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Mathematical analysis of some PDEs

By:

Hadjer Harkat

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Members of the jury:

Mr.	**Djamal Ahmed Chacha**	Prof. Kasdi Merbah University-Ouargla	Chairman
Mr.	**Abdallah Bensayah**	Prof. Kasdi Merbah University-Ouargla	Examiner
Mr.	**Ilyes Lacheheb**	M.C.B. Kasdi Merbah University-Ouargla	Supervisor

Dedication

Praise be to Allah, *who has granted me time and illuminated my path to appreciate this step in my academic journey. I entrust success and guidance to Him, and I seek His support in completing this work, the fruit of effort and achievement, by His grace.*

To the One who gave me life, hope, and an upbringing filled with a passion for learning and knowledge, To my dear parents, may Allah protect and preserve them, and to my entire family and my friends, to my dear friend Safa and to my fellow students. May God grant them success.

I dedicate this humble work to every teacher who has enriched us with the knowledge and wisdom granted by Allah, from the earliest stages of my education until this moment.

Acknowledgement

"**All praise and gratitude be to Allah**, who granted me patience to overcome the difficulties I faced in completing this humble work.

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I also extend my sincere thanks to professors **Chacha** and **Bensayah** for agreeing to report on this thesis and for their valuable comments and suggestions.

Lastly, I thank **all the professors in the Department of Mathematics** for their unwavering support."

Contents

Notations

- $\Omega \subset \mathbb{R}^n$: open subset of $\mathbb{R}^n$.
- $\partial\Omega = \Gamma$: The boundary of Ω.
- $\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial u_2}, ..., \frac{\partial u}{\partial u_n}\right)$: Gradient of the function u.
- $\frac{\partial u}{\partial n}$: outward normal derivative.
- $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$: Laplace of u.
- $\mathfrak{D}(\Omega)$: space of infinitely differential functions with compact support .
- $\mathfrak{D}^*(\Omega)$: space of distributions.
- V^* : the dual space of V.
- $\mathcal{L}(X)$: The space of linear, continuous mappings from X to X.
- $\dot{u} = \partial_t u$: Derivative with respect to time.
- $\langle , \rangle$: Inner product .
- $A_\lambda = AJ_\lambda$:Yosida approximation of the operator A.
- $|.|$:The euclidean norm in $\mathbb{R}^n$ and in the $\mathbb{C}$.

Introduction

Partial differential equations (PDEs) are fundamental mathematical tools for modeling natural and engineering phenomena.These equations are used to describe the behavior of systems that change over time and space, such as heat transfer, wave propagation, fluid flow, and quantum phenomena. Analytical solutions to these equations are of great importance because they provide a deep understanding of the behavior of complex systems without the need to rely entirely on numerical solutions. Additionally, evolution equations are a special class of partial differential equations that describe the change of systems over time. These equations are widely used in physics, engineering, and applied sciences to model dynamic phenomena.

In fact, most laws of nature in physics, such as Maxwell's equations, Newton's laws of motion, the Navier-Stokes equations, Schrödinger's equations in quantum mechanics, Kirchhoff's equations, and others, can be expressed in terms of partial differential equations. Partial derivatives describe natural phenomena; for example, the derivative with respect to time represents velocity, acceleration, and frictional force.

These equations are divided into two categories: those that do not involve time, such as Laplace's and Poisson's equations, and those that do, known as evolution equations. The latter include elliptic equations like the heat equation, hyperbolic equations(wave equations). Additionally, they can be classified based on several criteria, such as: linearity (e.g., the heat equation) versus nonlinearity (e.g., the Navier-Stokes equations).

It is necessary to specify initial conditions that define the initial state of the system and boundary conditions, such as Dirichlet conditions (specifying the value of the function on the boundary), Neumann conditions (specifying the derivative on the boundary), and Robin conditions (a combination of both), if applicable.

One of the most important methods for approximate the solution of PDEs (Partial Differential Equations) is the **finite element method** and **finite difference method**. Recently, another method has gained popularity, known as PINNs (physics informed neural networks) [27]. This is because the aforementioned methods can become more complex in finding approximate solutions as the dimensionality increases. As a result, the approximated systems using **neural networks**, which is now the most effective method for finding approximate systems for PDEs and the fastest due to its ease of control.

This is a focus of research for some scholars, and we will mention some of these studies below. The most important studies on PDEs focus on the existence and uniqueness of the solution, as well as the study of the solution's asymptotic behavior.This thesis follows the same approach by studying the existence of the solution and asymptotic behavior for the fractional Laplace wave equation with viscoelastic term.

Viscoelastic Materials:

In continuum mechanics, elastic materials and viscous fluids are mostly considered. An elastic material is a material in which at each material point the stress at the present time depends completely on the current value of the strain. For an incompressible viscous fluid, the stress at any given point depends on the value of the velocity gradient at that point. When a material exhibits both elastic and viscous behaviors it is called viscoelastic material. Precisely, for viscoelastic materials the stress at any given point depends on the present values of strain and velocity gradient. Examples of viscoelastic materials include, but not limited to, human tissue, disk in the human spine, wood, compressible gas, metals at very high temperature, concrete, plastic and polymeric materials. Some viscoelastic materials such as polymers, suspensions and emulsions can not be described in this way. For such materials, the stress at any given point does not depend only on the values of strain and velocity gradient at that point, but also on the entire history of the motion, that is, they posses a memory effect. Therefore, this type of viscoelastic behavior is modeled by equation with memory. Among the early contributors in this field are:Boltzmann, Maxwell, Kelvin and Voigt.

Consider a bar of uniform cross-section which occupies the unit interval $(0,1) \subset \mathbb{R}$ in unstressed state. A typical particle in $(0,1)$ is denoted by x, to describe the evolution of particles in $(0,1)$, we let $u(x,t)$ represents the displacement of the particle at time t and reference position x. The strain ϵ is given by

$$\epsilon(x,t) := u_x(x,t), \tag{1}$$

and the balance of linear momentum takes the form

$$u_{xx}(x,t) := \sigma_x(x,t) + f(x,t), \quad x \in (0.1), \quad t > 0, \tag{2}$$

where σ is the stress and f is an external force per unit mass. In 1874, Boltzmann [2] proposed that for material with memory, the constitutive relation for small deformation is given by

$$\sigma(x,t) = \beta\epsilon(x,t) + \int_{-\infty}^{t} g(t-s)(\epsilon(x,t) - \epsilon(x,s))ds, \tag{3}$$

where β is a non-negative constant and g is a positive non-increasing function defined on $[0,\infty)$. In the case where $g \in L^1(0,\infty)$, equation (3) takes the form

$$\sigma(x,t) = c^2\epsilon(x,t) - \int_{-\infty}^{t} g(t-s)\epsilon(x,s)ds, \tag{4}$$

where $c^2 := \beta + \int_0^\infty g(s)ds$ measures the instantaneous response of stress to strain. A

substitution of (4) into (2) yields

$$u_{xx}(x,t) - c^2u_{xx}(x,t) + \int_{-\infty}^{t} g(t-s)u_{xx}(x,s)ds = f(x,t), \quad x \in (0,1), \quad t > 0. \tag{5}$$

The function u is assumed to be known for any $t > 0$, that is, we have the following initial data:

$$u(x,t) = u_0(x,-t), \quad u_t(x,0) = u_1(x) \quad \forall x \in (0,1), \quad t \leq 0, \tag{6}$$

we further assume that $f \equiv 0$. In order to study system (5)-(6), Dafermos [5, 6] introduced a history function of the form

$$\eta^t(s) := u(t) - u(t-s), \quad \forall t, s > 0.$$

This allowed him to write problem (5)-(6) in the form of first-order evolution equation and took advantage of some powerful tools in the theory of dynamical systems. For more details on the theory of viscoelasticity, see [29] and [18].

There are analogous of these well-posedness results in the contexts of previous studies [19, 21] da Luz and Charoã [7] studied a semi-linear dissipative plate equation:

$$u_{tt} - \gamma\Delta u_{tt} + \Delta^2 u + u_t = f(u_t), \tag{7}$$

where $\gamma > 0$ and Δu_{tt} corresponds to the rotational inertia effects. They used an energy method to demonstrate the global existence and uniqueness of solutions and establish the decay rates under small initial data assumptions in the low space dimensions $1 \leq n \leq 5$.

Sugitani and Kawashima [16] studied the case of $\gamma = 1$ in Equation (7):

$$u_{tt} - \Delta u_{tt} + \Delta^2 u + u_t = f(u_t), \tag{8}$$

They used the time-weighted energy method in conjunction with a semigroup argument to show the global existence result for all space dimensions $n \geq 1$ and the asymptotic behavior of solutions.

Samuel Murray Rankine [28] also studied Semilinear Evolution Equations in Banach Spaces with Application to Parabolic Partial Differential Equations the class have the form

$$u'(t) + Au(t) = F(u(t)), \quad t > 0 \tag{9}$$

$$u(0) = \phi, \tag{10}$$

as a Cauchy problem in a Banach space X ; A is a closed linear operator which is densely defined and -A generates an analytic semigroup $T(t) : t > 0$.

The main study of this dissertation:

- In the first chapter 1, we present some definitions, properties, inequalities and spaces that will be used throughout this thesis.
- In chapter 2, we consider the heat equation. We prove the existence of the solution and its regularity using Faedo-Galerkin method and the semigroup theory. We based this work on references [13, 3, 4, 1]
- In chapter 3, we will study the fractional Laplace-wave equation with the influence of a memory term in $\mathbb{R}^n$, which represents the past history of the equation. We will find the solution formula for the equation by using another method, we relied on the Fourier and Laplace transforms. Then we proved the stability of the solution by multiplier technique and energy method. We based this work on references [33, 22]

Chapter 1

Mathematical preliminaries

We have dedicated this chapter to some basic concepts that we have picked it to help us address the following chapters.

1.1 Functional spaces

Definition 1.1.1 (Banach space) *A normed vector space $(X, \|\cdot\|)$ is called a Banach space if every Cauchy sequence in X converges to an element within X.*

Definition 1.1.2 (Hilbert space) *A prehilbert space $(X, \langle ., . \rangle)$ is called a Hilbert space if X is complete with respect to the norm induced by the inner product $\langle ., . \rangle$.*

Definition 1.1.3 (Separability) *Let (X, d) be a metric space. X is said to be separable if it contains a countable and dense subset.*

Lemma 1.1.4 (Continuity) *[13] Let E be a Banach space and F a Hilbert space such that $E \subset F$ with continuous embedding, and E dense in F. We identify F with F' (so that $E \subset F = F' \subset E'$). Let $u \in L^2_E([0,T])$, and suppose that $\partial_t u \in L^2_{E'}([0,T])$. Then*

$u \in C([0,T],F)$, and for all $t_1, t_2 \in [0,T]$ we have

$$\|u(t_1)\|_F^2 - \|u(t_2)\|_F^2 = 2\int_{t_1}^{t_2} \langle \partial_t u, u\rangle_{E',E}\, dt.$$

1.1.1 Lebesgue spaces $L^p(\Omega)$

Definition 1.1.5 *Let Ω be a subset of $\mathbb{R}^n$, and let $p \in \mathbb{R}$ with $1 \leq p < \infty$,we set:*

$$L^p(\Omega) = \{f : \Omega \longmapsto \mathbb{R}, \quad f \text{ is measurable } \quad \int_\Omega |f|^p dx < \infty\},$$

with

$$\|f\|_p = \|f\|_{L^p} = \Big(\int_\Omega |f|^p dx\Big)^{\frac{1}{p}},$$

if $p = 2$,$L^2(\Omega)$ is a Hilbert space equipped with the inner product

$$\langle f, g\rangle_{L^2(\Omega)} = \int_\Omega f(x)g(x)dx,$$

equipped with the norm

$$\|f\|_2 = \|f\|_{L^2} = \Big(\int_\Omega |f|^2 dx\Big)^{\frac{1}{2}},$$

Definition 1.1.6 *We recall that $L^\infty(\Omega)$ is the space of essentially bounded function on Ω,we set :*

$$L^\infty(\Omega) = \{f : \Omega \longmapsto \mathbb{R}, \quad f \text{ is measurable , and } \exists C > 0, |f(x)| \leq C \quad a.e \text{ on } \quad \Omega\}.$$

with

$$\|f\|_\infty = \|f\|_{L^\infty} = \inf\{C, C > 0, |f(x)| \leq C \quad a.e\}.$$

Theorem 1.1.7 (Fischer-Riesz) *[3]: L^p is a Banach space for any $1 \leq p \leq \infty$.*

Remark 1.1.8 *L^p spaces are reflexive for $1 < p < \infty$, and every Hilbert spaces are reflexive.*

1.1.2 Strong and weak convergence in Banach space

Definition 1.1.9 *A sequence* (u_n) *in Banach space* X *is said converge strongly to* u *iff*

$$\lim_{n \longrightarrow \infty} \|u_n - u\|_X = 0,$$

denoted by $[u_n \longrightarrow u \quad in\ X.]$

Definition 1.1.10 *A sequence* (u_n) *in Banach space* X *is said converge weakly to* u *iff*

$$\forall f \in X', \langle f, u_n \rangle_{X',X} \longrightarrow \langle f, u \rangle_{X',X},$$

denoted by

$$u_n \rightharpoonup u \quad in\ X.$$

Theorem 1.1.11 *[3] Assume that* X *is a reflexive Banach space and let* (x_n) *be a bounded sequence in* X*. Then there exists a subsequence* (x_{n_k}) *that converges in the weak topology* $\sigma(X, X^*)$*.*

Theorem 1.1.12 (Eberlein–Šmulian) *[3] Assume that* X *is a Banach space such that every bounded sequence in* X *admits a weakly convergent subsequence (in* $\sigma(X, X^*)$*). Then* X *is reflexive.*

1.1.3 The space $H^1(\Omega)$

Definition 1.1.13 *Let* Ω *be an open set of* $\mathbb{R}^n$*. The Sobolev space* $H^1(\Omega)$ *is defined by*

$$H^1(\Omega) = \left\{ v \in L^2(\Omega) \quad such\ that, \forall i \in \{1, ..., n\}, \frac{\partial v}{\partial x_i} \in L^2(\Omega) \right\},$$

where $\frac{\partial v}{\partial x_i}$ *is the weak partial derivative of* v*. Equipped with the norm*

$$\|u\|_{H^1(\Omega)} = \Big(\int_\Omega (|u(x)|^2 + |\nabla u(x)|^2) dx \Big)^{\frac{1}{2}}.$$

The sobolev space $H^1(\Omega)$ *is a Hilbert space.*

1.1.4 The space $H_0^1(\Omega)$

Let us define another Sobolev space which is a subspace of $H^1(\Omega)$ and which will be very useful for problem with Dirichlet boundary conditions.

Theorem 1.1.14 (Green's formula) *Let Ω be a regular open set of class $\mathcal{C}^1$. Let w be a $C^1(\bar{\Omega})$ function with bounded support in the closure $\bar{\Omega}$. Then ω satisfies Green's formula*

$$\int_\Omega \frac{\partial w}{\partial x_i}(x)dx = \int_{\partial\Omega} w(x)n_i(x)ds,$$

where n_i is the ith component of the unit outward normal to Ω.

Corollary 1.1.15 (Integration by parts formula) *Let Ω be a regular open set of class $\mathcal{C}^1$. Let u and v be two $C^1(\overline{\Omega})$ functions with bounded support in the closed set $\overline{\Omega}$. Then they satisfy the integration by parts formula*

$$\int_\Omega u(x)\frac{\partial v}{\partial x_i}(x)\,dx = -\int_\Omega v(x)\frac{\partial u}{\partial x_i}(x)\,dx + \int_{\partial\Omega} u(x)v(x)n_i(x)\,ds.$$

Proof. It is enough to take $w = uv$ in Theorem (1.1.14). ■

Proposition 1.1.16 *Let Ω be a regular open set of class $\mathcal{C}^1$.Let u be a function of $C^2(\bar{\Omega})$ and v a function of $C^1(\bar{\Omega})$, both with bounded support in the closed set $\bar{\Omega}$. Then they satisfy the integration by parts formula:*

$$\int_\Omega \Delta u(x)v(x)dx = -\int_\Omega \nabla u(x)\nabla v(x)dx + \int_{\partial\Omega} \frac{\partial u}{\partial n}(x)v(x)ds.$$

Proof. We apply corollary (1.1.15) to v and $\frac{\partial u}{\partial x_i}$ then we sum in i. ■

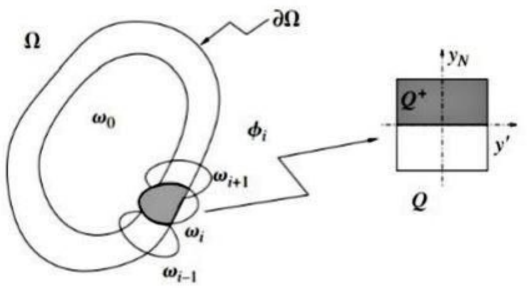

Figure 1.1: Definition of the regularity of an open set.

1.2 Some inequalities

Let $1 \le p \le \infty$, we denote by p' the conjugate exponent

$$\frac{1}{p} + \frac{1}{p'} = 1$$

Theorem 1.2.1 (Hölder's inequality) *Assume that* $f \in L^p(\Omega)$ *and* $g \in L^{p'}(\Omega)$ *with* $1 \le p \le \infty$ *. Then* $fg \in L^1(\Omega)$*and*

$$\int_\Omega |fg| \le \|f\|_p \|g\|_{p'}.$$

Theorem 1.2.2 (Cauchy-Schwarz inequality) $\forall (f,g) \in L^2(\Omega) \times L^2(\Omega)$, *we have*

$$\int_\Omega |f(x)g(x)|dx \le \Big(\int_\Omega |f|^2 dx\Big)^{\frac{1}{2}} \Big(\int_\Omega |g|^2 dx\Big)^{\frac{1}{2}},$$

which is

$$\|fg\|_{L^1(\Omega)} \le \|f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}.$$

Theorem 1.2.3 (Young's inequality with a parameter) *Let* ε *is a positive number, then* $\forall (a,b) \in \mathbb{R}^2$

$$|ab| \le \frac{\varepsilon}{2} a^2 + \frac{1}{2\varepsilon} b^2.$$

By putting $\varepsilon = 2\delta$ *we get:*

$$|ab| \le \delta a^2 + \frac{1}{4\delta} b^2.$$

Theorem 1.2.4 (Poincaré inequality) *Let Ω be an open set of $\mathbb{R}^N$ which is bounded in at a least one space direction. There exists a constant C>0 such that, for every function $u \in H_0^1(\Omega)$,*

$$\|u\|_{L^2(\Omega)} \leq C\|\nabla u\|^2_{L^2(\Omega)}.$$

1.3 Bochner's spaces

Definition 1.3.1 *We denote by $L^2(0,T,X)$ the space of functions of]0,T[in X such that the function $t \to \|v(t)\|_X$is measurable and square integrable, that is, to say that*

$$\|v\|^2_{L^2(]0,T[,X)} = \int_0^T \|v(t)\|^2_X dt < +\infty.$$

Such as, in addition the space $L^2(0,T,X)$ is a Banach space equipped with this norm.Further, if X is a Hilbert space, then $L^2(]0,T[,X)$ is a Hilbert space for the scalar product

$$\langle u, v\rangle_{L^2(]0,T[,X)} = \int_0^T \langle u(t), v(t)\rangle_X dt.$$

1.4 Fourier transform

Definition 1.4.1 *Let $f \in L^1(\mathbb{R}^N)$. We define the Fourier transform*

$$\hat{f} = \mathcal{F}[f] \in L^\infty(\mathbb{R}^N),$$

by formula

$$\mathcal{F}[f] = \hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-ix\xi} f(x)dx.$$

where $x.\xi = \sum_{j=1}^N x_j \xi_j$.
Its is immediate to see that $\mathcal{F}[f] : L^1(\mathbb{R}^N) \to L^\infty(\mathbb{R}^N)$is bounded linear operator, in fact we have from its definition

$$\|\hat{f}\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)}, \quad \text{for all } f \in L^1(\mathbb{R}^N),$$

and we denote its inverse transform as $\mathcal{F}^{-1}$.

Example 1.1 *Let's take the Dirac function δ, then the Fourier transform of this function defined by*

$$\mathcal{F}[\delta(x-a)](\xi) = \int_{-\infty}^{\infty} e^{-i\xi x}\delta(x-a)dx$$
$$= \int_{-\infty}^{\infty} e^{-i\xi a}dx$$

Since $a = 0$ we obtain:

$$\mathcal{F}[\delta(x)](\xi) = \int_{-\infty}^{\infty} e^{-i\xi x}\delta(x)dx$$
$$= \int_{-\infty}^{\infty} e^{-i\xi 0}dx$$
$$= 1.$$

Proposition 1.4.2 *Let $u \in L^2(\mathbb{R}^n)$. For all multi-index α is an N tuple integers $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N) \in \mathbb{N}^n$, we note $|\alpha| = \sum_{i=1}^{n} \alpha_i$, we have :*

$$\widehat{\partial^\alpha u} = i^{|\alpha|}\xi^\alpha \hat{u}.$$

Example 1.2 *The Fourier transform of the Laplace operator given by*

$$\widehat{\Delta u} = \widehat{\sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_i^2}}$$
$$= i^2 \sum_{k=1}^{n} \xi_k^2 \hat{u}$$
$$= -|\xi|^2 \hat{u}.$$

Proposition 1.4.3 *Let $u \in L^2(\mathbb{R}^n)$. Then:*
(i) $u \in H^m(\mathbb{R}^n)$ if and only if $(1+|\xi|)^{\frac{m}{2}}\hat{u} \in L^2(\mathbb{R}^n)$.
(ii)The norms

$$\|u\|_{H^m(\mathbb{R}^n)} \quad \text{and} \quad \|(1+|\xi|)^{\frac{m}{2}}\hat{u}\|_{L^2(\mathbb{R}^n)}$$

are equivalent

Theorem 1.4.4 (Plancherel's) *[24]*

If $f \in L^1 \cap L^2$, then $\hat{f} \in L^2$ and

$$\|f\|_2 = \|\hat{f}\|_2.$$

Theorem 1.4.5 (Hausdorff-Young inequality) *[26] Let $1 < p < 2$ and p,q are conjugate. Then we have*

$$\|\hat{v}\|_q \leq C_{p,q}\|v\|_p,$$

with a constant $C_{p,q} > 0$.

Theorem 1.4.6 (Young) *Let $f \in L^1(\mathbb{R}^n)$ and let $g \in L^p(\mathbb{R}^n)$ with $1 \leq p \leq \infty$. Then for a.e $x \in \mathbb{R}^n$ the function $y \mapsto f(x-y)g(y)$ is integrable on $\mathbb{R}^n$ and we define*

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy.$$

In addition $f * g \in L^p(\mathbb{R}^n)$ and,

$$\|f * g\|_p \leq \|f\|_1\|g\|_p.$$

Proof. See([3]. p, 104) ■

1.5 Laplace transform

Definition 1.5.1 *[31] For any function f on $\mathbb{R}$, that is locally integrable on $[0, \infty)$, we denote $\mathfrak{L}$ the Laplace transform of f defined by*

$$\mathfrak{L}[f](\lambda) = \int_0^\infty e^{-\lambda t} f(t)dt \quad (\lambda \in \mathbb{C}), \tag{1.1}$$

and we denote its inverse transform by

$$\mathfrak{L}^{-1}(f) = \int_0^\infty e^{\lambda t} f(\lambda)d\lambda.$$

The Laplace transform of $f(t)$ is said to exist if the integral (1.1) converges for some value of λ; otherwise it does not exist. For sufficient conditions under which the Laplace transform does exist.

Definition 1.5.2 (Functions of exponential order) *[23] If real constants $M > 0$ and γ exist such that for all $t > N$*

$$|e^{\gamma t}F(t)| < M \quad or \quad |F(t)| < Me^{\gamma t}.$$

We say that $F(t)$ is a function of exponential order γ as $t \to \infty$ or, briefly, is of exponential order.

Theorem 1.5.3 (Laplace transform of derivative) *[23]*

$$\mathfrak{L}[F'(t)] = \lambda \mathfrak{L}[F(t)] - F(0). \tag{1.2}$$

And

$$\mathfrak{L}[F''(t)] = \lambda^2 \mathfrak{L}[F(t)] - \lambda F(0) - F'(0). \tag{1.3}$$

Proof. Using integration by parts:

$$\begin{aligned}
\mathfrak{L}\{F'(t)\} &= \int_0^\infty e^{-\lambda t} F'(t)\, dt = \lim_{p\to\infty} \int_0^p e^{-\lambda t} F'(t) dt \\
&= \lim_{P\to\infty} \left[e^{-\lambda t} F(t) \Big|_0^P + \lambda \int_0^P e^{-\lambda t} F(t)\, dt \right] \\
&= \lim_{P\to\infty} \left[e^{-\lambda P} F(P) - F(0) + \lambda \int_0^\infty e^{-\lambda t} F(t)\, dt \right] \\
&= \lambda \int_0^\infty e^{-\lambda t} F(t) dt - F(0) \\
&= \lambda \mathfrak{L}[F(t)] - F(0).
\end{aligned}$$

We use (1.2) we get

$$\begin{aligned}
\mathfrak{L}\{F''(t)\} &= \lambda \mathcal{L}[F'(t)] - F'(0) \\
&= \lambda \left[\lambda \mathfrak{L}[F(t)] - F(0) \right] - F'(0) \\
&= \lambda^2 \mathfrak{L}[F(t)] - \lambda F(0) - F'(0)
\end{aligned}$$

■

Theorem 1.5.4 (Sufficient conditions for existence Laplace transforms) *[23] If $F(t)$ is sectionally continuous in every finite interval $0 \le t \le N$ and of exponential order γ for $t > N$, then its Laplace transform $f(\lambda)$ exists for all $\lambda > \gamma$.*

Theorem 1.5.5 (The Convolution Theorem for Laplace Transforms) *we have*

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy,$$

the Laplace transform defined by

$$\mathfrak{L}(f * g) = \mathfrak{L}(f)\mathfrak{L}(g).$$

Chapter 2

Existence and uniqueness of parabolic model

In this chapter, we first dedicate to the mathematical and numerical analysis of problems of evolution equation in time. More precisely we shall analysis two different types of partial differential equation: parabolic and hyperbolic.

Secondly, study the existence and uniqueness for a parabolic problem by Faedo-Galerkin method and Semi-group theory with steps .

2.1 Some classical models

In this section we describe some classical models to present the principal classes of PDEs, and to show that these equations play a very important role in diverse scientific areas. From now, we shall non dimensional size all the variable, which will allow us to set the constant in the models equal to 1.

2.2 Modeling and examples of parabolic equation

Let us present the principle parabolic problem and say a few words about its physical and mechanical origins. The archetype of these models is the heat flow equation .

2.2.1 Heat flow equation

Let Ω be an open set and bounded in $\mathbb{R}^n$ with boundary $\partial\Omega$. For Dirichlet boundary condition this model is written:

$$\begin{cases} \dfrac{du}{dt} - \Delta u = f & \Omega\times]0,T[\\ u = 0 & \partial\Omega\times]0,T[\\ u(.,0) = u_0 \end{cases} \tag{2.1}$$

The boundary value problem (2.1) describes the evolution of the temperature distribution $u(x,t)$ in a thermally conducting body occupying the domain Ω. The function u_0 represents the initial temperature distribution at time $t = 0$. The temperature on the boundary $\partial\Omega$ of the body is maintained at a constant value and used as the reference value (homogeneous Dirichlet condition $u(x,t) = 0$ on $\partial\Omega \times \mathbb{R}^+$). The given function f is called the heat source.

An obvious generalization of the heat flow equation is obtained when we replace the Laplacian with a more general second order elliptic-operator. For example if we study the propagation of heat in a nonhomogeneous material or in the presence of a convective effect. A second generalization concerns (less obvious) the system of time-dependent Stokes equation describing viscous fluid flow.

Denoted u the velocity and p the pressure of viscus fluid subject to the force f. This system is written

$$\begin{cases} \dfrac{du}{dt} + \nabla p - \mu\Delta u = f & \text{in } \Omega \times \mathbb{R}^+_* \\ divu = 0 & \text{in } \Omega \times \mathbb{R}^+_* \\ u = 0 & \text{on } \partial\Omega \\ u(x,0) = u_0(x) & \text{in } \Omega \end{cases} \tag{2.2}$$

$\mu > 0$ is the viscosity of the fluids, f external force density (N/m^3) and u_0 initial velocity distribution .

2.3 Modeling and examples of the hyperbolic equation

The following model is the wave equation that models propagation of waves or vibration. For example, in two space dimensions it is model to the study the vibration of stretched elastic membrane. In the one space dimension it is also called the vibration of string equation. At rest, the membrane occupies a plan domain Ω. Let Ω be an open bounded set of $\mathbb{R}^n$ with boundary $\partial\Omega$. For Dirichlet boundary conditions this model is written by:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = f & \text{in } \Omega \times \mathbb{R}_*^+ \\ u = 0 & \text{in } \Omega \times \mathbb{R}_*^+ \\ u(t=0) = u_0(x) & \text{in } \Omega \\ \dfrac{\partial u}{\partial t}(t=0) = u_1(x) & \text{in } \Omega \end{cases} \tag{2.3}$$

The second model is the elastodynamic model which is the time dependent version of the linearised elastoelasticity equation. By applying the fundamental principle of dynamic, the acceleration being the second time derivative of the displacement. We obtain an evolution problem which is the second order in time (2.3). The displacement $u(x,t)$ is the solution of the following elastodynamic system:

$$\begin{cases} \rho\frac{\partial^2 u}{\partial t^2} - \operatorname{div}\left(2\mu e(u) + \lambda \operatorname{tr}(e(u)) I\right) = f & \text{in } \Omega \times \mathbb{R}_*^+, \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}_*^+, \\ u(t=0) = u_0(x) & \text{in } \Omega, \\ \frac{\partial u}{\partial t}(t=0) = u_1(x) & \text{in } \Omega, \end{cases}$$

where:

- $\rho > 0$ is the constant density of the material,
- $\mu > 0$ and λ are the Lamé coefficients satisfying $2\mu + \lambda > 0$,
- $e(u) = \frac{\nabla u + (\nabla u)^t}{2}$ is the deformation tensor,
- u_0 is the initial displacement,
- u_1 is the initial velocity,

- $f(t, x)$ is the resultant of the exterior forces.

2.4 Existence and uniqueness in the parabolic case

We study the existence and uniqueness of problem (2.1) with $f = 0$. It means there is no external heat source or sink in the system. This form represents the diffusion of heat in a closed system or a material with no external heat addition or loss. It is written by

$$\begin{cases} \dfrac{du}{dt} - \Delta u = 0, & \Omega \times]0, T[\\ u = 0, & \partial\Omega \times]0, T[\\ u(., 0) = u_0, & \end{cases} \tag{2.4}$$

2.5 Weak Formulation

Multiply the problem (2.4) by $v \in H_0^1(\Omega)$, since $H_0^1(\Omega)$ is a Hilbert space then $L^2(0, T, H_0^1(\Omega))$ is Hilbert space equipped with inner product and we use Green formula, we define the weak formulation by

$$\int_0^T \langle u_t, v \rangle_{H^{-1}, H_0^1} + \int_0^T \int_\Omega \nabla u \nabla v dx dt = 0. \tag{2.5}$$

As we have mentioned at the beginning of this section to prove the existence and regularity of weak solutions, we use a Faedo-Galerkin method, which discretizes the evolution equation in space to obtain an ODE on a finite-dimensional space. We needed three steps to be completed. The first step project the PDE to an ODE on a finite-dimensional subspace and prove the existence of approximation solutions to these approximate problems. In the second step, we establish the fundamental a priori estimate on the energy and bounds on the approximation solutions, which allows us to extract a weakly convergent subsequence of (u_n). The third step, we show that the weak limit u of this subsequence is a weak solution. The uniqueness of the solution was proved by **contradiction**: we assume that there exist two distincts solutions and ultimately show that they are equal. Since $V = H_0^1(\Omega)$ is a separable, there exists a sequence (V_n) of spaces included in $H_0^1(\Omega)$

of a finite dimension, for example dim $V_n = n$, such that

$$\begin{cases} V_n \subset V_{n+1}, \\ \bigcup_{n=1}^{\infty} V_n \quad \text{is dense in } H_0^1(\Omega). \end{cases} \tag{2.6}$$

Firstly prove that $H_0^1(\Omega)$ has a Hilbert basis e_n.

Let $\{e_1, \dots, e_n\}$ be a basis of the finite-dimensional subspace V_n. We seek an approximate solution of the form:

$$u_n(t) = \sum_{i=1}^{n} \alpha_i(t) e_i,$$

where the coefficients $\alpha_i(t)$ are determined by the weak formulation.

Multiplying by test functions e_k and integrating over Ω, we obtain the system:

$$\sum_{i=1}^{n} \int_\Omega \alpha_i'(t) e_i e_k \, dx + \sum_{i=1}^{n} \int_\Omega \alpha_i(t) \nabla e_i \cdot \nabla e_k \, dx = 0, \quad \forall k \in \{1, \dots, n\}, \ t > 0.$$

The initial conditions $\alpha_i(0)$ are chosen such that:

$$u_n(0) = \sum_{i=1}^{n} \alpha_i(0) e_i \to u_0 \quad \text{in } L^2(\Omega) \text{ as } n \to +\infty.$$

For each $n \in \mathbb{N}^*$, this yields a system of n ordinary differential equations with initial conditions, for which we prove the existence of a solution. We then derive uniform estimates on u_n that allow us to pass to the limit as $n \to +\infty$, the limit u of the approximate solutions u_n satisfies:

$$\begin{cases} u \in L^2(]0,T[, H_0^1(\Omega)), u_t \in L^2(]0,T[, H^{-1}(\Omega)), u \in C([0,T], L^2(\Omega)), u(0) = u_0 \quad a.e. \\ \int_0^T \langle \partial_t u.v \rangle_{H_0^1, H^{-1}} dt + \int_0^T \Big(\int_\Omega \nabla u . \nabla v dx \Big) dt = 0 \\ \forall v \in L^2(]0,T[, H_0^1(\Omega)), \end{cases} \tag{2.7}$$

Theorem 2.5.1 (Faedo-Galerkin) *[13] Let Ω be a bounded open subset of $\mathbb{R}^N$, $T > 0$, and $u_0 \in L^2(\Omega)$, where we identify $L^2(\Omega)$ with its dual . Then there exists a unique u such that*

$$\begin{cases} u \in L^2(]0,T[, H_0^1(\Omega)), \ \partial_t u \in L^2(]0,T[, H^{-1}(\Omega)), \\ \int_0^T \langle \partial_t u(s), v(s) \rangle_{H^{-1}, H_0^1} \mathrm{d}s + \int_0^T \int_\Omega \nabla u(s) \cdot \nabla v(s) \mathrm{d}x \mathrm{d}s = 0 \\ u(0) = u_0 \ a.e. \end{cases} \tag{2.8}$$

(Recall that $u(t)$ (resp. $v(t)$) denotes the function $x \mapsto u(x,t)$ (resp. $v(x,t)$). Moreover, we have the following estimates for u and $\partial_t u$:

$$\|u\|_{L^2(]0,T[,H_0^1(\Omega))} \leq \|u_0\|_2,$$

$$\|\partial_t u\|_{L^2(]0,T[,H^{-1}(\Omega))} \leq \|u_0\|_2,$$

$$\|u(t)\|_2^2 \leq \|u_0\|_2^2 + \|\partial_t u\|^2_{L^2(]0,T[,H^{-1}(\Omega))} + \|u\|^2_{L^2(]0,T[,H_0^1(\Omega))}, \text{ for all } t \in [0,T].$$

Proof.

Using a Hilbert basis formed by eigen functions of the Laplacian, that is to say a Hilbert basis of $L^2(\Omega)$ denoted $\{e_n, \quad n \in \mathbb{N}\}$ such that e_n is a weak solution of the problem :

$$\begin{cases} -\Delta e_n = \lambda_n e_n & \text{in } \Omega \\ e_n = 0 & \partial\Omega, \end{cases}$$

the family $\{e_n\}$ is a Hilbert basis in $L^2(\Omega)$ verifies :

$$\begin{cases} \text{find} \quad e_n \in H_0^1(\Omega), \\ \displaystyle\int_\Omega \nabla e_n \nabla v dx = \lambda_n \int_\Omega e_n v dx \\ \forall v \in H_0^1(\Omega). \end{cases}$$

With $\lambda_n{>}0$, for all $n \in \mathbb{N}$ and $\lambda_n \longrightarrow \infty$, as n$\uparrow \infty$.

Since (e_n) is a Hilbert basis in $L^2(\Omega)$. So, for all $w \in L^2(\Omega)$

$$w = \sum_{n=1}^{n=\infty} (w, e_n)_2 e_n,$$

with the sense of the convergent in $L^2(\Omega)$. Then

$$\sum_{i=1}^{n} (w, e_i)_2 e_i \longrightarrow w \quad \text{in} \quad L^2(\Omega).$$

No, we prove that the family$\left(\dfrac{e_n}{\sqrt{\lambda_n}}\right)$ is a Hilbert basis of $H_0^1(\Omega)$. We remark that for all $n, m \geq 1$

$$\int_\Omega \nabla e_n . \nabla e_m dx, = \lambda_n \int_\Omega e_n e_m dx = \lambda_n \delta_{n,m}.$$

We deduce that $(e_n, e_m)_{H_0^1}$ =0 if $n \neq m$ and:

$$\left\| \frac{e_n}{\sqrt{\lambda_n}} \right\|^2_{H_0^1} = \int_\Omega \frac{\nabla e_n \nabla e_n}{\lambda_n} dx = 1.$$

We remark that the vector space generated by family denoted (e_n), for all n$\in \mathbb{N}$ is dense in $H_0^1(\Omega)$. Indeed, let $v \in H_0^1(\Omega)$ such that $(v|e_n)$=0 for all $n \in \mathbb{N}^\star$.

$$0 = (v|e_n)_{H_0^1(\Omega)} = \int_\Omega \nabla e_n \nabla v dx = \lambda_n \int_\Omega e_n v dx.$$

Since (e_n) is a Hilbert basis of $L^2(\Omega)$, we deduce that $v = 0$ a.e, this show that the orthogonal in $H_0^1(\Omega)$ is $\{0\}$, and so Span$\{e_n, n \in \mathbb{N}\}$ is dense in $H_0^1(\Omega)$.

Finally, we thus obtain that the family $\{e_n, n \in \mathbb{N}\}$ is a Hilbert basis of $H_0^1(\Omega)$. ■

Step 1:(Approximation problem)

We are looking for an approximate solution u_n in the form

$$u_n = \sum_{i=1}^{n} \alpha_i(t) e_i,$$

with $\alpha_i(t) \in \mathbb{C}$. We assume that α_i are derivable for all t (not true in general), then:

$$u'_n(t) = \sum_{i=1}^{n} \alpha'_i(t) e_i(x),$$

We have (By taking into account the embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$), for all $\varphi \in H_0^1(\Omega)$ and for all $t \in [0, T]$,

$$\langle u'_n, \varphi \rangle_{H^{-1}, H_0^1} = \sum_{i=1}^{n} \alpha'_i(t) \int_\Omega e_i \varphi dx,$$

on the other hand, for all $t \in [0, T]$, we have

$$-\Delta u_n(t) = -\sum_{i=1}^{n} \alpha_i(t) \Delta e_i = \sum_{i=1}^{n} \lambda_i \alpha_i(t) e_i \quad \text{in} \quad \mathfrak{D}^*(\Omega) \quad \text{and in} \quad H^{-1}(\Omega),$$

then for all $\varphi \in H_0^1(\Omega)$

$$\langle -\Delta u_n(t), \varphi \rangle_{H^{-1}, H_0^1} = \int_\Omega \nabla u_n(t) . \nabla \varphi dx = \sum_{i=1}^{n} \lambda_i \alpha_i(t) \int_\Omega e_i \varphi dx.$$

Finally, we obtain:

$$\langle u'_n - \Delta u_n, \varphi \rangle_{H^{-1}, H_0^1} = \sum_{i=1}^{n} (\alpha'_i(t) + \lambda_i \alpha_i(t)) \int_\Omega e_i \varphi.$$

To find u_n, we choose α_i such that:

$$\langle u_n' - \Delta u_n, \varphi\rangle_{H^{-1},H_0^1,} = 0,$$

this is equivalent for all $k \in \{1, ..., n\}$:

$$\alpha_i'(t) + \lambda_i \alpha_i(t) = 0 \qquad (ODE).$$

Taking into account the initial condition and setting $\alpha_i^{(0)} = (u_0, e_i)_2$. Therefore, taking

$$\alpha_i(t) = \alpha_i^{(0)} e^{-\lambda_i t}. \tag{2.9}$$

The functions α_i thus defined belong to $C([0,T],\mathbb{R})$ and we therefore have

$$u_n \in C([0,T],V_n) \subset C([0,T],H_0^1(\Omega)).$$

The functions α_i, are not necessarily differentiable, by definition we have u_n' is an element in $\mathfrak{D}_V^*$ with $V = H_0^1(\Omega)$, let $\varphi \in \mathfrak{D}(]0,T[,\Omega)$ we have :

$$\langle u_n', \varphi\rangle_{\mathfrak{D}^*,\mathfrak{D}} = -\int_0^T u_n(t)\varphi'(t)dt \quad \in V_n \subset H_0^1(\Omega),$$

therefore we have

$$\langle u_n', \varphi\rangle_{\mathfrak{D}^*,\mathfrak{D}} = -\sum_{i=1}^n \int_0^T \alpha_i(t) e_i \varphi'(t)dt = -\sum_{i=1}^n \Big(\int_0^T \alpha_i(t)\varphi'(t)dt\Big) e_i,$$

then we use (2.9) we get

$$\begin{aligned}\int_0^T \alpha_i(t)\varphi'(t)dt &= -\int_0^T \alpha_i'(t)\varphi(t)e_i dt\\ &= \lambda_i \int_0^T \alpha_i^{(0)} e^{-\lambda_i t}\varphi(t)e_i dt\\ &= \lambda_i \int_0^T \alpha_i(t)\varphi(t)e_i dt\end{aligned}$$

Since this equality holds for all $\varphi \in \mathfrak{D}(]0,T[,\mathbb{R})$ we have

$$u_n' = -\sum_{i=1}^n \lambda_i \alpha_i(t) e_i \qquad in \quad L^2(]0,T[,V_n).$$

Which can also be written

$$u_n' = \Delta u_n \qquad in \quad L^2(]0,T[,V_n) \subset L^2(]0,T[,H_0^1(\Omega)) \subset L^2(]0,T[,H^{-1}(\Omega)).$$

Now, let $v \in L^2(]0,T[,H_0^1(\Omega))$, since $\dot{u}_n \in L^2(]0,T[,H^{-1}(\Omega))$, we have

$$\langle u_n', v\rangle_{H^{-1},H_0^1} \in L^1(]0,T[),$$

and

$$\int_0^T \langle u_n', v\rangle_{H^{-1},H_0^1} dt = -\int_0^T \int_\Omega \nabla u_n \nabla v dx dt,$$

then the projected problem given by

$$\begin{cases} u_n \in L^2(]0,T[,H_0^1(\Omega)), u_n' \in L^2(]0,T[,H^{-1}(\Omega)), \\ \int_0^T \langle u_n'(t), v(t)\rangle + \int_0^T \int_\Omega \nabla u_n(t) \nabla v dx dt = 0 \\ u_n(0) = u_0 \end{cases} \tag{2.10}$$

then u_n is a solution of the projected problem (2.10), u_n in $C([0,T],H_0^1)$ and $u_n(0)$ is the projection of u_0.

Step 2: (apriori estimates of $\dot{u}_n, u_n, \nabla u_n$)

We have $u_n \in C(]0,T[,H_0^1) \subset L^2(]0,T[,H_0^1)$, and

$$u_n' = \Delta u_n \in L^2(]0,T[,H^{-1}).$$

On other hand, we have

$$\begin{aligned} \int_0^T \langle u_n', u_n\rangle_{H^{-1},H_0^1} &= \frac{1}{2}\int_0^T \frac{d}{dt}\|u_n\|_{L^2}^2 dt \\ &= \frac{1}{2}(\|u_n(T)\|_{L^2}^2 - \|u_n(0)\|_{L^2}^2), \end{aligned}$$

we take $v = u_n$ in (2.10) to find

$$\frac{1}{2}(\|u_n(T)\|_{L^2}^2 - \|u_n(0)\|_{L^2}^2) + \int_0^T \|\nabla u_n\|_{L^2}^2 dt = 0.$$

So

$$\int_0^T \|\nabla u_n\|_{L^2}^2 \leq \frac{1}{2}\|u_n(0)\|_{L^2}^2.$$

Thus ,

$$\|u_n\|^2_{L^2(]0,T[,H_0^1(\Omega))} \leq \|u_n(0)\|^2_{L^2}.$$

From (2.10), we have

$$\|u_n'\|_{L^2(]0,T[,H^{-1}(\Omega))} = \|\Delta u_n\|_{L^2(]0,T[,H^{-1}(\Omega))} = \Big(\int_0^T \|\nabla u_n\|^2_{L^2} dt\Big)^{\frac{1}{2}}$$
$$= \|u_n\|_{L^2(]0,T[,H_0^1(\Omega))} \leq \|u_n(0)\|_{L^2}.$$

So, the sequences $(u_n), (u_n')$ are bounded in $L^2(]0,T[, H_0^1(\Omega))$ and $L^2(]0,T[, H^{-1}(\Omega))$ respectively.

Passage to limits

We call $\{u_n\}$ a Galerkin approximation of the solution u.
Since $V = H_0^1$ is a reflexive space. We now use the theorem (1.1.11) and (1.1.12), and deduce that there exists a subsequence $(u_{n_k}) \in V_{n_k} \subset V_n$, such that

$$u_{n_k} \rightharpoonup u \quad \text{in} \quad L^2(0,T;H_0^1),$$
$$u_{n_k}' \rightharpoonup w \quad \text{in} \quad L^2(0,T;H^{-1}),$$
$$\nabla u_{n_k} \rightharpoonup \nabla u \quad \text{in} \quad L^2(\Omega),$$

We will prove that $w = u'$. It's sufficient to show that

$$\int_0^T w\phi dt = -\int_0^T u\phi' dt, \quad \forall \phi \in \mathfrak{D}(]0,T[,\mathbb{R}). \tag{2.11}$$

Since the left term of (2.11) is in H^{-1}, then the right term is in H_0^1. Therefore, this equation uses the fact $H_0^1(\Omega) \subset H^{-1}(\Omega)$.

We consider, for $\psi \in H_0^1(\Omega)$

$$f : L^2(]0,T[, H_0^1(\Omega)) \longrightarrow \mathbb{R}$$
$$f(v) = \int_\Omega \Big(-\int_0^T v\phi' dt\Big)\psi dx.$$

It's easy to see that f is linear continuous. So $f(u_{n_k}) \longrightarrow f(u)$. It means

$$-\langle\int_0^T u_{nk}\phi' dt, \psi\rangle \longrightarrow -\langle\int_0^T u\phi' dt, \psi\rangle.$$

Thus

$$\langle \int_0^T u'_{nk}\phi dt, \psi\rangle \longrightarrow -\langle \int_0^T u\phi' dt, \psi\rangle.$$

Also, $\bar{f} = \langle \int_0^T v\phi dt, \psi\rangle$ is linear continuous from $L^2(]0,T[, H^{-1}(\Omega))$ in $\mathbb{R}$.
So $\bar{f}(u_{n_k}) \longrightarrow \bar{f}(w)$. It means that

$$\langle \int_0^T u'_{nk}\phi dt, \psi\rangle \longrightarrow \langle \int_0^T w\phi dt, \psi\rangle_{H^{-1},H_0^1}.$$

Thus

$$-\int_0^T u\phi' dt \longrightarrow \int_0^T w\phi \quad \forall \phi \in \mathfrak{D}(]0,T[,\mathbb{R}),$$

so $u' = w$.

We just calculate the limit in (2.10), to prove that u is a solution of (2.10), which is the formulation in (2.8).

Now, we will prove that $u(0) = u_0$ a.e, i.e $u(0) = u_0$ in $L^2(\Omega)$. We know that

$$u_n(0) = \sum_{i=1}^{n} \alpha_i^0 e_i \longrightarrow u_0 \quad \text{in} \quad L^2(\Omega).$$

To deduce that $u(0) = u_0$, it suffices to show that u_n is relatively compact in $C(]0,T[, H^{-1}(\Omega))$.

By Ascoli's theorem, it suffices to show that

1. $\forall t \in [0,T], u_n$ is relatively compact in $H^{-1}(\Omega)$.

2. $\|u_n(t) - u_0(s)\|_{H^{-1}} \longrightarrow 0$ uniformly for $s \to t$ with respect of $n \in \mathbb{N}^*$.

To prove the first item, we use lemma (1.1.4), since

$$u_n \in L^2(]0,T[, H_0^1(\Omega)) \text{ and } u'_n \in L^2(]0,T[, H^{-1}(\Omega)),$$

we have for all $t, s \in [0,T]$

$$\|u_n(t)\|^2_{L^2} = \|u_n(s)\|_{L^2} + 2\int_s^t \langle u'_n(\lambda), u_n(\lambda)\rangle_{H^{-1},H_0^1} d\lambda,$$

then

$$\begin{aligned}\|u_n(t)\|^2_{L^2} &\le \|u_n(s)\|^2_{L^2} + 2\int_s^t \Big|\langle u'_n(\lambda), u_n(\lambda)\rangle\Big| d\lambda,\\ &\le \|u_n(s)\|^2_{L^2} + 2\|u'_n\|^2_{L^2(]0,T[,H^{-1})}\|u_n\|_{L^2(]0,T[,H_0^1)}.\end{aligned}$$

We integrate with respect of s on $[0.T]$, to find

$$T\|u_n(t)\|_{L^2}^2 \leq \|u_n(s)\|^2_{L^2(]0,T[,L^2)} + 2T\|u_n'\|^2_{L^2(]0,T[,H^{-1})}\|u_n\|.$$

It means that u_n is bounded sequence in $L^2(\Omega)$ for all $t \in [0,T]$ (and uniformly with respect of t). So, u_n is relatively compact in $H^{-1}(\Omega)$ for all $t \in [0,T]$.

Now, we will prove the second item. We use lemma 4.25 in [13].

Since $u_n \in L^1([0,T], H^{-1})$, then we have (in $H^{-1}(\Omega)$).

$\forall t_1, t_2 \in [0,T]$, $t_1 > t_2$

$$u_n(t_1) - u_n(t_2) = \int_{t_2}^{t_1} u_n'(s)ds,$$

so,

$$\begin{aligned}
\|u_n(t_1) - u_n(t_2)\|_{H^{-1}} &\leq \int_{t_2}^{t_1} \|u_n'(s)\|ds \\
&\leq \Big(\int_{t_2}^{t_1} ds\Big)^{\frac{1}{2}}\Big(\int_{t_2}^{t_1} \|u_n'(s)\|^2_{H^{-1}}ds\Big)^{\frac{1}{2}} \\
&\leq \sqrt{t_1 - t_2}\Big(\int_0^T \|u_n'(t)\|^2_{H^{-1}}dt\Big)^{\frac{1}{2}} \\
&= \sqrt{t_1 - t_2}\|u_n'\|_{L^2(]0,T[,H^{-1}(\Omega))}
\end{aligned}$$

Since u_n' is bounded in $L^2(]0,T[,H_0^1(\Omega))$, then we deduce that

$$\|u_n(t) - u_n(s)\|_{H^{-1}} \longrightarrow 0 \quad \text{uniformly with } n \in \mathbb{N}^* \text{and } t \in [0,T].$$

So, u_n (by using Ascoli's theorem) is relatively compact in $C([0,T], H^{-1})$. Because if u_n is relatively compact in $C([0,T], H^{-1})$, then there exist $w \in C([0,T], H^{-1})$ and subsequence u_{n_k} such that

$$u_{n_k} \longrightarrow w \quad \text{in} \quad H^{-1}(\Omega) \quad \text{uniformly with respect of } t \in [0,T] \text{ and also in } L^2(]0,T[,H_0^1).$$

So, we have

$$u_n(0) \longrightarrow w(0) \text{ and from an other hand } u_n(0) \longrightarrow u_0,$$

and hence $w(0) = u_0$.

On other hand we know that

$$\begin{aligned}
u_n &\rightharpoonup u \quad \text{in} \quad L^2(0,T;H_0^1), \\
u_n &\rightharpoonup u \quad \text{in} \quad L^2(0,T;H^{-1}).
\end{aligned}$$

By uniqueness of limit, then we get $u = w$, a.e on $[0, T]$. Since u and w are continuous on $[0, T]$, then

$$u(t) = w(t), \ \forall t \in [0, T],$$

and hence $u(0) = w(0) = u_0$.

Uniqueness:

Let u_1 and u_2 be two weak solutions of the problem (2.8) such that

$$\int_0^T \langle \dot{u}_1.v\rangle_{H^{-1},H_0^1} dt + \int_0^T \int_\Omega \nabla u_1 \nabla v(t) dx dt = 0.$$

And:

$$\int_0^T \langle \dot{u}_2.v\rangle_{H^{-1},H_0^1} dt + \int_0^T \int_\Omega \nabla u_2 \nabla v dx dt = 0.$$

Subtracting the two equations gives :

$$\int_0^T \langle \dot{u}_1 - \dot{u}_2, v\rangle_{H^{-1},H_0^1} ds + \int_0^T \int_\Omega (\nabla u_1 - \nabla u_2) \nabla v dx dt = 0.$$

We take $v = u_1 - u_2$, to get

$$\int_0^T \frac{1}{2} \frac{d}{dt} \|u_1 - u_2\|_2^2 + \int_0^T \|\nabla u_1(t) - \nabla u_2(t)\|_2^2 = 0$$

$$\|u_1(T) - u_2(T)\|_2^2 - \|u_1(0) - u_2(0)\|_2^2 = -2 \int_0^T \|\nabla u_1(t) - \nabla u_2(t)\|_2^2 \leq 0,$$

since $\|u_1(0) - u_2(0)\|_2 = 0$,

thus, $u_1 = u_2$. Then the problem has a unique solution.

In this part we will prove the well posedness of the solution to the problem (2.4) using Semigroup theory.

Firstly, we define some definitions and properties for helped.

2.6 Examples in the theory of partial differential equations

Let Ω be an open subset of $\mathbb{R}^n$, and let $Y = L^2(\Omega)$. Y is considered as a real Hilbert space, we define the linear operator A in Y by:

$$\begin{cases} D(A) = \Big\{ u \in H_0^1(\Omega); \Delta u \in L^2(\Omega) \Big\}; \\ Au = \Delta u, \quad u \in D(A). \end{cases} \tag{2.12}$$

The heat equation (without a source term) is given by:

$$\begin{cases} \dfrac{du}{dt} - \Delta u = 0, & \Omega \times]0,T[\\ u = 0, & \partial\Omega \times]0,T[\\ u(.,0) = u_0, \end{cases} \tag{2.13}$$

Definition 2.6.1 *Let X be a Banach space. A one parameter family $(S(t))_{t\geq 0}$ of $\mathcal{L}(X)$, is a semigroup of bounded linear operators on X if*

1. $S(0) = Id_X$.

2. $S(t+s) = S(t)S(s), \quad \forall t, s \geq 0.$

The linear operator A defined by:

$$D(A) = \Big\{ z \in X : \lim_{t \to 0^+} \frac{S(t)z - z}{t} \text{ exists} \Big\}$$

and

$$Az = \lim_{t \to 0^+} \frac{S(t)z - z}{t}, \quad \forall z \in D(A).$$

Is called the infinitesimal generator of the semigroup $(S(t))_{t\geq 0}$ and $D(A)$ is called the domain of A.

A semigroup $(S(t))_{t\geq 0}$ of bounded linear operators is called a strongly continuous (or a C_0-semigroup) if

$$\lim_{t \to 0^+} S(t)z = z, \quad \forall z \in X. \tag{2.14}$$

A strongly continuous $(S(t))_{t\geq 0}$ on X satisfying

$$\|S(t)\|_{\mathcal{L}(X)} \leq 1, \quad \forall t \geq 0,$$

is called a C_0-semigroup of contractions.

Definition 2.6.2 (Generator of a semi-group) *The generator of $(S(t))_{t\geq 0}$ is the linear operator L defined by*

$$D(A) = \left\{ x \in X : \frac{S(t)x - x}{h} \right\},$$

has a limit in X as $h \downarrow 0$,

and :

$$Lx = \lim_{h\downarrow 0} \frac{S(t)x - x}{h}$$

Proposition 2.6.3 *For the generator A of a C_0 semi-group $S(t)_{t\geq 0}$. The following conditions hold.*
(a) If $x \in X$ then $S(t)x \in D(A)$ and $AS(t)x = xAS(t)$ for all $t \geq 0$.
(b)The function $u : \mathbb{R}_+ \to X, t \to S(t)x$ is a unique solution.

2.7 Definition and main properties of m-dissipative operators:

Definition 2.7.1 *Let $A : D(A) \subset X \longrightarrow X$ be a (unbounded) linear operator. A is called dissipative if $R(Av, v)_X \leq 0$, $\forall v \in D(A)$. The dissipative operator$A$ is called m-dissipative if $\lambda I - A$ is surjective for some $\lambda > 0$.*

Definition 2.7.2 *An operator A in X is dissipative if and only if*

$$\|u - \lambda Au\| \geq \|u\|,$$

for all $u \in D(A)$ and all $\lambda > 0$.

Example 2.1 (The heat operator in $L^2(\Omega)$) *Let Ω be a bounded regular subset of $\mathbb{R}^n$, with a boundary Γ of class C^2. Set*

$$Y = L^2(\Omega), D(A) = H^2 \cap H_0^1(\Omega), Ay = \Delta y.$$

A is dissipative:

$$(Ay, y)_{L^2(\Omega)} = \int_\Omega \Delta yy = -\int_\Omega \nabla y \nabla y \leq 0$$

A is m-dissipative:

Let $\lambda > 0$. For all $f \in L^2(\Omega)$, the equation

$$\lambda y - \Delta y = f,$$

admits a unique solution in D(A).

Definition 2.7.3 *A bounded linear operator $A : D(A) \subset H \longrightarrow H$ is said to be monotone if it satisfies*

$$(Av, v) \geq 0 \quad \forall v \in D(A).$$

It is called maximal monotone if, in addition, $R(I + A) = H$, i.e., $\forall f \in H \quad \exists u \in D(A)$ such that

$$u + Au = f$$

Proposition 2.7.4 *Let A be a maximal monotone operator. Then*

(a) $D(A)$ *is dense in* H.

(b) A *is closed operator.*

(c) For every $\lambda > 0$, $(I + \lambda A)$*is bijective from* $D(A)$ *onto* H, $(I + \lambda A)^{-1}$ *is a bounded operator, and* $\|(I + \lambda A)^{-1}\|_{\mathcal{L}(H)} \leq 1$.

Proposition 2.7.5 *Let A be a maximal monotone operator. Then*

(a_1) $A_\lambda v = A(J_\lambda v) \quad \forall v \in H$ *and* $\forall \lambda > 0$,

(a_2) $A_\lambda v = J_\lambda(Av) \quad \forall v \in D(A)$ *and* $\forall \lambda > 0$,

(b) $|A_\lambda v| \leq |Av| \quad \forall v \in D(A)$ *and* $\forall \lambda > 0$,

(c) $\lim_{\lambda \to 0} J_\lambda v = v \quad \forall v \in H$,

(d) $\lim_{\lambda \to 0} A_\lambda v = Av \quad \forall v \in D(A)$,

(e) $(A_\lambda v, v) \geq 0 \quad \forall v \in H$ *and* $\forall \lambda > 0$,

(f) $|A_\lambda v| \leq (1/\lambda)|v| \quad \forall v \in H$ *and* $\forall \lambda > 0$.

Definition 2.7.6 *Let A be a maximal monotone operator. For every $\lambda > 0$, set*

$$\boxed{J_\lambda = (I + \lambda A)^{-1} \qquad \text{and} \qquad A_\lambda = \frac{1}{\lambda}(I - J_\lambda);}$$

J_λ is called the resolvent of A, and A_λ is the Yosida approximation (or regularization) of A. Keep in mind that $\|J_\lambda\|_{L(H)} \leq 1$.

Corollary 2.7.7 *If A is m-dissipative in X, then*

$$J_\lambda u \to u,$$

as $\lambda \downarrow 0$, for all $u \in X$,

and

$$A_\lambda u \to Au,$$

as $\lambda \downarrow 0$, and for all $u \in D(A)$.

2.8 Existence and uniqueness

Theorem 2.8.1 (Cauchy, Lipschitz, Picard) *Let E be a Banach space and let $F : E \to E$ be a Lipschitz map, i.e., there is a constant L such that*

$$\|Fu - Fv\| \leq L\|u - v\| \quad \forall u, v \in E.$$

Then given any $u_0 \in E$, there exists a unique solution $u \in C^1([0, +\infty); E)$ of the problem

$$\begin{cases} \frac{du}{dt}(t) = Fu(t) & \text{on } [0, +\infty), \\ u(0) = u_0. \end{cases}$$

u_0 is called the initial data.

Lemma 2.8.2 *[3] Let $w \in C^1([0, \infty); H)$ be a function satisfying*

$$\frac{dw}{dt} + A_\lambda w = 0 \quad \text{on} \quad [0, \infty). \tag{2.15}$$

Then the functions $t \mapsto |w(t)|$ and $t \mapsto \left|\frac{dw}{dt}\right| = |A_\lambda w|$ non-increasing on $[0, \infty)$

Theorem 2.8.3 (Hille-Yosida) *[3] Let A a maximal monotone operator. Then, given any $u_0 \in D(A)$ there exist a unique function*

$$u \in C^1([0,+\infty];H) \cap C([0,+\infty];D(A))$$

satisfying

$$\begin{cases} \dfrac{du}{dt} + Au = 0, \quad on \quad [0,\infty), \\ u(0) = u_0. \end{cases} \tag{2.16}$$

Moreover,

$$|u(t)| \leq |u_0| \quad and \quad \left|\frac{du}{dt}(t)\right| = |Au(t)| \leq |Au_0| \quad \forall t \geq 0.$$

Proof. Existence

Step 1: (Approximation problem)

We replace A by A_λ in (2.16) and apply Theorem (2.8.1) on the approximation problem, and then pass the limit as $\lambda \to 0$, using various estimates that are independent of λ. So, let u_λ is a solution of the problem

$$\begin{cases} \dfrac{du_\lambda}{dt} - \Delta_\lambda u_\lambda(t) = 0, \quad \text{on} \quad [0,\infty), \\ u_\lambda(0) = u_0 \in D(A) \end{cases} \tag{2.17}$$

and we have the estimate

$$|u_\lambda| \leq |u_0| \quad \forall t \geq 0, \quad \lambda > 0, \tag{2.18}$$

$$\left|\frac{du_\lambda}{dt}(t)\right| = |\Delta_\lambda u_\lambda(t)| \leq |\Delta u_0| \quad \forall t \geq 0, \quad \lambda > 0, \tag{2.19}$$

They follow directly from the lemma (2.8.2) and the fact that $|\Delta_\lambda u_\lambda(t)| \leq |\Delta u_0|$.

Step 2 (Convergence of u_λ)

We will prove here for every $t \geq 0, u_\lambda(t)$converges, as $\lambda \longrightarrow 0$, to some limit, denoted by $u(t)$. Moreover, the convergence is uniform on every bounded interval $[0,T]$.

For every $\lambda, \mu > 0$ we have

$$\frac{du_\lambda}{dt} - \frac{du_\mu}{dt} - \Delta_\lambda u_\lambda + \Delta_\mu u_\mu = 0,$$

and thus

$$\frac{1}{2}\frac{d}{dt}|u_\lambda - u_\mu|^2 + (\Delta_\mu u_\mu(t) - \Delta_\lambda u_\lambda(t), u_\mu(t) - u_\lambda(t)) = 0. \qquad (2.20)$$

Dropping t for simplicity, we write

$$\begin{aligned}(-\Delta_\lambda u_\lambda + \Delta_\mu u_\mu, u_\lambda - u_\mu)&\\ &= (-\Delta_\lambda u_\lambda + \Delta_\mu u_\mu, u_\lambda - J_\lambda u_\lambda + J_\lambda u_\lambda - J_\mu u_\mu + J_\mu u_\mu - u_\mu)\\ &= (-\Delta_\lambda u_\lambda + \Delta_\mu u_\mu, -\lambda\Delta_\lambda u_\lambda + \mu\Delta_\mu u_\mu)\\ &+ (-\Delta(J_\lambda u_\lambda - J_\mu u_\mu), J_\lambda u_\lambda - J_\mu u_\mu)\\ &\geq (-\Delta_\lambda u_\lambda + \Delta_\mu u_\mu, -\lambda\Delta_\lambda u_\lambda + \mu\Delta_\mu u_\mu). \qquad (2.21)\end{aligned}$$

It follows from (2.19), (2.20) and (2.21) that

$$\frac{1}{2}\frac{d}{dt}|u_\lambda - u_\mu|^2 \leq 2(\lambda+\mu)|\Delta u_0|^2.$$

Integrate this inequality, we obtain

$$|u_\lambda - u_\mu|^2 \leq 4(\lambda+\mu)t|\Delta u_0|^2,$$

i.e,

$$|u_\lambda - u_\mu| \leq 2\sqrt{(\lambda+\mu)t}|\Delta u_0|. \qquad (2.22)$$

It follows that for every fixed $t \geq 0$, u_λ is a Cauchy sequence as $\lambda \to 0$ and thus it converges to a limit, denoted by $u(t)$. Passage to the limit in (2.22) as $\mu \to 0$. We have

$$|u_\lambda - u(t)| \leq 2\sqrt{\lambda t}|\Delta u_0|$$

Therefore, the convergence is uniform in t on every bound interval [0,T] and so $u \in C([0,+\infty); H)$.

Step 3: Assuming, in addition, that $u_0 \in D(\Delta^2)$, i.e., $u_0 \in D(\Delta)$ and $\Delta u_0 \in D(A)$, we prove here that $\frac{du_\lambda}{dt}(t)$ converges, as $\lambda \to 0$, to some limit and that the convergence is uniform on every bounded interval $[0, T]$.

Set $v_\lambda = \frac{du_\lambda}{dt}$, so that $\frac{dv_\lambda}{dt} - \Delta_\lambda v_\lambda = 0$. Following the same argument as in Step 2, we see that

$$\frac{1}{2}\frac{d}{dt}|v_\lambda - v_\mu|^2 \leq (|\Delta_\lambda v_\lambda| + |\Delta_\mu v_\mu|)(\lambda|\Delta_\lambda v_\lambda| + \mu|\Delta_\mu v_\mu|). \qquad (2.23)$$

By lemma (2.8.2) we have

$$|\Delta_\lambda v_\lambda(t)| \leq |\Delta_\lambda v_\lambda(0)| = |\Delta_\lambda \Delta_\lambda u_0|, \tag{2.24}$$

and similarly

$$|\Delta_\mu v_\mu(t)| \leq |\Delta_\mu v_\mu(0)| = |\Delta_\mu \Delta_\mu u_0|. \tag{2.25}$$

Finally, since $\Delta u_0 \in D(A)$, we obtain

$$\Delta_\lambda \Delta_\lambda u_0 = J_\lambda \Delta J_\lambda \Delta u_0 = J_\lambda J_\lambda \Delta\Delta u_0 = J_\lambda^2 \Delta^2 u_0,$$

and thus

$$|\Delta_\lambda \Delta_\lambda u_0| \leq |\Delta^2 u_0|, \quad |\Delta_\mu \Delta_\mu u_0| \leq |\Delta^2 u_0|. \tag{2.26}$$

Combining (2.23), (2.24), (2.25), and (2.26), we are led to

$$\frac{1}{2}\frac{d}{dt}|v_\lambda - v_\mu|^2 \leq 2(\lambda+\mu)|\Delta^2 u_0|^2.$$

We conclude, as in Step 2, that $v_\lambda(t) = \frac{du_\lambda}{dt}(t)$ converges, as $\lambda \to 0$, to some limit and that the convergence is uniform on every bounded interval $[0,\, T]$.

Step 4 Assuming that $u_0 \in D(\Delta^2)$ we prove here that u is a solution of (2.16). By Steps 2 and 3 we know that for all $T < \infty$,

$$\begin{cases} u_\lambda(t) \to u(t), & \text{as } \lambda \to 0, \quad \text{uniformly on } [0,T], \\ \frac{du_\lambda}{dt}(t) & \text{converges, as } \lambda \to 0, \quad \text{uniformly on } [0,T]. \end{cases}$$

It follows easily that $u \in C^1([0,+\infty); H)$ and that $\frac{du_\lambda}{dt}(t) \to \frac{du}{dt}(t)$, as $\lambda \to 0$, uniformly on $[0,T]$. Rewrite (2.17) as

$$\frac{du_\lambda}{dt}(t) - \Delta(J_\lambda u_\lambda(t)) = 0. \tag{2.27}$$

Note that $J_\lambda u_\lambda(t) \to u(t)$ as $\lambda \to 0$, since

$$\begin{aligned} |J_\lambda u_\lambda(t) - u(t)| &\leq |J_\lambda u_\lambda(t) - J_\lambda u(t)| + |J_\lambda u(t) - u(t)| \\ &\leq |u_\lambda(t) - u(t)| + |J_\lambda u(t) - u(t)| \to 0. \end{aligned}$$

Applying the fact that A has a closed graph, we deduce from (2.27) that $u(t) \in D(A)$ $\forall t \geq 0$, and that

$$\frac{du}{dt}(t) - \Delta u(t) = 0.$$

Finally, since $u \in C^1([0,+\infty); H)$, the function $t \mapsto Au(t)$ is continuous from $[0,+\infty)$ into H and thus $u \in C([0,+\infty); D(A))$. Hence we have obtained a solution of (2.16) satisfying, in addition,

$$|u(t)| \leq |u_0| \quad \forall t \geq 0 \quad \text{and} \quad \left|\frac{du}{dt}(t)\right| = |Au(t)| \leq |Au_0| \quad \forall t \geq 0.$$

Uniqueness

Let u and $\bar{u}$ are two solutions of (2.13), we have

$$\Big(\frac{d}{dt}(u-\bar{u}), (u-\bar{u})\Big) = -\Big(A(u-\bar{u}), u-\bar{u}\Big) \leq 0.$$

Since

$$\frac{1}{2}\frac{d}{dt}|u(t)-\bar{u}(t)|^2 = \Big(\frac{d}{dt}(u(t)-\bar{u}(t)), u(t)-\bar{u}(t)\Big),$$

by (2.8.2) the function $t \to |u(t)-\bar{u}(t)|$ is non-increasing on $[0,\infty)$. Since $|u(0)-\bar{u}(0)| = 0$, it follows that

$$|u(t)-\bar{u}(t)| = 0.$$

Then $u = \bar{u}$, the problem (2.13) has a unique solution. ■

Chapter 3

The fractional Laplacian viscoelastic problem

In this chapter we will study the Cauchy problem for linear fractional wave equation with a viscoelastic term . Firstly we study the existence of the global solution using the Fourier and Laplace transforms. Then we investigate the stability of the solution using the energy method in the Fourier space, multiplier technique and Lyapunov functional.

We consider the following linear problem:

$$\begin{cases} u_{tt} + (-\Delta)^{\sigma} u + u + \displaystyle\int_0^t g(t-s)\Delta u(s,t)ds = 0, \quad x \in \mathbb{R}^n, t > 0, \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}^n, \\ u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{3.1}$$

where $\sigma \geq 1$ and $n \geq 1$, the last term of the problem is the memory term denoted by $\int_0^t g(t-s)\Delta u(s,t)ds$, and initial data (u_0, u_1).

Ting Xie and Han Yang [33] considered the initial value problem of the following semi-linear σ-evolution equation with memory term, using the energy method in the Fourier space, the decay estimates for the solutions to the corresponding linear problem are established. Additionally, assuming small initial data in suitable time-weighted Sobolev spaces, the global-in-time existence of the solutions to the semilinear issue is proved by contraction mapping:

$$u_{tt} + (-\Delta)^{\sigma} u + u + g * \Delta u = |u|^p, x \in \mathbb{R}^n, \quad t > 0 \tag{3.2}$$

under initial condition data

$$u(0, x) = u_0, u_t(0, t) = u_1, \quad x \in \mathbb{R}^n,$$

where $\sigma \geq 1, n \geq 1, p > 1$. The term $(g * \Delta u)(t, x) := \int_0^t g(t-s)\Delta u(s, x)ds$ denotes memory term. It was assumed that the kernel g satisfies the assumptions :

1. $g \in C^2(\mathbb{R}^+) \cap W^{1,2}(\mathbb{R}^+)$
2. $g(\tau) > 0, -C_0 g(\tau) \leq g'(\tau) \leq -C_1 g(\tau), |g'(\tau)| \leq C_2 g(\tau), \quad \forall \tau \in \mathbb{R}^+,$
3. $1 - \int_0^t g(\tau)d\tau \geq C_3, \quad \forall t \in \mathbb{R}^+,$

Where $C_i (i = 0, 1, 2, 3)$ are positive constants .

3.1 Preliminaries and Assumptions

We assume that the relaxation function g satisfies the following conditions:

$(\mathcal{H}1) g : [0; \infty) \to (0; \infty)$ is a strictly decreasing $\mathcal{C}^1$ function such that

$$1 - \int_0^{\infty} g(s)ds = l > 0. \tag{3.3}$$

$(\mathcal{H}2)$: There exists a positive non-increasing differentiable function $\eta(t)$ satisfying:

$$g'(t) \leq -\eta(t)g(t), \quad t > 0. \tag{3.4}$$

Lemma 3.1.1 *Assume that* $(\mathcal{H}1)$ *holds. Then for any* $v \in L^2_{loc}(\mathbb{R}_+, \mathbb{C})$,*we have :*

$$\left| \int_0^t g(t-s)(v(t) - v(s))ds \right|^2 \leq (1-l)(g \circ v)(t), \quad \forall t \geq 0, \tag{3.5}$$

and

$$\Big|\int_0^t g'(t-s)(v(t)-v(s))ds\Big|^2 \leq -g(0)(g'\circ v)(t), \quad \forall t \geq 0, \tag{3.6}$$

where

$$(g\circ v)(t) := \int_0^t g(t-s)|v(t)-v(s)|^2 ds. \tag{3.7}$$

Proof.

1. To prove the first inequality (3.5), we have

$$\Big|\int_0^t g(t-s)(v(t)-v(s))ds\Big|^2 = \Big|\int_0^t \sqrt{g(t-s)}\sqrt{g(t-s)}(v(t)-v(s))ds\Big|^2,$$

   (i) We apply Cauchy- Schwarz inequality we obtain :

$$\begin{aligned}\Big|\int_0^t g(t-s)(v(t)-v(s))ds\Big|^2 &= \Big|\int_0^t \sqrt{g(t-s)}\sqrt{g(t-s)}(v(t)-v(s))ds\Big|^2\\ &\leq \Big(\int_0^t \sqrt{g(t-s)}^2 ds\Big)\Big(\int_0^t \sqrt{g(t-s)}^2(v(t)-v(s))^2 ds\Big)\\ &= \Big(\int_0^t g(t-s)ds\Big)\Big(\int_0^t g(t-s)(v(t)-v(s))^2 ds\Big).\end{aligned}$$

   (ii) We prove that $\int_0^t g(s)ds = \int_0^t g(t-s)ds$ for all $t, s \in [0, \infty[$.

   To prove (ii) we use the change of variables .

   We put $z = t - s$, by derivation with respect to s we find:

$$dz = -ds,$$

$$-dz = ds,$$

   then, the boundary of integration are as follows:

   When $s = 0$, then $z = t$, and when $s = t$, then $z = 0$, and from it:

$$\int_t^0 g(z)(-dz) = \int_0^t g(z)dz,$$

   thus, we get

$$\int_0^t g(z)dz = \int_0^t g(s)ds.$$

Finally :

$$\int_0^t g(t-s)ds = \int_0^t g(s)ds.$$

(iii) By the assumption $(\mathcal{H}1)$, we have g is a continuous positive function then for $t \in [0, \infty[$:

$$\int_0^t g(s)ds \leq \int_0^{+\infty} g(s)ds, \tag{3.8}$$

Additionally, we have:

$$1 - \int_0^\infty g(s)ds = l > 0,$$

the integration from 0 to t

$$\int_0^t g(t) \leq 1 - l, \tag{3.9}$$

we use (i) and (ii),(iii) and (3.7), we finally find:

$$\begin{aligned}
\Big|\int_0^t g(t-s)(v(t)-v(s))ds\Big|^2 &= \Big|\int_0^t \sqrt{g(t-s)}\sqrt{g(t-s)}(v(t)-v(s))ds\Big|^2 \\
&\leq \Big(\int_0^t \sqrt{g(t-s)}^2 ds\Big)\Big(\int_0^t g(t-s)(v(t)-v(s))^2 ds\Big) \\
&\leq \int_0^t g(t-s)ds\Big(\int_0^t g(t-s)(v(t)-v(s))^2 ds\Big) \\
&= \int_0^t g(s)ds\Big(\int_0^t g(t-s)(v(t)-v(s))^2 ds\Big) \\
&\leq \int_0^\infty g(s)ds\Big(\int_0^t g(t-s)(v(t)-v(s))^2 ds\Big) \\
&\leq \int_0^\infty g(s)ds\Big(\int_0^t g(t-s)|v(t)-v(s)|^2 ds\Big) \\
&\leq (1-l)(g \circ v)(t).
\end{aligned}$$

The proof of (3.5) is completed.

2. Now, to prove the second inequality we use the assumption$(\mathcal{H}1)$:

   (a) Since g is a strictly decreasing $\mathcal{C}^1$ function it means $g'(t) < 0$ then $-g'(t) > 0$, after the same operation in (i) for the function $-g'(t)$ we obtain:

$$
\begin{aligned}
\Big|\int_0^t g'(t-s)(v(t)-v(s))ds\Big|^2 &= \Big|\int_0^t \sqrt{-g'(t-s)}\sqrt{-g'(t-s)}(v(t)-v(s))ds\Big|^2 \\
&\leq \Big(\int_0^t \sqrt{-g'(t-s)}^2 ds\Big)\Big(\int_0^t \sqrt{-g'(t-s)}^2 (v(t)-v(s))^2 ds\Big) \\
&= \Big(\int_0^t -g'(t-s)ds\Big)\Big(\int_0^t -g'(t-s)(v(t)-v(s))^2 ds\Big).
\end{aligned}
$$

(b) We prove that $\int_0^t g'(s)ds = \int_0^t g'(t-s)ds$ for all $t, s \in [0, \infty[$:

We put $z = t - s$, by derivation with respect to s we find

$$dz = -ds,$$

$$-dz = ds,$$

when $s = 0$, we find $z = t$, and when $s = t$, we find $z = 0$ then:

$$\int_t^0 g'(z)(-dz) = \int_0^t g'(z)dz,$$

so

$$
\begin{aligned}
\int_0^t g'(z)dz &= \int_0^t g'(s)ds \\
&= g(t) - g(0).
\end{aligned}
$$

(c) By the assumption($\mathcal{H}1$), g is a positive decreasing and continuous function in $[0, \infty[$,then it has a maximum value at point 0, or we say for all $t \in [0, \infty[$

$$g(0) = max\{g(t)\},$$

and always verified :

$$g(0) - g(t) \leq g(0).$$

By (a), (b) and (c) we find:

$$\begin{aligned}
\Big|\int_0^t g'(t-s)(v(t)-v(s))ds\Big|^2 &= \Big|\int_0^t \sqrt{-g'(t-s)}\sqrt{-g'(t-s)}(v(t)-v(s))ds\Big|^2 \\
&\leq \Big(\int_0^t \sqrt{-g'(t-s)}^2 ds\Big)\Big(\int_0^t -g'(t-s)(v(t)-v(s))^2 ds\Big) \\
&\leq \int_0^t -g'(t-s)ds\Big(\int_0^t -g'(t-s)(v(t)-v(s))^2 ds\Big) \\
&= \int_0^t -g'(s)ds\Big(\int_0^t -g'(t-s)(v(t)-v(s))^2 ds\Big) \\
&= (g(0)-g(t))\Big(\int_0^t -g'(t-s)(v(t)-v(s))^2 ds\Big) \\
&\leq -g(0)\Big(\int_0^t g'(t-s)(v(t)-v(s))^2 ds\Big) \\
&\leq -g(0)\Big(\int_0^t g'(t-s)|v(t)-v(s)|^2 ds\Big).
\end{aligned}$$

From (3.7) for the function g' we have

$$\Big(g' \circ v\Big)(t) = \int_0^t g'(t-s)|v(t)-v(s)|^2,$$

we find (3.6) .

The proof of lemma (3.1.1) is completed.

■

3.2 Existence of the Global solution

3.2.1 Fourier transform of the Cauchy problem

By taking the Fourier transform of the Cauchy problem, and using (1.4.2), we get the following problem:

$$\begin{cases}
\hat{u}_{tt} + (1+|\xi|^{2\sigma})\hat{u} - |\xi|^2 \int_0^t g(t-s)\hat{u}(s)ds = 0, \quad \xi \in \mathbb{R}^n, t>0, \\
\hat{u}(\xi,0) = \hat{u}_0, \\
\hat{u}_t(\xi,0) = \hat{u}_1,
\end{cases} \tag{3.10}$$

3.2.2 Solution formula

Applying the Laplace transform to (3.10), and we use (1.5.3), we obtain

$$\mathfrak{L}(\hat{u}_{tt}) + (1+|\xi|^{2\sigma})\mathfrak{L}(\hat{u}) - |\xi|^2\mathfrak{L}(g)\mathfrak{L}(\hat{u}) = 0,$$

$$\lambda^2\mathfrak{L}(\hat{u}) - \lambda\hat{u}_0 - \hat{u}_1 + (1+|\xi|^{2\sigma})\mathfrak{L}(\hat{u}) - |\xi|^2\mathfrak{L}(g)\mathfrak{L}(\hat{u}) = 0,$$

so

$$\mathfrak{L}(\hat{u})(\xi,t) = \frac{\lambda\hat{u}_0 + \hat{u}_1}{\lambda^2 + 1 + |\xi|^{2\sigma} - |\xi|^2\mathfrak{L}(g)(\lambda)}.$$

Then applying the Laplace inverse we find

$$\hat{u}(\xi,t) = \mathfrak{L}^{-1}\left[\frac{\lambda\hat{u}_0 + \hat{u}_1}{\lambda^2 + 1 + |\xi|^{2\sigma} - |\xi|^2\mathfrak{L}(g)(\lambda)}\right](\xi,t).$$

Taking $F(\lambda) = \lambda^2 + 1 + |\xi|^{2\sigma} - |\xi|^2\mathfrak{L}(g)(\lambda)$, therefore, the solution formula to (3.10) is

$$\hat{u}(\xi,t) = \hat{u}_0(\xi)\mathfrak{L}^{-1}\left[\frac{\lambda}{F(\lambda)}\right] + \hat{u}_1(\xi)\mathfrak{L}^{-1}\left[\frac{1}{F(\lambda)}\right], \tag{3.11}$$

As in Ref.[20], we can find the solution formula of the problem (3.1) .

To prove the existence of $\mathfrak{L}^{-1}\left[\frac{\lambda}{F(\lambda)}\right]$ and $\mathfrak{L}^{-1}\left[\frac{1}{F(\lambda)}\right]$, we use the following technique: Suppose that $G(x,t)$ is solution to the following problem

$$\begin{cases} G_{tt} + (1+(-\Delta)^\sigma)G + \int_0^t g(t-s)\Delta G(s,x)ds = 0, & x\in\mathbb{R}^n, t>0, \\ G(x,0) = \delta(x), & x\in\mathbb{R}^n, \\ G_t(x,0) = 0, & x\in\mathbb{R}^n, \end{cases} \tag{3.12}$$

Where δ denotes Dirac's delta function. Applying the Fourier transform to the space variable of (3.12), and we use (1.4.2), we get :

$$\begin{cases} \hat{G}_{tt} + (1+|\xi|^{2\sigma})\hat{G} - |\xi|^2\int_0^t g(t-s)\hat{G}(s)ds = 0, & \xi\in\mathbb{R}^n, t>0 \\ \hat{G}(\xi,0) = 1, & \xi\in\mathbb{R}^n, \\ \hat{G}_t(\xi,0) = 0, & \xi\in\mathbb{R}^n, \end{cases} \tag{3.13}$$

we use (1.5.3) in (3.13), to find

$$\mathfrak{L}[\hat{G}_{tt}](\lambda) = \lambda^2\mathfrak{L}[\hat{G}(\xi,t)] - \lambda\hat{G}(\xi,0) - \hat{G}_t(\xi,0) = \lambda^2\mathfrak{L}[\hat{G}] - \lambda.$$

So, the Laplace transform of (3.13) defined by the following :

$$\lambda^2\mathfrak{L}(\hat{G}) - \lambda + (1+|\xi|^{2\sigma})\mathfrak{L}(\hat{G}) - |\xi|^2\mathfrak{L}(g)\mathfrak{L}(\hat{G}) = 0.$$

On other hand, we suppose that $\hat{H}$ verifies

$$\begin{cases} H_{tt} + (1+(-\Delta)^\sigma)H + \int_0^t g(t-s)\Delta H(s,x)ds = 0, \quad x \in \mathbb{R}^n, \\ H(x,0) = 0, \quad x \in \mathbb{R}^n, \\ H_t(x,0) = \delta(x) \quad x \in \mathbb{R}^n, \end{cases} \tag{3.14}$$

Similarly to G, applying the Fourier transform to the space variable of (3.14), we have:

$$\begin{cases} \hat{H}_{tt} + (1+|\xi|^{2\sigma})\hat{H} - |\xi|^2 \int_0^t g(t-s)\hat{H}(s)ds = 0, \quad \xi \in \mathbb{R}^n, t > 0 \\ \hat{H}(\xi,0) = 0, \quad \xi \in \mathbb{R}^n, \\ \hat{H}_t(\xi,0) = 1, \quad \xi \in \mathbb{R}^n, \end{cases} \tag{3.15}$$

we use (1.5.3)in (3.15), we get

$$\mathfrak{L}[\hat{H}_{tt}](\lambda) = \lambda^2\mathfrak{L}[\hat{H}] - \lambda\hat{H}(\xi,0) - \hat{H}_t(\xi,0) = \lambda^2\mathfrak{L}[\hat{H}] - 1,$$

the Laplace transform of (3.15) defined by the following :

$$\lambda^2\mathfrak{L}(\hat{H}) - 1 + (1+|\xi|^{2\sigma})\mathfrak{L}(\hat{H}) - |\xi|^2\mathfrak{L}(g)\mathfrak{L}(\hat{H}) = 0.$$

Therefore, we can extract the expressions of $\hat{G}$ and$\hat{H}$,we obtain the following formulas :

$$\hat{G}(\xi,t) = \mathfrak{L}^{-1}\left[\frac{\lambda}{\lambda^2 + 1 + |\xi|^{2\sigma} - |\xi|^2\mathfrak{L}[g](\lambda)}\right](\xi,t),$$

$$\hat{H}(\xi,t) = \mathfrak{L}^{-1}\left[\frac{1}{\lambda^2 + 1 + |\xi|^{2\sigma} - |\xi|^2\mathfrak{L}[g](\lambda)}\right](\xi,t).$$

Then $\hat{G}$ and $\hat{H}$ given by:

$$\hat{G}(\xi,t) = \mathfrak{L}^{-1}\left[\frac{\lambda}{F(\lambda)}\right],$$

and

$$\hat{H}(\xi,t) = \mathfrak{L}^{-1}\left[\frac{1}{F(\lambda)}\right].$$

Where $F(\lambda) = \lambda^2 + 1 + |\xi|^{2\sigma} - |\xi|^2\mathfrak{L}(g)(\lambda)$.

The existence of $\hat{G}(\xi,t)$ and $\hat{H}(\xi,t)$ are deduce as follows.

Lemma 3.2.1 *$\hat{G}(\xi, t)$ and $\hat{H}(\xi, t)$ exist.*

Proof. The existence of $\hat{G}(\xi, t)$ and $\hat{H}(\xi, t)$ is proved by using the same arguments as in Liu and Ueda [30]. It is necessarily to find the inverse Laplace transform of the functions $\frac{\lambda}{F(\lambda)}$, similarly the same applies of $\hat{H}(\xi, t)$.

We identify the formula for inverse Laplace transform we can calculate $\hat{G}(\xi, t)$ and $\hat{H}(\xi, t)$

$$\mathfrak{L}^{-1}\left[\frac{\lambda}{F(\lambda)}\right] = \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\lambda}{F(\lambda)} e^{\lambda t} d\lambda.$$

This integral is called a complex integral in the Complex plane, meaning that is the entire integral is defined with in the set of complex numbers . It extends along a vertical line in the complex plane from $(-\infty)$ to $(+\infty)$ in the imaginary direction, but at a constant value γ in the real direction plane, and this located at $\mathcal{R}e\lambda = \gamma$, it means, the Bromwich inversion contour γ runs from$\gamma - i\infty$ to $\gamma + i\infty$ along a straight line. γ must lie to the right of all the singularities of $F(\lambda)$.

First, we need to prove the existence of singular points of the function, which are the values that cause the function to fail to be analytic. For the Laplace transform, these are typically the points where the function becomes undefined, meaning we need to solve the equation $F(\lambda) = 0$.

(i) For the function to be well-defined, we must prove that $\mathfrak{L}[g]$ exists. It means prove that the function g is a function of the exponential order and partially continuous.
$(\mathcal{H}1)$ gives that g is a function of $\mathcal{C}^1$ then it is continuous in $[0, \infty[$. Additionally, we have:

$$|\mathfrak{L}[g](\lambda)| = \Big|\int_0^\infty e^{-\lambda t} g(t)\Big| dt \leq \int_0^\infty |e^{-\lambda t}| g(t) dt = \int_0^\infty e^{-Re\lambda t} g(t) dt, \tag{3.16}$$

since $|e^{-\lambda t}| = |e^{-(\mu+iv)t)}| = |e^{-\mu t}||e^{-ivt}| = |e^{-\mu t}|\sqrt{cos^2 vt + sin^2 vt} = |e^{-Re(\lambda)t}|$. We know that $\eta(t) \geq \inf_t \eta(t)$. From $(\mathcal{H}2)$, we have

$$\frac{g'(t)}{g(t)} \leq -\inf_t \eta(t),$$

we integrate from 0 to t we get

$$\int_0^t \frac{g'(t)}{g(t)}\,dt \le -\inf_t \eta(t) \int_0^t dt,$$

thus

$$ln(g(t)) - ln(g(0)) \le -\inf_t \eta(t)t$$

$$ln\Big(\frac{g(t)}{g(0)}\Big) \le -\inf_t \eta(t)t$$

$$e^{ln\left(\frac{g(t)}{g(0)}\right)} \le e^{-\inf_t \eta(t)t},$$

finally we get

$$g(t) \le g(0)e^{-\inf_t \eta(t)t}.$$

Consequently, (3.16), implies

$$|\mathfrak{L}[g](\lambda)| \le g(0)\int_0^\infty e^{-(Re\lambda+\inf_t \eta(t))t} = \frac{g(0)}{Re\lambda+\inf_t \eta(t)}. \tag{3.17}$$

Thus, $\mathfrak{L}[g](\lambda)$ exists for $Re\lambda > -\inf_t \eta(t)$, where $\eta(t)$ is appeared in Assumption $[\mathcal{H}]$.

(ii) We need to consider the zero points of $F(\lambda)$. Let $\lambda = \mu + iv$, if we assume that $\lambda_1 = \mu_1 + iv_1$ represents the zero point of $F(\lambda)$ (is the root of the equation $F(\lambda)$), and $\mu_1 > -\inf_t \eta(t)$ then μ_1 and v_1 satisfy:

$$F(\lambda_1) = F(\mu_1 + iv_1) = 0.$$

Since $e^{-ivt} = cos(vt) - isin(vt)$, then, we have :

$$
\begin{aligned}
F(\mu_1 + iv_1) &= (\mu_1 + iv_1)^2 + 1 + |\xi|^{2\sigma} - |\xi|^2 \mathfrak{L}[g](\mu_1 + iv_1) \\
&= \mu_1^2 - v_1^2 + 2i\mu_1 v_1 + 1 + |\xi|^{2\sigma} - |\xi|^2 \int_0^\infty e^{-(\mu_1 + iv_1)t} g(t)dt. \\
&= \mu_1^2 - v_1^2 + 2i\mu_1 v_1 + 1 + |\xi|^{2\sigma} - |\xi|^2 \int_0^\infty e^{-\mu_1 t} e^{-iv_1 t} g(t)dt. \\
&= \mu_1^2 - v_1^2 + 2i\mu_1 v_1 + 1 + |\xi|^{2\sigma} - |\xi|^2 \int_0^\infty e^{-\mu_1 t} (cos(v_1 t) - isin(v_1 t)) g(t)dt \\
&= \mu_1^2 - v_1^2 + 2i\mu_1 v_1 + 1 + |\xi|^{2\sigma} - |\xi|^2 \int_0^\infty e^{-\mu_1 t} cos(v_1 t) g(t)dt \\
&+ |\xi|^2 \int_0^\infty e^{-\mu_1 t} isin(v_1 t) g(t)dt. \\
&= 0.
\end{aligned}
$$

So,

$$
\mathcal{R}eF(\lambda_1) = \mu_1^2 - v_1^2 + 1 + |\xi|^{2\sigma} - |\xi|^2 \int_0^\infty e^{-\mu_1 t} cos(v_1 t) g(t)dt = 0, \tag{3.18}
$$

and:

$$
\mathcal{I}mF(\lambda_1) = 2\mu_1 v_1 + |\xi|^2 \int_0^\infty sin(v_1 t) e^{-\mu_1 t} g(t)dt = 0. \tag{3.19}
$$

Case (1): when $\xi = 0$, from (3.18) and (3.19), we have $\mu_1 = 0, v_1 = \pm 1$.

Case (2): when $\xi \neq 0$ $\mathcal{R}e(\lambda_1) = \mu_1 < 0$. Prove it by contradiction .

Assume that $\mu_1 \geq 0$:

if $v_1 = 0$, then in view of assumption $\mathcal{H}1$, we know that:

$$
\int_0^\infty g(t)dt \leq 1 - l.
$$

- Since $v_1 = 0$, then $cos(v_1 t) = 1$.
- Since, $\mu_1 \geq 0$, for all $t \in [0, \infty[$, so we obtain

$$
e^{-\mu_1 t} = \frac{1}{e^{\mu_1 t}} \leq 1.
$$

- We know that, for all $\xi \in \mathbb{R}^n, |\xi|^2 \geq 0$.

By assumption $\mathcal{H}1$ and previous calculations, we get :

$$\int_0^\infty e^{-\mu_1 t} g(t)dt \leq \int_0^\infty g(t)dt \leq 1-l,$$

We multiply by $(-|\xi|^2)$ we obtain:

$$-|\xi|^2 \int_0^\infty e^{-\mu_1 t} g(t)dt \geq -|\xi|^2 \int_0^\infty g(t)dt \geq -|\xi|^2(1-l).$$

From (3.18), we have :

$$ReF(\lambda_1) = \mu_1^2 + 1 + |\xi|^{2\sigma} - |\xi|^2 \int_0^\infty e^{-\mu_1 t} g(t)dt \geq \mu_1^2 + 1 + |\xi|^{2\sigma} - |\xi|^2(1-l) \geq c,$$

it yields contradiction with (3.18).

If $v_1 \neq 0$, then we have :

$$\mathcal{I}mF(\lambda_1) = v_1\Big(2v_1 + |\xi|^2 \int_0^\infty \frac{sin(v_1 t)}{v_1} e^{-\mu_1 t} g(t)dt\Big)$$

we will prove that:

$$\int_0^\infty \frac{sin(v_1 t)}{v_1} e^{-\mu_1 t} g(t)dt > 0 \tag{3.20}$$

First, since sin is an odd function, we have

$$\frac{sin(v_1 t)}{v_1} = \frac{sin(-v_1 t)}{-v_1} = \frac{sin(|v_1| t)}{|v_1|},$$

so that, since $sin(|\mu_1|t + \pi) = -sin(|\mu_1|t)$, one get

$$\begin{aligned}
\int_0^\infty \frac{sin(v_1 t)}{v_1} e^{-\mu_1 t} g(t)dt &= \int_0^\infty \frac{sin(|v_1| t)}{|v_1|} e^{-\mu_1 t} g(t)dt \\
&= \frac{1}{|v_1|} \sum_{k=0}^\infty \int_{\frac{2k\pi}{|v_1|}}^{\frac{(2k+1)\pi}{|v_1|}} sin(|v_1|t) e^{-\mu_1 t} g(t)dt \\
&+ \frac{1}{|v_1|} \sum_{k=0}^\infty \int_{\frac{(2k+1)\pi}{|v_1|}}^{\frac{(2k+2)\pi}{|v_1|}} sin(|v_1|t) e^{-\mu_1 t} g(t)dt \\
&= \frac{1}{|v_1|} \sum_{k=0}^\infty \int_{\frac{2k\pi}{|v_1|}}^{\frac{(2k+1)\pi}{|v_1|}} sin(|v_1|t) \Big(e^{-\mu_1 t} g(t) - e^{\mu_1\left(t+\frac{\pi}{|v_1|}\right)} g\Big(t + \frac{\pi}{|v_1|}\Big)\Big)dt \\
&> 0.
\end{aligned}$$

This establishes (3.20), and it results that $\mathcal{I}mF(\lambda_1) \neq 0$, which contradicts (3.19). Therefore, we proved that $\mu_1 < 0$. Combining the two cases, we know that $\frac{\lambda}{F(\lambda)}$ is analytic in $\{\lambda \in \mathbb{C}; \mathcal{R}e\lambda > 0\}$ if $\xi = 0$, and in $\{\lambda \in \mathbb{C}; \mathcal{R}e\lambda > 0\}$if $\xi \neq 0$.

Taking $\lambda = \gamma + iv$, where $\gamma > 0$, then we have

$$d\lambda = idv.$$

So we can use Jordan's lemma states that this is true so long as $F(\lambda) \to 0$ as $|\lambda| \to \infty$ and then we calculate:

$$\begin{aligned}\mathfrak{L}^{-1}\Big[\frac{\lambda}{F(\lambda)}\Big](t) = \lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT}\frac{\lambda e^{\lambda t}}{F(\lambda)}d\lambda = \int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\lambda e^{\lambda t}}{F(\lambda)}d\lambda &= \int_{-\infty}^{+\infty}\frac{i(\gamma+iv)e^{\gamma+iv}}{F(\gamma+iv)}dv \\ &= \int_{|v|\le\varepsilon} + \int_{|v|\ge\varepsilon} := J_1 + J_2.\end{aligned}$$

For any fixed $\varepsilon > 0$, we now that

$$|J_1| \le (\gamma + r)e^{\gamma t}\int_{|v|\le\varepsilon}\Big|\frac{1}{F(\gamma+iv)}\Big|dv. \tag{3.21}$$

This ensures the convergence of J_1.

$$\begin{aligned}|J_2| &= |\int_{|v|\ge\varepsilon} e^{(\gamma+iv)t}\Big(\frac{1}{\gamma+iv} - \frac{1+|\xi|^2-|\xi|^{2\sigma}\mathfrak{L}(g)}{(\gamma+iv)F(\gamma+iv)}\Big)dv| \\ &\le e^{\gamma}\int_{|v|\ge\varepsilon}\Big(\frac{1}{\gamma+iv} + \frac{1+|\xi|^2+|\xi|^{2\sigma}|\mathfrak{L}(g)|}{|(\gamma+iv)F(\gamma+iv)|}\Big)dv,\end{aligned}$$

and we use (3.17), it is also not difficult to show that J_2 converges. Therefore, we conclude that $\hat{G}(\xi,t)$ exists. Similar arguments are used to prove the existence of $\hat{H}(\xi,t)$, we have

$$\begin{aligned}\mathfrak{L}^{-1}\Big[\frac{1}{F(\lambda)}\Big](t) = \lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT}\frac{e^{\lambda t}}{F(\lambda)}d\lambda = \int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\lambda t}}{F(\lambda)}d\lambda &= \int_{-\infty}^{+\infty}\frac{ie^{\gamma+iv}}{F(\gamma+iv)}dv \\ &= \int_{|v|\le\varepsilon} + \int_{|v|\ge\varepsilon}\end{aligned}$$

Thus, we complete the proof. ■

3.3 Energy method in the Fourier space

In this part of the section, we will focus on the **analytical study of the exponential stability of the solution** for (3.10) under conditions on the memory kernel g. We will use the **energy method in the Fourier space** . Before proceeding, we will prove the following steps of the work:

1. First, we derive the **energy expression E**.
2. Second, we define some auxiliary functions, such as F_1 and F_2.
3. Third, we define the **Lyapunov function** $\mathcal{L}$ and establish the equivalence between it and the energy function.
4. Finally, we show that the energy of the problem decays to zero **exponentially**.

Lemma 3.3.1 *Let $\hat{u}(\xi;t)$ be the solution of (3.10) and assume that $(\mathcal{H}1)$and $(\mathcal{H}2)$ hold. Then, the modified energy functional $\hat{E}(\xi;t)$ defined by:*

$$\hat{E}(\xi;t) = \frac{1}{2}\left[|\hat{u}_t|^2 + \left(1 + |\xi|^{2\sigma} - |\xi|^2 \int_0^t g(s)ds\right)|\hat{u}|^2 + |\xi|^2 (g \circ \hat{u})(t)\right]. \tag{3.22}$$

The energy is positive and non-increasing and satisfies

$$\hat{E}'(\xi;t) = \frac{|\xi|^2}{2}((g' \circ \hat{u})(t) - g(t)|\hat{u}|^2) \leq 0, \tag{3.23}$$

Proof. By multiplying the equation (3.10) by $\bar{\hat{u}}_t$ and taking the real part, we obtain

$$Re(\hat{u}_{tt}\bar{\hat{u}}_t) = \frac{1}{2}\frac{d}{dt}|\hat{u}_t|^2, \tag{3.24}$$

$$Re(\hat{u}\bar{\hat{u}}_t) = \frac{1}{2}\frac{d}{dt}|\hat{u}|^2, \tag{3.25}$$

we have:

$$\frac{1}{2}\frac{d}{dt}|\hat{u}_t|^2 + \frac{(1 + |\xi|^{2\sigma})}{2}\frac{d}{dt}|\hat{u}|^2 - |\xi|^2 Re \int_0^t g(t-s)\hat{u}(s)\bar{\hat{u}}_t ds = 0, \tag{3.26}$$

since $\int_0^t g(t-s)ds = \int_0^t g(s)ds$ we obtain

$$\begin{aligned}
-Re\Big((g*\hat{u})\bar{\hat{u}}_t\Big) &= -Re\Big(\int_0^t g(t-s)\hat{u}(s)\bar{\hat{u}}_t ds\Big) \\
&= -Re\Big(\int_0^t g(t-s)(\hat{u}(s)-\hat{u}(t)+\hat{u}(t))\bar{\hat{u}}_t ds\Big) \\
&= -Re\Big(\int_0^t g(t-s)(\hat{u}(s)-\hat{u}(t))\bar{\hat{u}}_t ds\Big) - \int_0^t g(t-s)Re(\hat{u}\bar{\hat{u}}_t) \\
&= \frac{1}{2}\int_0^t g(t-s)\frac{d}{dt}|\hat{u}(t)-\hat{u}(s)|^2 ds - \frac{1}{2}\int_0^t g(s)ds\frac{d}{dt}|\hat{u}|^2,
\end{aligned}$$

and also we have

$$\begin{aligned}
\int_0^t g(t-s)\frac{d}{dt}|\hat{u}(t)-\hat{u}(s)|^2 &= \int_0^t \frac{d}{dt}\Big[g(t-s)|\hat{u}(t)-\hat{u}(s)|^2\Big]ds - \int_0^t g'(t-s)|\hat{u}(t)-\hat{u}(s)|^2 \\
&= \frac{d}{dt}\int_0^t g(t-s)|\hat{u}(t)-\hat{u}(s)|^2 ds - \int_0^t g'(t-s)|\hat{u}(t)-\hat{u}(s)|^2.
\end{aligned}$$

And we have

$$\int_0^t g(s)ds\frac{d}{dt}|\hat{u}|^2 = \frac{d}{dt}\Big(\int_0^t g(s)ds|\hat{u}|^2\Big) - \int_0^t \frac{d}{dt}g(s)ds|\hat{u}|^2 = \frac{d}{dt}\Big(\int_0^t g(s)ds|\hat{u}|^2\Big) - g(t)dt|\hat{u}|^2.$$

We use previous calculations, to get

$$
\begin{aligned}
-Re\Big((g*\hat{u})\bar{\hat{u}}_t\Big) &= -Re\Big(\int_0^t g(t-s)\hat{u}(s)\bar{\hat{u}}_t ds\Big)\\
&= Re\int_0^t g(t-s)(\hat{u}(t)-\hat{u}(s))\bar{\hat{u}}_t ds - \int_0^t g(s)ds Re(\hat{u}\bar{\hat{u}}_t)\\
&= \frac{1}{2}\int_0^t g(t-s)\frac{d}{dt}|\hat{u}(t)-\hat{u}(s)|^2 ds - \frac{1}{2}\int_0^t g(s)ds\frac{d}{dt}|\hat{u}|^2\\
&= \frac{1}{2}\int_0^t \frac{d}{dt}g(t-s)|\hat{u}(t)-\hat{u}(s)|^2 ds - \frac{1}{2}\int_0^t g'(t-s)|\hat{u}(t)-\hat{u}(s)|^2 ds\\
&\quad - \frac{1}{2}\int_0^t g(s)ds\frac{d}{dt}|\hat{u}|^2\\
&= \frac{1}{2}\frac{d}{dt}\int_0^t g(t-s)|\hat{u}(t)-\hat{u}(s)|^2 ds - \frac{1}{2}\int_0^t g'(t-s)|\hat{u}(t)-\hat{u}(s)|^2 ds\\
&\quad - \frac{1}{2}\frac{d}{dt}\Big(\int_0^t g(s)ds|\hat{u}|^2\Big) + \frac{1}{2}g(t)|\hat{u}|^2\\
&= \frac{1}{2}\frac{d}{dt}(g\circ\hat{u})(t) - \frac{1}{2}(g'\circ\hat{u})(t) - \frac{1}{2}\frac{d}{dt}\Big(\int_0^t g(s)ds|\hat{u}|^2\Big) + \frac{1}{2}g(t)|\hat{u}|^2.
\end{aligned}
$$

We inserted this result into (3.26), we obtain:

$$
\frac{1}{2}\frac{d}{dt}|\hat{u}_t|^2 + \frac{(1+|\xi|^{2\sigma})}{2}\frac{d}{dt}|\hat{u}|^2 + |\xi|^2\Bigg(\frac{1}{2}\frac{d}{dt}(g\circ\hat{u})(t) - \frac{1}{2}(g'\circ\hat{u})(t) - \frac{1}{2}\frac{d}{dt}\Big(\int_0^t g(s)ds|\hat{u}|^2\Big) + \frac{1}{2}g(t)|\hat{u}|^2\Bigg) = 0
$$

then,

$$
\frac{1}{2}\frac{d}{dt}\Big(|\hat{u}_t|^2 + \Big(1+|\xi|^{2\sigma} - |\xi|^2\int_0^t g(s)ds\Big)|\hat{u}|^2 + |\xi|^2(g\circ\hat{u})(t)\Big) = \frac{|\xi|^2}{2}((g'\circ\hat{u})(t) - g(t)|\hat{u}|^2).
$$

Hence

$$
\hat{E}'(\xi;t) = \frac{|\xi|^2}{2}((g'\circ\hat{u})(t) - g(t)|\hat{u}|^2).
$$

Since $(g'\circ\hat{u}) \le 0$, then

$$
\hat{E}'(\xi;t) \le 0.
$$

Let's prove that $\hat{E}(\xi;t)$ is non-negative:

We use the assumption $(\mathcal{H}1)$, it is easy to see that, for all $t \ge 0$,

$$
1+|\xi|^{2\sigma} - |\xi|^2\int_0^t g(s)ds \ge 1+|\xi|^{2\sigma} - |\xi|^2\int_0^\infty g(s)ds, \quad \forall t \ge 0
$$

$$= 1 + |\xi|^{2\sigma} - |\xi|^2(1-l) > 0, \tag{3.27}$$

consequently, $\hat{E}(\xi;t) \geq 0$.

■

Then, we find the following auxiliary functions by the following lemma.

Lemma 3.3.2 *Under the assumption* $(\mathcal{H}1)$*, the functional :*

$$F_1(\xi,t) := Re(\hat{u}_t\bar{\hat{u}}), \tag{3.28}$$

satisfies, along the solution of (3.10) *and for any* $\delta_1 > 0$*, the estimate*

$$F_1'(\xi,t) \leq |\hat{u}_t|^2 - \left(1-\frac{\delta_1}{l}\right)\left(1+|\xi|^{2\sigma}-|\xi|^2\int_0^t g(s)ds\right)|\hat{u}|^2 + \frac{1-l}{4\delta_1}(1+|\xi|^{2\sigma})(g\circ\hat{u})(t). \tag{3.29}$$

Proof. Taking the derivative of F_1, and exploiting (3.10) we get

$$\begin{aligned}
F_1'(\xi;t) &= Re(\hat{u}_{tt}\bar{\hat{u}}) + |\hat{u}_t|^2 \\
&= -(1+|\xi|^{2\sigma})|\hat{u}|^2 + |\xi|^2 Re\int_0^t g(t-s)(\hat{u}(s)-\hat{u}(t))\bar{\hat{u}}ds \\
&+ |\xi|^2|\hat{u}|^2\int_0^t g(s)ds + |\hat{u}_t|^2 \\
&= |\hat{u}_t|^2 - \Big(1+|\xi|^{2\sigma} - \int_0^t |\xi|^2 g(s)ds\Big)|\hat{u}|^2 + |\xi|^2 Re\Big(\bar{\hat{u}}\int_0^t g(t-s)(\hat{u}(s)-\hat{u}(t))\Big)ds
\end{aligned}$$

Since $|\hat{u}| = |\bar{\hat{u}}|$, and we use Young's inequality in the last term we get

$$\Big|\bar{\hat{u}}\int_0^t g(t-s)(\hat{u}(s)-\hat{u}(t))\Big|ds \leq \delta_1|\hat{u}|^2 + \frac{1}{4\delta_1}\int_0^t |g(t-s)\hat{u}(s)-\hat{u}(t)|^2 ds,$$

on other hand by (3.27), we have

$$1 + |\xi|^{2\sigma} - |\xi|^2 + l|\xi|^2 \leq 1 + |\xi|^{2\sigma} - |\xi|^2\int_0^t g(s)ds.$$

And

$$l|\xi|^2 \leq 1 + |\xi|^{2\sigma} - |\xi|^2\int_0^t g(s)ds. \tag{3.30}$$

$$|\xi|^2 \leq \frac{1+|\xi|^{2\sigma}-|\xi|^2\int_0^t g(s)ds}{l}.$$

So,

$$\begin{aligned}
F_1'(\xi;t) &= Re(\hat{u}_{tt}\bar{\hat{u}}) + |\hat{u}_t|^2 \\
&= -(1+|\xi|^{2\sigma})|\hat{u}|^2 + |\xi|^2 Re\int_0^t g(t-s)(\hat{u}(s)-\hat{u}(t))\bar{\hat{u}}ds \\
&+ |\xi|^2|\hat{u}^2|\int_0^t g(s)ds + |\hat{u}_t|^2 \\
&\leq |\hat{u}_t|^2 - \Big(1+|\xi|^{2\sigma} - \int_0^t |\xi|^2 g(s)ds\Big)|\hat{u}|^2 + |\xi|^2 Re\Big(\bar{\hat{u}}\int_0^t g(t-s)(\hat{u}(s)-\hat{u}(t))\Big)ds \\
&\leq |\hat{u}_t|^2 - \Big(1+|\xi|^{2\sigma} - \int_0^t |\xi|^2 g(s)ds\Big)|\hat{u}|^2 + \delta_1|\xi|^2|\hat{u}|^2 + \frac{|\xi|^2}{4\delta_1}\Big|\int_0^t g(t-s)(\hat{u}(s)-\hat{u}(t))\Big|^2 ds \\
&\leq |\hat{u}_t|^2 - \Big(1+|\xi|^{2\sigma} - \int_0^t |\xi|^2 g(s)ds\Big)|\hat{u}|^2 + \delta_1\frac{1+|\xi|^{2\sigma}-\int_0^t|\xi|^2 g(s)ds}{l}|\hat{u}|^2 \\
&+ \frac{|\xi|^2}{4\delta_1}|\int_0^t g(t-s)(\hat{u}(s)-\hat{u}(t))|^2 ds \\
&= |\hat{u}_t|^2 - \Big(1+|\xi|^{2\sigma} - \int_0^t |\xi|^2 g(s)ds - \delta_1\frac{1+|\xi|^{2\sigma}-\int_0^t|\xi|^2 g(s)ds}{l}\Big)|\hat{u}|^2 \\
&+ \frac{|\xi|^2}{4\delta_1}\Big|\int_0^t g(t-s)(\hat{u}(s)-\hat{u}(t))\Big|^2 ds \\
&= |\hat{u}_t|^2 - \Big(1-\frac{\delta_1}{l}\Big)\Big(1+|\xi|^{2\sigma} - \int_0^t |\xi|^2 g(s)\Big)|\hat{u}|^2 ds + \frac{|\xi|^2}{4\delta_1}\Big|\int_0^t g(t-s)(\hat{u}(s)-\hat{u}(t))\Big|^2 ds.
\end{aligned}$$

Finally,

$$F_1'(\xi;t) \leq |\hat{u}_t|^2 - \Big(1-\frac{\delta_1}{l}\Big)\left(1+|\xi|^{2\sigma}-|\xi|^2\int_0^t g(s)ds\right)|\hat{u}|^2 + \frac{|\xi|^2}{4\delta_1}\Big|\int_0^t g(t-s)(\hat{u}(s)-\hat{u}(t))\Big|^2 ds$$

By using (3.5) and (3.7), we get (3.29).

■

Lemma 3.3.3 *Let $\hat{u}(\xi;t)$ be a solution of (3.10) assume that $(\mathcal{H}1)$ holds. Then, the second auxiliary function F_2 defined by:*

$$F_2(\xi;t) := -Re\Big(\hat{u}_t\int_0^t g(t-s)(\bar{\hat{u}}(t)-\bar{\hat{u}}(s))ds\Big), \tag{3.31}$$

satisfies, for any $\delta_2, \delta_3 > 0$,

$$
\begin{aligned}
F_2'(\xi;t) \leq \delta_2\Big(1+|\xi|^{2\sigma}-|\xi|^2\int_0^t g(s)ds\Big)|\hat{u}|^2 - \Big(\int_0^t g(s)ds - \delta_3\Big)|\hat{u}_t| \\
+ c\Big(1+\frac{1}{\delta_2}\Big)(1+|\xi|^{2\sigma})(g\circ\hat{u})(t) - \frac{g(0)}{4\delta_3}(g'\circ\hat{u})(t). \qquad (3.32)
\end{aligned}
$$

Proof.

$$
F_2'(\xi;t) = -Re\Big(\hat{u}_{tt}\int_0^t g(t-s)(\bar{\hat{u}}(t)-\bar{\hat{u}}(s))ds\Big) - Re\Big(\hat{u}_t\int_0^t g'(t-s)(\bar{\hat{u}}(t)-\bar{\hat{u}}(s))ds\Big)
$$

$$
-|\hat{u}_t|^2\int_0^t g(s)ds.
$$

By exploiting (3.10) we get

$$
\begin{aligned}
F_2'(\xi;t) &= (1+|\xi|^{2\sigma})Re\Big(\hat{u}\int_0^t g(t-s)(\bar{\hat{u}}(t)-\bar{\hat{u}}(s))ds\Big) - Re\Big(\hat{u}_t\int_0^t g'(t-s)(\bar{\hat{u}}(t)-\bar{\hat{u}}(s))ds\Big) \\
&- |\hat{u}_t|^2\int_0^t g(s)ds + |\xi|^2 Re\Big(\int_0^t g(t-s)(\hat{u}(t)-\hat{u}(s))ds\int_0^t g(t-s)(\bar{\hat{u}}(t)-\bar{\hat{u}}(s))ds\Big) \\
&- |\xi|^2\int_0^t g(s)ds Re\Big(\hat{u}\int_0^t g(t-s)(\bar{\hat{u}}(t)-\bar{\hat{u}}(s))ds\Big) \\
&= \Big(1+|\xi|^{2\sigma}-|\xi|^2\int_0^t g(s)ds\Big)Re\Big(\hat{u}\int_0^t g(t-s)(\bar{\hat{u}}(t)-\bar{\hat{u}}(s))ds\Big) - |\hat{u}_t|^2\int_0^t g(s)ds \\
&+ |\xi|^2\Big|\int_0^t g(t-s)(\hat{u}(t)-\hat{u}(s))ds\Big|^2 - Re\Big(\hat{u}_t\int_0^t g'(t-s)(\bar{\hat{u}}(t)-\bar{\hat{u}}(s))ds\Big)
\end{aligned}
$$

Using Young's inequality and lemma (3.1.1), and $(\mathcal{H}1)$, we obtain :

$$
\begin{aligned}
F_2'(\xi;t) \leq \delta_2\Big(1+|\xi|^{2\sigma}-|\xi|^2\int_0^t g(s)ds\Big)|\hat{u}|^2 + \frac{1-l}{4\delta_2}\Big(1+|\xi|^{2\sigma}-|\xi|^2\int_0^t g(s)ds\Big)(g\circ\hat{u})(t) \\
+ (1-l)|\xi|^2(g\circ\hat{u})(t) - \Big(\int_0^t g(s)ds-\delta_3\Big)|\hat{u}_t|^2 - \frac{g(0)}{4\delta_3}(g'\circ\hat{u})(t)
\end{aligned}
$$

we have

$$
|\xi|^2 \leq \begin{cases} 1, & if \quad \xi \leq 1, \\ |\xi|^{2\sigma}, & if \quad \xi \geq 1. \end{cases}
$$

Then

$$
|\xi|^2 \leq 1+|\xi|^{2\sigma}, \quad \forall \xi \in \mathbb{R}^n \qquad (3.33)
$$

we use (3.33) to arrive at (3.32) ■

Lemma 3.3.4 *The functional $\mathcal{L}$, defined by :*

$$\mathcal{L}(\xi,t) := N(1+|\xi|^{2\sigma})^2\hat{E}(\xi,t) + |\xi|^{2(\sigma+1)}F_1(\xi,t) + N_1|\xi|^{2(\sigma+1)}F_2(\xi,t), \tag{3.34}$$

Then for any positive constants N_1 there existe a sufficiently large N and two constants C_1, C_2 satisfies

$$C_1(1+|\xi|^{2\sigma})^2\hat{E}(\xi,t) \le \mathcal{L}(\xi,t) \le C_1(1+|\xi|^{2\sigma})^2\hat{E}(\xi,t). \tag{3.35}$$

And for any t_0, there exists a positive λ such that :

$$\mathcal{L}'(\xi,t) \le -\lambda|\xi|^{2(\sigma+1)}\hat{E}(\xi,t) + (c+\lambda)(1+|\xi|^{2\sigma})^2|\xi|^2(g\circ\hat{u})(t), \quad \forall t \ge t_0, \tag{3.36}$$

Proof. First, notice that:

$$\begin{aligned}\Big|\mathcal{L}(\xi,t) - N(1+|\xi|^{2\sigma})^2\hat{E}(\xi,t)\Big| &\le |\xi|^{2(\sigma+1)}|Re(u_t u)| \\ &\quad + N_1|\xi|^{2(\sigma+1)}\Big|Re(u_t\int_0^t g(t-s)(\bar{\hat{u}}(s)-\bar{\hat{u}}(t))ds)\Big|,\end{aligned}$$

we use Young's inequality, we get

$$|\hat{u}_t\bar{\hat{u}}| \le \frac{1}{2}|\hat{u}_t|^2 + \frac{1}{2}|\hat{u}|^2,$$

and

$$|\hat{u}_t\int_0^t g(t-s)(\bar{\hat{u}}(t)-\bar{\hat{u}}(s))ds \le \frac{1}{2}|\hat{u}_t|^2 + \frac{1}{2}\Big|\int_0^t g(t-s)(u(t)-u(s))ds\Big|^2.$$

Then, we use (3.5) and previous calculate, we get :

$$\begin{aligned}\Big|\mathcal{L}(\xi,t) - N(1+|\xi|^{2\sigma})^2\hat{E}(\xi,t)\Big| &\le \frac{1+N_1}{2}|\xi|^{2(\sigma+1)}|\hat{u}_t|^2 + \frac{1}{2}|\xi|^{2(\sigma+1)}|\hat{u}|^2 \\ &\quad + \frac{N_1}{2}|\xi|^{2(\sigma+1)}\Big|\int_0^t g(t-s)(\hat{u}(t)-\hat{u}(s))ds\Big|^2 \\ &\le \frac{1+N_1}{2}|\xi|^{2(\sigma+1)}|\hat{u}_t|^2 + \frac{1}{2}|\xi|^{2(\sigma+1)}|\hat{u}|^2 \\ &\quad + \frac{1-l}{2}N_1|\xi|^{2(\sigma+1)}(g\circ\hat{u})(t).\end{aligned}$$

By recalling (3.22), (3.30) and (3.33), we have :

$$|\xi|^2 \leq \frac{1+|\xi|^{2\sigma} - |\xi|^2 \int_0^t g(s)ds}{l}.$$

And for all $\sigma \geq 1$ we have

$$|\xi|^{2\sigma} \leq 1 + |\xi|^{2\sigma},$$

then, we arrive at

$$\begin{aligned}
\Big|\mathcal{L}(\xi,t) - N(1+|\xi|^{2\sigma})^2\hat{E}(\xi,t)\Big| &\leq \frac{1+N_1}{2}|\xi|^{2(\sigma+1)}|\hat{u}_t|^2 \\
&+ \frac{|\xi|^{2\sigma}}{2l}\Big(1+|\xi|^{2\sigma} - |\xi|^2\int_0^t g(s)ds\Big)|\hat{u}|^2 \\
&+ cN_1|\xi|^{2(\sigma+1)}(g\circ\hat{u})(t). \\
&\leq \frac{1+N_1}{2}(1+|\xi|^{2\sigma})^2|\hat{u}_t|^2 \\
&+ \frac{(1+|\xi|^{2\sigma})^2}{2l}\Big(1+|\xi|^{2\sigma} - |\xi|^2\int_0^t g(s)ds\Big)|\hat{u}|^2 \\
&+ cN_1(1+|\xi|^{2\sigma})^2|\xi|^2(g\circ\hat{u})(t) \\
&\leq c_{N_1,l}(1+|\xi|^{2\sigma})^2\hat{E}
\end{aligned}$$

Where $c_{N_1,l} = \frac{1+N_1}{2} + \frac{1}{2l} + cN_1$,

Consequently, by choosing N sufficiently large ($N > c_{N_1,l}$), (3.35) is established.

Now, we prove (3.36) .

From, (3.27), we have

$$\mathcal{L}'(\xi,t) := N(1+|\xi|^{2\sigma})^2\hat{E}'(\xi,t) + |\xi|^{2(\sigma+1)}F_1'(\xi,t) + N_1|\xi|^{2(\sigma+1)}F_2'(\xi,t), \tag{3.37}$$

Substituting (3.23), (3.29) and (3.32), and we use $|\xi|^{2\sigma} \leq (1+|\xi|^{2\sigma})^2$ we get for any $\delta_2, \delta_3 > 0$ in to (3.37)

$$\begin{aligned}
\mathcal{L}'(\xi,t) \leq &\Big(N_1\Big(\int_0^t g(s)ds - \delta_3\Big) - 1\Big)|\xi|^{2(\sigma+1)}|\hat{u}_t|^2 \\
&- \Big(1 - \frac{\delta_1}{l} - \delta_2 N_1\Big)\Big(1+|\xi|^{2\sigma} - |\xi|^2\int_0^t g(s)ds\Big)|\xi|^{2(\sigma+1)}|\hat{u}|^2 \\
&+ c\Big(N_1(1+\frac{1}{\delta_2}) + \frac{1}{\delta_1}\Big)(1+|\xi|^{2\sigma})^2|\xi|^2(g\circ\hat{u})(t) \\
&+ \Big(\frac{N}{2} - \frac{g(0)N_1}{4\delta_3}\Big)(1+|\xi|^{2\sigma})^2|\xi|^2(g'\circ\hat{u})(t).
\end{aligned} \tag{3.38}$$

Let $g_0 = \int_0^{t_0} g(s)ds$ and take $\delta_1 = \frac{1}{4}, \delta_2 = \frac{1}{4N_1}, \delta_3 = \frac{1}{N_1}$, to find, for any $t \geq t_0$,

$$\mathcal{L}'(\xi,t) \leq -(N_1 g_0 - 2)|\xi|^{2(\sigma+1)}|\hat{u}_t|^2 - \frac{1}{2}\Big(1 + |\xi|^{2\sigma} - |\xi|^2 \int_0^t g(s)ds\Big)|\xi|^{2(\sigma+1)}|\hat{u}|^2$$
$$+ c_{N_1}(1 + |\xi|^{2\sigma})^2|\xi|^2(g \circ \hat{u})(t) + \Big(\frac{N}{2} - \frac{g(0)}{4}N_1^2\Big)(1 + |\xi|^{2\sigma})^2|\xi|^2(g' \circ \hat{u})(t)). \quad (3.39)$$

Now, we choose N_1 large enough such that

$$N_1 g_0 - 2 > 0,$$

then, select N so large that (3.35) remains valid and, furthermore,

$$\frac{N}{2} - \frac{g(0)}{4}N_1^2 > 0,$$

Consequently, (3.39)becomes, for a positive λ,

$$\mathcal{L}'(\xi,t) \leq -\lambda|\xi|^{2(\sigma+1)}\Big[|\hat{u}_t|^2 + \Big(1 + |\xi|^{2\sigma} - |\xi|^2 \int_0^t g(s)ds\Big)|\hat{u}|^2 + |\xi|^2(g \circ \hat{u})(t) - |\xi|^2(g \circ \hat{u})(t)\Big]$$
$$+ c(1 + |\xi|^{2\sigma})^2|\xi|^2(g \circ \hat{u})(t))$$
$$= -\lambda|\xi|^{2(\sigma+1)}\Big[|\hat{u}_t|^2 + \Big(1 + |\xi|^{2\sigma} - |\xi|^2 \int_0^t g(s)ds\Big)|\hat{u}|^2 + |\xi|^2(g \circ \hat{u})(t)\Big] + \lambda|\xi|^{2\sigma+2}|\xi|^2(g \circ \hat{u})$$
$$+ c(1 + |\xi|^{2\sigma})^2|\xi|^2(g \circ \hat{u})(t)), \quad \forall t \geq t_0,$$

So, from (3.22) we arrive (3.36). ■

Theorem 3.3.5 *let $\hat{u}$ is a solution of (3.10), then, for any $t_0 > 0$, there exist two positive constants k_1, k_2 such that*

$$\hat{E}(\xi,t) \leq k_1\hat{E}(\xi,0)e^{-k_2\rho(\xi)\int_0^t \eta(s)ds}, \quad \forall t \geq t_0, \quad (3.40)$$

where $\rho(\xi) = \frac{|\xi|^{2(\sigma+1)}}{(1+|\xi|^{2\sigma})}$.

Proof.

We have already proven that $\int_0^t g(s)ds = \int_0^t g(t-s)ds$ and we know that $\int_0^t g'(s) = \int_0^t g'(t-s)$, $\eta(\xi)$ is a positive function. Then by $(\mathcal{H}2)$

$$\int_0^t g'(t-s)ds \leq -\int_0^t \eta(t-s)g(t-s)ds.$$

Multiplying by $-|\hat{u}(t)-\hat{u}(s)|^2$ and using inequality (3.7) we get:

$$-(g'\circ\hat{u}) \geq \int_0^t \eta(t-s)g(t-s)|\hat{u}(t)-\hat{u}(s)|^2ds.$$

Multiplying the inequality (3.36) by $\eta(t)$ and using the previous calculations. Since $\eta(s)$ is a non-increasing function, then for $t>s$, we have $\eta(t)\leq\eta(t-s)$, so we get for some $\lambda_1,\lambda_2>0$

$$\begin{aligned}\eta(t)\mathcal{L}'(\xi,t) &\leq -\lambda_1\eta(t)|\xi|^{2(\sigma+1)}\hat{E}(\xi,t)+(c+\lambda)(1+|\xi|^{2\sigma})^2|\xi|^2\\ &\quad\times\int_0^t\eta(t-s)g(t-s)|\hat{u}(t)-\hat{u}(s)|^2ds\\ &\leq -\lambda_1\eta(t)|\xi|^{2(\sigma+1)}\hat{E}(\xi,t)-(c+\lambda)(1+|\xi|^{2\sigma})^2|\xi|^2(g'\circ\hat{u})\\ &\leq -\lambda_1\eta(t)|\xi|^{2(\sigma+1)}\hat{E}(\xi,t)-\lambda_2(1+|\xi|^{2\sigma})^2\hat{E}'(\xi,t),\quad \forall t\geq t_0,\end{aligned}$$

Recalling that $\eta'(t)<0$ and setting

$$L(\xi,t):=\eta(t)\mathcal{L}(\xi,t)+\lambda_2(1+|\xi|^{2\sigma})^2\hat{E}(\xi,t),$$

we get

$$L'(\xi,t)\leq-\lambda_1\eta(t)|\xi|^{2(\sigma+1)}\hat{E}(\xi,t),\quad \forall t\geq t_0.$$

Since $\eta(t)$ is bounded and from (3.35), we deduce that there exist two constants C_3, C_4 such that

$$C_3(1+|\xi|^{2\sigma})^2\hat{E}(\xi,t)\leq L(\xi,t)\leq C_4(1+|\xi|^{2\sigma})^2\hat{E}(\xi,t). \tag{3.41}$$

Since $\mathcal{L}\sim\hat{E}$, then $L\sim\hat{E}$, so by (3.35)

$$\begin{aligned}L'(\xi,t) &\leq -\lambda_1\eta(t)|\xi|^{2(\sigma+1)}\frac{L(\xi,t)}{C_4(1+|\xi|^{2\sigma})^2},\\ &= -k_2\rho(\xi)\eta(t)L(\xi,t),\quad \forall t\geq t_0\end{aligned} \tag{3.42}$$

and for some k_2, devising by $L(\xi,t)$ and integration the last inequality over (t_0,t) we have :

$$\frac{L'(\xi,t)}{L(\xi,t)}\leq-k_2\rho(\xi)\eta(t),$$

$$\int_{t_0}^{t} \frac{L'(\xi,t)}{L(\xi,t)}dt \leq -k_2\rho(\xi)\int_{t_0}^{t}\eta(s)ds,$$

$$ln(L(\xi,t)) - ln(L(\xi,t_0)) \leq -k_2\rho(\xi)\int_{t_0}^{t}\eta(s)ds$$

so,

$$ln\left(\frac{L(\xi,t)}{L(\xi,t_0)}\right) \leq -k_2\rho(\xi)\int_{t_0}^{t}\eta(s)ds$$

$$e^{ln\left(\frac{L(\xi,t)}{L(\xi,t_0)}\right)} \leq e^{-k_2\rho(\xi)\int_{t_0}^{t}\eta(s)ds}$$

$$\frac{L(\xi,t)}{L(\xi,t_0)} \leq e^{-k_2\rho(\xi)\int_{t_0}^{t}\eta(s)ds},$$

we get

$$L(\xi,t) \leq L(\xi,t_0)e^{-k_2\rho(\xi)\int_{t_0}^{t}\eta(s)ds},$$

since $L(\xi,t)$ is a non-increasing function then, for all, $0 \leq t_0 \leq t$ is verified:

$$L(\xi,t_0) \leq L(\xi,0),$$

and for any positive constant c is chosen we have

$$L(\xi,t_0) \leq cL(\xi,0),$$

finally we obtain

$$L(\xi,t_0) \leq cL(\xi,0)e^{-k_2\rho(\xi)\int_0^{t_0}\eta(s)ds}e^{\int_{t_0}^{t}\eta(s)ds} = cL(\xi,0)e^{-k_2\rho(\xi)\int_0^{t}\eta(s)ds},$$

by(3.41) we have

$$\begin{aligned}
\hat{E}(\xi,t) &\leq \frac{L(\xi,t)}{C_3(1+|\xi|^{2\sigma})^2} \\
&\leq \frac{c}{C_3(1+|\xi|^{2\sigma})^2}L(\xi,0)e^{-k_2\rho(\xi)\int_0^{t}\eta(s)ds} \\
&\leq \frac{c}{C_3(1+|\xi|^{2\sigma})^2}C_4(1+|\xi|^{2\sigma})^2\hat{E}(\xi,0)e^{-k_2\rho(\xi)\int_0^{t}\eta(s)ds}. \\
&= c\hat{E}(\xi,0)e^{-k_2\rho(\xi)\int_0^{t}\eta(s)ds}.
\end{aligned}$$

then the estimate (3.40) is established. ■

3.4 Decay estimates of solution

In this section, we discus the decay estimates of solutions for Cauchy problem (3.1). We start with the following lemma :

Lemma 3.4.1 *Assume that ($\mathcal{H}2$) hold. Then there exists $c > 0$ such that*

$$\left\| |\xi|^l e^{-c|\xi|^{2m}\int_0^t \eta(s)ds} \right\|_{L^p(\mathbb{R}^n)} \leq C\Big(1 + \int_0^t \eta(s)ds\Big)^{-\frac{l}{2m}-\frac{n}{2mp}}, \quad \forall t \geq 0, \tag{3.43}$$

where C and m are positive constants, $l \geq 0$ and $1 \leq p < \infty$

Proof. We use direct calculation as in [32] to get, for $t_0, c_1 > 0$

$$\begin{aligned} I &= \left\| |\xi|^l e^{-c|\xi|^{2\sigma}\int_0^t \eta(s)ds} \right\|^p_{L^p(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \Big(|\xi|^p e^{-c|\xi|^{2\sigma}\int_0^t \eta(s)ds} \Big)^p d\xi \\ &\leq c_1 \int_0^\infty |\xi|^{lp} e^{-cp|\xi|^{2\sigma}\int_0^t \eta(s)ds} |\xi|^{n-1} d|\xi| \\ &= c_1 \int_0^\infty |\xi|^{lp+n-1} e^{-cp|\xi|^{2\sigma}\int_0^t \eta(s)ds} d|\xi| = c_1 \int_0^\infty |\xi|^{2\sigma\left(\frac{lp+n}{2\sigma}\right)-1} e^{-cp|\xi|^{2\sigma}\int_0^t \eta(s)ds} d|\xi| \\ &\leq c_1 \int_0^\infty |\xi|^{2\sigma\left(\frac{lp+n}{2\sigma}-1\right)} |\xi|^{2\sigma-1} \left(\frac{\int_0^t \eta(s)ds}{\int_0^t \eta(s)ds} \right)^{\frac{lp+n}{2\sigma}-1} e^{-cp|\xi|^{2\sigma}\int_0^t \eta(s)ds}, \quad t \geq t_0 \\ &\leq \frac{c_1}{2\sigma} \int_0^\infty \Big(|\xi|^{2\sigma} \int_0^t \eta(s)ds \Big)^{\frac{lp+n}{2\sigma}-1} e^{-cp|\xi|^{2\sigma}\int_0^t \eta(s)ds} \Big(\int_0^t \eta(s)ds \Big)^{\frac{lp+n}{2\sigma}-1} 2\sigma|\xi|^{2\sigma-1} d|\xi| \end{aligned} \tag{3.44}$$

By putting $\mu = |\xi|^{2\sigma}\int_0^t \eta(s)ds$, we get

$$\begin{aligned} I &\leq \frac{c_1}{2\sigma} \int_0^\infty \mu^{\frac{lp+n}{2\sigma}} e^{-cp\mu} d\mu \Big(\int_0^t \eta(s)ds \Big)^{-\frac{lp+n}{2\sigma}} \\ &\leq \frac{c_1}{2\sigma} (cp)^{1-\frac{lp+n}{2\sigma}} \int_0^\infty (cp\mu)^{\frac{lp+n}{2\sigma}-1} e^{-cp\mu} d\mu \Big(\int_0^t \eta(s)ds \Big)^{-\frac{lp+n}{2\sigma}} \end{aligned}$$

Observe that $\int_0^\infty (cp\mu)^{\frac{lp+n}{2\sigma}-1} e^{-cp\mu} d\mu = \Gamma\left(\frac{lp+n}{2\sigma}\right) < \infty$, where Γ is the gamma function. Then we obtain

$$\left\| |\xi|^l e^{-c|\xi|^{2\sigma}\int_0^t \eta(s)ds} \right\|_{L^p(\mathbb{R}^n)}^p \leq C\left(\int_0^t \eta(s)ds\right)^{-\frac{lp+n}{2\sigma}}, \quad \forall t \geq t_0. \tag{3.45}$$

It is clear, for any $t \geq t_0$, that

$$\int_0^t \eta(s)ds \geq \frac{1}{2}\int_0^{t_0} \eta(s)ds + \frac{1}{2}\int_0^t \eta(s)ds = C + \frac{1}{2}\int_0^t \eta(s)ds \geq C\left(1 + \int_0^t \eta(s)ds\right).$$

So

$$\left(\int_0^t \eta(s)ds\right)^{-\frac{lp+n}{2\sigma}} \leq C\left(1 + \int_0^t \eta(s)ds\right)^{-\frac{lp+n}{2\sigma}}, \quad \forall t \geq t_0.$$

For $t \in [0, t_0]$, by virtue of boundedness of $\eta(t)$ and from (3.44), we obtain

$$\begin{aligned}
\left\| |\xi|^p e^{-c|\xi|^2 t} \int_0^t \eta(s)ds \right\|_{L^p(\mathbb{R}^n)}^p &\leq c_1 \int_0^\infty |\xi|^{2\sigma\left(\frac{lp+n}{2\sigma}\right)-1} e^{-c_2 p|\xi|^{2\sigma}} d|\xi| \\
&= c_1 \int_0^\infty |\xi|^{2\sigma\left(\frac{lp+n}{2\sigma}-1\right)} e^{-c_2|\xi|^{2\sigma}} |\xi|^{2\sigma-1} d|\xi| \\
&= c_1 \int_0^\infty v^{\left(\frac{lp+n}{2\sigma}-1\right)} e^{-c_1 p v} dv \leq c_1 \Gamma\left(\frac{lp+n}{2\sigma}\right) < \infty,
\end{aligned}$$

where $\nu = |\xi|^{2a}$. Then, for any $t \in [0, t_0]$

$$\begin{aligned}
\left\| |\xi|^p e^{-c|\xi|^{2\sigma}\int_0^t \eta(s)ds} \right\|_{L^p(\mathbb{R}^n)}^p &\leq c_3 \left(1 + \int_0^t \eta(s)ds\right)^{\frac{lp+n}{2\sigma}} \left(1 + \int_0^t \eta(s)ds\right)^{-\frac{lp+n}{2\sigma}} \\
&\leq C\left(1 + \int_0^t \eta(s)ds\right)^{-\frac{lp+n}{2\sigma}}
\end{aligned}$$

■

3.5 Main result

Our main result is stated as follows :

Theorem 3.5.1 *Let r be a non-negative integer. Assume that $(\mathcal{H}1)$and $(\mathcal{H}2)$ hold, and that*

$$U_0 = (u_1, u_0, (-\Delta)^{\frac{\sigma}{2}} u_0)^T \in H^r(\mathbb{R}^n) \cap L^p(\mathbb{R}^n), \quad 1 \le p \le 2.$$

Then $U = (u_t, u, (-\Delta)^{\frac{\sigma}{2}} u)^T$satisfies, for all $t \ge 0$, the following decay estimate

- *For the case $\sigma > 1$:*

$$\|\nabla^k U\|_2 \le C\Big(1 + \int_0^t \eta(s) ds\Big)^{-\frac{n}{2(\sigma+1)}(\frac{1}{p}-\frac{1}{2}) - \frac{k}{2(\sigma+1)}} \|U_0\|_p + C\Big(1 + \int_0^t \eta(s) ds\Big)^{-\frac{l}{2(\sigma-1)}} \|\nabla^{k+l} U_0\|_2, \tag{3.46}$$

where C is positive constant and $0 \le k + l \le r$.

- *For the case $\sigma = 1$*

$$\|\nabla^k U\|_2 \le C\Big(1+\int_0^t \eta(s) ds\Big)^{-\frac{n}{2(\sigma+1)}(\frac{1}{p}-\frac{1}{2}) - \frac{k}{2(\sigma+1)}} \|U_0\|_p + Ce^{-c\int_0^t \eta(s)ds} \|\nabla^k U_0\|_2, \tag{3.47}$$

where C is positive constant and $0 \le k \le r$.

Proof. Let

$$\hat{E}_2(\xi, t) = \frac{1}{2}(|\hat{u}_t|^2 + (1 + |\xi|^{2\sigma})|\hat{u}|^2). \tag{3.48}$$

Noting that $|\hat{U}(\xi, t)|^2$ and $\hat{E}_2$ are equivalent, and $\hat{E}_2(\xi, t) \le c\hat{E}(\xi, t), \forall t \ge 0$, then

$$|\hat{U}(\xi, t)|^2 \sim \hat{E},$$

or

$$|\hat{U}(\xi, t)|^2 \le c\hat{E}(\xi, t),$$

by applying the Plancherel theorem we have

$$\|U\|_2 = \|\hat{U}\|_2,$$

and

$$\|\nabla^k U\|_2 = \|\widehat{\nabla^k U}\|_2,$$

and we use (3.40) ,we find

$$\begin{aligned}
\|\nabla^k U\|_2^2 &= \int_{\mathbb{R}^n} |\xi|^{2k} |\hat{U}(\xi,t)|^2 \\
&\le c\int_{\mathbb{R}^n} |\xi|^{2k} \hat{E}_2(\xi,t)|^2 \\
&\le c\int_{\mathbb{R}^n} |\xi|^{2k} e^{-k_2\rho(\xi)\int_0^t \eta(s)ds} \hat{E}(\xi,0) \\
&\le c\int_{\mathbb{R}^n} |\xi|^{2k} e^{-k_2\rho(\xi)\int_0^t \eta(s)ds} |\hat{U}(\xi,0)|^2 \\
&= c\int_{|\xi|\le 1} |\xi|^{2k} e^{-k_2\rho(\xi)\int_0^t \eta(s)ds} |\hat{U}(\xi,0)|^2 d\xi + c\int_{|\xi|\ge 1} |\xi|^{2k} e^{-k_2\rho(\xi)\int_0^t \eta(s)ds} |\hat{U}(\xi,0)|^2 d\xi \\
&= I_1 + I_2. \qquad (3.49)
\end{aligned}$$

Now, we estimate I_1. It is clear $\rho(\xi) \ge \frac{1}{2}|\xi|^{2(\sigma+1)}$, for $|\xi| \le 1$, where $\rho(\xi)$ is given in (3.40). Then, by applying Holder's inequality and (3.30), we get .

$$\begin{aligned}
I_1 &\le c\int_{|\xi|\le 1} |\xi|^{2k} e^{-\frac{k_2}{4}|\xi|^{2(\sigma+1)}\int_0^t \eta(s)ds} |\hat{U}|^2 d\xi \quad \le c\Big\| |\xi| e^{-\frac{k_2}{4}|\xi|^{2(\sigma+1)}\int_0^t \eta(s)ds}\Big\|_{\frac{q}{2}} \Big(\int_{|\xi|\le 1} |\hat{U}|^{p'} d\xi\Big)^{\frac{2}{p'}} \\
&\le c\Big(1+\int_0^t \eta(s)ds\Big)^{-\frac{n}{(\sigma+1)q}-\frac{k}{(\sigma+1)}} \|\hat{U}_0\|_{p'}^2
\end{aligned} \qquad (3.50)$$

where $\frac{1}{p}+\frac{1}{p'}=\frac{1}{2}$, applying Hausdorff-Young's inequality,we obtain

$$I_1 \le \Big(1+\int_0^t \eta(s)ds\Big)^{-\frac{n}{(\sigma+1)q}-\frac{k}{(\sigma+1)}} \|\hat{U}_0\|_{p'}^2, \qquad (3.51)$$

for $1 \le p \le 2$, where $\frac{1}{p}+\frac{1}{p'}=1$. Next, we estimate I_2. So, for $|\xi| \ge 1$, we have $2|\xi|^{2\sigma} \ge 1+|\xi|^{2\sigma}$, therefore $\rho(\xi) \ge \frac{1}{4|\xi|^{2(\sigma-1)}}$ and, hence

* for $\sigma > 1$, then

$$\begin{aligned}
I_2 &\le c\int_{|\xi|\ge 1} |\xi|^{2k} e^{-c|\xi|^{2(1-\sigma)}\int_0^t \eta(s)ds} |\hat{U}_0|^2 d\xi \\
&\le c\sup_{|\xi|\ge 1}\Big(|\xi|^{-2l} e^{-c|\xi|^{2(1-\sigma)}\int_0^t \eta(s)ds}\Big)\int_{|\xi|\ge 1} |\xi|^{2(k+l)} |\hat{U}_0|^2 d\xi.
\end{aligned}$$

Using $e^{\gamma} \geq \gamma + 1, \forall \gamma \in \mathbb{R}$, and, $|\xi| \geq 1$we have

$$\begin{aligned}
\sup_{|\xi|\geq 1}\left(|\xi|^{-2l} e^{-c|\xi|^{2(1-\sigma)}\int_0^t \eta(s)ds}\right) &= c\sup_{|\xi|\geq 1}\left(|\xi|^{2(\sigma-1)} e^{\frac{(\sigma-1)c}{l}|\xi|^{2(1-\sigma)}\int_0^t \eta(s)ds}\right)^{-\frac{l}{\sigma-1}} \\
&\leq \sup_{|\xi|\geq 1}\left(|\xi|^{2(\sigma-1)} e^{c_1|\xi|^{2}(1-\sigma)\int_0^t \eta(s)ds}\right)^{-\frac{l}{\sigma-1}} \\
&\leq \sup_{|\xi|\geq 1}\left(|\xi|^{2(\sigma-1)}\left(1 + c_1|\xi|^{2(1-\sigma)}\int_0^t \eta(s)ds\right)\right)^{-\frac{l}{\sigma-1}} \\
&= \sup_{|\xi|\geq 1}\left(|\xi|^{2(\sigma-1)} + c_1|\xi|^{2(1-\sigma)}\int_0^t \eta(s)ds\right)^{-\frac{l}{\sigma-1}} \\
&\leq \left(1 + c_1\int_0^t \eta(s)ds\right)^{-\frac{l}{\sigma-1}} \leq c_2\left(1 + \int_0^t \eta(s)ds\right)^{-\frac{l}{\sigma-1}},
\end{aligned}$$

where $c_2 = \min\{1, c_1^{-\frac{l}{\sigma-1}}\}$. Thus,

$$I_2 \leq c\left(1+\int_0^t \eta(s)ds\right)^{-\frac{l}{\sigma-1}}\int_{|\xi|\geq 1}|\xi|^{2(k+l)}|\hat{U}_0|^2 d\xi \leq c\left(1+\int_0^t \eta(s)ds\right)^{-\frac{l}{\sigma-1}}\|\nabla^{k+l}U_0\|_2^2, \tag{3.52}$$

for $k + l \leq r$.Substituting (3.51) and (3.52) in (3.49), we get (3.46).

* If $\sigma = 1$, then

$$\begin{aligned}
I_2 &\leq ce^{-c\int_0^t \eta(s)ds}\int_{|\xi|\geq 1}|\xi|^{2k}|\hat{U}_0|^2 d\xi \\
&\leq e^{-c\int_0^t \eta(s)ds}\int_{\mathbb{R}^n}|\xi|^{2k}|\hat{U}_0|^2 d\xi.
\end{aligned}$$

Again the Placherel theorem yields

$$I_2 \leq ce^{-c\int_0^t \eta(s)ds}\|\nabla^k U_0\|_2^2, \quad \forall t \geq 0. \tag{3.53}$$

Substituting (3.51) and (3.53) in (3.49), we arrive at (3.47).

■

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Abstract

In this thesis, we study the asymptotic behavior of solutions to a class of hyperbolic equations. **Firstly**, we introduce some notations and we review some mathematical concepts that will be used throughout this thesis. **In chapter 2**, we investigate the existence and uniqueness of solution to the heat equation using two methods: Faedo-Galerkin method and the semigroup theory, based on references [3, 13].

Finally, **in chapter 3**, we study the asymptotic behavior of solution to the Fractional-Laplace-wave equation with viscoelastic term. We use the Fourier and Laplace transforms to derive the explicit solution. For the stability, we employ the energy method, multiplier technique and the Lyapunov function, relying on [22, 33].
Keywords: Faedo-Galerkin method, Semigroup, Energy method, Lyapunov functional, Fourier transform, Asymptotic behavior.

Résumé

Dans ce mémoire, nous étudions le comportement asymptotique de la solution d'une classe d'équations hyperbolique. **Tout d'abord**, nous introduisons quelques notations et passons en revue des concepts mathématiques qui seront utilisés tout au long de ce travail. **Au chapitre 2**, nous étudions l'existence et l'unicité de la solution de l'équation de la chaleur en utilisant deux méthodes : la méthode de Faedo-Galerkin et la théorie des semi-groupes, en nous basant sur [3, 13].

Au chapitre 3, nous examinons le comportement asymptotique de la solution de l'équation des ondes Laplace fractionnaire avec terme viscoélastique. Les transformations de Fourier et Laplace sont utilisées pour obtenir une expression explicite de la solution. Pour le stabilité, nous appliquons la méthode de l'énergie, la technique des multiplicateurs ainci que la fonction de Lyapunov, en nous appuyant sur les références [22, 33].
Mots-clés: Méthode de Faedo-Galerkin, Semi-groupe, Méthode de l'énergie, Fonction de Lyapunov, Transformation de Fourier, Comportement asymptotique

الملخص

في هذه الأطروحة، نتناول دراسة السلوك التقاربي للحل لإحدى مسائل المعادلات الزائدية. بداية، نقدم بعض الرموز، التعاريف والنظريات التي سنحتاجها في باقي الأطروحة. أما في الفصل الثاني، نبرهن على وجود ووحدانية الحل لمعادلة الحرارة بطريقتين: طريقة فادو-غالركين Faedo-Galerkin ونظرية أنصاف الزمر Semigroup، واعتمدنا على المرجعين [13,3].

أخيرا، في الفصل الثالث، ندرس السلوك التقاربي لمعادلة الأمواج برتبة كسرية لمؤثر لابلاس مع وجود حد المرونة اللزجة. نستخدم تحويل فوريي ولابلاس لايجاد عبارة الحل. بالنسبة لدراسة الاستقرار نستخدم طريقة دالة الطاقة، دالة ليابونوف Lyapunov وطريقة المضروبات. نعتمد في ذلك على المرجعين [34,22].

الكلمات المفتاحية: طريقة فادو-غالركين، نظرية أنصاف الزمر، طريقة دالة الطاقة، دالة ليابونوف، تحويل فوري، السلوك التقاربي.