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Dedication

I dedicate this humble work to my parents and to all my family may Allah protect us, and to everyone who has helped and supported me.

Acknowledgement

Firstly, all praise and thanks be to Allah who has assisted me in everything in life, especially this work.

I thank all my family.

I sincerely thank ***Dr. Djemaa Messaoudi*** for her great help and guidance.

Without forgetting thanking some persons for helping me to LaTeXing this work.

Contents

List of Figures

Notations

- $\partial_t u$: Partial derivative of u with respect to time t.
- ∇: Gradient operator.
- Δ: Laplacian operator.
- $L^p(\Omega)$: Lebesgue space of p-integrable functions on Ω,

$$L^p(\Omega) = \{v : \Omega \to \mathbb{R} \text{ measurable; } \|v\|_p < \infty\}.$$

- $L^\infty(\Omega)$: Space of essentially bounded functions on Ω.
- $L^2(\Omega)$: Hilbert space of square-integrable functions on Ω, with:

$$\|v\|_{L^2(\Omega)} = \left(\int_\Omega |v(x)|^2 \, dx \right)^{1/2}.$$

- $H^k(\Omega)$ or $W^{k,p}(\Omega)$: Sobolev spaces with weak derivatives up to order k in L^p.
- $H^1(\Omega)$: Sobolev space,

$$H^1(\Omega) = \{v \in L^2(\Omega) : \nabla v \in (L^2(\Omega))^d\}.$$

- $\|v\|_{1,\Omega} = \left(\|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)} \right)^{1/2}.$

- $H_0^1(\Omega)$: Functions in $H^1(\Omega)$ with zero trace on the boundary:

$$H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}.$$

- $H^{-1}(\Omega)$: The dual space of $H_0^1(\Omega)$.
- L^m Hilbert space, with $m \in R$.
- H^m Sobolev space, with $m \in R$.
- $\|u\|_{H^m(0,T;B)} = \{\sum_{k=0}^{m} \int_0^T \|u^k(t)\|_B^2 \; dt\}^{1/2}$.
- $H^m(0,T;B) = \{u(t) \in B$ for a. e $t \in (0,T)$ and $\|u\|_{H^m(0,T;B)} < \infty\}$.
- $\rightharpoonup$ Weak convergence.
- $\rightarrow$ Strong convergence.

Introduction

The study of inverse problems began in the early 20^{th} century [14, 15]. At that time, scientists focused mainly on problems in the physical sciences, such as quantum scattering, electrical prospecting, seismology, and potential theory. These early works laid the foundation for methods used to determine the unknown values or causes from incomplete or indirect data. Inverse problems were also explored in astronomy, where they tried to learn about stars and planets from limited observations.

With the development of powerful computers in the second half of the 20^{th} century, inverse problems became more widely used in many areas of science and engineering [2, 8]. Today, they appear in fields such as medical imaging, remote sensing, fluid mechanics, machine learning, finance, and environmental science. These problems are especially useful when we cannot measure something directly but still want to estimate it using mathematical models. For more detailed background, see [3, 4, 5, 11, 17, 18, 21].

A major area where inverse problems play an important role is heat transfer, especially in inverse heat conduction problems (IHCPs). These problems aim to estimate things like heat flux or thermal boundary conditions based on how temperature changes over time. However, IHCPs are known to be ill-posed, which means that small errors in temperature data can cause large errors in the results. This makes them very sensitive and difficult to solve [25].

Because of this, solutions to inverse heat problems are not exact. Instead, they rely on regularization methods that provide stable and approximate answers even when the input data is noisy. Engineers and scientists often want to estimate heat-related parameters such as heat flux, Robin coefficients (which describe convective heat transfer), or internal temperatures. These are useful in many practical situations, such as testing thermal insulation, designing cooling systems for electronics, or modeling heat transfer in medical applications [6].

In recent years, researchers have shown growing interest in solving inverse problems to estimate heat transfer coefficients, especially those involving Robin-type boundary conditions. For example, Haó [12] used a nonlinear conjugate gradient method with a boundary element solver to find heat transfer coefficients that vary over space or time. Marián [23] worked on estimating a time-dependent Robin coefficient using a technique called Rothe's method and provided mathematical proof for how well the method works.

Another study by Slodicka et al. [22] developed a full mathematical model for estimating a time-dependent Robin coefficient in a one-dimensional heat problem. They also proved that a unique solution exists. More recently, Jiang and Talaat [16] used advanced computational methods like the Levenberg-Marquardt algorithm and surrogate functionals to recover both the Robin coefficient and the heat flux at the same time in an elliptic system.

Together, these studies show how important mathematical tools and numerical methods are in solving heat transfer inverse problems. They also emphasize the need for stable solution techniques, error control, and efficient algorithms to make sure the results are accurate and useful in real-world applications.

The objective of this work is to study the inverse problem of the identification of boundary coefficients in a parabolic system. Our work will be divided into three chapters, as outlined below:

In Chapter one, we begin with some background and literature review.

In Chapter two, we study an inverse problem where we aim to identify two unknown functions: the Robin boundary coefficient $g(x)$ and the heat flux $q(x)$, using temperature measurements. To address the ill-posed nature of the problem, we apply Tikhonov regularization.

We also prove the well-posedness of the model and develop a numerical algorithm based on the finite element method in space and backward Euler in time. The solution is computed using the Levenberg–Marquardt method combined with a surrogate functional strategy.

In Chapter three, we test the algorithm from Chapter two on numerical examples using MATLAB. Synthetic data with added noise is used to simulate measurements. The method successfully recovers the unknown functions $g(x)$ and $q(x)$.

Chapter 1

Generalities

In this chapter, we provide the basic concepts and motivation behind inverse problems. We begin by explaining what inverse problems are and why they are important in science and engineering. We then present several real-world examples, such as in heat conduction, medical imaging, astronomy, and geophysics. The chapter also introduces the challenges of ill-posed problems and discusses two common tools for solving them: Tikhonov regularization and the Levenberg–Marquardt method.

1.1 Background and motivation

1.1.1 Inverse problems

Generally, causes and effects are examined in a specific order. The term "inverse problem" derives from physics, when our objective is to recover information by observing the effects and then try to figure out what caused them.

Let X and Y be normed spaces, $K : X \longrightarrow Y$ a (linear or nonlinear) mapping. Then, given the mathematical model

$$K(x) = y,$$

where x is a vector of unknowns and y is a vector of measurements, the direct problem is to find y given x, while the inverse problem is to find x given y. In practice, the unknown could be parameters in our mathematical model or the source term or boundaries or a combination of these. Due to its indirect nature, solving the inverse problem is usually

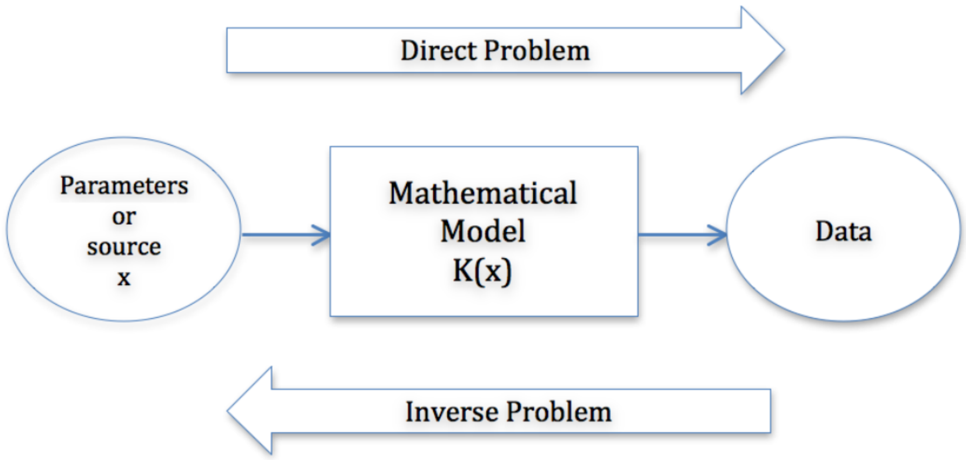

Figure 1.1: Inverse problem via direct problem

very difficult. In fact, solving such an inverse problem by standard methods numerically is difficult and often yields unstable results, even when the data are exact. Therefore, to obtain a stable approximation of the solution, we have to use special techniques.

1.1.2 Some inverse problems in real life

This section discusses inverse problems and provides some examples across various application fields.

Inverse problems in heat conduction

The classical (well-posed) problem in heat conduction is the direct problem of computing the temperature evolution in a body with known thermal parameters, given the initial temperature and the temperature or heat flux on the entire boundary. Several important types of inverse problems arise:

- Determining the initial temperature from later measurements – Mathematically, this involves solving the backward heat equation.
- Estimating thermal parameters of the material from temperature measurements – This requires inferring material properties from observed thermal behavior.
- Reconstructing the temperature on an inaccessible part of the boundary – This is done using heat flux and temperature measurements from other boundary sections.

Inverse problems in astronomy

Astronomy is fundamentally an inverse problem, as it deduces the structure and dynamics of celestial objects from indirect observations. Examples include:

- Radio astronomy – Reconstructing the shape of radio wave-emitting objects from telescope data.
- Exoplanet detection – Inferring planetary properties from light curves and radial velocity measurements.
- Cosmological parameter estimation – Determining dark matter distribution or the universe's expansion rate from observational data.

These problems often involve large-scale computations and require robust statistical methods.

Inverse problems in medical imaging

Medical imaging relies heavily on solving inverse problems to reconstruct internal body structures from indirect measurements. Key modalities include:

- Computed Tomography (CT) – Reconstructing cross-sectional images from X-ray projections.

- Magnetic Resonance Imaging (MRI) – Recovering tissue properties from radio frequency signals.
- Positron Emission Tomography (PET) – Mapping metabolic activity from detected gamma rays.

Solutions require regularization to address ill-posedness and noise.

Inverse problems in geophysics

Geophysical inverse problems aim to infer Earth's subsurface properties from surface measurements:

- Seismic tomography – Reconstructing Earth's interior structure from seismic wave data.
- Gravimetric inversion – Estimating density distributions from gravity field measurements.
- Electromagnetic induction – Mapping underground resistivity from electromagnetic responses.

These are critical for oil exploration, earthquake studies, and groundwater detection.

Remark 1.1.1 *Due to their special properties, most inverse problems are ill-posed.*

1.1.3 What is an ill-posed problem?

The French mathematician Jacques Hadamard introduced the concept of a wellposed problem in his paper of 1902 on boundary-value problems for partial differential equations and their physical interpretation [13].

Definition 1.1.2 *Based on Hadamard's definition, a mathematical problem is well-posed if it satisfies the following properties:*
1. Existence: For all (suitable) data, there exists a solution of the problem (in an appropriate sense).

2. *Uniqueness: For all (suitable) data, the solution is unique.*
3. *Stability: The solution depends continuously on the data.*

A problem is ill-posed if one of these three conditions is violated.

Remark 1.1.3 *The main concern when studying the inverse problem is the violation of the third condition, that is, the solution does not depend continuously on the data.*

In general, to solve ill-posed problems, we follow the schema below:

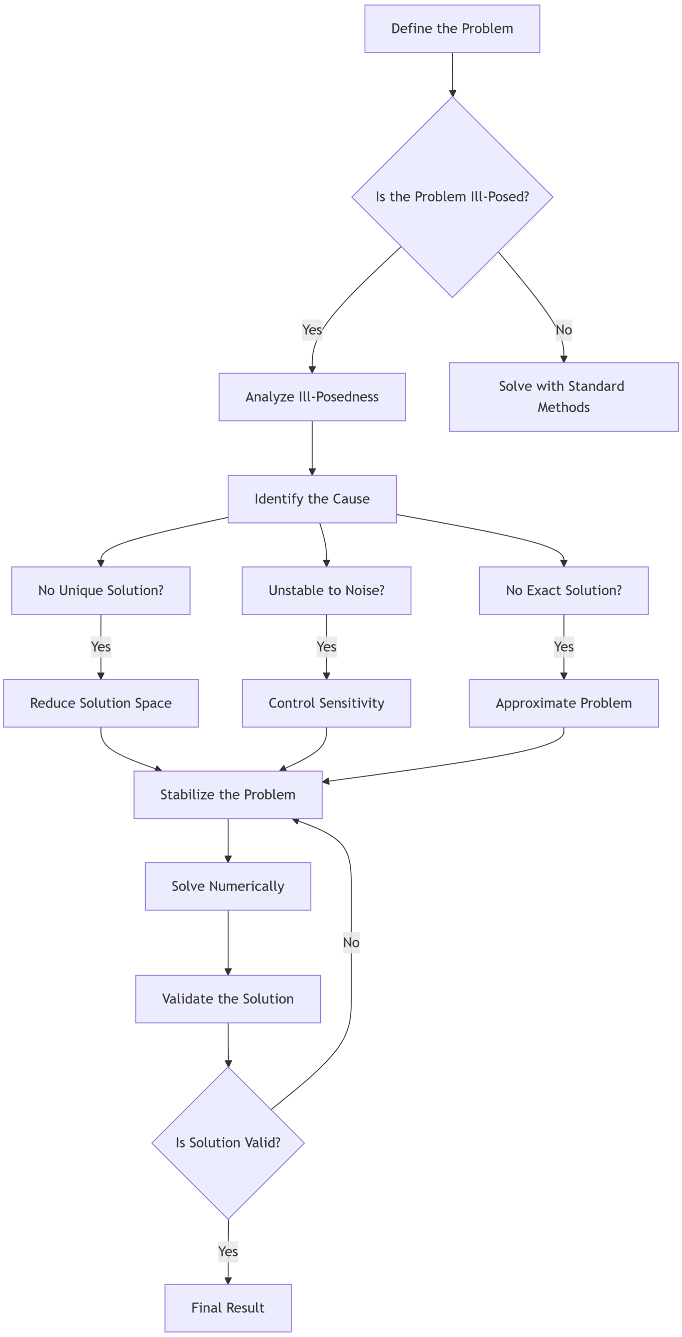

Figure 1.2: Techniques for Solving ill-Posed Problems

1.2 Tikhonov Regularization

The core idea of regularization methods is to transform an ill-posed problem into a well-posed one by introducing additional information or constraints. This transformation is typically achieved by incorporating a regularization operator that reflects available prior knowledge about the exact solution.

Among these methods, Tikhonov regularization—named after the Russian mathematician Andrey Tikhonov and introduced in the early 20th century—is one of the most widely used techniques for stabilizing ill-posed inverse problems. The Tikhonov method modifies the original problem by adding a penalty term to the objective functional. This regularization term imposes smoothness or other desirable properties on the solution, thereby enhancing stability and mitigating sensitivity to noise or incomplete data. In doing so, it also helps prevent overfitting to noisy observations, leading to more robust and physically meaningful solutions.

1.3 Levenberg–Marquardt (LM) Method

The **Levenberg–Marquardt (LM) method** [1, 16]is a powerful and widely used optimization algorithm for solving *nonlinear least-squares problems*. These problems frequently arise in fields such as parameter estimation, inverse problems, and *data fitting*, where the goal is to determine unknown parameters by minimizing the discrepancy between model predictions and observed data.

The method was originally introduced in 1944 by *Kenneth Levenberg* [19] and later in 1963 refined by *Donald Marquardt* [20]. It is often described as a hybrid algorithm that combines the advantages of two classical optimization techniques:

- The **Gauss–Newton method**, which converges rapidly when the current estimate is close to the true solution, but can be unstable or divergent for poor initial guesses or ill-conditioned problems.

- The **gradient descent method**, which is more stable and converges from a wider range of initial guesses, though typically slower near the solution.

The LM method smoothly interpolates between these two methods by adjusting a damping parameter that controls the update step. When far from the solution, it behaves more like gradient descent to ensure stability. As the iterations progress and the algorithm approaches the solution, it behaves more like the Gauss–Newton method to gain speed and accuracy.

This adaptive behavior makes the Levenberg–Marquardt method particularly well-suited for nonlinear models that may be sensitive to noise or difficult to solve using standard techniques.

1.4 Important tools

Surrogate functional method [9] The surrogate functional method is a powerful numerical technique used especially in inverse problems to stabilize and accelerate the convergence of iterative schemes. It is particularly helpful when dealing with ill-posed problems, such as those governed by partial differential equations (PDEs), where direct minimization is difficult or unstable. It replaces a difficult minimization problem with a sequence of simpler, better-behaved approximations. Next Chapter, we will see how to use it.

Theorem 1.4.1 (Banach-Alaoglu) *[7] In a normed space X, the closed unit ball of the dual space X* is weak*-compact. For reflexive spaces, this implies every bounded sequence has a weakly convergent subsequence.*

Lemma 1.4.2 *(Aubin-Lions Lemma, [24], p.189) Let X_0, X be two Banach spaces and X_1 be a Hilbert space with $X_0 \subset X \subset X_1$, the injections being continuous and the injection of X_0 into X being compact. Then the injection of $\mathcal{Y}(0,T;\alpha_0,\alpha_1;X_0,X_1)$ into $L^{\alpha_0}(0,T;X)$ is compact for any finite number $\alpha_0 > 1$, where*

$$\mathcal{Y}(0,T;\alpha_0,\alpha_1;X_0,X_1) = \left\{ v \in L^{\alpha_0}(0,T;X_0);\ v' = \frac{dv}{dt} \in L^{\alpha_1}(0,T;X_1) \right\}.$$

Chapter 2

Study of the Inverse Problem

In this chapter, we study an inverse problem involving a parabolic partial differential equation. The goal is to identify two unknown boundary functions: the Robin coefficient and the heat flux, using noisy measurements. We present the mathematical model, prove its well-posedness, and apply regularization to handle the instability of the problem. A numerical method based on finite elements and the Levenberg–Marquardt algorithm is developed for solving the problem efficiently.

2.1 Problem Formulation

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ is a bounded domain with boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. We consider the following parabolic initial-boundary value problem:

$$\begin{cases} \partial_t u - \Delta u = f(x,t) & \text{in } \Omega \times (0,T], \\ \frac{\partial u}{\partial n} + \gamma(x) u = g(x) h(x,t) & \text{on } \Gamma_1 \times (0,T], \\ \frac{\partial u}{\partial n} = q(x) R(x,t) & \text{on } \Gamma_2 \times (0,T], \\ \frac{\partial u}{\partial n} = 0 & \text{on } \Gamma_3 \times (0,T], \\ u(x,0) = 0 & \text{on } \Omega, \end{cases} \tag{2.1}$$

where:

- $f(x,t) \in L^2(0,T;L^2(\Omega))$ is a known internal source function.
- $\gamma(x) \in L^\infty(\Gamma_1)$ is a known Robin coefficient satisfying $0 < \gamma_1 \leq \gamma(x) \leq \gamma_2$ almost everywhere on Γ_1.
- $h(x,t) \in L^2(0,T;L^\infty(\Gamma_1))$ and $R(x,t) \in L^2(0,T;L^\infty(\Gamma_2))$ are known time-dependent boundary weight functions.
- $g(x)$ is the unknown Robin boundary coefficient to be determined, assumed to lie in the admissible set:
$$K_1 = \{g \in L^2(\Gamma_1) \mid g_1 \leq g(x) \leq g_2\}.$$
- $q(x)$ is the unknown Neumann heat flux function to be identified on Γ_2, assumed to satisfy:
$$K_2 = \{q \in L^2(\Gamma_2) \mid q_1 \leq q(x) \leq q_2\}.$$

This study focuses on an inverse problem governed by a time-dependent parabolic partial differential equation (PDE). The primary goal is to simultaneously identify two unknown spatially varying functions: the Robin-type boundary coefficient $g(x)$, defined on a part of the boundary denoted by Γ_1, and the Neumann boundary flux $q(x)$, defined on another portion of the boundary denoted by Γ_2.
The available data consist of boundary observations of the solution $u(x,t)$ on the subset Γ_3 of the boundary, over the interval $(0,T]$. These measurements are denoted by $z^\delta(x,t)$ and may be contaminated by noise of level $\delta > 0$. That is,

$$z^\delta(x,t) = u(x,t) + \text{noise}, \quad \text{for } (x,t) \in \Gamma_3 \times (0,T].$$

The inverse problem can thus be formulated as follows:

Given the measured data z^δ on $\Gamma_3 \times (0,T]$, determine the unknown functions $g(x)$ on Γ_1 and $q(x)$ on Γ_2 such that the solution $u(x,t)$ of the parabolic system matches the observed data as accurately as possible.

This inverse problem is inherently ill-posed in the sense of Hadamard: small perturbations in the data may cause large deviations in the recovered parameters. Therefore, to ensure stability and uniqueness of the solution, regularization techniques are required.

2.2 Regularization and Well-posedness

In this section, we first give a preliminary lemma for stating the well-posedness of forward solution u of system (2.1). Then, we introduce a Tikhonov regularization framework that transforms the inverse problem into a well-posed variational optimization problem.

Lemma 2.2.1 *[10]*
Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ *be an open bounded domain with* C^1 *boundary* $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. *Assume:*

1. $\gamma(x) \in L^\infty(\Gamma_1)$ *with* $0 < \gamma_1 \leq \gamma(x) \leq \gamma_2$
2. $g(x) \in K_1 = \{g \in L^2(\Gamma_1) : g_1 \leq g(x) \leq g_2\}$
3. $q(x) \in K_2 = \{q \in L^2(\Gamma_2) : q_1 \leq q(x) \leq q_2\}$
4. $f(x,t) \in L^2(0,T; L^2(\Omega))$
5. $h(x,t) \in L^2(0,T; L^\infty(\Gamma_1))$
6. $R(x,t) \in L^2(0,T; L^\infty(\Gamma_2))$

Then there exists a unique solution $u \in L^2(0,T; H^1(\Omega))$ *to system* (2.1) *satisfying the estimate:*

$$\|u\|_{1,\Omega} \leq C \left(\|f\|_{L^2(0,T;L^2(\Omega))} + \|gh\|_{L^2(0,T;L^2(\Gamma_1))} + \|qR\|_{L^2(0,T;L^2(\Gamma_2))} \right)$$

2.2.1 Tikhonov's regularization

As the inverse problem is ill-posed, we formulate it into a mathematically stabilized minimization system with Tikhonov's regularization. The goal is to find the unknown functions $g(x)$ and $q(x)$ such that the solution of the PDE matches the observed data as closely as possible, while also ensuring that the solution remains stable and physically meaningful.

To achieve this, we define a cost functional that includes two key components:

1. A term that measures how close the computed solution $u(g, q)$ is to the observed data z^δ on the boundary Γ_3.

2. A regularization term that penalizes large or unstable values of g and q to enforce smoothness and stability.

The regularized cost functional is written as:

$$J(g,q) = \int_0^T \|u(g,q) - z^\delta\|_{\Gamma_3}^2 dt + \beta\|g\|_{\Gamma_1}^2 + \eta\|q\|_{\Gamma_2}^2,$$

where:

- $u(g,q)$ is the solution of the forward problem corresponding to g and q,
- $z^\delta(x,t)$ is the (possibly noisy) measurement data on Γ_3,
- $\beta > 0$ and $\eta > 0$ are regularization parameters that control the strength of the penalization,
- $\|g\|^2$ and $\|q\|^2$ ensure that the solution remains stable and does not overfit noisy data.

The regularized inverse problem is therefore formulated as:

$$\text{Find } (g,q) \in K_1 \times K_2 \text{ such that } J(g,q) \text{ is minimized.}$$

That is mean,

$$\min_{(g,q)\in K_1\times K_2} J(g,q) = \min_{(g,q)} \left[\int_0^T \|u(g,q) - z^\delta\|_{\Gamma_3}^2 dt + \beta\|g\|_{\Gamma_1}^2 + \eta\|q\|_{\Gamma_2}^2 \right] \tag{2.2}$$

In the following, we shall justify the regularizing effects of the nonlinear optimization system (2.2) that it always has solutions and its solutions are stable with respect to the noise error in the observation data z delta. The first theorem establishes the existence of solutions.

Theorem 2.2.2 *Under the assumptions of Lemma 2.2.1, then the minimization problem (2.2) has at least one solution.*

Proof.

Since $J(g,q) \geq 0$, there exists a minimizing sequence $\{(g_k, q_k)\} \subset K_1 \times K_2$ such that:

$$\lim_{k\to\infty} J(g_k, q_k) = \inf_{(g,q)\in K_1\times K_2} J(g,q).$$

The regularization terms $\beta\|g_k\|^2_{\Gamma_1} + \eta\|q_k\|^2_{\Gamma_2}$ ensure $\{(g_k, q_k)\}$ is bounded in $L^2(\Gamma_1) \times L^2(\Gamma_2)$. Then, by Theorem 1.4.1, there exists a subsequence (still denoted $\{(g_k, q_k)\}$) and $(g^*, q^*) \in K_1 \times K_2$ such that:

$$g_k \rightharpoonup g^* \text{ in } L^2(\Gamma_1), \quad q_k \rightharpoonup q^* \text{ in } L^2(\Gamma_2).$$

Let $u_k \equiv u(g_k, q_k)$. From Lemma 2.2.1, $\{u_k\}$ is bounded in $L^2(0, T; H^1(\Omega))$. By Lemma 1.4.2, there exists a subsequence $u_k \to u^*$ strongly in $L^2(\Gamma_3 \times (0, T))$.

The weak formulation of the parabolic system shows $u^* = u(g^*, q^*)$.

Now by using lower Semi-continuity of the norm, we get:

$$\liminf_{k\to\infty} \|g_k\|^2_{\Gamma_1} \geq \|g^*\|^2_{\Gamma_1}, \quad \liminf_{k\to\infty} \|q_k\|^2_{\Gamma_2} \geq \|q^*\|^2_{\Gamma_2}.$$

Combined with the strong convergence of u_k, we obtain:

$$J(g^*, q^*) \leq \liminf_{k\to\infty} J(g_k, q_k) = \inf_{(g,q)\in K_1\times K_2} J(g, q).$$

Thus, (g^*, q^*) is a minimizer. ■

Theorem 2.2.3 *Let $\{z^\delta_n\}$ be a sequence in $L^2(0, T; L^2(\Gamma_3))$ such that $z^\delta_n \to z^\delta$ as $n \to \infty$. Let $\{(g^{(n)}, q^{(n)})\}$ be minimizers of the regularized problems (2.2). Then there exists a subsequence of $\{(g^{(n)}, q^{(n)})\}$ converging weakly in $L^2(\Gamma_1) \times L^2(\Gamma_2)$ to a minimizer (g^*, q^*) of the limiting problem with data z^δ.*

Proof.

For each n, $(g^{(n)}, q^{(n)})$ satisfies:

$$J(g^{(n)}, q^{(n)}) \leq J(g, q) \quad \forall (g, q) \in K_1 \times K_2.$$

Thus, $\{(g^{(n)}, q^{(n)})\}$ is bounded in $L^2(\Gamma_1) \times L^2(\Gamma_2)$. So, we can extract a subsequence (still denoted $\{(g^{(n)}, q^{(n)})\}$) such that:

$$g^{(n)} \rightharpoonup g^* \text{ in } L^2(\Gamma_1), \quad q^{(n)} \rightharpoonup q^* \text{ in } L^2(\Gamma_2).$$

Since $z^\delta_n \to z^\delta$ in $L^2(\Gamma_3 \times (0, T))$ and $u(g^{(n)}, q^{(n)}) \to u(g^*, q^*)$ strongly:

$$\lim_{n\to\infty} \|u(g^{(n)}, q^{(n)}) - z^\delta_n\|^2_{L^2(\Gamma_3\times(0,T))} = \|u(g^*, q^*) - z^\delta\|^2_{L^2(\Gamma_3\times(0,T))}.$$

By lower semi-continuity:

$$J(g^*, q^*) \leq \liminf_{n\to\infty} J(g^{(n)}, q^{(n)}) \leq \liminf_{n\to\infty} J(g, q) = J(g, q) \quad \forall (g, q) \in K_1 \times K_2.$$

Hence, (g^*, q^*) is a minimizer. ■

2.3 Sensitivity and Adjoint Method Theory

In this section, we study how small changes in the unknown boundary functions $g(x)$ and $q(x)$ affect the solution $u(x, t)$ of the parabolic system. This is important for solving the inverse problem using optimization methods like the Levenberg–Marquardt algorithm.

To do this, we use what's called Gateaux derivatives, which help us understand how the solution changes in specific directions.

2.3.1 Sensitivity Analysis

Definition 2.3.1 (Gateaux Derivative) *For the solution operator $u(g, q)$, the Gateaux derivatives in directions $(d, p) \in L^2(\Gamma_1) \times L^2(\Gamma_2)$ are defined as:*

$$u'_g(g, q)d \simeq \lim_{\varepsilon\to 0} \frac{u(g + \varepsilon d, q) - u(g, q)}{\varepsilon}$$

$$u'_q(g, q)p \simeq \lim_{\varepsilon\to 0} \frac{u(g, q + \varepsilon p) - u(g, q)}{\varepsilon}$$

Lemma 2.3.2 .

1- Let $d \in L^2(\Gamma_1)$ be a small change (or direction) in the function $g(x)$. The resulting change in the solution, written as $u'_g(g, q)d$, satisfies the following system:

$$\begin{cases} \partial_t u'_g - \Delta u'_g = 0 & \text{in } \Omega \times (0, T], \\ \frac{\partial u'_g}{\partial n} + \gamma(x) u'_g = d \cdot h(x, t) & \text{on } \Gamma_1 \times (0, T], \\ \frac{\partial u'_g}{\partial n} = 0 & \text{on } (\Gamma_2 \cup \Gamma_3) \times (0, T], \\ u'_g(x, 0) = 0 & \text{in } \Omega. \end{cases}$$

2- Now let $p \in L^2(\Gamma_2)$ be a small change in $q(x)$. The effect on the solution, written as $u'_q(g,q)p$, is given by:

$$\begin{cases} \partial_t u'_q - \Delta u'_q = 0 & \text{in } \Omega \times (0,T], \\ \frac{\partial u'_q}{\partial n} + \gamma(x) u'_q = 0 & \text{on } \Gamma_1 \times (0,T], \\ \frac{\partial u'_q}{\partial n} = p \cdot R(x,t) & \text{on } \Gamma_2 \times (0,T], \\ \frac{\partial u'_q}{\partial n} = 0 & \text{on } \Gamma_3 \times (0,T], \\ u'_q(x,0) = 0 & \text{in } \Omega. \end{cases}$$

Proof. We prove both parts step by step.

Part (i): Sensitivity with Respect to the Robin Coefficient $g(x)$

1. **Perturbed Problem:** Let $\varepsilon > 0$ and consider the perturbed function $g^\varepsilon = g + \varepsilon d$. Let $u^\varepsilon = u(g^\varepsilon, q)$ solve the forward problem:

$$\begin{cases} \partial_t u^\varepsilon - \Delta u^\varepsilon = f(x,t) & \text{in } \Omega \times (0,T], \\ \frac{\partial u^\varepsilon}{\partial n} + \gamma(x) u^\varepsilon = g^\varepsilon(x) h(x,t) & \text{on } \Gamma_1 \times (0,T], \\ \frac{\partial u^\varepsilon}{\partial n} = q(x) R(x,t) & \text{on } \Gamma_2 \times (0,T], \\ \frac{\partial u^\varepsilon}{\partial n} = 0 & \text{on } \Gamma_3 \times (0,T], \\ u^\varepsilon(x,0) = 0 & \text{in } \Omega. \end{cases}$$

2. **Difference Quotient:** Define the difference quotient $w^\varepsilon = \frac{u^\varepsilon - u}{\varepsilon}$. Then w^ε satisfies:

$$\begin{cases} \partial_t w^\varepsilon - \Delta w^\varepsilon = 0 & \text{in } \Omega \times (0,T], \\ \frac{\partial w^\varepsilon}{\partial n} + \gamma(x) w^\varepsilon = d(x) h(x,t) & \text{on } \Gamma_1 \times (0,T], \\ \frac{\partial w^\varepsilon}{\partial n} = 0 & \text{on } \Gamma_2 \cup \Gamma_3 \times (0,T], \\ w^\varepsilon(x,0) = 0 & \text{in } \Omega. \end{cases}$$

3. **Limit Passage:** As $\varepsilon \to 0, w^\varepsilon \to u'_g$. Then u'_g satisfies:

$$\begin{cases} \partial_t u'_g - \Delta u'_g = 0 & \text{in } \Omega \times (0,T], \\ \frac{\partial u'_g}{\partial n} + \gamma(x) u'_g = d(x) h(x,t) & \text{on } \Gamma_1 \times (0,T], \\ \frac{\partial u'_g}{\partial n} = 0 & \text{on } \Gamma_2 \cup \Gamma_3 \times (0,T], \\ u'_g(x,0) = 0 & \text{in } \Omega. \end{cases}$$

Part (ii): Sensitivity with Respect to the Neumann Flux $q(x)$

1. **Perturbed Problem:** Let $\varepsilon > 0$ and define $q^\varepsilon = q + \varepsilon p$. Let $u^\varepsilon = u(g, q^\varepsilon)$ solve:

$$\begin{cases} \partial_t u^\varepsilon - \Delta u^\varepsilon = f(x,t) & \text{in } \Omega \times (0,T], \\ \frac{\partial u^\varepsilon}{\partial n} + \gamma(x) u^\varepsilon = g(x) h(x,t) & \text{on } \Gamma_1 \times (0,T], \\ \frac{\partial u^\varepsilon}{\partial n} = q^\varepsilon(x) R(x,t) & \text{on } \Gamma_2 \times (0,T], \\ \frac{\partial u^\varepsilon}{\partial n} = 0 & \text{on } \Gamma_3 \times (0,T], \\ u^\varepsilon(x,0) = 0 & \text{in } \Omega. \end{cases}$$

2. **Difference Quotient:** Define $w^\varepsilon = \frac{u^\varepsilon - u}{\varepsilon}$. Then w^ε satisfies:

$$\begin{cases} \partial_t w^\varepsilon - \Delta w^\varepsilon = 0 & \text{in } \Omega \times (0,T], \\ \frac{\partial w^\varepsilon}{\partial n} + \gamma(x) w^\varepsilon = 0 & \text{on } \Gamma_1 \times (0,T], \\ \frac{\partial w^\varepsilon}{\partial n} = p(x)R(x,t) & \text{on } \Gamma_2 \times (0,T], \\ \frac{\partial w^\varepsilon}{\partial n} = 0 & \text{on } \Gamma_3 \times (0,T], \\ w^\varepsilon(x,0) = 0 & \text{in } \Omega. \end{cases}$$

3.**Limit Passage:** As $\varepsilon \to 0$, $w^\varepsilon \rightharpoonup u'_q$ in $L^2(0,T;H^1(\Omega))$. Then u'_q satisfies:

$$\begin{cases} \partial_t u'_q - \Delta u'_q = 0 & \text{in } \Omega \times (0,T], \\ \frac{\partial u'_q}{\partial n} + \gamma(x) u'_q = 0 & \text{on } \Gamma_1 \times (0,T], \\ \frac{\partial u'_q}{\partial n} = p(x)R(x,t) & \text{on } \Gamma_2 \times (0,T], \\ \frac{\partial u'_q}{\partial n} = 0 & \text{on } \Gamma_3 \times (0,T], \\ u'_q(x,0) = 0 & \text{in } \Omega. \end{cases}$$

This completes the proof. ■

Remark 2.3.3 .

1- *Understanding these sensitivities helps us improve the estimates of $g(x)$ and $q(x)$ in the inverse problem. These results are later used in the Levenberg-Marquardt (LM) method to efficiently compute the directional derivatives, which makes the optimization process faster and more accurate.*

2- *(Regularity) Both sensitivity equations have unique solutions in $L^2(0,T;H^1(\Omega))$ by the same arguments as in Lemma 2.2.1.*

2.3.2 Adjoint Operators

The sensitivity equations induce corresponding adjoint operators:

$$\begin{aligned} u'^*_g &: L^2(\Gamma_3) \to L^2(\Gamma_1) \\ u'^*_q &: L^2(\Gamma_3) \to L^2(\Gamma_2) \end{aligned} \tag{2.3}$$

defined via the duality relations:

$$\langle u'_g d, \omega \rangle_{\Gamma_3} = \langle d, u'^*_g \omega \rangle_{\Gamma_1}, \quad \langle u'_q p, \omega \rangle_{\Gamma_3} = \langle p, u'^*_q \omega \rangle_{\Gamma_2}$$

Adjoint Formulations

Lemma 2.3.4 (Adjoint Relations) *For any measurement weights $\omega, v \in L^2(\Gamma_3)$, define adjoint operators through the following terminal-boundary value problems:*

***(i) Robin Coefficient Adjoint** ($u_g'^*\omega = \varphi$):*

$$\begin{cases} -\partial_t\varphi - \Delta\varphi = 0 & \text{in } \Omega \times (0,T] \\ \frac{\partial\varphi}{\partial n} + \gamma\varphi = 0 & \text{on } \Gamma_1 \times (0,T] \\ \frac{\partial\varphi}{\partial n} = 0 & \text{on } \Gamma_2 \times (0,T] \\ \frac{\partial\varphi}{\partial n} = -\omega & \text{on } \Gamma_3 \times (0,T] \\ \varphi(x,T) = 0 & \text{in } \Omega \end{cases} \tag{2.4}$$

***(ii) Flux Adjoint** ($u_q'^*v = \psi$):*

$$\begin{cases} -\partial_t\psi - \Delta\psi = 0 & \text{in } \Omega \times (0,T] \\ \frac{\partial\psi}{\partial n} + \gamma\psi = 0 & \text{on } \Gamma_1 \times (0,T] \\ \frac{\partial\psi}{\partial n} = 0 & \text{on } \Gamma_2 \times (0,T] \\ \frac{\partial\psi}{\partial n} = -v & \text{on } \Gamma_3 \times (0,T] \\ \psi(x,T) = 0 & \text{in } \Omega \end{cases} \tag{2.5}$$

Then the following adjoint relations hold:

$$\left\langle \int_0^T u_g' d\, dt, \omega \right\rangle_{\Gamma_3} = \left\langle d, \int_0^T h(x,t)\varphi\, dt \right\rangle_{\Gamma_1} \tag{2.6}$$

$$\left\langle \int_0^T u_q' p\, dt, v \right\rangle_{\Gamma_3} = \left\langle p, \int_0^T R(x,t)\psi\, dt \right\rangle_{\Gamma_2} \tag{2.7}$$

for all $d \in L^2(\Gamma_1)$ and $p \in L^2(\Gamma_2)$.

Proof.

Part (i): Robin Coefficient Adjoint Relation

Let $u_g'(x,t)$ satisfy the sensitivity equation:

$$\begin{aligned} \partial_t u_g' - \Delta u_g' &= 0 \quad \text{in } \Omega \times (0,T], \\ \frac{\partial u_g'}{\partial n} + \gamma(x)u_g' &= d(x)h(x,t) \quad \text{on } \Gamma_1 \times (0,T], \\ \frac{\partial u_g'}{\partial n} &= 0 \quad \text{on } \Gamma_2 \cup \Gamma_3 \times (0,T], \\ u_g'(x,0) &= 0 \quad \text{in } \Omega. \end{aligned}$$

Let $\varphi(x,t)$ be the solution of the adjoint problem:

$$\begin{aligned} -\partial_t\varphi - \Delta\varphi &= 0 \quad \text{in } \Omega\times(0,T], \\ \frac{\partial\varphi}{\partial n} + \gamma(x)\varphi &= 0 \quad \text{on } \Gamma_1\times(0,T], \\ \frac{\partial\varphi}{\partial n} &= 0 \quad \text{on } \Gamma_2, \\ \frac{\partial\varphi}{\partial n} &= -\omega(x,t) \quad \text{on } \Gamma_3, \\ \varphi(x,T) &= 0 \quad \text{in } \Omega. \end{aligned}$$

Step 1: Multiply and integrate Multiply the sensitivity equation by φ and integrate over $\Omega\times(0,T)$:

$$\int_0^T\int_\Omega(\partial_t u_g' - \Delta u_g')\varphi\,dx\,dt = 0.$$

Step 2: Apply integration by parts Using integration by parts in both time and space and applying the initial and terminal conditions ($u_g'(x,0)=0$, $\varphi(x,T)=0$), we obtain:

$$\int_0^T\int_\Omega -u_g'\partial_t\varphi + \nabla u_g'\cdot\nabla\varphi\,dx\,dt + \int_0^T\int_{\partial\Omega} u_g'\frac{\partial\varphi}{\partial n}\,ds\,dt - \int_0^T\int_{\partial\Omega}\varphi\frac{\partial u_g'}{\partial n}\,ds\,dt = 0.$$

Step 3: Boundary evaluation On Γ_1:

$$\begin{aligned} \frac{\partial u_g'}{\partial n} &= -\gamma u_g' + dh, \\ \frac{\partial\varphi}{\partial n} &= -\gamma\varphi. \end{aligned}$$

This gives:

$$\int_0^T\int_{\Gamma_1} u_g'(-\gamma\varphi) - \varphi(-\gamma u_g' + dh)\,ds\,dt = \int_0^T\int_{\Gamma_1}\varphi dh\,ds\,dt.$$

On Γ_3:

$$\frac{\partial\varphi}{\partial n} = -\omega, \quad \frac{\partial u_g'}{\partial n} = 0,$$

thus,

$$\int_0^T\int_{\Gamma_3} u_g'(-\omega)\,ds\,dt = -\int_0^T\int_{\Gamma_3} u_g'\omega\,ds\,dt.$$

Step 4: Final identity Bringing all terms together:

$$\int_0^T \int_{\Gamma_3} u'_g \omega \, ds \, dt = \int_0^T \int_{\Gamma_1} \varphi dh \, ds \, dt,$$

which proves the adjoint relation:

$$\left\langle \int_0^T u'_g dt, \, \omega \right\rangle_{\Gamma_3} = \left\langle d, \int_0^T h(x,t)\varphi(x,t) dt \right\rangle_{\Gamma_1}.$$

Part (ii): Flux Adjoint Relation

Let u'_q solve the sensitivity equation:

$$\begin{aligned} \partial_t u'_q - \Delta u'_q &= 0 \quad \text{in } \Omega \times (0,T], \\ \frac{\partial u'_q}{\partial n} + \gamma(x) u'_q &= 0 \quad \text{on } \Gamma_1, \\ \frac{\partial u'_q}{\partial n} &= p(x)R(x,t) \quad \text{on } \Gamma_2, \\ \frac{\partial u'_q}{\partial n} &= 0 \quad \text{on } \Gamma_3, \\ u'_q(x,0) &= 0. \end{aligned}$$

Let $\psi(x,t)$ solve the adjoint problem:

$$\begin{aligned} -\partial_t \psi - \Delta \psi &= 0 \quad \text{in } \Omega \times (0,T], \\ \frac{\partial \psi}{\partial n} + \gamma(x)\psi &= 0 \quad \text{on } \Gamma_1, \\ \frac{\partial \psi}{\partial n} &= 0 \quad \text{on } \Gamma_2, \\ \frac{\partial \psi}{\partial n} &= -v(x,t) \quad \text{on } \Gamma_3, \\ \psi(x,T) &= 0. \end{aligned}$$

Step 1: Multiply and integrate Multiply the sensitivity equation by ψ and integrate over $\Omega \times (0,T)$:

$$\int_0^T \int_\Omega (\partial_t u'_q - \Delta u'_q)\psi \, dx \, dt = 0.$$

Step 2: Apply integration by parts As before, boundary contributions arise only from Γ_2 and Γ_3:

- On Γ_2: $\frac{\partial u'_q}{\partial n} = pR$, $\frac{\partial \psi}{\partial n} = 0$.

$$\int_0^T \int_{\Gamma_2} -\psi \cdot pR \, ds \, dt.$$

- On Γ_3: $\frac{\partial u'_q}{\partial n} = 0$, $\frac{\partial \psi}{\partial n} = -v$.

$$\int_0^T \int_{\Gamma_3} u'_q \cdot v \, ds \, dt.$$

Step 3: Final identity

$$\int_0^T \int_{\Gamma_3} u'_q v \, ds \, dt = \int_0^T \int_{\Gamma_2} p(x) R(x,t) \psi(x,t) \, ds \, dt,$$

which gives the adjoint relation:

$$\left\langle \int_0^T u'_q dt,\, v \right\rangle_{\Gamma_3} = \left\langle p, \int_0^T R(x,t)\psi(x,t) dt \right\rangle_{\Gamma_2}.$$

■

Remark 2.3.5 *(Well-posedness) Both adjoint problems are well-posed backward parabolic equations with unique solutions $\varphi, \psi \in L^2(0,T;H^1(\Omega))$ by standard theory.*

Proposition 2.3.6 (Adjoint Operator Properties) *The adjoint operators satisfy:*

1. *Boundedness:* $\|u'^*_g\|_{\mathcal{L}(L^2(\Gamma_3), L^2(\Gamma_1))} \leq C\|h\|_{L^\infty}$.

2. *Linearity:* $u'^*_g(a\omega_1 + b\omega_2) = a u'^*_g \omega_1 + b u'^*_g \omega_2$.

3. *The adjoint of the adjoint returns the original sensitivity operator.*

2.4 Numerical Solution via L-M and Surrogate Functional

In this section, we mainly propose an efficient algorithm for the nonlinear minimization problem (2.2). Since the forward solution $u(g, q)$ is nonlinear with respect to both $g(x)$ and $q(x)$, the minimization (2.2) is also non-convex, which is generally more difficult to solve than a convex minimization problem. Therefore, we first reformulate the non-convex minimization (2.2) into a sequence of convex minimization problems using the Levenberg–Marquardt iterative method. Then, the surrogate functional method is applied to the resulting convex problems in order to obtain explicit expressions for the minimizers.
We shall linearize the forward solution, then apply the Levenberg-Marquardt method to solve (2.2), which changes the non-convex minimization (2.2) into convex minimization. For a given $(\bar{g}, \bar{q}) \in K_1 \times K_2$, we linearize the forward operator $u(g, q)$ around $(\bar{g}, \bar{q})$ using the first-order Taylor expansion:

$$u(g, q) \approx u(\bar{g}, \bar{q}) + u'_g(\bar{g}, \bar{q})(g - \bar{g}) + u'_q(\bar{g}, \bar{q})(q - \bar{q}), \tag{2.8}$$

where u'_g and u'_q are the Fréchet derivatives of u with respect to g and q, respectively. and solve the minimization system (2.2) by the following Levenberg-Marquardt iterative minimization

$$\begin{aligned} J(g^{k+1}, q^{k+1}) &= \min_{(g,q)\in K_1\times K_2} J(g, q) \\ &\equiv \int_0^T \big\| u'_g(g^k, q^k)(g - g^k) + u'_q(g^k, q^k)(q - q^k) \\ &\qquad - [z^\delta - u(g^k, q^k)] \big\|^2_{L^2(\Gamma_3)} dt + \beta \|g\|^2_{L^2(\Gamma_1)} + \eta \|q\|^2_{L^2(\Gamma_2)}. \end{aligned} \tag{2.9}$$

As equation (2.9) is a simple convex quadratic minimization problem, we can write its necessary optimality conditions as follows:

$$\frac{\partial J}{\partial g}(g^{k+1}, q^{k+1})d = 0 \quad \text{for any } d \in L^2(\Gamma_1), \quad \text{and} \quad \frac{\partial J}{\partial q}(g^{k+1}, q^{k+1})p = 0 \quad \text{for any } p \in L^2(\Gamma_2).$$

$$\begin{aligned} \frac{\partial J}{\partial g}(g; q)d = 2\int_0^T & \langle u'_g(g^k; q^k)(g - g^k) + u'_q(g^k; q^k)(q - q^k) - (z^\delta - u(g^k; q^k)), \\ & u'_g(g^k; q^k)d\rangle_{\Gamma_3}\, dt + 2\beta\langle g, d\rangle_{\Gamma_1}, \end{aligned}$$

$$\frac{\partial J}{\partial q}(g;q)p = 2\int_0^T \langle u_g'(g^k;q^k)(g-g^k) + u_q'(g^k;q^k)(q-q^k) - (z^\delta - u(g^k;q^k)),\\ u_q'(g^k;q^k)p\rangle_{\Gamma_3}\,dt + 2\eta\langle q,p\rangle_{\Gamma_2}.$$

One may see that we need to use some iterative method to get the minimizer (g^{k+1}, q^{k+1}) from the above systems, which is still difficult. Therefore, we would like to apply the surrogate functional technique to further simplify the numerical solution of the minimization (2.2).

Now, we introduce a surrogate functional $J^s(g,q)$ of $J(g,q)$:

$$J^s(g,q) = J(g,q) + A\|g-g^k\|_{\Gamma_1}^2 + B\|q-q^k\|_{\Gamma_2}^2 \\ - \int_0^T \|u_g'(g^k,q^k)(g-g^k) + u_q'(g^k,q^k)(q-q^k)\|_{\Gamma_3}^2\,dt, \tag{2.10}$$

where A and B are two positive constants such that

$$A\|d\|_{\Gamma_1}^2 + B\|p\|_{\Gamma_2}^2 - \int_0^T \|u_g'(g^k,q^k)d + u_q'(g^k,q^k)p\|_{\Gamma_3}^2\,dt \geq 0,$$

for any $d \in L^2(\Gamma_1)$, $p \in L^2(\Gamma_2)$, to guarantee the positivity of $J^s(\gamma;h)$. Indeed, we only need to choose

$$A \geq 2\|u_g'(g^k,q^k)\|_{\mathcal{L}(L^2(\Gamma_1),L^2(\Gamma_3))}^2, \quad B \geq 2\|u_q'(g^k,q^k)\|_{\mathcal{L}(L^2(\Gamma_2),L^2(\Gamma_3))}^2.$$

Here, $\|\cdot\|_{\mathcal{L}(U,V)}$ denotes the norm of a bounded linear operator from space U to V. The boundedness of u_g' and u_q' follows from Lemma 3.1.

Using the adjoint relations (3.5) and (3.6), we can rewrite $J^s(\gamma,h)$ as follows:

$$\begin{aligned} J^s(g,q) = &-2\int_0^T \langle u_g'(g^k,q^k)(g-g^k) + u_q'(g^k,q^k)(q-q^k), z^\delta - u(g^k,q^k)\rangle_{\Gamma_3}\,dt \\ &+ \beta\|g\|_{\Gamma_1}^2 + \eta\|q\|_{\Gamma_2}^2 + A\|g-g^k\|_{\Gamma_1}^2 + B\|q-q^k\|_{\Gamma_2}^2 \\ &+ \int_0^T \|z^\delta - u(g^k,q^k)\|_{\Gamma_3}^2\,dt \\ = &-2\langle g-g^k, \int_0^T h(x,t)u_g'(g^k,q^k)^*(z^\delta - u(g^k,q^k))\,dt\rangle_{\Gamma_1} \\ &- 2\langle q-q^k, \int_0^T R(x,t)u_q'(g^k,q^k)^*(z^\delta - u(g^k,q^k))\,dt\rangle_{\Gamma_2} \\ &+ (\beta+A)\|g-g^k\|_{\Gamma_1}^2 + (\eta+B)\|q-q^k\|_{\Gamma_2}^2 \\ &+ \int_0^T \|z^\delta - u(g^k,q^k)\|_{\Gamma_3}^2\,dt. \end{aligned} \tag{2.11}$$

As this is a quadratic minimization with respect to g and q, so we use the necessary conditions $\dfrac{\partial J^s}{\partial g}(g^*, q^*) = \dfrac{\partial J^s}{\partial q}(g^*, q^*) = 0$ to find the exact minimizer (g^*, q^*):

$$
\begin{aligned}
g^{k+1} &= g^k + \frac{1}{\beta + A} \int_0^T h(x,t) \cdot u'_g(g^k, q^k)^* (z^\delta - u(g^k, q^k))\, dt, \quad x \in \Gamma_1, \\
q^{k+1} &= q^k + \frac{1}{\eta + B} \int_0^T R(x,t) \cdot u'_q(g^k, q^k)^* (z^\delta - u(g^k, q^k))\, dt, \quad x \in \Gamma_2,
\end{aligned}
$$

Now, we are ready to propose an efficient algorithm for the simultaneous reconstruction of g and q.

Algorithm 2.1.

Step 1. Initialize g^0, q^0, set tolerance parameters $\varepsilon_1, \varepsilon_2$.

Step 2. At iteration k, solve the forward problem to get $u(g^k, q^k)$.

Step 3. Solve the adjoint problems to get $u'_g(g^k, q^k)^*(z^\delta - u(g^k, q^k))$ and $u'_q(g^k, q^k)^*(z^\delta - u(g^k, q^k))$.

Step 4. Update g^{k+1} and q^{k+1} using the explicit formulas.

Step 5. If the relative error satisfies

$$
\frac{\|g^{k+1} - g^k\|_{L^2(\Gamma_1)}}{\|g^k\|} < \varepsilon_1 \quad \text{and} \quad \frac{\|q^{k+1} - q^k\|_{L^2(\Gamma_2)}}{\|q^k\|} < \varepsilon_2,
$$

then stop; otherwise, set $k \leftarrow k + 1$ and repeat.

Chapter 3

Numerical Experiments

In this chapter, we apply algorithm 2.1 that was proposed in the previous chapter to simultaneously identify the Robin boundary coefficient $g(x)$ and the heat flux function $q(x)$ in the parabolic system (2.1) by using MATLAB. The computational domain is chosen as $\Omega = (0,1) \times (0,2)$, with the boundary split as follows:

$$\Gamma_1 = \{(x,y) \in \partial\Omega \,|\, x = 1,\, 0 \le y \le 2\}, \quad \Gamma_2 = \{(x,y) \in \partial\Omega \,|\, y = 0,\, 0 \le x \le 1\},$$

$$\Gamma_3 = \partial\Omega \setminus (\Gamma_1 \cup \Gamma_2).$$

The spatial domain is discretized using a structured triangular mesh derived from dividing each rectangular cell along its diagonal. The forward and adjoint problems are solved using linear finite element methods.

We take the source term, time-dependent heat flux factor, and boundary temperature function as:

$$f(x,t) = (xy+1)t + t, \qquad R(x,t) = x^2 + t, \qquad h(x,t) = 1 - y - 3t.$$

Noisy measurement data z^δ on Γ_3 is generated by solving the forward problem with exact parameters and adding 2% uniform random noise:

$$z^\delta = u + \delta R u, \quad \text{where } \delta = 0.01.$$

We fix regularization parameters $\beta = \eta = 10^{-4}$, surrogate weights $A = B = 1$, and stopping tolerances $\varepsilon_1 = \varepsilon_2 = 2\times 10^{-4}$. Then we present two numerical tests for simultaneous reconstruction of the Robin boundary coefficient $g(x)$ and the heat flux function $q(x)$ in the parabolic system (2.1), where we take $\gamma(x) = 1$ and the initial guesses are chosen as:

$$g^0(x) = g_{\text{exact}}(x), \quad q^0(x) = q_{\text{exact}}(x) \quad \text{for all mesh points.}$$

Example 1:

We define the exact functions to be recovered as:

$$g_{\text{exact}}(x) = 1 + \frac{1}{4}y^2, \quad q_{\text{exact}}(x) = 1 + \frac{1}{2}\sin\left(\frac{\pi}{2}x\right).$$

Figure 3.1 shows the exact coefficients and the numerical simultaneously recovered coefficients by Algorithms 2.1.

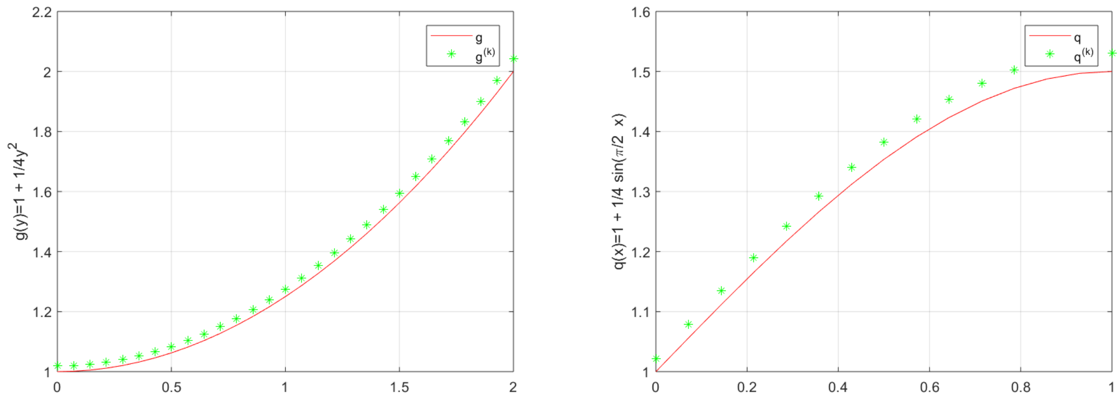

Figure 3.1: Exact and reconstructed Robin coefficient (left) and flux coefficient (right)(Example 1)

After 11 iterations of Algorithm 2.1, the relative errors between the exact and reconstructed coefficients are:

$$\frac{\|g^{11} - g_{\text{exact}}\|_{L^2(\Gamma_1)}}{\|g_{\text{exact}}\|_{L^2(\Gamma_1)}} = 0.0201, \qquad \frac{\|q^{11} - q_{\text{exact}}\|_{L^2(\Gamma_2)}}{\|q_{\text{exact}}\|_{L^2(\Gamma_2)}} = 0.0210.$$

Example 2:

In this example, we choose more nonlinear target functions:

$$g_{\text{exact}}(x) = 2 - \frac{1}{4}y^2, \quad q_{\text{exact}}(x) = 2 + (x-1)^2.$$

Figure 3.2 shows the exact coefficients and the numerical simultaneously recovered coefficients by Algorithms 2.1.

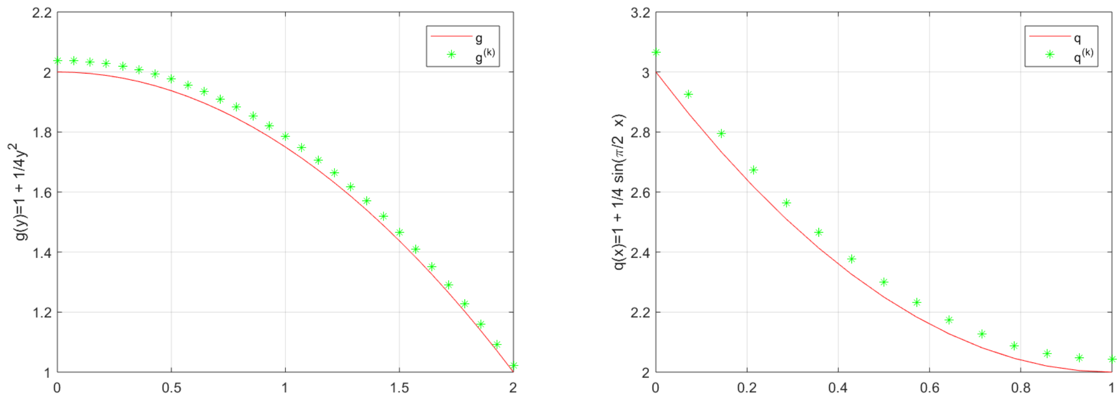

Figure 3.2: Exact and reconstructed Robin coefficient (left) and flux coefficient (right) (Example 2)

After 13 iterations of Algorithm 2.1, the relative errors between the exact and reconstructed coefficients are:

$$\frac{\|g^{13} - g_{\text{exact}}\|_{L^2(\Gamma_1)}}{\|g_{\text{exact}}\|_{L^2(\Gamma_1)}} = 0.0199, \qquad \frac{\|q^{13} - q_{\text{exact}}\|_{L^2(\Gamma_2)}}{\|q_{\text{exact}}\|_{L^2(\Gamma_2)}} = 0.0217.$$

3.1 REMARKS

Remark 3.1.1 *Since the norms $|u'_g(g^k, q^k)|^2_{L^2(\Gamma_1)\to L^2(\Gamma_3)}$ and $|u'_q(g^k, q^k)|^2_{L^2(\Gamma_2)\to L^2(\Gamma_3)}$ are difficult to estimate directly, we ensure the positivity of $J^s(g, q)$ by requiring the surrogate constants A and B to satisfy:*

$$A \geq 2\|u'_g(g^k, q^k)\|^2_{L^2(\Gamma_1)\to L^2(\Gamma_3)}$$
$$B \geq 2\|u'_q(g^k, q^k)\|^2_{L^2(\Gamma_2)\to L^2(\Gamma_3)}$$

In practice, we initialize with $A_0 = B_0 = 1$ *and adaptively increase these values until convergence is achieved.*

Conclusion

In this work, we have investigated the numerical identification of boundary coefficients in a parabolic system, focusing on the simultaneous recovery of the Robin boundary coefficient $g(x)$ and the heat flux $q(x)$ from noisy boundary measurements. The study addressed the inherent ill-posedness of the inverse problem by employing Tikhonov regularization and the Levenberg-Marquardt (LM) method combined with a surrogate functional strategy.

First, we established the well-posedness of the forward problem and analyzed the stability and convergence of the regularized formulation. Second, we developed a computational algorithm based on the finite element method for spatial discretization and the backward Euler scheme for time integration. The proposed method demonstrated robustness in handling noisy data and achieved accurate reconstructions of the unknown boundary functions.

Numerical experiments validated the effectiveness of the approach, showing successful recovery of $g(x)$ and $q(x)$ with relative errors below 2.2% in both test cases. These results demonstrate that regularization and optimization methods can effectively solve challenging inverse problems in heat transfer.

Future Work

This work leads to several important next questions for future research:

- **Time-dependent coefficients:** Develop techniques for reconstructing time-varying boundary coefficients $g(x,t)$ and $q(x,t)$.
- **Machine learning enhancements:** Investigate hybrid approaches combining traditional regularization with neural networks for improved coefficient prediction.

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ملخص

يقدم هذا العمل منهجية لإعادة بناء معاملين حدوديين بشكل متزامن في مسألة عكسية. لمعالجة عدم الاستقرار الجوهري في المسألة، نطبق Tikhonov regularization لضمان حلول موثوقة وإثبات وجود هذه الحلول. يتم حل مشكلة التحسين غير الخطية باستخدام طريقة Levenberg-Marquardt التي تحول المسألة إلى صيغة محدبة، بينما توفر surrogate functionals حلولاً صريحة في كل تكرار. تظهر التجارب العددية كفاءة المنهجية الحسابية وفعاليتها حتى في وجود بيانات مشوشة.

الكلمات المفتاحية: مسألة عكسية ، Tikhonov regularization، Levenberg-Marquardt method، surrogate functional

Résumé

Ce travail présente une méthode pour la reconstruction simultanée de deux coefficients aux limites dans un problème inverse. Pour remédier à l'instabilité inhérente, nous appliquons une régularisation de Tikhonov afin d'obtenir des solutions fiables et démontrons l'existence de solutions. L'optimisation non linéaire est résolue à l'aide de la méthode de Levenberg-Marquardt, qui transforme le problème en une forme convexe, tandis que des fonctionnelles substitutives fournissent des solutions explicites à chaque itération. Les tests numériques démontrent l'efficacité computationnelle de cette approche et sa robustesse même avec des données bruitées.

Mots-clés : problème inverse, régularisation de Tikhonov, méthode de Levenberg-Marquardt, fonctionnelle substitutive.

Abstract

This work presents a method for simultaneously reconstructing two boundary coefficients in an inverse problem. To address the inherent instability, we apply Tikhonov regularization to ensure reliable solutions and prove the existence of solutions. The nonlinear optimization is solved using the Levenberg-Marquardt method, which converts the problem to a convex form, while surrogate functionals provide explicit solutions at each iteration. Numerical experiments demonstrate the approach's computational efficiency and effectiveness even with noisy data.

Key words: inverse problem**,** Tikhonov regularization, Levenberg-Marquardt method, surrogate functional.