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TITLE::

Application of q-Homotopy Analysis Transform Method for Solving a Fractional Singular One-Dimensional Thermo-Elasticity Coupled System

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Before the jury composed of::

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Acknowledgment

قال الله تعالى:
" لَئِن شَكَرْتُمْ لَأَزِيدَنَّكُمْ "
وقال رسول الله صلى الله عليه وسلم:
" لا يشكر الله من لا يشكر الناس ".

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Dedication

In the Name of Allah, the Most Gracious, the Most Merciful
"My Lord, inspire me to be grateful for Your favor which You have bestowed upon me and upon my parents."

To my dear father...
My steadfast shelter, you who nourished my roots with your determination until they bore fruit - to you I offer every fruit of pride and thankfulness.

To my beloved mother...
The eternal garden of my heart, you who planted the seeds of knowledge in me before I even knew the meaning of letters - to you I offer every flower of gratitude and appreciation.

To my siblings...
Hisham, Amira, Ahmed, Adam
The strong branches of my tree that sway resiliently with winds of challenge - thank you for transforming every stumble into a success story.

To my fiance ...
Abd Elhak
My second pillar and the undimming light of my path - thank you for turning every hardship into a beautiful moment and every achievement into a fragrant memory.

To my dearest friends...
My journey companions, my heart's support, the flowers of my life who stood by me at every step, supported me in every moment, and rejoiced in every success - thank you for making the road more beautiful and the accomplishment more radiant.

And **to everyone who taught me even a single letter...**
To you I extend my highest gratitude, for through your favor this success was achieved, and through your efforts my path was illuminated.

In conclusion...
"This success would not have been possible without Allah's grace first, then a mother's sleepless prayers, a father's tireless efforts, siblings' encouragement, a dearà support, and friends' companionship."

Madiha

Abstract

This research investigated the existence and uniqueness of the strong solution for a coupled fractional pseudo-hyperbolic differential equation system with initial conditions, Neumann boundary conditions, and nonlocal integral conditions. The theoretical analysis relied exclusively on the ”a priori estimates method” to examine the solvability of the problem and study solution characteristics.

For the applied component, the q-Homotopy Analysis Transform Method (q-HATM) was employed to obtain numerical solutions. Practical examples were then presented to demonstrate the effectiveness of this (q-HATM) approach in solving fractional-order coupled differential equation systems.

Keywords: Coupled fractional systems, Strong solutions, A priori estimates, q-Homotopy Analysis Transform Method (q-HATM), Caputo fractional derivatives, Nonlocal boundary conditions.

الملخص

تم في هذه الدراسة التحقق من وجود ووحدانية الحل القوي لنظام المعادلات التفاضلية الكسرية الزائدية المقترنة، مع شروط ابتدائية و حدودية من نوع نيومان وشروط تكاملية غير محلية. اعتمد الجانب النظري على "طريقة التقديرات المسبقة" لتحليل إمكانية حل المسألة ودراسة خصائص الحل. أما الجانب التطبيقي، فقد تم استخدام طريقة تحليل التجانس-q التحويلي للحصول على حلول عددية .ثم تقديم أمثلة عملية لشرح فعالية هذه الطريقة $(q-HATM)$ في حل نظام المعادلات التفاضلية ذات الرتبة الكسرية.

الكلمات المفتاحية: أنظمة المعادلات الكسرية المقترنة، الحل القوية التقديرات المسبقة، طريقة تحليل التجانس-q التحويلي ، المشتقات الكسرية بطريقة كابوتو، الشروط الحدية غير المحلية.

Notation

${}^{c}\partial_{0t}^{\beta}$	The fractional Caputo derivative.
$D_t^{-\beta}$	The Riemann-Liouville integral.
$\mathcal{L}, l$	Linear operators.
$D(L)$	Domain of definition of the operator L.
$\mathcal{L}_{\rho}^{2}(Q)$	Space of square integrable function u with weight function ρ, defined on Q.
$\mathrm{R}(L)$	Image of the operator L.
HAM	Homotopy analysis method.
q-HATM	q-Homotopy Analysis Transform Method.

Introduction

Fractional differential equations have become one of the most important modern research topics, witnessing remarkable development in recent decades. They have emerged as a powerful mathematical tool for modeling complex phenomena in physics, engineering, and applied sciences [1]. This growing interest stems from their ability to describe systems with long-term memory and non-local effects, which classical models fail to accurately represent. Among the prominent practical applications of these equations are modeling heat transfer in heterogeneous materials and analyzing the motion of charged particles in complex electromagnetic fields, where fractional solutions provide deeper insights than those offered by traditional models [2].

Amidst this progress, the study of differential equations with integral (non-local) conditions has garnered significant attention. These conditions represent a crucial enhancement to mathematical models, particularly when direct boundary measurements are unfeasible or when only averaged or aggregated values are available. One of the earliest studies in this field was conducted by Cannon [3], who employed integral conditions to investigate heat transfer in metallic materials using heat section equations. This was followed by numerous works, most notably those of Lonkin and Yorshuk [4, 5], which utilized advanced methods such as the Fourier method and energy estimates to analyze the existence and uniqueness of solutions in such systems. Recent studies, such as [6], have expanded these analyses to include nonlinear fractional systems.

From a theoretical perspective, researchers have made substantial efforts to study the properties of these equations, with a particular focus on proving the existence and uniqueness of solutions using diverse mathematical methodologies. These methodologies include the fixed-point principle and Lax-Milgram [7], as well as energy estimate methods that first appeared in the works of Petrovsky [8] and became foundational for analyzing systems with integral boundary conditions. This approach was later extended to analyze a wide range of equations, as seen in the works of [9].

To overcome the challenges in solving non-local fractional systems, a set of advanced numerical methods has recently been developed to obtain highly accurate approximate solutions. These methods can be classified into two main categories: classical methods such as the Finite Element Method (FEM) and the Finite Difference Method (FDM), and modern methods including the Adomian Decomposition Method (ADM) [10], the Homotopy Analysis Method (HAM) [11], and the Homotopy Analysis Transform Method (HATM), along with its improved version known as q-HATM, which combines homotopy techniques with q-transforms. The q-HATM method represents a qualitative leap in solving fractional equations, embodying a modern development that merges the advantages of traditional homotopy analysis with q-transform techniques. This method relies on transforming the original equation into a simplified form using the q-transform, facilitating the application of a series of iterative functions to reach the solution. This technique is distinguished by its ability to accelerate convergence, enhance the accuracy of numerical solutions, and provide great flexibility in handling complex and nonlinear systems, making it an effective tool in this important research field [12].

In this research, we adopt a methodology that combines numerical and theoretical aspects. Numerically, we employ the q-HATM method to test its computational efficiency in handling dual fractional systems. Theoretically, we rely on the energy method, which provides a robust analytical framework for studying the existence, uniqueness, and dynamic stability of solutions, enabling a deep understanding of their behavior and long-term stability or divergence [13].
Below, we study the following system:

$$\begin{cases} {}^{c}\partial_{0t}^{\beta} u - \frac{1}{x}\left(xu_x\right)_x - \frac{1}{x}\frac{\partial}{\partial t}\left(xu_x\right)_x + z_1 v + \alpha_1 u_t = f\left(x,t,u,v,u_x,v_x\right), \\ {}^{c}\partial_{0t}^{\gamma} v - \frac{1}{x}\left(xv_x\right)_x - \frac{1}{x}\frac{\partial}{\partial t}\left(xv_x\right)_x + z_2 u + \alpha_2 v_t = g\left(x,t,u,v,u_x,v_x\right), \\ u(x,0) = \varphi_1(x), \quad u_t(x,0) = \varphi_2(x), \\ v(x,0) = \psi_1(x), \quad v_t(x,0) = \psi_2(x), \\ u_x(b,t) = 0, \quad v_x(b,t) = 0, \\ \int_0^b xudx = 0, \quad \int_0^b xvdx = 0. \end{cases} \tag{1}$$

where: $Q = \Omega \times [0,T] = \{(x,t) : 0 < x < b, 0 \le t \le T\}$.
The functions f and g are $L^2\left(0,T;L^2_\rho(\Omega)\right.$ given Lipschitzian functions, that is there exist two positive constants δ_1 and δ_2 such that

$$\begin{aligned} \left|f\left(x,t,u_1,v_1,w_1,d_1\right) - f\left(x,t,u_2,v_2,w_2,d_2\right)\right| &\le \delta_1\left(|u_1-u_2| + |v_1-v_2| + |w_1-w_2| + |d_1-d_2|\right), \\ \left|g\left(x,t,u_1,v_1,w_1,d_1\right) - g\left(x,t,u_2,v_2,w_2,d_2\right)\right| &\le \delta_2\left(|u_1-u_2| + |v_1-v_2| + |w_1-w_2| + |d_1-d_2|\right), \end{aligned}$$

for all $(x,t) \in Q$. The functions $\varphi_1, \psi_1, \varphi_2$ and ψ_2 are in $H^1_\rho(\Omega)$, and z_1, z_2, α_1, and α_2 are positive constants. The operator ${}^{C}\partial_{0t}^{\beta}$ denotes the left Caputo fractional derivative, defined in the second section, where $1 < \beta, \gamma \le 2$.
Which is a model for coupled fractional systems with initial and integrated boundary conditions. This system is used as a case study to apply the q-HATM method for deriving approximate solutions, as well as the energy method for analyzing the existence, uniqueness, and stability of the solution. The aim is to highlight the effectiveness of the methodology used in dealing with complex fractional systems.
Chapter One: In this research, we establish the theoretical foundations by presenting the required functional spaces and formulating the fundamental equations in fractional calculus, while outlining the Energy Inequality method.
Chapter Two: This chapter is dedicated to studying the uniqueness of the solution for the fractional system. We reformulate the fractional system within the linear operational framework, which allows us not only to prove the uniqueness of the solution for the linear fractional system but also to discuss the implications of the derived energy estimate for the solution. We then focus on proving the solvability of the associated linear problem.
Chapter Three: In this chapter, we apply the q- Homotopy Analysis Transform Method (q-HATM) to develop a comprehensive methodology for solving coupled fractional system models. We then present numerical computation results that provide a series of illustrative examples to evaluate the effectiveness of the developed methodology. Finally, we review the key findings and conclusions.

CHAPTER

1

PRELIMINARIES AND FUNCTION SPACES

In this chapter, we recall a collection of function spaces and some basic tools related to them.

1.1 Functions spaces

DEFINITION 1.1: (see[21])

Let $L^2(\Omega)$ denoted the space of square-integrable function on Ω.A complex-valued function u defined on Ω is square-integrable if it is measurable and $|u|^2 \in L^1(\Omega)$. The norm on $L^2(\Omega)$ is defined as :

$$\|u\|_{L^2(\Omega)} = \left(\int_\Omega |u(x)|^2 dx \right)^{\frac{1}{2}}.$$

DEFINITION 1.2: (see[28, 23])

To investigote the posed problem, we introduce the needed function spaces. We denoted by $L^2_\rho(\Omega)$ the Hilbert space of weighted square integrable functions with inner product

$$(u, v)_{L^2_\rho(\Omega)} = (xu, v)_{L^2(\Omega)} = \int_\Omega xuvdx, \tag{1.1}$$

And with associated norm

$$\|u\|^2_{L^2_\rho(\Omega)} = \|\sqrt{x}u\|^2_{L^2(\Omega)} = \left(\int_\Omega xu^2 dx \right)^{\frac{1}{2}}, \tag{1.2}$$

And denoted by $H^1_\rho(\Omega)$ the weighted sobolev space with:

$$\|u\|^2_{H^1_\rho(\Omega)} = \|u\|^2_{L^2_\rho(\Omega)} + \|u_x\|^2_{L^2_\rho(\Omega)} < \infty, \tag{1.3}$$

Let $L^2(0, T; L^2_\rho(\Omega))$ be the space consisting of all mesurable functions:
$u : [0, T] \longrightarrow L^2_\rho(\Omega)$ with scalar product:

$$(u,v)_{L^2(0,T;L^2_\rho(\Omega))} = \int_0^T (u,v)_{L^2_\rho(\Omega)} dt, \tag{1.4}$$

And with associated finite norm

$$\|u\|_{L^2(0,T;L^2_\rho(\Omega))} = \int_0^T \|u\|^2_{L^2_\rho(\Omega)}, \tag{1.5}$$

And we denote by $L^2(0,T;H^1_\rho(\Omega))$ the space of functions which are square integrable in the Bochner sense, with the inner product

$$(u,v)_{L^2(0,T;H^1_\rho(\Omega))} = \int_0^T (u(.,t),v(.,t))_{H^1_\rho(\Omega)} dt, \tag{1.6}$$

And the associated norm is

$$\|u\|_{L^2(0,T;H^1_\rho(\Omega))} = \int_0^T \|u(.,t)\|^2_{L^2_\rho(\Omega)} dt + \int_0^T \|u_x(.,t)\|^2_{L^2_\rho(\Omega)} dt. \tag{1.7}$$

1.2 Preliminaries

1.2.1 Special Functions

DEFINITION 1.3

The Gamma function is denoted by $\Gamma(.)$ and defined by:

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt, \qquad (x \in \mathbb{C}, Re(x) > 0).$$

this integral is convergente if $Re(x) > 0$.

PROPOSITION 1.1

For every $x \in \mathbb{R}^*_+$ we have:

$$\Gamma(x+1) = x\Gamma(x).$$

In particular $\forall n \in \mathbb{N}^*$

$$\Gamma(n) = (n-1)!.$$

DEFINITION 1.4

The Mittag-Leffler function is defined by:

$$E_\alpha(x) = \sum_{k=0}^{+\infty} \frac{x^k}{\Gamma(\alpha k + 1)}, \qquad (\alpha > 0), \qquad x \in \mathbb{C}$$

The generalized Mittag-Leffler function is defined by:

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{+\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \qquad (\alpha, \beta > 0),$$

1.2.2 The fractional caputo derivative

We recall some definitions of fractional derivatives and fractional integral(see[19, 22]). For any positive integer n where: $n - 1 < \alpha < n$, the Caputo derivative, Riemann-Liouville integral, and fractional integral of order α are respectively defined as:

left Caputo derivative:

$$^{C}\partial^{\alpha}_{ot} v(x,t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{v^{(n)}(x,\tau)\mathrm{d}\tau}{(t-\tau)^{\alpha-n+1}}, \quad \forall t \in [0,T], \tag{1.8}$$

right Caputo derivative:

$$^{C}\partial^{\alpha}_{tT} v(x,t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^T \frac{v^{(n)}(x,\tau)\mathrm{d}\tau}{(\tau-t)^{\alpha-n+1}}, \quad \forall t \in [0,T], \tag{1.9}$$

Riemann-Liouville integral:

$$I^{\alpha}_{0t} v(x,t) = D^{-\alpha}_{0t} v(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{v(x,\tau)\mathrm{d}\tau}{(t-\tau)^{1-\alpha}}, \quad \forall t \in [0,T]. \tag{1.10}$$

LEMMA 1.1: (see[23])

Let a nonnegative absolutely continuous function $y(x,t)$ satisfy the inequality

$$^{C}\partial^{\beta}_{0t} y(x,t) \leq c y(x,t) + k(x,t), \quad 0 < \beta < 1,$$

for almost all $t \in [0,T]$, where c is positive and $k(x,t)$ is an integrable nonnegative function on $[0,T]$. Then

$$y(x,t) \leq y(x,0) E_\beta(ct^\beta) + \mathit{\Gamma}(\beta) E_{\beta,\beta}(ct^\beta) D^{-\beta}_{0t} k(x,t).$$

LEMMA 1.2: (see[23])

For any absolutely continuous function $v(x,t)$ on the interval $[0,T]$, the following inequality holds

$$v(x,t)^C\partial_{0t}^{\alpha}v(x,t) \geq \frac{1}{2}\,{}^C\partial_{0t}^{\alpha}v^2(x,t), \quad 0<\alpha<1.$$

LEMMA 1.3: Gronwall's lemma(see[23])

If f_1, f_2 and f_3 are nonnegative functions on the interval $[0,T]$. f_1 and f_2 are integrable on $[0,T]$, and f_3 is bounded nondecreasing on $[0,T]$, and C is a positive constant, then:

$$\int_0^t f_1(\tau)d\tau + f_2(t) \leqslant e^{Ct}f_3(t), \tag{1.11}$$

is a direct consequence of the inequality

$$\int_0^t f_1(\tau)d\tau + f_2(t) \leqslant f_3(t) + C\int_0^t f_2(\tau)d\tau. \tag{1.12}$$

We need to this **fractional inequality**(see[24],page 9) :

$$D_{0t}^{-\alpha}\|V\|^2_{L^2_\rho(\Omega)} \leq \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^t \|V\|^2_{L^2_\rho(\Omega)}d\tau. \tag{1.13}$$

and we have this **equality** :

$$\int_0^t \partial_{0t}^{\beta}\|V\|^2_{L^2_\rho(\Omega)}\,d\tau = D_{0t}^{\beta-1}\|V\|^2_{L^2_\rho(\Omega)} - \frac{t^{1-\beta}}{\Gamma(\beta)}\|V(x,0)\|^2_{L^2_\rho(\Omega)}\,. \tag{1.14}$$

Proof :

$${}^C\partial_{0t}^{\alpha}V(\tau) = {}^R\,\partial_{0t}^{\alpha}V(\tau) - \sum_{k=0}^{n-1}\frac{V^{(k)}(0)t^{k-\alpha}}{\Gamma(k-\alpha+1)}, \quad \text{where: } n-1<\alpha<n,$$

put: $\alpha=\beta-1$, we get :

$${}^C\partial_{0t}^{\beta-1}V(\tau) = {}^R\,\partial_{0t}^{\beta-1}V(\tau) - \sum_{k=0}^{n-1}\frac{V^{(k)}(0)t^{k-\beta+1}}{\Gamma(k-\beta+2)}, \quad \text{where: } 0<\beta-1<1,$$

then

$${}^C\partial_{0\tau}^{\beta-1}V(\tau) = {}^R\,\partial_{0\tau}^{\beta-1}V(\tau) - \frac{V(0)t^{1-\beta}}{\Gamma(2-\beta)},$$

change t to τ, and integrate the equation from 0 to t :

$$\int_0^t {}^C\partial_{0\tau}^{\beta-1}V(\tau)d\tau = \int_0^t {}^R\partial_{0\tau}^{\beta-1}V(\tau)d\tau - \frac{V(0)t^{2-\beta}}{(2-\beta)\Gamma(2-\beta)},$$

$$\int_0^t \ {}^C\partial_{0\tau}^{\beta-1} V(\tau)d\tau = \partial^{-1} \circ^R \partial_{0\tau}^{\beta-1} V(\tau) - V(0)\frac{t^{2-\beta}}{\Gamma(3-\beta)},$$

put $V(\tau) = V_\tau(\tau)$, we get :

$$\int_0^t \ {}^C\partial_{0\tau}^{\beta-1} V_\tau(\tau)d\tau =^R \partial_{0\tau}^{\beta-2} V_\tau(\tau) - V_\tau(0)\frac{t^{2-\beta}}{\Gamma(3-\beta)},$$

but $\beta - 2 < 0$ so the derivative $\partial^{\beta-2}$ is the integral $\mathrm{D}^{\beta-2}$ in the Rieman-Liouville sense:

$$\int_0^t \ {}^C\partial_{0\tau}^{\beta-1} V_\tau(\tau)d\tau = \mathrm{D}^{\beta-2} V_\tau(\tau) - V_\tau(0)\frac{t^{2-\beta}}{\Gamma(3-\beta)},$$

And **the Cauchyε-inequality**(see[23])

$$ab \leq \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2, \quad \forall \varepsilon > 0,$$

where a and b are positive numbers.
A Poincare type inequalities (see[23])

$$\|\Im_x(\xi u)\|^2_{L^2(\Omega)} \leqslant \frac{b^3}{2}\|u(.,t)\|^2_{L^2_\rho(\Omega)}, \tag{1.15}$$

$$\left\|\Im_x^2(\xi u)\right\|^2_{L^2(\Omega)} \leqslant \frac{b^2}{2}\left\|\Im_x(\xi u)\right\|^2_{L^2(\Omega)}, \tag{1.16}$$

$$\|\Im_x(\xi u)\|^2_{L^2_\rho(\Omega)} \leqslant b\left\|\Im_x(\xi u)\right\|^2_{L^2(\Omega)}, \tag{1.17}$$

where:

$$\Im_x(\xi v) = \int_0^x \xi v(\xi, t)d\xi, \tag{1.18}$$

and

$$\Im_x^2(\xi v) = \int_0^x \int_0^\xi \eta v(\eta, t)d\eta. \tag{1.19}$$

1.2.3 The relationship between orthogonality and density in Hilbert spaces

PROPOSITION 1.2

Let M be a vector subspace of a Hilbert space H. Then, M is dense in H **if and only if** $M^\perp = \{0\}$.

Proof:

Part 1: Assume M is dense in H. Let $f \in H$, and suppose $\{f_n\}$ is a sequence in M converging to f. For every $n \in \mathbb{N}$, we have:

$$\langle f, f_n \rangle_H = 0.$$

Taking the limit as $n \to \infty$, we obtain $\langle f, f \rangle_H = 0$, which implies $f = 0$. Thus, $M^\perp = \{0\}$.
Part 2: Conversely, assume $M^\perp = \{0\}$. Then:

$$(M^\perp)^\perp = \{0\}^\perp = H.$$

Since $M \subseteq (M^\perp)^\perp$, it follows that $M \subseteq H$. Moreover, $(M^\perp)^\perp \subseteq M$ implies $(M^\perp)^\perp = M$. Combining these results, we have:

$$M^\perp \subseteq M \implies H \subseteq M.$$

Therefore, M is dense in H.

1.3 The Energy Inequality Method

(see[32])The Energy Inequality method, also known as "A Priori Estimates," is an important mathematical framework for solving boundary value problems for differential equations, particularly Cauchy problems for hyperbolic equations. This method was developed by E. G. Petrovsky. The fundamental idea is based on transforming the problem into an operator form and analyzing the behavior of the solution through norm estimates.

First, the problem is formulated in the operator form:

$$Lu = \mathcal{F}, \quad \forall u \in D(L), \tag{1.20}$$

where the operator L maps from a Banach space B to a Hilbert space H. Then, the uniqueness of the solution is studied. This is done by establishing the following energy inequality:

$$\|u\|_B \leq c\|Lu\|_H, \tag{1.21}$$

This inequality bounds the norm of the solution u based on the norm of the equation Lu, and is typically obtained by multiplying the studied equation by an operator Mu (containing u, its derivatives, and a specific weight function) and applying computational operations. The choice of Mu depends heavily on the equation and the boundary conditions.
Afterwards, it is shown that L has a closure $\overline{L}$, so the solution u is considered a strong solution to the equation:

$$\overline{L}u = \mathcal{F}, \quad u \in D(\overline{L}), \tag{1.22}$$

The inequality (1.21) is extended to the closure $\overline{L}$ to obtain:

$$\|u\|_B \leq c\|\overline{L}u\|_H, \tag{1.23}$$

which confirms the uniqueness of the solution to equation (1.22).
To guarantee the existence of the solution, the image of the operator $\overline{L}$ must be closed and dense in H.
Explanation of the Existence Proof: To prove existence, we use the energy inequality (1.23) and the properties of the operator $\overline{L}$.
We analyze the image of $\overline{L}$ (the set $Im(\overline{L})$ of the outputs). If this image is closed in the space H (meaning any sequence of outputs that converges in H belongs to the image) and is dense (meaning it is "sufficiently present" throughout all of H), then this ensures that the data $\mathcal{F}$ for which we want to solve the equation $\overline{L}u = \mathcal{F}$ is "compatible" with the nature of the operator $\overline{L}$. The energy inequality, together with these properties, ensures that the mapping $u \mapsto \overline{L}u$ has a bounded inverse (bounded inverse) on its image. This allows us to

transform the equation $\overline{L}u = \mathcal{F}$ into a suitable form (such as applying the Cauchy-Lipschitz theorem or the Frechet-Kolmogorov theorem) which guarantees the existence of a solution $u \in D(\overline{L})$. In simple terms, if the data $\mathcal{F}$ is “consistent” with the nature of the process $\overline{L}$ (which is expressed by the closedness and density), then the energy inequality ensures the existence of a solution that is not “too large.”

CHAPTER

2

EXISTENCE AND UNIQUENESS OF SOLUTIONS

2.1 Reformulation of the Linear Problem

We consider a fractional coupled system of the form:

$$\begin{cases} \mathcal{L}_1(u,v) =^C \partial_{0t}^{\beta} u + -\frac{1}{x}\left(xu_x\right)_x - \frac{1}{x}\left(xu_x\right)_{xt} + v = f\left(x,t\right), \\ \mathcal{L}_2(u,v) =^C \partial_{0t}^{\gamma} v + -\frac{1}{x}\left(xv_x\right)_x - \frac{1}{x}\left(xv_x\right)_{xt} + u = g\left(x,t\right), \end{cases} \tag{2.1}$$

associate with the initial conditions:

$$\begin{aligned} \ell_1 u = u(x,0) = \varphi_1(x), \quad \ell_2 u = u_t(x,0) = \varphi_2(x), \\ \ell_3 v = v(x,0) = \psi_1(x), \quad \ell_4 v = v_t(x,0) = \psi_2(x), \end{aligned} \tag{2.2}$$

the classical and integral (non-local) boundary conditions:

$$\begin{aligned} u_x(b,t) = 0, \quad v_x(b,t) = 0, \\ \int_0^b xu dx = 0, \quad \int_0^b xv dx = 0. \end{aligned} \tag{2.3}$$

We assume that there exists a solution $(u,v) \in (C^{2,2}(\overline{Q}))^2$, the set of functions together with their partial derivatives of order 2 in x and t, are continuous on $\overline{Q}$.
The solution of system (2.1)-(2.3) can be regarded as the solution of operator equation

$$LW = \mathcal{F}, \tag{2.4}$$

where W, LW and $\mathcal{F}$ are respectively the pairs:

$$\begin{aligned} W &= (u,v), \\ LW &= \left(L_1 u, L_2 v\right), \\ \mathcal{F} &= \left(\mathcal{F}_1, \mathcal{F}_2\right), \end{aligned} \tag{2.5}$$

where

$$L_1 u = \left\{\mathcal{L}_1 u, \ell_1 u, \ell_2 u\right\}, \quad L_2 v = \left\{\mathcal{L}_2 v, \ell_3 v, \ell_4 v\right\}, \tag{2.6}$$

and

$$\mathcal{F}_1 = \left\{f, \varphi_1, \varphi_2\right\}, \quad \mathcal{F}_2 = \left\{g, \psi_1, \psi_2\right\}. \tag{2.7}$$

The operator L :

$$L : B = (B_1 \times B_2) \longrightarrow H = (H_1 \times H_2)$$

, where B is a Banach space consisting of all functions $(u, v) \in \left(L^2_\rho(Q)\right)^2$ satisfying conditions (2.3) and having the finite norm:

$$\|(u,v)\|^2_B = \sup_{0\leq t\leq T} \left\{ D^{\beta-2}_{0t} \left(\|u_t\|^2_{L^2_p(\Omega)} + \|\Im_x(\xi u_t)\|^2_{L^2_p(\Omega)} \right) + D^{\gamma-2}_{0t} \left(\|v_t\|^2_{L^2_p(\Omega)} + \|\Im_x(\xi v_t)\|^2_{L^2_p(\Omega)} \right) \right.$$
$$\left. + \|u\|^2_{L^2(0,T;H^1_\rho(\Omega))} + \|v\|^2_{L^2(0,T;H^1_\rho(\Omega))} \right\}, \tag{2.8}$$

and $H = H_1 \times H_2$ is the completion of the Hilbert space $\left\{ L^2_\rho(Q) \times L^2(0,T; H^1_\rho(\Omega)) \times L^2_\rho(\Omega) \right\}^2$ with respect to the norm

$$\|\mathcal{F}\|^2_H = \|f\|^2_{L^2_\rho(Q)} + \|\varphi_1\|^2_{L^2(0,T;H^1_\rho(\Omega))} + \|\varphi_2\|^2_{L^2_\rho(\Omega)} + \|g\|^2_{L^2_\rho(Q)} + \|\psi_1\|^2_{L^2(0,T;H^1_\rho(\Omega))} + \|\psi_2\|^2_{L^2_\rho(\Omega)} . \tag{2.9}$$

Let $D(L)$, be the domain of definition of the operator L, defined by:

$$D(L) = \left\{ (u,v) \in \left(L^2_\rho(Q)\right)^2 / {}^C\partial^\beta_{0t} u, {}^C\partial^\gamma_{0t} v, u_x, v_x, u_{xx}, v_{xx}, u_{tx}, v_{tx}, u_{txx}, v_{txx} \in L^2_\rho(Q) \right\}, \tag{2.10}$$

satisfying conditions (2.3).

2.2 Uniqueness of Solutions

In this section, we prove the uniqueness result for the fractional coupled system (2.1)-(2.3), that is we establish an energy inequality for the operator L and we give some of its consequences.

THEOREME 2.1

For any $(u, v) \in D(L)$, and $f(x,t), g(x,t) \in C(\overline{D})$ the solution of the problem (2.1)-(2.3) verify the inequality

$$\|(u,v)\|^2_B \leq C_7 \|\mathcal{F}\|^2_H, \tag{2.11}$$

where

$$C_0 = \frac{3 + \frac{b^6}{4}}{\min(1, \frac{1}{b})},$$
$$C_1 = \max\left\{ C_0, \left(1 + \frac{b^4}{2}\right) \frac{T^{2-\beta}}{\Gamma(3-\beta)}, \left(1 + \frac{b^4}{2}\right) \frac{T^{2-\gamma}}{\Gamma(3-\gamma)} \right\},$$
$$C_2 = \frac{b^2 C_1}{min\left\{1, \frac{3b^2}{4}\right\}},$$
$$C_3 = C_2 \exp(C_2 T),$$
$$C_4 = C_3 \Gamma(\beta - 1) E_{\beta-1,\beta-1}(C_3 T^{\beta-1}) \max\left(1, \frac{T^{\beta-1}}{\Gamma(\beta)}\right),$$

$$
\begin{aligned}
C_5 &= C_3 \max\left\{1 + C_4, 1 + \frac{C_4 T^{\beta-1}}{\Gamma(\beta)}\right\},\\
C_6 &= C_5 \Gamma(\gamma-1) E_{\gamma-1,\gamma-1}(C_5 T^{\gamma-1}) \max\left(1, \frac{T^{\gamma-1}}{\Gamma(\gamma)}\right),\\
C_7 &= C_5 \max\left\{1 + C_6, 1 + \frac{C_6 T^{\gamma-1}}{\Gamma(\gamma)}\right\}.
\end{aligned}
$$

---

proof

Taking the inner products in $L^2_\rho(\Omega)$, of the fractional partial differential equations in (2.1) and the operators $\mathcal{M}_1 u = u_t - \Im^2_x(\xi u_t)$, $\mathcal{M}_2 v = v_t - \Im^2_x(\xi v_t)$ respectively, we have

$$
\begin{aligned}
&\left({}^C\partial^\beta_{0t}u, u_t\right)_{L^2_p(\Omega)} - \left({}^C\partial^\beta_{0t}u, \Im^2_x(\xi u_t)\right)_{L^2_p(\Omega)} - \left(\frac{1}{x}\left(xu_x\right)_x, u_t\right)_{L^2_p(\Omega)} + \left(\frac{1}{x}\left(xu_x\right)_x, \Im^2_x(\xi u_t)\right)_{L^2_p(\Omega)}\\
&- \left(\frac{1}{x}\left(xu_x\right)_{xt}, u_t\right)_{L^2_p(\Omega)} + \left(\frac{1}{x}\left(xu_x\right)_{xt}, \Im^2_x(\xi u_t)\right)_{L^2_p(\Omega)} + (v, u_t)_{L^2_p(\Omega)} - \left(v, \Im^2_x(\xi u_t)\right)_{L^2_p(\Omega)}\\
&+ \left({}^C\partial^\gamma_{0t}v, v_t\right)_{L^2_p(\Omega)} - \left({}^C\partial^\gamma_{0t}v, \Im^2_x(\xi v_t)\right)_{L^2_p(\Omega)} - \left(\frac{1}{x}\left(xv_x\right)_x, v_t\right)_{L^2_p(\Omega)} + \left(\frac{1}{x}\left(xv_x\right)_x, \Im^2_x(\xi v_t)\right)_{L^2_p(\Omega)}\\
&- \left(\frac{1}{x}\left(xv_x\right)_{xt}, v_t\right)_{L^2_p(\Omega)} + \left(\frac{1}{x}\left(xv_x\right)_{xt}, \Im^2_x(\xi v_t)\right)_{L^2_p(\Omega)} + (u, v_t)_{L^2_p(\Omega)} - \left(u, \Im^2_x(\xi v_t)\right)_{L^2_p(\Omega)}\\
&= (f, u_t)_{L^2_\rho(\Omega)} - \left(f, \Im^2_x(\xi u_t)\right)_{L^2_\rho(\Omega)} + (g, v_t)_{L^2_\rho(\Omega)} - (g, \Im^2_x(\xi v_t))_{L^2_\rho(\Omega)}.
\end{aligned}
\tag{2.12}
$$

Using the conditions (2.3), and the standard inegration by parts of some terms in the LHS of (2.12), as follows:

$$
\begin{aligned}
-\left({}^C\partial^\beta_{0t}u, \Im^2_x(\xi u_t)\right)_{L^2_p(\Omega)} &= -\left(x^C\partial^\beta_{0t}u, \Im^2_x(\xi u_t),\right)_{L^2(\Omega)}\\
&= \left({}^C\partial^\beta_{0t}\left(\int_0^x \xi u\right), \Im_x(\xi u_t)\right)_{L^2(\Omega)}\\
&= \left({}^C\partial^\beta_{0t}\left(\Im_x(\xi u)\right), \Im_x(\xi u_t)\right)_{L^2(\Omega)}.
\end{aligned}
\tag{2.13}
$$

$$
\begin{aligned}
-\left(\frac{1}{x}\left(xu_x\right)_x, u_t\right)_{L^2_p(\Omega)} &= -\left(\left(xu_x\right)_x, u_t\right)_{L^2(\Omega)}\\
&= \left(xu_x, u_{tx}\right)_{L^2(\Omega)}\\
&= \frac{1}{2}\left(x, \left(u_x^2\right)_t\right)_{L^2(\Omega)},\\
&= \frac{1}{2}\frac{\partial}{\partial t}\|u_x\|^2_{L^2_p(\Omega)}.
\end{aligned}
\tag{2.14}
$$

$$\begin{aligned} +\left(\frac{1}{x}\left(xu_x\right)_x,\Im_x^2(\xi u_t)\right)_{L_p^2(\Omega)}, &= \left(\left(xu_x\right)_x,\Im_x^2(\xi u_t)\right)_{L^2(\Omega)}, \\ &= -\left(xu_x,\Im_x(\xi u_t)\right)_{L^2(\Omega)}, \\ &= -\left(u_x,\Im_x(\xi u_t)\right)_{L_p^2(\Omega)}. \end{aligned} \tag{2.15}$$

$$\begin{aligned} -\left(\frac{1}{x}\left(xu_x\right)_{xt},u_t\right)_{L_p^2(\Omega)}, &= -\left(\left(xu_x\right)_{xt},u_t\right)_{L^2(\Omega)}, \\ &= \left(xu_{tx},u_{tx}\right)_{L^2(\Omega)} \\ &= \|u_{xt}\|_{L_p^2(\Omega)}^2. \end{aligned} \tag{2.16}$$

$$\begin{aligned} +\left(\frac{1}{x}\left(xu_x\right)_{xt},\Im_x^2(\xi u_t)\right)_{L_p^2(\Omega)} &= +\left(\left(xu_x\right)_{xt},\Im_x^2(\xi u_t)\right)_{L^2(\Omega)}, \\ &= -\left(\left(xu_x\right)_t,\Im_x(\xi u_t)\right)_{L^2(\Omega)}, \\ &= -\left(u_{xt},\Im_x(\xi u_t)\right)_{L_p^2(\Omega)}. \end{aligned} \tag{2.17}$$

$$-\left(v,\Im_x^2(\xi u_t)\right)_{L_p^2(\Omega)} = -\left(\Im_x^2(\xi v),u_t\right)_{L_p^2(\Omega)}. \tag{2.18}$$

For the first term of(2.12), using lemma (1.2), we obtain

$$\left({}^C\partial_{0t}^{\beta}u,u_t\right)_{L_p^2(\Omega)} = \left({}^C\partial_{0t}^{\beta-1}u_t,u_t\right)_{L_p^2(\Omega)} \geq \frac{1}{2}\ {}^C\partial_{0t}^{\beta-1}\|u_t\|_{L_p^2(\Omega)}^2. \tag{2.19}$$

For equation (2.13), using lemma (1.2), then inequality (1.17), we get :

$$\left({}^C\partial_{0t}^{\beta}\left(\Im_x(\xi u)\right),\Im_x(\xi u_t)\right)_{L^2(\Omega)} = \left({}^C\partial_{0t}^{\beta-1}\left(\Im_x(\xi u_t)\right),\Im_x(\xi u_t)\right)_{L^2(\Omega)} \geq \frac{1}{2b}\ {}^C\partial_{0t}^{\beta-1}\|\Im_x(\xi u_t)\|_{L_p^2(\Omega)}^2. \tag{2.20}$$

Substitution of (2.13)-(2.20) into (2.12), and using the symmetry in this problem, to obtain:

$$\begin{aligned} &\frac{1}{2}\ {}^C\partial_{0t}^{\beta-1}\|u_t\|_{L_p^2(\Omega)}^2 + \frac{1}{2b}\ {}^C\partial_{0t}^{\beta-1}\|\Im_x(\xi u_t)\|_{L_p^2(\Omega)}^2 + \frac{1}{2}\ {}^C\partial_{0t}^{\gamma-1}\|v_t\|_{L_p^2(\Omega)}^2 + \frac{1}{2b}\ {}^C\partial_{0t}^{\gamma-1}\|\Im_x(\xi v_t)\|_{L_p^2(\Omega)}^2 \\ &\frac{1}{2}\frac{\partial}{\partial t}\|u_x\|_{L_p^2(\Omega)}^2 + \frac{1}{2}\frac{\partial}{\partial t}\|v_x\|_{L_p^2(\Omega)}^2 + \|u_{xt}\|_{L_p^2(\Omega)}^2 + \|v_{xt}\|_{L_p^2(\Omega)}^2 \\ &\leq \left(f,u_t\right)_{L_\rho^2(\Omega)} - \left(f,\Im_x^2(\xi u_t)\right)_{L_\rho^2(\Omega)} + \left(g,v_t\right)_{L_\rho^2(\Omega)} - \left(g,\Im_x^2(\xi v_t)\right)_{L_\rho^2(\Omega)} \\ &- \left(v,u_t\right)_{L_p^2(\Omega)} + \left(u_x,\Im_x(\xi u_t)\right)_{L_p^2(\Omega)} + \left(u_{xt},\Im_x(\xi u_t)\right)_{L_p^2(\Omega)} + \left(\Im_x^2(\xi v),u_t\right)_{L_p^2(\Omega)} \\ &- \left(u,v_t\right)_{L_p^2(\Omega)} + \left(v_x,\Im_x(\xi v_t)\right)_{L_p^2(\Omega)} + \left(v_{xt},\Im_x(\xi v_t)\right)_{L_p^2(\Omega)} + \left(\Im_x^2(\xi u),v_t\right)_{L_p^2(\Omega)}. \end{aligned} \tag{2.21}$$

By applying **Cauchy-ε-inequality** and Poincare type inequalities (1.15)-(1.17), for the terms in RHS of (2.21), gives:

$$(f, u_t)_{L^2_\rho(\Omega)} \leq \frac{1}{2}\|f\|^2_{L^2_p(\Omega)} + \frac{1}{2}\|u_t\|^2_{L^2_p(\Omega)}. \tag{2.22}$$

$$\begin{aligned} -\left(f, \Im^2_x(\xi u_t)\right)_{L^2_\rho(\Omega)} &\leq \frac{1}{2}\|f\|^2_{L^2_p(\Omega)} + \frac{1}{2}\|\Im^2_x(\xi u_t)\|^2_{L^2_p(\Omega)}, \\ &\leq \frac{1}{2}\|f\|^2_{L^2_p(\Omega)} + \frac{b}{2}\|\Im^2_x(\xi u_t)\|^2_{L^2(\Omega)}, \\ &\leq \frac{1}{2}\|f\|^2_{L^2_p(\Omega)} + \frac{b^3}{4}\|\Im_x(\xi u_t)\|^2_{L^2(\Omega)} \\ &\leq \frac{1}{2}\|f\|^2_{L^2_p(\Omega)} + \frac{b^6}{8}\|u_t\|^2_{L^2_p(\Omega)}. \end{aligned} \tag{2.23}$$

$$-(v, u_t)_{L^2_p(\Omega)} \leq \frac{1}{2}\|v\|^2_{L^2_p(\Omega)} + \frac{1}{2}\|u_t\|^2_{L^2_p(\Omega)}, \tag{2.24}$$

$$(u_x, \Im_x(\xi u_t))_{L^2_p(\Omega)} \leq \frac{1}{2}\|u_x\|^2_{L^2_p(\Omega)} + \frac{1}{2}\|\Im_x(\xi u_t)\|^2_{L^2_p(\Omega)}. \tag{2.25}$$

$$(u_{xt}, \Im_x(\xi u_t))_{L^2_p(\Omega)} \leq \|u_{xt}\|^2_{L^2_p(\Omega)} + \frac{1}{4}\|\Im_x(\xi u_t)\|^2_{L^2_p(\Omega)}. \quad (\mathit{choose} \;\; \epsilon = 2) \tag{2.26}$$

$$\begin{aligned} \left(\Im^2_x(\xi v), u_t\right)_{L^2_p(\Omega)} &\leq \frac{1}{2}\|\Im^2_x(\xi v)\|^2_{L^2_p(\Omega)} + \frac{1}{2}\|u_t\|^2_{L^2_p(\Omega)}, \\ &\leq \frac{b}{2}\|\Im^2_x(\xi v)\|^2_{L^2(\Omega)} + \frac{1}{2}\|u_t\|^2_{L^2_p(\Omega)}, \\ &\leq \frac{b^3}{2}\|\Im_x(\xi v)\|^2_{L^2(\Omega)} + \frac{1}{2}\|u_t\|^2_{L^2_p(\Omega)}, \\ &\leq \frac{b^6}{8}\|v\|^2_{L^2_p(\Omega)} + \frac{1}{2}\|u_t\|^2_{L^2_p(\Omega)}. \end{aligned} \tag{2.27}$$

Insertion of (2.22)-(2.27) into(2.21), yields

$$\begin{aligned} &\frac{1}{2}\ {}^C\partial^{\beta-1}_{0t}\left(\|u_t\|^2_{L^2_p(\Omega)} + \frac{1}{b}\|\Im_x(\xi u_t)\|^2_{L^2_p(\Omega)}\right) + \frac{1}{2}\ {}^C\partial^{\gamma-1}_{0t}\left(\|v_t\|^2_{L^2_p(\Omega)} + \frac{1}{b}\|\Im_x(\xi v_t)\|^2_{L^2_p(\Omega)}\right) \\ &+ \frac{1}{2}\frac{\partial}{\partial t}\|u_x\|^2_{L^2_p(\Omega)} + \frac{1}{2}\frac{\partial}{\partial t}\|v_x\|^2_{L^2_p(\Omega)} \\ &\leq \left(\frac{3}{2} + \frac{b^6}{8}\right)\|u_t\|^2_{L^2_p(\Omega)} + \frac{3}{4}\|\Im_x(\xi u_t)\|^2_{L^2_p(\Omega)} + \frac{1}{2}\|u_x\|^2_{L^2_p(\Omega)} + \left(\frac{b^6}{8} + \frac{1}{2}\right)\|v\|^2_{L^2_p(\Omega)} + \|f\|^2_{L^2_p(\Omega)} \\ &+ \left(\frac{3}{2} + \frac{b^6}{8}\right)\|v_t\|^2_{L^2_p(\Omega)} + \frac{3}{4}\|\Im_x(\xi v_t)\|^2_{L^2_p(\Omega)} + \frac{1}{2}\|v_x\|^2_{L^2_p(\Omega)} + \left(\frac{b^6}{8} + \frac{1}{2}\right)\|u\|^2_{L^2_p(\Omega)} + \|g\|^2_{L^2_p(\Omega)}, \end{aligned}$$

then:

$$\begin{aligned}
&{}^{C}\partial_{0t}^{\beta-1}\left(\|u_t\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi u_t)\|^2_{L^2_p(\Omega)}\right)+\ {}^{C}\partial_{0t}^{\gamma-1}\left(\|v_t\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi v_t)\|^2_{L^2_p(\Omega)}\right)\\
&+\frac{\partial}{\partial t}\|u_x\|^2_{L^2_p(\Omega)}+\frac{\partial}{\partial t}\|v_x\|^2_{L^2_p(\Omega)}\\
&\leq C_0\left(\|u_t\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi u_t)\|^2_{L^2_p(\Omega)}+\|u_x\|^2_{L^2_p(\Omega)}+\|v\|^2_{L^2_p(\Omega)}+\|f\|^2_{L^2_p(\Omega)}\right.\\
&\left.+\|v_t\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi v_t)\|^2_{L^2_p(\Omega)}+\|v_x\|^2_{L^2_p(\Omega)}+\|u\|^2_{L^2_p(\Omega)}+\|g\|^2_{L^2_p(\Omega)}\right),
\end{aligned}\tag{2.28}$$

where

$$C_0=\frac{3+\frac{b^6}{4}}{\min(1,\frac{1}{b})}.$$

replacing t by τ and integrating both sides of (2.28) with respect to τ from 0 to t, we have:

$$\begin{aligned}
&\int_0^t {}^{C}\partial_{0\tau}^{\beta-1}\left(\|u_\tau\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi u_\tau)\|^2_{L^2_p(\Omega)}\right)d\tau+\int_0^t {}^{C}\partial_{0\tau}^{\gamma-1}\left(\|v_\tau\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi v_\tau)\|^2_{L^2_p(\Omega)}\right)d\tau\\
&+\int_0^t\frac{\partial}{\partial\tau}\|u_x\|^2_{L^2_p(\Omega)}d\tau+\int_0^t\frac{\partial}{\partial\tau}\|v_x\|^2_{L^2_p(\Omega)}d\tau\\
&\leq C_0\int_0^t\left(\|u_\tau\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi u_\tau)\|^2_{L^2_p(\Omega)}+\|u_x\|^2_{L^2_p(\Omega)}+\|v\|^2_{L^2_p(\Omega)}+\|f\|^2_{L^2_p(\Omega)}\right.\\
&\left.+\|v_\tau\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi v_\tau)\|^2_{L^2_p(\Omega)}+\|v_x\|^2_{L^2_p(\Omega)}+\|u\|^2_{L^2_p(\Omega)}+\|g\|^2_{L^2_p(\Omega)}\right)d\tau,
\end{aligned}$$

using the equality (1.14), leads to:

$$\begin{aligned}
&D_{0t}^{\beta-2}\left(\|u_t\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi u_t)\|^2_{L^2_p(\Omega)}\right)+D_{0t}^{\gamma-2}\left(\|v_t\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi v_t)\|^2_{L^2_p(\Omega)}\right)+\|u_x\|^2_{L^2_p(\Omega)}-\|\varphi_{1x}\|^2_{L^2_p(\Omega)}\\
&+\|v_x\|^2_{L^2_p(\Omega)}-\|\psi_{1x}\|^2_{L^2_p(\Omega)}\\
&\leq C_0\left[\int_0^t\left(\|u_\tau\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi u_\tau)\|^2_{L^2_p(\Omega)}+\|u_x\|^2_{L^2_p(\Omega)}+\|v\|^2_{L^2_p(\Omega)}+\|f\|^2_{L^2_p(\Omega)}\right.\right.\\
&\left.\left.+\|v_\tau\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi v_\tau)\|^2_{L^2_p(\Omega)}+\|v_x\|^2_{L^2_p(\Omega)}+\|u\|^2_{L^2_p(\Omega)}+\|g\|^2_{L^2_p(\Omega)}\right)d\tau\right]\\
&+\frac{t^{2-\beta}}{\Gamma(2-\beta)}\|u_t(x,0)\|^2_{L^2_p(\Omega)}+\frac{t^{2-\beta}}{\Gamma(2-\beta)}\|\Im_x(\xi u_t(x,0))\|^2_{L^2_p(\Omega)}+\frac{t^{2-\gamma}}{\Gamma(2-\gamma)}\|v_t(x,0)\|^2_{L^2_p(\Omega)}\\
&+\frac{t^{2-\gamma}}{\Gamma(2-\gamma)}\|\Im_x(\xi v_t(x,0))\|^2_{L^2_p(\Omega)},
\end{aligned}$$

using initail condition (2.2) and poincare inqualities (1.15)-(1.17), we have:

$$\begin{aligned}
&D_{0t}^{\beta-2}\left(\|u_t\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi u_t)\|^2_{L^2_p(\Omega)}\right)+D_{0t}^{\gamma-2}\left(\|v_t\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi v_t)\|^2_{L^2_p(\Omega)}\right)+\|u_x\|^2_{L^2_p(\Omega)}+\|v_x\|^2_{L^2_p(\Omega)}\\
&\leq C_0\int_0^t\left(\|u_\tau\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi u_\tau)\|^2_{L^2_p(\Omega)}+\|u_x\|^2_{L^2_p(\Omega)}+\|v\|^2_{L^2_p(\Omega)}+\|f\|^2_{L^2_p(\Omega)}\right.\\
&\left.+\|v_\tau\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi v_\tau)\|^2_{L^2_p(\Omega)}+\|v_x\|^2_{L^2_p(\Omega)}+\|u\|^2_{L^2_p(\Omega)}+\|g\|^2_{L^2_p(\Omega)}\right)d\tau+\|\varphi_{1x}\|^2_{L^2_p(\Omega)}+\|\psi_{1x}\|^2_{L^2_p(\Omega)}\\
&+\frac{T^{2-\beta}}{\Gamma(2-\beta)}\|\varphi_2\|^2_{L^2_p(\Omega)}+\frac{b^4T^{2-\beta}}{2\Gamma(2-\beta)}\|\varphi_2\|^2_{L^2_p(\Omega)}+\frac{T^{2-\gamma}}{\Gamma(2-\gamma)}\|\psi_2\|^2_{L^2_p(\Omega)}\\
&+\frac{b^4T^{2-\gamma}}{2\Gamma(2-\gamma)}\|\psi_2\|^2_{L^2_p(\Omega)},
\end{aligned}$$

then:

$$D_{0t}^{\beta-2}\left(\|u_t\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi u_t)\|^2_{L^2_p(\Omega)}\right)+D_{0t}^{\gamma-2}\left(\|v_t\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi v_t)\|^2_{L^2_p(\Omega)}\right)+\|u_x\|^2_{L^2_p(\Omega)}+\|v_x\|^2_{L^2_p(\Omega)}$$
$$\leq C_1\left[\int_0^t\left(\|u_\tau\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi u_\tau)\|^2_{L^2_p(\Omega)}\right)d\tau+\int_0^t\left(\|u_x\|^2_{L^2_p(\Omega)}+\|u\|^2_{L^2_p(\Omega)}\right)d\tau+\int_0^t\|f\|^2_{L^2_p(\Omega)}d\tau\right.$$
$$+\int_0^t\left(\|v_\tau\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi v_\tau)\|^2_{L^2_p(\Omega)}\right)d\tau+\int_0^t\left(\|v_x\|^2_{L^2_p(\Omega)}+\|v\|^2_{L^2_p(\Omega)}\right)d\tau+\int_0^t\|g\|^2_{L^2_p(\Omega)}d\tau$$
$$\left.+\|\varphi_2\|^2_{L^2_p(\Omega)}+\|\psi_2\|^2_{L^2_p(\Omega)}+\|\varphi_{1x}\|^2_{L^2_p(\Omega)}+\|\psi_{1x}\|^2_{L^2_p(\Omega)}\right],\tag{2.29}$$

where:

$$C_1=\max\left\{C_0,\left(1+\frac{b^4}{2}\right)\frac{T^{2-\beta}}{\Gamma(3-\beta)},\left(1+\frac{b^4}{2}\right)\frac{T^{2-\gamma}}{\Gamma(3-\gamma)}\right\}.$$

Multiplying (2.29) by b^2, then adding the following elementary inequalities :

$$\|u(x,t)\|^2_{L^2_\rho(\Omega)}\leq\frac{b^2}{4}\|u_x(x,t)\|^2_{L^2_\rho(\Omega)},\tag{2.30}$$

$$\|v(x,t)\|^2_{L^2_\rho(\Omega)}\leq\frac{b^2}{4}\|v_x(x,t)\|^2_{L^2_\rho(\Omega)},\tag{2.31}$$

Inequality (2.29) become:

$$b^2D_{0t}^{\beta-2}\left(\|u_t\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi u_t)\|^2_{L^2_p(\Omega)}\right)+b^2D_{0t}^{\gamma-2}\left(\|v_t\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi v_t)\|^2_{L^2_p(\Omega)}\right)$$
$$+b^2\|u_x(x,t)\|^2_{L^2_p(\Omega)}+\|u(x,t)\|^2_{L^2_\rho(\Omega)}+b^2\|v_x(x,t)\|^2_{L^2_p(\Omega)}+\|v(x,t)\|^2_{L^2_\rho(\Omega)}$$
$$\leq b^2C_1\left[\int_0^t\left(\|u_\tau\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi u_\tau)\|^2_{L^2_p(\Omega)}\right)d\tau+\int_0^t\left(\|u_x\|^2_{L^2_p(\Omega)}+\|u\|^2_{L^2_p(\Omega)}\right)d\tau+\int_0^t\|f\|^2_{L^2_p(\Omega)}d\tau\right.$$
$$+\int_0^t\left(\|v_\tau\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi v_\tau)\|^2_{L^2_p(\Omega)}\right)d\tau+\int_0^t\left(\|v_x\|^2_{L^2_p(\Omega)}+\|v\|^2_{L^2_p(\Omega)}\right)d\tau+\int_0^t\|g\|^2_{L^2_p(\Omega)}d\tau$$
$$\left.+\|\varphi_2\|^2_{L^2_p(\Omega)}+\|\psi_2\|^2_{L^2_p(\Omega)}+\|\varphi_{1x}\|^2_{L^2_p(\Omega)}+\|\psi_{1x}\|^2_{L^2_p(\Omega)}\right]+\frac{b^2}{4}\|u_x(x,t)\|^2_{L^2_\rho(\Omega)}+\frac{b^2}{4}\|v_x(x,t)\|^2_{L^2_\rho(\Omega)},$$

return the lost two terms to the LHS:

$$b^2D_{0t}^{\beta-2}\left(\|u_t\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi u_t)\|^2_{L^2_p(\Omega)}\right)+b^2D_{0t}^{\gamma-2}\left(\|v_t\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi v_t)\|^2_{L^2_p(\Omega)}\right)$$
$$+\left(b^2-\frac{b^2}{4}\right)\|u_x(x,t)\|^2_{L^2_p(\Omega)}+\|u(x,t)\|^2_{L^2_\rho(\Omega)}+\left(b^2-\frac{b^2}{4}\right)\|v_x(x,t)\|^2_{L^2_p(\Omega)}+\|v(x,t)\|^2_{L^2_\rho(\Omega)}$$
$$\leq b^2C_1\left[\int_0^t\left(\|u_\tau\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi u_\tau)\|^2_{L^2_p(\Omega)}\right)d\tau+\int_0^t\left(\|u_x\|^2_{L^2_p(\Omega)}+\|u\|^2_{L^2_p(\Omega)}\right)d\tau+\int_0^t\|f\|^2_{L^2_p(\Omega)}d\tau\right.$$
$$+\int_0^t\left(\|v_\tau\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi v_\tau)\|^2_{L^2_p(\Omega)}\right)d\tau+\int_0^t\left(\|v_x\|^2_{L^2_p(\Omega)}+\|v\|^2_{L^2_p(\Omega)}\right)d\tau+\int_0^t\|g\|^2_{L^2_p(\Omega)}d\tau$$
$$\left.+\|\varphi_2\|^2_{L^2_p(\Omega)}+\|\psi_2\|^2_{L^2_p(\Omega)}+\|\varphi_{1x}\|^2_{L^2_p(\Omega)}+\|\psi_{1x}\|^2_{L^2_p(\Omega)}\right],$$

then:

$$D_{0t}^{\beta-2}\left(\|u_t\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi u_t)\|^2_{L^2_p(\Omega)}\right)+D_{0t}^{\gamma-2}\left(\|v_t\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi v_t)\|^2_{L^2_p(\Omega)}\right)+\|u\|^2_{L^2(0,T;H^1_\rho(\Omega))}$$
$$+\|v\|^2_{L^2(0,T;H^1_\rho(\Omega))}$$
$$\leq C_2\left[\int_0^t\left(\|u_\tau\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi u_\tau)\|^2_{L^2_p(\Omega)}\right)d\tau+\int_0^t\left(\|u\|^2_{L^2(0,T;H^1_\rho(\Omega))}+\|v\|^2_{L^2(0,T;H^1_\rho(\Omega))}\right)d\tau\right.$$
$$+\int_0^t\|f\|^2_{L^2_p(\Omega)}d\tau+\int_0^t\left(\|v_\tau\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi v_\tau)\|^2_{L^2_p(\Omega)}\right)d\tau+\int_0^t\|g\|^2_{L^2_p(\Omega)}d\tau+\|\varphi_2\|^2_{L^2_p(\Omega)}+\|\psi_2\|^2_{L^2_p(\Omega)}$$
$$\left.+\|\varphi_{1x}\|^2_{L^2_p(\Omega)}+\|\psi_{1x}\|^2_{L^2_p(\Omega)}\right],\tag{2.32}$$

where:

$$C_2 = \frac{b^2 C_1}{min\left\{1, \frac{3b^2}{4}\right\}}.$$

Using **the Gronwall's lemma** (1.3), by taking:

$$f_2(t) = \|u\|^2_{L^2(0,T;H^1_\rho(\Omega))} + \|v\|^2_{L^2(0,T;H^1_\rho(\Omega))},$$

it leads to:

$$\begin{aligned}
&D^{\beta-2}_{0t}\left(\|u_t\|^2_{L^2_p(\Omega)} + \|\Im_x(\xi u_t)\|^2_{L^2_p(\Omega)}\right) + D^{\gamma-2}_{0t}\left(\|v_t\|^2_{L^2_p(\Omega)} + \|\Im_x(\xi v_t)\|^2_{L^2_p(\Omega)}\right) + \|u\|^2_{L^2(0,T;H^1_\rho(\Omega))} \\
&+ \|v\|^2_{L^2(0,T;H^1_\rho(\Omega))} \\
&\leq C_3\left[\int_0^t \left(\|u_\tau\|^2_{L^2_p(\Omega)} + \|\Im_x(\xi u_\tau)\|^2_{L^2_p(\Omega)}\right)d\tau + \int_0^t \left(\|v_\tau\|^2_{L^2_p(\Omega)} + \|\Im_x(\xi v_\tau)\|^2_{L^2_p(\Omega)}\right)d\tau\right. \\
&\left.+ \int_0^t \|g\|^2_{L^2_p(\Omega)}d\tau + \int_0^t \|f\|^2_{L^2_p(\Omega)}d\tau + \|\varphi_2\|^2_{L^2_p(\Omega)} + \|\psi_2\|^2_{L^2_p(\Omega)} + \|\varphi_1\|^2_{L^2(0,T;H^1_\rho(\Omega))} + \|\psi_1\|^2_{L^2(0,T;H^1_\rho(\Omega))}\right],
\end{aligned} \tag{2.33}$$

where:

$$C_3 = C_2 \exp(C_2 T).$$

Then, by omitting some terms on the LHS in inequality (2.33), we obtain:

$$\begin{aligned}
&D^{\beta-2}_{0t}\left(\|u_t\|^2_{L^2_p(\Omega)} + \|\Im_x(\xi u_t)\|^2_{L^2_p(\Omega)}\right) \\
&\leq C_3\left[\int_0^t \left(\|u_\tau\|^2_{L^2_p(\Omega)} + \|\Im_x(\xi u_\tau)\|^2_{L^2_p(\Omega)}\right)d\tau + \int_0^t \left(\|v_\tau\|^2_{L^2_p(\Omega)} + \|\Im_x(\xi v_\tau)\|^2_{L^2_p(\Omega)}\right)d\tau\right. \\
&\left.+ \int_0^t \|g\|^2_{L^2_p(\Omega)}d\tau + \int_0^t \|f\|^2_{L^2_p(\Omega)}d\tau + \|\varphi_2\|^2_{L^2_p(\Omega)} + \|\psi_2\|^2_{L^2_p(\Omega)} + \|\varphi_1\|^2_{L^2(0,T;H^1_\rho(\Omega))} + \|\psi_1\|^2_{L^2(0,T;H^1_\rho(\Omega))}\right],
\end{aligned} \tag{2.34}$$

by **Lemma** (1.1), where :

$$\begin{aligned}
y(t) &= \int_0^t \left(\|u_\tau\|^2_{L^2_p(\Omega)} + \|\Im_x(\xi u_\tau)\|^2_{L^2_p(\Omega)}\right)d\tau, \quad y(0) = 0, \\
\partial^{\beta-1}_{0t} y(t) &= D^{\beta-2}_{0t}\left(\|u_t\|^2_{L^2_p(\Omega)} + \|\Im_x(\xi u_t)\|^2_{L^2_p(\Omega)}\right),
\end{aligned}$$

and from (2.34), we obtain the inequality:

$$\begin{aligned}
&\int_0^t \left(\|u_\tau\|^2_{L^2_p(\Omega)} + \|\Im_x(\xi u_\tau)\|^2_{L^2_p(\Omega)}\right)d\tau \\
&\leq C_3\Gamma(\beta-1)E_{\beta-1,\beta-1}(C_3 T^{\beta-1}) D^{1-\beta}_{0t}\left[\int_0^t \left(\|v_\tau\|^2_{L^2_p(\Omega)} + \|\Im_x(\xi v_\tau)\|^2_{L^2_p(\Omega)}\right)d\tau\right. \\
&\left.+ \int_0^t \|g\|^2_{L^2_p(\Omega)}d\tau + \int_0^t \|f\|^2_{L^2_p(\Omega)}d\tau + \|\varphi_2\|^2_{L^2_p(\Omega)} + \|\psi_2\|^2_{L^2_p(\Omega)} + \|\varphi_1\|^2_{L^2(0,T;H^1_\rho(\Omega))} + \|\psi_1\|^2_{L^2(0,T;H^1_\rho(\Omega))}\right],
\end{aligned}$$

then:

$$\begin{aligned}
&\int_0^t \left(\|u_\tau\|^2_{L^2_p(\Omega)} + \|\Im_x(\xi u_\tau)\|^2_{L^2_p(\Omega)}\right)d\tau \\
&\leq C_3\Gamma(\beta-1)E_{\beta-1,\beta-1}(C_3 T^{\beta-1})\left[D^{-\beta}_{0t}\|f\|^2_{L^2_p(\Omega)} + D^{-\beta}_{0t}\|g\|^2_{L^2_p(\Omega)} + D^{-\beta}_{0t}\left(\|v_\tau\|^2_{L^2_p(\Omega)} + \|\Im_x(\xi v_\tau)\|^2_{L^2_p(\Omega)}\right)\right. \\
&\left.+\frac{T^{\beta-1}}{\Gamma(\beta)}\|\varphi_2\|^2_{L^2_p(\Omega)} + \frac{T^{\beta-1}}{\Gamma(\beta)}\|\psi_2\|^2_{L^2_p(\Omega)} + \frac{T^{\beta-1}}{\Gamma(\beta)}\|\varphi_1\|^2_{L^2(0,T;H^1_\rho(\Omega))} + \frac{T^{\beta-1}}{\Gamma(\beta)}\|\psi_1\|^2_{L^2(0,T;H^1_\rho(\Omega))}\right],
\end{aligned}$$

we get:

$$\begin{aligned}
&\int_0^t \left(\|u_\tau\|^2_{L^2_p(\Omega)} + \|\Im_x(\xi u_\tau)\|^2_{L^2_p(\Omega)}\right) d\tau \le C_4 \left(D^{-\beta}_{0t}\|f\|^2_{L^2_p(\Omega)} + D^{-\beta}_{0t}\|g\|^2_{L^2_p(\Omega)}\right. \\
&\left. + D^{-\beta}_{0t}\left(\|v_\tau\|^2_{L^2_p(\Omega)} + \|\Im_x(\xi v_\tau)\|^2_{L^2_p(\Omega)}\right) + \|\varphi_2\|^2_{L^2_p(\Omega)} + \|\psi_2\|^2_{L^2_p(\Omega)} + \|\varphi_1\|^2_{L^2(0,T;H^1_\rho(\Omega))} + \|\psi_1\|^2_{L^2(0,T;H^1_\rho(\Omega))}\right),
\end{aligned} \tag{2.35}$$

where:

$$C_4 = C_3\Gamma(\beta-1)E_{\beta-1,\beta-1}(C_3T^{\beta-1})\max\left(1, \frac{T^{\beta-1}}{\Gamma(\beta)}\right).$$

Combination of (2.33) and (2.35) yields:

$$\begin{aligned}
&D^{\beta-2}_{0t}\left(\|u_t\|^2_{L^2_p(\Omega)} + \|\Im_x(\xi u_t)\|^2_{L^2_p(\Omega)}\right) + D^{\gamma-2}_{0t}\left(\|v_t\|^2_{L^2_p(\Omega)} + \|\Im_x(\xi v_t)\|^2_{L^2_p(\Omega)}\right) + \|u\|^2_{L^2(0,T;H^1_\rho(\Omega))} \\
&+ \|v\|^2_{L^2(0,T;H^1_\rho(\Omega))} \\
&\le C_3\left[\int_0^t \left(\|v_\tau\|^2_{L^2_p(\Omega)} + \|\Im_x(\xi v_\tau)\|^2_{L^2_p(\Omega)}\right) d\tau + \int_0^t \|g\|^2_{L^2_p(\Omega)} d\tau + \int_0^t \|f\|^2_{L^2_p(\Omega)} d\tau + \|\varphi_2\|^2_{L^2_p(\Omega)} + \|\psi_2\|^2_{L^2_p(\Omega)}\right. \\
&\left. + \|\varphi_1\|^2_{L^2(0,T;H^1_\rho(\Omega))} + \|\psi_1\|^2_{L^2(0,T;H^1_\rho(\Omega))}\right] + C_3C_4\left[D^{-\beta}_{0t}\|f\|^2_{L^2_p(\Omega)} + D^{-\beta}_{0t}\|g\|^2_{L^2_p(\Omega)} + \|\varphi_2\|^2_{L^2_p(\Omega)}\right. \\
&\left. + D^{-\beta}_{0t}\left(\|v_\tau\|^2_{L^2_p(\Omega)} + \|\Im_x(\xi v_\tau)\|^2_{L^2_p(\Omega)}\right) + \|\psi_2\|^2_{L^2_p(\Omega)} + \|\varphi_1\|^2_{L^2(0,T;H^1_\rho(\Omega))} + \|\psi_1\|^2_{L^2(0,T;H^1_\rho(\Omega))}\right],
\end{aligned} \tag{2.36}$$

In virtue of the inequality (1.17), we can simplify (2.36) as follows:

$$\begin{aligned}
&D^{\beta-2}_{0t}\left(\|u_t\|^2_{L^2_p(\Omega)} + \|\Im_x(\xi u_t)\|^2_{L^2_p(\Omega)}\right) + D^{\gamma-2}_{0t}\left(\|v_t\|^2_{L^2_p(\Omega)} + \|\Im_x(\xi v_t)\|^2_{L^2_p(\Omega)}\right) + \|u\|^2_{L^2(0,T;H^1_\rho(\Omega))} \\
&+ \|v\|^2_{L^2(0,T;H^1_\rho(\Omega))} \\
&\le C_3\left[\int_0^t \left(\|v_\tau\|^2_{L^2_p(\Omega)} + \|\Im_x(\xi v_\tau)\|^2_{L^2_p(\Omega)}\right) d\tau + \int_0^t \|g\|^2_{L^2_p(\Omega)} d\tau + \int_0^t \|f\|^2_{L^2_p(\Omega)} d\tau\right. \\
&\left. + \|\varphi_2\|^2_{L^2_p(\Omega)} + \|\psi_2\|^2_{L^2_p(\Omega)} + \|\varphi_1\|^2_{L^2(0,T;H^1_\rho(\Omega))} + \|\psi_1\|^2_{L^2(0,T;H^1_\rho(\Omega))}\right] + C_3C_4\left[\int_0^t \|f\|^2_{L^2_p(\Omega)}\, d\tau\right. \\
&+ \int_0^t \|g\|^2_{L^2_p(\Omega)} d\tau + \int_0^t \left(\|v_\tau\|^2_{L^2_p(\Omega)} + \|\Im_x(\xi v_\tau)\|^2_{L^2_p(\Omega)}\right) d\tau + \frac{T^{\beta-1}}{\Gamma(\beta)}\|\varphi_2\|^2_{L^2_p(\Omega)} + \frac{T^{\beta-1}}{\Gamma(\beta)}\|\psi_2\|^2_{L^2_p(\Omega)} \\
&\left. + \frac{T^{\beta-1}}{\Gamma(\beta)}\|\varphi_1\|^2_{L^2(0,T;H^1_\rho(\Omega))} + \frac{T^{\beta-1}}{\Gamma(\beta)}\|\psi_1\|^2_{L^2(0,T;H^1_\rho(\Omega))}\right],
\end{aligned}$$

then:

$$\begin{aligned}
&D^{\gamma-2}_{0t}\left(\|v_t\|^2_{L^2_p(\Omega)} + \|\Im_x(\xi v_t)\|^2_{L^2_p(\Omega)}\right) + D^{\beta-2}_{0t}\left(\|u_t\|^2_{L^2_p(\Omega)} + \|\Im_x(\xi u_t)\|^2_{L^2_p(\Omega)}\right) + \|u\|^2_{L^2(0,T;H^1_\rho(\Omega))} \\
&+ \|v\|^2_{L^2(0,T;H^1_\rho(\Omega))} \\
&\le C_5\left[\int_0^t \|f\|^2_{L^2_p(\Omega)} d\tau + \int_0^t \|g\|^2_{L^2_p(\Omega)} d\tau + \int_0^t \left(\|v_\tau\|^2_{L^2_p(\Omega)} + \|\Im_x(\xi v_\tau)\|^2_{L^2_p(\Omega)}\right) d\tau\right. \\
&\left. + \|\varphi_2\|^2_{L^2_p(\Omega)} + \|\psi_2\|^2_{L^2_p(\Omega)} + \|\varphi_1\|^2_{L^2(0,T;H^1_\rho(\Omega))} + \|\psi_1\|^2_{L^2(0,T;H^1_\rho(\Omega))}\right].
\end{aligned} \tag{2.37}$$

where:

$$C_5 = C_3\max\left\{1 + C_4, 1 + \frac{C_4T^{\beta-1}}{\Gamma(\beta)}\right\}$$

.

Now, we discard the last third terms in LHS of (2.37), we have

$$\begin{aligned}&D_{0t}^{\gamma-2}\left(\|v_t\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi v_t)\|^2_{L^2_p(\Omega)}\right)\\&\leq C_5\Big[\int_0^t\|f\|^2_{L^2_p(\Omega)}d\tau+\int_0^t\|g\|^2_{L^2_p(\Omega)}d\tau+\int_0^t\left(\|v_\tau\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi v_\tau)\|^2_{L^2_p(\Omega)}\right)d\tau\\&+\|\varphi_2\|^2_{L^2_p(\Omega)}+\|\psi_2\|^2_{L^2_p(\Omega)}+\|\varphi_1\|^2_{L^2(0,T;H^1_\rho(\Omega))}+\|\psi_1\|^2_{L^2(0,T;H^1_\rho(\Omega))}\Big],\end{aligned}\tag{2.38}$$

by applying **Lemma (1.1)**, where :

$$\begin{aligned}y(t)&=\int_0^t\left(\|v_\tau\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi v_\tau)\|^2_{L^2_p(\Omega)}\right)d\tau,\quad y(0)=0,\\\partial_{0t}^{\gamma-1}y(t)&=D_{0t}^{\gamma-2}\left(\|v_t\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi v_t)\|^2_{L^2_p(\Omega)}\right),\end{aligned}$$

from (2.38), we obtain the inequality :

$$\begin{aligned}&\int_0^t\left(\|v_t\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi v_t)\|^2_{L^2_p(\Omega)}\right)\\&\leq C_5\Gamma(\gamma-1)E_{\gamma-1,\gamma-1}(C_5t^{\gamma-1})D_{0t}^{1-\gamma}\Big[\int_0^t\|f\|^2_{L^2_p(\Omega)}d\tau+\int_0^t\|g\|^2_{L^2_p(\Omega)}d\tau+\|\varphi_2\|^2_{L^2_p(\Omega)}\\&+\|\psi_2\|^2_{L^2_p(\Omega)}+\|\varphi_1\|^2_{L^2(0,T;H^1_\rho(\Omega))}+\|\psi_1\|^2_{L^2(0,T;H^1_\rho(\Omega))}\Big],\end{aligned}$$

is results in :

$$\begin{aligned}&\int_0^t\left(\|v_\tau\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi v_\tau)\|^2_{L^2_p(\Omega)}\right)d\tau\\&\leq C_5\Gamma(\gamma-1)E_{\gamma-1,\gamma-1}(C_5t^{\gamma-1})\Bigg(D_{0t}^{-\gamma}\|f\|^2_{L^2_p(\Omega)}+D_{0t}^{-\gamma}\|g\|^2_{L^2_p(\Omega)}+\frac{t^{\gamma-1}}{\Gamma(\gamma)}\|\varphi_2\|^2_{L^2_p(\Omega)}\\&+\frac{t^{\gamma-1}}{\Gamma(\gamma)}\|\psi_2\|^2_{L^2_p(\Omega)}+\frac{t^{\gamma-1}}{\Gamma(\gamma)}\|\varphi_1\|^2_{L^2(0,T;H^1_\rho(\Omega))}+\frac{t^{\gamma-1}}{\Gamma(\gamma)}\|\psi_1\|^2_{L^2(0,T;H^1_\rho(\Omega))}\Bigg),\end{aligned}$$

then :

$$\begin{aligned}&\int_0^t\left(\|v_\tau\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi v_\tau)\|^2_{L^2_p(\Omega)}\right)d\tau\\&\leq C_6\Bigg(D_{0t}^{-\gamma}\|f\|^2_{L^2_p(\Omega)}+D_{0t}^{-\gamma}\|g\|^2_{L^2_p(\Omega)}+\|\varphi_2\|^2_{L^2_p(\Omega)}+\|\psi_2\|^2_{L^2_p(\Omega)}+\|\varphi_1\|^2_{L^2(0,T;H^1_\rho(\Omega))}\\&+\|\psi_1\|^2_{L^2(0,T;H^1_\rho(\Omega))}\Bigg),\end{aligned}\tag{2.39}$$

where:

$$C_6=C_5\Gamma(\gamma-1)E_{\gamma-1,\gamma-1}(C_5T^{\gamma-1})\max\left(1,\frac{T^{\gamma-1}}{\Gamma(\gamma)}\right).$$

Combination of (2.39) and (2.37), then using the inequality (2.14) to find the estimation:

$$
\begin{aligned}
&D_{0t}^{\gamma-2}\left(\|v_t\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi v_t)\|^2_{L^2_p(\Omega)}\right)+D_{0t}^{\beta-2}\left(\|u_t\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi u_t)\|^2_{L^2_p(\Omega)}\right)+\|u\|^2_{L^2(0,T;H^1_\rho(\Omega))}\\
&+\|v\|^2_{L^2(0,T;H^1_\rho(\Omega))}\\
&\leq C_7\Bigg(\int_0^t\|f\|^2_{L^2_p(\Omega)}d\tau+\int_0^t\|g\|^2_{L^2_p(\Omega)}d\tau+\|\varphi_2\|^2_{L^2_p(\Omega)}+\|\psi_2\|^2_{L^2_p(\Omega)}+\|\varphi_1\|^2_{L^2(0,T;H^1_\rho(\Omega))}\\
&+\|\psi_1\|^2_{L^2(0,T;H^1_\rho(\Omega))}\Bigg),\\
&\leq C_7\Bigg(\int_0^T\|f\|^2_{L^2_p(\Omega)}d\tau+\int_0^T\|g\|^2_{L^2_p(\Omega)}d\tau+\|\varphi_2\|^2_{L^2_p(\Omega)}+\|\psi_2\|^2_{L^2_p(\Omega)}\\
&+\|\varphi_1\|^2_{L^2(0,T;H^1_\rho(\Omega))}+\|\psi_1\|^2_{L^2(0,T;H^1_\rho(\Omega))}\Bigg),
\end{aligned}
\tag{2.40}
$$

where:

$$C_7=C_5\max\left\{1+C_6,1+\frac{C_6T^{\gamma-1}}{\Gamma(\gamma)}\right\}.$$

The RHS side of the above inequality (2.40) does not depend on t. By taking the supremum with respect to t over 0 to T, we get :

$$
\begin{aligned}
&\sup_{0\leqslant t\leqslant T}\Big\{D_{0t}^{\gamma-2}\left(\|v_t\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi v_t)\|^2_{L^2_p(\Omega)}\right)+D_{0t}^{\beta-2}\left(\|u_t\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi u_t)\|^2_{L^2_p(\Omega)}\right)\\
&+\|u\|^2_{L^2(0,T;H^1_\rho(\Omega))}+\|v\|^2_{L^2(0,T;H^1_\rho(\Omega))}\Big\}\\
&\leq C_7\Bigg(\int_0^T\|f\|^2_{L^2_p(\Omega)}d\tau+\int_0^T\|g\|^2_{L^2_p(\Omega)}d\tau+\|\varphi_2\|^2_{L^2_p(\Omega)}+\|\psi_2\|^2_{L^2_p(\Omega)}+\|\varphi_1\|^2_{L^2(0,T;H^1_\rho(\Omega))}\\
&+\|\psi_1\|^2_{L^2(0,T;H^1_\rho(\Omega))}\Bigg),
\end{aligned}
\tag{2.41}
$$

so :

$$\|(u,v)\|^2_B\leqslant C_7\|\mathcal{F}\|^2_H.$$

□

COROLLARY 2.1

The solution of problem (2.1)-(2.3) is unique.

proof

Let (u_1,v_1) and (u_2,v_2) be two solution of problem (2.1).

$$\left.\begin{array}{l}L(u_1,v_1)=\mathcal{F}\\L(u_2,v_2)=\mathcal{F}\end{array}\right\}\Rightarrow L(u_1,v_1)-L(u_2,v_2)=0,$$

we have that L is linear operator ,meaning it satisfies :

$$L\left((u_1,v_1)-(u_2,v_2)\right)=0,$$

and based on statement (2.11), we find :

$\|(u_1,v_1)-(u_2,v_2)\|_B^2 \leqslant 0 \Rightarrow \|(u_1,v_1)-(u_2,v_2)\|_B^2 = 0 \Rightarrow (u_1,v_1)-(u_2,v_2) = 0 \Rightarrow (u_1,v_1) = (u_2,v_2).$

therefore, the coupled system has a unique solution. ■

2.3 Existence of Solutions

To prove the existence of the solution,using the energy inequality method we study the propreties of operateur L.
The proof of existence relies on the following three steps:

1. The operator $L : B \to H$ is closed.
2. $\overline{L}u = H$ and $\mathrm{R}(\overline{L}) = H$ is a closed subset in H, and $\mathrm{R}(L) = \mathrm{R}(\overline{L})$ and $\overline{L}^{-1} = \overline{L^{-1}}$.
3. $\mathrm{R}(L)$ is dense in H.

PROPOSITION 2.1

The operator $L : B \longrightarrow H$ is closable .

These are some consequences of theorem (2.1) .
Let $\bar{L}$ be the closure of L ,and $D(\bar{L})$ the domain of definition of $\bar{L}$.

COROLLARY 2.2

There exists a positive constant C such that

$$\|(u,v)\|_B \leq C\|\bar{L}(u,v)\|_H, \quad \forall (u,v) \in D(\bar{L}), \tag{2.42}$$

where: $C = \sqrt{C_7}$.and (u,v) is solution of $\bar{L}(u,v) = \mathcal{F}$
The above inequality (2.41) can be obtained, after passing to limit in (2.11).

.

COROLLARY 2.3

A strong solution of system (2.1)-(2.3) is unique and depends continuously on

$$\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2) \in H,$$

where $\mathcal{F}_1 = \{f, \varphi_1, \varphi_2\}$ and $\mathcal{F}_2 = \{g, \psi_1, \psi_2\}$.

COROLLARY 2.4

The range $R(\overline{L})$ of $\overline{L}$ is closed in H and $R(\overline{L}) = \overline{R(L)}$. This last corollary shows that in order to prove that system (2.1)-(2.3) has a strong solution for arbitrary $(\mathcal{F}_1, \mathcal{F}_2) \in H$, it is sufficient to prove that the range of L is dense in H, that is $\overline{R(L)} = H$.

PROPOSITION 2.2

If for some function: $Y(x,t) = (y_1, y_2) \in (L_p^2(Q))^2$, and for all $W(x,t) = (u,v) \in D_0(L) = \{W/W \in D(L) : \ell_1 u = 0, \ell_2 u = 0, \ell_3 v = 0, \ell_4 v = 0\}$, (satisfying homogeneous initial conditions), we have:

$$(\mathcal{L}W, Y)_{L_p^2(Q)} = (\mathcal{L}_1 u, y_1)_{L_\rho^2(Q)} + (\mathcal{L}_2 v, y_2)_{L_\rho^2(Q)} = 0, \tag{2.43}$$

then Y is vanishes "a.e" in the domain Q.

proof

we set

$$W = (u,v) = (\Im_t^2(p_1), \Im_t^2(p_2)), \tag{2.44}$$

$$Y = (y_1, y_2) = (\Im_t(p_1) - \Im_x^2\left(\xi \Im_t(p_1)\right), \Im_t(p_2) - \Im_x^2\left(\xi \Im_t(p_2)\right)), \tag{2.45}$$

where

$$\Im_t(p_i) = \int_0^t p_i(x,s)ds, \tag{2.46}$$

$$\Im_t^2(p_i) = \int_0^t \int_0^s p_i(x,z)dzds, \tag{2.47}$$

$$\Im_x^2\left(\xi \Im_t(p_i)\right) = \int_0^x \int_0^\xi \int_0^t \eta \; p_i(\eta,s)dsd\eta, \quad i = 1,2. \tag{2.48}$$

We suppose that the function $p_i(x,t)$ satisfying conditions (2.3) and such that :

$$p_i, p_{ix}, \Im_t(p_i), \Im_t^2(p_i), x\Im_t^2(p_{ix}), \Im_x^2\left(\xi \Im_t(p_i)\right), {}^C\partial_{0t}^\beta p_i, {}^C\partial_{0t}^\gamma p_i \in L^2(Q), i = 1,2. \tag{2.49}$$

Now replacing (2.44) and (2.45) in the relation (2.43), we obtain :

$$\begin{aligned}
&\left({}^C\partial_{0t}^\beta\left(\Im_t^2(p_1)\right), \Im_t(p_1)\right)_{L_p^2(\Omega)} - \left(\left(x\left(\Im_t^2(p_{1x})\right)\right)_x, \Im_t(p_1)\right)_{L^2(\Omega)} - \left(\left(x\left(\Im_t^2(p_{1x})\right)\right)_{xt}, \Im_t(p_1)\right)_{L^2(\Omega)} \\
&+ \left(\Im_t^2(p_2), \Im_t(p_1)\right)_{L_p^2(\Omega)} - ({}^C\partial_{0t}^\beta\left(\Im_t^2(p_1)\right), \Im_x^2\left(\xi \Im_t(p_1)\right))_{L_p^2(\Omega)} + \left(\left(x\left(\Im_t^2(p_{1x})\right)\right)_x, \Im_x^2\left(\xi \Im_t(p_1)\right)\right)_{L^2(\Omega)} \\
&+ \left(\left(x\left(\Im_t^2(p_{1x})\right)\right)_{xt}, \Im_x^2\left(\xi \Im_t(p_1)\right)\right)_{L^2(\Omega)} - \left(\Im_t^2(p_2), \Im_x^2\left(\xi \Im_t(p_1)\right)\right)_{L_p^2(\Omega)} \\
&+ \left({}^C\partial_{0t}^\gamma\left(\Im_t^2(p_2)\right), \Im_t(p_2)\right)_{L_p^2(\Omega)} - \left(\left(x\left(\Im_t^2(p_{2x})\right)\right)_x, \Im_t(p_2)\right)_{L^2(\Omega)} - \left(\left(x\left(\Im_t^2(p_{2x})\right)\right)_{xt}, \Im_t(p_2)\right)_{L^2(\Omega)} \\
&+ \left(\Im_t^2(p_1), \Im_t(p_2)\right)_{L_p^2(\Omega)} - \left({}^C\partial_{0t}^\gamma\left(\Im_t^2(p_2)\right), \Im_x^2\left(\xi \Im_t(p_2)\right)\right)_{L_p^2(\Omega)} + \left(\left(x\left(\Im_t^2(p_{2x})\right)\right)_x, \Im_x^2\left(\xi \Im_t(p_2)\right)\right)_{L^2(\Omega)} \\
&+ \left(\left(x\left(\Im_t^2(p_{2x})\right)\right)_{xt}, \Im_x^2\left(\xi \Im_t(p_2)\right)\right)_{L^2(\Omega)} - \left(\Im_t^2(p_1), \Im_x^2\left(\xi \Im_t(p_2)\right)\right)_{L_p^2(\Omega)} = 0.
\end{aligned} \tag{2.50}$$

Put in mind this **Poincare inequality**:

$$\|\Im_t^2(p_i)\|_{L_p^2(\Omega)}^2 \leq \frac{T^2}{2}\|\Im_t(p_i)\|_{L_p^2(\Omega)}^2, \; i = 1,2.$$

Using conditions (2.3), and computation each term of (2.50), gives :

$$
\begin{aligned}
\left({}^{C}\partial_{0t}^{\beta}\left(\Im_t^2(p_1)\right),\Im_t(p_1)\right)_{L_p^2(\Omega)} &= \left({}^{C}\partial_{0t}^{\beta-1}\left(\Im_t(p_1)\right),\Im_t(p_1)\right)_{L_p^2(\Omega)} \\
&\geq \frac{1}{2}\ {}^{C}\partial_{0t}^{\beta-1}\|\Im_t(p_1)\|^2_{L_p^2(\Omega)}, && (2.51) \\
-\left(\left(x\left(\Im_t^2(p_{1x})\right)\right)_x,\Im_t(p_1)\right)_{L^2(\Omega)} &= \frac{1}{2}\frac{\partial}{\partial t}\|\Im_t^2(p_{1x})\|^2_{L_p^2(\Omega)}, && (2.52) \\
-\left(\left(x\left(\Im_t^2(p_{1x})\right)\right)_{xt},\Im_t(p_1)\right)_{L^2(\Omega)} &= \|\Im_t(p_{1x})\|^2_{L_p^2(\Omega)}, && (2.53) \\
-\left(\Im_t^2(p_2),\Im_t(p_1)\right)_{L_p^2(\Omega)} &\leq \frac{1}{2}\|\Im_t^2(p_2)\|^2_{L_p^2(\Omega)}+\frac{1}{2}\|\Im_t(p_1)\|^2_{L_p^2(\Omega)} \\
&\leq \frac{T^2}{4}\|\Im_t(p_2)\|^2_{L_p^2(\Omega)}+\frac{1}{2}\|\Im_t(p_1)\|^2_{L_p^2(\Omega)}, && (2.54) \\
-\left({}^{C}\partial_{0t}^{\beta}\left(\Im_t^2(p_1)\right),\Im_x^2\left(\xi\Im_t(p_1)\right)\right)_{L_p^2(\Omega)} &= -\left({}^{C}\partial_{0t}^{\beta-1}\left(\Im_t(p_1)\right),\Im_x^2\left(\xi\Im_t(p_1)\right)\right)_{L_p^2(\Omega)} \\
&= \left({}^{C}\partial_{0t}^{\beta-1}\left(\Im_x\left(\xi\Im_t(p_1)\right)\right),\Im_x\left(\xi\Im_t(p_1)\right)\right)_{L^2(\Omega)} \\
&\geq \frac{1}{2b}\ {}^{C}\partial_{0t}^{\beta-1}\|\Im_x\left(\xi\Im_t(p_1)\right)\|^2_{L_p^2(\Omega)}, && (2.55) \\
+\left(\left(x\left(\Im_t^2(p_{1x})\right)\right)_x,\Im_x^2\left(\xi\Im_t(p_1)\right)\right)_{L^2(\Omega)} &= -\left(\Im_t^2(p_{1x}),\Im_x\left(\xi\Im_t(p_1)\right)\right)_{L_p^2(\Omega)} \\
&\leq \frac{1}{T^2}\|\Im_t^2(p_{1x})\|^2_{L_p^2(\Omega)}+\frac{T^2}{4}\|\Im_x\left(\xi\Im_t(p_1)\right)\|^2_{L_p^2(\Omega)} \\
&\leq \frac{1}{2}\|\Im_t(p_{1x})\|^2_{L_p^2(\Omega)}+\frac{T^2}{4}\|\Im_x\left(\xi\Im_t(p_1)\right)\|^2_{L_p^2(\Omega)}, && (2.56) \\
+\left(\left(x\left(\Im_t^2(p_{1x})\right)\right)_{xt},\Im_x^2\left(\xi\Im_t(p_1)\right)\right)_{L^2(\Omega)} &= -\left(\Im_t(p_{1x}),\Im_x\left(\xi\Im_t(p_1)\right)\right)_{L_p^2(\Omega)} \\
&\leq \frac{1}{2}\|\Im_t(p_{1x})\|^2_{L_p^2(\Omega)}+\frac{1}{2}\|\Im_x\left(\xi\Im_t(p_1)\right)\|^2_{L_p^2(\Omega)}, && (2.57) \\
+\left(\Im_t^2(p_2),\Im_x^2\left(\xi\Im_t(p_1)\right)\right)_{L_p^2(\Omega)} &\leq \frac{1}{2}\|\Im_t^2(p_2)\|^2_{L_p^2(\Omega)}+\frac{1}{2}\|\Im_x^2\left(\xi\Im_t(p_1)\right)\|^2_{L_p^2(\Omega)} \\
&\leq \frac{T^2}{4}\|\Im_t(p_2)\|^2_{L_p^2(\Omega)}+\frac{b^6}{8}\|\Im_t(p_1)\|^2_{L_p^2(\Omega)}, && (2.58)
\end{aligned}
$$

Insertion of equations (2.51)–(2.58), into (2.50), we get :

$$
\begin{aligned}
&\frac{1}{2}\ {}^{C}\partial_{0t}^{\beta-1}\left(\|\Im_t(p_1)\|^2_{L_p^2(\Omega)}+\frac{1}{b}\|\Im_x\left(\xi\Im_t(p_1)\right)\|^2_{L_p^2(\Omega)}\right)+\frac{1}{2}\ \frac{\partial}{\partial t}\|\Im_t^2(p_{1x})\|^2_{L_p^2(\Omega)}+\|\Im_t(p_2x)\|^2_{L_p^2(\Omega)} \\
&+\frac{1}{2}\ {}^{C}\partial_{0t}^{\gamma-1}\left(\|\Im_t(p_2)\|^2_{L_p^2(\Omega)}+\frac{1}{b}\ \|\Im_x\left(\xi\Im_t(p_2)\right)\|^2_{L_p^2(\Omega)}\right)+\frac{1}{2}\ \frac{\partial}{\partial t}\|\Im_t^2(p_{2x})\|^2_{L_p^2(\Omega)}+\|\Im_t(p_1x)\|^2_{L_p^2(\Omega)} \\
&\leqslant \frac{T^2}{4}\|\Im_t(p_2)\|^2_{L_p^2(\Omega)}+\frac{1}{2}\|\Im_t(p_1)\|^2_{L_p^2(\Omega)}+\|\Im_t(p_1x)\|^2_{L_p^2(\Omega)}+\frac{T^2}{4}\|\Im_x\left(\xi\Im_t(p_1)\right)\|^2_{L_p^2(\Omega)} \\
&+\frac{T^2}{4}\|\Im_t(p_2)\|^2_{L_p^2(\Omega)}+\frac{1}{2}\|\Im_x\left(\xi\Im_t(p_1)\right)\|^2_{L_p^2(\Omega)}+\frac{b^6}{8}\|\Im_t(p_1)\|^2_{L_p^2(\Omega)}+\frac{T^2}{4}\|\Im_t(p_1)\|^2_{L_p^2(\Omega)} \\
&+\frac{1}{2}\|\Im_t(p_2)\|^2_{L_p^2(\Omega)}+\|\Im_t(p_2x)\|^2_{L_p^2(\Omega)}+\frac{T^2}{4}\|\Im_x\left(\xi\Im_t(p_2)\right)\|^2_{L_p^2(\Omega)} \\
&+\frac{T^2}{4}\|\Im_t(p_1)\|^2_{L_p^2(\Omega)}+\frac{1}{2}\|\Im_x\left(\xi\Im_t(p_2)\right)\|^2_{L_p^2(\Omega)}+\frac{b^6}{8}\|\Im_t(p_2)\|^2_{L_p^2(\Omega)}.
\end{aligned}
$$

if we multiply the last equation by 2 and introduce **min** on the left hand side and **max** on the right side , we find :

$$
\begin{aligned}
&{}^{C}\partial_{0t}^{\beta-1}\left(\|\Im_t(p_1)\|^2_{L^2_p(\Omega)}+\|\Im_x\left(\xi\Im_t(p_1)\right)\|^2_{L^2_p(\Omega)}\right)+\frac{\partial}{\partial t}\|\Im_t^2(p_{1x})\|^2_{L^2_p(\Omega)}\\
&{}^{C}\partial_{0t}^{\gamma-1}\left(\|\Im_t(p_2)\|^2_{L^2_p(\Omega)}+\|\Im_x\left(\xi\Im_t(p_2)\right)\|^2_{L^2_p(\Omega)}\right)+\frac{\partial}{\partial t}\|\Im_t^2(p_{2x})\|^2_{L^2_p(\Omega)}\\
&\leq M_1\left[\left(\|\Im_t(p_1)\|^2_{L^2_p(\Omega)}+\|\Im_x\left(\xi\Im_t(p_1)\right)\|^2_{L^2_p(\Omega)}\right)+\left(\|\Im_t(p_2)\|^2_{L^2_p(\Omega)}+\|\Im_x\left(\xi\Im_t(p_2)\right)\|^2_{L^2_p(\Omega)}\right)\right],
\end{aligned}
\tag{2.59}
$$

where:

$$
M_1=\frac{\max\left\{T^2;1+\frac{b^6}{4};1+\frac{T^2}{2}\right\}}{\min\left\{1;\frac{1}{b}\right\}}.
$$

taking in account the homogeneous initial conditions in $D_0(L)$, then replacing t by τ in(2.59) and integrate with respect to τ from 0 to t, we have:

$$
\begin{aligned}
&D_{0t}^{\beta-2}\left(\|\Im_t(p_1)\|^2_{L^2_p(\Omega)}+\|\Im_x\left(\xi\Im_t(p_1)\right)\|^2_{L^2_p(\Omega)}\right)+\|\Im_t^2(p_{1x})\|^2_{L^2_p(\Omega)}\\
&D_{0t}^{\gamma-2}\left(\|\Im_t(p_2)\|^2_{L^2_p(\Omega)}+\|\Im_x\left(\xi\Im_t(p_2)\right)\|^2_{L^2_p(\Omega)}\right)+\|\Im_t^2(p_{2x})\|^2_{L^2_p(\Omega)}\\
&\leq M_1\left[\int_0^t\left(\|\Im_\tau(p_1)\|^2_{L^2_p(\Omega)}+\|\Im_x\left(\xi\Im_\tau(p_1)\right)\|^2_{L^2_p(\Omega)}\right)d\tau\right.\\
&\left.+\int_0^t\left(\|\Im_\tau(p_2)\|^2_{L^2_p(\Omega)}+\|\Im_x\left(\xi\Im_\tau(p_2)\right)\|^2_{L^2_p(\Omega)}\right)d\tau\right].
\end{aligned}
\tag{2.60}
$$

Now, we discard the last three terms in LHS of (2.60), and applying lemma(1.1) ,where:

$$
y(t)=\int_0^t\left(\|\Im_\tau(p_1)\|^2_{L^2_p(\Omega)}+\|\Im_x\left(\xi\Im_\tau(p_1)\right)\|^2_{L^2_p(\Omega)}\right)d\tau.
$$

$$
{}^{C}\partial_{0t}^{\beta-1}y(t)=D_{0t}^{\beta-2}\left(\|\Im_\tau(p_1)\|^2_{L^2_p(\Omega)}+\|\Im_x\left(\xi\Im_\tau(p_1)\right)\|^2_{L^2_p(\Omega)}\right)
$$

then using **fractional inequality** , we get :

$$
\begin{aligned}
&\int_0^t\left(\|\Im_\tau(p_1)\|^2_{L^2_p(\Omega)}+\|\Im_x\left(\xi\Im_\tau(p_1)\right)\|^2_{L^2_p(\Omega)}\right)d\tau\\
&\leq M_2D_{0t}^{-\beta}\left(\|\Im_\tau(p_2)\|^2_{L^2_p(\Omega)}+\|\Im_x\left(\xi\Im_\tau(p_2)\right)\|^2_{L^2_p(\Omega)}\right)\\
&\leq M_3\left(\int_0^t\left(\|\Im_\tau(p_2)\|^2_{L^2_p(\Omega)}+\|\Im_x\left(\xi\Im_\tau(p_2)\right)\|^2_{L^2_p(\Omega)}\right)d\tau\right),
\end{aligned}
\tag{2.61}
$$

where :

$$
\begin{aligned}
M_2&=M_1\Gamma(\beta-1)E_{\beta-1,\beta-1}(M_1T^{\beta-1}),\\
M_3&=M_2\frac{T^{\beta-1}}{\Gamma(\beta)}.
\end{aligned}
$$

Substitution of (2.61) into (2.60) yields :

$$
\begin{aligned}
&D_{0t}^{\gamma-2}\left(\|\Im_t(p_2)\|^2_{L^2_p(\Omega)}+\|\Im_x\left(\xi\Im_t(p_2)\right)\|^2_{L^2_p(\Omega)}\right)+\|\Im_t^2(p_{2x})\|^2_{L^2_p(\Omega)}\\
&D_{0t}^{\beta-2}\left(\|\Im_t(p_1)\|^2_{L^2_p(\Omega)}+\|\Im_x\left(\xi\Im_t(p_1)\right)\|^2_{L^2_p(\Omega)}\right)+\|\Im_t^2(p_{1x})\|^2_{L^2_p(\Omega)}\\
&\leq M_4\left[\int_0^t\left(\|\Im_\tau(p_2)\|^2_{L^2_p(\Omega)}+\|\Im_x\left(\xi\Im_\tau(p_2)\right)\|^2_{L^2_p(\Omega)}\right)d\tau\right],
\end{aligned}
\tag{2.62}
$$

where:

$$M_4 = M_1(M_3+1). \tag{2.63}$$

If we discard the last three terms in LHS of (2.62), and applying lemma(1.1),where :

$$y(t) = \int_0^t \left(\|\Im_\tau(p_2)\|^2_{L^2_p(\Omega)} + \|\Im_x\left(\xi\Im_\tau(p_2)\right)\|^2_{L^2_p(\Omega)}\right) d\tau$$

$${}^C\partial_{0t}^{\gamma-1} y(t) = D_{0t}^{\gamma-2}\left(\|\Im_t(p_2)\|^2_{L^2_p(\Omega)} + \|\Im_x\left(\xi\Im_t(p_2)\right)\|^2_{L^2_p(\Omega)}\right)$$

we obtain:

$$\int_0^t \left(\|\Im_\tau(p_2)\|^2_{L^2_p(\Omega)} + \|\Im_x\left(\xi\Im_\tau(p_2)\right)\|^2_{L^2_p(\Omega)}\right) d\tau \leq M_5\left(D_{0t}^{-\gamma}(0)\right) = 0, \tag{2.64}$$

where:

$$M_5 = \Gamma(\gamma-1)E_{\gamma-1,\gamma-1}(M_4T^{\gamma-1}).$$

Return and using the inequality (2.61), we deduce that $Y = (y_1, y_2) = (0,0)$ almost everywhere in the domain Q. ■

THEOREME 2.2

For any $(f,g) \in \left(L^2_\rho(Q)\right)^2$ and any $(\varphi_1,\varphi_2), (\psi_1,\psi_2) \in \left\{L^2(0,T;H^1_\rho(\Omega)) \times L^2_\rho(Q)\right\}$, there exists a unique strong solution $W = \overline{L}^{-1}\mathcal{F} = \overline{L^{-1}}\mathcal{F}$ of the system (2.1)-(2.3), where $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2) \in H$, $\mathcal{F}_1 = \{f, \varphi_1, \varphi_2\}$, $\mathcal{F}_2 = \{g, \psi_1, \psi_2\}$, $W = (u,v)$ and

$$\|W\|_B \leq C\|LW\|_H, \tag{2.65}$$

for a positive constant C, independent of W.

proof

It is sufficient to prove that the range $R(L)$ of L is dense in $H = H_1 \times H_2$. Suppose that for some: $Y = (\mathcal{Y}_1, \mathcal{Y}_2) = (\{y_1,y_2,y_3\}, \{y_4,y_5,y_6\}) \in H$, the orthogonal of $R(L)$, so that

$$\begin{aligned}
(LW,Y)_H &= \left(\{L_1u, L_2v\}, \{\mathcal{Y}_1, \mathcal{Y}_2\}\right)_H \\
&= \left(\{(\mathcal{L}_1u, \ell_1u, \ell_2u), (\mathcal{L}_2v, \ell_3v, \ell_4v)\}, \{(y_1,y_2,y_3), (y_4,y_5,y_6)\}\right)_H \\
&= (\mathcal{L}_1u, y_1)_{L^2_\rho(Q)} + (\ell_1u, y_2)_{L^2(0,T;H^1_\rho(\Omega))} + (\ell_2u, y_3)_{L^2_\rho(\Omega)} \\
&+ (\mathcal{L}_2v, y_4)_{L^2_\rho(Q)} + (\ell_3v, y_5)_{L^2(0,T;H^1_\rho(\Omega))} + (\ell_4v, y_6)_{L^2_\rho(\Omega)} = 0.
\end{aligned} \tag{2.66}$$

We must show that: $Y = \mathbf{0}$a.e in$Q$. Putting $W \in D_0(L)$ in (2.66), we get :

$$(\mathcal{L}_1u, y_1)_{L^2_\rho(Q)} + (\mathcal{L}_2v, y_4)_{L^2_\rho(Q)} = 0, \quad \forall W \in D(L), \tag{2.67}$$

hence proposition (2.2) implies that: $y_1 = y_4 = 0$.
The relation (2.67), implies that :

$$(\ell_1u, y_2)_{L^2(0,T;H^1_\rho(\Omega))} + (\ell_2u, y_3)_{L^2_\rho(\Omega)} + (\ell_3v, y_5)_{L^2(0,T;H^1_\rho(\Omega))} + (\ell_4v, y_6)_{L^2_\rho(\Omega)} = 0, \quad \forall W \in D(L), \tag{2.68}$$

since the four quantities in (2.68) vanish independently and since the ranges of the trace operators ℓ_1, ℓ_3 and ℓ_2, ℓ_4 are respectively everywhere dense in the spaces $L^2(0,T;H^1_\rho(\Omega))$ and $L^2_\rho(\Omega)$, therefore it follows, from(2.67), that $y_2 = y_3 = y_5 = y_6 = 0$a.e in$Q$. Hence $\overline{R(L)} = H$.

Now, let's consider the general case. Based on the fact that R(L) is dense in H, we conclude that we can prove that R(L) is dense in H by means of the continuation method along the parameter (see [20]). This is what leads to **Proposition (2.3)**.

PROPOSITION 2.3

Assume that the necessary conditions hold. Then, for all$\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2), \in H$, there exists a unique strong solution $u = \overline{L}^{-1}\mathcal{F} = \overline{L^{-1}}\mathcal{F}$ of problem (2.1)-(2.3).

CHAPTER

3

Q−HOMOTOPY ANALISIS TRONSFORM METHOD

The Homotopy Analysis Method (HAM) was first proposed in 1992 by **Liao** as a flexible tool for solving nonlinear equations, gaining widespread attention due to its ability to control the form of the solution and its convergence speed. With the growing need for more efficient methods, an improved version known as ***q*-HATM** emerged, which combines **HAM**, the Laplace transform, and the parameter **q**, helping to simplify computations and enhance the accuracy of results, particularly when dealing with fractional-order equations.

3.1 Homotopy Analysis Method (HAM)

In this section , we present the Homotopy Analysis Method (HAM), introduced by Liao in 1992 for solving both linear and nonlinear equations [Liao, 1992]. Liao later provided a systematic explanation in 2003, which enhanced its understanding and application [Liao, 2003]. Here, we aim to review the basic principles of HAM and extend them to fractional partial differential equations. HAM has since proven effective in solving complex nonlinear equations, including fractional-order differential equations [Liao, 2005; Liao, 2006].

DEFINITION 3.1: Homotopy Function

Let $f, g : X \to Y$ be two continuous functions. A homotopy between f and g is a continuous function:

$$H : X \times [0,1] \to Y$$

such that:

$$H(x,0) = f(x) \quad \text{and} \quad H(x,1) = g(x) \quad \forall x \in X.$$

Here, H is viewed as a continuous deformation of f into g, where $t \in [0,1]$ acts as a time parameter controlling the deformation process[see [16]].

We consider the following fractional coupled system (see[26]):

$$\begin{aligned} \mathcal{N}_1[u(x,t)] &= 0, \\ \mathcal{N}_2[v(x,t)] &= 0, \end{aligned} \tag{3.1}$$

where $\mathcal{N}_1$,$\mathcal{N}_2$ is a linear or nonlinear operators, x and t denotes independent variables, $u(x,t)$ and $v(x,t)$ is an unknowns functions. For simplicity, we ignore all boundary or initial conditions, which can be treated in the same way. Based on the constructed zero-order deformation equation by Liao (see[27]), we give the following **zero-order deformation equation** in the similar way:

$$\begin{aligned}(1-q)\mathcal{L}[\phi(x,t;q)-u_0(x,t)] =& qh\mathcal{N}_1(\phi(x,t;q)),\\ (1-q)\mathcal{L}[\psi(x,t;q)-v_0(x,t)] =& qh\mathcal{N}_2(\psi(x,t;q)),\end{aligned} \tag{3.2}$$

where $q \in [0,1]$ is the embedding parameter,h is a nonzero auxiliary parameter, $\mathcal{L}$ is an auxiliary linear non-integer order operator which possesses the property $\mathcal{L}(C) = 0$, $u_0(x,t), v_0(x,t)$ is an initial guess of $u(x,t), v(x,t)$ and $\phi(x,t;q)$ and $\psi(x,t;q)$ is an unknowns functions on independent variables x,t,q. It is important to note that one has great freedom to choose the auxiliary parameter h in HAM. If $q=0$ and $q=1$, then we have:

$$\begin{aligned}\phi(x,t;0) = u_0(x,t), \qquad & \textit{and} \qquad \phi(x,t;1) = u(x,t),\\ \psi(x,t;0) = v_0(x,t), \qquad & \textit{and} \qquad \psi(x,t;1) = v(x,t),\end{aligned} \tag{3.3}$$

respectively. Thus as q increases from 0 to 1, the solutions $\phi(x,t;q), \psi(x,t;q)$ varies from the initial guess $u_0(x,t)$, $v_0(x,t)$ to the solutions $u(x,t)$ and $v(x,t)$. Expanding $\phi(x,t;q)$ and $\psi(x,t;q)$ in Taylor series with respect to q, we have :

$$\begin{aligned}\phi(x,t;q) =& u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)q^m,\\ \psi(x,t;q) =& v_0(x,t) + \sum_{m=1}^{\infty} v_m(x,t)q^m,\end{aligned} \tag{3.4}$$

where:

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;q)}{\partial q^m}|_{q=0}, \qquad \textit{and} \qquad v_m(x,t) = \frac{1}{m!}\frac{\partial^m \psi(x,t;q)}{\partial q^m}|_{q=0}, \tag{3.5}$$

If the auxiliary linear operator, the initial guess, the auxiliary h, and the auxiliary function are so properly chosen, the series (3.4) converges at $q=1$, then we have:

$$\begin{aligned}u(x,t) =& u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t),\\ v(x,t) =& v_0(x,t) + \sum_{m=1}^{\infty} v_m(x,t),\end{aligned} \tag{3.6}$$

which must be one of the solution of the original linearor nonlinear equation as proved by [27]. As $h=-1$, the equations (3.2) becomes :

$$\begin{aligned}(1-q)\mathcal{L}\left[\phi(x,t;q)-u_0(x,t)\right] + q\mathcal{N}_1(\phi(x,t;q)) = 0,\\ (1-q)\mathcal{L}\left[\psi(x,t;q)-v_0(x,t)\right] + q\mathcal{N}_2(\psi(x,t;q)) = 0,\end{aligned} \tag{3.7}$$

According to (3.4), the governing equation can be deduced from the zero-order deformation (3.2). Define the vectors :

$$\begin{aligned}\vec{u_m}(x,t) = \left\{u_0(x,t), u_1(x,t), u_2(x,t), u_3(x,t), ..., u_m(x,t)\right\},\\ \vec{v_m}(x,t) = \left\{v_0(x,t), v_1(x,t), v_2(x,t), v_3(x,t), ..., v_m(x,t)\right\},\end{aligned} \tag{3.8}$$

Differentiating (3.2) m times with respect to the embedding parameter q, then setting $q = 0$, and finally dividing them by $m!$, we have the so-called **mth-order deformations equations**:

$$\begin{aligned} \mathcal{L}\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] &= h\mathcal{R}_1(u_{m-1}(x,t)), \\ \mathcal{L}\left[v_m(x,t) - \chi_m v_{m-1}(x,t)\right] &= h\mathcal{R}_2(v_{m-1}(x,t)), \end{aligned} \tag{3.9}$$

where :

$$\begin{aligned} \mathcal{R}_1(u_{m-1}(x,t)) &= \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}_1(\phi(x,t;q))}{\partial q^{m-1}}\bigg|_{q=0}, \\ \mathcal{R}_2(v_{m-1}(x,t)) &= \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}_2(\psi(x,t;q))}{\partial q^{m-1}}\bigg|_{q=0}, \end{aligned} \tag{3.10}$$

and

$$\mathcal{X}_m = \begin{cases} 0 & m \leq 1, \\ 1 & m > 1, \end{cases} \tag{3.11}$$

Applying $\mathcal{L}^{-1}$ both sides of (3.9), we get:

$$\begin{aligned} u_m(x,t) &= \chi_m u_{m-1}(x,t) + h\mathcal{L}^{-1}\left[\mathcal{R}_1(u_{m-1}(x,t))\right]. \\ v_m(x,t) &= \chi_m v_{m-1}(x,t) + h\mathcal{L}^{-1}\left[\mathcal{R}_2(v_{m-1}(x,t))\right]. \end{aligned} \tag{3.12}$$

In this way, it is easily to obtain u_m,v_m, for $m \geq 1$, at mth-order, we have:

$$u(x,t) = \sum_{m=1}^{M} u_m(x,t), \qquad v(x,t) = \sum_{m=1}^{M} v_m(x,t), \tag{3.13}$$

when $M \to \infty$ we get an accurate approximation of the original equation(3.1). For the convergence of the above method we refer the reader Liao's work. If equation (3.1) admits unique solution, then this method will produce the unique solution. If equation (3.1) does not possess unique solution, the HAM will give a solution among many other (possible) solutions.

3.2 q-Homotopy Analysis Transform Method

The q-Homotopy Analysis Transform Method (q-HATM) was introduced as an innovative approach to solving nonlinear partial differential equations. It was built upon the foundation of the Homotopy Analysis Method (HAM), first proposed by Liao in 1992 (see[11]) and later developed further in his book Beyond Perturbation (see[15]). Over time, this method has been enhanced by integrating additional techniques, such as the Laplace transform (see[18]), which has contributed to its increased efficiency and expanded range of applications.

DEFINITION 3.2

The Laplace transform $\mathcal{L}$ of the Caputo fractional derivative $\partial_t^\beta \theta(x,t), n-1 < \beta < n$, of a function $\theta(x,t)$ is defined as:

$$\mathcal{L}\left[\frac{\partial^\beta}{\partial t^\beta}\theta(x,t)\right] = s^\beta \mathcal{L}\left\{\theta(x,t)\right\} - \sum_{k=0}^{n-1} s^{\beta-k-1}\theta^{(k)}\left(x,0^+\right) \tag{3.14}$$

Thus, consider the following diffirential equation of the form :

$$\frac{\partial^\beta}{\partial t^\beta}u(x,t) + R(u(x,t)) + \tilde{N}(u(x,t)) = F(x,t), n-1 < \beta \leq n, \tag{3.15}$$

where $u(x,t)$ is a differentiable function, $\frac{\partial^\beta}{\partial t^\beta}$ is the Caputo derivative of order β, R is a linear differential operator, $\tilde{N}$ denotes a nonlinear differential operator, and $F(x,t)$ is a known function.

Then, applying Laplace transform to both sides of equation (3.15) gives:

$$s^\beta \mathcal{L}\{u(x,t)\} - \sum_{k=0}^{n-1} s^{\beta-k-1}u^{(k)}(x,0) + \mathcal{L}\{R(u(x,t)) + \tilde{N}(u(x,t))\} = \mathcal{L}\{F(x,t)\},$$

we divide both sides of the equation by s^β :

$$\mathcal{L}\{u(x,t)\} - \sum_{k=0}^{n-1} \frac{1}{s^{k+1}}u^{(k)}(x,0) + \frac{1}{s^\beta}\mathcal{L}\left\{R(u(x,t)) + \tilde{N}(u(x,t)) - F(x,t)\right\} = 0. \tag{3.16}$$

Next, according to the HAM method (see[17]), we define an operator $\mathcal{N}$ as follows:

$$\begin{aligned}\mathcal{N}[\varphi(x,t;q)] =& \mathcal{L}\{\varphi(x,t;q)\} - \sum_{k=0}^{n-1} \frac{1}{s^{k+1}}\varphi^{(k)}(x,0;q) + \frac{1}{s^\beta}\mathcal{L}\left\{R(\varphi(x,t;q))\right. \\ & \left. + \tilde{N}(\varphi(x,t;q)) - F(x,t)\right\},\end{aligned} \tag{3.17}$$

where $q \in \left[0, \frac{1}{n}\right], n \geq 1$, and φ is a real valued function in x, t and q. Thus, we take the zeroth-order deformation equation to be:

$$(1-nq)\mathcal{L}\left[\varphi(x,t;q) - u_0(x,t)\right] = q\hbar\mathcal{N}[\varphi(x,t;q)] \tag{3.18}$$

where $\hbar$ is a non-vanishing auxiliary parameter, which is used to control and adjust the convergence region of the desired series solution, $q \in \left[0, \frac{1}{n}\right]$ is an embedding parameter, $\mathcal{L}$ denotes the traditional Laplace transform operator, $u_0(x,t)$ is an initial guess for the exact solution $u(x,t)$, and $\varphi(x,t;q)$ is an unknown function.

It is clear that at $q=0$ and $q=\frac{1}{n}$, equation (3.18) implies:

$$\varphi(x,t;0) = u_0(x,t) \qquad \text{and} \qquad \varphi\left(x,t;\frac{1}{n}\right) = u(x,t).$$

Thus, as q moves continuously from 0 to $\frac{1}{n}$, the function $\varphi(x,t;q)$ deforms from the initial approximation $u_0(x,t)$ to the exact solution $u(x,t)$.

Next, the Taylor series expansion of $\varphi(x,t;q)$ in powers of q implies:

$$\varphi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)q^m. \tag{3.19}$$

where

$$u_m(x,t) = \frac{1}{m!}\left\{\frac{\partial^m \varphi(x,t;q)}{\partial q^m}\right\}\Bigg|_{q=0}.$$

Now, if the auxiliary parameter $\hbar$ and the initial guess $u_0(x,t)$ are properly chosen, then the power series (3.19) would converge at $q=\frac{1}{n}$ to one of the solutions of the above problem, and it is given as:

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty}\left(\frac{1}{n}\right)^m u_m(x,t). \tag{3.20}$$

In fact, the existence of the factor $\left(\frac{1}{n}\right)^m$ in the series (3.20) accelerates the convergence in the q-HATM compared with the HAM.

Differentiating the zeroth order deformation equation (3.18) m-times with respect to q, dividing by $m\,!$ and then setting $q=0$, gives the following m th order deformation equation:

$$\mathcal{L}\left\{u_m(x,t)-\chi_m u_{m-1}(x,t)\right\}=\hbar\Re\left(\vec{u}_{m-1}\right), \tag{3.21}$$

where

$$\vec{u}_k(x,t)=\left[u_0(x,t),u_1(x,t),\ldots,u_k(x,t)\right],$$

and

$$\Re\left(\vec{u}_{m-1}\right)=\frac{1}{(m-1)!}\left\{\frac{\partial^{m-1}}{\partial q^{m-1}}\mathcal{N}[\varphi(x,t;q)]\right\}\Bigg|_{q=0}$$

3.3 Application of the q-HATM

The q-HATM method, also known as the ”Modified Homotopy Analysis Method with the Parameter”, is considered one of the most efficient and accurate techniques for solving mathematical models of integer or fractional orders. It is distinguished by its ability to address various problems across different scientific fields, as highlighted in references [[26], [27], [29],[31]].

To solve a problem (2.1)-(2.3), we start by **applying the Laplace transform** to both sides of each equation in the system,we obtain:

$$\begin{aligned}
\mathcal{L}[u(x,t)]-&\sum_{k=0}^{1}\frac{1}{s^{k+1}}u^{(k)}(x,0)-\frac{1}{s^{\beta}}\mathcal{L}\left[\frac{d_1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}u(x,t)\right)+\frac{d_2}{x}\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial}{\partial x}u(x,t)\right)\right.\\
&\left.-cv(x,t)\right]=\mathcal{L}[f(x,t)],\\
\mathcal{L}[v(x,t)]-&\sum_{k=0}^{1}\frac{1}{s^{k+1}}u^{(k)}(x,0)-\frac{1}{s^{\gamma}}\mathcal{L}\left[\frac{d_1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}v(x,t)\right)+\frac{d_2}{x}\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial}{\partial x}v(x,t)\right)\right.\\
&\left.-cu(x,t)\right]=\mathcal{L}[g(x,t)],
\end{aligned} \tag{3.22}$$

then :

$$\begin{aligned}
\mathcal{L}[u(x,t)]-&\frac{1}{s}u(x,0)-\frac{1}{s^2}u_t(x,0)-\frac{1}{s^{\beta}}\mathcal{L}\left[\frac{d_1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}u(x,t)\right)+\frac{d_2}{x}\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial}{\partial x}u(x,t)\right)\right.\\
&\left.-cv(x,t)+f(x,t)\right]=0,\\
\mathcal{L}[v(x,t)]-&\frac{1}{s}v(x,0)-\frac{1}{s^2}v_t(x,0)-\frac{1}{s^{\gamma}}\mathcal{L}\left[\frac{d_1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}v(x,t)\right)+\frac{d_2}{x}\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial}{\partial x}v(x,t)\right)\right.\\
&\left.-cu(x,t)+g(x,t)\right]=0.
\end{aligned} \tag{3.23}$$

Next, we define two operators $\mathcal{N}_1$ and $\mathcal{N}_2$ as follows:

$$
\begin{aligned}
\mathcal{N}_1\left[\phi(x,t;q),\psi(x,t;q)\right] =& \mathcal{L}\left\{\phi(x,t;q)\right\} - \frac{1}{s}u(x,0) - \frac{1}{s^2}u_t(x,0) - \frac{1}{s^\beta}\mathcal{L}\left\{\frac{d_1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\phi(x,t)\right)\right.\\
&\left. + \frac{d_2}{x}\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial}{\partial x}\phi(x,t)\right) - c\psi(x,t) + f(x,t)\right\},\\
\mathcal{N}_2\left[\phi(x,t;q),\psi(x,t;q)\right] =& \mathcal{L}\left\{\phi(x,t;q)\right\} - \frac{1}{s}v(x,0) - \frac{1}{s^2}v_t(x,0) - \frac{1}{s^\gamma}\mathcal{L}\left\{\frac{d_1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\psi(x,t)\right)\right.\\
&\left. + \frac{d_2}{x}\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial}{\partial x}\psi(x,t)\right) - c\phi(x,t) + g(x,t)\right\}.
\end{aligned}
\tag{3.24}
$$

Then, we define the zeroth-order deformation equation as follows:

$$
\begin{aligned}
(1-nq)\mathcal{L}\left[\phi(x,t;q) - u_0(x,t)\right] &= q\hbar_1\mathcal{N}_1\left[\phi(x,t;q),\psi(x,t;q)\right]\\
(1-nq)\mathcal{L}\left[\psi(x,t;q) - v_0(x,t)\right] &= q\hbar_2\mathcal{N}_2\left[\phi(x,t;q),\psi(x,t;q)\right],
\end{aligned}
\tag{3.25}
$$

which leads to the following mth-order deformation equation:

$$
\begin{aligned}
\mathcal{L}\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] &= \hbar_1\Re_1\left(\vec{u}_{m-1},\vec{v}_{m-1}\right),\\
\mathcal{L}\left[v_m(x,t) - \chi_m v_{m-1}(x,t)\right] &= \hbar_2\Re_2\left(\vec{u}_{m-1},\vec{v}_{m-1}\right),
\end{aligned}
\tag{3.26}
$$

where:

$$
\begin{aligned}
&\Re_1\left(\vec{u}_{m-1},\vec{v}_{m-1}\right) = \mathcal{L}\left\{u_{m-1}(x,t)\right\} - \left(1-\frac{\chi_m}{n}\right)\left(\frac{1}{s}u(x,0) + \frac{1}{s^2}u_t(x,0)\right) - \frac{1}{s^\beta}\mathcal{L}\left\{\frac{d_1}{x}\right.\\
&\left.\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}u_{m-1}(x,t)\right) + \frac{d_2}{x}\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial}{\partial x}u_{m-1}(x,t)\right) - cv_{m-1}(x,t) + \left(1-\frac{\chi_m}{n}\right)f(x,t)\right\},\\
&\Re_2\left(\vec{u}_{m-1},\vec{v}_{m-1}\right) = \mathcal{L}\left\{v_{m-1}(x,t)\right\} - \left(1-\frac{\chi_m}{n}\right)\left(\frac{1}{s}v(x,0) + \frac{1}{s^2}v_t(x,0)\right) - \frac{1}{s^\gamma}\mathcal{L}\left\{\frac{d_1}{x}\right.\\
&\left.\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}v_{m-1}(x,t)\right) + \frac{d_2}{x}\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial}{\partial x}v_{m-1}(x,t)\right) - cu_{m-1}(x,t) + \left(1-\frac{\chi_m}{n}\right)g(x,t)\right\},
\end{aligned}
\tag{3.27}
$$

You can calculate the terms of the recursive series using an iterative method, where each term is calculated based on the previous terms, and we rely on the following relationship.

$$
\begin{aligned}
u_m(x,t) &= \chi_m u_{m-1}(x,t) + \hbar_1\mathcal{L}^{-1}\left[\Re_1\left(\vec{u}_{m-1},\vec{v}_{m-1}\right)\right], \qquad m \geq 1\\
v_m(x,t) &= \chi_m v_{m-1}(x,t) + \hbar_2\mathcal{L}^{-1}\left[\Re_2\left(\vec{u}_{m-1},\vec{v}_{m-1}\right)\right], \qquad m \geq 1
\end{aligned}
\tag{3.28}
$$

and the solution will be given as:

$$
\begin{aligned}
u(x,t) &= u_0(x,t) + \sum_{m=1}^{\infty}\left(\frac{1}{n}\right)^m u_m(x,t)\\
v(x,t) &= v_0(x,t) + \sum_{m=1}^{\infty}\left(\frac{1}{n}\right)^m v_m(x,t)
\end{aligned}
\tag{3.29}
$$

3.4 Numerical Test

In this section, we implement the iterative computational scheme (3.28) derived from the q-HATM technique to solve numerical test cases. The objective is to examine the efficacy and computational performance of this method in solving the fractional-order systems (2.1)-(2.3).

Example 1. We consider a fractional coupled system with $1 < \beta, \gamma \leq 2$:

$$\begin{cases} {}^{C}\partial^{\beta}_{0t}u - \frac{1}{x}\left(xu_x\right)_x - \frac{1}{x}\left(xu_x\right)_{xt} + v = 2\left(t+2\right) + \left(\frac{-1}{2}x^2 + \ln(x) + \frac{3}{4}\right)t, \\ {}^{C}\partial^{\gamma}_{0t}v - \frac{1}{x}\left(xv_x\right)_x - \frac{1}{x}\left(xv_x\right)_{xt} + u = 2\left(t+1\right) + \left(\frac{-1}{2}x^2 + \ln(x) + \frac{3}{4}\right)\left(t+1\right), \end{cases} \tag{3.30}$$

with:

$$\begin{aligned} u_0(x,t) &= u(x,0) = u_t(x,0) = \left(\frac{-1}{2}x^2 + \ln(x) + \frac{3}{4}\right), \\ v_0(x,t) &= v(x,0) = 0, \quad v_t(x,0) = \left(\frac{-1}{2}x^2 + \ln(x) + \frac{3}{4}\right), \end{aligned} \tag{3.31}$$

This functions satisfies :

$$u_x(1,t) = 0, \quad v_x(1,t) = 0,$$

and

$$\int_0^1 xu(x,t)\,dx = 0, \quad \int_0^1 xv(x,t)\,dx = 0.$$

Put $m = 1$ in equations (3.28), we get :

$$\begin{aligned} u_1(x,t) &= \chi_1 u_0(x,t) + \hbar_1 \mathcal{L}^{-1}\Big[\Re_1\left(u_0, v_0\right)\Big], \\ &= \hbar_1 \mathcal{L}^{-1}\Bigg[\mathcal{L}\Big\{u_0(x,t)\Big\} - \left(1 - \frac{\chi_1}{n}\right)\left(\frac{1}{s}u(x,0) + \frac{1}{s^2}u_t(x,0)\right) \\ &\quad - \frac{1}{s^\beta}\mathcal{L}\Bigg\{\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}u_0(x,t)\right) + \frac{1}{x}\frac{\partial^2}{\partial x \partial t}\left(x\frac{\partial}{\partial x}u_0(x,t)\right) \\ &\quad - v_0(x,t) + \left(1 - \frac{\chi_1}{n}\right)\left\{2\left(t+2\right) + \left(\frac{-1}{2}x^2 + \ln(x) + \frac{3}{4}\right)t\right\}\Bigg\}\Bigg], \\ &= \hbar_1 \mathcal{L}^{-1}\Bigg[\mathcal{L}\left\{\left(\frac{-1}{2}x^2 + \ln(x) + \frac{3}{4}\right)\right\} - \frac{1}{s}\left(\frac{-1}{2}x^2 + \ln(x) + \frac{3}{4}\right) \\ &\quad - \frac{1}{s^2}\left(\frac{-1}{2}x^2 + \ln(x) + \frac{3}{4}\right) - \frac{1}{s^\beta}\mathcal{L}\Bigg\{\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\left(\frac{-1}{2}x^2 + \ln(x) + \frac{3}{4}\right)\right) \\ &\quad + \frac{1}{x}\frac{\partial^2}{\partial x \partial t}\left(x\frac{\partial}{\partial x}\left(\frac{-1}{2}x^2 + \ln(x) + \frac{3}{4}\right)\right) + 2\left(t+2\right) + \left(\frac{-1}{2}x^2 + \ln(x) + \frac{3}{4}\right)t\Bigg\}\Bigg], \\ &= -\hbar_1 \mathcal{L}^{-1}\left[\left(\frac{-1}{2}x^2 + \ln(x) + \frac{3}{4}\right)\left\{\frac{1}{s^2} + \frac{1}{s^{2+\beta}}\right\} + 2\left(\frac{1}{s^{2+\beta}} + \frac{1}{s^{1+\beta}}\right)\right], \\ &= -\hbar_1\left(\left(-\frac{1}{2}x^2 + \ln(x) + \frac{3}{4}\right)\left\{t + \frac{t^{\beta+1}}{\Gamma(\beta+2)}\right\} + 2\left\{\frac{t^{1+\beta}}{\Gamma(2+\beta)} + \frac{t^\beta}{\Gamma(1+\beta)}\right\}\right). \end{aligned} \tag{3.32}$$

We now proceed to calculate $v_1(x,t)$:

$$\begin{aligned}
v_1(x,t) &= \chi_1 v_0(x,t) + \hbar_2\mathcal{L}^{-1}\Big[\Re_2\left(u_0,v_0\right)\Big],\\
&= \chi_1 v_0(x,t) + \hbar_2\mathcal{L}^{-1}\Bigg[\mathcal{L}\Big\{v_0(x,t)\Big\} - \Big(1-\frac{\chi_1}{n}\Big)\Big(\frac{1}{s}v(x,0)+\frac{1}{s^2}v_t(x,0)\Big)\\
&\quad - \frac{1}{s^\gamma}\mathcal{L}\Bigg\{\frac{1}{x}\frac{\partial}{\partial x}\Big(x\frac{\partial}{\partial x}v_0(x,t)\Big) + \frac{1}{x}\frac{\partial^2}{\partial x\partial t}\Big(x\frac{\partial}{\partial x}v_0(x,t)\Big)\\
&\quad - u_0(x,t) + \Big(1-\frac{\chi_1}{n}\Big)\Bigg\{2\left(t+1\right)+\Big(\frac{-1}{2}x^2+\ln(x)+\frac{3}{4}\Big)\left(t+1\right)\Bigg\}\Bigg\}\Bigg],\\
&= \hbar_2\mathcal{L}^{-1}\Bigg[-\frac{1}{s^2}\Big(\frac{-1}{2}x^2+\ln(x)+\frac{3}{4}\Big) - \frac{1}{s^\gamma}\mathcal{L}\Bigg\{2\left(t+1\right)+\Big(\frac{-1}{2}x^2+\ln(x)+\frac{3}{4}\Big)t\Bigg\}\Bigg],\\
&= -\hbar_2\Bigg(\Big(-\frac{1}{2}x^2+\ln(x)+\frac{3}{4}\Big)\Bigg\{t+\frac{t^{\gamma+1}}{\Gamma(\gamma+2)}\Bigg\} + 2\Bigg\{\frac{t^{1+\gamma}}{\Gamma(2+\gamma)}+\frac{t^{\gamma}}{\Gamma(1+\gamma)}\Bigg\}\Bigg).
\end{aligned} \tag{3.33}$$

Take $m=2$, we get :

$$\begin{aligned}
u_2(x,t) &= \chi_2 u_1(x,t) + \hbar_1\mathcal{L}^{-1}\Big[\Re_1\left(u_1,v_1\right)\Big],\\
&= \chi_2 u_1(x,t) + \hbar_1\mathcal{L}^{-1}\Bigg[\mathcal{L}\Big\{u_1(x,t)\Big\} - \Big(1-\frac{\chi_2}{n}\Big)\Big(\frac{1}{s}u(x,0)+\frac{1}{s^2}u_t(x,0)\Big)\\
&\quad - \frac{1}{s^\beta}\mathcal{L}\Bigg\{\frac{1}{x}\frac{\partial}{\partial x}\Big(x\frac{\partial}{\partial x}u_1(x,t)\Big) + \frac{1}{x}\frac{\partial^2}{\partial x\partial t}\Big(x\frac{\partial}{\partial x}u_1(x,t)\Big)\\
&\quad - v_1(x,t) + \Big(1-\frac{\chi_2}{n}\Big)\Bigg\{2\left(t+2\right)+\Big(\frac{-1}{2}x^2+\ln(x)+\frac{3}{4}\Big)t\Bigg\}\Bigg],\\
&= \chi_2 u_1(x,t) + \hbar_1\mathcal{L}^{-1}\Bigg[\mathcal{L}\Big\{u_1(x,t)\Big\} - \frac{-\hbar_1}{s^\beta}\mathcal{L}\Bigg\{-2t-2\frac{t^{1+\beta}}{\Gamma(\beta+2)}-\frac{t^{\beta}}{\Gamma(\beta+1)}-2\\
&\quad - \Big(\frac{-1}{2}x^2+\ln(x)+\frac{3}{4}\Big)\Bigg\{t+\frac{t^{\gamma+1}}{\Gamma(\gamma+2)}\Bigg\}+2\Bigg\{\frac{t^{\gamma}}{\Gamma(\gamma+1)}+\frac{t^{1+\gamma}}{\Gamma(\gamma+2)}\Bigg\}\Bigg\}\Bigg],\\
&= (1+\hbar_1)u_1(x,t) + \hbar_1^2\mathcal{L}^{-1}\Bigg[\frac{1}{s^\beta}\Bigg\{-2\Bigg\{\frac{1}{s}+\frac{\Gamma(2)}{s^2}+\frac{1}{s^{\gamma+2}}+\frac{1}{s^{\gamma+1}}+\frac{1}{s^{\beta+1}}+\frac{1}{s^{\beta+2}}\Bigg\}\\
&\quad - \Big(\frac{-1}{2}x^2+\ln(x)+\frac{3}{4}\Big)\Bigg\{\frac{\Gamma(2)}{s^2}+\frac{1}{s^{\gamma+2}}\Bigg\}\Bigg\}\Bigg],\\
&= (1+\hbar_1)u_1(x,t) + \hbar_1^2\Bigg[-2\Bigg\{\frac{t^{\beta}}{\Gamma(\beta+1)}+\frac{t^{\beta+1}}{\Gamma(\beta+2)}+\frac{t^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)}+\frac{t^{\beta+\gamma+1}}{\Gamma(\beta+\gamma+2)}
\end{aligned}$$

$$\frac{t^{2\beta}}{\Gamma(2\beta+1)}+\frac{t^{2\beta+1}}{\Gamma(2\beta+2)}\Bigg\}-\left(\frac{-1}{2}x^{2}+\ln(x)+\frac{3}{4}\right)\left\{\frac{t^{\beta+\gamma+1}}{\Gamma(\beta+\gamma+2)}+\frac{t^{\gamma+1}}{\Gamma(\gamma+2)}\right\}\Bigg]. \tag{3.34}$$

The next step involves finding $v_2(x,t)$:

$$\begin{aligned}
v_2(x,t) &= \chi_2 v_1(x,t)+\hbar_2\mathcal{L}^{-1}\Big[\Re_2\left(u_1,v_1\right)\Big],\\
&= \chi_2 v_1(x,t)+\hbar_2\mathcal{L}^{-1}\Bigg[\mathcal{L}\Big\{v_1(x,t)\Big\}-\left(1-\frac{\chi_2}{n}\right)\left(\frac{1}{s}v(x,0)+\frac{1}{s^2}v_t(x,0)\right)\\
&\quad-\frac{1}{s^{\gamma}}\mathcal{L}\Bigg\{\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}v_1(x,t)\right)+\frac{1}{x}\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial}{\partial x}v_1(x,t)\right)\\
&\quad-u_1(x,t)+\left(1-\frac{\chi_2}{n}\right)\left\{2\left(t+1\right)+\left(\frac{-1}{2}x^2+\ln(x)+\frac{3}{4}\right)\left(t+1\right)\right\}\Bigg\}\Bigg],\\
&= \chi_2 v_1(x,t)+\hbar_2\mathcal{L}^{-1}\Bigg[\mathcal{L}\Big\{v_1(x,t)\Big\}-\frac{-\hbar_2}{s^{\gamma}}\mathcal{L}\Bigg\{-2t-2-\left(\frac{-1}{2}x^2+\ln(x)+\frac{3}{4}\right)t\\
&\quad-\frac{2t^{\gamma}}{\Gamma(\gamma+1)}-\frac{2t^{1+\gamma}}{\Gamma(\gamma+2)}+2\left\{\frac{t^{\beta}}{\Gamma(\beta+1)}+\frac{t^{1+\beta}}{\Gamma(\beta+2)}\right\}\Bigg\}\Bigg],\\
&= (1+\hbar_2)v_1(x,t)+\hbar_2^2\mathcal{L}^{-1}\Bigg[\frac{1}{s^{\gamma}}\Bigg\{-2\Bigg\{\frac{1}{s}+\frac{\Gamma(2)}{s^2}+\frac{\Gamma(\beta+2)}{s^{\beta+2}}+\frac{\Gamma(\beta+1)}{s^{\beta+1}}+\frac{\Gamma(\gamma+1)}{s^{\gamma+1}}\\
&\quad+\frac{\Gamma(\gamma+2)}{s^{\gamma+2}}\Bigg\}-\left(\frac{-1}{2}x^2+\ln(x)+\frac{3}{4}\right)\left\{\frac{\Gamma(\beta+2)}{s^{\beta+2}}+\frac{\Gamma(2)}{s^2}\right\}\Bigg\}\Bigg],\\
&= (1+\hbar_2)v_1(x,t)+\hbar_2^2\Bigg[-2\Bigg\{\frac{t^{\gamma}}{\Gamma(\gamma+1)}+\frac{t^{\gamma+1}}{\Gamma(\gamma+2)}+\frac{t^{\gamma+\beta}}{\Gamma(\gamma+\beta+1)}+\frac{t^{\gamma+\beta+1}}{\Gamma(\gamma+\beta+2)}\\
&\quad+\frac{t^{2\gamma}}{\Gamma(2\gamma+1)}+\frac{t^{2\gamma+1}}{\Gamma(2\gamma+2)}\Bigg\}-\left(\frac{-1}{2}x^2+\ln(x)+\frac{3}{4}\right)\left\{\frac{t^{\gamma+1}}{\Gamma(\gamma+2)}+\frac{t^{\gamma+\beta+1}}{\Gamma(\gamma+\beta+2)}\right\}\Bigg].
\end{aligned} \tag{3.35}$$

Take $m=3$, we get:

$$\begin{aligned}
u_3(x,t) &= \chi_3 u_2(x,t)+\hbar_1\mathcal{L}^{-1}\Big[\Re_1\left(u_2,v_2\right)\Big],\\
&= \chi_1 u_2(x,t)+\hbar_1\mathcal{L}^{-1}\Bigg[\mathcal{L}\Big\{u_2(x,t)\Big\}-\left(1-\frac{\chi_2}{n}\right)\left(\frac{1}{s}u(x,0)+\frac{1}{s^2}u_t(x,0)\right)\\
&\quad-\frac{1}{s^{\beta}}\mathcal{L}\Bigg\{\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}u_2(x,t)\right)+\frac{1}{x}\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial}{\partial x}u_2(x,t)\right)-v_2(x,t)\\
&\quad+\left(1-\frac{\chi_3}{n}\right)\left\{2\left(t+1\right)+\left(\frac{-1}{2}x^2+\ln(x)+\frac{3}{4}\right)t\right\}\Bigg\}\Bigg],
\end{aligned}$$

$$
\begin{aligned}
&= \chi_3 u_2(x,t) + \hbar_1 \mathcal{L}^{-1}\Bigg[\mathcal{L}\Big\{u_2(x,t)\Big\} - \frac{1}{s^{\beta}}\mathcal{L}\Bigg\{2t(1+\hbar_1)\hbar_1 - 2\hbar_1^2\Bigg\{\frac{t^{\beta+1}}{\Gamma(\beta+2)} + \frac{t^{2\beta+1}}{\Gamma(2\beta+2)}\Bigg\} \\
&\quad + 2\hbar_1(1+\hbar_1) - 2\hbar_1^2\Bigg\{\frac{t^{\beta}}{\Gamma(\beta+1)} + \frac{t^{2\beta}}{\Gamma(2\beta+1)}\Bigg\} - (1+\hbar_2)u_1(x,t) + \hbar_1^2\Bigg\{2\Bigg\{\frac{t^{\gamma}}{\Gamma(\gamma+1)} \\
&\quad + \frac{t^{\gamma+1}}{\Gamma(\gamma+2)} + \frac{t^{\gamma+\beta+1}}{\Gamma(\gamma+\beta+2)} + \frac{t^{\gamma+\beta}}{\Gamma(\gamma+\beta+1)}\Bigg\} + \frac{t^{\gamma+1}}{\Gamma(\gamma+2)}\left(\frac{-1}{2}x^2 + \ln(x) + \frac{3}{4}\right)\Bigg\}\Bigg\}\Bigg], \\
&= (1+\hbar_1)u_2(x,t) - \hbar_1^2\Bigg[2\Bigg\{(1+\hbar_1)\frac{t^{\beta+1}}{\Gamma(\beta+2)} + \hbar_1\frac{t^{2\beta+1}}{\Gamma(2\beta+2)} + (1+\hbar_1)\frac{t^{\beta}}{\Gamma(\beta+1)} + (1+\hbar_2) \\
&\quad \Bigg\{\frac{t^{\gamma}}{\Gamma(\gamma+1)} + \frac{t^{\gamma+1}}{\Gamma(\gamma+2)}\Bigg\} + \hbar_1\Bigg\{\frac{t^{\gamma}}{\Gamma(\gamma+1)} + \frac{t^{\gamma+1}}{\Gamma(\gamma+2)} + \frac{t^{\gamma+\beta}}{\Gamma(\gamma+\beta+1)} + \frac{t^{\gamma+\beta+1}}{\Gamma(\gamma+\beta+2)}\Bigg\} \\
&\quad + \frac{t^{\beta+1}}{\Gamma(\beta+2)}\left(\frac{-1}{2}x^2 + \ln(x) + \frac{3}{4}\right)(1+\hbar_2) + \frac{\hbar_1 t^{\gamma+\beta+1}}{\Gamma(\gamma+\beta+2)}\left(\frac{-1}{2}x^2 + \ln(x) + \frac{3}{4}\right)\Bigg].
\end{aligned}
\tag{3.36}
$$

let us now determine the value of $v_3(x,t)$:

$$
\begin{aligned}
v_3(x,t) &= \chi_3 v_2(x,t) + \hbar_2 \mathcal{L}^{-1}\Big[\Re_2\left(u_2, v_2\right)\Big], \\
&= \chi_1 v_2(x,t) + \hbar_2 \mathcal{L}^{-1}\Bigg[\mathcal{L}\Big\{v_2(x,t)\Big\} - \left(1 - \frac{\chi_2}{n}\right)\left(\frac{1}{s}v(x,0) + \frac{1}{s^2}v_t(x,0)\right) \\
&\quad - \frac{1}{s^{\gamma}}\mathcal{L}\Bigg\{\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}v_2(x,t)\right) + \frac{1}{x}\frac{\partial^2}{\partial x \partial t}\left(x\frac{\partial}{\partial x}v_2(x,t)\right) - u_2(x,t)\Bigg\}\Bigg], \\
&= \chi_3 v_2(x,t) + \hbar_2 \mathcal{L}^{-1}\Bigg[\mathcal{L}\Big\{v_2(x,t)\Big\} - \frac{1}{s^{\gamma}}\mathcal{L}\Bigg\{2\hbar_2^2\Bigg\{\frac{t^{\gamma+1}}{\Gamma(\gamma+2)} + \frac{t^{\gamma+\beta+1}}{\Gamma(\gamma+\beta+2)} \\
&\quad + \frac{t^{\gamma}}{\Gamma(\gamma+1)} + \frac{t^{\gamma+\beta}}{\Gamma(\gamma+\beta+1)}\Bigg\} - (1+\hbar_1)u_1(x,t) - \hbar_1^2\Bigg\{2\Bigg\{\frac{t^{\beta}}{\Gamma(\beta+1)} + \frac{t^{\beta+1}}{\Gamma(\beta+2)} \\
&\quad + \frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{t^{2\beta+1}}{\Gamma(2\beta+2)} - \frac{t^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)} - \frac{t^{\beta+\gamma+1}}{\Gamma(\beta+\gamma+2)}\Bigg\} \\
&\quad - \left(\frac{-1}{2}x^2 + \ln(x) + \frac{3}{4}\right)\left\{\frac{t^{\beta+1}}{\Gamma(\beta+2)} + \frac{t^{\beta+\gamma+1}}{\Gamma(\beta+\gamma+2)}\right\}\Bigg\}\Bigg\}\Bigg],
\end{aligned}
$$

$$\begin{aligned}
&= (1+\hbar_2)v_2(x,t) - \hbar_2^3\Bigg\{2\Bigg\{\frac{t^{2\gamma+1}}{\Gamma(2\gamma+2)} + \frac{t^{2\gamma+\beta+1}}{\Gamma(2\gamma+\beta+2)} \\
&+ \frac{t^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{t^{2\gamma+\beta}}{\Gamma(2\gamma+\beta+1)}\Bigg\}\Bigg\} - \hbar_2\hbar_1^2\Bigg\{-2\Bigg\{\frac{t^{\beta+\gamma}}{\Gamma(\gamma+\beta+1)} + \frac{t^{\gamma+\beta+1}}{\Gamma(\gamma+\beta+2)} \\
&+ \frac{t^{2\beta+\gamma}}{\Gamma(2\beta+\gamma+1)} + \frac{t^{2\beta+\gamma+1}}{\Gamma(2\beta+\gamma+2)}\Bigg\}\Bigg\} + \hbar_2(1+\hbar_1)u_1(x,t) \\
&- \hbar_2\hbar_1^2\left(\frac{-1}{2}x^2+\ln(x)+\frac{3}{4}\right)\left\{\frac{t^{\beta+\gamma+1}}{\Gamma(\gamma+\beta+2)} + \frac{t^{\beta+2\gamma+1}}{\Gamma(2\gamma+\beta+2)}\right\}.
\end{aligned} \tag{3.37}$$

Thus, the solution to the coupled system is given by:

$$\begin{aligned}
u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + ... \\
&= \left(-\frac{1}{2}x^2+\ln(x)+\frac{3}{4}\right) - \hbar_1\Bigg(\left(-\frac{1}{2}x^2+\ln(x)+\frac{3}{4}\right)\left\{t+\frac{t^{\beta+1}}{\Gamma(\beta+2)}\right\} \\
&+ 2\left\{\frac{t^{1+\beta}}{\Gamma(2+\beta)} + \frac{t^{\beta}}{\Gamma(1+\beta)}\right\}\Bigg) + (1+\hbar_1)u_1(x,t) + \hbar_1^2\Bigg[-2\Bigg\{\frac{t^{\beta}}{\Gamma(\beta+1)} + \frac{t^{\beta+1}}{\Gamma(\beta+2)} \\
&+ \frac{t^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)} + \frac{t^{\beta+\gamma+1}}{\Gamma(\beta+\gamma+2)}\frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{t^{2\beta+1}}{\Gamma(2\beta+2)}\Bigg\} \\
&\quad - \left(\frac{-1}{2}x^2+\ln(x)+\frac{3}{4}\right)\left\{\frac{t^{\beta+\gamma+1}}{\Gamma(\beta+\gamma+2)} + \frac{t^{\gamma+1}}{\Gamma(\gamma+2)}\right\}\Bigg] + ...
\end{aligned} \tag{3.38}$$

$$\begin{aligned}
v(x,t) &= v_0(x,t) + v_1(x,t) + v_2(x,t) + v_3(x,t) + ... \\
&= -\hbar_2\left(\left(-\frac{1}{2}x^2+\ln(x)+\frac{3}{4}\right)\left\{t+\frac{t^{\gamma+1}}{\Gamma(\gamma+2)}\right\} + 2\left\{\frac{t^{1+\gamma}}{\Gamma(2+\gamma)} + \frac{t^{\gamma}}{\Gamma(1+\gamma)}\right\}\right) \\
&+ (1+\hbar_2)v_1(x,t) + \hbar_2^2\Bigg[-2\Bigg\{\frac{t^{\gamma}}{\Gamma(\gamma+1)} + \frac{t^{\gamma+1}}{\Gamma(\gamma+2)} + \frac{t^{\gamma+\beta}}{\Gamma(\gamma+\beta+1)} + \frac{t^{\gamma+\beta+1}}{\Gamma(\gamma+\beta+2)} \\
&+ \frac{t^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{t^{2\gamma+1}}{\Gamma(2\gamma+2)}\Bigg\} - \left(\frac{-1}{2}x^2+\ln(x)+\frac{3}{4}\right)\left\{\frac{t^{\gamma+1}}{\Gamma(\gamma+2)} + \frac{t^{\gamma+\beta+1}}{\Gamma(\gamma+\beta+2)}\right\}\Bigg] + ...
\end{aligned} \tag{3.39}$$

When substituting $\beta = \gamma = 2$ and $\hbar_1 = \hbar_2 = -1$, we obtain a series of the form:

$$\begin{aligned}
u_m &= \left(\frac{-1}{2}x^2+\ln(x)+\frac{3}{4}\right)\left\{1+t+\frac{t^{m\beta+1}}{\Gamma(m\beta+2)}\right\} + 2\frac{t^{m\beta}}{\Gamma(m\beta+1)}\mathrm{E}_{1,m\beta+1}(t), \\
v_m &= \left(\frac{-1}{2}x^2+\ln(x)+\frac{3}{4}\right)\left\{t+\frac{t^{m\gamma+1}}{\Gamma(m\gamma+2)}\right\} + 2\frac{t^{m\gamma}}{\Gamma(m\gamma+1)}\mathrm{E}_{1,m\gamma+1}(t),
\end{aligned} \tag{3.40}$$

When m tends to infinity

$$\frac{t^{m\gamma}}{\Gamma(m\gamma+1)}\mathrm{E}_{1,m\gamma+1}(t) \to 0, \qquad \text{and} \qquad \frac{t^{m\gamma+1}}{\Gamma(m\gamma+2)} \to 0,$$

So we get the exact solutions of the coupled system :

$$\begin{cases} u(x,t) = & \left(\frac{-1}{2}x^2 + \ln(x) + \frac{3}{4}\right)(1+t), \\ v(x,t) = & \left(\frac{-1}{2}x^2 + \ln(x) + \frac{3}{4}\right)t, \end{cases} \tag{3.41}$$

The table **(01)**, presents a coparison of the numerical solution $u(x,t), v(x,t)$ and the approximate solution $u_m(x,t), v_m(x,t)$,including their error. Notably,as m increases, the approximate solution approaches the exact solution $u(x,t), v(x,t)$.

t	x	m	$(u^{(m)}, v^{(m)})$	(u_{exact}, v_{exact})	$error$
0.1	0.1	1	-1.71287	-1.71334	4.75126×10^{-4}
			0.621554	0.623034	1.4801×10^{-3}
		2	-1.71301	-1.71334	3.33298×10^{-4}
			0.623030	0.623034	2.54247×10^{-6}
		3	-1.71334	-1.71334	2.6322×10^{-10}
			0.623034	0.623034	2.45596×10^{-9}
0.5	0.4	1	-0.245731	-0.443323	1.97592×10^{-1}
			0.467676	0.492581	2.49058×10^{-2}
		2	-2.79729	-0.443323	3.09619×10^{-2}
			0.491922	0.492581	6.59955×10^{-4}
		3	-0.443292	-0.443323	3.08603×10^{-5}
			0.492591	0.492581	9.83397×10^{-6}
1	0.6	1	0.495755	0.118349	3.77406×10^{-1}
			-0.192032	-0.236698	4.46656×10^{-2}
		2	-0.00978826	0.118349	1.28137×10^{-1}
			-0.232852	-0.236698	3.84537×10^{-3}
		3	0.118317	0.118349	3.16228×10^{-5}
			-0.236884	-0.236698	1.86167×10^{-4}

Table 01: Comparison between approximate and exact solutions for parameters $h = -1$, $\beta = 1.9$, $\gamma = 1.7$, $n = 1$.

The numerical results demonstrate that the proposed algorithm exhibits both high efficiency and stable convergence toward the exact solution as the approximation order (m) increases. While initial solution estimates at lower-order approximations $(m = 1, 2)$ show some deviation, the method achieves remarkable accuracy (with errors below 10^{-5}) at $m = 3$. This behavior confirms the algorithm's robustness and its optimal balance between computational efficiency and numerical precision. These characteristics make the method particularly suitable for practical engineering applications where both solution reliability and convergence rate are critical.

$$\text{Error} = \|\mathbf{u}_{\text{num}} - \mathbf{u}_{\text{exact}}\| < 10^{-5} \quad \text{for} \quad m \geq 3 \tag{3.42}$$

Example 2:
We consider a fractional coupled system with $1 < \beta, \gamma \leq 2$:

$$\begin{cases} {}^{C}\partial_{0t}^{\beta} u - \frac{1}{x}\left(xu_x\right)_x - \frac{1}{x}\left(xu_x\right)_{xt} + v = f(x,t), \;\; 0 < x < 1 \\ {}^{C}\partial_{0t}^{\gamma} v - \frac{1}{x}\left(xv_x\right)_x - \frac{1}{x}\left(xv_x\right)_{xt} + u = g(x,t), \;\; 0 < t < T \end{cases} \tag{3.43}$$

where:

$$\begin{aligned} f(x,t) &= \Big(-4\ln(x) + 2x^2 - 3\Big)\left(\frac{2t^{2-\beta}}{\Gamma(3-\beta)} + t + 1\right) - 8\left(t+1\right)^2, \\ g(x,t) &= \Big(2x^2 - 3 - 4\ln(x)\Big)\Big(t^2 + 1\Big) - 8\left(t+2\right), \end{aligned} \tag{3.44}$$

with:

$$\begin{aligned} &u_0(x,t) = u(x,0) = \Big(-4\ln(x) + 2x^2 - 3\Big), \quad u_t(x,0) = 0 \\ &v_0(x,t) = v(x,0) = \Big(2x^2 - 3 - 4\ln(x)\Big), \quad v_t(x,0) = \Big(2x^2 - 3 - 4\ln(x)\Big), \end{aligned} \tag{3.45}$$

The function satisfies :

$$u_x(1,t) = 0, \quad v_x(1,t) = 0,$$
$$\int_0^1 xu(x,t)\;dx = 0, \quad \int_0^1 xv(x,t)\;dx = 0,$$

Then, put $m = 1$ in equations (3.28), we get :

$$\begin{aligned} u_1(x,t) &= \chi_1 u_0(x,t) + \hbar_1 \mathcal{L}^{-1}\Big[\Re_1\left(u_0, v_0\right)\Big], \\ &= \hbar_1 \mathcal{L}^{-1}\Bigg[\mathcal{L}\Big\{u_0(x,t)\Big\} - \left(1 - \frac{\chi_1}{n}\right)\left(\frac{1}{s}u(x,0) + \frac{1}{s^2}u_t(x,0)\right) \\ &\quad - \frac{1}{s^\beta}\mathcal{L}\Bigg\{\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}u_0(x,t)\right) + \frac{1}{x}\frac{\partial^2}{\partial x \partial t}\left(x\frac{\partial}{\partial x}u_0(x,t)\right) \\ &\quad - v_0(x,t) + \left(1 - \frac{\chi_1}{n}\right)\Bigg\{\Big(-4\ln(x) + 2x^2 - 3\Big)\left(\frac{2t^{2-\beta}}{\Gamma(3-\beta)} + t + 1\right) - 8\left(t+1\right)^2\}\Bigg\}\Bigg], \\ &= \hbar_1 \mathcal{L}^{-1}\Bigg[\mathcal{L}\left\{\Big(-4\ln(x) + 2x^2 - 3\Big)\right\} - \frac{1}{s}\Big(-4\ln(x) + 2x^2 - 3\Big) \\ &\quad - \frac{1}{s^2}\Big(-4\ln(x) + 2x^2 - 3\Big) - \frac{1}{s^\beta}\mathcal{L}\Bigg\{\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\Big(-4\ln(x) + 2x^2 - 3\Big)\right) \\ &\quad + \frac{1}{x}\frac{\partial^2}{\partial x \partial t}\left(x\frac{\partial}{\partial x}\Big(-4\ln(x) + 2x^2 - 3\Big)\right) + \Big(-4\ln(x) + 2x^2 - 3\Big)\left(\frac{2t^{2-\beta}}{\Gamma(3-\beta)} + t + 1\right) \\ &\quad - 8\left(t+1\right)^2\Bigg\}\Bigg], \\ &= -\hbar_1 \mathcal{L}^{-1}\Bigg[\Big(-4\ln(x) + 2x^2 - 3\Big)\left\{\frac{2}{s^2} + \frac{1}{s^{2+\beta}}\right\} - 8\left(\frac{2}{s^{3+\beta}} + \frac{2}{s^{2+\beta}}\right)\Bigg], \\ &= -\hbar_1\Bigg(\Big(-4\ln(x) + 2x^2 - 3\Big)\left\{t^2 + \frac{t^{\beta+1}}{\Gamma(\beta+2)}\right\} - 8\left\{\frac{2t^{2+\beta}}{\Gamma(3+\beta)} + 2\frac{t^{\beta+1}}{\Gamma(2+\beta)}\right\}\Bigg). \end{aligned} \tag{3.46}$$

Moving on to $v_1(x,t)$:

$$\begin{aligned}
v_1(x,t) =& \chi_1 v_0(x,t) + \hbar_2 \mathcal{L}^{-1}\Big[\Re_2\left(u_0, v_0\right)\Big],\\
=& \chi_1 v_0(x,t) + \hbar_2 \mathcal{L}^{-1}\Bigg[\mathcal{L}\Big\{v_0(x,t)\Big\} - \left(1 - \frac{\chi_1}{n}\right)\left(\frac{1}{s}v(x,0) + \frac{1}{s^2}v_t(x,0)\right)\\
&- \frac{1}{s^\gamma}\mathcal{L}\Bigg\{\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}v_0(x,t)\right) + \frac{1}{x}\frac{\partial^2}{\partial x \partial t}\left(x\frac{\partial}{\partial x}v_0(x,t)\right) - u_0(x,t)\\
&+ \left(1 - \frac{\chi_1}{n}\right)\Bigg\{\Big(2x^2 - 3 - 4\ln(x)\Big)\Big(t^2 + 1\Big) - 8\left(t + 2\right)\Bigg\}\Bigg\}\Bigg],\\
=& \hbar_2 \mathcal{L}^{-1}\Bigg[-\frac{1}{s^2}\Big(2x^2 - 3 - 4\ln(x)\Big) - \frac{1}{s^\gamma}\mathcal{L}\Bigg\{-8\left(t+1\right) + \Big(2x^2 - 3 - 4\ln(x)\Big)t^2\Bigg\}\Bigg],\\
=& -\hbar_2\left(\Big(2x^2 - 3 - 4\ln(x)\Big)\left\{t + \frac{2t^{\gamma+2}}{\Gamma(\gamma+3)}\right\} - 8\left\{\frac{t^{1+\gamma}}{\Gamma(2+\gamma)} + \frac{t^\gamma}{\Gamma(1+\gamma)}\right\}\right).
\end{aligned} \tag{3.47}$$

Take $m = 2$, we get :

$$\begin{aligned}
u_2(x,t) =& \chi_2 u_1(x,t) + \hbar_1 \mathcal{L}^{-1}\Big[\Re_1\left(u_1, v_1\right)\Big],\\
=& \chi_2 u_1(x,t) + \hbar_1 \mathcal{L}^{-1}\Bigg[\mathcal{L}\Big\{u_1(x,t)\Big\} - \left(1 - \frac{\chi_2}{n}\right)\left(\frac{1}{s}u(x,0) + \frac{1}{s^2}u_t(x,0)\right)\\
&- \frac{1}{s^\beta}\mathcal{L}\Bigg\{\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}u_1(x,t)\right) + \frac{1}{x}\frac{\partial^2}{\partial x \partial t}\left(x\frac{\partial}{\partial x}u_1(x,t)\right)\\
&- v_1(x,t) + \left(1 - \frac{\chi_2}{n}\right)\Bigg\{\Bigg\{\Big(-4\ln(x) + 2x^2 - 3\Big)\left(\frac{2t^{2-\beta}}{\Gamma(3-\beta)} + t + 1\right) - 8\left(t+1\right)^2\Bigg\}\Bigg\}\Bigg],\\
=& \chi_2 u_1(x,t) + \hbar_1 \mathcal{L}^{-1}\Bigg[\mathcal{L}\Big\{u_1(x,t)\Big\} - \frac{-\hbar_1}{s^\beta}\mathcal{L}\Bigg\{8t^2 + 8\frac{t^{1+\beta}}{\Gamma(\beta+2)} + 8\frac{t^\beta}{\Gamma(\beta+1)} + 16t\\
&- \Big(-4\ln(x) + 2x^2 - 3\Big)\left\{t + \frac{2t^{\gamma+2}}{\Gamma(\gamma+3)}\right\} + 8\left\{\frac{t^\gamma}{\Gamma(\gamma+1)} + \frac{t^{1+\gamma}}{\Gamma(\gamma+2)}\right\}\Bigg\}\Bigg],\\
=& (1 + \hbar_1)u_1(x,t) + \hbar_1^2 \mathcal{L}^{-1}\Bigg[\frac{1}{s^\beta}\Bigg\{8\Bigg\{\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s^{\gamma+2}} + \frac{1}{s^{\gamma+1}} + \frac{1}{s^{\beta+1}} + \frac{1}{s^{\beta+2}}\Bigg\}\\
&- \Big(-4\ln(x) + 2x^2 - 3\Big)\left\{\frac{2}{s^2} + \frac{2}{s^{\gamma+3}}\right\}\Bigg\}\Bigg],
\end{aligned}$$

$$
\begin{aligned}
&= (1+\hbar_1)u_1(x,t) + \hbar_1^2\Bigg[8\Bigg\{\frac{2t^{\beta+1}}{\Gamma(\beta+2)} + \frac{t^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)} + \frac{t^{\beta+\gamma+1}}{\Gamma(\beta+\gamma+2)} + \frac{2t^{\beta+2}}{\Gamma(\beta+3)} + \frac{t^{2\beta}}{\Gamma(2\beta+1)} \\
&+ \frac{t^{2\beta+1}}{\Gamma(2\beta+2)}\Bigg\} - \Big(-4\ln(x) + 2x^2 - 3\Big)\left\{\frac{2t^{\beta+\gamma+2}}{\Gamma(\beta+\gamma+3)} + \frac{t^{\beta+1}}{\Gamma(\beta+2)}\right\}\Bigg].
\end{aligned}
\tag{3.48}
$$

Now, we calculate $v_2(x,t)$:

$$
\begin{aligned}
v_2(x,t) &= \chi_2 v_1(x,t) + \hbar_2 \mathcal{L}^{-1}\Big[\Re_2\left(u_1, v_1\right)\Big], \\
&= \chi_2 v_1(x,t) + \hbar_2 \mathcal{L}^{-1}\Bigg[\mathcal{L}\Big\{v_1(x,t)\Big\} - \left(1 - \frac{\chi_2}{n}\right)\left(\frac{1}{s}v(x,0) + \frac{1}{s^2}v_t(x,0)\right) \\
&- \frac{1}{s^\gamma}\mathcal{L}\Bigg\{\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}v_1(x,t)\right) + \frac{1}{x}\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial}{\partial x}v_1(x,t)\right) - u_1(x,t) \\
&+ \left(1 - \frac{\chi_2}{n}\right)\left\{\Big(2x^2 - 3 - 4\ln(x)\Big)\Big(t^2+1\Big) - 8\left(t+2\right)\right\}\Bigg\}\Bigg], \\
&= \chi_2 v_1(x,t) + \hbar_2 \mathcal{L}^{-1}\Bigg[\mathcal{L}\Big\{v_1(x,t)\Big\} - \frac{-\hbar_2}{s^\gamma}\mathcal{L}\Bigg\{8t + 8 - \Big(2x^2 - 3 - 4\ln(x)\Big)t^2 \\
&+ 8\frac{2t^{2+\gamma}}{\Gamma(\gamma+3)} + 8\frac{2t^{1+\gamma}}{\Gamma(\gamma+2)} + 8\left\{\frac{t^\beta}{\Gamma(\beta+1)} + \frac{t^{1+\beta}}{\Gamma(\beta+2)}\right\} - \frac{t^{1+\beta}}{\Gamma(\beta+2)}\Big(2x^2 - 3 - 4\ln(x)\Big)\Bigg\}\Bigg], \\
&= (1+\hbar_2)v_1(x,t) + \hbar_2^2 \mathcal{L}^{-1}\Bigg[\frac{1}{s^\gamma}\Bigg\{8\Bigg\{\frac{1}{s} + \frac{\Gamma(2)}{s^2} + \frac{1}{s^{\beta+2}} + \frac{1}{s^{\beta+1}} + \frac{2}{s^{\gamma+3}} + \frac{2}{s^{\gamma+2}}\Bigg\} \\
&- \Big(2x^2 - 3 - 4\ln(x)\Big)\left\{\frac{1}{s^{\beta+2}} + \frac{2}{s^3}\right\}\Bigg\}\Bigg], \\
&= (1+\hbar_2)v_1(x,t) + \hbar_2^2\Bigg[8\Bigg\{\frac{t^\gamma}{\Gamma(\gamma+1)} + \frac{t^{\gamma+1}}{\Gamma(\gamma+2)} + \frac{t^{\gamma+\beta}}{\Gamma(\gamma+\beta+1)} + \frac{t^{\gamma+\beta+1}}{\Gamma(\gamma+\beta+2)} \\
&+ \frac{2t^{2+\gamma}}{\Gamma(\gamma+3)} + \frac{2t^{2\gamma+1}}{\Gamma(2\gamma+2)}\Bigg\} - \Big(2x^2 - 3 - 4\ln(x)\Big)\left\{\frac{2t^{\gamma+2}}{\Gamma(\gamma+3)} + \frac{t^{\gamma+\beta+1}}{\Gamma(\gamma+\beta+2)}\right\}\Bigg].
\end{aligned}
\tag{3.49}
$$

If $m = 3$, we get:

$$
\begin{aligned}
u_3(x,t) &= \chi_3 u_2(x,t) + \hbar_1 \mathcal{L}^{-1}\Big[\Re_1\left(u_2, v_2\right)\Big], \\
&= \chi_1 u_2(x,t) + \hbar_1 \mathcal{L}^{-1}\Bigg[\mathcal{L}\Big\{u_2(x,t)\Big\} - \left(1 - \frac{\chi_2}{n}\right)\left(\frac{1}{s}u(x,0) + \frac{1}{s^2}u_t(x,0)\right) \\
&- \frac{1}{s^\beta}\mathcal{L}\Bigg\{\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}u_2(x,t)\right) + \frac{1}{x}\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial}{\partial x}u_2(x,t)\right) - v_2(x,t) \\
&+ \left(1 - \frac{\chi_3}{n}\right)\left\{\Big(-4\ln(x) + 2x^2 - 3\Big)\left(\frac{2t^{2-\beta}}{\Gamma(3-\beta)} + t + 1\right) - 8\left(t+1\right)^2\right\}\Bigg\}\Bigg],
\end{aligned}
$$

In the same way and steps, we find $u_3(x,t)$:

$$\begin{aligned}
u_3(x,t) = {} & (1+\hbar_1)u_2(x,t) + \hbar_1^2(1+\hbar_1)\Bigg\{8\Bigg\{\frac{2t^{\beta+1}}{\Gamma(\beta+2)} + \frac{t^{2\beta+1}}{\Gamma(2\beta+2)} + \frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{t^{\gamma+\beta}}{\Gamma(\gamma+\beta+1)} \\
& + \frac{t^{\beta+\gamma+1}}{\Gamma(\beta+\gamma+2)} + \frac{2t^{\beta+2}}{\Gamma(\beta+3)}\Bigg\} - \Big(-4\ln(x) + 2x^2 - 3\Big)\left\{\frac{t^{\beta+1}}{\Gamma(\beta+2)} + \frac{2t^{\gamma+\beta+2}}{\Gamma(\gamma+\beta+3)}\right\}\Bigg\} \\
& - \hbar_1^3 + \Bigg\{-8\Bigg\{\frac{t^{2\beta+1}}{\Gamma(2\beta+2)} + \frac{t^{2\beta+1}}{\Gamma(2\beta+2)} + \frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{t^{\gamma+\beta}}{\Gamma(\gamma+\beta+1)} + \frac{t^{\beta+\gamma+1}}{\Gamma(\beta+\gamma+2)} \\
& + \frac{2t^{2\beta+\gamma+2}}{\Gamma(\gamma+2\beta+3)} + \frac{t^{\gamma+2\beta+1}}{\Gamma(\gamma+2\beta+2)} + \frac{t^{2\gamma+\beta+2}}{\Gamma(2\gamma+\beta+3)} + \frac{t^{\gamma+2\beta}}{\Gamma(\gamma+2\beta+1)}\Bigg\}\Bigg\} \\
& + \Big(-4\ln(x) + 2x^2 - 3\Big)\left\{\frac{2t^{\gamma+\beta+2}}{\Gamma(\gamma+\beta+3)} + \frac{t^{\gamma+2\beta+1}}{\Gamma(\gamma+2\beta+2)}\right\}.
\end{aligned} \tag{3.50}$$

Now, we find $v_3(x,t)$:

$$\begin{aligned}
v_3(x,t) = {} & (1+\hbar_2)v_2(x,t) + \hbar_2^2(1+\hbar_2)\Bigg\{8\Bigg\{\frac{t^{\gamma+1}}{\Gamma(\gamma+2)} + \frac{2t^{2\gamma+1}}{\Gamma(2\gamma+2)} + \frac{t^{\gamma}}{\Gamma(\gamma+1)} + \frac{2t^{2+\gamma+\beta}}{\Gamma(\gamma+\beta+3)} \\
& + \frac{2t^{\beta+\gamma+1}}{\Gamma(\beta+\gamma+2)} + \frac{2t^{2\gamma+2}}{\Gamma(2\gamma+3)}\Bigg\} - \Big(2x^2 - 4\ln(x) - 3\Big)\left\{\frac{t^{\gamma+\beta+1}}{\Gamma(\gamma+\beta+2)} + \frac{t^{\gamma+2}}{\Gamma(\gamma+3)}\right\}\Bigg\} \\
& - \hbar_2^3 + \Bigg\{-8\Bigg\{\frac{t^{2\beta+\gamma}}{\Gamma(2\beta+\gamma+1)} + \frac{2t^{\gamma+\beta+1}}{\Gamma(\gamma+\beta+2)} + \frac{2t^{2+\beta+\gamma}}{\Gamma(3+\beta+\gamma)} + \frac{t^{2\gamma+\beta+1}}{\Gamma(2\gamma+\beta+2)} \\
& + \frac{t^{\beta+2\gamma}}{\Gamma(\beta+2\gamma+1)} + \frac{t^{\gamma+2\beta+1}}{\Gamma(\gamma+2\beta+2)}\Bigg\} + \Big(2x^2 - 4\ln(x) - 3\Big)\Bigg\{\frac{2t^{2\gamma+\beta+2}}{\Gamma(2\gamma+\beta+3)} \\
& + \frac{t^{\gamma+\beta+1}}{\Gamma(\gamma+\beta+2)}\Bigg\}\Bigg\}.
\end{aligned} \tag{3.51}$$

Thus, the solution to the coupled system is given by:

$$\begin{aligned}
u(x,t) & = u_0(x,t) + \left(\frac{1}{2}\right)u_1(x,t) + \left(\frac{1}{2}\right)^2 u_2(x,t) + \left(\frac{1}{2}\right)^3 u_3(x,t) + ... \\
& = -\hbar_1\Bigg(\Big(-4\ln(x) + 2x^2 - 3\Big)\left\{1 + t^2 + \frac{t^{\beta+1}}{\Gamma(\beta+2)}\right\} - 8\left\{\frac{2t^{2+\beta}}{\Gamma(3+\beta)} + 2\frac{t^{\beta+1}}{\Gamma(2+\beta)}\right\}\Bigg) \\
& \quad + (1+\hbar_1)u_1(x,t) + \hbar_1^2\Bigg[8\Bigg\{\frac{2t^{\beta+1}}{\Gamma(\beta+2)} + \frac{t^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)} + \frac{t^{\beta+\gamma+1}}{\Gamma(\beta+\gamma+2)} + \frac{2t^{\beta+2}}{\Gamma(\beta+3)} \\
& \quad + \frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{t^{2\beta+1}}{\Gamma(2\beta+2)}\Bigg\} - \Big(-4\ln(x) + 2x^2 - 3\Big)\left\{\frac{2t^{\beta+\gamma+2}}{\Gamma(\beta+\gamma+3)} + \frac{t^{\beta+1}}{\Gamma(\beta+2)}\right\}\Bigg] + ...,
\end{aligned} \tag{3.52}$$

And

$$
\begin{aligned}
v(x,t) &= v_0(x,t) + \left(\frac{1}{2}\right) v_1(x,t) + \left(\frac{1}{2}\right)^2 v_2(x,t) + \left(\frac{1}{2}\right)^3 v_3(x,t) + ..., \\
&= -\hbar_2 \left(\Big(2x^2 - 3 - 4\ln(x)\Big) \left\{ 1 + t + \frac{2t^{\gamma+2}}{\Gamma(\gamma+3)} \right\} - 8 \left\{ \frac{t^{1+\gamma}}{\Gamma(2+\gamma)} + \frac{t^{\gamma}}{\Gamma(1+\gamma)} \right\} \right) \\
&\quad + (1+\hbar_2) v_1(x,t) + \hbar_2^2 \Bigg[8 \Bigg\{ \frac{t^{\gamma}}{\Gamma(\gamma+1)} + \frac{t^{\gamma+1}}{\Gamma(\gamma+2)} + \frac{t^{\gamma+\beta}}{\Gamma(\gamma+\beta+1)} + \frac{t^{\gamma+\beta+1}}{\Gamma(\gamma+\beta+2)} \\
&\quad + \frac{2t^{2+\gamma}}{\Gamma(\gamma+3)} + \frac{2t^{2\gamma+1}}{\Gamma(2\gamma+2)} \Bigg\} - \Big(2x^2 - 3 - 4\ln(x)\Big) \left\{ \frac{2t^{\gamma+2}}{\Gamma(\gamma+3)} + \frac{t^{\gamma+\beta+1}}{\Gamma(\gamma+\beta+2)} \right\} \Bigg] + ...
\end{aligned}
\tag{3.53}
$$

When substituting $\beta = \gamma = 2$ and $\hbar_1 = \hbar_2 = -1$, we obtain a series of the form:

$$
\begin{aligned}
u_m &= \Big(2x^2 - 3 - 4\ln(x)\Big) \left\{ 1 + t^2 + (-1)^{m+1} \cdot \frac{t^{m\beta+1}}{\Gamma(m\beta+2)} \right\} + (-1)^m \Bigg\{ \frac{2t^{2m+1}}{\Gamma(2m+2)} \\
&\quad + \frac{2t^{2m}}{\Gamma(2m+1)} + \frac{t^{2m+2}}{\Gamma(2m+3)} \Bigg\},
\end{aligned}
\tag{3.54}
$$

$$
v_m = \Big(2x^2 - 3 - 4\ln(x)\Big) \Big\{ 1 + t + S_m \Big\}
$$

Where:

$$
S_m = \begin{cases} \dfrac{2t^{(m-1)\gamma+\beta+\left\lfloor \frac{m}{2} \right\rfloor+1}}{\Gamma\left((m-1)\gamma+\beta+\left\lfloor \frac{m}{2} \right\rfloor+2\right)} & \text{if } m \text{ is an odd number,} \\ -\dfrac{t^{(m-1)\gamma+\beta+\left\lfloor \frac{m}{2} \right\rfloor}}{\Gamma\left((m-1)\gamma+\beta+\left\lfloor \frac{m}{2} \right\rfloor+1\right)} & \text{if } m \text{ is an even number.} \end{cases}
$$

When m tends to infinity

$$
S_m \longrightarrow 0,
$$

$$
(-1)^{m+1} \cdot \frac{t^{m\beta+1}}{\Gamma(m\beta+2)} \longrightarrow 0,
$$

$$
(-1)^m \left\{ \frac{2t^{2m+1}}{\Gamma(2m+2)} \frac{2t^{2m}}{\Gamma(2m+1)} + \frac{t^{2m+2}}{\Gamma(2m+3)} \right\} \longrightarrow 0,
$$

So we get the exact solutions of the coupled system :

$$
\begin{cases} u(x,t) = (2x^2 - 3 - 4\ln(x))\,(1+t^2). \\ v(x,t) = (2x^2 - 3 - 4\ln(x))\,(1+t). \end{cases}
\tag{3.55}
$$

Figure 3.1 shows the convergence of numerical approximations to the analytical solutions of $u(x,t)$ and $v(x,t)$ under varying m at $x = 0.1$, with $n = 2$, $\beta = 1.3$, $\gamma = 1.7$, $h = -1$.

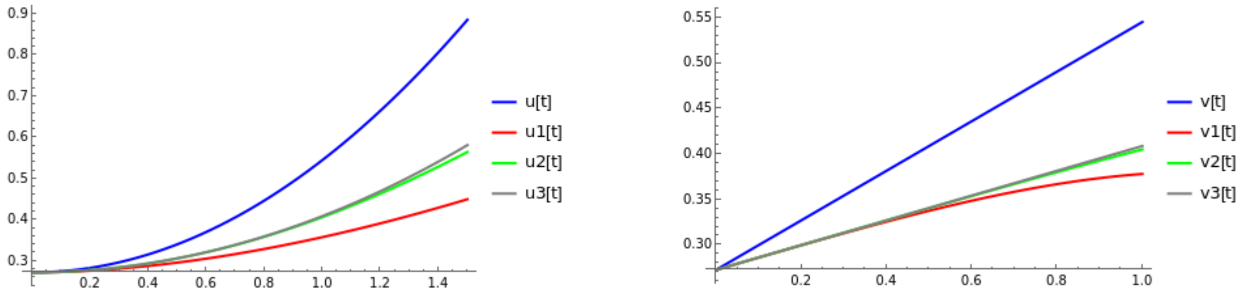

Figure 3.1:
Approximate solutions of $u(x,t)$ and $v(x,t)$ for varying parameter m.

The figures **1** and **2** demonstrate the performance of the numerical method in providing approximate solutions for the functions $u[t]$ and $v[t]$. The curves show that the numerical solutions $u_1[t]$, $u_2[t]$, $u_3[t]$, and $v_1[t]$, $v_2[t]$, $v_3[t]$ rapidly converge toward the exact solutions as the approximation levels increase.

In the case of $u[t]$, the results exhibit nearly perfect agreement by the third level ($m = 3$), confirming the accuracy and effectiveness of the algorithm. For $v[t]$, the approximate solutions show clear convergence toward the exact values, with only minor discrepancies at certain points that could be further refined by increasing the number of iterations or adjusting the algorithm's parameters.

Overall, the results validate the efficiency of the numerical method in delivering precise and reliable solutions, while also highlighting its potential for further optimization to achieve even greater accuracy in practical applications.

Conclusion

This study focuses on the numerical resolution of systems of fractional partial differential equations with fractional orders, using the q-homotopy analysis transform method, along with integral-type boundary conditions. This study appears to be one of the first attempts to investigate a system of equations of this form using this specific method. The research process involved applying this method and evaluating its efficiency in solving a model representing a system of fractional non-local mixed problems with integral boundary conditions. The study observed the effectiveness and efficiency of this method through its application to specific test examples, which demonstrated its ability to solve systems of fractional partial differential equations.

The research reaffirmed the validity of the generalized solution based on a previous study concerning the existence and uniqueness of this solution, which relied on the method of a priori estimation. This study highlights the importance of using the q-homotopy analysis transform method as an effective tool for studying systems of fractional partial differential equations with integral boundary conditions.

Solving fractional differential equations numerically, using methods like the one reviewed, requires high accuracy. We hope this study contributes to a better understanding of this type of problem and encourages the development of additional computational techniques with higher accuracy for solving other systems of fractional differential equations in the future.

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