this boundary term is zero [31][35].

Now we turn to calculating δS_2. Using the fact that, for any square matrix A with $\det A \neq 0$, we have

$$\ln(\det A) = \mathrm{Tr}(\ln A) \ , \tag{1.48}$$

and the variation of each side

$$\frac{\delta(\det A)}{\det A} = \mathrm{Tr}\left(A^{-1}\delta A\right) \implies \delta(\det A) = \det A \ \mathrm{Tr}\left(A^{-1}\delta A\right) \ . \tag{1.49}$$

The variation of the determinant of the metric, $g = \det g_{\mu\nu}$, can now be written as

$$\begin{aligned} \delta g &= g \ (g^{\mu\nu}\delta g_{\nu\mu}) \\ &= -g \ (g_{\nu\mu}\delta g^{\mu\nu}) \ , \end{aligned} \tag{1.50}$$

where we have used the fact that Kronecker delta do not change under variations

$$\delta(\delta^{\nu}_{\mu}) = 0 = \delta(g_{\nu\sigma}g^{\sigma\nu}) = \delta g_{\nu\sigma}g^{\sigma\nu} + g_{\nu\sigma}\delta g^{\sigma\nu} \ . \tag{1.51}$$

This allows us to find the variation appearing in δS_2

$$\begin{aligned} \delta\sqrt{-g} &= -\frac{1}{2}\frac{1}{\sqrt{-g}}\delta g \\ &= \frac{1}{2}\frac{1}{\sqrt{-g}}g \ \left(g_{\nu\mu}\delta g^{\mu\nu}\right) \\ &= -\frac{1}{2}\sqrt{-g}g_{\nu\mu}\delta g^{\mu\nu} \ . \end{aligned} \tag{1.52}$$

Now, using (1.52) and the fact that the contribution of $\delta S_1$is zero, the variation (1.42) becomes

$$\begin{aligned} \delta S_{HE} &= \gamma\int dx^4\left(\sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu} - \frac{1}{2}R\sqrt{-g}g_{\nu\mu}\delta g^{\mu\nu}\right) \\ &= \gamma\int dx^4\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\nu\mu}\right)\delta g^{\mu\nu} \ . \end{aligned} \tag{1.53}$$

Since $\delta g^{\mu\nu}$ is an arbitrary variation, stationary points of the action S_{HE} satisfy

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0 \quad \Longrightarrow \quad R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0 \ , \tag{1.54}$$

which are the Einstein field equations in vacuum.

However, the equations just derived are the Einstein equations in vacuum since we only considered the gravitational part of the action S. In order to obtain the Einstein equations in the presence of matter, we need to add another term S_m that represent matter, so that the total action becomes

$$S = S_H + S_m \ .$$

Following the same procedure, the variation of S leads to

$$\frac{\delta S}{\delta g^{\mu\nu}} = \gamma\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) + \frac{\delta S_m}{\delta g^{\mu\nu}} \ . \tag{1.55}$$

Defining the energy-momentum tensor as [16][22]

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} \ , \tag{1.56}$$

the principle of least action $\delta S/\delta g^{\mu\nu} = 0$ gives

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{1}{2\gamma}T_{\mu\nu} \ . \tag{1.57}$$

At the weak field limit, equations (1.57) must reduce to those of Poisson, i.e., (1.27). This will fix the proportionality constant to $\gamma = 1/16\pi G$, a ultimately give the complete Einstein equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu} \ . \tag{1.58}$$

1.4.3 Properties of Einstein Field Equations

Now, we will summarize some interesting properties about Einstein field equations:

The constant Λ appearing in (1.34) is called the *cosmological constant*. Few years after he published his equations, Einstein inserted the term of Λ in order to obtain static cosmological solutions, in accordance with the observations at that time. Later, after new observations confirmed the expansion of the universe, he rejected the term and regret he had ever invented [18][32]. Recently, astronomical observations revealed that the universe expansion is accelerating, and hence the need for the Λ term again to account for this acceleration.

Einstein field equations (1.33) can be derived also from the principle of least action [22][35]. The corresponding action then is the Einstein-Hilbert action S_{EH}, which has the following expression in terms of the metric determinant g and the Ricci scalar R

$$S_{EH} = \int \sqrt{-g} R dx^4 \ . \tag{1.59}$$

Since the tensors $R^{\mu\nu}$, $g^{\mu\nu}$ and $T^{\mu\nu}$are all symmetric, there are only ten independent Einstein equation. When we take into account the four equations of the Bianchi identity in the form $\nabla^\mu R_{\rho\mu} - (1/2)\nabla_\rho R = 0$, which represent four constraints on $R_{\rho\mu}$, the number of independent Einstein equations reduce to only six [18][42].

Einstein equations are nonlinear, which make them more difficult to be solved analytically, even in vacuum where $T^{\mu\nu} = 0$. Usually, simplifying assumptions are made, such as symmetries of spacetime or weak field, to make them less complex.

From a physical point of view, the nonlinearity of the Einstein equations is due to the coupling of the gravitational field to itself [16][42]. Unlike GR, field equations of Newtonian gravity, Poisson's equations, are linear. This means that the gravitational field due to the mass of the system Sun-Earth at some point is merely the sum of the gravitational field of each mass separately. However, in GR, the gravitational field due to the same system must be less than the sum of the field of each mass separately. This is because of the negative gravitational binding energy that make the inertial mass of that system is less than the sum of each mass. Therefore, according to the equivalence principle, the gravitational mass of the system is less, and so is hence the produced gravitational field.

Unlike Newtonian gravity, in GR pressure p can be a source of the field. Suppose for example the source of the field is a perfect fluid, which is a good approximation for many astrophysical application. The energy-momentum tensor then is

$$T^{\mu\nu} = (\rho + p)\, U^\mu U^\nu + p g^{\mu\nu} \ , \tag{1.60}$$

where U^μ is the fluid four-velocity and ρ and p are energy and pressure in the fluid rest-frame. The pressure is hence a source of the gravitational field in GR. Pressure arises in the random velocities of the fluid particles. If those random velocities are very small compared to the speed of light, pressure becomes negligible compared to the total mass density, $p \ll \rho$. This is the reason we do not see the effect of pressure in the solar system for example, where the Newtonian limit ($v \ll 1$) of Einstein equations is sufficient for studying most planets. Consequently, the pressure do not appear as a field source in the Newtonian limit of Einstein, i.e., Poisson's equation (1.27) [32].

Although the undisputed success of GR, it is still making one of the most difficult questions in physics now: how to make GR compatible with quantum mechanics? Gravity is maintaining the only yet nonquantized force of nature. Enormous theoretical efforts are being made in this concern, but till now, there is no robust theory of quantum gravity.

Chapter 2

Black Holes

In this chapter we will explore the concept of black hole, one of the most enigmatic predictions of General Relativity. We will see what are their types, structure and direct observational evidence. But first, let us start with the historical context.

2.1 Newtonian Black Stars

Black holes are spacetime regions where the gravitational field, i.e., the curvature, is so strong that nothing can escape it, not even light. Although it is a result of GR, the idea of black object that even light cannot escape it isn't in fact utterly new, but in fact goes back at least to the 17th century [18][32]. In Newtonian theory, the minimum velocity that would allow a particle to escape the gravitational field of a body is known as the escape velocity. For a spherical star of radius R, mass M and having a gravitational field $\phi = -GM/r$, consider a particle launched vertically from the surface. The energy conservation law enables us to get the escape velocity

$$\frac{1}{2}v^2 = GM/r \implies v = \sqrt{2GM/R}\ . \tag{2.1}$$

Obviously, this speed is inversely proportional to the star radius. In the late 1700's, the British physicist John Michell and French mathematician and physicist Pierre Laplace, independently, had considered the possibility of a star that whose escape velocity is larger than the speed of light, a black star. They understand that if the star can somehow be made denser, that is with less radius but the same mass, the escape velocity can be then larger than the velocity of light. The star therefore will be dark. Hence, setting $v = c$ in that last equation give us the minimum radius of a start to be a black star

$$R = \frac{2GM}{c^2}\ . \tag{2.2}$$

Surprisingly, this is exactly the relativistic formula for the radius of a the simplest type of black holes, the Schwarzschild black hole. Using the then known speed of light and mass of the Sun, the calculations of Michell and Laplace wound up with a star as massive as the Sun but compacted to the size of few kilometers. At that time, this seemed to be absurdly small, the fact that led them to abandon the idea later.

Today, the idea of astronomical objects, that are so dense that even light cannot escape, has a robust theoretical basis, and strong observational evidences. Their size range from the order of few kilometers to thousands of light years. They range in mass from few solar masses to billions. Black holes are not only being discovered all over the universe, but also being imaged to an unprecedented level. In 2022, the Event Horizon Telescope team remarkably captured the first image of the Sagittarius A*, the supermassive black hole at the heart of the Milky Way galaxy [10].

Although both seem to be black, Newtonian black stars and relativistic black holes are fundamentally different. For the black star, the surface was still there shining light. The light is constantly leaving the surface, but it cannot escape to infinity since gravity will eventually pull it back. In relativity however, the edge of a black hole, is not a material surface, but rather empty space left behind a collapsing matter that formed the star. Beyond this surface, gravity is so strong that light never leave it.

2.2 Relativistic Black Holes

After discussing the idea of black holes from a Newtonian perspective, we shall now examine black holes from a relativistic perspective.

2.2.1 Gravitational Collapse

The life of a star begins when clouds of interstellar, consisting generally of hydrogen and helium, collapse under the influence of gravity [17][18]. The accumulation of compressed clouds of these gases will raise the temperature at the core until the thermonuclear reaction ignites. Hydrogen atoms fuse to produce helium and release a large amount of energy, some of which in the form of radiation, and the star shines. At this stage, a balance is established between the gravitational contraction and the thermonuclear pressure. However, after most of the hydrogen in the star's core is exhausted, the gravitational contraction resumes, heating up the core gain. Eventually, the core temperature gets high enough that helium atoms start fusing, making other heavier elements and releasing large amount of energy. Again, a balanced state is reached between the gravitational contraction and the thermonuclear pressure. This process will continue until ^{56}Fe atoms are produced in the core. Since iron have the highest binding energy, it cannot be burn to produce heavier elements would absorb energy rather than releasing it. At this stage, the star's thermonuclear fuel run out and the corresponding thermal pressure cannot stop the gravitational contraction. At this point, two scenarios can happen.

The first scenario occurs when the star reach a new equilibrium state, supported against gravitational collapse by nonthermal sources of pressure. Since two electrons cannot occupy the same quantum state, this generate a pressure called the *electron Fermi pressure*. Stars ends up in equilibrium between the gravitational collapse and the electron Fermi pressure are called *White Dwarfs*. Although their mass is of the same order of the Sun, white dwarfs are more compact, having a radius of thousands of kilometers. If the star is however sufficiently massive, more than the *Chandrasekhar limit* (about 1.4 solar mass), the electron pressure cannot support the star, so that it will shrink more under its gravity. However, stars can still reach another equilibrium state. In this state, the star is supported by *neutron Fermi pressure* and nuclear forces. Stars in this state are called a *Neutron stars*. A neutron star having a mass of the order of the mass of the Sun could be $10km$ radius [16][22].

The second scenario happen when the star is more massive so that nothing can stop the gravitational collapse. The ongoing gravitational collapse will eventually lead to a black hole. These are the most compact objects in the universe, with masses ranging from few solar masses, called stellar-mass black hole, up to billions of solar masses, called supermassive black holes.

2.2.2 The No Hair Theorem

Spherical symmetry enormously simplify the study of the gravitational field generated by a spherical planet, star or black hole. As Birkhoff's theorem showed, this gravitational field do only depend of the mass of the central body. However, real astrophysical bodies are not perfectly spherical. This will entail a considerable complexities. In the case of a planet for example, the external gravitational field will depend on the density and profile of all mountains and valleys on the planet. The exact description of the external field then would need a huge, if not infinite, number of parameters.

Black holes, on the other hand, are extremely simple. Surprisingly, you need very few parameters to describe any stationary black hole solution in GR. This is stated by the following no-hair theorem: *stationary, asymptotically flat black hole solutions to GR coupled to electromagnetism that are nonsingular outside the event horizon are fully characterized by the parameters of mass , electric and magnetic charge, and angular momentum* [17][18][42]. A stationary solution means that the metric possesses a Killing vector that is timelike at infinity. Asymptotic flatness means that the metric can be becomes Minkowsian as we move well outside the black hole. According to this theorem, the four types of stationary black holes in GR are

	Non rotating	Rotating
Non charged	Schwarzschild	Kerr
Charged	Reissner-Nordström	Kerr-Newman

Table 2.1 – Types of black holes according to the no-hair theorem.

We will consider each case separately later.

2.2.3 Event Horizons

The most important features of any black hole are its event horizon and the singularity hidden there. If you cross the event horizon of a black hole, you will inevitably meet the singularity and you are totally doomed. However, if there is a singularity with out a horizon covering it, no matter how close you get from it, you can accelerate and get escape it. Hence, the most important feature of a black hole is the event horizon rather than the singularity inside it. In fact, a singularity without event horizon is not considered as a black hole. It is rather called naked singularity.

An event horizon is a hypersurface separating those spacetime points that are connected to infinity by a timelike path from those that are not. To give a more technical definition, we will need the following definitions. The causal past J^- of a region is the set of points we can reach from that region by moving along past-directed timelike paths. The future null infinity is the set of points that are approached asymptotically by future-directed null paths. Then, the *future event horizon is defined as the boundary of the causal past of future null infinity*, denoted by $\partial J^-(I^+)$ [22][17].

The importance of horizons in GR comes from the fact that they are almost inevitable in GR [35][42]. This is a result of the singularity theorems and the cosmic censorship conjecture. In short, *singularity theorems* (Penrose and Hawking) show that the evolution to a singularity in gravitational collapse is inevitable. On the other hand, according to the *cosmic censorship conjecture* (Penrose), singularities that form in gravitational collapse are typically hidden within an event horizon. If a singularity is not hidden behind an event horizon, signals from it can reach null future infinity. This is known as *naked singularity.* Thus, the cosmic censorship conjecture rule out the possibility of the formation of naked singularity in a gravitational collapse.

Now let us address the question of how to locate the event horizon? Generally, this is a difficult question if we are given a metric in an arbitrary coordinate system. How ever, if the metric is stationary, which is the case for the four types black hole, and written in a cleverly chosen coordinate system (t, r, θ, ϕ) [16], the event horizon r_H is the null hypersurface of constant r switches sign from positive to negative. Since the normal to this surface is $N_\mu = \nabla_\mu r = \partial_\mu r$, the null hypersurface of $r = r_H$ is

$$N_\mu N_\nu g^{\mu\nu} \mid_{r_H} = 0 \quad \Longrightarrow \quad g^{rr} \mid_{r_H} = 0 \ . \tag{2.3}$$

This equation will be sufficient to locate the event horizon for the four types of black holes.

2.3 Schwarzschild Black Hole

Schwarzschild black hole is going to be the first black hole we are going to explore. In Schwarzschild spherical coordinates (t, r, θ, ϕ), the corresponding metric is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1} \mathrm{d}r^2 + r^2 d\Omega^2 \ , \tag{2.4}$$

where M is the total mass of the spherical gravitating body, as we will see later. The gravitating body can in fact be, not only a black hole, but even a static spherically symmetric body, like a star of planet. However, in the latter case, the metric (2.4) can be applied only outside of the body, since it is a solution for Einstein equation in vacuum.

According *Birkhoff's theorem*, Schwarzschild metric is the unique spherically symmetric vacuum solution for Einstein's equations [17][35]. We will show that spherical symmetry also imply the solution is static. In differential geometry jargon, this means that the metric possesses a timelike Killing vector that is orthogonal to a family of spacelike hypersurfaces. Practically, this means that we can find a coordinate system where all the metric components are independent of time t and that there are no cross terms of the form $\mathrm{d}t\mathrm{d}x^i$. We shall see this in more detail when we derive the metric.

2.3.1 Derivation of Schwarzschild Metric

Geometrically, spherical symmetry means that the spacetime has the same killing vectors as those of a two dimensional sphere S^2, for which we know the metric is $d\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2$. Practically, in a general coordinate system $(a,\ b,\ \theta,\ \phi)$, this means that the metric should has the following form

$$ds^2 = g_{aa}(a,b)\mathrm{d}a^2 + g_{ab}(a,b)(\mathrm{d}a\mathrm{d}b + \mathrm{d}b\mathrm{d}a) + g_{bb}(a,b) + r^2(a,b)d\Omega^2 \ , \tag{2.5}$$

where g_{aa}, g_{ab}, g_{bb}, r are arbitrary functions of the coordinates a and b [16]. We can make the metric simpler by changing coordinates from $(a,\ b)$ to $(a,\ r)$ (or equivalently from $(a,\ b)$ to $(b,\ r)$), by solving $r(a,b)$ for b, than substituting in g_{aa}, g_{ab}, g_{bb}. Hence, in the coordinate system $(a,\ r,\ \theta,\ \phi)$, the metric will take the form

$$ds^2 = A(a,r)\mathrm{d}a^2 + B(a,r)(\mathrm{d}a\mathrm{d}r + \mathrm{d}r\mathrm{d}a) + C(a,r)dr^2 + r^2 d\Omega^2 \ . \tag{2.6}$$

Moreover, we can always get rid of the cross terms $(\mathrm{d}a\mathrm{d}b{+}\mathrm{d}b\mathrm{d}a)$ by doing the following. We defining a new function $t(a,r)$ so that

$$\mathrm{d}t = \frac{\partial t}{\partial a}\mathrm{d}a + \frac{\partial t}{\partial r}\mathrm{d}r \ ,$$
$$\mathrm{d}t^2 = \left(\frac{\partial t}{\partial a}\right)^2 \mathrm{d}a^2 + \left(\frac{\partial t}{\partial a}\right)\left(\frac{\partial t}{\partial r}\right)(\mathrm{d}a\mathrm{d}r + \mathrm{d}r\mathrm{d}a) + \left(\frac{\partial t}{\partial r}\right)^2 \mathrm{d}r^2.$$

We will enables us to rewrite our metric in another coordinate system (t, r, θ, ϕ), so that

$$\begin{aligned} ds^2 &= m(a,r)\mathrm{d}t^2 + n(a,r)\mathrm{d}r^2 + r^2 d\Omega^2 \\ &= m(a,r)\left(\frac{\partial t}{\partial a}\right)^2 \mathrm{d}a^2 + m(a,r)\left(\frac{\partial t}{\partial a}\right)\left(\frac{\partial t}{\partial r}\right)(\mathrm{d}a\mathrm{d}r + \mathrm{d}r\mathrm{d}a) \\ &\qquad + \left[m(a,r)\left(\frac{\partial t}{\partial r}\right)^2 + n(a,r)\right]\mathrm{d}r^2 + r^2 d\Omega^2 \ . \end{aligned}$$

Comparing with (2.6), we get the following three equations for three unknown functions $t(a,r)$, $m(a,r)$ and $n(a,r)$

$$\begin{aligned} m(a,r)\left(\frac{\partial t}{\partial a}\right)^2 &= A(a,r) \ , \\ m(a,r)\left(\frac{\partial t}{\partial a}\right)\left(\frac{\partial t}{\partial r}\right) &= B(a,r) \ , \\ m(a,r)\left(\frac{\partial t}{\partial r}\right)^2 + n(a,r) &= C(a,r) \ . \end{aligned}$$

Therefore, given the functions $A(a,r)$, $B(a,r)$ and $C(a,r)$, we can find the functions $t(a,r)$, $m(a,r)$ and $n(a,r)$, up to initial conditions for t. Using $t(a,r)$, we solve for $a(t,r)$ and substitute in $m(a,r)$ and $n(a,r)$, so that we get the desired form of the metric

$$ds^2 = m(t,r)\mathrm{d}t^2 + n(t,r)\mathrm{d}r^2 + r^2 d\Omega^2 \tag{2.7}$$

Next, we will chose t to be the time coordinate. Since our spacetime is Lorentzian, let us assume $m(t,r)$ must be negative whereas $n(t,r)$ (in fact this assumption is not always true, as shall see later inside the event horizon). This assumption can be made easier if we replace functions m and n with new ones α and β, so that the metric (2.8) becomes

$$ds^2 = -e^{2\alpha(t,r)}\mathrm{d}t^2 + e^{2\beta(t,r)}\mathrm{d}r^2 + r^2 d\Omega^2 \ . \tag{2.8}$$

That's everything we can relying on spherical symmetry. The rest of the derivation will rely on the Einstein's equations in vacuum. To use these equations, we need first to calculate the Christoffel symbols, Riemann and Ricci tensor the metric (2.8), and then insert them into Einstein's equations. Using the definition (see appendix), the nonvanishing Christoffel symbols corresponding to (2.8) are

$$\begin{aligned} &\Gamma^t_{tt} = \partial_t\alpha && \Gamma^t_{tr} = \partial_r\alpha && \Gamma^t_{rr} = e^{2(\beta-\alpha)}\partial_t\beta \\ &\Gamma^r_{tt} = e^{2(\alpha-\beta)}\partial_r\alpha && \Gamma^r_{tr} = \partial_t\beta && \Gamma^r_{rr} = \partial_r\beta \\ &\Gamma^\theta_{r\theta} = \frac{1}{r} && \Gamma^r_{\theta\theta} = -re^{-2\beta} && \Gamma^\phi_{r\phi} = \frac{1}{r} \\ &\Gamma^r_{\phi\phi} = -re^{-2\beta}\sin^2\theta && \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta && \Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta} \ , \end{aligned}$$

where the dot and prime imply derivative with respect to t and r respectively. The corresponding nonvanishing Riemann tensor components are

$$\begin{aligned}
R^t{}_{rtr} &= e^{2(\beta-\alpha)}\left[(\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta + \partial_t^2\beta\right] + \left[\partial_r\alpha\partial_r\beta - \partial_r^2\alpha - (\partial_r\alpha)^2\right] \\
R^t{}_{\theta t\theta} &= -r\partial_r\alpha e^{-2\beta} \\
R^t{}_{\phi t\phi} &= -r\partial_r\alpha e^{-2\beta}\sin^2\theta \\
R^t{}_{\theta r\theta} &= -r\partial_t\beta e^{-2\alpha} \\
R^t{}_{\phi r\phi} &= -r\partial_t\beta e^{-2\alpha}\sin^2\theta \\
R^r{}_{\theta r\theta} &= r\partial_r\beta e^{-2\beta} \\
R^r{}_{\phi r\phi} &= r\partial_r\beta e^{-2\beta}\sin^2\theta \\
R^\theta{}_{\phi\theta\phi} &= \sin^2\theta\left[1 - e^{-2\beta}\right] \ ,
\end{aligned}$$

whereas the nonvanishing Ricci tensor components are

$$\begin{aligned}
R_{tt} &= -\left[(\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta + \partial_t^2\beta\right] + e^{2(\alpha-\beta)}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\alpha\right] \\
R_{rr} &= e^{2(\beta-\alpha)}\left[(\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta + \partial_t^2\beta\right] + \left[\partial_r\alpha\partial_r\beta - \partial_r^2\alpha - (\partial_r\alpha)^2 + \frac{2}{r}\partial_r\beta\right] \\
R_{\theta\theta} &= e^{-2\beta}\left[(\partial_r\beta - \partial_r\alpha)\, r - 1\right] + 1 \\
R_{\phi\phi} &= R_{\theta\theta}\sin^2\theta \\
R_{tr} &= \frac{2}{r}\partial_t\beta \ .
\end{aligned} \tag{2.9}$$

These components must solve the field equations in vacuum,

$$R_{\mu\nu} = 0\,. \tag{2.10}$$

From $R_{tr} = 0$, we get $\partial_t\beta = 0$. Taking the time derivative of $R_{\theta\theta} = 0$, with $\partial_t\beta = 0$ yields

$$e^{-2\beta}\left(\partial_t\partial_r\alpha\right) r = 0 \quad \Longrightarrow \partial_t\partial_r\alpha = 0 \ , \tag{2.11}$$

which is true only when the temporal and radial dependence of the function α are separated. Thus, we can set

$$\beta(t,r) = \beta(r)\,, \tag{2.12}$$

$$\alpha(t,r) = f(r) + g(t)\,, \tag{2.13}$$

The $t-$term of the metric (2.8) becomes $-e^{2f(r)}e^{2g(t)}\mathrm{d}t^2$. However, we are always free to redefine the temporal coordinate to get rid of the factor $e^{2g(t)}$ so that $e^{2g(t)}\mathrm{d}t^2 \longrightarrow \mathrm{d}t^2$ and $\alpha(t,r) = f(r) = \alpha(r)$. therefore, the components of the metric (2.8) becomes time-independent [16][22]

$$ds^2 = -e^{2\alpha(r)}\mathrm{d}t^2 + e^{2\beta(r)}\mathrm{d}r^2 + r^2 d\Omega^2 \ . \tag{2.14}$$

At this point, we can draw the following remark: any spherically symmetric vacuum metric possesses a timelike Killing vector, since its components do not depend on t coordinate. this means that our metric (2.14) is *stationary*. In addition to this, the absence of cross terms of the form $\mathrm{d}t\mathrm{d}x^i$ means that our metric is *static*. Roughly speaking, a stationary metric can be thought of as "doing the same thing every time". On the other hand, static is a more restrictive property. A static metric can be thought of as "not doing any thing at all".

Now, to obtain the metric (2.4), we need to go back to (2.8) with the fact that β and α depend on r only

$$R_{tt} = e^{2(\alpha-\beta)}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\alpha\right] \tag{2.15}$$

$$R_{rr} = \left[\partial_r\alpha\partial_r\beta - \partial_r^2\alpha - (\partial_r\alpha)^2 + \frac{2}{r}\partial_r\beta\right] \ . \tag{2.16}$$

Using the field equations in vacuum $R_{\mu\nu} = 0$, we have

$$0 = e^{2(\beta-\alpha)}R_{tt} + R_{rr} = \frac{2}{r}\left[\partial_r\alpha + \partial_r\beta\right] \quad \Longrightarrow \quad \alpha = -\beta + C \ ,$$

where C is an integration constant. We can again absorb C by rescaling the time coordinate, $e^{2C}\mathrm{d}t^2 \longrightarrow \mathrm{d}t^2$, so that we obtain

$$\alpha = -\beta. \tag{2.17}$$

Now, from (2.9) and $\alpha = -\beta$, the $R_{\theta\theta} = 0$ equation yields

$$\begin{aligned} e^{2\alpha}\left[2\partial_r\alpha r + 1\right] = 1 &\Longrightarrow \ \partial_r\left[e^{2\alpha}r\right] = 1 \\ &\Longrightarrow \ e^{2\alpha}r = r - R_S \\ &\Longrightarrow \ e^{2\alpha} = 1 - \frac{R_S}{r} \\ &\Longrightarrow \ e^{2\beta} = \left(1 - \frac{R_S}{r}\right)^{-1} \ , \end{aligned}$$

where R_S is an integration constant called the *Schwarzschild radius*. Inserting these equations in (2.14), the metric take the form

$$ds^2 = -\left(1 - \frac{R_S}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{R_S}{r}\right)^{-1}\mathrm{d}r^2 + r^2 d\Omega^2 \ . \tag{2.18}$$

The only thing left is to interpret the constant R_S. From weak-field limit, i.e., when $r \gg 2M$, we know that the $tt-$component of the metric outside a spherical body of mass M is

$$g_{tt} = -\left(1 - \frac{2M}{r}\right) \qquad \text{(weak-field limit).} \tag{2.19}$$

Taking the limit $r \gg 2M$, g_{tt} for the metric (2.18) has the same form as g_{tt} of the weak-field limit above, and the constant R_S is

$$R_S = 2M \ .$$

Inserting this into (2.18), we get the desired metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2 d\Omega^2 \ , \tag{2.20}$$

the static spherically symmetric vacuum solution for Einstein's equations.

2.3.2 Properties of Schwarzschild Metric

In the following, we shall see the main properties of the Schwarzschild metric [17][22][35]

- *Asymptotic flatness*: at large r, the metric (2.20) becomes progressively Minkowskian. We say thus that the Schwarzschild metric is asymptotically flat.

- *Symmetries*: by construction, the Schwarzschild metric is spherically symmetric, hence is invariant under rotation about the three spatial axes. This imply the existence of three Killing vectors. More directly, since the metric (2.20) is independent of t and ϕ, the associated timelike and one rotational Killing vectors, respectively, are

$$\xi^\mu = (1,0,0,0)\,, \tag{2.21}$$
$$\psi^\mu = (0,0,0,1)\,. \tag{2.22}$$

Notice that the vector normal to t =constant hypersurfaces is given by

$$N^\mu = g^{\mu\nu}\nabla_\nu\,(t) = g^{\mu\nu}\partial_\nu\,(t) = g^{\mu\nu}\delta^t_\nu = g^{\mu t} = g^{tt}(1,0,0,0), \tag{2.23}$$

meaning that N^μ is proportional to ξ^μ. Hence, the existence of a time translation Killing vector ξ^μ that is normal to t =constant hypersurfaces confirms the fact that Schwarzschild spacetime is static [22][35]. Moreover, for a particle of four-velocity U^μ, the existence of two Killing vectors imply the existence of two conserved quantities E and L along the geodesics defined as

$$E = -\xi^\mu U_\mu = \left(1-\frac{2M}{r}\right)\frac{dt}{d\tau}\;, \tag{2.24}$$
$$L = \psi^\mu U_\mu = r^2\sin^2\theta\frac{d\phi}{d\tau}\;. \tag{2.25}$$

Since at large r the conserved constant E becomes $E = -dt/d\tau = -U^0 = E_{tot}/m$, where m is the rest mass and E_{tot} is total energy of the test particle, the conserved quantity E is interpreted as the total energy per unit rest mass. In addition, at low velocities, $\frac{dx^i}{dt} \ll 1$ and $\frac{dt}{d\tau} \approx 1$, which means $L \approx r^2\sin^2\theta\frac{d\phi}{dt}$. Hence, the conserved constant L is called the angular momentum per unit rest mass.

- *Mass parameter M*: the M appearing in (2.20) is the only parameter that characterize the spherically symmetric Schwarzschild black hole. Generally, M is not the Newtonian mass of the whatever body is curving spacetime, in the sense that it is merely the sum of the masses of the constituents of this body. The parameter M is rather the *total mass* of that body, in the sense that it includes all sources of energy, including the electromagnetic, nuclear, and even the gravitational binding energy of the constituents of the gravitating body. However, for a static source at large distances, the weak field limit is applied and the parameter M in (2.20) hence coincides with the Newtonian mass, so that it can be measured, for instance, using Kepler's third law. When source of curvature vanishes, the limit $M \longrightarrow 0$, the metric (2.20) becomes Minkowskian., exactly as expected.

- *Singularity r=0*: Obviously, some components of the metric (2.20) becomes infinite at $r = 0$. We know that the components of any tensor are coordinate-dependant. If a singularity is a result of a breakdown of the chosen coordinate system in some region, it is called coordinate singularity. But if the singularity is independent of the chosen coordinates, it is then a singularity in spacetime itself, a curvature singularity. Since curvature is measured by the Riemann tensor, a sufficient, though not necessary, condition for a curvature singularity is to test coordinate independent quantities constructed from $R^{\alpha}{}_{\beta\mu\nu}$. For the metric (2.20), the Kretschmann scalar is

$$R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} = \frac{48M}{r^6}\ . \tag{2.26}$$

This is sufficient to see that $r = 0$ is a curvature singularity. Any thing that falls there will be destroyed by the infinite gravitational forces.

- *Singularity r=2M:* at this surface, the g_{rr} component blows up, meaning it is a singularity. From the discussion above, we saw that is it is difficult to know whether this is a coordinate or curvature singularity. The fact that coordinate independent quantity (2.26) behave well at $r = 2M$ does not mean necessarily it is a coordinate and not curvature singularity. However, we will see later that, in a more appropriate coordinates, the surface $r = 2M$. For a spherically symmetric object, this surface is usually deep inside, and hence does not matter since Schwarzschild metric is valid only in vacuum. However, for a black hole, it turns out that this surface is of significant importance, since it represent its even horizon, as we shall see next.

2.3.3 The Surface $r = 2M$ in Schwarzschild Coordinates

In order to examine and understand the surface $r = 2M$ we will consider the radial null curves, for which $d\theta = d\phi = 0$. From (2.20) and for $r > 2M$, radial null curves are specified by

$$ds^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 = 0 \tag{2.27}$$

$$\implies \frac{dt}{dr} = +\left(1 - \frac{2M}{r}\right)^{-1} \qquad \text{outgoing light rays,} \tag{2.28}$$

$$\implies \frac{dt}{dr} = -\left(1 - \frac{2M}{r}\right)^{-1} \qquad \text{ingoing light rays,} \tag{2.29}$$

This is the slope of light rays world line in Schwarzschild coordinates. The plus and minus signs represent an outgoing and ingoing light rays, respectively. For large r, this slope is $dt/dr = \pm 1$, so that light cones are at 45°from the time axis as expected, since the metric is asymptotically flat. However, when ingoing light rays approach $r = 2M$, the slope goes to infinity $dt/dr = \pm 1$. As portrayed in Fig. 2.1, light cones closes up as we approach $r = 2M$. This means that it is impossible for any particle to cross the surface $r = 2M$. We will see that this apparent impossibility is just a coordinate illusion, so that particles can cross that surface without any problem [17][18].

There is another important phenomenon that we see in this coordinates. Consider an observer that is radially falling into the black hole and sending light signals back to another observer who is stationary very far away, at $r = \infty$. Suppose the light signals are sent separated by equal proper time intervals $\Delta\tau$ according to the falling observer's clock. Since spacetime there is flat for the stationary far away observer, the proper time is just the time coordinate t. From Fig.

2.1 we can see that, as the infalling observer approaches $r = 2M$, light cones get narrower, so that light signals will take larger and larger time intervals Δt to reach the observer at infinity. In other words, the observer at infinity will see, not only that the infalling observer never cross $r = 2M$, but also that light signals from him are infinitely redshifted.

Now let us examine the time taken by an infalling observer, but as measure by an observer at infinity. For a radially infalling observer, $d\theta/d\tau = d\phi/d\tau = 0$ and the four-velocity is $U^{\mu} =$

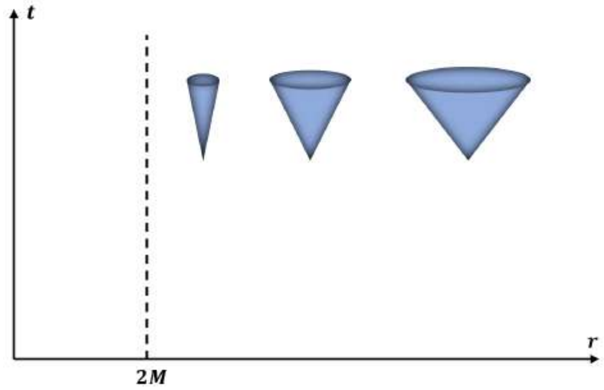


Fig. 2.1 – Light cones of Schwarzschild black hole in Schwarzschild coordinates. Notice that they close up near the surface $2M$.

$(dt/d\tau, dr/d\tau, 0, 0)$. Using (2.24), we obtain $U^t = dt/d\tau = E\,(-g_{tt})^{-1}$, and the normalization equation is

$$g_{\mu\nu}U^\mu U^\nu = g_{tt}e^2\,(-g_{tt})^{-2} + g_{rr}\,(U^r)^2 = -1 \tag{2.30}$$

$$\implies \quad (U^r)^2 = -\frac{E^2 + g_{tt}}{g_{tt}g_{rr}} = E^2 + g_{tt} \tag{2.31}$$

$$\implies \quad U^r = -\left(E^2 + g_{tt}\right)^{1/2} \;, \tag{2.32}$$

where the minus sign refers to an infalling observer, $dr/d\tau < 0$. Thus, we have

$$\frac{dr}{dt} = \frac{dr/d\tau}{dt/d\tau} = \frac{U^r}{U^t} = \frac{g_{tt}\,(E^2 + g_{tt})^{1/2}}{E} \;. \tag{2.33}$$

For simplicity, suppose that this observer started from infinity with a unit energy per rest mass, $E = 1$. The last equation becomes

$$\frac{dr}{dt} = -\left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r}\right)^{1/2} \quad \implies \quad dt = -\frac{dr}{(2M/r)^{1/2}\,(1 - 2M/r)}. \tag{2.34}$$

Setting $z^2 = r/2M$ and hence $dr = 4Mzdz$, the integral take the following form

$$\begin{aligned}
dt &= -\frac{4Mz}{z^{-1}\,(1 - z^{-2})}dz \quad \implies \quad dt = -4M\frac{z^4}{z^2 - 1}dz \;, \\
t &= -4M\int \frac{(z^2 - 1)\,(z^2 + 1) + 1}{z^2 - 1}dz \\
&= -4M\int \left[z^2 + 1 + \frac{1}{z^2 - 1}\right] dz \\
&= -4M\int \left[z^2 + 1 + \frac{1/2}{z - 1} - \frac{1/2}{z + 1}\right] dz \\
&= -4M\left[\frac{z^3}{3} + z + \frac{1}{2}\ln\left|\frac{z - 1}{z + 1}\right|\right] + t_0 \\
t &= -4M\left[\frac{1}{3}\left(\frac{r}{2M}\right)^{3/2} + \left(\frac{r}{2M}\right)^{1/2} + \frac{1}{2}\ln\left|\frac{(r/2M)^{1/2} - 1}{(r/2M)^{1/2} + 1}\right|\right] + t_0 \;,
\end{aligned} \tag{2.35}$$

where t_0 is the time coordinate at $r = 0$. At large r, the time coordinate t is well behaved. But when $r \longrightarrow 2M$, the third term blows up and $\Delta t \longrightarrow \infty$. Therefore, for an observer at infinity, where proper time is the $t-$coordinate, that the infalling observer never crossed the surface $r = 2M$.

Let us examine the elapsed time from the point of view of an infalling observer. As we have seen, for a radially infalling observer, the radial component of the four-velocity is $U^r = -\left(e^2 + g_{tt}\right)^{1/2}$. Similarly, assuming that $e = 1$, we obtain

$$\frac{dr}{d\tau} = U^r = -\sqrt{1^2 - \left(1 - \frac{2M}{r}\right)} = -\sqrt{\frac{2M}{r}}$$
$$\Longrightarrow \quad d\tau = -\sqrt{\frac{r}{2M}}dr \ . \tag{2.36}$$

Integrating from some finite $r > 2M$ to $r = 2M$ yields

$$\Delta\tau = \sqrt{\frac{2}{9M}}\left[r^{3/2} - (2M)^{3/2}\right] \ . \tag{2.37}$$

Clearly, the infalling observer experience a finite proper time interval $\Delta\tau$ as it reach $r = 2M$. This means that the singularity at $r = 2M$ is merely an artifact of the chosen coordinate system.

Now what about photons emitted by the infalling observer towards the one at infinity? To answer this question, let us rewrite (2.24) in the following form

$$dt = E\left(1 - \frac{2M}{r}\right)^{-1} d\tau \ , \tag{2.38}$$

where E is again the energy per unite rest mass of the falling observer when he was at infinity. Obviously, as we approach $r = 2M$, for any finite proper time $d\tau$ for the falling observer, the far away observer measure a larger and larger proper time dt. Since photon's frequency is inversely proportional to periods of time, frequency of light signals measured by the observer at infinity are redshifted as we approach $r = 2M$.

2.3.4 The Surface $r = 2M$ in Eddington-Finkelstein Coordinates

In order to show that the surface $r = 2M$ is a mere coordinate singularity, let us transform from Schwarzschild (t, r, θ, ϕ) coordinates to Eddington-Finkelstein (v, r, θ, ϕ) coordinates. This transformation is defined by [17][13]

$$t = v - r - 2M\log\left|\frac{r}{2M} - 1\right| \tag{2.39}$$
$$\Longrightarrow \ \mathrm{d}t = \mathrm{d}v - \left(1 - \frac{2M}{r}\right)^{-1} \mathrm{d}r \ . \tag{2.40}$$

Substituting dt from the last equation into (2.20), we get the Schwarzschild metric in the Eddington-Finkelstein coordinates

$$ds^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2\mathrm{d}\Omega^2 \ . \tag{2.41}$$

Now, even if g_{vv} vanish at $r = 2M$, the determinant of the metric is $\det\left[g_{\mu\nu}\right] = -r^4\sin^2\theta$. This means the metric is invertible, and none of the components of it of its inverse blows up at $r = 2M$.

Hence, it was just a coordinate singularity that has been removed by a coordinate transformation. However, the surface $r = 0$ is still a singularity in the new coordinates, which is expected, since it is a genuine one.

Now let us examine the paths of radial light rays in this coordinates. Setting $d\theta = d\phi = 0$ and $ds^2 = 0$, we get

$$-\left(1 - \frac{2M}{r}\right) \mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r = 0 \ . \tag{2.42}$$

Solution for this equation are the following curves

$$dv = 0 \qquad \text{ingoing light rays,} \tag{2.43}$$

$$\frac{dv}{dr} = 2\left(1 - \frac{2M}{r}\right)^{-1} \qquad \text{ingoing for } r < 2M \tag{2.44}$$

$$\frac{dv}{dr} = 2\left(1 - \frac{2M}{r}\right)^{-1} \qquad \text{outgoing for } r < 2M \ . \tag{2.45}$$

The first solutions $dv = 0$ are the curves v =constant. These describe ingoing light rays, since (2.39) imply that, as t increase, r must decrease in order to keep v constant. For the second solution, replacing dv/dr in (2.40) gives $dt/dr = (1 - 2M/r)^{-1}$, which represent an ingoing light ray for $r < 2M$, but an outgoing light ray for $r > 2M$. Radial light rays and some light cones are portrayed in Fig. 2.2. An important solution for (2.42) is the surface $r = 2M$, which describes stationary light rays that are hovering at the Schwarzschild radius, between those ingoing and those outgoing light rays.

From Fig. 2.2, we notice that light cones are not closing up at $r = 2M$, indicating that this surface is a regular one. However, light cones inside $r = 2M$ are qualitatively different from outside ones. As we approach the surface $r = 2M$, light cones gradually tilt in the decreasing direction of r. As long as $r > 2M$, any particle can still be able to escape to infinity. But in the region $r < 2M$, at every point, future light cones are directed towards the singularity $r = 0$, and any particle that cross this surface, can never come back to infinity. This is not surprising, if we remember that $r = 2M$ is a null surface, and null surface can only crossed in one direction by timelike curves. Since nothing can escape the surface $r = 2M$, not even light, it is impossible to see inside it. For this reason, this surface is called the *Schwarzschild event horizon*, and the region inside it is the *Schwarzschild black hole*.

2.3.5 Geometry of the Horizon and Singularity

Using the metric (2.20), surface of constant r have the following normal vector [16]

$$r = \text{constant} \quad \Longrightarrow \quad N^\mu = g^{\mu\nu}\nabla_\nu(r) = g^{\mu\nu}\delta^r_\nu = g^{\mu r}$$
$$\Longrightarrow \ N^\mu = \left[0, \left(1 - \frac{2M}{r}\right), 0, 0\right] \ . \tag{2.46}$$

The norm of N^μ is

$$N^\mu N_\mu = N^\mu N^\nu g^{\mu\nu} = N^r N^r g_{rr} = \left(1 - \frac{2M}{r}\right)^2 \left(1 - \frac{2M}{r}\right)^{-1}$$
$$\Longrightarrow \ N^\mu N_\mu = \left(1 - \frac{2M}{r}\right) \ . \tag{2.47}$$

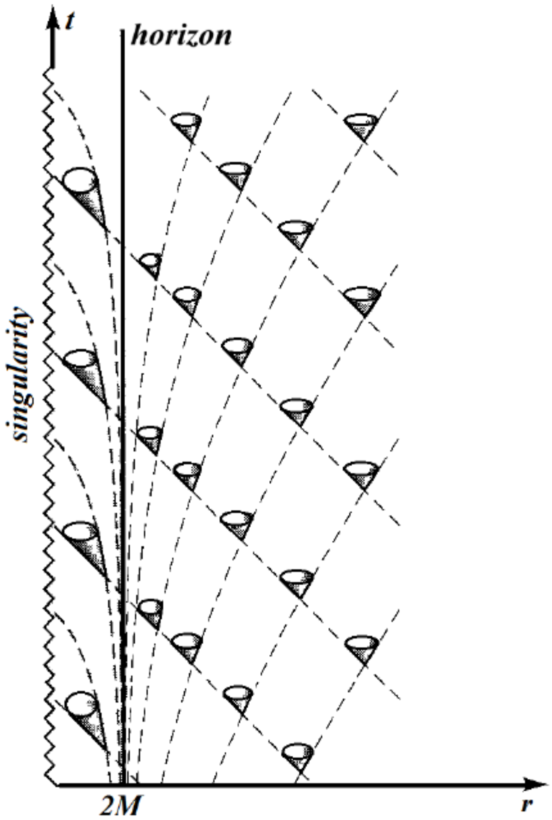


Fig. 2.2 – Light cones of Schwarzschild black hole in Eddington-Finkelstein coordinates. Notice that they do not close up.

Hence the horizon $r = 2M$, is a three dimensional null surface since $N^{\mu}N_{\mu} = 0$. Similar to null surfaces in flat spacetime of SR, the horizon is a one-way surface for timelike paths: once crossed, you cannot return back. As we saw from (2.42), it is consisted of those light rays that neither fall into the black hole nor escape to infinity. At any time, the line element on the horizon is obtained by setting $r = 2M$ and t =constant in (2.40) and (2.41). This gives $ds^2 = d\Sigma^2 = (2M)^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$, which is only the line element on a sphere of area $A = 16\pi M^2$ that is called the *area of the horizon*. This area do not change with time, but can change if matter fall into the black hole in a spherically symmetric way, since then the mass M increases and the area A increase too.

In flat spacetime, where the metric is $ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$, $r = 0$ is a timelike curve, i.e., a place in space at all times. However, in the Schwarzschild geometry, inside $r = 2M$ the situation is interesting. In the region $r < 2M$, putting r =constant (2.41) always yields $ds^2 > 0$, hence surfaces of constant r are spacelike. We can see this from (2.47).This means that inside the event horizon, r is the timelike coordinate. In other words, inside $r = 2M$ surfaces of constant r are moments in time rather than places in space. This is why, inside the horizon, all timelike paths are directed towards the singularity $r = 0$, as Fig. 2.2 shows.

2.3.6 Black Hole Formation

Consider a spherically symmetric star ,consisting of pressureless particles, that is undergoing a gravitational collapse [17][32]. If the star is so massive that the neutron Fermi pressure cannot support it, particles forming the surface of the star will move on some radial timelike paths towards the center. Once the Schwarzschild radius $r = 2M$ is crossed, all the surface particles will move in the direction of decreasing r, heading towards the singularity $r = 0$. Integrating (2.36) from $r = 2M$ to $r = 0$, the proper time needed for an observer on the star surface to travel from the horizon to the singularity is only $\Delta\tau = 4M/3$. For a star as massive as our Sun, this is of the order $\Delta\tau \sim 10^{-5}s$. If this observer a rocket inside the horizon, he will no longer be moving on a geodesic, where proper time is maximum, so he will hit the singularity in a shorter proper time. He will live the longest if he does not struggle. Eventually, there is nothing left inside the horizon, since all the matter has been concentrated in the zero size infinite density singularity. The formation of singularity is inevitable.

As discussed earlier, before the star surface reach $r = 2M$, from the point of view of a far away observer, clocks on the star surface increasingly slow down. Hence, he never sees the star cross the radius $r = 2M$. Moreover, this observer sees photons coming from the star surface to be increasingly redshifted. Since the star luminosity is proportional to photon's frequency, the collapsing star appears dimmer from infinity. Once the star cross $r = 2M$, its last light rays reach the far observer with infinite redshift and the star appears to be turned off.

2.3.7 Frequency Shift of Photons Emitted by Equatorial Massive Test Particles

Now, let us consider the frequency of photons emitted by test particles orbiting a Schwarzschild black hole in the equatorial plane $\theta = \pi/2$. We saw earlier that the conserved quantities associated with the Schwarzschild spacetime are the energy per unite mass E and the angular momentum per unite mass L. For a particle of four-velocity U^μ, this quantities are defined by

$$E = -\xi^\mu U_\mu = \left(1 - \frac{2M}{r}\right) U^t = -U_t \ , \tag{2.48}$$

$$L = \psi^\mu U_\mu = r^2 \sin^2\theta U^\phi = U_\phi \ . \tag{2.49}$$

For a massless particle with wave four-vector k^μ, the analogous conserved quantities E_γ and L_γ are given in a similar way

$$E_\gamma = -\xi^\mu k_\mu = \left(1 - \frac{2M}{r}\right) k^t = -k_t \ , \tag{2.50}$$

$$L_\gamma = \psi^\mu k_\mu = r^2 \sin^2\theta k^\phi = k_\phi \ . \tag{2.51}$$

Since angular momentum is conserved, a test particle motion is restricted to one single plane. If we initially chose this plane to be the meridional plane $\phi = 0$, and orient the particle three-velocity $\vec{U}$ to be tangent to this plane so that $U^\phi = d\phi/d\tau = 0$, equation (2.49) tell us that U^ϕ will be zero everywhere on the geodesic. We can also rotate the coordinates so that the motion will be on the equatorial plane with $\theta = \pi/2$ instead of the meridional plane. Therefore, the condition for motion in the equatorial plane can be written as

$$\begin{cases} \theta = \pi/2 \\ U^\theta = 0 \end{cases} \qquad \text{motion in the equatorial plane.} \tag{2.52}$$

Equatorial Geodesics for Massive Particles

Along side with the normalization equation $U_\mu U^\mu = -1$, these three equations represent the equations of motion of a relativistic particle. Using the Schwarzschild metric, for a particle moving in the equatorial plane, we have

$$-\left(1 - \frac{2M}{r}\right)\left(U^t\right)^2 + \left(1 - \frac{2M}{r}\right)^{-1}\left(U^r\right)^2 + r^2\left(U^\phi\right)^2 = -1 \ . \tag{2.53}$$

Using (2.48) and (2.49) to eliminate U^t and U^ϕ, the last equation take the form

$$-\left(1 - \frac{2M}{r}\right)^{-1} E^2 + \left(1 - \frac{2M}{r}\right)^{-1}\left(U^r\right)^2 + \frac{L^2}{r^2} = -1 \ . \tag{2.54}$$

Multiplying both sides by $\left(1 - 2M/r\right)/2$, we get

$$\frac{E^2}{2} = \frac{1}{2}\left(U^r\right)^2 + \frac{1}{2} - + \frac{L^2}{2r^2} - \frac{ML^2}{r^3} \ . \tag{2.55}$$

It is more convenient to rewrite the last equation in a form analogous to Newtonian mechanics

$$\frac{E^2}{2} = \frac{1}{2}\left(U^r\right)^2 + V\left(r\right) \ , \tag{2.56}$$

where

$$V\left(r\right) = \frac{1}{2} - \frac{M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3} \ . \tag{2.57}$$

The last term in$V(r)$ represent the relativistic correction brought by GR to the Newtonian case.

Stable Circular Orbits in the Equatorial Plane

Now let us consider stable circular orbits for massive particles in the equatorial plane. A circular orbit is the one having r fixed along the geodesic, meaning that $U^r = 0$ and $dU^r/d\tau = 0$. Alternatively, differentiating equation (2.56) with respect to τ yields

$$\begin{aligned} 0 &= \frac{dU^r}{d\tau}U^r + \frac{dV(r)}{d\tau} \\ 0 &= \frac{dU^r}{d\tau}\frac{dr}{d\tau} + \frac{dV(r)}{dr}\frac{dr}{d\tau} \\ 0 &= \frac{dU^r}{d\tau} + \frac{dV(r)}{dr} \ . \end{aligned}$$

Hence, the conditions for circular motion are

$$\begin{cases} U^r = 0 & \Longrightarrow \quad V(r) = E^2/2 \\ \dfrac{dU^r}{d\tau} = 0 & \Longrightarrow \quad \dfrac{dV(r)}{dr} = 0 \end{cases} \tag{2.58}$$

These couple of equation depend on the coordinate r and the conserved quantities E and L. We can thus use them to find expression of E and L for massive particles in an equatorial stable circular orbits. Since the potential $V(r)$ for massive particles is

$$V(r) = \frac{1}{2} - \frac{M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3} \ ,$$

the second condition in (2.58) becomes

$$\begin{aligned} \frac{dV(r)}{dr} &= \frac{M}{r^2} - \frac{L^2}{r^3} + 3\frac{ML^2}{r^4} \\ 0 &= \frac{1}{r^4}\left(Mr^2 - L^2 r + 3ML^2\right) \ , \end{aligned} \tag{2.59}$$

which can be solved for r to obtain the radius of equatorial circular orbits r_{CO} in the Schwarzschild spacetime

$$r_{\pm CO} = \frac{L^2}{2M}\left[1 \pm \sqrt{1 - 12\left(\frac{M}{L}\right)^2}\right] \ , \tag{2.60}$$

with the condition $L^2 \geq 12M^2$. In this equation, the radii r_+ correspond to a minima of $V(r)$, while r_- correspond to a maxima. Of course, *stable* circular orbits corresponds to minima of the potential $V(r)$, or $d^2V(r)/dr^2 > 0$. On the other hand, solving (2.59) for L gives

$$L_{CO} = \frac{M^{1/2} r}{\sqrt{r - 3M}} \ , \tag{2.61}$$

the expression of angular momentum per unite mass in a stable circular orbits in the equatorial plane around a Schwarzschild black hole. Substituting (2.61) in the second condition for stable circular orbits, the firs equation in (2.58), we get

$$\begin{aligned} E^2/2 = V(r) &= \frac{1}{2} - \frac{M}{r} + L^2\left(\frac{1}{2r^2} - \frac{M}{r^3}\right) \\ E^2/2 &= \frac{1}{2} - \frac{M}{r} + \frac{Mr^2}{r - 3M}\left(\frac{r - 2M}{2r^3}\right) \ . \end{aligned}$$

Simplifying and solving this equation for E, we find

$$E_{CO} = \frac{r-2M}{\sqrt{r^2-3Mr}} \ , \tag{2.62}$$

which is the expression of the energy per unite mass for a massive particle in an equatorial circular orbit around the Schwarzschild black hole.

Kinematic Frequency Shifts Measured by a Distant Observer

Consider a photon of wave four-vector k^ν propagating in spacetime. In GR, the frequency of this photon measured by an observer with four-velocity U^ν at point P is given by

$$\omega_C = -k_\mu U^\mu|_P \quad . \tag{2.63}$$

Hence, the frequency of this photon measured by two observers, one at the emission point (e) and another one at the detection point are respectively

$$\omega_e = -k_\mu U^\mu|_e = \left(-k_t U^t - k_r U^r - k_\theta U^\theta - L\gamma U^\phi\right)|_e \quad , \tag{2.64}$$
$$\omega_d = -k_\mu U^\mu|_d = \left(-k_t U^t - k_r U^r - k_\theta U^\theta - L\gamma U^\phi\right)|_d \quad . \tag{2.65}$$

the expression of the frequency shift of this photon subjected by between the emission and detection points is

$$1+z = \frac{\omega_e}{\omega_d} = \frac{\left(-k_t U^t - k_r U^r - k_\theta U^\theta - L\gamma U^\phi\right)|_e}{\left(-k_t U^t - k_r U^r - k_\theta U^\theta - L\gamma U^\phi\right)|_d} \ . \tag{2.66}$$

This is the general expression of frequency shift.

Now, we will restrict ourselves to massive particles moving in stable circular orbits ($U^r = 0$) in the equatorial plane ($\theta = \pi/2$ and $U^\theta = 0$). Using equations (2.48 - 2.51), we know that $k_t = -E_\gamma$ and $k_\phi = L_\gamma$ for photons. Therefore, the expression above of the frequency shift of photons emitted by massive particles moving in equatorial stable circular orbits is

$$1+z = \frac{\left(E_\gamma U^t - L_\gamma U^\phi\right)|_e}{\left(E_\gamma U^t - L_\gamma U^\phi\right)|_d} = \frac{U_e^t - bU_e^\phi}{U_d^t - bU_d^\phi} \ , \tag{2.67}$$

where we have introduced the impact parameter $b \equiv \frac{L_\gamma}{E_\gamma}$. Since the quantities E_γ and L_γ are conserved along the photons trajectory, which is a geodesics, the impact parameter will be conserved along it, i.e., we have $b_e = b_d$. That is why we have dropped the subscript e and d in the last step in (2.67).

Now, we will restrict ourselves to only kinematical frequency shift z_{kin} which is defined by

$$z_{kin} \equiv z - z_c \quad , \tag{2.68}$$

where z is the total shift, given by (2.67), and z_c is the shift of photons emitted by a static particle located at $b = 0$. From (2.67), the expression of z_c is

$$1 + z_c = \frac{U_e^t}{U_d^t} \ , \tag{2.69}$$

so that the expression of z_{kin} defined above will be

$$\begin{aligned} z_{kin} &= \frac{U_e^t - bU_e^\phi}{U_d^t - bU_d^\phi} - \frac{U_e^t}{U_d^t} \\ \Longrightarrow z_{kin} &= \frac{U_e^t U_d^\phi b - U_d^t U_e^\phi b}{U_d^t \left(U_d^t - bU_d^\phi\right)} \; . \end{aligned} \tag{2.70}$$

Let us suppose now that the detecting observer is far away from the central black hole. The angular velocity of this observer is

$$\Omega_d \equiv \frac{d\phi}{dt} = \frac{d\phi/d\tau}{dt/d\tau} = \frac{U_d^\phi}{U_d^t} \Longrightarrow U_d^\phi = \Omega_d U_d^t \; , \tag{2.71}$$

where τ is the observer proper time. Inserting this into equation (2.70), we get

$$z_{kin} = \frac{b\left(U_e^t\Omega_d - U_e^\phi\right)}{U_d^t\left(1 - \Omega_d b\right)} = \frac{\left(U_e^t\Omega_d - U_e^\phi\right)}{U_d^t\left(\frac{1}{b} - \Omega_d\right)} \; . \tag{2.72}$$

The next step is to maximize z_{kin}, following [47]. Thus, only equatorial photons emitted from particles orbiting either sides of the source will be considered, i.e. photons with $k^\theta = k^r = 0$. Thus, the photon wave four-vector normalization equation $k^\mu k_\mu = 0$ takes the form

$$\begin{aligned} k^\nu k_\nu &= k^t k_t + k^\phi k_\phi = -k^t E_\gamma + k^\phi L_\gamma = 0 \\ \Longrightarrow \quad b &= \frac{L_\gamma}{E_\gamma} = \frac{k^t}{k^\phi} \; , \end{aligned} \tag{2.73}$$

where we have used (2.50) and (2.51). We can as well write the equation $k^\mu k_\mu = 0$ in the following alternative form

$$\begin{aligned} k^\nu k_\nu &= g_{tt}\left(k^t\right)^2 + g_{\phi\phi}\left(k^\phi\right)^2 = 0 \\ 0 &= g_{tt}\left(\frac{k^t}{k^\phi}\right)^2 + g_{\phi\phi} \\ 0 &= g_{tt} b^2 + g_{\phi\phi} \\ \Longrightarrow \quad b_\pm &= \pm\sqrt{-\frac{g_{\phi\phi}}{g_{tt}}} \; . \end{aligned} \tag{2.74}$$

Inserting these two values for b in (2.72) will give rise to two different shifts z_1 and z_2, corresponding respectively to a receding and an approaching particle on both sides of the central body as seen by a distant observer

$$z_1 = \frac{\left(U_e^t\Omega_d - U_e^\phi\right)}{U_d^t\left(1/b_- - \Omega_d\right)} \; , \tag{2.75}$$

$$z_2 = \frac{\left(U_e^t\Omega_d - U_e^\phi\right)}{U_d^t\left(1/b_+ - \Omega_d\right)} \; . \tag{2.76}$$

After all this have been done, the last step is to express all the quantities U^t, Ω_d, $b_\pm$, U^ϕ in terms of the Schwarzschild coordinate. The metric and inverse metric components for equatorial

($\theta = \pi/2$, $d\theta = 0$) circular motion ($dr = 0$) are

$$g_{tt} = -\left(1 - \frac{2M}{r}\right) \quad , \quad g_{\phi\phi} = r^2 \ ,$$
$$g^{tt} = -\left(1 - \frac{2M}{r}\right)^{-1} \quad , \quad g^{\phi\phi} = \frac{1}{r^2} \ .$$

Using this and the fact that $U_t = -E$ and $U_\phi = L$ from equations (2.48, 2.49), the expressions of U^t and U^ϕ becomes

$$U^t = g^{t\mu}U_\mu = g^{tt}U_t = \left(1 - \frac{2M}{r}\right)^{-1} E \ , \tag{2.77}$$
$$U^\phi = g^{\phi\mu}U_\mu = g^{\phi\phi}U_\phi = \frac{1}{r^2}L \ . \tag{2.78}$$

Substituting E and L from (2.62) and (2.61), we obtain

$$U^t = \left(\frac{r}{r - 2M}\right)\frac{r - 2M}{\sqrt{r^2 - 3Mr}} = \frac{r^{1/2}}{\sqrt{r - 3M}} \ , \tag{2.79a}$$
$$U^\phi = \pm\frac{1}{r^2}\frac{M^{1/2}r}{\sqrt{r - 3M}} = \pm\frac{M^{1/2}}{r\sqrt{r - 3M}} \ , \tag{2.79b}$$

where the $(+)$ and $(-)$ signs correspond to rotating in the direction of increasing or decreasing of coordinate ϕ, respectively. Using the last couple of equations and (2.71), we can calculate Ω_d and obtain

$$\Omega_d = \frac{U_d^\phi}{U_d^t} = \pm\frac{M^{1/2}}{r\sqrt{r - 3M}}\frac{\sqrt{r - 3M}}{r^{1/2}} = \pm\frac{M^{1/2}}{r^{3/2}} \ , \tag{2.80}$$

As for the expression of $b_\pm$, equation (2.74) yields

$$b_\pm = \pm\sqrt{-\frac{g_{\phi\phi}}{g_{tt}}} = \pm\sqrt{-r^2\left(\frac{r}{r - 2M}\right)} = \pm\frac{r^{3/2}}{\sqrt{r - 2M}} \tag{2.81}$$

Finally, inserting equations (2.79a - 2.81) into the denominator of (2.75), we get

$$\begin{aligned} U_e^t\Omega_d - U_e^\phi &= \frac{r_e^{1/2}}{\sqrt{r_e - 3M}}\frac{\pm M^{1/2}}{r_d^{3/2}} - \frac{\pm M^{1/2}}{r_e\sqrt{r_e - 3M}} \\ &= \frac{\pm M^{1/2}}{\sqrt{r_e - 3M}}\left[\frac{r_e^{3/2} - r_d^{3/2}}{r_d^{3/2}r_e}\right] \ , \end{aligned} \tag{2.82}$$

while the numerator gives

$$\begin{aligned} U_d^t\left(1/b_- - \Omega_d\right) &= \frac{r_d^{1/2}}{\sqrt{r_d - 3M}}\left(-\frac{\sqrt{r_e - 2M}}{r_e^{3/2}} - \frac{\pm M^{1/2}}{r_d^{3/2}}\right) \\ &= -\frac{r_d^{3/2}\sqrt{r_e - 2M} \pm r_e^{3/2}M^{1/2}}{r_e^{3/2}r_d\sqrt{r_d - 3M}} \ . \end{aligned} \tag{2.83}$$

Putting the numerator and the denominator together, equation (2.75) becomes

$$\begin{aligned} z_1 &= \frac{\left(U_e^t \Omega_d - U_e^\phi\right)}{U_d^t \left(1/b_- - \Omega_d\right)} \\ &= \frac{\pm M^{1/2}}{\sqrt{r_e - 3M}} \left[\frac{r_e^{3/2} - r_d^{3/2}}{r_d^{3/2} r_e}\right] \times \left[-\frac{r_e^{3/2} r_d \sqrt{r_d - 3M}}{r_d^{3/2}\sqrt{r_e - 2M} \pm r_e^{3/2} M^{1/2}}\right] \end{aligned}$$

$$\implies z_1 = \pm \frac{M^{1/2}}{\sqrt{r_e - 3M}} \left[\frac{r_d^{3/2} - r_e^{3/2}}{r_d^{1/2}}\right] \times \frac{r_e^{1/2}\sqrt{r_d - 3M}}{r_d^{3/2}\sqrt{r_e - 2M} \pm r_e^{3/2} M^{1/2}} \,. \tag{2.84}$$

This is the expression of redshifts of photons emitted by particles moving in equatorial stable circular orbits and detected by a far away observer, given only in terms the emission and detection radii r_e and r_d respectively, and the mass of the black hole. Following the same steps, we find the expression of the blueshift. The denominator in (2.76) is the same as in (2.75)

$$U_e^t \Omega_d - U_e^\phi = \frac{\pm M^{1/2}}{\sqrt{r_e - 3M}} \left[\frac{r_e^{3/2} - r_d^{3/2}}{r_d^{3/2} r_e}\right] \,. \tag{2.85}$$

However, the numerator in (2.76) gives

$$\begin{aligned} U_d^t \left(1/b_+ - \Omega_d\right) &= \frac{r_d^{1/2}}{\sqrt{r_d - 3M}} \left(+\frac{\sqrt{r_e - 2M}}{r_e^{3/2}} - \frac{\pm M^{1/2}}{r_d^{3/2}}\right) \\ &= \frac{r_d^{3/2}\sqrt{r_e - 2M} \mp r_e^{3/2} M^{1/2}}{r_e^{3/2} r_d \sqrt{r_d - 3M}} \,. \end{aligned} \tag{2.86}$$

Substituting the numerator and the denominator in (2.76), we obtain

$$\begin{aligned} z_2 &= \frac{\left(U_e^t \Omega_d - U_e^\phi\right)}{U_d^t \left(1/b_+ - \Omega_d\right)} \\ &= \frac{\pm M^{1/2}}{\sqrt{r_e - 3M}} \left[\frac{r_e^{3/2} - r_d^{3/2}}{r_d^{3/2} r_e}\right] \times \left[\frac{r_e^{3/2} r_d \sqrt{r_d - 3M}}{r_d^{3/2}\sqrt{r_e - 2M} \mp r_e^{3/2} M^{1/2}}\right] \end{aligned}$$

$$\implies z_2 = \pm \frac{M^{1/2}}{\sqrt{r_e - 3M}} \left[\frac{r_d^{3/2} - r_e^{3/2}}{r_d^{1/2}}\right] \times \frac{-r_e^{1/2}\sqrt{r_d - 3M}}{r_d^{3/2}\sqrt{r_e - 2M} \mp r_e^{3/2} M^{1/2}} \,. \tag{2.87}$$

Therefore, this expressions of z_1 and z_2 can help estimating the mass M of a Schwarzschild black hole by measuring the red/blue shifts of photons emitted by particles moving in an equatorial stable circular orbits. However, we now that Schwarzschild black holes are just the simplest case, and that real black holes can be rotating and charged. Later in this thesis, we will derive these expressions for the red/blue shifts, but for the Kerr-Newman black hole, which is the most general black hole.

2.4 Reissner-Nordström Black Holes

The simplest black hole next to Schwarzschild is the charged Reissner-Nordström (RN) black hole. It is the spherically symmetric solution of the coupled equations of Einstein and Maxwell [16][18].

As stated by the no-hair theorem, it is a stationary, asymptotically flat solution characterized by the mass M, electric charge Q and magnetic charge P. Experimentally, there is no evidence of the existence of isolated magnetic charges P, called also monopoles. Theoretically, their existence is still possible. For the sake of simplicity, we will assume $P = 0$ in the rest of this chapter. Practically, this means that the electromagnetic field strength tensor will no have a magnetic field components ($B_i = 0$). If, moreover, we consider the non charged case and set $Q = 0$, RN metric must reduce to Schwarzschild. From an astrophysical point of view, charged black holes are not significantly important, since any macroscopic charged body would interact with matter in its surrounding and be quickly neutralized. However, from a theoretical point of view, the study charged black holes is very informative.

2.4.1 Derivation of Reissner-Nordström Metric

As we saw in the derivation of Schwarzschild metric, we know that any spherically symmetric metric must have the following general form [17][44]

$$ds^2 = -e^{2\alpha(t,r)}\mathrm{d}t^2 + e^{2\beta(t,r)}\mathrm{d}r^2 + r^2 d\Omega^2 \ , \tag{2.88}$$

for which the following non vanishing Christoffel symbols

$$\begin{aligned}
&\Gamma^t_{tt} = \partial_t\alpha && \Gamma^t_{tr} = \partial_r\alpha && \Gamma^t_{rr} = e^{2(\beta-\alpha)}\partial_t\beta \\
&\Gamma^r_{tt} = e^{2(\alpha-\beta)}\partial_r\alpha && \Gamma^r_{tr} = \partial_t\beta && \Gamma^r_{rr} = \partial_r\beta \\
&\Gamma^\theta_{r\theta} = \frac{1}{r} && \Gamma^r_{\theta\theta} = -re^{-2\beta} && \Gamma^\phi_{r\phi} = \frac{1}{r} \\
&\Gamma^r_{\phi\phi} = -re^{-2\beta}\sin^2\theta && \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta && \Gamma^\phi_{\theta\phi} = \frac{\cos\theta}{\sin\theta}.
\end{aligned} \tag{2.89}$$

The only non vanishing Ricci tensor components are

$$\begin{aligned}
R_{tt} &= -\left[(\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta + \partial_t^2\beta\right] + e^{2(\alpha-\beta)}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\alpha\right] \\
R_{rr} &= e^{2(\beta-\alpha)}\left[(\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta + \partial_t^2\beta\right] + \left[\partial_r\alpha\partial_r\beta - \partial_r^2\alpha - (\partial_r\alpha)^2 + \frac{2}{r}\partial_r\beta\right] \\
R_{\theta\theta} &= e^{-2\beta}\left[(\partial_r\beta - \partial_r\alpha)\, r - 1\right] + 1 \\
R_{\phi\phi} &= R_{\theta\theta}\sin^2\theta \\
R_{tr} &= \frac{2}{r}\partial_t\beta \ .
\end{aligned} \tag{2.90}$$

Moreover, spherical symmetry implies that the electromagnetic strength tensor $F_{\mu\nu}$ will have only a radial component of the electric field that do not depend on θ or ϕ. That is, the only non vanishing components are $F_{rt} = -F_{tr} = E_r(t,r)$, so that

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_r(t,r) & 0 & 0 \\ E_r(t,r) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ . \tag{2.91}$$

As we said earlier, the RN metric is the solution to the coupled Einstein and Maxwell equations. We start with the Einstein field equations

$$R_{\mu\nu} = 8\pi\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right), \tag{2.92}$$

where $T_{\mu\nu}$ and T are the stress-energy tensor and trace of the source, respectively. For the electromagnetic field, $T_{\mu\nu}$ reads

$$T_{\mu\nu} = F_{\mu}{}^{\sigma} F_{\nu\sigma} - \frac{1}{4} g_{\mu\nu} F_{\sigma\rho} F^{\sigma\rho} \ . \tag{2.93}$$

The $T_{\mu\nu}$ for the electromagnetic field is traceless

$$T = T_{\mu\nu} g^{\mu\nu} = F_{\mu}{}^{\sigma} F^{\mu}{}_{\sigma} - \frac{1}{4} \delta^{\mu}_{\mu} F_{\sigma\rho} F^{\sigma\rho} = F_{\mu\sigma} F^{\sigma\mu} - \frac{1}{4} 4 F_{\sigma\rho} F^{\sigma\rho} = 0 \ , \tag{2.94}$$

so that the Einstein equations takes the easier form [35][44]

$$R_{\mu\nu} = 8\pi T_{\mu\nu} \ . \tag{2.95}$$

Let us now simplify $T_{\mu\nu}$ further. Using (2.91), the first term in (2.93) becomes

$$F_{\mu}{}^{\sigma} F_{\nu\sigma} = g_{\mu\lambda} F^{\lambda\sigma} F_{\nu\sigma} = g_{\mu\lambda} \left(F^{\lambda t} F_{\nu t} + F^{\lambda r} F_{\nu r} \right) = g_{\mu r} F^{rt} F_{\nu t} + g_{\mu t} F^{tr} F_{\nu r} \ . \tag{2.96}$$

The second term becomes

$$-\frac{1}{4} g_{\mu\nu} F_{\sigma\rho} F^{\sigma\rho} = -\frac{1}{4} g_{\mu\nu} \left(F_{tr} F^{tr} + F_{rt} F^{rt} \right) = -\frac{1}{4} g_{\mu\nu} \left(2 F_{rt} F^{rt} \right) = -\frac{1}{2} g_{\mu\nu} F_{rt} F^{rt} \ , \tag{2.97}$$

and the stress-energy tensor takes the form

$$T_{\mu\nu} = g_{\mu r} F^{rt} F_{\nu t} + g_{\mu t} F^{tr} F_{\nu r} - \frac{1}{2} g_{\mu\nu} F_{rt} F^{rt} \ . \tag{2.98}$$

The only nonvanishing components of $T_{\mu\nu}$ are the diagonal ones

$$\begin{aligned} T_{tt} &= g_{tr} F^{rt} F_{tt} + g_{tt} F^{tr} F_{tr} - \frac{1}{2} g_{tt} F_{rt} F^{rt} = g_{tt} \left(F^{tr} F_{tr} - \frac{1}{2} F_{rt} F^{rt} \right) \\ &= g_{tt} \left(\frac{1}{2} F_{rt} F^{rt} \right) = -\frac{1}{2} e^{2\alpha(t,r)} F_{rt} F^{rt} \ , \end{aligned} \tag{2.99}$$

$$\begin{aligned} T_{rr} &= g_{rr} F^{rt} F_{rt} + g_{rt} F^{tr} F_{rr} - \frac{1}{2} g_{rr} F_{rt} F^{rt} = g_{rr} \left(F^{rt} F_{rt} - \frac{1}{2} F_{rt} F^{rt} \right) \\ &= g_{rr} \left(\frac{1}{2} F_{rt} F^{rt} \right) = \frac{1}{2} e^{2\beta(t,r)} F_{rt} F^{rt} \ , \end{aligned} \tag{2.100}$$

$$T_{\theta\theta} = g_{\theta r} F^{rt} F_{\theta t} + g_{\theta t} F^{tr} F_{\theta r} - \frac{1}{2} g_{\theta\theta} F_{rt} F^{rt} = -\frac{1}{2} r^2 F_{rt} F^{rt} \ , \tag{2.101}$$

$$T_{\phi\phi} = g_{\phi r} F^{rt} F_{\phi t} + g_{\phi t} F^{tr} F_{\phi r} - \frac{1}{2} g_{\phi\phi} F_{rt} F^{rt} = -\frac{1}{2} r^2 \sin^2\theta F_{rt} F^{rt} \ . \tag{2.102}$$

Since we have non diagonal components of $T_{\mu\nu}$ are zero, using (2.90) in the rt Einstein equation yields

$$R_{rt} = 8\pi T_{rt} = 0 \quad \Longrightarrow \quad \frac{2}{r} \partial_t \beta = 0 \quad \Longrightarrow \quad \beta(t,r) = \beta(r),$$

meaning that the function β depend only on r. From (2.102) we notice that

$$\frac{T_{tt}}{e^{2\alpha}} + \frac{T_{rr}}{e^{2\beta}} = 0 \quad \Longrightarrow \quad \frac{R_{tt}}{e^{2\alpha}} + \frac{R_{rr}}{e^{2\beta}} = 0 \ .$$

Using the fact that $\beta(r)$ is independent of time, the last equation gives

$$e^{-2\beta}\left[\frac{2}{r}\left(\partial_r\alpha+\partial_r\beta\right)\right]=0 \quad\Longrightarrow\quad \partial_r\left(\alpha+\beta\right)=0 \;\Longrightarrow\; \alpha(r,t)+\beta(r)=f(t)\ ,$$

where $f(t)$ is some function that may depend on t. Thus, the first term of the metric (2.88) is $-e^{2\alpha(t,r)}\mathrm{d}t^2 = -e^{-2\beta(r)}e^{2f(t)}\mathrm{d}t^2$. The function $f(t)$ can be absorbed by redefining the time coordinate as follows $e^{f(t)}\,\mathrm{d}t \to \mathrm{d}t$, so that

$$\alpha(r)=-\beta(r)\ . \tag{2.103}$$

meaning that all metric components are now independent of t. Consequently, the metric (2.88) becomes

$$ds^2=-e^{-2\beta(r)}\mathrm{d}t^2+e^{2\beta(r)}\mathrm{d}r^2+r^2d\Omega^2\ . \tag{2.104}$$

Now, let us examine Maxwell's equations in vacuum [17][44]

$$\nabla_\beta F^{\alpha\beta}=\partial_\beta F^{\alpha\beta}+\Gamma^{\alpha}_{\beta\sigma}F^{\sigma\beta}+\Gamma^{\beta}_{\beta\sigma}F^{\alpha\sigma}=0\ , \tag{2.105}$$

$$\nabla_{[\alpha}F_{\beta\mu]}=\nabla_\alpha F_{\beta\mu}+\nabla_\beta F_{\mu\alpha}+\nabla_\mu F_{\alpha\beta}=0\ . \tag{2.106}$$

Since we considered the case where magnetic poles do not exist, we don't have a magnetic field ($B_i=0$), the second equation is identically zero, and hence do not add any information. As for the first equation, the second term vanish because $\Gamma^{\alpha}_{\beta\sigma}$ is symmetric while $F^{\sigma\beta}$ is antisymmetric in β and σ, so that

$$\nabla_\beta F^{\alpha\beta}=\partial_\beta F^{\alpha\beta}+\Gamma^{\beta}_{\beta\sigma}F^{\alpha\sigma}=0\ . \tag{2.107}$$

Since the only non vanishing component of $F^{\alpha\beta}$ are $F_{rt}=-F_{tr}=E_r(t,r)$, setting $\alpha=r$ in the last equation yields

$$\begin{aligned}
0&=\partial_\beta F^{r\beta}+\Gamma^{\beta}_{\beta\sigma}F^{r\sigma}\\
0&=\partial_t F^{rt}+\Gamma^{\beta}_{\beta t}F^{rt}\\
0&=\partial_t F^{rt}+\left(\Gamma^{t}_{tt}+\Gamma^{r}_{rt}\right)F^{rt}\\
0&=\partial_t F^{rt}+\left(\partial_t\alpha+\partial_t\beta\right)F^{rt}\\
0&=\partial_t F^{rt}\\
&\Longrightarrow\quad F^{rt}=F^{rt}(r)\ ,
\end{aligned} \tag{2.108}$$

where we have substituted $\Gamma^{\beta}_{\beta t}$ from (2.89) and the fact that $\alpha(r)$ and $\beta(r)$ are independent of t. Thus, using (2.104), we conclude that the radial component of the electric field E_r depend only on r,

$$\begin{aligned}
E_r&=F_{rt}=g_{rr}g_{tt}F^{rt}=F^{rt}(r)\ ,\\
&\Longrightarrow\quad E_r=E_r(r)\ .
\end{aligned} \tag{2.109}$$

Now, repeating the same steps with $\alpha=t$ in (2.107) yields

$$\begin{aligned}
0&=\partial_\beta F^{t\beta}+\Gamma^{\beta}_{\beta\sigma}F^{t\sigma}\\
0&=\ \partial_r F^{tr}+\Gamma^{\beta}_{\beta r}F^{tr}\\
0&=\ \partial_r F^{tr}+\left(\Gamma^{t}_{tr}+\Gamma^{r}_{rr}+\Gamma^{\theta}_{\theta r}+\Gamma^{\phi}_{\phi r}\right)F^{tr}\\
0&=\ \partial_r F^{tr}+\left(\partial_r\alpha+\partial_r\beta+\frac{1}{r}+\frac{1}{r}\right)F^{tr}\\
0&=\ \partial_r F_{tr}+\left(\frac{2}{r}\right)F_{tr}\ ,
\end{aligned} \tag{2.110}$$

where we have used $\alpha(r) = -\beta(r)$ to get the last step. The last line is an ordinary differential equation that can be solved straightforwardly to obtain

$$F_{tr}(r) = E_r(r) = \frac{A}{r^2} \ , \tag{2.111}$$

where A is an integration constant. Using Gauss's flux theorem for an electric field generated by a charge Q, we can identify the integration constant A. At infinity, the two-sphere Σ of constant t and r has the two normals

$$n^\mu = \nabla^\mu t = (1,0,0,0) \ , \qquad m^\mu = \nabla^\mu r = (0,1,0,0), \qquad \text{at infinity}$$

and the induced metric $\gamma^{(S^2)}_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = r^2 \mathrm{d}r^2 + r^2 \sin^2\theta \mathrm{d}\phi^2$. The Gauss's flux theorem gives then

$$\begin{aligned} Q &= -\oint_\Sigma dx^2 \sqrt{|\gamma^{(S^2)}|} n^\mu m^\nu F_{\mu\nu} \\ Q &= -\oint_\Sigma dr d\phi r^2 \sin^2\theta F_{tr} \\ Q &= +4\pi r^2 E_r \\ \implies \quad E_r &= \frac{Q}{4\pi r^2} \ . \end{aligned} \tag{2.112}$$

Identifying with (2.111), we found $A = Q/4\pi$.

Now we go back to Einstein equations, $R_{\mu\nu} = 8\pi T_{\mu\nu}$. Using $\alpha(r) = -\beta(r)$ in (2.90), the left hand side of the $\theta - \theta$ equation becomes

$$\begin{aligned} R_{\theta\theta} &= e^{-2\beta}\left[(\partial_r\beta - \partial_r\alpha) r - 1\right] + 1 \\ &= e^{2\alpha}\left[(-2\partial_r\alpha) r - 1\right] + 1 \\ &= -\partial_r\left(re^{2\alpha}\right) + 1 \ . \end{aligned} \tag{2.113}$$

On the other hand, using (2.102), the right hand side of the $\theta - \theta$ Einstein equation is

$$8\pi T_{\theta\theta} = -8\pi \frac{1}{2} r^2 F_{rt} F^{rt} = +8\pi \frac{1}{2} r^2 \left(F_{rt}\right)^2 = 4\pi r^2 \left(\frac{Q}{4\pi r^2}\right)^2 = \frac{Q^2}{4\pi r^2} \ . \tag{2.114}$$

Hence, the Einstein equation $R_{\theta\theta} = 8\pi T_{\theta\theta}$ can be integrated to give

$$\begin{aligned} -\partial_r\left(re^{2\alpha}\right) + 1 &= \frac{Q^2}{4\pi r^2} \ , \\ -re^{2\alpha} + r &= -\frac{Q^2}{4\pi r} + C \ , \\ e^{2\alpha} &= 1 - \frac{C}{r} + \frac{Q^2}{4\pi r^2} \ , \end{aligned} \tag{2.115}$$

where C is an integration constant. To determine this constant, let us remember that Schwarzschild metric is a special case of RN metric when there is no electric charge. Setting $Q = 0$ in the last equation, then identifying with Schwarzschild metric reveal that C is just the Schwarzschild radius [17][35]

$$C = R_s = 2M \ .$$

Setting $R_Q = Q/4\pi$, the final expression of the RN metric is

$$ds^2 = -\Delta \mathrm{d}t^2 + \Delta^{-1}\mathrm{d}r^2 + r^2 d\Omega^2 \ , \tag{2.116}$$

where

$$\Delta = 1 - \frac{R_S}{r} + \frac{R_Q}{r^2} \, . \tag{2.117}$$

In the case of existence of magnetic monopoles, we will have a non zero magnetic field. In terms of the Levi-Civita tensor

$$\epsilon^{\mu\nu\alpha\beta} = |-g|^{-1/2} \, \tilde{\epsilon}^{\mu\nu\alpha\beta} \ ,$$

$\tilde{\epsilon}$ being the Levi-Civita symbol, the magnetic field B^i is related to the electromagnetic strength tensor by

$$B^i = \frac{1}{2}\epsilon^{0i\alpha\beta}F_{\alpha\beta} \ , \tag{2.118}$$

For a black hole of total magnetic charge P, spherical symmetry means that the only non zero component of the magnetic field is the radial one, given by [16]

$$B_r = \frac{P}{r^2} \ , \tag{2.119}$$

where B_r is related to $F_{\alpha\beta}$ by

$$\begin{aligned} B_r = B_1 &= \frac{1}{2} g_{00} g_{11} \epsilon^{01\alpha\beta} F_{\alpha\beta} \\ &= -\frac{1}{2} \, |-g|^{-1/2} \, \tilde{\epsilon}^{01\alpha\beta} F_{\alpha\beta} \\ &= -\frac{1}{2} \left| -r^4 \sin^2\theta \right|^{-1/2} \tilde{\epsilon}^{01\alpha\beta} F_{\alpha\beta} \\ &= -\frac{1}{2r^2 \sin\theta} \left(\tilde{\epsilon}^{0123} F_{23} + \tilde{\epsilon}^{0132} F_{32} \right) \\ &= -\frac{F_{23}}{r^2 \sin\theta} \ . \end{aligned} \tag{2.120}$$

Following similar steps as for the case $P = 0$, we eventually end up with the same expression for the metric (2.116), with the charge Q replaced by the sum of both electric and magnetic charges $Q^2 \longrightarrow Q^2 + P^2$. For the sake of simplicity, we will assume that $P = 0$ in the rest of these dissertation.

2.4.2 Properties of Reissner-Nordström Spacetime

Now, let us see the main properties of the RN metric [17][35]

- *Asymptotic flatness*: at large r, the metric (2.116) becomes progressively Minkowskian.
- *Symmetries*: the RN metric is spherically symmetric, hence is invariant under rotation about the three spatial axes, implying the existence of three Killing vectors.

  From the explicit form (2.116), the metric is independent of t and ϕ, hence possesses the following Killing vectors

$$\xi^\mu = (1, 0, 0, 0) \ , \tag{2.121}$$
$$\psi^\mu = (0, 0, 0, 1) \ . \tag{2.122}$$

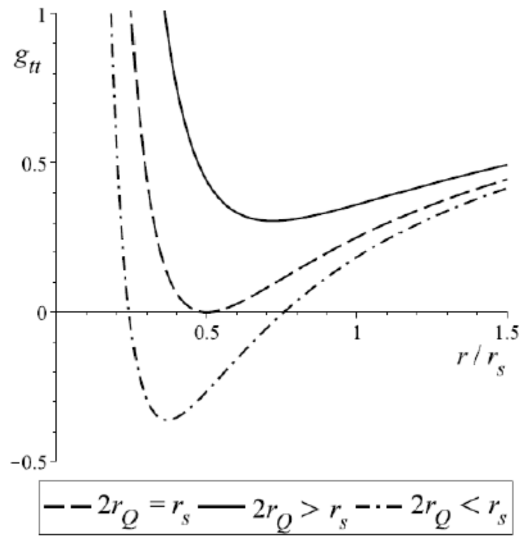


Fig. 2.3 – The function $\Delta(r)$ for the RN solutions. The existence of event horizons depend on the values of M and Q.

Notice that the vector normal to to t =constant hypersurfaces is given by

$$N^\mu = g^{\mu\nu}\nabla_\nu\,(t) = g^{\mu\nu}\partial_\nu\,(t) = g^{\mu\nu}\delta^t_\nu = g^{\mu t} = g^{tt}(1,0,0,0), \tag{2.123}$$

meaning that N^μ is proportional to ξ^μ. The existence of a time translation Killing vector ξ^μ that is normal to t =constant hypersurfaces means that Reissner-Nordström spacetime is static. The associated conserved quantities of the Killing vectors ξ^μ and ψ^μ are the energy E and angular momentum L per unit rest mass, respectively

$$E = -\xi^\mu U_\mu = \Delta\frac{dt}{d\tau}\ , \tag{2.124}$$

$$L = \psi^\mu U_\mu = r^2\sin^2\theta\frac{d\phi}{d\tau}\ . \tag{2.125}$$

- *Mass and charge parameters*: the parameters M and Q appearing in (2.116) are the Komar total mass and total electric charge of the black hole.

- *Curvature singularity:* Obviously, the metric (2.116) is singular at $r = 0$. This turns our to be a real singularity. As we did with Schwarzschild metric, a sufficient, though not necessary, criteria for curvature singularity is that a coordinate independent quantity constructed from the curvature $R^\alpha{}_{\beta\mu\nu}$ blows up. The Kretschmann scalar $R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$ can do the job. For the metric (2.116), we have

$$R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} = \frac{56R_Q^4}{r^8} - \frac{48R_Q^2R_S}{r^7} + \frac{12R_S^2}{r^6}\ . \tag{2.126}$$

  This is sufficient to see that $r = 0$ is a curvature singularity. Unlike the Schwarzschild case, the hypersurface $r = 0$ is timelike rather than spacelike. This means that particles having enough acceleration can avoid hitting the singularity, as we shall see later.

- *Event horizons:* The other singularities occur when the function $g^{rr} = -g_{tt} = \Delta(r)$ in (2.116) vanishes. These are coordinate singularities, and can be removed hence by a suitable

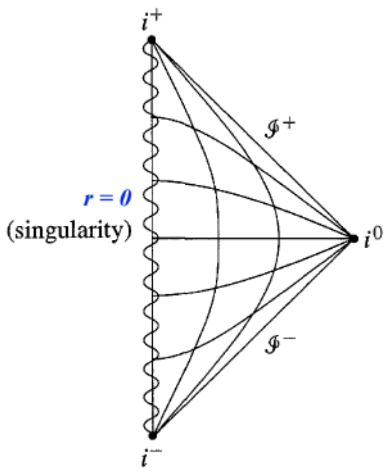


Fig. 2.4 – Conformal diagram for RN solution with $R_S^2 < R_Q^2$. There is a naked singularity at $r = 0$.

change of coordinates. The solutions for $\Delta(r) = 0$ are

$$r_\pm = \frac{1}{2}\left(R_S \pm \sqrt{R_S^2 - R_Q^2}\right) . \tag{2.127}$$

From the discussion preceding equation (2.3), we saw that, in a suitably chosen coordinates, hypersurfaces of constant r at which $g^{rr} = 0$ represent event horizons. As we can see in Fig. 2.3, according to the relative values of R_S and R_Q, we will have three cases [17][35]

- First case: $R_S^2 < R_Q^2 \implies$ no horizons,
- Second case: $R_S^2 = R_Q^2 \implies$ one horizons,
- Third case: $R_S^2 > R_Q^2 \implies$ two horizons.

We will consider each case separately.

First case $R_S^2 < R_Q^2$

As shown in Fig. 2.3, in this case the function $\Delta(r) = g^{rr}$ is always positive, and the metric components are regular all the way down to the singularity $r = 0$. Since we always have $g_{tt} < 0$ and $g_{rr} > 0$, the coordinate t is always timelike while r is always spacelike. This means that there is no even horizon to hide the singularity from the rest of spacetime; a naked singularly. Observers approaching the singularity are no longer forced to hit it. It is possible for an observer to approach the singularity and then return back to infinity. The conformal diagram of this case in portrayed in Fig. 2.4, where we can see that the singularity is a timelike line.

Second case: $R_S^2 = R_Q^2$

Solution of this case is called the extreme Reissner-Nordström. The function $\Delta(r) = g^{rr}$ vanish at the hypersurface $r = R_S/2 = M$, but is positive elsewhere, as shown in Fig. 2.3. The hypersurface $r = R_S/2$ hence represent an event horizon hiding the singularity. Moreover, since $\Delta(r) = -g_{tt} = g_{rr}^{-1}$, the coordinate t is always timelike, and r is always spacelike. Since $r = 0$ is a timelike line, an observer crossing the event horizon can either continue to hit the singularity $r = 0$, or move back to infinity and avoid it. However, an observer crossing the event horizon in the direction

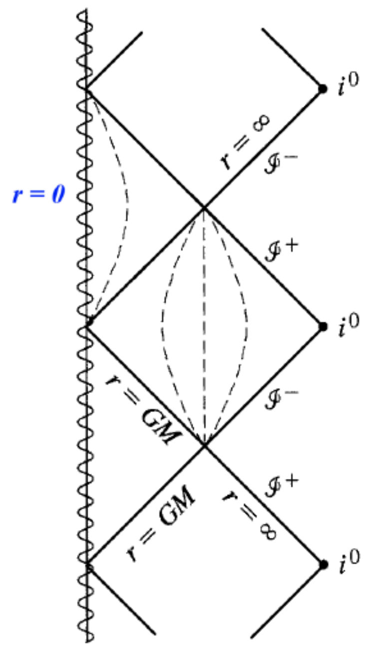


Fig. 2.5 – Conformal diagram for RN solution with $R_S^2 = R_Q^2$, where there is a naked singularity at $r = 0$, and infinite number of asymptotically flat regions.

of increasing r will this time emerge in another asymptotically flat region, different than the one he started from at the beginning. In fact, there are infinitely many flat regions outside the black hole. The conformal diagram shown in Fig. 2.5 summarizes what have been said.

Third case: $R_S^2 > R_Q^2$

In this case, the function $\Delta(r) = g^{rr}$ has two roots, and hence can be written in the form

$$\Delta(r) = g^{rr} = \frac{1}{r^2}\left(r - r_+\right)\left(r - r_-\right) \ . \tag{2.128}$$

where

$$r_+ = \frac{1}{2}\left(R_S + \sqrt{R_S^2 - R_Q^2}\right), \qquad r_- = \frac{1}{2}\left(R_S - \sqrt{R_S^2 - R_Q^2}\right) \ , \tag{2.129}$$

with $r_+ > r_-$. There are hence two distinct event horizons. For an observer coming from infinity, once crossing the outer horizon r_+, the coordinate r switches from being spacelike to timelike. Consequently, particles are forced to move in the direction of decreasing r, until reaching r_-. We can show that a far away observer will see infalling particles to slow down and increasingly redshifted, in a similar way to the Schwarzschild case.

Once the infalling observer crosses the inner horizon r_-, the r coordinate switches back to be spacelike. This means he is no longer forced to move in the direction of decreasing r, and the timelike singularity at $r = 0$ can be avoided [16][35].

The observer choose to accelerate and avoid the singularity until he crosses again the inner horizon r_-. Then, the coordinate r will becomes timelike, forcing him to continue moving in direction of increasing r, until appearing from r_+ in another asymptotically flat region. At this point, it is up to the observer to continue his journey towards infinity, or to repeat the previous experience by recrossing r_+. However, his experience this time will be in a completely another hole, different than the previous one, as shown in Fig. 2.6.

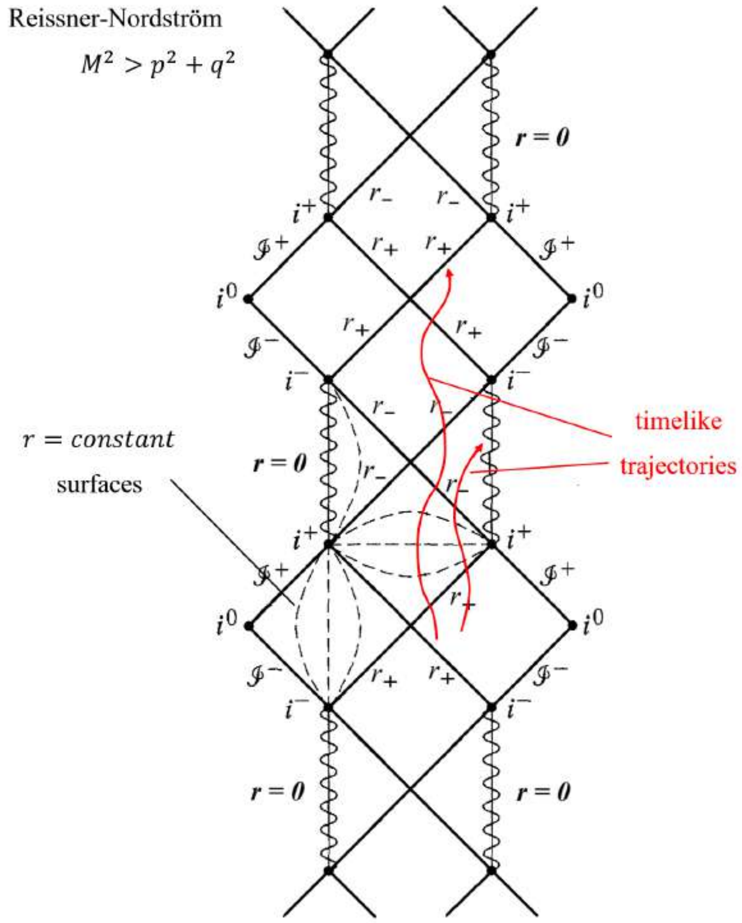


Fig. 2.6 – Conformal diagram for RN solution with $R_S^2 > R_Q^2$, where there are an infinite number of asymptotically flat regions.

2.5 Kerr Black Holes

Schwarzschild solution is the most general static vacuum solution in GR to Einstein equations. However, this solution only describe non rotating object, which is not the case with almost all astrophysical real bodies. The most general stationary vacuum solution is Kerr solution [17][35][24]. It is the vacuum stationary solution to Einstein equations with axial symmetry that is asymptotically flat. This solution is not only parametrized by the mass M, but also by the angular momentum J. For this reason, Kerr solution can be seen as the rotating generalization of the Schwarzschild solution. Although Schwarzschild and Reissner-Nordström solutions were discovered short after GR was published, Kerr solution was only discovered in 1963. When generalized to include electrically charged rotating bodies, another constant is needed to parametrize the solution which is the electric charge Q, and the solution is then called Kerr-Newman. We are not going to derive the Kerr metric in this dissertation for reasons that will be clear later. However, we will review some intresting steps that lead to its derivation.

The discovery of Kerr solutions took about half a century after Einstein equations had been published, in 1915. This was mainly due to its complexity: the transition from spherical to axial symmetry entails enormous made the calculations much more difficult. The best thing physicist knew was the form of the metric of a rotating body at weak-field approximation, i.e., very far away from the source

$$\mathrm{d}s^2 = -\left[1 - \frac{2M}{r} + O\left(\frac{1}{r^2}\right)\right]\mathrm{d}t^2 - \left[\frac{4J\sin^2\theta}{r} + O\left(\frac{1}{r^2}\right)\right]\mathrm{d}t\mathrm{d}\phi \tag{2.130}$$

$$+ \left[1 + \frac{2M}{r} + O\left(\frac{1}{r^2}\right)\right]\left[\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right] \ . \tag{2.131}$$

This is called the Lense-Thirring metric [13][18]. Obviously, for the limit $J \longrightarrow 0$, this reduce to the Schwarzschild metric. Although the results obtained from this metric was excellent when applied to solar system tests of GR, it was not enough for situations like neutron stars or rotating black holes, where a strong-field are present.

After lot of efforts that took 48 years, the sought after solution was published by Roy Kerr in 1963. Assuming a vacuum solution to Einstein equations that is axially symmetric, stationary and asymptotically flat, Kerr managed to obtain the desired metric. In his original work, in the Kerr-Schild coordinate (t, x, y, z) is [24]

$$\begin{aligned}\mathrm{d}s^2 &= -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \\ &+ \frac{2Mr^3}{r^4 + a^2z^2}\left[\mathrm{d}t + \frac{r\left(x\mathrm{d}x + y\mathrm{d}y\right) + a\left(y\mathrm{d}x - x\mathrm{d}y\right)}{r^2 + a^2} + \frac{z}{r}\mathrm{d}z\right] \ ,\end{aligned}$$

where $r \equiv r(x, y, z)$ is a function of the coordinates defined as

$$x^2 + y^2 + z^2 = r^2 + a^2\left[1 - \frac{z^2}{r^2}\right] \ .$$

In the limit $M \rightarrow 0$, the metric reduce to the flat Minkowski metric in Cartesian coordinates $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$.

However, the derivation of the solution was not clear so that some physicist believed there was no way to derive the metric from its general properties: stationary, axial symmetric and vacuum solution. In his book *Classical fields* (1975), Landau stated that "there is no constructive analytic derivation of the Kerr metric that is adequate in its physical ideas and even a check of this solution of Einstein's equations involves cumbersome calculations"[17]. Twenty years later, in his book *The Mathematical Theory of Black Holes* (1983), Chandrasekhar stated that "Contrary to this statement, we shall find that, once the basic equations have been properly written and reduced, the derivation of the Kerr metric is really very simple" [17]. In spite of the great details he made, not everyone would find Chandrasekhar's derivation very simple since it is really lengthy and still cumbersome. Elsewhere in his book, he add "The nature of developments simply does not allow a presentation that can be followed in details with modest efforts: the reductions that are required to go from one step to another are often very elaborate and, on occasion, may require as many as ten, twenty, or even fifty pages". Today, this kind of calculations concerning the Kerr metric are easily done using computer symbolic softwares such as Maple or Mathematica.

Another approach to derive Kerr solution is the Newman-Janis algorithm, which is shorter than the one presented by Chandrasekhar. The basic idea is that we can obtain the Kerr metric by performing a complex coordinate transformation on the Schwarzschild metric [41]. Recent works are suggestion other approaches to derive the Kerr metric, which are not as lengthy as the one developed by Chandrasekhar.

2.5.1 Properties of Kerr metric

The line element of the Kerr spacetime in the *Boyer-Lindquist* coordinates (t, r, θ, ϕ) is [17][18][35]

$$ds^2 = \frac{-1}{\rho^2}\left[r^2 + a^2\cos^2\theta - 2Mr\right]\mathrm{d}t^2 - \frac{2Mar\sin^2\theta}{\rho^2}(\mathrm{d}t\mathrm{d}\phi + \mathrm{d}\phi\mathrm{d}t) + \frac{\rho^2}{\Delta}\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[\left(r^2 + a^2\right)^2 - a^2\Delta\sin^2\theta\right]\mathrm{d}\phi^2 , \tag{2.132}$$

where

$$\Delta(r) = r^2 - 2Mr + a^2, \tag{2.133}$$

$$\rho^2(r, \theta) = r^2 + a^2\cos^2\theta . \tag{2.134}$$

The parameters M and $a = J/M$ are the Komar energy and Komar angular momentum of the black hole, respectively. In order to describe a rotating charged black hole, and hence include the electric charge Q, we do the replacement $M \rightarrow 2Mr - Q^2$. The resulting metric is called *Kerr-Newman.*

- *Asymptotic flatness:* it is obvious that, at large r, the metric (2.132) becomes progressively Minkowskian.

- *Generalization of Schwarzschild Metric:* notice that if we set $a = 0$ in (2.132), we recover Schwarzschild metric, which is expected since Kerr metric is the rotating generalization of Schwarzschild black hole. On the other hand, setting $M = 0$ in (2.132), we recover the Minkowski flat spacetime, where the spacial part is not in the ordinary polar coordinates, but rather in the ellipsoidal coordinate

$$ds^2 = -\mathrm{d}t^2 + \frac{\left(r^2 + a^2\cos^2\theta\right)^2}{(r^2 + a^2)}\mathrm{d}r^2 + \left(r^2 + a^2\cos^2\theta\right)^2\mathrm{d}\theta^2 + \sin^2\theta\left[r^2 + a^2\right]\mathrm{d}\phi^2 . \tag{2.135}$$

  These spacial ellipsoidal coordinate, shown in Fig. 2.7, are related to Cartesian coordinates of flat space by

$$\begin{aligned} x &= (r^2 + a^2)^{1/2}\sin\theta\cos\phi , \\ y &= (r^2 + a^2)^{1/2}\sin\theta\sin\phi , \\ z &= r\cos\theta . \end{aligned} \tag{2.136}$$

- *Symmetries*: Kerr metric is not static, and is therefore not invariant under time reversal $t \longrightarrow -t$. However, since it represent stationary axisymmetric solution to Einstein equations, it is independent of the coordinates t and ϕ. Hence, the metric (2.132) possesses two killing vectors

$$\xi^\mu = (\partial_t)^\mu = (1, 0, 0, 0) \quad , \tag{2.137}$$

$$\psi^\mu = (\partial_\phi)^\mu = (0, 0, 0, 1) \quad . \tag{2.138}$$

  The vector $\psi^\mu = (\partial_\phi)^\mu$ is expected, since it represent the axial symmetry of a black hole. The vector normal to to t =constant hypersurfaces is given by

$$N^\mu = g^{\mu\nu}\nabla_\nu(t) = g^{\mu\nu}\partial_\nu(t) = g^{\mu\nu}\delta^t_\nu = g^{\mu t} = (g^{tt}, 0, 0, g^{\phi t}),$$

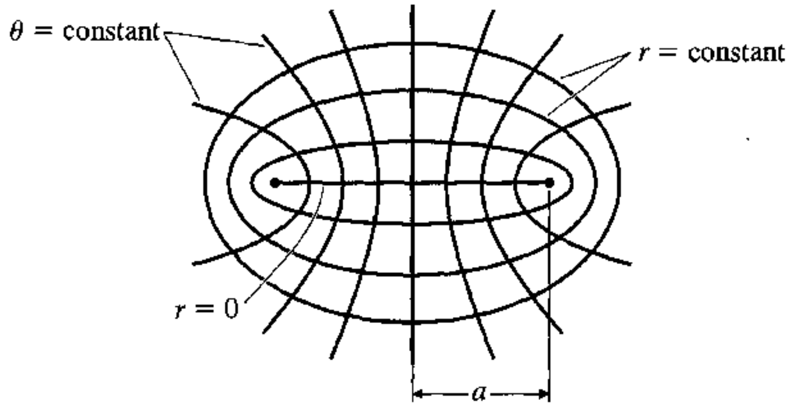


Fig. 2.7 – Ellipsoidal coordinates (r, θ), used in the Kerr metric. $r = 0$ is a two-dimensional disk, while the intersection of $r = 0$ with $\theta = \pi/2$ is the ring at the boundary of this disk.

meaning that N^μ is not proportional to ξ^μ for Kerr spacetime. In fact, it is possible to prove that ξ^μ is not normal to any hypersurface. Hence, the existence of a time translation Killing vector ξ^μ that is normal to $t =$constant hypersurfaces confirms the fact that Schwarzschild spacetime is static [16]. Hence, unlike the case of Schwarzschild and RN, Kerr space time is not static, but only stationary.

In addition to ξ^μ and ψ^μ, Kerr spacetime has the following Killing tensor

$$\sigma_{\mu\nu} = 2\rho^2 l_{(\mu} n_{\nu)} + r^2 g_{\mu\nu}, \tag{2.139}$$

where the one-forms l_μ and n_ν are given by their vector counterparts

$$l^\mu = \frac{1}{\Delta}\left(r^2 + a^2, \Delta, 0, a\right) \ , \tag{2.140}$$

$$n^\mu = \frac{1}{2\rho}\left(r^2 + a^2, -\Delta, 0, a\right) \ . \tag{2.141}$$

The constants of motion associated with the above Killing vectors are

$$E = -g_{\mu\nu}\xi^\mu P^\nu = -P_t \ , \tag{2.142a}$$

$$L = g_{\mu\nu}\psi^\mu P^\nu = P_\phi \ , \tag{2.142b}$$

$$C = P_\theta^2 + \cos^2\theta \left[a^2\left(\mu^2 - P_t^2\right) + \frac{P_\phi^2}{\sin^2\theta}\right] \ , \tag{2.142c}$$

where E and L are, as usual, the conserved energy per rest mass and angular momentum per rest mass respectively, while C is the constant corresponding to the Killing tensor $\sigma_{\mu\nu}$, called Carter constant [17].

Moreover, Kerr metric is invariant under the simultaneous transformations

$$\begin{cases} t \longrightarrow -t \\ \phi \longrightarrow -\phi \end{cases} ,$$

This is expected since rotating a body in opposite direction is equivalent to time reversal it. Another symmetry is the transformation

$$\begin{cases} a \longrightarrow -a \\ \phi \longrightarrow -\phi \end{cases} ,$$

which leave the Kerr metric unchanged, since reversing angular momentum is the same as rotating in the opposite direction.

2.5.2 Curvature Singularity

From (2.132) is singular when $\rho = 0$, which will turns out to be a real singularity. As in cases of Schwarzschild and RN metrics, a sufficient, though not necessary, criteria for curvature singularity is that a coordinate independent quantity constructed from the curvature $R^{\alpha}{}_{\beta\mu\nu}$ blows up. As in the case of Schwarzschild and RN spacetimes, the Kretschmann scalar for Kerr metric is [46]

$$R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} = \frac{8}{\left(r^2 + a^2\cos^2\theta\right)^6}\left[6M^2\left(r^6 - 15a^2r^4\cos^2\theta + 15a^4r^2\cos^4\theta - a^6\cos^6\theta\right)\right] \ , \tag{2.143}$$

which obviously blows up when

$$\begin{aligned} \rho = r^2 + a^2\cos^2\theta = 0 \\ \implies \quad r = 0 \ , \quad \theta = \frac{\pi}{2} \ . \end{aligned} \tag{2.144}$$

According to (2.136), $r = 0$ is not a point in space but a disk, while $r = 0$, $\theta = \pi/2$ is the ring at the boundary of this disk. Hence, unlike the case of Schwarzschild, the curvature singularity of the Kerr metric is a ring rather than a point in space. The rotation of the black hole have have extended and spread the singularity into a ring. Therefore, an observer moving towards $r = 0$ is not going necessarily to hit the ring singularity, but her can rather avoid it by passing through it, i.e., through the surface $(r = 0, \theta \neq \pi/2)$, and emerge in a different spacetime. Indeed, the analytic continuation of Kerr space time shows that if you move in the direction of decreasing r, all the way through the ring singularity $(r = 0, \theta \neq \pi/2)$, you will exit to another asymptotically flat spacetime that is described by Kerr metric but with $r < 0$ [35]. Consequently, there are no horizons in the new asymptotically flat region, since $\Delta(r)$ does not vanish for $r < 0$. Such a world line is the one illustrated on the right in Fig. 2.8.

Now, we say that a spacetime is inextendible if all causal curves (null and timelike) can be arbitrarily extended to large values of the affine parameter or hit the ring singularity [42]. Intuitively, an inextendible spacetime can be thought of as the one that is as large as it can be [43]. If we require Kerr spacetime to be inextendible, an infinite number of copies of the asymptotically flat region must be patched together, as the conformal diagram in Fig. 2.8 shows.

However, it is believed that these asymptotically flat region with $r < 0$ are unlikely to occur in real astrophysical scenarios where rotating black hole are formed from in a gravitational callapse or in the coalescence of compact bodies [35].

2.5.3 Event Horizons

The metric (2.132) is singular also when $\Delta = 0$, or equivalently $g^{rr} = 0$. This turns out to be a coordinate singularity. From the discussion preceding equation (2.3), we saw that, in a suitably chosen coordinates, hypersurfaces of constant r at which $g^{rr} = 0$ represent event horizons. Boyer-Lindquist coordinates in which the metric (2.132) is written are just suitable enough, so that we have

$$g^{rr} = \frac{\Delta}{\rho^2} = 0 \quad \implies \quad r_{\pm} = M \pm \sqrt{M^2 - a^2}. \tag{2.145}$$

Hence, in a similar way to RN metric, according to the values of M and a, we can distinguish three cases [16][17]

- First case: $M^2 < a^2 \implies$ *no horizons,*
- Second case: $M^2 = a^2 \implies$ *one horizons,*
- Third case: $M^2 > a^2 \implies$ *two horizons.*

Let us consider each case in more details.

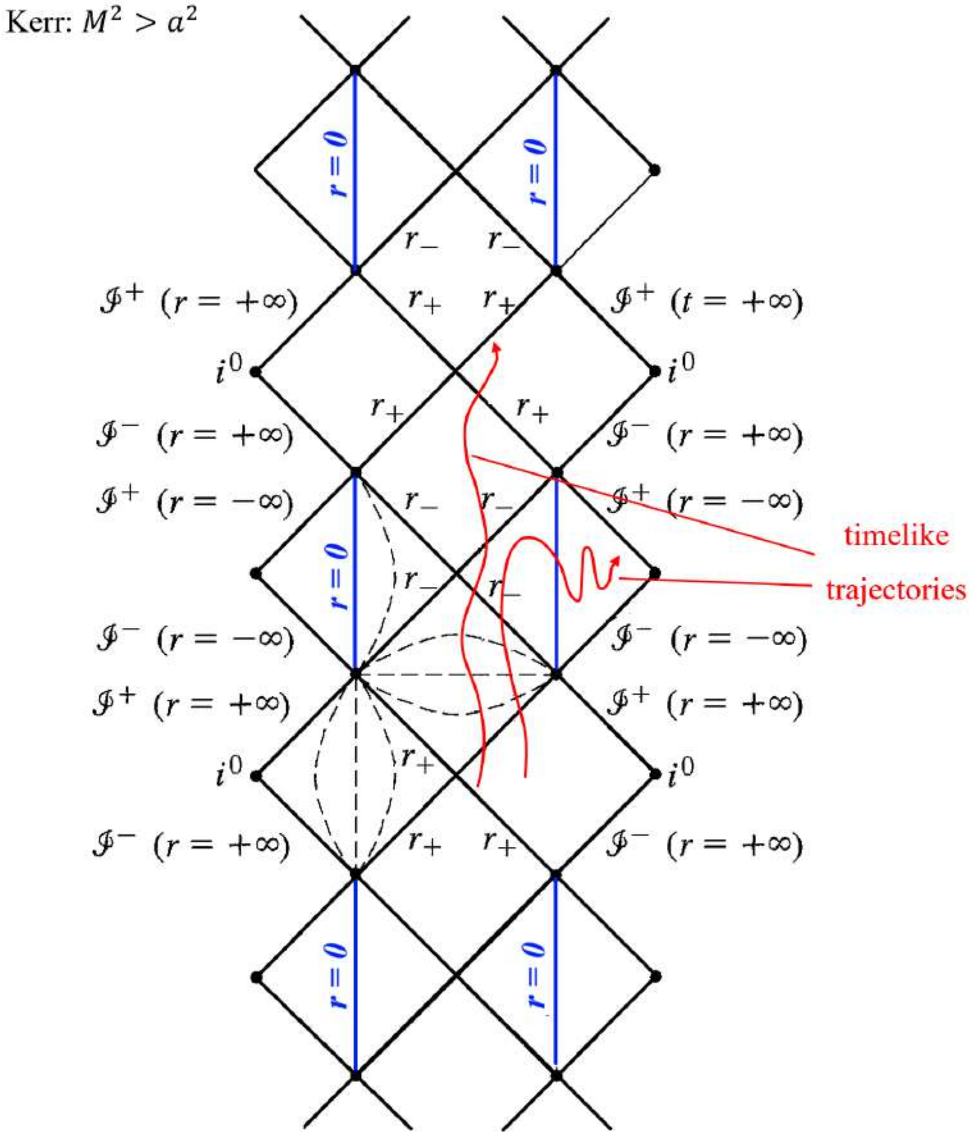


Fig. 2.8 – Conformal diagram for the Kerr solution with $M^2 > a^2$. There are infinite copies of asymptotically flat regions, not only outside the black hole, but also beyond the ring singularity.

- *First case:* $M^2 < a^2$

In this case, the function $g^{rr} = \Delta/\rho^2$ is always positive and has no roots, and hence there are no horizons. The normal N^μ to the hypersurface $r = 0$ has the following norm

$$\begin{aligned} N_\mu N_\nu g^{\mu\nu} &= \partial_\mu(r)\partial_\nu(r)g_{\mu\nu} = \delta^r_\mu \delta^r_\nu g_{\mu\nu} = g_{rr} = \frac{\Delta}{\rho^2} \\ &= \frac{r^2 + a^2 - 2Mr}{r^2 + a^2\cos^2\theta} \ , \end{aligned} \tag{2.146}$$

which is obviously always positive. Hence, the naked singularity $r = 0$ is timelike, and the Conformal diagram would be much like that for RN naked singularity. As we have seen earlier, the Cosmic Censorship Conjecture prohibit the formation of naked singularity, rendering the first case less important for realistic scenarios.

- *Second case:* $M^2 = a^2$

The second case is less important as well in realistic situations, since it is unstable: any small amount of matter falling into the black hole will make $M^2 > a^2$, carrying it to case three. In this case, the equation $\Delta(r) = g^{rr} = 0$ has one root $r = M$, which represent the horizon.

- *Third case:* $M^2 > a^2$

This is the most important case for real astrophysical situations. In this case, the equation $\Delta(r) = g^{rr} = 0$ has two roots

$$r_\pm = M \pm \sqrt{M^2 - a^2}\ , \tag{2.147}$$

with $r_+ > r_-$, meaning that there is two distinct horizons. The corresponding conformal diagram, shown in Fig. 2.8, it is very similar to that of RN solution when $R_S^2 > R_Q^2$ in that it contains an infinite copies of asymptotically flat regions, but with the exception that singularity now is a ring. This means that an observer there can pass through ring singularity towards another asymptotically flat region that do not contain horizons.

2.5.4 Frame Dragging

For an observer having angular momentum P^μ in Kerr spacetime. In Boyer-Lindquist coordinates, the observer's angular velocity as measured from infinity $\Omega \equiv d\phi/dt$, where time coordinate coincide with proper time, can be written as

$$\Omega \equiv \frac{d\phi}{dt} = \frac{d\phi/d\tau}{dt/d\tau} = \frac{P^\phi}{P^t} = \frac{g^{\phi t}P_t + g^{\phi\phi}P_\phi}{g^{tt}P_t + g^{t\phi}P_\phi}\ . \tag{2.148}$$

Now, suppose that the observer is falling into the black hole from infinity with zero angular momentum. Using equation (2.142b), this means

$$L = P_\phi = 0\ , \quad \text{along the geodesic.}$$

Such an observer is called ZAMO, which is an abbreviation for zero angular momentum observer. At infinity $r \to \infty$, the spacetime is asymptotically flat, and the metric is Minkowskian, i.e., $g_{\mu\nu} = \eta_{\mu\nu}$. Substituting $P_\phi = 0$ and $\eta^{\mu\nu} = diag(-1, +1, +1, +1)$ in (3.32), we obtain

$$\Omega = \frac{\eta^{\phi\phi}P_\phi}{\eta^{tt}P_t} = 0\ , \qquad \text{at infinity.} \tag{2.149}$$

However, along the ZAMO's trajectory, the metric is no longer Minkowskian, i.e., $g_{\mu\nu} \neq \eta_{\mu\nu}$, and equation (3.32) for the angular velocity becomes

$$\Omega = \frac{g^{\phi t}}{g^{tt}}\ , \tag{2.150}$$

where we put $P_\phi = 0$ since it is constant along the geodesic. Consequently, the ZAMO, which started with $\Omega = 0$, acquired an angular velocity while is falling in the black hole. Using the fact that a ZAMO has $L = P_\phi = 0$, we get

$$P_\phi = g_{\phi t}P^t + g_{\phi\phi}P^\phi = 0 \quad \Longrightarrow \quad \Omega = \frac{P^\phi}{P^t} = -\frac{g_{\phi t}}{g_{\phi\phi}}\ , \tag{2.151}$$

where we have used $\Omega = P^\phi/P^t$ from (3.32). Substituting $g_{\mu\nu}$ from (2.132), the ZAMO angular velocity take the form

$$\Omega = \frac{P^\phi}{P^t} = -\frac{g_{\phi t}}{g_{\phi\phi}}$$

$$\Omega = \frac{2Mar}{(r^2 + a^2)^2 - a^2\sin^2\theta\Delta}\ . \tag{2.152}$$

You can see that, as $r \to \infty$, the angular velocity vanishes. Moreover, notice that

$$r^4 + a^4 + 2Mr > a^4 + a^2r^2 - 2Mr = a^2\Delta(r) \; ,$$
$$\Longrightarrow \left(r^2 + a^2\right)^2 > a^2\Delta \sin^2\theta \; . \tag{2.153}$$

Hence, using the fact that $a \equiv J/M$ and equation (3.37), we get

$$\frac{\Omega}{J} = \frac{\Omega}{Ma} = \frac{2r}{(r^2 + a^2)^2 - a^2 \sin^2\theta\Delta} > 0 \; , \tag{2.154}$$

which means that the angular velocity of the ZAMO Ω and the black hole angular momentum J have always the same sign, i.e., they are always rotating in the same direction. In other words, although the ZAMO started at infinity with zero angular velocity $\Omega = 0$, it will soon acquire an angular velocity because of the dragg of the black hole which forces the ZAMO to corotate with it [17][35]. Frame dragging of different paths of ZAMOs moving on the equatorial plane are shown in Fig. 2.9.

2.5.5 Ergosphere

Since Kerr spacetime is not static, its horizon is not a Killing horizon for the time translation Killing vector $\xi^\mu = (\partial_t)^\mu$, i.e., ξ^μ is not null on the horizons $r_\pm$ [16][42]. Indeed, a simple calculations reveals that

$$\xi^\mu \xi^\nu g_{\mu\nu} = g_{tt} = \frac{-1}{\rho^2}\left(r^2 + a^2\cos^2\theta - 2Mr\right) = \frac{-1}{\rho^2}\left(\Delta - a^2\sin^2\theta\right) \; , \tag{2.155}$$

where we have used (2.132) for the tt component or the metric. At the horizons $r_\pm$, we have $\Delta(r_\pm) = 0$, and hence we obtain

$$\xi^\mu \xi_\mu \,|_{r_\pm} = \frac{a^2\sin^2\theta}{\rho^2} \geqslant 0 \; . \tag{2.156}$$

The hypersurface where $\xi^\mu\xi_\mu = 0$ is given by

$$\xi^\mu\xi_\mu = g_{tt} = 0 \quad \Longrightarrow \quad -r^2 - a^2\cos^2\theta + 2Mr = 0$$
$$\Longrightarrow \quad r_{s(\pm)} = M \pm \sqrt{M^2 - a^2\cos^2\theta} \; . \tag{2.157}$$

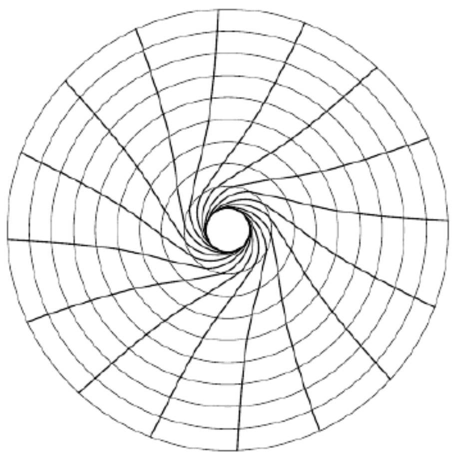


Fig. 2.9 – Observers starting from infinity with zero angular velocity (ZAMOs) are dragged by the black hole and froced to corotate with it.

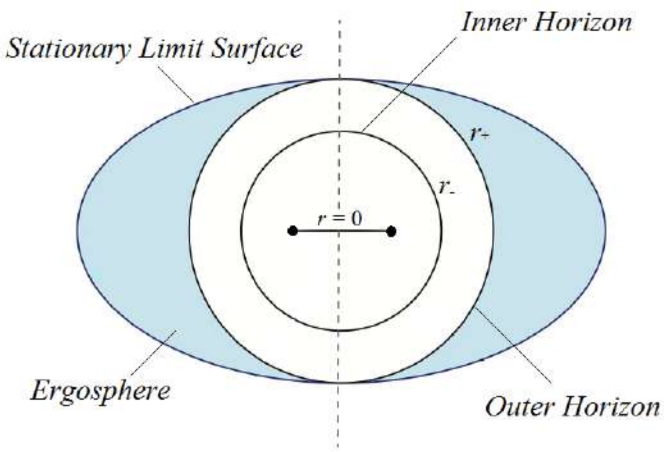


Fig. 2.10 – Structure of Kerr black hole (side view) where the singularity ring, horizons and stationary limit surfaces are illustrated.

Surfaces defined by $r = r_{s(\pm)}$ are called *stationary limit surfaces*, for a reason that will be explained later. Using the fact that $M^2 - a^2 \leqslant M^2 - a^2 \cos^2\theta$, and equations (2.147), (2.157), it follows that

$$r_{s(-)} \leqslant r_- < r_+ \leqslant r_{s(+)}, \tag{2.158}$$

which means that both horizons lies between the stationary limit surfaces. On the symmetry axis, where $\theta = 0, \pi$, equation (2.157) gives $r_{s(-)} = r_-$ and $r_+ = r_{s(+)}$.

The sign of the tt component of the metric is

$$\begin{cases} g_{tt} < 0 \;, & \text{for} \quad r < r_{s(-)} \;, \; r_{s(+)} < r \\ g_{tt} > 0 \;, & \text{for} \quad r_{s(-)} < r < r_{s(+)} \end{cases} . \tag{2.159}$$

Now, consider a stationary observer ($dx^i/d\tau = 0$) in the Kerr spacetime whose four-velocity is $U^\mu = (U^t, 0, 0, 0)$. The normalization of U^μ yields

$$U^\mu U_\mu = \left(U^t\right)^2 g_{tt} = -1 \quad \Longrightarrow \quad U^t = \left(-g_{tt}\right)^{-1/2} \;, \tag{2.160}$$

which means that U^t can be real only when $g_{tt} < 0$, i.e, when $r < r_{s(-)}$ or $r_{s(+)} < r$. Therefore, a *stationary observer only exists outside the hypersurface* $r_{s(+)}$ [35]. For this reason, this hypersurface is called the stationary limit surface while the region inside it is called Ergosphere.

Notice that for $a = 0$, the horizon r_+ coincides with stationary limit surface, which means that it is the rotation of the black hole made this region forbidden for any stationary observer.

However, we can consider the special observer for which the geometry does not change. This observer must have a normalized four velocity that is proportional to the spacetime Killing vectors $U^\mu = a\xi^\mu + b\psi^\mu$, where a and b are coefficients. We can write U^μ in the form

$$U^\mu = \left(U^t, 0, 0, U^\phi\right) = U^t \left(1, 0, 0, \Omega\right) \;, \tag{2.161}$$

where $\Omega \equiv d\phi/dt = U^\phi/U^t$ is the angular velocity of the observer as measured from infinity. Since $U^r = dr/d\tau = 0$ and $U^\theta = d\theta/d\tau = 0$, such an observer is moving on circular orbits with angular velocity Ω. The normalization of the four velocity $U^\mu U_\mu = -1$ can be written as

$$\begin{aligned} U^\mu U_\mu &= g_{tt}\left(U^t\right)^2 + 2g_{t\phi}U^tU^\phi + g_{\phi\phi}\left(U\phi\right)^2 = -1 \;, \\ &\Longrightarrow \left(U^t\right)^2 \left[g_{tt} + 2g_{t\phi}\Omega + g_{\phi\phi}\Omega^2\right] = -1 \;, \\ &\Longrightarrow \left[g_{tt} + 2g_{t\phi}\Omega + g_{\phi\phi}\Omega^2\right] < 0 \;. \end{aligned} \tag{2.162}$$

The quantity between brackets has the following roots

$$\Omega_{\pm} = \frac{-g_{t\phi} \pm \sqrt{(g_{t\phi})^2 - g_{\phi\phi}g_{tt}}}{g_{\phi\phi}} = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \left(\frac{g_{tt}}{g_{\phi\phi}}\right)} \ , \tag{2.163}$$

This roots can be rewritten in the form

$$\Omega_{\pm} = \frac{-g_{t\phi} \pm \sin\theta\sqrt{\Delta}}{g_{\phi\phi}} \ , \tag{2.164}$$

where we have used the fact that $(g_{t\phi})^2 - g_{\phi\phi}g_{tt} = \sin^2\theta\Delta^2$. Thus, the last inequality holds only if

$$\Omega_- < \Omega < \Omega_+ \ \ , \tag{2.165}$$

This means that there is a limited range of allowed angular velocity for our observer. Notice that for $r > 0$, we have always $g_{t\phi} < 0$ while $g_{\phi\phi} > 0$.

Now, using (2.164), in the region between horizons $r_- < r < r_+$, we have $g_{t\phi} < 0$, $g_{\phi\phi} > 0$, and $\Delta < 0$. Thus $\Omega_{\pm}$ is not real, and there is no observer in circular motion with around the symmetry axis in this region.

Using (2.163), in the region outside the Ergosphere $r > r_{s(+)}$, we have $g_{t\phi} < 0$, $g_{\phi\phi} > 0$, and $g_{tt} < 0$, meaning that the solutions to (2.163) have the following signs

$$\begin{aligned} &\Omega_+ > 0 \ , \quad \textit{Corotate with the black hole,} \\ &\Omega_- < 0 \ , \quad \textit{Counter rotate with the black hole.} \end{aligned}$$

Hence, in this region, an observer which is in circular motion around the symmetry axis can either corotate with the black hole ($\Omega_+ > 0$), or counter rotate with it ($\Omega_- < 0$).

In the region between the outer stationary limit surface and the outer horizon $r_+ < r < r_{s(+)}$, we have $g_{t\phi} < 0$, $g_{\phi\phi} > 0$, and $g_{tt} > 0$, which means that the solutions (2.163) are both positive

$$\begin{aligned} &\Omega_+ > 0 \ , \quad \textit{Corotate with the black hole,} \\ &\Omega_- > 0 \ , \quad \textit{Corotate with the black hole,} \end{aligned}$$

which means that an observer in circular orbit around the axis of symmetry in this region of the Ergosphere always corotate with the black hole.

2.5.6 Causality Violation

Another bizarre property in the Kerr spacetime is the existence of *closed timelike curves* (CTCs), which are curves that can represent an observer's world line, so that he can meet himself in his own past. In other words, this observer would interfere in his own past and hence violating causality [35][42]. To see how CTCs can happen in Kerr spacetime, consider an observer who manages to reach the asymptotically flat spacetime with $r < 0$ that we discussed earlier. This observer will start his journey from outside the black hole and moves in the direction of decreasing r, crossing both horizons, go all the way through the ring singularity ($r = 0, \theta \neq \pi/2$) until he emerges in the asymptotic region of $r < 0$. Then, he need to start moving on a closed curve γ having the form a ring, just outside the ring singularity, in the equatorial plane with $t = const$ and $r = const < 0$, as portrayed in Fig. 2.11. This curve γ is defined as

$$\gamma = (t = const, r = const, \theta = \pi/2, 0 \leqslant \phi \leqslant 2\pi) \ , \qquad \text{with } |r| \ll M \ , r < 0 \ . \tag{2.166}$$

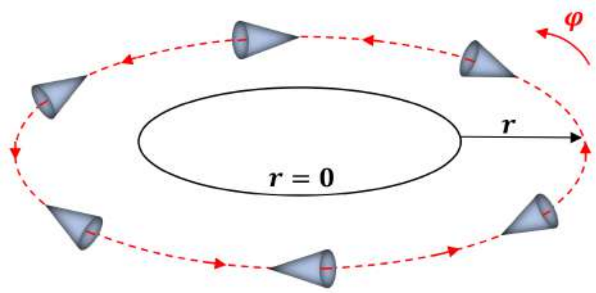


Fig. 2.11 – A closed time like curve (CTC) with $|r| \ll M$ and $r < 0$, just outside the ring singularity, in the asymptotically flat region with $r < 0$. Future light cones are illustrated along the curve.

The observer's four velocity, i.e., the tangent vector to this curve, is obviously the Killing vector $\psi^\mu = (\partial_\phi)^\mu$. Using (2.132), the norm of ψ^μ is

$$\begin{aligned}\psi^\mu \psi^\nu g_{\mu\nu} = g_{\phi\phi} &= \frac{1}{r^2}\left[\left(r^2+a^2\right)^2 - a^2(r^2+a^2-2Mr)\right] \\ &= r^2 + a^2 + \frac{2a^2 M}{r} \ .\end{aligned} \tag{2.167}$$

Under the condition $|r| \ll M$, $r < 0$, we have $\psi^\mu \psi^\nu < 0$, which means that the curve γ is timelike. Since γ closed and timelike, it is a CTC, which violates causality, and therefore is considered as a pathology for Kerr solution. This curve is shown in Fig. 2.11. However, it is believed that CTCs and even the existence of the asymptotically flat region with $r < 0$ are unlikely to occur in real astrophysical scenarios.

2.6 Direct observational evidence of black holes

Today, there are many pieces of evidence of the existence of black holes, though all of them are in fact indirect. This is due to the fact that black holes do not emit light in any wave length. Here we give some of those observations.

- The detection of *ultraluminous X-ray sources* is a strong piece of evidence of the existence of black holes. When the gas of the accretion disk falls into the black hole due to friction, it heats up and radiates a tremendous amount of energy [12], mostly in the form of X-ray. One of these ultraluminous sources are *quasars*, the brightest known objects in the universe. Their extreme luminosity, thousands of times greater than that of average galaxies like ours, is believed to be powered by supermassive black holes. The reason behind this extreme brightness is that up to 40% of the rest mass of the infalling matter is converted from gravitational potential energy to radiation [13]. This is a very high conversion rate compared to only 0.7% of nuclear fusion at the heart of stars like the Sun.

- Other similar sources are *X-ray binaries*. These are systems composed of two bodies orbiting their common center of mass. The first body is a star while the second is either a black hole or neutron star. When matter falls from the companion star into the black hole, a tremendous amount of energy is radiated in the form of X-ray.

- *Trajectories of stars orbiting the center of galaxies* strongly suggest the existence of black holes. For many years, astronomers have been tracking the motion of stars orbiting the center of the Milky Way galaxy [15]. By fitting their trajectories to Keplerian orbits, they deduced that the gravitating body at the center was several millions more massive than our Sun while contained in a relatively small radius. The only known astrophysical object that has these characteristics is the black hole.

- *The detection of the gravitational waves* in 2015 on the 100th anniversary of General Relativity provided another highly accurate test of the theory. This significant achievement is expected to open the doors for a new era in astronomical observation. These gravitational waves were traveling ripples in the fabric of spacetime, produced by the merger of two stellar-mass black holes. The detected waves suggested that the separation between the two merging massive objects was only 350 km [8][9]. This indicated that the massive objects were extremely compact, making black holes the only known candidates. Therefore, the detection of gravitational waves is considered the most compelling piece of evidence of the existence of black holes.

- *The first direct image of a black hole ever* was a groundbreaking astronomical achievement. In 2019, the Event Horizon Telescope collaboration [10] released the image of the supermassive black hole at the center of Messier 87 galaxy. Two years later, the same team released the first image of Sagittarius A*, the supermassive black hole at the center of the Milky Way galaxy. The images showed the black hole's accretion disks with details aligning with the theoretical predictions, hence providing compelling evidence of the existence of black holes.

2.7 Black holes unsolved mysteries

The extreme and enigmatic nature of black holes continues to pose significant problems to physicists. Despite decades of research, our understanding of these objects remains incomplete.

- The *absence of a quantum theory of gravity* [16][48], which represents the first and foremost unsolved black hole problem. Physicists believe most of other problems boil down to this one. Inside black holes, the effect of both General Relativity and Quantum Mechanics becomes important. Consequently, this raises the necessity of a quantum theory of gravity. Unfortunately, there is no complete consistent quantum theory of gravity. However, several candidate theories are proposed, among which are *String Theory*, *Loop Quantum Gravity*, and *Noncommutative Geometry*. These theories face a number of challenges, such as experimentally testing their predictions.

- The existence of *naked singularities* poses another challenge to the current understanding [16][49]. These are singularities not hidden by an event horizon from the rest of the spacetime. Their existence is supposedly forbidden by the debatable *cosmic censorship conjecture*. However, the correctness of this conjecture itself remains debatable.

- Another problem is the existence of *closed timelike curves* in some regions of Kerr black holes [35][50], as we discussed previously in this chapter. An observer traveling along such curves would be able to meet and interact with himself in the past. This leads to the *Grandfather paradox*, in which an observer travels to the past and prevent somehow his birth. According to the math of General Relativity, nothing forbids an observer from traveling along such curves. However, the existence of such curves is a serious pathology since it is in contradiction with causality.

- The math of General Relativity predicts the existence of *white holes*, which represents another challenge to our understanding [51]. Roughly speaking, these are the reverse of black holes, i.e., regions of spacetime with singularity and event horizon that cannot be entered from outside. These hypothetical objects have not yet been observed and their existence and formation is an open question.

- The *information paradox* is one of the hardest black holes' problems [27]. The laws of Quantum Mechanics state that quantum information (the mass, the spin) of every particle falling into the black hole must be preserved somehow inside, even if it is behind the event horizon, beyond our reach. The problem comes when we combine the effects of Quantum Mechanics with General Relativity, which leads to the *Hawking radiation.* This hypothetical radiation suggests that black holes gradually evaporate and lose mass after a considerably long time. Since the radiated photons carry no information when they leave it, the black hole and the entire quantum information inside it could be permanently erased. This would violate the laws of Quantum Mechanics.

- Observation of distant *supermassive black holes* at the center of galaxies raised another difficult question about their formation [52]. Unlike stellar-mass black holes, which form after the collapse of stars, the formation of supermassive black holes remains an unsolved problem. To explain this, some proposed mechanisms suggest that they formed in the early universe and grew gradually through the accretion of matter and the merger with other black holes, a process that would last billions of years. Surprisingly, over the last decades, observations showed the existence of quasars powered by supermassive black holes with high redshift. This means that they grew and formed in the early universe, way earlier than we thought they could.

Chapter 3

Frequency Shift of Photons in Kerr-Newman Black Hole

As we have seen before, the Schwarzschild and RN black holes are not very likely to represent real situations in reality, since we expect real black holes to be nor stationary neither charged. This is why the rotating Kerr black holes provide a better description in this concern. However, the most general case is the Kerr-Newman black hole representing a rotating charged black hole, characterized by the three parameter mass M, angular momentum J and electric charge Q. Although the charge Q is not expected to survive for a long time, since it will be neutralized by interacting with the surrounding matter, the existence of a Kerr-Newman black hole still possible. Moreover, from a theoretical perspective, its study is very informative and rich. Therefore, this chapter will be dedicated to the study of Kerr-Newman black hole three parameters M, K and Q in terms of frequency shift of photons emitted by particles orbiting it. But before we delve into the details, we need explore some properties of this black hole.

3.1 The Kerr-Newman metric

The Kerr-Newman metric in the Boyer-Lindquist coordinates is given by:

$$g_{\mu\nu} = \begin{pmatrix} g_{tt} & 0 & 0 & g_{t\phi} \\ 0 & g_{rr} & 0 & 0 \\ 0 & 0 & g_{\theta\theta} & 0 \\ g_{\phi t} & 0 & 0 & g_{\phi\phi} \end{pmatrix} \quad ,$$

where

$$g_{tt} = -\left(1 - \frac{2Mr - Q^2}{\Sigma}\right), \tag{3.1}$$

$$g_{\phi t} = g_{t\phi} = -a\sin^2\theta\left(\frac{2Mr - Q^2}{\Sigma}\right), \quad g_{rr} = \frac{\Sigma}{\Delta}, \quad g_{\theta\theta} = \Sigma\ , \tag{3.2}$$

$$g_{\phi\phi} = \sin^2\theta\left(r^2 + a^2 - \frac{a^2\left(2Mr - Q^2\right)\sin^2\theta}{\Sigma}\right) \tag{3.3}$$

and

$$\Delta \equiv r^2 + a^2 - 2Mr + Q^2\ , \qquad \Sigma \equiv r^2 + a^2\cos^2\theta\ , \qquad a = \frac{J}{M}\ . \tag{3.4}$$

One useful relation between the metric components is

$$
\begin{aligned}
g_{tt}g_{\phi\phi} - g_{t\phi}^2 &= -\left(1 - \frac{2Mr - Q^2}{\Sigma}\right) \sin^2\theta \left(r^2 + a^2 - \frac{a^2\left(2Mr - Q^2\right)\sin^2\theta}{\Sigma}\right) \\
&\quad -a^2 \sin^4\theta \left(\frac{2Mr - Q^2}{\Sigma}\right)^2 \\
&= -\left(r^2 + a^2 - \frac{a^2\left(2Mr - Q^2\right)\sin^2\theta}{\Sigma}\right)\sin^2\theta + \frac{2Mr - Q^2}{\Sigma} r^2 \sin^2\theta + \frac{2Mr - Q^2}{\Sigma} a^2 \sin^2\theta \\
&\quad +a^2 \left(\frac{2Mr - Q^2}{\Sigma}\right)^2 \sin^4\theta - a^2 \sin^4\theta \left(\frac{2Mr - Q^2}{\Sigma}\right)^2 \\
&= \sin^2\theta \left[-r^2 - a^2 + \frac{a^2\left(2Mr - Q^2\right)\sin^2\theta}{\Sigma} + \frac{2Mr - Q^2}{\Sigma} r^2 + \frac{2Mr - Q^2}{\Sigma} a^2\right] \\
&= \sin^2\theta \left[-r^2 - a^2 + \frac{\left(2Mr - Q^2\right)}{\Sigma}\left(-a^2\sin^2\theta + r^2 + a^2\right)\right] \\
&= \sin^2\theta \left[-r^2 - a^2 + \frac{\left(2Mr - Q^2\right)}{\Sigma}\left(a^2\cos^2\theta + r^2\right)\right] \\
&= \sin^2\theta \left[-r^2 - a^2 + 2Mr - Q^2\right] = -\sin^2\theta\Delta
\end{aligned}
$$

In order to obtain the components of the inverse metric $g^{\mu\nu}$, we only need to calculate the inverse of the $t\phi$ block, while the inversion of the $r\theta$ block is trivial. The $t\phi$ block is

$$
\tilde{g}_{\mu\nu} = \begin{pmatrix} g_{tt} & g_{t\phi} \\ g_{t\phi} & g_{\phi\phi} \end{pmatrix} \quad \text{and} \quad \det\left[g_{\mu\nu}\right] = g_{tt}g_{\phi\phi} - g_{t\phi}^2 = -\sin^2\theta\Delta \quad ,
$$

while it's inverse is

$$
\tilde{g}^{\mu\nu} = \frac{1}{-\sin^2\theta\Delta} \begin{pmatrix} g_{\phi\phi} & -g_{t\phi} \\ -g_{t\phi} & g_{tt} \end{pmatrix} \quad .
$$

Hence, the inverse metric is

$$
g^{\mu\nu} = \begin{pmatrix} g^{tt} & 0 & g^{tt} & g^{t\phi} \\ 0 & \frac{\Delta}{\Sigma} & 0 & 0 \\ 0 & 0 & \frac{1}{\Sigma} & 0 \\ g^{t\phi} & 0 & 0 & g^{\phi\phi} \end{pmatrix} \quad ,
$$

where

$$
g^{tt} = -\frac{A}{\Delta\Sigma} \quad , \tag{3.5}
$$

$$
g^{t\phi} = g^{\phi t} = -a\frac{\left(2Mr - Q^2\right)}{\Delta\Sigma} \quad , \tag{3.6}
$$

$$
g^{\phi\phi} = \frac{\Delta - a^2\sin^2\theta}{\Delta\Sigma\sin^2\theta} \quad , \tag{3.7}
$$

and

$$
A \equiv \left(r^2 + a^2\right)^2 - \Delta a^2 \sin^2\theta \quad . \tag{3.8}
$$

3.1.1 The conserved quantities

The independence of the Kerr-Newman metric of the t and ϕ imply the existence of the following Killing vectors

$$\xi^\mu = (1, 0, 0, 0) \quad , \tag{3.9}$$

$$\psi^\mu = (0, 0, 0, 1) \quad . \tag{3.10}$$

In addition, the Kerr metric possesses a Killing tensor field [54], which means the existence of another conserved quantity called the Carter constant, denoted by C. Thus, The corresponding constant of motion are

$$E = -g_{\mu\nu}\xi^\mu P^\nu = -P_t = \textit{total energy}, \tag{3.11a}$$

$$L = g_{\mu\nu}\psi^\mu P^\nu = P_\phi = \textit{angular momentum}, \tag{3.11b}$$

$$C = P_\theta^2 + \cos^2\theta \left[a^2\left(\mu^2 - P_t^2\right) + \frac{P_\phi^2}{\sin^2\theta}\right] \tag{3.11c}$$

$$= P_\theta^2 + \cos^2\theta \left[a^2\left(\mu^2 - E^2\right) + \frac{L^2}{\sin^2\theta}\right] \tag{3.11d}$$

where μ is the mass of the test particle which we assume is neutral.

3.1.2 Equations of motions of neutral test particles

Along side with the equation of normalization of the particle's momentum $P^\nu P_\nu = -\mu^2$, the previous equations constitute the equations of motions of the particle.

Equation (3.11a) yields

$$\begin{aligned}
P^t &= g^{t\mu}P_\mu = g^{tt}P_t + g^{t\phi}P_\phi = -\frac{A}{\Delta\Sigma}(-E) - a\frac{(2Mr - Q^2)}{\Delta\Sigma}L \\
\Sigma P^t &= \frac{AE - a(2Mr - Q^2)L}{\Delta} = \frac{\left[(r^2 + a^2)^2 - a^2\sin^2\theta\Delta\right]E}{\Delta} - \frac{a(2Mr - Q^2)L}{\Delta} \\
\Sigma P^t &= \frac{(r^2 + a^2)^2 E}{\Delta} - a^2\sin^2\theta E - \frac{a(r^2 + a^2 - \Delta)L}{\Delta} \\
\Sigma P^t &= \frac{(r^2 + a^2)^2 E}{\Delta} - a^2\sin^2\theta E - \frac{a(r^2 + a^2)L}{\Delta} + aL \\
\Sigma P^t &= -a\left(a\sin^2\theta E - L\right) + \left(r^2 + a^2\right)\frac{\left[(r^2 + a^2)E - aL\right]}{\Delta} \\
\Sigma P^t &= -a\left(a\sin^2\theta E - L\right) + \left(r^2 + a^2\right)\frac{T}{\Delta} \quad ,
\end{aligned}$$

where

$$T = \left(r^2 + a^2\right)E - aL \quad .$$

Equation (3.11b) yields

$$\begin{aligned}
P^{\phi} &= g^{\phi\mu}P_{\mu} = g^{\phi t}P_t + g^{\phi\phi}P_{\phi} \\
&= -a\frac{(2Mr-Q^2)}{\Delta\Sigma}(-E) + \frac{\Delta - a^2\sin^2\theta}{\Delta\Sigma\sin^2\theta}L \\
\Sigma P^{\phi} &= \frac{a\left(r^2+a^2-\Delta\right)E}{\Delta} + \frac{L}{\sin^2\theta} - \frac{a^2}{\Delta}L \\
&= -aE + \frac{L}{\sin^2\theta} + a\frac{\left(r^2+a^2\right)E - aL}{\Delta} \\
\Sigma P^{\phi} &= -\left(aE - \frac{L}{\sin^2\theta}\right) + a\frac{T}{\Delta} \quad .
\end{aligned}$$

Equation (3.11c) yields

$$\begin{aligned}
P_{\theta}^2 &= C - \cos^2\theta\left[a^2\left(\mu^2-E^2\right) + \frac{L^2}{\sin^2\theta}\right] \quad , \\
P^{\theta} &= g^{\theta\mu}P_{\mu} = g^{\theta\theta}P_{\theta} = \frac{1}{\Sigma}P_{\theta} \quad , \\
\Sigma P^{\theta} &= \left[C - \cos^2\theta\left[a^2\left(\mu^2-E^2\right) + \frac{L^2}{\sin^2\theta}\right]\right]^{1/2} \quad .
\end{aligned}$$

The normalization of the particle's momentum lead to

$$\begin{aligned}
P^{\nu}P_{\nu} &= P^tP_t + P^rP_r + P^{\theta}P_{\theta} + P^{\phi}P_{\phi} = -\mu^2 \\
P^rP_r &= g_{rr}\left(P^r\right)^2 = -\mu^2 - P^tP_t - g^{\theta\theta}\left(P_{\theta}\right)^2 - P^{\phi}P_{\phi} \\
\frac{\Sigma}{\Delta}\left(P^r\right)^2 &= -\mu^2 - P^t(-E) - g^{\theta\theta}\left(P_{\theta}\right)^2 - P^{\phi}L
\end{aligned}$$

Multiplying by Σ, then using the obtained expressions of P^t, P^{θ} and P^{ϕ}, we find

$$\begin{aligned}
\frac{\left(\Sigma P^r\right)^2}{\Delta} &= -\Sigma\mu^2 + \Sigma P^tE - \Sigma\frac{1}{\Sigma}\left(P_{\theta}\right)^2 - \Sigma P^{\phi}L \\
&= -\left(r^2+a^2\cos^2\theta\right)\mu^2 - a\left(a\sin^2\theta E - L\right)E + \left(r^2+a^2\right)\frac{T}{\Delta}E \\
&\quad -C + \cos^2\theta\left[a^2\left(\mu^2-E^2\right) + \frac{L^2}{\sin^2\theta}\right] + \left(aE - \frac{L}{\sin^2\theta}\right)L - a\frac{T}{\Delta}L \\
&= -r^2\mu^2 - a^2\mu^2\cos^2\theta - a^2E^2\sin^2\theta + aLE + \left(r^2+a^2\right)\frac{T}{\Delta}E \\
&\quad -C + a^2\mu^2\cos^2\theta - a^2E^2\cos^2\theta + \cos^2\theta\frac{L^2}{\sin^2\theta} + aEL - \frac{L^2}{\sin^2\theta} - a\frac{T}{\Delta}L \\
&= -r^2\mu^2 - a^2E^2\left(\sin^2\theta + \cos^2\theta\right) + 2aLE + \left[\left(r^2+a^2\right)E - aL\right]\frac{T}{\Delta} \\
&\quad -C + \left(\cos^2\theta - 1\right)\frac{L^2}{\sin^2\theta} \\
&= -r^2\mu^2 - a^2E^2 + 2aLE + \frac{T^2}{\Delta} - C - L^2 \\
&= -r^2\mu^2 - (aE-L)^2 + \frac{T^2}{\Delta} - C \\
\left(\Sigma P^r\right)^2 &= T^2 - \Delta\left(\mu^2r^2 + (aE-L)^2 + C\right) \\
\Sigma P^r &= \left[T^2 - \Delta\left(\mu^2r^2 + (aE-L)^2 + C\right)\right]^{\frac{1}{2}} \quad .
\end{aligned}$$

Hence, the four equations of motion of a neutral test particle in the Kerr-Newman Black hole are

$$
\begin{aligned}
\Sigma P^t &= -a\left(a\sin^2\theta E - L\right) + \left(r^2+a^2\right)\frac{T}{\Delta} \quad , && (3.12)\\
\Sigma P^r &= \left[T^2 - \Delta\left(\mu^2 r^2 + (aE-L)^2 + C\right)\right]^{\frac{1}{2}} = V(r)\,, && (3.13)\\
\Sigma P^\theta &= \left[C - \cos^2\theta\left(a^2\left(\mu^2 - E^2\right) + \frac{L^2}{\sin^2\theta}\right)\right]^{1/2} = \Theta(\theta), && (3.14)\\
\Sigma P^\phi &= -\left(aE - \frac{L}{\sin^2\theta}\right) + a\frac{T}{\Delta} = \Phi(r,\theta). && (3.15)
\end{aligned}
$$

3.1.3 Circular equatorial motion of neutral test particles

For the a motion initially in the equatorial plane ($\theta = \pi/2$) and remain in that plane, equation (3.14) lead to $C = 0$. For circular orbits in the equatorial plane at some radius r, the radial velocity $dr/d\tau = P^r/\mu$ must vanish instantaneously and at subsequent times. Hence, equation (3.13) gives the following conditions

$$V(r) = 0 \quad \text{and} \quad \dot{V}(r) = 0 \qquad \text{(for circular orbits)}, \tag{3.16}$$

where the dot denotes derivative with respect to the radial coordinate. the $V(r) = 0$ condition can be written as

$$
\begin{aligned}
0 &= T^2 - \Delta\left(\mu^2 r^2 + (aE-L)^2\right)\\
0 &= \left[\left(r^2+a^2\right)E - aL\right]^2 - \left(r^2+a^2-2Mr+Q^2\right)\left[\mu^2 r^2 + (aE-L)^2\right]\\
0 &= \left(r^2+a^2\right)^2 E^2 + a^2L^2 - 2\left(r^2+a^2\right)EaL - \mu^2 r^4 - a^2E^2r^2 - L^2r^2 + 2aELr^2\\
&\quad -a^2\mu^2r^2 - a^4E^2 - a^2L^2 + 2a^3EL + 2M\mu^2r^3 + 2Ma^2E^2r + 2ML^2r - 4MaELr\\
&\quad -\mu^2Q^2r^2 - a^2E^2Q^2 - L^2Q^2 + 2aELQ^2\\
0 &= E^2\left[\left(r^2+a^2\right)^2 - a^2r^2 - a^4 + 2Ma^2r - a^2Q^2\right] + L^2\left[a^2 - r^2 - a^2 + 2Mr - Q^2\right]\\
&\quad +2EL\left[-\left(r^2+a^2\right)a + ar^2 + a^3 - 2Mar + aQ^2\right] - \mu^2\left[r^4 + a^2r^2 - 2Mr^3 + Q^2r^2\right]\\
0 &= E^2\left[r^4 + a^2r^2 + 2Ma^2r - a^2Q^2\right] + L^2\left[-r^2 + 2Mr - Q^2\right] + 2EL\left[-2Mar + aQ^2\right]\\
&\quad -\mu^2r^2\left[r^2 + a^2 - 2Mr + Q^2\right] \quad .
\end{aligned}
$$

On the other hand, the $\dot{V}(r) = 0$ condition can be written as

$$
\begin{aligned}
0 &= E^2\left[4r^3 + 2a^2r + 2Ma^2\right] + L^2\left[-2r + 2M\right] + 2EL\left[-2Ma\right]\\
&\quad -\mu^2\left[4r^3 + 2a^2r - 6Mr^2 + 2Q^2r\right] \quad .
\end{aligned}
$$

Solving the above couple of equations for E and L yields [56]

$$
\begin{aligned}
E/\mu &= \frac{r^2 - 2Mr + Q^2 \pm a\left(Mr - Q^2\right)^{\frac{1}{2}}}{r\left[r^2 - 3Mr + 2Q^2 \pm 2a\left(Mr - Q^2\right)^{\frac{1}{2}}\right]^{1/2}} \quad , && (3.17)\\
L/\mu &= \pm\frac{\left(Mr - Q^2\right)^{\frac{1}{2}}\left[r^2 + a^2 \mp 2a\left(Mr - Q^2\right)^{\frac{1}{2}}\right] \mp aQ^2}{r\left[r^2 - 3Mr + 2Q^2 \pm 2a\left(Mr - Q^2\right)^{\frac{1}{2}}\right]^{1/2}} \quad . && (3.18)
\end{aligned}
$$

In these formulae, the (+) correspond to a co-rotating test particle with respect to the black hole rotation, while the (-) correspond to a counter rotating one.

Notice that if we substitute $Q = 0$ in the last couple of equation reduce to the energy and angular momentum expressions for circular orbits in Kerr black hole [47].

3.2 Equations of motion for photons

The equations of motion for photons in the Kerr-Newman metric can be obtained in a similar way to those of massive particles. The only difference is the normalization equation $k^\nu k_\nu = 0$, where $k^\nu$is the momentum four-vector of the photon. Hence, the constants of motion for photons are

$$\begin{aligned} E_\gamma &= -g_{\mu\nu}\xi^\mu k^\nu = -k_t = \textit{total energy}, && (3.19a)\\ L_\gamma &= g_{\mu\nu}\psi^\mu k^\nu = k_\phi = \textit{angular momentum}, && (3.19b)\\ C &= k_\theta^2 + \cos^2\theta\left[-a^2E_\gamma^2 + \frac{L_\gamma^2}{\sin^2\theta}\right] \quad , && (3.19c)\\ k^\nu k_\nu &= k^t k_t + k^r k_r + k^\theta k_\theta + k^\phi k_\phi = 0 \quad . && (3.19d) \end{aligned}$$

These equations can be written in the form

$$\begin{aligned} \Sigma k^t &= -a\left(aE_\gamma \sin^2\theta - L_\gamma\right) + \left(r^2 + a^2\right)\frac{T_\gamma}{\Delta} \quad , && (3.20a)\\ \Sigma k^r &= \left[T_\gamma^2 - \Delta\left((E_\gamma a - L_\gamma)^2 + C\right)\right]^{1/2} \quad , && (3.20b)\\ \Sigma k^\theta &= \left[C - \cos^2\theta\left(-a^2E_\gamma^2 + \frac{L_\gamma^2}{\sin^2\theta}\right)\right]^{1/2} \quad , && (3.20c)\\ \Sigma k^\phi &= -\left(aE_\gamma - \frac{L_\gamma}{\sin^2\theta}\right) + a\frac{T_\gamma}{\Delta} \quad , && (3.20d) \end{aligned}$$

where $T_\gamma = E_\gamma\left(r^2 + a^2\right) - E_\gamma a$.

3.3 The red/blue shifts of photons

3.3.1 General Expression of the red/blue shifts of photons

In general, the frequency of a photon of momentum 4-vector k^ν, measured by an observer with 4-velocity U^ν at point P is

$$\omega = -k_\mu U^\mu|_P \quad . \tag{3.21}$$

Thus, the frequency of photons measured by an observer at the emission point (e) and absorption point respectively are

$$\begin{aligned} \omega_e &= -k_\mu U^\mu|_e \quad ,\\ \omega_d &= -k_\mu U^\mu|_d \quad . \end{aligned}$$

Hence, the expression of the frequency shift subjected by between the emission and detection point is

$$1+z = \frac{\omega_e}{\omega_d} = \frac{-k_\mu U^\mu|_e}{-k_\mu U^\mu|_d} = \frac{-\left(k_t U^t + k_r U^r + k_\theta U^\theta + k_\phi U^\phi\right)|_e}{-\left(k^t U^t + k_r U^r + k_\theta U^\theta + k^\phi U^\phi\right)|_d} \tag{3.22}$$

$$= \frac{\left(E\gamma U^t - k_r U^r - k_\theta U^\theta - L\gamma U^\phi\right)|_e}{\left(E\gamma U^t - k_r U^r - k_\theta U^\theta - L\gamma U^\phi\right)|_d} \tag{3.23}$$

$$= \frac{\left(E k^t - U_r k^r - U_\theta k^\theta - L k^\phi\right)|_e}{\left(E k^t - U_r k^r - U_\theta k^\theta - L k^\phi\right)|_d} \quad , \tag{3.24}$$

where we have used equations (3.19a,3.19b) for the constants $E\gamma$ and $L\gamma$ in the second line, and equations (3.11a,3.11b) for the constants E and L in the second line.

3.4 Kinematical red/blue shift of photons emitted by particles in equatorial circular motion

From now on, we shall restrict ourselves to the case circular equatorial neutral test particles

- Circular orbit: $U^r = 0$ and $\dot{U}^r = 0$
- Equatorial plan: $\theta = \pi/2, \quad U^\theta = 0, \;\; Q = 0$

In this case, the previous expression of frequency shift, equation (3.22), takes the form

$$1+z = \frac{\omega_e}{\omega_d} = \frac{\left(E\gamma U^t - L\gamma U^\phi\right)|_e}{\left(E\gamma U^t - L\gamma U^\phi\right)|_d} = \frac{U_e^t - b_e U_e^\phi}{U_d^t - b_d U_d^\phi} \;, \tag{3.25}$$

where we have introduced the impact parameter $b \equiv L\gamma/E\gamma$. Since the quantities $L\gamma$ and $E\gamma$ are conserved along the photons trajectory, which is a geodesic, the impact parameter b is the same in the points of emission and detection respectively, i.e. we have $b_e = b_d = b$.

Now, we shall consider the kinematical shift z_{kin} of photons defined as follows [47]

$$z_{kin} \equiv z - z_c \quad , \tag{3.26}$$

where z is defined by equation (3.25) while z_c is the frequency shift of photons emitted by a static particle located at $b = 0$. Substituting $b = 0$ in equation (3.25), we obtain

$$1 + z_c = \frac{U_e^t}{U_d^t} \quad . \tag{3.27}$$

Thus, the expression of z_{kin} is

$$z_{kin} = z - z_c = \frac{U_e^t - bU_e^\phi}{U_d^t - bU_d^\phi} - \frac{U_e^t}{U_d^t} = \frac{U_d^t U_e^t - bU_d^t U_e^\phi - U_d^t U_e^t + bU_d^\phi U_e^t}{U_d^t\left(U_d^t - bU_d^\phi\right)}$$

$$\rightarrow \quad z_{kin} = \frac{U_e^t U_d^\phi b - U_d^t U_e^\phi b}{U_d^t\left(U_d^t - bU_d^\phi\right)} \quad . \tag{3.28}$$

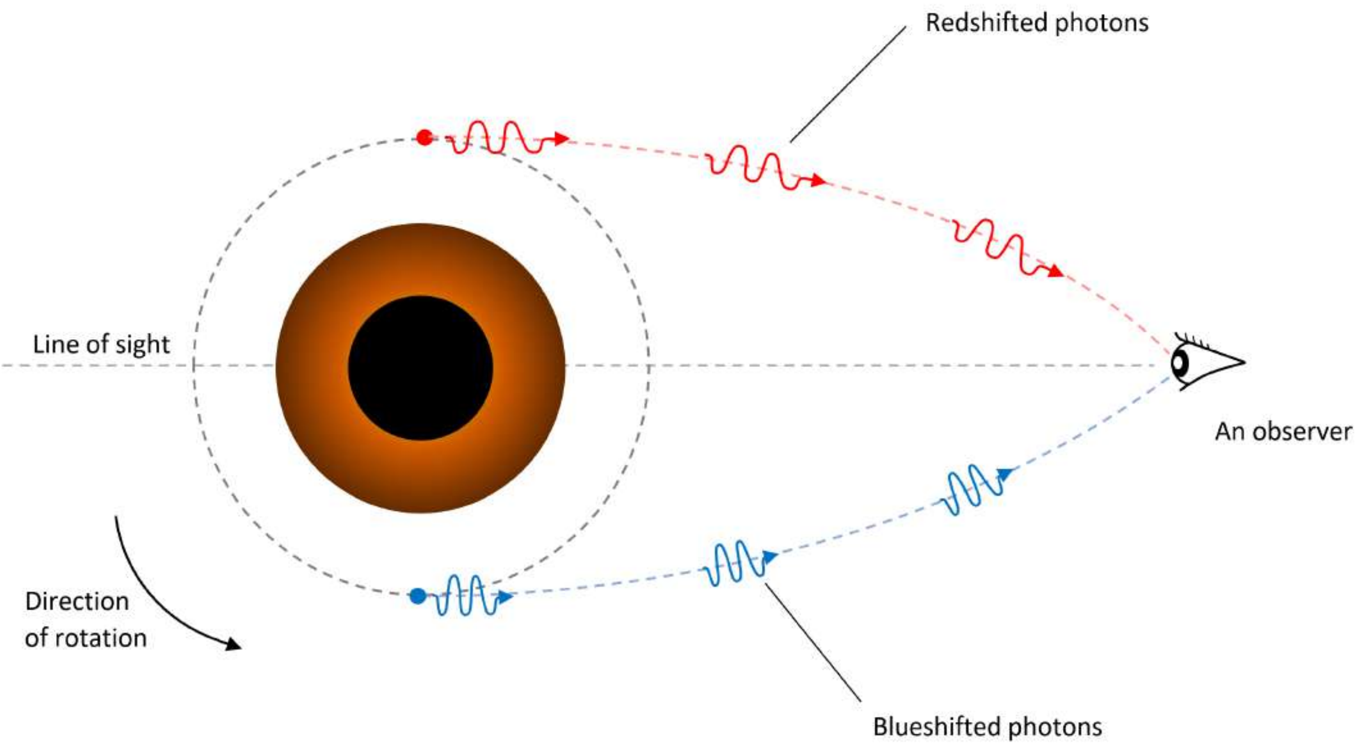


Fig. 3.1 – An axial view of photons emitted by particles in circular motion around the Kerr-Newman black hole. We considered only particles on the sides of the black hole. For photons, only those emitted with $k^r = k^\theta = 0$ are presented.

In the following, we will follow the author of [47] when they chose to maximize z. This means b must be maximized too. This can be done if we substitute $k^r = k^\theta = 0$ in equation (3.19d). On one hand we have

$$\begin{aligned} k^\nu k_\nu &= k^t k_t + k^r k_r = -k^t E_\gamma + k^\phi L_\gamma = 0 \\ &\rightarrow \quad b = \frac{L_\gamma}{E_\gamma} = \frac{k^t}{k^\phi} \quad . \end{aligned} \tag{3.29}$$

On the other hand, we can write

$$k^\nu k_\nu = g_{tt}\left(k^t\right)^2 + 2g_{t\phi}k^t k^\phi + g_{\phi\phi}\left(k^\phi\right)^2 = 0 \quad .$$

Dividing by $\left(k^\phi\right)^2$ and taking into account equation (3.29), we get the equation relating the impact parameter b with the metric components

$$g_{tt}\left(b\right)^2 + 2g_{t\phi}b + g_{\phi\phi} = 0 \quad ,$$

for which, the solutions are

$$\rightarrow b_\pm = -\frac{g_{t\phi} \pm \sqrt{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}}{g_{tt}} \quad . \tag{3.30}$$

These two different values of b will give rise to two different values of the frequency shift, z_1 and z_2, corresponding respectively to a receding and an approaching object with respect to a far away positioned observer

$$z_1 = \frac{U_e^t U_d^\phi b_- - U_d^t U_e^\phi b_-}{U_d^t \left(U_d^t - b_- U_d^\phi\right)} \quad , \tag{3.31a}$$

$$z_2 = \frac{U_e^t U_d^\phi b_+ - U_d^t U_e^\phi b_+}{U_d^t \left(U_d^t - b_+ U_d^\phi\right)} \quad , \tag{3.31b}$$

These photons are illustrated in Fig. 3.1, alongside with the emitting particles. Redshifted photons are represented in red, while blueshifted photons are represented in blue in the colored version of this dissertation. For an observer located far away from the photons source, the angular velocity is defined by

$$\Omega_d \equiv \frac{d\phi}{dt} = \frac{d\phi/d\tau}{dt/d\tau} = \frac{U_d^\phi}{U_d^t} \quad , \tag{3.32}$$

where τ is the detector's proper time. Using this expression to substitute U_d^ϕ in the frequency shift, equations (3.31), we get

$$z_1 = \frac{\left(U_e^t \Omega_d - U_e^\phi\right) b_-}{U_d^t \left(1 - \Omega_d b_-\right)} \quad , \tag{3.33}$$

$$z_2 = \frac{\left(U_e^t \Omega_d - U_e^\phi\right) b_+}{U_d^t \left(1 - \Omega_d b_+\right)} \quad . \tag{3.34}$$

3.5 Expressions of red/blue shifts in terms of the Boyer-Lindquist coordinates

Now, in order to get the expressions of the frequency shift in terms of the Boyer-Lindquist coordinates, we need to start write the following quantities first:

$$U_d^t, \quad U_d^\phi, \quad \Omega_d \text{ and } b_\pm$$

3.5.1 Expression of U_d^t and U_d^ϕ in Boyer-Lindquist coordinates

For $\theta = \pi/2$, equations (3.8) and (3.4) respectively gives

$$\begin{aligned} A &\equiv \left(r^2 + a^2\right)^2 - a^2 \left(r^2 + a^2 - 2Mr + Q^2\right) \\ &= r^4 + a^2 r^2 + a^2 \left(2M - Q^2\right) \ , \\ \Delta\Sigma &= r^2 \left(r^2 + a^2 - 2Mr + Q^2\right) \quad . \end{aligned}$$

Using these results, the t component of the 4-velocity reads

$$\begin{aligned} U^t\left(r, \pi/2\right) &= g^{t\nu} U_\nu = g^{t\nu}\left(P_\nu/\mu\right) = g^{tt}\left(P_t/\mu\right) + g^{t\phi}\left(P_\phi/\mu\right) \\ &= -\frac{A}{\Delta\Sigma}\left(-E/\mu\right) - a\left(\frac{2Mr - Q^2}{\Delta\Sigma}\right)\left(L/\mu\right) \\ &= \frac{\left[r^4 + a^2 r^2 + a^2\left(2Mr - Q^2\right)\right]}{r^2\left(r^2 + a^2 - 2Mr + Q^2\right)}\left(E/\mu\right) - \frac{a\left(2Mr - Q^2\right)}{r^2\left(r^2 + a^2 - 2Mr + Q^2\right)}\left(L/\mu\right) \quad , \end{aligned}$$

whereas the ϕ component of the 4-velocity reads

$$\begin{aligned}
U^{\phi}(r,\pi/2) &= g^{\phi\nu}U_{\nu} = g^{\phi\nu}(P_{\nu}/\mu) = g^{\phi t}(P_t/\mu) + g^{\phi\phi}(P_{\phi}/\mu) \\
&= -a\left(\frac{2Mr-Q^2}{\Delta\Sigma}\right)(-E/\mu) + \frac{\Delta - a^2}{\Delta\Sigma}(L/\mu) \\
&= \frac{a(2Mr-Q^2)}{r^2(r^2+a^2-2Mr+Q^2)}(E/\mu) + \frac{r^2-2Mr+Q^2}{r^2(r^2+a^2-2Mr+Q^2)}(L/\mu) \quad .
\end{aligned}$$

Now, using equations (3.17) and (3.18), U^t and U^{ϕ} become

$$\begin{aligned}
U^t(r,\pi/2) &= \frac{[r^4+a^2r^2+a^2(2Mr-Q^2)]}{r^2(r^2+a^2-2Mr+Q^2)}(E/\mu) - \frac{a(2Mr-Q^2)}{r^2(r^2+a^2-2Mr+Q^2)}(L/\mu) \\
&= \frac{[r^4+a^2r^2+a^2(2Mr-Q^2)]}{r^2(r^2+a^2-2Mr+Q^2)}\frac{r^2-2Mr+Q^2 \pm a(Mr-Q^2)^{\frac{1}{2}}}{r\left[r^2-3Mr+2Q^2 \pm 2a(Mr-Q^2)^{\frac{1}{2}}\right]^{1/2}} \\
&\quad -\frac{a(2Mr-Q^2)}{r^2(r^2+a^2-2Mr+Q^2)}\frac{\pm(Mr-Q^2)^{\frac{1}{2}}\left[r^2+a^2 \mp 2a(Mr-Q^2)^{\frac{1}{2}}\right] - aQ^2}{r\left[r^2-3Mr+2Q^2 \pm 2a(Mr-Q^2)^{\frac{1}{2}}\right]^{1/2}} \quad .
\end{aligned}$$

The numerator of this expression is

$$\begin{aligned}
&\left[r^4+a^2r^2+2a^2Mr-a^2Q^2\right]\left[r^2-2Mr+Q^2 \pm a\left(Mr-Q^2\right)^{\frac{1}{2}}\right] \\
&+\left(-2aMr+aQ^2\right)\left[\pm\left(Mr-Q^2\right)^{\frac{1}{2}}\left(r^2+a^2 \mp 2a\left(Mr-Q^2\right)^{\frac{1}{2}}\right) - aQ^2\right] \\
=\;& r^6-2Mr^5+Q^2r^4 \pm ar^4\left(Mr-Q^2\right)^{\frac{1}{2}} + a^2r^4 - 2a^2Mr^3 + a^2Q^2r^2 \pm a^3r^2\left(Mr-Q^2\right)^{\frac{1}{2}} + 2a^2Mr^3 \\
&-4a^2M^2r^2 + 2a^2MQ^2r \pm 2a^3Mr\left(Mr-Q^2\right)^{\frac{1}{2}} - a^2Q^2r^2 + 2a^2Q^2Mr - a^2Q^4 \mp a^3Q^2\left(Mr-Q^2\right)^{\frac{1}{2}} \\
&\pm\left(Mr-Q^2\right)^{\frac{1}{2}}\left[-2aMr^3 - 2a^3Mr \pm 4a^2Mr\left(Mr-Q^2\right)^{\frac{1}{2}} + aQ^2r^2 + a^3Q^2 \mp 2a^2Q^2\left(Mr-Q^2\right)^{\frac{1}{2}}\right] \\
&+2a^2MQ^2r - a^2Q^4 \\
=\;& r^6-2Mr^5+r^4\left(Q^2+a^2\right) - 4a^2M^2r^2 + 6a^2MQ^2r - 2a^2Q^4 \\
&\pm\left(Mr-Q^2\right)^{\frac{1}{2}}\left[ar^4+a^3r^2+2a^3Mr-a^3Q^2-2aMr^3-2a^3Mr+aQ^2r^2+a^3Q^2\right] \\
&+\left(Mr-Q^2\right)\left[4a^2Mr-2a^2Q^2\right] \\
=\;& r^4\left[r^2-2Mr+Q^2+a^2\right] - 4a^2M^2r^2 + 6a^2MQ^2r - 2a^2Q^4 + 4a^2M^2r^2 - 2a^2MQ^2r - 4a^2MQ^2r \\
&+2a^2Q^4 \pm \left(Mr-Q^2\right)^{\frac{1}{2}}\left[ar^4+a^3r^2-2aMr^3+aQ^2r^2\right] \\
=\;& r^4\left[r^2-2Mr+Q^2+a^2\right] \pm ar^2\left(Mr-Q^2\right)^{\frac{1}{2}}\left[r^2+a^2-2M+Q^2\right] \\
=\;& \left[r^4 \pm ar^2\left(Mr-Q^2\right)^{\frac{1}{2}}\right]\left[r^2+a^2-2M+Q^2\right] \quad .
\end{aligned}$$

Inserting the numerator in the formula of U^t, we find

$$\begin{aligned}
U^t(r,\pi/2) &= \frac{\left[r^4 \pm ar^2(Mr-Q^2)^{\frac{1}{2}}\right][r^2+a^2-2M+Q^2]}{r^3(r^2+a^2-2Mr+Q^2)\left[r^2-3Mr+2Q^2 \pm 2a(Mr-Q^2)^{\frac{1}{2}}\right]^{1/2}} \\
&\to\; U^t(r,\pi/2) = \frac{r^2 \pm a(Mr-Q^2)^{\frac{1}{2}}}{r\left[r^2-3Mr+2Q^2 \pm 2a(Mr-Q^2)^{\frac{1}{2}}\right]^{1/2}} \quad . \qquad (3.35)
\end{aligned}$$

Following the same steps with U^{ϕ} leads to

$$\begin{aligned} U^{\phi}(r,\pi/2) &= \frac{a\left(2Mr-Q^2\right)}{r^2\left(r^2+a^2-2Mr+Q^2\right)}\left(E/\mu\right)+\frac{r^2-2Mr+Q^2}{r^2\left(r^2+a^2-2Mr+Q^2\right)}\left(L/\mu\right) \\ &= \frac{a\left(2Mr-Q^2\right)}{r^2\left(r^2+a^2-2Mr+Q^2\right)}\frac{r^2-2Mr+Q^2\pm a\left(Mr-Q^2\right)^{\frac{1}{2}}}{r\left[r^2-3Mr+2Q^2\pm 2a\left(Mr-Q^2\right)^{\frac{1}{2}}\right]^{1/2}} \\ &+\frac{r^2-2Mr+Q^2}{r^2\left(r^2+a^2-2Mr+Q^2\right)}\frac{\pm\left(Mr-Q^2\right)^{\frac{1}{2}}\left[r^2+a^2\mp 2a\left(Mr-Q^2\right)^{\frac{1}{2}}\right]-aQ^2}{r\left[r^2-3Mr+2Q^2\pm 2a\left(Mr-Q^2\right)^{\frac{1}{2}}\right]^{1/2}} \quad . \end{aligned}$$

The numerator of this expression is

$$\begin{aligned} & \left(2aMr-aQ^2\right)\left[r^2-2Mr+Q^2\pm a\left(Mr-Q^2\right)^{\frac{1}{2}}\right] \\ & +\left(r^2-2Mr+Q^2\right)\left[\pm\left(Mr-Q^2\right)^{\frac{1}{2}}\left(r^2+a^2\mp 2a\left(Mr-Q^2\right)^{\frac{1}{2}}\right)-aQ^2\right] \\ = \; & 2aMr^3-4aM^2r^2+2aMrQ^2\pm 2a^2Mr\left(Mr-Q^2\right)^{\frac{1}{2}}-aQ^2r^2+2aQ^2Mr-aQ^4 \\ & \mp a^2Q^2\left(Mr-Q^2\right)^{\frac{1}{2}}\pm\left(Mr-Q^2\right)^{\frac{1}{2}}\left[r^4+a^2r^2-2Mr^3-2a^2Mr++Q^2r^2+Q^2a^2\right] \\ & -\left(r^2-2Mr+Q^2\right)2a\left(Mr-Q^2\right)-aQ^2\left(r^2-2Mr+Q^2\right) \\ = \; & 2aMr^3-4aM^2r^2+2aMrQ^2-aQ^2r^2+2aQ^2Mr-aQ^4-2aMr^3+4aM^2r^2 \\ & -2aMrQ^2+2aQ^2r^2-4aQ^2Mr+2aQ^4-aQ^2r^2+2aQ^2Mr-aQ^4 \\ & \pm\left(Mr-Q^2\right)^{\frac{1}{2}}\left[2a^2Mr-a^2Q^2+r^4+a^2r^2-2Mr^3-2a^2Mr++Q^2r^2+Q^2a^2\right] \\ = \; & \pm\left(Mr-Q^2\right)^{\frac{1}{2}}r^2\left[r^2+a^2-2Mr+Q^2\right] \quad . \end{aligned}$$

Inserting the numerator in the formula of U^t, we find

$$U^{\phi}(r,\pi/2) = \frac{\pm\left(Mr-Q^2\right)^{\frac{1}{2}}r^2\left[r^2+a^2-2Mr+Q^2\right]}{r^3\left(r^2+a^2-2Mr+Q^2\right)\left[r^2-3Mr+2Q^2\pm 2a\left(Mr-Q^2\right)^{\frac{1}{2}}\right]^{1/2}}$$

$$\rightarrow \quad U^{\phi}(r,\pi/2) = \frac{\pm\left(Mr-Q^2\right)^{\frac{1}{2}}}{r\left[r^2-3Mr+2Q^2\pm 2a\left(Mr-Q^2\right)^{\frac{1}{2}}\right]^{1/2}} \tag{3.36}$$

3.5.2 Expressions of Ω and $b_{\pm}$ in Boyer-Lindquist coordinates

With the formulae of U^t and U^{ϕ} at hand, it's straightforward to write the angular velocity of an observer moving in a circular orbit in the equatorial plane, equation (3.32), in terms of the Boyer-Lindquist coordinates

$$\Omega = \frac{U^{\phi}}{U^t} = \frac{\pm\left(Mr-Q^2\right)^{\frac{1}{2}}}{r^2\pm a\left(Mr-Q^2\right)^{\frac{1}{2}}} \tag{3.37}$$

where again, the (+) and (-) signs respectively correspond to co-rotating and counter-rotating objects with respect to the black hole.

In addition to that, we can express the impact parameter, equation (3.30), in terms of the Boyer-Lindquist coordinates (with $\theta = \pi/2$)

$$\begin{aligned} g_{t\phi} &= -a\sin^2\theta\left(\frac{2Mr-Q^2}{r^2}\right) = -a\left(\frac{2Mr-Q^2}{r^2}\right) \quad , \\ g_{t\phi}^2 - g_{tt}g_{\phi\phi} &= \sin^2\theta\Delta = r^2 + a^2 - 2Mr + Q^2 \quad , \\ g_{tt} &= -\left(1 - \frac{2Mr-Q^2}{r^2+a^2\cos^2\theta}\right) = -\frac{r^2-2Mr+Q^2}{r^2} \end{aligned}$$

$$\begin{aligned} b_\pm &= -\frac{g_{t\phi} \pm \sqrt{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}}{g_{tt}} = -\frac{-a\left(\frac{2Mr-Q^2}{r^2}\right) \pm \sqrt{r^2+a^2-2Mr+Q^2}}{-\frac{r^2-2Mr+Q^2}{r^2}} \\ &\to \; b_\pm = \frac{-a\left(2Mr-Q^2\right) \pm r^2\sqrt{r^2+a^2-2Mr+Q^2}}{r^2-2Mr+Q^2} \quad . \end{aligned} \tag{3.38}$$

3.5.3 Expressions of z_1 and z_2

Having done all this, we can now write the kinematic frequency shift in terms of the Boyer-Lindquist coordinates. Inserting equations (3.35) and (3.36) in (3.33) yields

$$\begin{aligned} z_1 &= \frac{\left(U_e^t\Omega_d - U_e^\phi\right)b_-}{U_d^t\left(1-\Omega_d b_-\right)} \\ &= \frac{\left(\left[r_e^2 \pm a\left(Mr_e-Q^2\right)^{\frac{1}{2}}\right]\Omega_d - \pm\left(Mr_e-Q^2\right)^{\frac{1}{2}}\right)b_-}{r_e\left[r_e^2 - 3Mr_e + 2Q^2 \pm 2a\left(Mr_e-Q^2\right)^{\frac{1}{2}}\right]^{1/2}} \frac{r_d\left[r_d^2 - 3Mr_d + 2Q^2 \pm 2a\left(Mr_d-Q^2\right)^{\frac{1}{2}}\right]^{1/2}}{\left[r_d^2 \pm a\left(Mr_d-Q^2\right)^{\frac{1}{2}}\right]\left(1-\Omega_d b_-\right)} \\ &= \frac{r_d\left[r_d^2 - 3Mr_d + 2Q^2 \pm 2a\left(Mr_d-Q^2\right)^{\frac{1}{2}}\right]^{1/2}}{r_e\left[r_e^2 - 3Mr_e + 2Q^2 \pm 2a\left(Mr_e-Q^2\right)^{\frac{1}{2}}\right]^{1/2}} \frac{\left(\left[r_e^2 \pm a\left(Mr_e-Q^2\right)^{\frac{1}{2}}\right]\Omega_d - \pm\left(Mr_e-Q^2\right)^{\frac{1}{2}}\right)b_-}{\left[r_d^2 \pm a\left(Mr_d-Q^2\right)^{\frac{1}{2}}\right]\left(1-\Omega_d b_-\right)} \end{aligned}$$

From (3.37), we have

$$\pm\left(Mr_e-Q^2\right)^{\frac{1}{2}} = \left[r_e^2 \pm a\left(Mr_e-Q^2\right)^{\frac{1}{2}}\right]\Omega_e \; .$$

Inserting this in the previous expression, we find

$$\begin{aligned} z_1 &= \frac{r_d\left[r_d^2 - 3Mr_d + 2Q^2 \pm 2a\left(Mr_d-Q^2\right)^{\frac{1}{2}}\right]^{1/2}}{r_e\left[r_e^2 - 3Mr_e + 2Q^2 \pm 2a\left(Mr_e-Q^2\right)^{\frac{1}{2}}\right]^{1/2}} \times \\ &\quad \frac{\left(\left[r_e^2 \pm a\left(Mr_e-Q^2\right)^{\frac{1}{2}}\right]\Omega_d - \left[r_e^2 \pm a\left(Mr_e-Q^2\right)^{\frac{1}{2}}\right]\Omega_e\right)b_-}{\left[r_d^2 \pm a\left(Mr_d-Q^2\right)^{\frac{1}{2}}\right]\left(1-\Omega_d b_-\right)} \\ &= \frac{r_d\left[r_d^2 - 3Mr_d + 2Q^2 \pm 2a\left(Mr_d-Q^2\right)^{\frac{1}{2}}\right]^{1/2}}{r_e\left[r_e^2 - 3Mr_e + 2Q^2 \pm 2a\left(Mr_e-Q^2\right)^{\frac{1}{2}}\right]^{1/2}} \times \frac{\left[r_e^2 \pm a\left(Mr_e-Q^2\right)^{\frac{1}{2}}\right]\left(\Omega_d-\Omega_e\right)b_-}{\left[r_d^2 \pm a\left(Mr_d-Q^2\right)^{\frac{1}{2}}\right]\left(1-\Omega_d b_-\right)} \end{aligned}$$

Using (3.37) again, yields

$$\begin{aligned} z_1 &= \frac{r_d\left[r_d^2-3Mr_d+2Q^2\pm 2a\left(Mr_d-Q^2\right)^{\frac{1}{2}}\right]^{1/2}}{r_e\left[r_e^2-3Mr_e+2Q^2\pm 2a\left(Mr_e-Q^2\right)^{\frac{1}{2}}\right]^{1/2}} \times \frac{\pm\left(Mr_e-Q^2\right)^{\frac{1}{2}}/\Omega_e}{\pm\left(Mr_d-Q^2\right)^{\frac{1}{2}}/\Omega_d} \times \frac{\left(\Omega_d-\Omega_e\right)b_-}{\left(1-\Omega_d b_-\right)} \\ &= \frac{r_d\left[r_d^2-3Mr_d+2Q^2\pm 2a\left(Mr_d-Q^2\right)^{\frac{1}{2}}\right]^{1/2}}{r_e\left[r_e^2-3Mr_e+2Q^2\pm 2a\left(Mr_e-Q^2\right)^{\frac{1}{2}}\right]^{1/2}} \times \frac{\Omega_d\left(\Omega_d-\Omega_e\right)b_-}{\Omega_e\left(1-\Omega_d b_-\right)} \times \left[\frac{Mr_e-Q^2}{Mr_d-Q^2}\right]^{1/2} \end{aligned} \tag{3.39}$$

Following the same steps with z_2, gives

$$z_2 = \frac{r_d\left[r_d^2-3Mr_d+2Q^2\pm 2a\left(Mr_d-Q^2\right)^{\frac{1}{2}}\right]^{1/2}}{r_e\left[r_e^2-3Mr_e+2Q^2\pm 2a\left(Mr_e-Q^2\right)^{\frac{1}{2}}\right]^{1/2}} \times \frac{\Omega_d\left(\Omega_d-\Omega_e\right)b_+}{\Omega_e\left(1-\Omega_d b_+\right)} \times \left[\frac{Mr_e-Q^2}{Mr_d-Q^2}\right]^{1/2} . \tag{3.40}$$

We can further substitute the expressions of Ω and b in z_1 and z_2 to obtain an expression of kinematic frequency shift that depend only on the Kerr black hole parameters a and M, the detector and emitter radii r_d and r_e respectively. Using equations (3.37) and (3.38), the second factor of (3.39) takes the form

$$\begin{aligned} \frac{\Omega_d\left(\Omega_d-\Omega_e\right)b_-}{\Omega_e\left(1-\Omega_d b_+\right)} &= \frac{\Omega_d\left(\Omega_d-\Omega_e\right)b_-}{\Omega_e\left(\frac{1}{b_-}-\Omega_d\right)b_-} \\ &= \frac{\pm\left(Mr_d-Q^2\right)^{\frac{1}{2}}}{r_d^2\pm a\left(Mr_d-Q^2\right)^{\frac{1}{2}}} \times \frac{r_e^2\pm a\left(Mr_e-Q^2\right)^{\frac{1}{2}}}{\pm\left(Mr_e-Q^2\right)^{\frac{1}{2}}} \times \frac{\left(\Omega_d-\Omega_e\right)b_-}{\left(\frac{1}{b_-}-\Omega_d\right)b_-} \\ &= \frac{r_e^2\pm a\left(Mr_e-Q^2\right)^{\frac{1}{2}}}{r_d^2\pm a\left(Mr_d-Q^2\right)^{\frac{1}{2}}} \times \left[\frac{Mr_d-Q^2}{Mr_e-Q^2}\right]^{1/2} \times \frac{\left(\Omega_d-\Omega_e\right)}{\left(\frac{1}{b_-}-\Omega_d\right)} \\ &= \frac{r_e^2\pm a\left(Mr_e-Q^2\right)^{\frac{1}{2}}}{r_d^2\pm a\left(Mr_d-Q^2\right)^{\frac{1}{2}}} \times \left[\frac{Mr_d-Q^2}{Mr_e-Q^2}\right]^{1/2} \\ &\quad \times \left(\frac{\pm\left(Mr_d-Q^2\right)^{\frac{1}{2}}}{r_d^2\pm a\left(Mr_d-Q^2\right)^{\frac{1}{2}}} - \frac{\pm\left(Mr_e-Q^2\right)^{\frac{1}{2}}}{r_e^2\pm a\left(Mr_e-Q^2\right)^{\frac{1}{2}}}\right) \\ &\quad \times \left(\frac{r_e^2-2Mr_e+Q^2}{-a\left(2Mr_e-Q^2\right)-r_e^2\sqrt{r_e^2+a^2-2Mr_e+Q^2}} - \frac{\pm\left(Mr_d-Q^2\right)^{\frac{1}{2}}}{r_d^2\pm a\left(Mr_d-Q^2\right)^{\frac{1}{2}}}\right)^{-1} \\ &= A\times B\times C \quad , \end{aligned} \tag{3.41}$$

where

$$A = \frac{r_e^2\pm a\left(Mr_e-Q^2\right)^{\frac{1}{2}}}{r_d^2\pm a\left(Mr_d-Q^2\right)^{\frac{1}{2}}} \times \left[\frac{Mr_d-Q^2}{Mr_e-Q^2}\right]^{1/2} , \tag{3.42}$$

$$B = \frac{\pm\left(Mr_d-Q^2\right)^{\frac{1}{2}}}{r_d^2\pm a\left(Mr_d-Q^2\right)^{\frac{1}{2}}} - \frac{r_e^2\pm a\left(Mr_e-Q^2\right)^{\frac{1}{2}}}{\pm\left(Mr_e-Q^2\right)^{\frac{1}{2}}} , \tag{3.43}$$

$$C = \left(\frac{r_e^2-2Mr_e+Q^2}{-a\left(2Mr_e-Q^2\right)-r_e^2\sqrt{\Delta(r_e)}} - \frac{\pm\left(Mr_d-Q^2\right)^{\frac{1}{2}}}{r_d^2\pm a\left(Mr_d-Q^2\right)^{\frac{1}{2}}}\right)^{-1} , \tag{3.44}$$

with $\Delta(r_e) = r_e^2 + a^2 - 2Mr_e + Q^2$. Hence, calculating B yields

$$\begin{aligned}
B &= \frac{\pm\left(Mr_d - Q^2\right)^{\frac{1}{2}}}{r_d^2 \pm a\left(Mr_d - Q^2\right)^{\frac{1}{2}}} - \frac{\pm\left(Mr_e - Q^2\right)^{\frac{1}{2}}}{r_e^2 \pm a\left(Mr_e - Q^2\right)^{\frac{1}{2}}} \\
&= \frac{\pm\left(Mr_d - Q^2\right)^{\frac{1}{2}}\left[r_e^2 \pm a\left(Mr_e - Q^2\right)^{\frac{1}{2}}\right] \mp \left(Mr_e - Q^2\right)^{\frac{1}{2}}\left[r_d^2 \pm a\left(Mr_d - Q^2\right)^{\frac{1}{2}}\right]}{\left[r_d^2 \pm a\left(Mr_d - Q^2\right)^{\frac{1}{2}}\right]\left[r_e^2 \pm a\left(Mr_e - Q^2\right)^{\frac{1}{2}}\right]} \\
&= \pm\frac{r_e^2\left(Mr_d - Q^2\right)^{\frac{1}{2}} - r_d^2\left(Mr_e - Q^2\right)^{\frac{1}{2}}}{\left[r_d^2 \pm a\left(Mr_d - Q^2\right)^{\frac{1}{2}}\right]\left[r_e^2 \pm a\left(Mr_e - Q^2\right)^{\frac{1}{2}}\right]} \quad .
\end{aligned}$$

Moreover, simplifying C gives the following

$$\begin{aligned}
C &= \left(\frac{r_e^2 - 2Mr_e + Q^2}{-a\left(2Mr_e - Q^2\right) - r_e^2\sqrt{\Delta(r_e)}} - \frac{\pm\left(Mr_d - Q^2\right)^{\frac{1}{2}}}{r_d^2 \pm a\left(Mr_d - Q^2\right)^{\frac{1}{2}}}\right)^{-1} \\
&= \left(\frac{\left(r_e^2 - 2Mr_e + Q^2\right)\left[r_d^2 \pm a\left(Mr_d - Q^2\right)^{\frac{1}{2}}\right] \mp \left(Mr_d - Q^2\right)^{\frac{1}{2}}\left[-a\left(2Mr_e - Q^2\right) - r_e^2\sqrt{\Delta(r_e)}\right]}{\left[-a\left(2Mr_e - Q^2\right) - r_e^2\sqrt{\Delta(r_e)}\right]\left[r_d^2 \pm a\left(Mr_d - Q^2\right)^{\frac{1}{2}}\right]}\right)^{-1} \\
&= \frac{\left[-a\left(2Mr_e - Q^2\right) - r_e^2\sqrt{\Delta(r_e)}\right]\left[r_d^2 \pm a\left(Mr_d - Q^2\right)^{\frac{1}{2}}\right]}{\left(r_e^2 - 2Mr_e + Q^2\right)\left[r_d^2 \pm a\left(Mr_d - Q^2\right)^{\frac{1}{2}}\right] \mp \left(Mr_d - Q^2\right)^{\frac{1}{2}}\left[-a\left(2Mr_e - Q^2\right) - r_e^2\sqrt{\Delta(r_e)}\right]} \\
&= \frac{\left[-a\left(2Mr_e - Q^2\right) - r_e^2\sqrt{\Delta(r_e)}\right]\left[r_d^2 \pm a\left(Mr_d - Q^2\right)^{\frac{1}{2}}\right]}{D} \quad ,
\end{aligned}$$

where

$$\begin{aligned}
D &= \left(r_e^2 - 2Mr_e + Q^2\right)\left[r_d^2 \pm a\left(Mr_d - Q^2\right)^{\frac{1}{2}}\right] \mp \left(Mr_d - Q^2\right)^{\frac{1}{2}}\left[-a\left(2Mr_e - Q^2\right) - r_e^2\sqrt{\Delta(r_e)}\right] \\
&= r_d^2\left(r_e^2 - 2Mr_e + Q^2\right) \pm a\left(Mr_d - Q^2\right)^{\frac{1}{2}}\left(r_e^2 - 2Mr_e + Q^2\right) \\
&\quad \pm \left(Mr_d - Q^2\right)^{\frac{1}{2}}\left[+2aMr_e - aQ^2 + r_e^2\sqrt{\Delta(r_e)}\right] \\
&= r_d^2\left(r_e^2 - 2Mr_e + Q^2\right) \pm \left(Mr_d - Q^2\right)^{\frac{1}{2}}\left[ar_e^2 - 2aMr_e + aQ^2 + 2aMr_e - aQ^2 + r_e^2\sqrt{\Delta(r_e)}\right] \\
&= r_d^2\left(r_e^2 - 2Mr_e + Q^2\right) \pm \left(Mr_d - Q^2\right)^{\frac{1}{2}} r_e^2\left[a + \sqrt{\Delta(r_e)}\right] \quad .
\end{aligned}$$

Now, inserting A,B,C and D in the equation (3.41), we obtain

$$\begin{aligned}
\frac{\Omega_d\left(\Omega_d-\Omega_e\right)b_-}{\Omega_e\left(1-\Omega_d b_+\right)} &= \pm\frac{r_e^2\pm a\left(Mr_e-Q^2\right)^{\frac{1}{2}}}{r_d^2\pm a\left(Mr_d-Q^2\right)^{\frac{1}{2}}}\times\left[\frac{Mr_d-Q^2}{Mr_e-Q^2}\right]^{1/2}\\
&\times\frac{r_e^2\left(Mr_d-Q^2\right)^{\frac{1}{2}}-r_d^2\left(Mr_e-Q^2\right)^{\frac{1}{2}}}{\left[r_d^2\pm a\left(Mr_d-Q^2\right)^{\frac{1}{2}}\right]\left[r_e^2\pm a\left(Mr_e-Q^2\right)^{\frac{1}{2}}\right]}\\
&\times\frac{\left[-a\left(2Mr_e-Q^2\right)-r_e^2\sqrt{\Delta(r_e)}\right]\left[r_d^2\pm a\left(Mr_d-Q^2\right)^{\frac{1}{2}}\right]}{r_d^2\left(r_e^2-2Mr_e+Q^2\right)\pm\left(Mr_d-Q^2\right)^{\frac{1}{2}}r_e^2\left[a+\sqrt{\Delta(r_e)}\right]}\\
&= \pm\left[\frac{Mr_d-Q^2}{Mr_e-Q^2}\right]^{1/2}\times\frac{r_d^2\left(Mr_e-Q^2\right)^{\frac{1}{2}}-r_e^2\left(Mr_d-Q^2\right)^{\frac{1}{2}}}{\left[r_d^2\pm a\left(Mr_d-Q^2\right)^{\frac{1}{2}}\right]}\\
&\times\frac{\left[a\left(2Mr_e-Q^2\right)+r_e^2\sqrt{\Delta(r_e)}\right]}{r_d^2\left(r_e^2-2Mr_e+Q^2\right)\pm\left(Mr_d-Q^2\right)^{\frac{1}{2}}r_e^2\left[a+\sqrt{\Delta(r_e)}\right]}\,. \qquad (3.45)
\end{aligned}$$

Inserting this relation in (3.39) yields to the expression of z_1 in terms of the Boyer-Lindquist coordinates

$$\begin{aligned}
z_{kin1} &= \pm\frac{r_d\left[r_d^2-3Mr_d+2Q^2\pm 2a\left(Mr_d-Q^2\right)^{\frac{1}{2}}\right]^{1/2}}{r_e\left[r_e^2-3Mr_e+2Q^2\pm 2a\left(Mr_e-Q^2\right)^{\frac{1}{2}}\right]^{1/2}}\times\frac{r_d^2\left(Mr_e-Q^2\right)^{\frac{1}{2}}-r_e^2\left(Mr_d-Q^2\right)^{\frac{1}{2}}}{\left[r_d^2\pm a\left(Mr_d-Q^2\right)^{\frac{1}{2}}\right]}\\
&\times\frac{\left[a\left(2Mr_e-Q^2\right)+r_e^2\sqrt{\Delta(r_e)}\right]}{r_d^2\left(r_e^2-2Mr_e+Q^2\right)\pm r_e^2\left(Mr_d-Q^2\right)^{\frac{1}{2}}\left[a+\sqrt{\Delta(r_e)}\right]} \qquad (3.46)
\end{aligned}$$

Following the same steps, we find the expression of z_2 in terms of the Boyer-Lindquist coordinates

$$\begin{aligned}
z_{kin2} &= \frac{r_d\left[r_d^2-3Mr_d+2Q^2\pm 2a\left(Mr_d-Q^2\right)^{\frac{1}{2}}\right]^{1/2}}{r_e\left[r_e^2-3Mr_e+2Q^2\pm 2a\left(Mr_e-Q^2\right)^{\frac{1}{2}}\right]^{1/2}}\times\frac{r_d^2\left(Mr_e-Q^2\right)^{\frac{1}{2}}-r_e^2\left(Mr_d-Q^2\right)^{\frac{1}{2}}}{\left[r_d^2\pm a\left(Mr_d-Q^2\right)^{\frac{1}{2}}\right]}\\
&\times\frac{\left[a\left(2Mr_e-Q^2\right)-r_e^2\sqrt{\Delta(r_e)}\right]}{r_d^2\left(r_e^2-2Mr_e+Q^2\right)\pm r_e^2\left(Mr_d-Q^2\right)^{\frac{1}{2}}\left[a-\sqrt{\Delta(r_e)}\right]}\,. \qquad (3.47)
\end{aligned}$$

These are the expression of the kinematic red/blue shifts of photons emitted by particles in an equatorial stable circular motion at r_e around a Kerr-Newman black hole, and received by a far away observer at r_d. This expressions are of significant important in investigating the black hole parameters M, Q and a.

In order to better understand the content of formulas (3.46) and (3.47), we need to investigate the behavior of the frequency shift $z_{kin1(FA)}$ and $z_{kin2(FA)}$ for an emitter at $r_e = 10M$ and a far away observer ($r_d \to \infty$), in terms of the rotation parameter a and the electric charge Q. In Fig. 3.2 and Fig. 3.3, kinematic red and blue shifts from (3.46) and (3.47) are respectively plotted in terms of the rotation parameter a, for a black hole of mass $M = 1$ and different values of the

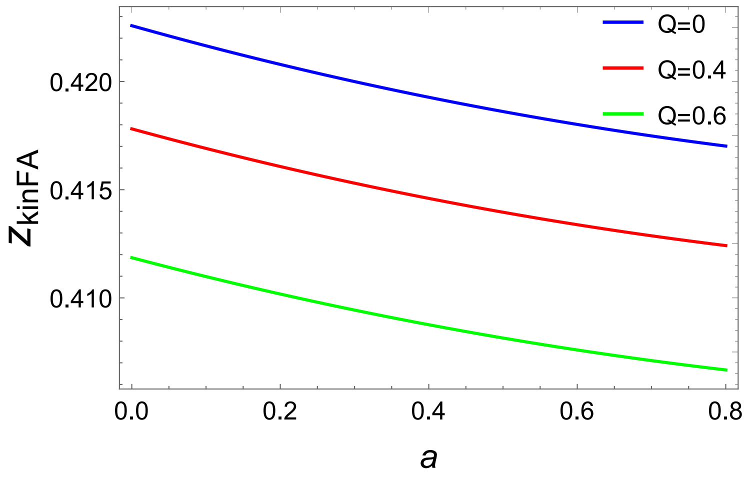


Fig. 3.2 – Kinematic red shift $z_{kin(FA)}$ from (3.46) with $(r_e = 10, r_d \to \infty)$, is plotted in terms of the rotation parameter a, for a black hole of mass $M = 1$ and different values of the charge $Q = 0$, $Q = 0.4$, and $Q = 0.6$.

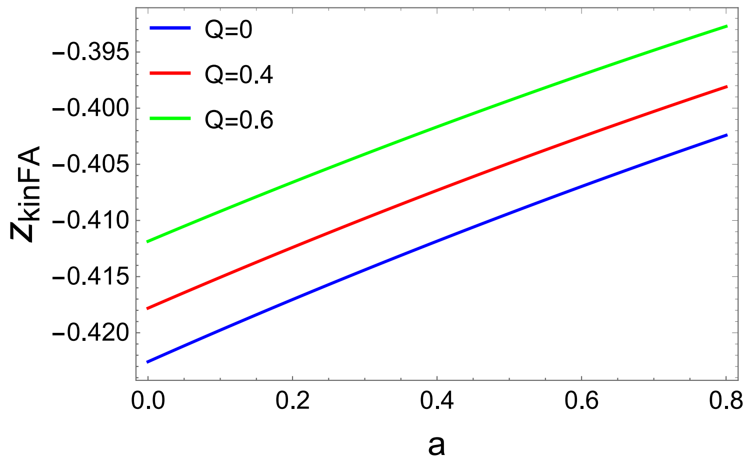


Fig. 3.3 – Kinematic blue shift z_{kin} from (3.47) with $(r_e = 10, r_d \to \infty)$, is plotted in terms of the rotation parameter a, for a black hole of mass $M = 1$ and different values of the charge $Q = 0$, $Q = 0.4$, and $Q = 0.6$.

charge $Q = 0$, $Q = 0.4$, and $Q = 0.6$. For a fixed value of electric charge Q, the kinematic red shift z_{kin1} decreases as the rotation parameter a increases. For the kinematic blue shift z_{kin1}, the opposite happens.

On the other hand, Fig. 3.4 and Fig. 3.5 represent, respectively, the kinematic red and blue shifts $z_{kin(FA)}$ from (3.46) and (3.47) for an emitter at $r_e = 10M$ and a far away observer $(r_d \to \infty)$. The shift $z_{kin(FA)}$ is plotted in terms of the electric charge Q, for a black hole of mass $M = 1$ and different values of the rotation parameter $a = 0$, $a = 0.25$, and $a = 0.50$. Again, for a fixed rotation parameter value, the red shift z_{kin1} decrease as the electric charge Q increases.

3.5.4 The Kerr and Schwarzschild case

In order for our results of z_1 and z_1, obtained above, to be true, they must reduce to the same expressions for a rotating uncharged black hole, i.e., Kerr black hole, and the simplest case of a non rotating uncharged case, i.e., the Schwarzschild black hole. To show that the expressions

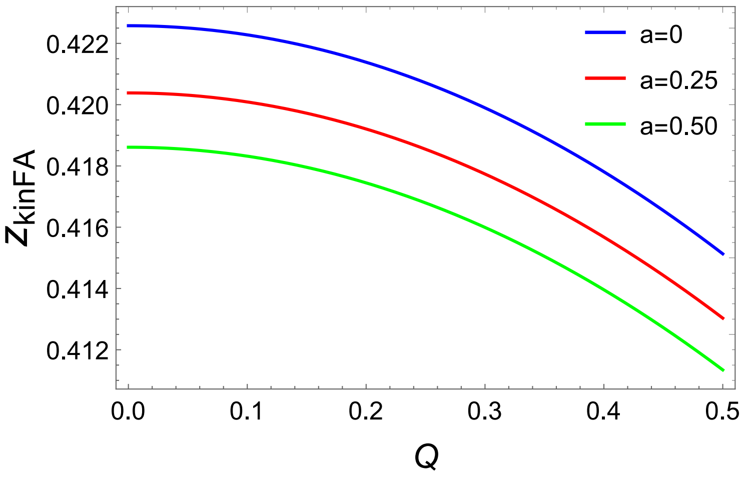


Fig. 3.4 – Kinematic red shift z_{kin} from (3.46) with ($r_e = 10, r_d \to \infty$), is plotted in terms of the electric charge Q, for a black hole of mass $M = 1$ and different values of the rotation parameter $a = 0$, $a = 0.25$, and $a = 0.50$.

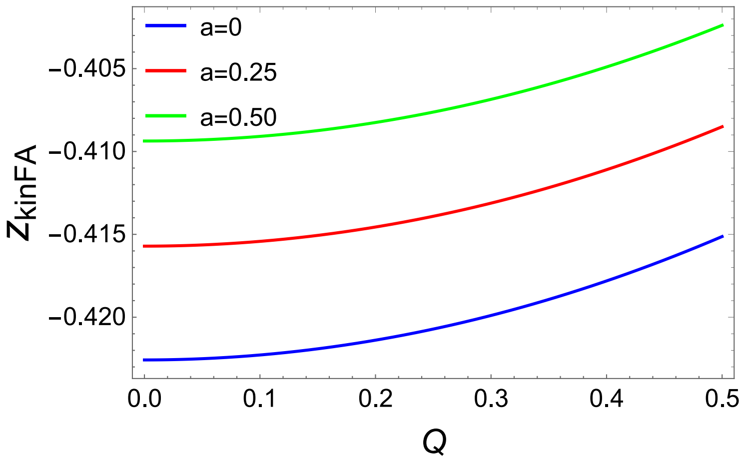


Fig. 3.5 – Kinematic blue shift z_{kin} from (3.47) with ($r_e = 10, r_d \to \infty$), is plotted in terms of the electric charge Q, for a black hole of mass $M = 1$ and different values of the rotation parameter $a = 0$, $a = 0.25$, and $a = 0.50$.

obtained above reduce to those of the Kerr black hole if we put $Q = 0$ in z_1:

$$\begin{aligned}
z_{kin1} &= \pm\frac{r_d\left[r_d^2 - 3Mr_d \pm 2aM^{1/2}r_d^{1/2}\right]^{1/2}}{r_e\left[r_e^2 - 3Mr_e \pm 2aM^{1/2}r_d^{1/2}\right]^{1/2}} \times \frac{r_d^2 M^{1/2} r_d^{1/2} - r_e^2 M^{1/2} r_d^{1/2}}{\left[r_d^2 \pm aM^{1/2} r_d^{1/2}\right]} \\
&\quad \times \frac{\left[a\left(2Mr_e\right) + r_e^2\sqrt{r_e^2 + a^2 - 2Mr_e}\right]}{r_d^2\left(r_e^2 - 2Mr_e\right) \pm r_e^2 M^{1/2} r_d^{1/2}\left[a + \sqrt{r_e^2 + a^2 - 2Mr_e}\right]} \\
&= \pm M^{\frac{1}{2}}\left(\frac{r_d^{1/2}}{r_e^{1/2}}\right)\frac{r_d^{3/4}\left[r_d^{3/2} - 3Mr_d^{1/2} \pm 2aM^{1/2}\right]^{1/2}}{r_e^{3/4}\left[r_e^{3/2} - 3Mr_e^{1/2} \pm 2aM^{1/2}\right]^{1/2}} \times \left(\frac{r_d^{1/2} r_e^{1/2}}{r_d^{1/2}}\right)\frac{r_d^{3/2} - r_e^{3/2}}{\left[r_d^{3/2} \pm aM^{1/2}\right]} \\
&\quad \times \left(\frac{r_e}{r_d^{1/2} r_e}\right)\frac{\left[a\left(2M\right) + r_e\sqrt{r_e^2 + a^2 - 2Mr_e}\right]}{r_d^{3/2}\left(r_e - 2M\right) \pm r_e M^{1/2}\left[a + \sqrt{r_e^2 + a^2 - 2Mr_e}\right]},
\end{aligned}$$

which finally leads to

$$z_{kin1} = \pm M^{\frac{1}{2}} \frac{r_d^{3/4}\left[r_d^{3/2} - 3Mr_d^{1/2} \pm 2aM^{1/2}\right]^{1/2}}{r_e^{3/4}\left[r_e^{3/2} - 3Mr_e^{1/2} \pm 2aM^{1/2}\right]^{1/2}} \times \frac{r_d^{3/2} - r_e^{3/2}}{\left[r_d^{3/2} \pm aM^{1/2}\right]}$$
$$\times \frac{\left[a\left(2M\right) + r_e\sqrt{r_e^2 + a^2 - 2Mr_e}\right]}{r_d^{3/2}\left(r_e - 2M\right) \pm r_e M^{1/2}\left[a + \sqrt{r_e^2 + a^2 - 2Mr_e}\right]} . \tag{3.48}$$

Following the same steps, we can obtain a similar expression for z_2. which is exactly the expression obtained in [47], as expected.

Moreover, if we set $Q = 0$ and $a = 0$, which is the case of a non rotating uncharged black hole, we will obtain the expressions of red/blue shifts of photon emitted by particles in an equatorial stable orbit around a Schwarzschild black hole. Setting $Q = a = 0$ in (3.48), we get

$$z_{kin1} = \pm \frac{M^{1/2}}{\sqrt{r_e - 3M}}\left[\frac{r_d^{3/2} - r_e^{3/2}}{r_d^{1/2}}\right] \times \frac{r_e^{1/2}\sqrt{r_d - 3M}}{r_d^{3/2}\sqrt{r_e - 2M} \pm r_e^{3/2}M^{1/2}} . \tag{3.49}$$

Following the same steps, we can obtain a similar expression for z_2. Again, this is exactly the same expression (2.84) we established for the case of a Schwarzschild black hole in the previous chapter.

3.5.5 Equation for r_d in terms of a, Q and b_e

The fact that E_γ and L_γ are constant along the photons path imply that the impact parameter b is constant too. In particular, b has the same value regardless of being measured at the emitter or detector position: $b_e = b_d$. Using (3.38), we get

$$\begin{aligned}
b_e &= b_d \ , \\
b_e &= \frac{-a\left(2Mr_d - Q^2\right) \pm r_d^2\sqrt{r_d^2 + a^2 - 2Mr_d + Q^2}}{r_d^2 - 2Mr_d + Q^2} \\
\pm r_d^2\sqrt{r_d^2 + a^2 - 2Mr_d + Q^2} &= b_e r_d^2 - 2b_e M r_d + b_e Q^2 + 2aMr_d - aQ^2 \\
r_d^4\left(r_d^2 + a^2 - 2Mr_d + Q^2\right) &= \left[b_e r_d^2 - 2b_e M r_d + b_e Q^2 + 2aMr_d - aQ^2\right]^2
\end{aligned}$$

$$r_d^6 - 2Mr_d^5 + \left(-b_e + a^2 + Q^2\right)r_d^4 + 4b_e\left(b_e - a\right)r_d^3 + 2\left[2M^2\left(b_e - a\right)^2 - b_e Q^2\left(b_e - a\right)\right]r_d^2$$
$$+4MQ^2\left(b_e - a\right)^2 r_d - Q^4\left(b_e - a\right)^2 = 0 \, . \tag{3.50}$$

Solving this equation will determine the value of r_d in terms of a, Q and b_e. Putting $Q = 0$ reduce this equation to the one obtained in [47] for the Kerr black hole:

$$r_d^4 - 2Mr_d^3 + \left(a^2 - b_e^2\right)r_d^2 + 4Mb_e\left(b_e - a\right)r_d + 4M^2\left(b_e - a\right)^2 = 0 \quad . \tag{3.51}$$

The authors of [47] solved this equation and obtained an expression of r_d in terms of a, M and b_e (see equation (52) in [47]).

3.5.6 Expression of z_1 and z_2 for a far away observer

In the particular case when the detector is located far away from the source and the following conditions $r_d \gg M \geq \sqrt{a^2+Q^2}$ (the Kerr-Newman black hole horizon condition) and $r_d \gg r_e$, we have equation (3.46) yields

$$\begin{aligned}
\left[r_d^2 - 3Mr_d + 2Q^2 \pm 2a\left(Mr_d - Q^2\right)^{\frac{1}{2}}\right]^{1/2} &\longrightarrow r_d\sqrt{1+0} \quad , \\
\frac{r_d^2\left(Mr_e - Q^2\right)^{\frac{1}{2}} - r_e^2\left(Mr_d - Q^2\right)^{\frac{1}{2}}}{\left[r_d^2 \pm a\left(Mr_d - Q^2\right)^{\frac{1}{2}}\right]} &\longrightarrow \frac{r_d^2\left[\left(Mr_e - Q^2\right)^{\frac{1}{2}} - 0\right]}{r_d^2\left[1 \pm 0\right]} = \left(Mr_e - Q^2\right)^{\frac{1}{2}} \quad , \\
r_d^2\left(r_e^2 - 2Mr_e + Q^2\right) \pm r_e^2\left(Mr_d - Q^2\right)^{\frac{1}{2}}\left[a + \sqrt{\Delta(r_e)}\right] &\longrightarrow r_d^2\left[\left(r_e^2 - 2Mr_e + Q^2\right) + 0\right] \quad .
\end{aligned}$$

Substituting this in equation (3.46) yields

$$z_1 = \pm \frac{\left(Mr_e - Q^2\right)^{\frac{1}{2}}}{r_e\left[r_e^2 - 3Mr_e + 2Q^2 \pm 2a\left(Mr_e - Q^2\right)^{\frac{1}{2}}\right]^{1/2}} \times \frac{a\left(2Mr_e - Q^2\right) + r_e^2\sqrt{\Delta(r_e)}}{\left(r_e^2 - 2Mr_e + Q^2\right)} \tag{3.52}$$

In the same way, equation (3.47) takes the form

$$z_2 = \pm \frac{\left(Mr_e - Q^2\right)^{\frac{1}{2}}}{r_e\left[r_e^2 - 3Mr_e + 2Q^2 \pm 2a\left(Mr_e - Q^2\right)^{\frac{1}{2}}\right]^{1/2}} \times \frac{a\left(2Mr_e - Q^2\right) - r_e^2\sqrt{\Delta(r_e)}}{\left(r_e^2 - 2Mr_e + Q^2\right)} \tag{3.53}$$

In this particular case, it is important to express the Kerr-Newman black hole parameters a, M and Q in terms of the frequency shifts (3.52) and (3.53). The last couple of equations give

$$\alpha \equiv \left(z_1 + z_2\right)^2 = \frac{\left(Mr_e - Q^2\right)4a^2\left(2Mr_e - Q^2\right)^2}{r_e^2\left[r_e^2 - 3Mr_e + 2Q^2 \pm 2a\left(Mr_e - Q^2\right)^{\frac{1}{2}}\right]\left(r_e^2 - 2Mr_e + Q^2\right)^2} \quad , \tag{3.54}$$

$$\beta \equiv \left(z_1 - z_2\right)^2 = \frac{\left(Mr_e - Q^2\right)4r_e^4\Delta(r_e)}{r_e^2\left[r_e^2 - 3Mr_e + 2Q^2 \pm 2a\left(Mr_e - Q^2\right)^{\frac{1}{2}}\right]\left(r_e^2 - 2Mr_e + Q^2\right)^2} \tag{3.55}$$

$$\begin{aligned}
\frac{\alpha}{\beta} &= \frac{\left(z_1 + z_2\right)^2}{\left(z_1 - z_2\right)^2} = \frac{a^2\left(2Mr_e - Q^2\right)^2}{r_e^4\Delta(r_e)} \\
&= \frac{a^2\left(4M^2r_e^2 + Q^4 - 4MQ^2r_e\right)}{r_e^4\left(r_e^2 + a^2 - 2Mr_e + Q^2\right)} \\
&= \frac{a^2\left(4M^2r_e^2 + Q^4 - 4MQ^2r_e\right)}{r_e^6 + a^2r_e^4 - 2Mr_e^5 + Q^2r_e^4} \\
\frac{\alpha}{\beta}\left(r_e^6 - 2Mr_e^5 + Q^2r_e^4\right) + \frac{\alpha}{\beta}a^2r_e^4 &= a^2\left(4M^2r_e^2 + Q^4 - 4MQ^2r_e\right) \\
\frac{\alpha}{\beta}\left(r_e^6 - 2Mr_e^5 + Q^2r_e^4\right) &= a^2\left(4M^2r_e^2 + Q^4 - 4MQ^2r_e - \frac{\alpha}{\beta}r_e^4\right)
\end{aligned}$$

Solving for a^2 give

$$
\begin{aligned}
a^2 &= \frac{\frac{\alpha}{\beta}\left(r_e^6 - 2Mr_e^5 + Q^2 r_e^4\right)}{4M^2 r_e^2 + Q^4 - 4MQ^2 r_e - \frac{\alpha}{\beta} r_e^4} \\
&= \frac{\alpha\left(r_e^6 - 2Mr_e^5 + Q^2 r_e^4\right)}{\left(4M^2 r_e^2 + Q^4 - 4MQ^2 r_e\right)\beta - \alpha r_e^4} \\
a^2 &= \frac{\left(z_1+z_2\right)^2\left(r_e^6 - 2Mr_e^5 + Q^2 r_e^4\right)}{\left(4M^2 r_e^2 + Q^4 - 4MQ^2 r_e\right)\left(z_1-z_2\right)^2 - \left(z_1+z_2\right)^2 r_e^4} \quad . \qquad (3.56)
\end{aligned}
$$

If we put $Q = 0$ to reduce the Kerr-Newman black hole to Kerr black hole, we obtain the following equation

$$
\begin{aligned}
a^2 &= \frac{\left(z_1+z_2\right)^2\left(r_e^6 - 2Mr_e^5\right)}{\left(4M^2 r_e^2\right)\left(z_1-z_2\right)^2 - \left(z_1+z_2\right)^2 r_e^4} \\
&= \frac{\left(z_1+z_2\right)^2 r_e^3\left(r_e - 2M\right)}{4M^2\left(z_1-z_2\right)^2 - \left(z_1+z_2\right)^2 r_e^2} \quad , \qquad (3.57)
\end{aligned}
$$

or

$$
a^2 = \frac{\alpha r_e^3\left(r_e - 2M\right)}{4M^2\beta - \alpha r_e^2} \qquad (3.58)
$$

Inserting this formula for a^2 in (3.54) and putting $Q = 0$, we get

$$
\begin{aligned}
\alpha &= \frac{16a^2M^3 r_e^3}{r_e^2\left[r_e^2 - 3Mr_e \pm 2aM^{1/2} r_e^{1/2}\right]\left[r_e^2 - 2Mr_e\right]^2} \\
\alpha &= \frac{16a^2M^3}{r_e^{3/2}\left[r_e^{3/2} - 3Mr_e^{1/2} \pm 2aM^{1/2}\right]\left[r_e - 2M\right]^2} \\
16a^2M^3 &= \alpha r_e^{3/2}\left[r_e^{3/2} - 3Mr_e^{1/2} \pm 2aM^{1/2}\right]\left[r_e - 2M\right]^2 \\
16M^3\alpha r_e^3\frac{\left(r_e - 2M\right)}{4M^2\beta - \alpha r_e^2} &= \alpha r_e^{3/2}\left[r_e^{3/2} - 3Mr_e^{1/2} \pm 2aM^{1/2}\right]\left[r_e - 2M\right]^2 \\
16M^3 r_e^{3/2} &= \left(4M^2\beta - \alpha r_e^2\right)\left[r_e^{3/2} - 3Mr_e^{1/2} \pm 2aM^{1/2}\right]\left[r_e - 2M\right]
\end{aligned}
$$

$$
\begin{aligned}
\left[16M^3 r_e^{3/2} - \left(4M^2\beta - \alpha r_e^2\right)\left(r_e - 2M\right)\left(r_e^{3/2} - 3Mr_e^{1/2}\right)\right]^2 &= 4a^2M\left[r_e - 2M\right]^2 \\
&\quad \times\left(4M^2\beta - \alpha r_e^2\right)^2
\end{aligned}
$$

$$
\begin{aligned}
\left[16M^3 r_e^{3/2} - \left(4M^2\beta - \alpha r_e^2\right)\left(r_e - 2M\right)\left(r_e^{3/2} - 3Mr_e^{1/2}\right)\right]^2 &= 4\alpha r_e^3\frac{\left(r_e - 2M\right)}{4M^2\beta - \alpha r_e^2}M\left[r_e - 2M\right]^2 \\
&\quad \times\left(4M^2\beta - \alpha r_e^2\right)^2 \\
\left[16M^3 r_e^{3/2} - \left(4M^2\beta - \alpha r_e^2\right)\left(r_e - 2M\right)\left(r_e^{3/2} - 3Mr_e^{1/2}\right)\right]^2 &= 4\alpha M r_e^3\left(r_e - 2M\right)^3\left(4M^2\beta - \alpha r_e^2\right) \\
\left[16M^3 r_e - \left(4M^2\beta - \alpha r_e^2\right)\left(r_e - 2M\right)\left(r_e - 3M\right)\right]^2 &= 4\alpha M r_e^2\left(r_e - 2M\right)^3\left(4M^2\beta - \alpha r_e^2\right)
\end{aligned}
$$

This is exactly the equation obtained in [47]. We think that this equation can be used to determine the the emitter radius r_e in terms of the black hole parameters.

Conclusion

Without a doubt, black holes are one of the most appealing yet mysterious objects in the universe. This makes them a hot topic not only at the scientific level but also for artistic and literature ones. In physics, black holes were first predicted by Karl Schwarzschild in 1916, two years after Einstein published his theory of General Relativity. Their existence was doubted or even dismissed by prominent physicists in the following decades. However, in the 1960s, the observational data started accumulating, making them more accepted among physicists. In recent decades, the detection of gravitational waves by LIGO collaboration and the release of the first images of a black hole by EHT collaboration have provided the most solid evidence for their existence.

In this dissertation, we explored the theory of General Relativity, which represents the mathematical framework for black holes. This theory was published ten years after the development of Special Relativity by Einstein, Lorentz and others. The postulates of Special Relativity were i) relativity, that is, the laws of physics are invariant for all observers, and ii) that the speed of light is the same for all observers. One of the main consequences of the theory is that space and time are now components of the same fabric called spacetime rather than two separate physical quantities. General Relativity revolutionized physics and led to new phenomena such as time dilation and length contraction. However, one physical law was incompatible with Special Relativity, which is Newton's gravitational law. This problem pushed Einstein to think about a theory of relativity that involved gravity, an endeavor that culminated in the formulation of General Relativity.

The cornerstone of General Relativity is the equivalence principle. According to this principle, it is impossible to detect the presence of the gravitational field by means of a local experiment. In differential geometry jargon, this is equivalent to saying that free particles move along geodesics and gravity is the spacetime curvature. We have shown in the first chapter that Einstein field equations can be derived from a least action principle. The resulting equations are the relativistic equivalent of Poisson's equation: they tell us how matter determine spacetime curvature. On the other hand, the relativistic equivalent to Newtons second law in a gravitational field is the geodesic equation: it tells us how curvature governs the motion of matter.

The advent of General Relativity gave rise to many unprecedented predictions. This includes gravitational redshift, gravitational lensing, and the precession of the perihelion of Mercury. Black holes are certainly among the most important and enigmatic predictions. These are regions of spacetime where the gravitational field is so immense that nothing, not even light, can escape. In the second chapter, we showed that a similar idea of black holes originated in the 18th century under the name of black stars. However, the geometrical approach of General Relativity revealed that black holes can have a rich structure and weird properties. The event horizon is the most important element in a black hole structure, which can be thought of as a one-way surface in spacetime, you cannot cross it twice. At the center, there is the singularity, a region of infinite curvature and density. The existence of naked singularities, wich are singularities without an event horizon, is a debatable topic. Accretion disks are also a common element in black hole structures, which are crucial for observations. Other structures such as the ergosphere can only be found in rotating black holes. While having this complex geometrical structure, only three numbers are needed to describe any black hole. According to the no-hair theorem, these numbers are the mass, the electric charge, and the angular momentum of the black hole. Therefore, we have seen the four types of black holes: i) neutral and static Schwarzschild, ii) charged and static Reissner-Nordström, iii) neutral and rotating Kerr, and finally iv) charged and rotating Kerr-Newmann. Starting from the Einstein field equations and other assumptions, we derived the Schwarzschild solution. In this spacetime, we investigated the red/blue shifts of photons emitted by particles in circular motion around the black hole. Then, we solved the Einstein-Maxwell field equations in the charged static case and obtained the Reissner-Nordström solutions. We have discussed the structure of the three first types in detail. We additionally examine some pathologies of the Kerr black holes such as the closed timelike curves. Moreover, in this chapter we gave some direct evidence of the existence of black holes in astrophysics. This includes the detection of ultraluminous X-ray sources like quasars and X-ray binaries, observation of the gravitational waves, and the first direct image of a black hole ever. We ended the chapter by exploring some of the unsolved problems of black holes. The first and foremost one is the absence of a consistent theory of quantum gravity.

Many black hole issues remain open research questions. At the theoretical level, this includes the existence of closed timelike curves the information paradox. Some theoretical predictions such as white holes and naked singularities are still speculative and neither confirmed nor dismissed by observations. At the observational level, the unsolved problems include the formation of supermassive black holes and the mechanism behind the relativistic jets from them. To solve these problems, several theories have been proposed, both within General Relativity and beyond it. It turns out that measuring the black hole in three parameters M, Q, and a are crucial testing predictions of those theories. Several approaches have been suggested to measure those parameters. One such method involves analyzing the red/blue shifts of photons emitted by particles orbiting the black hole. In 2015, a formula of the red/blue of photons emitted by particles in Kerr black hole [47]. In our paper, we followed a similar approach and obtained expressions of the frequency shifts for photons orbiting a Kerr-Newman black hole. Although charged black holes have not been observed before, their existence remains an open question. We believe that considering the electric charge is of significant importance on the theoretical level. Our work will allow testing the effect of the charge on the frequency shifts. The derived formulas of the total and kinematic frequency shifts z and z_{kin} depend on the Kerr-Newmann three parameters M, Q, and a, as well as on the radii r_e and r_d of the emitter and detector respectively. Results showed that, for a fixed value of electric charge Q, the kinematic redshift z_{kin} decreases as the rotation parameter a increases. For the kinematic blue shift z_{kin}, the opposite happens. On the other hand, for a fixed value of rotation parameter a, the redshift z_{kin1} decreases as the electric charge Q increases. Moreover, in cases where the angular velocity of the detector vanishes $\Omega_d = 0$ or the charge of the Kerr-Newman black hole take the value $Q^2 = Mr_e$, the red/blue shift mentioned above vanished. Putting the electric charge $Q = 0$, the obtained expressions reduce to those obtained in [47]. In addition, the cases of Reissner-Nordström and Schwarzschild can be easily obtained by $a = 0$ then $Q = 0$. We believe our results can contribute to a more comprehensive picture of black holes' properties in general, and offer a deeper understanding of their formation and evolution.

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الملخص:

في هذه الأطروحة، نتطرق إلى طريقة نسبية لقياس خصائص الثقب الأسود، انطلاقا من معطيات رصدية. حسب النسبية العامة، هنالك أربعة أنواع رئيسية للثقوب السوداء هي شوارزشيلد، رايسنر-نوردستروم، كيير وكيير-نيومان التي تنص عليها نظرية اللاشعر (no-hair). هذه الأنواع الأربعة تتميز عن بعضها من حيث الكتلة، الشحنة الكهربائية، والعزم الحركي. قياس هذه المقادير وتحديد مجال تغيرها يعتبر مهما جدا لفهم الثقوب السوداء ودورها في المجرات وفي الكون عموما. في هذه الأطروحة نقوم بإيجاد عبارة تربط بين مقدار رصدي وهو إنزياح التردد للفوتونات الصادرة عن جسيمات تدور حول الثقب الأسود من نوع كيير-نيومان، والذي يتميز بكتلة *m*، شحنة كهربائية *Q* وعزم حركي *a* معا. نعتبر أن مسار الجسيمات المصدرة والراصدة هو دائري مستقر في المستوي الاستوائي، وهو حال الكثير من الأجرام السماوية التي تدور حول الثقوب السوداء. انطلاقا من هذه العبارة، يمكن اشتقاق العبارة المماثلة لثقب أسود يدور من نوع كيير، وهو مايتطابق مع دراسات سابقة. قمنا أيضا باستخدام هذه العبارة عليها للحصول على عبارة انزياح التردد لثقب أسود مشحون من نوع رايسنر-نوردستروم وكذا ثقب أسود متعادل كهربائيا وساكن من نوع شوارزشيلد. نتوقع أن هذه العبارة ستكون مفيدة كاختبار آخر للنسبية العامة وكذا من أجل تحديد قيم هذه المقادير عند دراسة الأجسام التي تدور حول ثقوب سوداء مثل النجوم.

Abstract:

This dissertation addresses a relativistic method to measure of the parameters of black holes. According to General Relativity, there are four types of black holes: Schwarzschild, Reissner-Nordström, Kerr and Kerr-Newman. These four types are distinguished by their parameters: the mass, the electric charge, and the angular momentum. The measurement of these parameters and their range is extremely important for understanding black holes and their role in the universe. In this dissertation we developed formulae relating a measurable quantity, which is frequency shift of photons emitted by massive particles, in terms of the Kerr-Newman black holes parameters: the mass m, electric charge Q, and the angular momentum a. The trajectories of these particles are considered to be circular and stable in the equatorial plane, which is the case for many celestial objects revolving black holes. Since Kerr-Newman black hole represent the most general case, we have used the obtained formulae to reproduce the case of Kerr black hole, which is in total agreement with previous studies. We could as well reproduce the charged nonrotating case of Reissner-Nordström and the neutral static case of Schwarzschild. We think the obtained results can be used as another test for General Relativity and used to determine the range of black holes parameters when observing celestial objects revolving black holes such as stars.

Résumé

Cette thèse présente une méthode relativiste pour mesurer les paramètres des trous noirs. Selon la Relativité Générale, il existe quatre types de trous noirs : Schwarzschild, Reissner-Nordström, Kerr et Kerr-Newman. Ces quatre types se distinguent par leurs paramètres : la masse, la charge électrique et le moment cinétique. La mesure de ces paramètres et leur plage sont extrêmement importantes pour comprendre les trous noirs et leur rôle dans l'univers. Dans cette thèse, nous avons développé des formules reliant une quantité mesurable, le décalage de fréquence des photons émis par des particules massives, aux paramètres des trous noirs de Kerr-Newman : la masse m, la charge électrique Q et le moment cinétique a. Les trajectoires de ces particules sont considérées comme circulaires et stables dans le plan équatorial, ce qui est le cas pour de nombreux objets célestes en orbite autour des trous noirs. Étant donné que les trous noirs de Kerr-Newman représentent le cas le plus général, nous avons utilisé les formules obtenues pour reproduire le cas du trou noir de Kerr, ce qui est en total accord avec les études précédentes. Nous pouvons également reproduire le cas non rotatif chargé de Reissner-Nordström et le cas neutre statique de Schwarzschild. Nous pensons que les résultats obtenus peuvent être utilisés comme un autre test de la Relativité Générale et pour déterminer la plage des paramètres des trous noirs lors de l'observation d'objets célestes en orbite autour des trous noirs, tels que les étoiles.