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Bernoulli numbers and Riemann zeta function

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Dedication

I dedicate this humble work, the fruit of my effort and years of perseverance, to those who planted the seeds of ambition in my heart and nurtured them with their love and prayers until they bore the fruit of success... To my dear parents, with all gratitude and loyalty.

To my wonderful brothers and sisters, who shared with me the dream, the hardship, and the joy you are part of this achievement.

To my esteemed family, who were a warm embrace and constant support I thank you deeply for your unwavering encouragement.

To my honorable teachers, who generously shared their knowledge and guidance you have my utmost respect and appreciation.

Hoping that success will always accompany our paths, by the grace of God.

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Abstract

In this thesis, we aim to compute the values of the zeta function at positive even integers using Bernoulli numbers, and to analytically continue it in order to obtain its values at negative even integers.

Keywords: zeta function, Bernoulli numbers, Bernoulli polynomials.

Résumé

Dans ce mémoire, nous visons à calculer les valeurs de la fonction zêta aux entiers pairs positifs en utilisant les nombres de Bernoulli, et à l'étendre analytiquement afin d'en déduire ses valeurs aux entiers pairs négatifs.

Mots-clés : Fonction zêta, les nombres de Bernoulli, les polynômes de Bernoulli.

ملخص

في هذه المذكرة نهدف إِلَى حسَاب قيم الدَالة زيتَا عند الأَعدَاد الصحيحة الموجبة الزوجية بَاستعمَال أَعدَاد برنولي، وتمديدهَا تحليليَا مع استخلَاص جذورهَا الزوجية السَالبة.

الكلمَات المفتَاحية: دَالة زيتَا، أَعدَاد برنولي، كثيرَات حدود برنولي.

Contents

Introduction

Bernoulli polynomials are mathematical functions expressed as polynomials, and they appear in summation formulas and expansions in mathematical analysis. They are widely used in approximating integrals and calculating errors. Bernoulli numbers are special values of these polynomials and play a vital role in mathematics.

On the other hand, the Riemann zeta function is one of the most prominent functions in complex analysis and number theory. Remarkably, this function is closely related to Bernoulli numbers.

This study aims to compute the values of the Riemann zeta function using Bernoulli numbers through three interconnected phases:

In the first chapter, we present the definition of Bernoulli numbers and their polynomials using the recursive relation that characterizes them.

The second chapter is dedicated to studying Bernoulli functions and their properties.

The third chapter focuses on how Bernoulli numbers are used to compute the values of the Riemann zeta function, providing the proof and related formulas.

This study adopts an analytical approach based on the precise formulation of mathematical relations, supported by examples and proofs, and makes use of tools from complex analysis and infinite series.

Chapter 1

Bernoulli Numbers And Polynomials Via Recurrence Relation

In this chapter, we aim to introduce the Bernoulli numbers and polynomials, along with some of their distinctive properties, the most important theorem concerning the axis and center of symmetry of Bernoulli polynomials, and the study of specific Bernoulli polynomials on the interval $[0, 1]$, representing them graphically for further clarification. Let us begin with the following theorem:

1.1 Bernoulli Numbers and Polynomials

Theorem 1.1.1. *[Theorem and Definition][2, 4]*

There exists an unique sequence of real polynomials $\{p_r(x)\}_{r\in\mathbb{N}}$ *that satisfies the following conditions:*

$$\begin{cases} p_0(x) = 1, \\ p_r'(x) = rp_{r-1}(x) \quad or \quad p_r(x) = r\int_0^x p_{r-1}(t)dt + B_r, \quad (r \geq 1) \\ \int_0^1 p_r(t)dt = 0, \quad (r > 0) \end{cases} \tag{1.1}$$

for every natural number r, *the polynomial* $p_r(x)$ *is monic of degree* r.

B_r *and* $p_r(x)$ *are called, respectively, the Bernoulli number and Bernoulli polynomial of index* r.

Proof.

(A) Existence. The Bernoulli numbers and polynomials are constructed starting from $p_0(x) = 1$ using Equation (1.1), as follows:

$$p_0(x) = 1, \quad B_0 = 1,$$

$$p_1(x) = x - \frac{1}{2}, \quad B_1 = p_1(0) = -\frac{1}{2},$$

$$p_2(x) = x^2 - x + \frac{1}{6}, \quad B_2 = p_2(0) = \frac{1}{6},$$

$$p_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \quad B_3 = p_3(0) = 0,$$

$$p_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}, \quad B_4 = p_4(0) = -\frac{1}{30},$$

$$p_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x, \quad B_5 = p_5(0) = 0,$$

$$p_6(x) = x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42}, \quad B_6 = p_6(0) = \frac{1}{42}, \quad \cdots$$

(B) Uniqueness. We assume the existence of a sequence of polynomials $\{q_r(x)\}_{r\in\mathbb{N}} \subset \mathbb{R}[x]$ that satisfies the previously mentioned conditions. We want to prove by induction on r the following:

$$q_r(x) = p_r(x) \quad (\forall r \in \mathbb{N}).$$

- For r= 0, we have $q_0(x) = p_0(x) = 1 \quad$ (holds true).
- For r= 1: $q_1(x) = p_1(x) = x - \frac{1}{2} \quad$ (holds true).

Now, we assume that $q_r(x) = p_r(x)$. Thus,

$$\begin{aligned} q_{r+1}(x) &= (r+1)\int_0^x q_r(t)dt + C_{r+1} \\ &= (r+1)\int_0^x p_r(t)dt + C_{r+1}. \end{aligned}$$

On the other hand, we have

$$p_{r+1}(x) = (r+1)\int_0^x p_r(t)dt + B_{r+1}.$$

From the last condition in (1.1), we find:

$$\begin{aligned} C_{r+1} &= \int_0^1 \left(-(r+1)\int_0^x p_r(t)dt\right) dx \\ &= B_{r+1}. \end{aligned}$$

Therefore $p_{r+1}(x) = q_{r+1}(x)$.

Thus

$$p_r(x) = q_r(x) \quad (\forall r \in \mathbb{N}).$$

(C) $p_r(x)$ **is a monic of degree** r**.** It is clear that $p_0(x) = 1$ is monic of degree 0. Similarly, $p_1(x) = x - \frac{1}{2}$ is monic of degree 1.

Assume that $p_r(x)$ is monic of degree r. Thus, $p_r(x)$ can be written as:

$$\begin{aligned} p_r(x) &= x^r + a_{r-1}x^{r-1} + \cdots + a_1 x + a_0 \\ &= x^r + \sum_{i=1}^{r} a_{r-i}x^{r-i}. \end{aligned}$$

We know that:

$$p_{r+1}(x) = (r+1)\int_0^x p_r(t)dt + B_{r+1}.$$

Thus,

$$\begin{aligned}
p_{r+1}(x) &= (r+1)\int_0^x t^r dt + (r+1)\left(\sum_{i=1}^{r} a_{r-i}\int_0^x t^{r-i}dt\right) + B_{r+1}\\
&= r+1\left[\frac{t^{r+1}}{r+1}\right]_0^x + (r+1)\left(\sum_{i=1}^{r} a_{r-i}\int_0^x t^{r-i}dt\right) + B_{r+1}\\
&= (r+1)\frac{x^{r+1}}{(r+1)} + (r+1)\left(\sum_{i=1}^{r} a_{r-i}\left[\frac{t^{r-i+1}}{r-i+1}\right]_0^x\right) + B_{r+1}\\
&= x^{r+1} + \sum_{i=1}^{r}\frac{(r+1)a_{r-i}x^{r-i+1}}{r-i+1} + B_{r+1}.
\end{aligned}$$

Hence, $p_{r+1}(x)$ is monic of degree $r+1$. □

Lemma 1.1.1. *[4]*

Let $f(x)$ be a continuous function from $\mathbb{R}$ to $\mathbb{R}$. Then

1. *If the line $x = \frac{1}{2}$ is a symmetry axis for $f(x)$, then:*

$$\int_0^{\frac{1}{2}+x} f(t)dt = -\int_0^{\frac{1}{2}-x} f(t)dt + \int_0^1 f(t)dt, \quad (\forall x \in \mathbb{R}).$$

2. *If $s = \left(\frac{1}{2}, 0\right)$ is a symmetry point for $f(x)$, Then:*

$$\int_0^{\frac{1}{2}+x} f(t)dt = \int_0^{\frac{1}{2}-x} f(t)dt + \int_0^1 f(t)dt, \quad (\forall x \in \mathbb{R}).$$

Proof.

1. We have

$$\int_0^{\frac{1}{2}+x} f(t)dt = \int_{-\frac{1}{2}}^{x} f\left(\frac{1}{2}+u\right)du, \tag{1.2}$$

where $t = \frac{1}{2} + u$,

$$\int_0^{\frac{1}{2}-x} f(t)dt = -\int_{\frac{1}{2}}^{x} f\left(\frac{1}{2}-u\right)du,$$

where $t = \frac{1}{2} - u$, and

$$\int_0^{1} f(t)dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} f\left(\frac{1}{2}+u\right)du,$$

where $t = \frac{1}{2} + u$.

Since $x = \frac{1}{2}$ is a symmetry axis for $f(x)$, we get

$$f\left(\frac{1}{2} + u\right) = f\left(\frac{1}{2} - u\right), \quad (\forall u \in \mathbb{R}).$$

Thus,

$$\int_0^{\frac{1}{2}-x} f(t)dt = -\int_{\frac{1}{2}}^{x} f\left(\frac{1}{2} + u\right) du = \int_x^{\frac{1}{2}} f\left(\frac{1}{2} + u\right) du. \tag{1.3}$$

From (1.2) and (1.3) we find that

$$\begin{aligned}
\int_0^{\frac{1}{2}+x} f(t)dt + \int_0^{\frac{1}{2}-x} f(t)dt &= \int_{-\frac{1}{2}}^{x} f\left(\frac{1}{2} + u\right) du + \int_x^{\frac{1}{2}} f\left(\frac{1}{2} + u\right) du \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} f\left(\frac{1}{2} + u\right) du \\
&= \int_0^1 f(t)dt.
\end{aligned}$$

Thus,

$$\int_0^{\frac{1}{2}+x} f(t)dt = -\int_0^{\frac{1}{2}-x} f(t)dt + \int_0^1 f(t)dt \quad (\forall x \in \mathbb{R}).$$

2. If $t = \frac{1}{2} + u$, then we have

$$\int_0^{\frac{1}{2}+x} f(t)dt = \int_{-\frac{1}{2}}^{x} f\left(\frac{1}{2} + u\right) du$$

and

$$\int_0^1 f(t)dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} f\left(\frac{1}{2} + u\right) du.$$

Also

$$\begin{aligned}
\int_0^{\frac{1}{2}-x} f(t)dt &= -\int_{\frac{1}{2}}^{x} f\left(\frac{1}{2} - u\right) du \\
&= \int_x^{\frac{1}{2}} f(\frac{1}{2} - u)du,
\end{aligned}$$

where $t = \frac{1}{2} - u$. Thus

$$\int_0^{\frac{1}{2}+x} f(t)dt - \int_0^{\frac{1}{2}-x} f(t)dt = \int_{-\frac{1}{2}}^{x} f\left(\frac{1}{2} + u\right) du - \int_x^{\frac{1}{2}} f(\frac{1}{2} - u)du.$$

Since the point $s=\left(\frac{1}{2},0\right)$ the symmetry center of $f(x)$ we obtain

$$f\left(\frac{1}{2}+u\right)=-f\left(\frac{1}{2}-u\right)\quad(\forall u\in\mathbb{R}).$$

Thus

$$\int_0^{\frac{1}{2}+x}f(t)dt-\int_0^{\frac{1}{2}-x}f(t)dt=\int_{-\frac{1}{2}}^{x}f\left(\frac{1}{2}+u\right)du+\int_x^{\frac{1}{2}}f(\frac{1}{2}+u)du$$
$$=\int_{-\frac{1}{2}}^{\frac{1}{2}}f\left(\frac{1}{2}+u\right)du=\int_0^1 f(t).$$

The proof is finish. □

Theorem 1.1.2. *[1, 4]*

1. *For all $r\in\mathbb{N}-\{0,1\}$: $p_r(0)=p_r(1)=B_r$.*
2. *For all $r\in\mathbb{N}^*$: $p_r(x+1)-p_r(x)=rx^{r-1}$.*
3. *For each $r\in\mathbb{N}$: the polynomial $p_{2r}(x)$ has the line $x=\frac{1}{2}$ as an axis of symmetry, the polynomial $p_{2r+1}(x)$ has the point $s=\left(\frac{1}{2},0\right)$ as a center of symmetry with*
$$B_{2r+1}=p_{2r+1}(0)=p_{2r+1}(1)=p_{2r+1}\left(\frac{1}{2}\right)=0,\quad(\forall r\in\mathbb{N}^*).$$
4. *For all $r\in\mathbb{N}$: $p_r(x)=(-1)^r p_r(1-x)$.*
5. *For all $r\in\mathbb{N}$:*
$$p_r(x)=2^{r-1}\left[p_r\left(\frac{x}{2}\right)+p_r\left(\frac{x+1}{2}\right)\right].$$
6. *For all $r\in\mathbb{N}-\{0,1\}$:*
$$B_r=B_1+\sum_{k=2}^{r-1}\frac{(-1)^{r-k+1}C_r^{r-k}}{(r-k+1)}B_k+\frac{(-1)^r r}{r+1}.$$
7. *For all $r\in\mathbb{N}$:*
$$p_r(x)=\sum_{k=0}^{r}C_r^k B_k x^{r-k}$$
and for all $r\in\mathbb{N}-\{0,1\}$:
$$B_r=-\frac{1}{r+1}\sum_{k=0}^{r-1}C_{r+1}^k B_k.$$

Proof.

1. For every Bernoulli polynomial, we have the following relation: $p'_r(t) = rp_{r-1}(t)$ $(r \geq 2)$. By integrating over the interval $[0, 1]$, we obtain

$$\int_0^1 p'_r(t)dt = r\int_0^1 p_{r-1}(t)dt.$$

Thus, $p_r(1) - p_r(0) = 0$, which implies that:

$$p_r(0) = p_r(1) = B_r, \quad \forall r \geq 2.$$

2. If we put $H_r : p_r(x+1) - p_r(x) = rx^{r-1}$, then

$$H_1 : p(x+1) - p(x) = ((x+1) - \frac{1}{2}) - (x - \frac{1}{2}) = 1 \quad \text{(holds true)}.$$

Now, assume that H_r is true and we prove the validity of H_{r+1}.

H_r is true, then $p_r(x+1) - p_r(x) = rx^{r-1}$. By integrating over the interval $[0, x]$, we obtain:

$$\int_0^x p_r(t+1)dt - \int_0^x p_r(t)dt = \int_0^x rt^{r-1}dt.$$

Thus,

$$\int_1^{x+1} p_r(t)dt - \int_0^x p_r(t)dt = r\int_0^x t^{r-1}dt,$$

so

$$\int_1^{x+1} p_r(t)dt - \int_0^1 p_r(t)dt - \int_1^x p_r(t)dt = \int_0^x rt^{r-1}dt.$$

Hence

$$\int_x^{x+1} p_r(t)dt = x^r \Rightarrow p_{r+1}(x+1) - p_{r+1}(x) = (r+1)x^r. \tag{1.4}$$

Thus, H_{r+1} is true, so for all $r \in \mathbb{N}^*$:

$$p_r(x+1) - p_r(x) = rx^{r-1}.$$

3. We prove the property by induction on r. The property holds for $r = 0, 1, 2, 3$. It is clear that $x = \frac{1}{2}$ is the axis of symmetric for $p_0(x) = 1$, and the point $s = (\frac{1}{2}, 0)$ is the center0. of symmetry for $p_1(x) = x - \frac{1}{2}$. Likewise, we see that the polynomials $p_2(x) = x^2 - x + \frac{1}{2}$ and $p_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$ satisfy property 3. This is due to their satisfaction of the following equation:

$$p_2(\frac{1}{2} - x) = p_2(\frac{1}{2} + x) = x^2 - \frac{1}{4} \quad (x \text{ axis of symmetry on } \mathbb{R}).$$

$$p_3(\frac{1}{2}+x) = -p_3(\frac{1}{2}-x) = x^3 - \frac{1}{4}x \quad (s \text{ center of symmetry on } \mathbb{R}).$$

Thus, we prove the validity of the following two implications:

$$x = \frac{1}{2} \text{ is the axis of symmetric for } p_{2r}(x) \overset{1}{\Longrightarrow} s = (\frac{1}{2}, 0) \text{ is the point of symmetry for } p_{2r+1}$$
$$\overset{2}{\Longrightarrow} x = \frac{1}{2} \text{ is the axis of symmetric for } p_{2r+2}.$$

1. Proof of the first implication

   Since $x = \frac{1}{2}$ is the axis of symmetry for $p_{2r}(x)$ and also

$$\int_0^1 p_{2r}(t)dt = 0, \quad (r \geq 1),$$

   then according to Lemma 1.1.1, we have the following:

$$\int_0^{\frac{1}{2}+x} p_{2r}(t)dt = -\int_0^{\frac{1}{2}-x} p_{2r}(t)dt,$$

   it requires that

$$p_{2r+1}\left(\frac{1}{2}+x\right) = (2r+1)\int_0^{\frac{1}{2}+x} p_{2r}(t)dt + B_{2r+1}$$

   and we also know that

$$p_{2r+1}\left(\frac{1}{2}-x\right) = (2r+1)\int_0^{\frac{1}{2}-x} p_{2r}(t)dt + B_{2r+1},$$

   adding the other two equations side to side,

$$\begin{aligned}
p_{2r+1}\left(\frac{1}{2}-x\right) + p_{2r+1}\left(\frac{1}{2}+x\right) &= (2r+1)\int_0^{\frac{1}{2}-x} p_{2r}(t)dt + B_{2r+1} + (2r+1)\int_0^{\frac{1}{2}+x} p_{2r}(t)dt + B_{2r+1} \\
&= (2r+1)\left(\int_0^{\frac{1}{2}-x} p_{2r}(t)dt + \int_0^{\frac{1}{2}+x} p_{2r}(t)dt\right) + 2B_{2r+1} \\
&= (2r+1)\left(\int_0^{\frac{1}{2}-x} p_{2r}(t)dt - \int_0^{\frac{1}{2}-x} p_{2r}(t)dt\right) + 2B_{2r+1} \\
&= 2B_{2r+1}.
\end{aligned}$$

   We obtain

$$p_{2r+1}\left(\frac{1}{2}-x\right) + p_{2r+1}\left(\frac{1}{2}+x\right) = 2B_{2r+1}. \tag{1.5}$$

By integrating over the interval $[0, \frac{1}{2}]$,

$$\int_0^{\frac{1}{2}} p_{2r+1}\left(\frac{1}{2}-x\right)dx + \int_0^{\frac{1}{2}} p_{2r+1}\left(\frac{1}{2}+x\right)dx = \int_0^{\frac{1}{2}} 2B_{2r+1}dx$$
$$= 2B_{2r+1}\int_0^{\frac{1}{2}} dx$$
$$= 2B_{2r+1}[x]_0^{\frac{1}{2}} = B_{2r+1}.$$

We obtain

$$\int_0^{\frac{1}{2}} p_{2r+1}\left(\frac{1}{2}-x\right)dx + \int_0^{\frac{1}{2}} p_{2r+1}\left(\frac{1}{2}+x\right)dx = B_{2r+1}. \tag{1.6}$$

Then

$$\int_0^{\frac{1}{2}} p_{2r+1}(\frac{1}{2}+x)dx = \int_{\frac{1}{2}}^{1} p_{2r+1}(u)du \qquad \textit{where} \quad u = \frac{1}{2}+x.$$

Likewise

$$\int_0^{\frac{1}{2}} p_{2r+1}(\frac{1}{2}-x)dx = -\int_{\frac{1}{2}}^{0} p_{2r+1}(u)du \qquad \textit{where} \quad u = \frac{1}{2}-x.$$

Thus, it follows by using (1.6):

$$B_{2r+1} = \int_0^{\frac{1}{2}} p_{2r+1}(u)du + \int_{\frac{1}{2}}^{1} p_{2r+1}(u)du$$
$$= \int_0^1 p_{2r+1}(u)du = 0.$$

Thus

$$B_{2r+1} = p_{2r+1}(0) = p_{2r+1}(1) = 0$$

Hence, Equation (1.5) becomes as follows

$$p_{2r+1}\left(\frac{1}{2}-x\right) + p_{2r+1}\left(\frac{1}{2}+x\right) = 0 \quad \Longrightarrow \quad p_{2r+1}\left(\frac{1}{2}+x\right) = -p_{2r+1}\left(\frac{1}{2}-x\right).$$

This last equation is valid for every natural number ($r \geqslant 1$). This proves that the point $s = \left(\frac{1}{2}, 0\right)$ is the center of symmetry for $p_{2r+1}(x)$.

2. Proof of the second implication.

   Since point $s = \left(\frac{1}{2}, 0\right)$ is center of symmetry for p_{2r+1} and since,

$$\int_0^1 p_{2r+1}(x)dx = 0$$

which is true for every natural number $r \geqslant 1$. Lemma 1.1.1, allows us to get

$$\int_0^{\frac{1}{2}+x} p_{2r+1}(t)dt = \int_0^{\frac{1}{2}-x} p_{2r+1}(t)dt,$$

which implies that

$$\begin{aligned} p_{2r+2}\left(\frac{1}{2}+x\right) &= (2r+2)\int_0^{\frac{1}{2}+x} p_{2r+1}(t)dt + B_{2r+2} \\ &= p_{2r+2}\left(\frac{1}{2}-x\right). \end{aligned}$$

Thus

$$p_{2r+2}\left(\frac{1}{2}+x\right) = p_{2r+2}\left(\frac{1}{2}-x\right), \quad (\forall r \in \mathbb{N}).$$

This proves that the line $x = \frac{1}{2}$ axis of symmetry of p_{2r+2}. We have according to the above

$$p_{2r+1}\left(\frac{1}{2}+x\right) = -p_{2r+1}\left(\frac{1}{2}-x\right), \quad (\forall r \in \mathbb{N}^*) \quad (\forall x \in \mathbb{R}).$$

By taking $x = 0$, we find

$$p_{2r+1}\left(\frac{1}{2}\right) = 0, \text{i.e., } p_{2r+1}\left(\frac{1}{2}\right) = -p_{2r+1}\left(\frac{1}{2}\right)$$

and by taking $x = \frac{1}{2}$ we find

$$p_{2r+1}(1) = -p_{2r+1}(0) = -p_{2r+1}(1).$$

Thus,

$$B_{2r+1} = p_{2r+1}(0) = p_{2r+1}\left(\frac{1}{2}\right) = p_{2r+1}(1) = 0, \quad (\forall r \in \mathbb{N}^*).$$

4. We define for every natural number r: $q_r(x) = (-1)^r p_r(1-x)$. We have $q_0(x) = 1$ and $\forall r \in \mathbb{N}^*$:

$$\begin{aligned} q_r'(x) &= (-1)^{r+1} p_r'(1-x) \\ &= r(-1)^{r-1} p_{r-1}(1-x) \\ &= r q_{r-1}(x). \end{aligned}$$

On the other hand, $\forall r \in \mathbb{N}^*$:

$$\begin{aligned} \int_0^1 q_r(t)dt &= (-1)^r \int_0^1 p_r(1-t)dt \\ &= (-1)^r \int_0^1 p_r(u)du = 0. \end{aligned}$$

From the fact that $\{p_r(x)\}_{r\geqslant 0}$ is unique, we find that: $p_r(x) = q_r(x) \quad (\forall r \in \mathbb{N})$.

5. We define for every natural number $r \in \mathbb{N}$:

$$q_r(x) = 2^{r-1}\left[p_r\left(\frac{x}{2}\right) + p_r\left(\frac{x+1}{2}\right)\right].$$

We have

- For $r = 0$,

$$q_0(x) = \frac{1}{2}[1+1] = 1.$$

- $\forall r \in \mathbb{N}^*$:

$$\begin{aligned} q_r'(x) &= 2^{r-1}\left[\frac{1}{2}p_r'\left(\frac{x}{2}\right) + \frac{1}{2}p_r'\left(\frac{x+1}{2}\right)\right] \\ &= 2^{r-2}\left[rp_{r-1}\left(\frac{x}{2}\right) + rp_{r-1}\left(\frac{x+1}{2}\right)\right] \\ &= r2^{r-2}\left[p_{r-1}\left(\frac{x}{2}\right) + p_{r-1}\left(\frac{x+1}{2}\right)\right] \\ &= r2^{r-2}q_{r-1}(x). \end{aligned}$$

- $\forall r \in \mathbb{N}^*$:

$$\begin{aligned} \int_0^1 q_r(t)dt &= 2^{r-1}\int_0^1 p_r\left(\frac{t}{2}\right)dt + \int_0^1 p_r\left(\frac{t+1}{2}\right)dt \\ &= 2^{r-1}\left[\int_0^{\frac{1}{2}} p_r(u)du + \int_{\frac{1}{2}}^1 p_r(u)du\right] = 0. \end{aligned}$$

From the fact that $\{p_r(x)\}_{r\in\mathbb{N}}$ is unique, we find that:

$$(\forall r \in \mathbb{N}): \quad p_r(x) = q_r(x).$$

Thus,

$$(\forall r \in \mathbb{N}) \quad p_r(x) = 2^{r-1}\left[p_r\left(\frac{x}{2}\right) + p_r\left(\frac{x+1}{2}\right)\right].$$

6. We have $B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}$, and $B_{2r+1} = 0$ $(\forall r \in \mathbb{N}^*)$. Similarly, according to Theorem 1.1.1, we have

$$B_r = -r\int_0^1 \left(\int_0^x p_{r-1}(t)dt\right)dx \quad (r \geqslant 4).$$

To determine the expression for B_r, we use integration by parts. We set

$$\frac{dv}{dx} = 1 \qquad u(x) = \int_0^x p_{r-1}(t)dt,$$
$$v(x) = x \qquad \frac{du}{dx} = p_{r-1}(x),$$

accordingly, we write

$$\begin{aligned} B_r &= -r\left[x\int_0^x p_{r-1}(t)dt\right]_{x=0}^{x=1} + r\int_0^1 xp_{r-1}(t)dx \\ &= r\int_0^1 xp_{r-1}(t)dx, \end{aligned}$$

we calculate the value of the last integral using integration by parts $(r-2)$ times.

- The first stage. We set

$$\frac{dv}{dx} = x \qquad u(x) = p_{r-1}(x),$$
$$v(x) = \frac{x^2}{2} \qquad \frac{du}{dx} = (r-1)p_{r-2}(x),$$

accordingly, we get

$$\begin{aligned} B_r &= r\left[p_{r-1}(x).\frac{x^2}{2}\right]_{x=0}^{x=1} - \frac{r(r-1)}{2}\int_0^1 x^2 p_{r-2}(x)dx \\ &= \frac{r}{2}B_{r-1} - \frac{r(r-1)}{2}\int_0^1 x^2 p_{r-2}(x)dx \end{aligned}$$

since $p_r(1) = B_r$ $(\forall r \geqslant 2)$.

- The second stage. We set

$$\frac{dv}{dx} = x^2 \qquad u(x) = p_{r-2}(x),$$
$$v(x) = \frac{x^3}{3} \qquad \frac{du}{dx} = (r-2)p_{r-3}(x).$$

Thus,

$$\begin{aligned} B_r &= \frac{r}{2}B_{r-1} - x^3\frac{r(r-1)}{6}B_{r-2} + \frac{r(r-1)(r-2)}{6}\int_0^1 x^3 p_{r-3}(x)dx \\ &= \frac{C_r^1}{2}B_{r-1} - \frac{C_r^2}{3}B_{r-2} + \frac{(r-2)C_r^2}{3}\int_0^1 x^3 p_{r-3}(x)dx. \end{aligned}$$

Since $\frac{(n-k)C_n^k}{(k+1)} = C_n^{k+1}$ for $n, k \in \mathbb{N}$ and $0 \leq k \leq n$, then B_r is written as follows:

$$B_r = \frac{C_r^1}{2}B_{r-1} - \frac{C_r^2}{3}B_{r-2} + C_r^3\int_0^1 x^3 p_{r-3}(x)dx.$$

- The third stage. We set

$$\frac{dv}{dx} = x^3 \qquad u(x) = p_{r-3}(x),$$

$$v(x) = \frac{x^4}{4} \qquad \frac{du}{dx} = (r-3)p_{r-4}(x),$$

we find the expression of B_r is written as follows:

$$B_r = \frac{C_r^1}{2}B_{r-1} - \frac{C_r^2}{3}B_{r-2} + \frac{C_r^3}{4}B_{r-3} - C_r^4\int_0^1 x^4 p_{r-4}(x)dx$$

and so on, successively up to stage $(r-2)$, then we find that the expression of B_r can be written as:

$$\begin{aligned} B_r &= \frac{C_r^1}{2}B_{r-1} - \frac{C_r^2}{3}B_{r-2} + \frac{C_r^3}{4}B_{r-3} - \cdots + (-1)^r C_r^{r-1}\int_0^1 x^{r-1}p_1(x)dx \\ &= \sum_{k=1}^{r-2}\frac{(-1)^{k+1}C_r^k}{(k+1)}B_{r-k} + (-1)^r r\int_0^1 x^{r-1}p_1(x)dx. \end{aligned}$$

We assign the value of the last integral

$$\begin{aligned} (-1)^r r\int_0^1 P_1(x)x^{r-1}dx &= (-1)^r r\int_0^1 x^{r-1}\left(x-\frac{1}{2}\right)dx \\ &= (-1)^r r\left[\frac{1}{r}\left(\frac{-1}{2}\right) + \frac{1}{r+1}\right] \\ &= \frac{(-1)^r C_r^{r-1}}{r}B_1 + (-1)^r\frac{r}{r+1}. \end{aligned}$$

Thus,

$$\begin{aligned} B_r &= \sum_{k=1}^{r-1}\frac{(-1)^{k+1}C_r^k}{(k+1)}B_{r-k} + (-1)^r\frac{r}{r+1} \\ &= \sum_{k=1}^{r-1}\frac{(-1)^{r-k+1}C_r^{r-k}}{(r-k+1)}B_k + (-1)^r\frac{r}{r+1} \\ &= (-1)^r B_1 + \sum_{k=2}^{r-1}\frac{(-1)^{r-k+1}C_r^{r-k}}{(r-k+1)}B_k + (-1)^r\frac{r}{r+1} \\ &= (-1)^r B_1 + \frac{(-1)^{r-1}C_r^{r-2}}{(r-1)}B_2 + \cdots + \frac{(-1)^2 C_r^1}{2}B_{r-1} + (-1)^r\frac{r}{r+1}. \end{aligned}$$

7. If we put $H_r : p_r(x) = \sum\limits_{k=0}^{r} C_r^k B_k x^{r-k}$, then

$$\begin{aligned} H_1 : p_1(x) = x - \frac{1}{2} &= \sum_{k=0}^{1} C_1^k B_k x^{1-k} \\ &= C_0^1 B_0 x + C_1^1 B_1 \\ &= x - \frac{1}{2}, \end{aligned}$$

so H_1 is correct.

Now, suppose that H_r true, and we prove that H_{r+1} is true. Since H_r is true, then

$$p_r(x) = \sum_{k=0}^{r} C_r^k B_k x^{r-k}.$$

From which,

$$\begin{aligned} p_{r+1}(x) &= (r+1)\int_0^x p_r(t)dt + B_{r+1} \\ &= (r+1)\int_0^x \left(\sum_{k=0}^{r} C_r^k B_k t^{r-k}\right) dt + B_{r+1} \\ &= \sum_{k=0}^{r} \frac{(r+1)C_r^k}{(r+1-k)} B_k x^{r+1-k} + B_{r+1} \\ &= \sum_{k=0}^{r} C_{r+1}^k B_k x^{r+1-k} + C_{r+1}^{r+1} B_{r+1} x^{(r+1)-(r+1)} \\ &= \sum_{k=0}^{r} C_{r+1}^k B_k x^{r+1-k}. \end{aligned}$$

Hence H_{r+1} is true, i.e.,

$$\forall r \in \mathbb{N} : \quad p_r(x) = \sum_{k=0}^{r} C_r^k B_k x^{r-k}.$$

The last equality implies that

$$p_{r+1}(x) = \sum_{k=0}^{r+1} C_{r+1}^k B_k x^{r+1-k}.$$

Thus

$$\begin{aligned} p_{r+1}(1) = B_{r+1} = \sum_{k=0}^{r+1} C_{r+1}^{k} B_k &= \sum_{k=0}^{r} C_{r+1}^{k} B_k + B_{r+1} \\ \Longrightarrow \sum_{k=0}^{r} C_{r+1}^{k} B_k &= 0 \\ \Longrightarrow \sum_{k=0}^{r-1} C_{r+1}^{k} B_k + (r+1)B_r &= 0. \end{aligned}$$

Therefore

$$B_r = -\frac{1}{r+1}\sum_{k=0}^{r-1} C_{r+1}^{k} B_k, \quad (\forall r \geqslant 2),$$

as example, the first 16 nonzero Bernoulli numbers are given as follows:

$$B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30},$$
$$B_{10} = \frac{5}{66}, \quad B_{12} = -\frac{691}{2730}, \quad B_{14} = \frac{7}{6}, \quad B_{16} = -\frac{3617}{510},$$
$$B_{18} = \frac{43867}{798}, \quad B_{20} = -\frac{174611}{330}, B_{22} = \frac{854513}{138},$$
$$B_{24} = -\frac{263364091}{2730}, \quad B_{26} = \frac{8553103}{6},$$
$$B_{28} = -\frac{23749461029}{870}, \quad B_{30} = \frac{8615841276005}{14322}.$$

The proof is finish. □

Remark

From Equation (1.4) we have

$$\forall r \in \mathbb{N}: \quad \int_x^{x+1} p_r(t)dt = x^r.$$

Hence

$$\begin{aligned} \sum_{x=0}^{n} x^r = \sum_{x=0}^{n} \int_x^{x+1} p_r(t)dt &= \int_0^{n+1} p_r(t)dt \\ &= \left[\frac{p_{r+1}(t)}{r+1}\right]_{t=0}^{t=n+1} \\ &= \frac{p_{r+1}(n+1) - p_{r+1}(0)}{r+1}. \end{aligned}$$

From which the following sums can be calculated

$$\sum_{k=1}^{n} k = \frac{p_2(n+1) - p_2(0)}{2}$$
$$= \frac{1}{2}\left((n+1)^2 - (n+1) + \frac{1}{6} - \frac{1}{6}\right)$$
$$= \frac{n(n+1)}{2}$$

and

$$\sum_{k=1}^{n} k^2 = \frac{p_3(n+1) - p_3(0)}{3}$$
$$= \frac{1}{3}\left((n+1)^3 - \frac{3}{2}(n+1)^2 + \frac{1}{2}(n+1)\right)$$
$$= \frac{n(n+1)(2n+1)}{6}.$$

1.2 Bernoulli Polynomials on the Interval [0,1]

Theorem 1.2.1. *[4]*

1. *For every $r \in \mathbb{N}^*$, the polynomial $p_{2r}(x)$ has a unique maximum value in the interval $[0,1]$ at $x = \frac{1}{2}$ and its sign is opposite to the sign of $p_{2r}(1) = p_{2r}(0)$, and the polynomial $p_{2r+1}(x)$ has exactly three zeros in the interval $[0,1]$ at the values $0, \frac{1}{2}$, and 1. It attains two extreme values of opposite signs in the interval $[0,1]$, which are symmetric with respect to the point $s = \left(\frac{1}{2}, 0\right)$.*

2. *For every $r \in \mathbb{N}^*$, we have for all $x \in [0,1]$*

$$(-1)^{r-1}(B_{2r} - p_{2r}(x)) \geqslant 0 \text{ and } (-1)^{r-1}B_{2r} > 0.$$

Proof.

1. We first verify that the polynomials

$$p_2(x) = x^2 - x + \frac{1}{6},\ p_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x,$$

$$p_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30},\ \text{and } p_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x$$

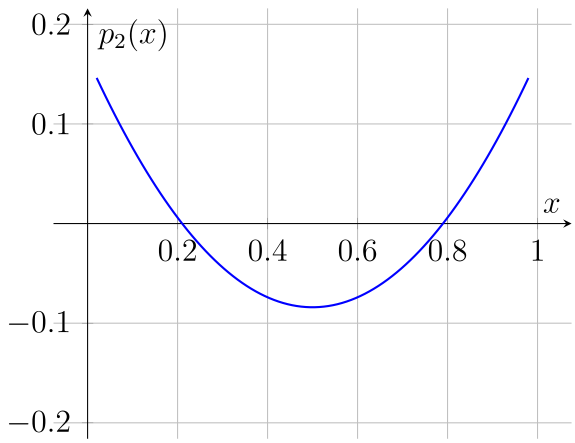
verify the previously mentioned condition in Theorem 1.2.1.

Here are its variation table and graphical representations on the interval [0,1].

$$p_2(x) = x^2 - x + \frac{1}{6} \quad \Longrightarrow \quad p_2'(x) = 2x - 1$$

x	0		$\frac{1}{2}$		1
$p_2'(x)$		-	0	$+$	
$p_2(x)$	$B_2 = \frac{1}{6}$	$\searrow$	$p_2(\frac{1}{2}) = -\frac{1}{12}$	$\nearrow$	$B_2 = \frac{1}{6}$

Graphical representations of $p_2(x)$ on the interval [0,1]:

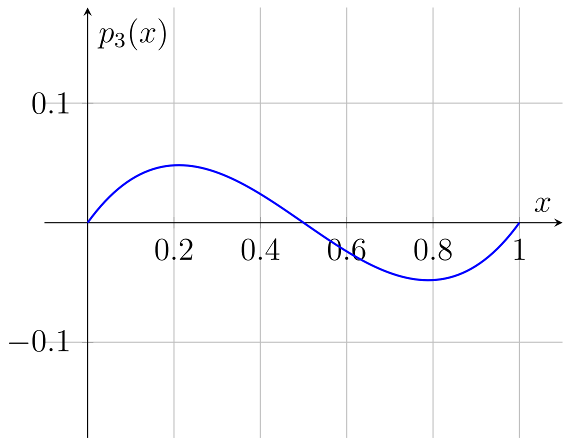


$Figure(1)$

$$p_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x \quad \Longrightarrow \quad p_3'(x) = 3x^2 - 3x + \frac{1}{2}$$

x	0		a		$\frac{1}{2}$		b		1
$p_3'(x)$		$+$		-		-		$+$	
$p_3(x)$	$B_5 = 0$	$\nearrow$	$p_3(a) > 0$	$\searrow$	$p_3(\frac{1}{2}) = 0$	$\searrow$	$p_3(b) < 0$	$\nearrow$	$B_3 = 0$

Graphical representations of $p_3(x)$ on the interval [0,1]:

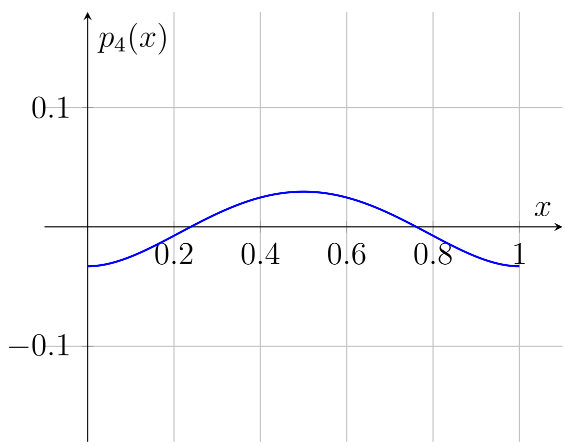


$Figure(2)$

$$p_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30} \quad \Longrightarrow \quad p_4'(x) = 4x^3 - 6x^2 + 2x$$

x	0		$\frac{1}{2}$		1
$p_4'(x)$		+	0	-	
$p_4(x)$	$B_4 = -\frac{1}{30}$	$\nearrow$	$p_4(\frac{1}{2}) = \frac{7}{240}$	$\searrow$	$B_4 = -\frac{1}{30}$

Graphical representations of $p_4(x)$ on the interval [0,1]:

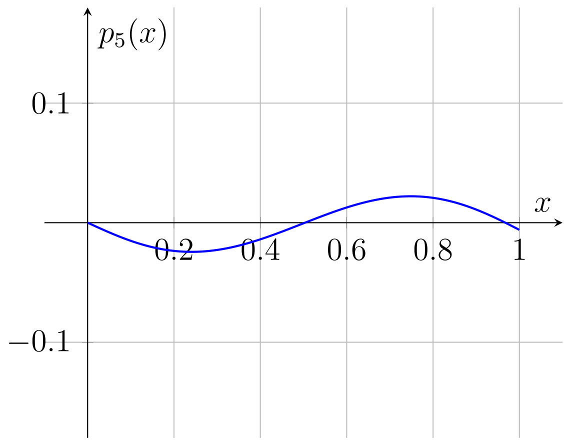


$Figure(3)$

$$p_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x \quad \Longrightarrow \quad p_5'(x) = 5x^4 - 10x^3 + 5x^2 - \frac{1}{6}$$

x	0		c		$\frac{1}{2}$		d		1
$p_5'(x)$		-		+		+		-	
$p_5(x)$	$B_5 = 0$	$\searrow$	$p_5(c) < 0$	$\nearrow$	$p_5(\frac{1}{2}) = 0$	$\nearrow$	$p_5(d) > 0$	$\searrow$	$B_5 = 0$

Graphical representations of $p_5(x)$ on the interval [0,1]:



$Figure(4)$

Now, by using

$$p'_r(x) = rp_{r-1}(x)\ (r \geq 2)$$

we prove that the following implications hold.

$p_{2r}(x)$ satisfies the condition of hypothesis $\overset{1}{\Longrightarrow} p_{2r+1}(x)$ also satisfies $\overset{2}{\Longrightarrow} p_{2r+2}(x)$ it also satisfies.

A)$\overset{1}{\Longrightarrow}$

Since $p_{2r}(x)$ has a single nonzero maximum at $x = \frac{1}{2}$ and its sign is opposite to that of $p_{2r}(0)$, then $p_{2r}(x)$ has exactly two in the interval [0,1] at two symmetric points with respect to the line $x = \frac{1}{2}$. Since $p'_{2r+1}(x) = (2r+1)p_{2r}(x)$ also vanishes exactly twice in the interval $]0,1[$ therefore, $p'_{2r+1}(x)$ has two extreme values in the interval $]0,1[$, which are symmetric with respect to the point $s = (\frac{1}{2}, 0)$, and it changes its sign exactly once in the interval $]0,1[$. Hence it vanishes three times at the values $x = 0, x = \frac{1}{2}$, and $x = 1$.

B) $\overset{2}{\Longrightarrow}$

Since $p'_{2r+2}(x) = (2r+2)p_{2r+1}(x)$ vanishes exactly once at $x = \frac{1}{2}$ in the interval $]0,1[$, then $p_{2r+2}(x)$ has a single maximum value in this interval at $x = \frac{1}{2}$.

We now prove that $p_{2r+2}(\frac{1}{2})$ is nonzero and sign opposite to that of $p_{2r+2}(0) = p_{2r+2}(1)$ by using proof by contrapositive. We assume that: $p_{2r+2}(\frac{1}{2}) \geqslant 0$ and $p_{2r+2}(0) = p_{2r+2}(1) \geqslant 0$, this implies that

$$p_{2r+2}(x) = \frac{p'_{2r+3}(x)}{2r+3} \geqslant 0$$

and this proves that $p_{2r+3}(x)$ is increasing on the interval [0,1]. Thus,

$$p_{2r+3}(0) \leqslant p_{2r+3}(x) \leqslant p_{2r+3}(1) = p_{2r+3}(0) \Longrightarrow p_{2r+3}(x) = 0 \quad (\forall x \in \mathbb{R})$$

and thus $p_{2r+3}(x)$ is vanishes in the interval [0,1], which is a contradiction. We reach the same contradiction if we assume that $p_{2r+2}(\frac{1}{2}) \leqslant 0$ and $p_{2r+2}(0) = p_{2r+2}(1) \leqslant 0$. Thus, necessarily $p_{2r+2}(\frac{1}{2}) \neq 0$ and has a sign opposite to that $p_{2r+2}(0)$. Thus $p_{2r+2}(x)$ vanishes exactly twice in the interval [0,1].

conclusion: In the interval [0,1] and for $r \geqslant 1$, the polynomials $p_{4r}, p_{4r+2}, p_{4r+1}$, and p_{4r+3} take the following polynomials forms in this order p_4, p_2, p_5, and p_3.

2. We prove the inequality:

$$(\forall r \in \mathbb{N}^*) \qquad (-1)^{r-1} B_{2r} > 0.$$

By induction the hypothesis is true for $r = 1$, because $(-1)^{1-1} B_{2r} = \frac{1}{6} > 0$.

We assume that $(-1)^{r-1} B_{2r} > 0$ and show that $(-1)^r B_{2r+2} > 0$. For that, we consider two cases.

- **The first case** $(-1)^{r-1} > 0$ **and** $B_{2r} > 0$**.** In this case, the polynomial $p_{2r}(x)$ takes the same form as $p_2(x)$. Thus, $p_{2r+1}(x)$ takes the shape of $p_3(x)$ and $p_{2r+2}(x)$ takes the shape of $p_4(x)$, hence $B_{2r+2} < 0$.
  Since $(-1)^r < 0$, then $(-1)^r B_{2r+2} > 0$.

- **The second case** $(-1)^{r-1} < 0$ **and** $B_{2r} < 0$**.** In this case, the polynomial $p_{2r}(x)$ takes the same shape as $p_4(x)$. Thus, $p_{2r+1}(x)$ takes the shape of $p_5(x)$, and $p_{2r+2}(x)$ takes the shape of $p_2(x)$. Thus $B_{2r+2} > 0$ and since $(-1)^r > 0$, then $(-1)^r B_{2r+2} > 0$.
  The proof is completed.

□

Chapter 2

Bernoulli Functions

In this chapter, we will address the definition of Bernoulli function along with some of their distinctive properties their expansion using the Fourier series, and the introduction of the most important theorem. Let us begin with following definition:

2.1 Definition

Definition 2.1.1. *For every natural number r we let*

$$B_r(x) := p_r(\{x\}) \quad (x \in \mathbb{R}),$$

where $p_r(x)$ is the Bernoulli polynomial of order r and $\{x\}$ denotes the fractianal part of the real x. We call $B_r(x)$ the Bernoulli function of order r.

2.2 Some Properties of Bernoulli Functions

Theorem 2.2.1. *[4]*

1. *For every nonzero natural number r, the function $B_r(x)$ is periodic with period 1.*

2. *The function $B_1(x)$ is discontinuous at $x \in \mathbb{Z}$, and equal to $p_1(x)$ in the half-open interval $[0, 1[$.*

3. *For every natural number $r \geq 2$, the function $B_r(x)$ is continuous on $\mathbb{R}$, and equal to $p_r(x)$ in the interval $[0, 1]$.*

4. *For every real number x and every nonzero natural number r, we have*

$$(-1)^{r-1}(B_{2r} - B_{2r}(x)) \geq 0.$$

5. *For every $x \in \mathbb{R}$, the functions $B_r(x)$, satisfy:*

$$\begin{cases} B_0(x) = 1, \\ B_r(x) = r\int_0^x B_{r-1}(t)dt + B_r \Longleftrightarrow B_r'(x) = rB_{r-1}(x), \quad (r \geqslant 1) \\ \int_0^1 B_r(x)dx = 0, \quad (r \geqslant 1) \end{cases} \tag{2.1}$$

Example: $B_1(x)$ ***and*** $B_2(x)$ ***in the interval[0,1].***

$B_1(x) = P_1(\{x\}) = \{x\} - \frac{1}{2} \quad \Longrightarrow \quad B_1(0) = -\frac{1}{2},\ B_1(1) = -\frac{1}{2}.$

$B_2(x) = P_2(\{x\}) = \{x\}^2 - \{x\} + \frac{1}{6} \quad \Longrightarrow \quad B_2(0) = \frac{1}{6},\ B_2(1) = \frac{1}{6}.$

6. *The Fourier series for the function* $B_1(x)$ *is given by:*

$$\frac{B_1(x_-) + B_1(x_+)}{2} = \frac{-1}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2\pi n x)}{n} \quad (x \in \mathbb{R}),$$

where $B_1(x_+) = \lim\limits_{\mathrm{x} \overset{>}{\to} x_0} B_1(x)$.

7. *For all* $r \in \mathbb{N}^*$*, the Fourier series for the functions* $B_{2r}(x)$ *and* $B_{2r+1}(x)$ *are given by:*

$$B_{2r}(x) = (-1)^{r-1} 2 \frac{(2r)!}{(2\pi)^{2r}} \sum_{n=1}^{\infty} \frac{\cos(2\pi n x)}{n^{2r}} \quad (x \in \mathbb{R}).$$

$$B_{2r+1}(x) = (-1)^{r-1} 2 \frac{(2r+1)!}{(2\pi)^{2r+1}} \sum_{n=1}^{\infty} \frac{\sin(2\pi n x)}{n^{2r+1}} \quad (x \in \mathbb{R}).$$

8. *For all* $r \in \mathbb{N}^*$*:*

$$|B_{2r}(x)| \leq |B_{2r}(0)| = |B_{2r}| = 2 \frac{(2r)!}{(2\pi)^{2r}} \sum_{n=1}^{\infty} \frac{1}{n^{2r}} \quad (x \in \mathbb{R})$$

and

$$B_{2r} \sim (-1)^{r-1} \frac{2(2r)!}{(2\pi)^{2r}}, \quad \left(|B_{2r}| \longrightarrow +\infty \text{ for } (r \longrightarrow +\infty)\right).$$

Proof.

1. We know that for every $x \in \mathbb{R} : [x] \leqslant x < [x] + 1$, and also $\{x\} = x - [x]$.

   We define the function $g(x)$ by $g : x \longmapsto \{x\}$.

   The function $g(x)$ is periodic with a period of one because:

$$g(x+k) = x + k - [x+k] = x + k - [x] - k = g(x), \quad (k \in \mathbb{Z}).$$

   and this implies that, $\forall k \in \mathbb{Z}$

$$B_r(x+k) = p_r(g(x+k)) = p_r(g(x)) = B_r(x).$$

   The smallest positive value that satisfies $B_r(x) = B_r(x+T)$ is $T = 1$, and it is called the period of $B_r(x)$.

2. lt is clear that for all $0 < x < 1$ we have $g(x) = x$, which means that $B_1(x) = p_1(x)$.

   and also,

$$B_1(x_+) = \lim_{x\to 0} B_1(x) = \lim_{x\to 0} p_1(x) = \frac{1}{2}.$$

$$B_1(x_-) = \lim_{x\to 1} B_1(x) = \lim_{x\to 1} p_1(x) = -\frac{1}{2}.$$

   Then $\quad B_1(x_+) = \frac{1}{2}, B_1(x_-) = -\frac{1}{2} \quad (\forall x \in \mathbb{Z}).$

   So $B_1(x)$ is discrete at $x \in \mathbb{Z}$.

3. For every natural number $r \geqslant 2$, the function $B_r(x)$ is continuous at all point $x \notin \mathbb{Z}$.

   We still need to prove continuity at the integer points only. To do so, let $n \in \mathbb{Z}$,

   we have,

   - $\lim\limits_{\mathrm{x}\overset{>}{\to} n} B_r(x) = \lim\limits_{\mathrm{x}\overset{>}{\to} n} p_r(g(x)) = p_r\left(\lim\limits_{\mathrm{x}\overset{>}{\to} n} g(x)\right) = p_r\left(\lim\limits_{\mathrm{x}\overset{>}{\to} n}(x-n)\right) = p_r(0) = B_r.$
   - $\lim\limits_{\mathrm{x}\overset{<}{\to} n} B_r(x) = \lim\limits_{\mathrm{x}\overset{<}{\to} n} p_r(g(x)) = p_r\left(\lim\limits_{\mathrm{x}\overset{<}{\to} n} g(x)\right) = p_r\left(\lim\limits_{\mathrm{x}\overset{<}{\to} n}(x-(n-1))\right) = p_r(1) = B_r.$

   Then

$$\lim_{\mathrm{x}\overset{>}{\to} n} B_r(x) = \lim_{\mathrm{x}\overset{<}{\to} n} B_r(x) = B_r. \quad (\forall n \in \mathbb{Z}).$$

   Now, we prove that $B_r(x) = p_r(x)$ on the interval $[0, 1]$.

   ○ Firstly: for $0 \leqslant x < 1$ we have $g(x) = x$, and thus $\quad B_r(x) = p_r(g(x)) = p_r(x)$.

   ○ Secondly: for $x = 1$, we observe that $\quad B_r(1) = p_r(1) = p_r(0) = B_r$. Then

$$\forall x \in [0, 1] : \quad B_r(x) = p_r(x).$$

4. Form Theorem(1.2.1) we have

$$(-1)^{r-1}(B_{2r} - p_{2r}(x)) \geqslant 0 \quad (\forall r \geqslant 1) \quad (\forall x \in [0, 1]).$$

   Since $B_{2r}(x) = p_{2r}(x) \quad (\forall r \geqslant 1)$ on the interval $[0, 1]$.

   Then,

$$(-1)^{r-1}(B_{2r} - B_{2r}(x)) \geqslant 0 \quad (\forall r \geqslant 1) \quad (\forall x \in [0, 1]).$$

   Since $B_{2r}(x)$ is periodic and its period equal to 1, then the last inequality is true for every $\forall x \in \mathbb{R}$, and every natural number $r \geqslant 1$.

5. We have $B_0(x) = p_0(x) = 1$, since $p_0(x)$ is a constant polynomial. Let $r \geqslant 1$, we have

$$B_r'(x) = g'(x)P_r'(g(x)) = g'(x)(rP_{r-1}(g(x))),$$

because $(\forall r \geqslant 1) : P_r'(x) = rP_{r-1}(x)$.

Similarly for every $x \in \mathbb{R}$, there is unique integer such that $[x] \leqslant x < [x] + 1$. Therefore,

$$g(x) = x - [x] \quad \Longrightarrow \quad g'(x) = 1.$$

Then,

$$B_r'(x) = rP_{r-1}(g(x)) = rB_{r-1}(x)$$

and

$$B_r'(x) = rB_{r-1}(x) \quad \Longleftrightarrow \quad B_r(x) = r\int_0^x B_{r-1}(t)dt + B_r, \quad (B_r(0) = B_r).$$

Finally, for all $r \geq 1$, we obtain

$$\int_0^1 B_r(x)dx = \int_0^1 \frac{B_{r+1}'(x)}{r+1}dx = \frac{1}{r+1}(B_{r+1}(1) - B_{r+1}(0)) = 0.$$

6. The function $B_1(x)$ periodic with period 1, so it can be expanded as a Fourier series as follows:

$$\frac{B_1(x_+) + B_1(x_-)}{2} = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(2\pi nx) + b_n \sin(2\pi nx)\right),$$

where:

$$a_0 = \int_0^1 B_1(x)dx = 0,$$

$$\begin{aligned} a_n &= 2\int_0^1 B_1(x)\cos(2\pi nx)dx \quad (n = 0, 1, 2, 3, \dots) \\ &= 2\int_0^1 \left(x - \frac{1}{2}\right)\cos(2\pi nx)dx \\ &= 0, \end{aligned}$$

$$\begin{aligned} b_n &= 2\int_0^1 B_1(x)\sin(2\pi nx)dx \quad (n = 1, 2, 3, \dots) \\ &= 2\int_0^1 \left(x - \frac{1}{2}\right)\sin(2\pi nx)dx \\ &= -\frac{1}{n\pi}. \end{aligned}$$

Thus, we have

$$\frac{B_1(x_+) + B_1(x_-)}{2} = -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2\pi nx)}{n}, \quad (x \in \mathbb{R}).$$

The series $\sum\limits_{n=1}^{\infty}\frac{\sin(2\pi nx)}{n}$ is convergent on $\mathbb{R}$ and vanishes for all $(x \in \mathbb{Z})$ .It is uniformly convergent on every interval [a,b] where $(a < b)$ and $[a, b] \cap \mathbb{Z} = \emptyset$

7. The Fourier series of function $B_2(x)$ is given by:

$$B_2(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos(2\pi nx) + b_n \sin(2\pi nx)\right),$$

where:

$$a_0 = \int_0^1 B_2(x)dx = \int_0^1 (x^2 - x + \frac{1}{6})dx = 0,$$

$$\begin{aligned} a_n &= 2\int_0^1 B_2(x)\cos(2\pi nx)dx \\ &= \int_0^1 (2x^2 - 2x + \frac{1}{3})\cos(2\pi nx)dx \\ &= \frac{1}{n^2\pi^2}, \end{aligned}$$

$$b_n = 2\int_0^1 (x^2 - x + \frac{1}{6})\sin(2\pi nx)dx = 0.$$

Thus, $B_2(x) = \frac{1}{\pi^2}\sum\limits_{n=1}^{\infty}\frac{\cos(2\pi nx)}{n^2}$ and it is uniformly convergent on $\mathbb{R}$.
By integrating $(2r + 1)$ times $B_2(x)$, where $(r \geqslant 1)$, we obtain

$$B_{2r+1}(x) = (-1)^{r-1}2\frac{(2r+1)!}{(2\pi)^{2r+1}}\sum_{n=1}^{\infty}\frac{\sin(2\pi nx)}{n^{2r+1}}, \quad (x \in \mathbb{R})$$

and we also integrating $(2r)$ times $B_2(x)$, we obtain

$$B_{2r}(x) = (-1)^{r-1}2\frac{(2r)!}{(2\pi)^{2r}}\sum_{n=1}^{\infty}\frac{\cos(2\pi nx)}{n^{2r}}, \quad (x \in \mathbb{R})$$

8. For every $x \in \mathbb{R}$ and every natural number $r \geqslant 1$, we have

$$|B_{2r}(x)| = \left|(-1)^{r-1}2\frac{(2r)!}{(2\pi)^{2r}}\sum_{n=1}^{\infty}\frac{\cos(2\pi nx)}{n^{2r}}\right|.$$

$$|B_{2r}(x)| \leq 2\frac{(2r)!}{(2\pi)^{2r}}\sum_{n=1}^{\infty}\frac{1}{n^{2r}} = |B_{2r}(0)| = |B_{2r}|.$$

we have

$$\sum_{n=1}^{\infty}\frac{1}{n^{2r}}=1+\sum_{n=2}^{\infty}\frac{1}{n^{2r}}<1+\int_{1}^{+\infty}\frac{dx}{x^{2r}}=1+\frac{1}{2r-1}\overset{(r\to+\infty)}{\to}1 .$$

So

$$B_{2r}=B_{2r}(0)=\left((-1)^{r-1}2\frac{(2r)!}{(2\pi)^{2r}}\right)\sum_{n=1}^{\infty}\frac{1}{n^{2r}}\sim(-1)^{r-1}2\frac{(2r)!}{(2\pi)^{2r}}.$$

Thus,

$$|B_{2r}|\to+\infty \quad \text{as } (r\to+\infty).$$

The proof is finish. □

Theorem 2.2.2. *[4]*

Let $f(x)$ be a real, positive, and strictly decreasing function on $]0,+\infty[$ such that

$$\int_{0}^{+\infty}f(x)dx<\infty.$$

1. *For every $N,r\in\mathbb{N}$, we have*

$$(-1)^{r-1}\int_{0}^{+\infty}B_{2r+1}(x)f(x)dx>0.$$

2. *If $f'(x)$ has the same properties as $f(x)$, then we have*

$$\int_{N}^{+\infty}|B_{2r}-B_{2r}(x)|\,f(x)dx<|B_{2r}|\int_{N}^{+\infty}f(x)dx,\quad (\forall N\in\mathbb{N}),(\forall r\in\mathbb{N}^*).$$

Proof.

1. For simplicity, we let $I_1=\int_0^{+\infty}h(x)f(x)dx$ and $h(x)=(-1)^{r-1}B_{2r+1}(x)$.

   - We first study the case when $(r\geqslant 1)$:

     Since $(-1)^{r-1}B_{2r}(x)>0$ where $(r\in\mathbb{N}^*)$.then the function $(-1)^{r-1}B_{2r}(x)$ is of the form $B_2(x)$, and the function $h(x)$ is of the form $B_3(x)$.

     Thus $h(x)$ satisfies the following properties of the function $B_3(x)$:

$$\begin{cases} h(x) \text{ is periodic with period 1.}\\ h\left(-x+\frac{1}{2}\right)=-h\left(x+\frac{1}{2}\right), \quad (\forall x\in\mathbb{R}).\\ h(x)<0 \quad \text{for} \quad \frac{1}{2}<x<1 \quad \text{and} \quad h(x)>0 \quad \text{for} \quad 0<x<\frac{1}{2}.\end{cases}$$

- In the first stage: we use the periodicity of $h(x)$, and write I_1 in the following form:

$$I_1 = \int_0^{+\infty} h(t+N)f(t+N)dt = \int_0^{+\infty} h(x)f(t+N)dt. \qquad (x = t+N)$$

- In the second stage: we write I_1 as the following series:

$$\sum_{n=0}^{+\infty} \int_n^{n+1} h(x)f(x+N)dx.$$

and we set,

$$u_n = \int_n^{n+1} h(x)f(x+N)dx.$$

- In the third stage: we prove that $u_n > 0$ for every $n \geqslant 0$.

The periodicity of $h(x)$ allows us to write u_n in the following form, where we take $x = u + n$:

$$\begin{aligned} u_n &= \int_0^1 h(u+n)f(u+n+N)du \\ &= \int_0^1 h(u)f(u+n+N)du \\ &= \int_0^{\frac{1}{2}} h(x)f(x+n+N)dx + \int_{\frac{1}{2}}^1 h(x)f(x+n+N)dx. \end{aligned}$$

Now, the property $h\left(x+\frac{1}{2}\right) = -h(x)$ allows us to obtain,

$$\int_{\frac{1}{2}}^1 h(x)f(x+n+N)dx = -\int_{\frac{1}{2}}^1 h(x)f(x+\frac{1}{2}+n+N)dx,$$

from which

$$u_n = \int_0^{\frac{1}{2}} h(x)\left(f(x+n+N) - f(x+\frac{1}{2}+n+N)\right)dx.$$

Since that $h(x) > 0$ in interval $(0, \frac{1}{2})$ and $f(x) > 0$ for $x > 0$ and is strictly decreasing then:

$$u_n = \int_0^{\frac{1}{2}} h(x)\left(f(x+n+N) - f(x+\frac{1}{2}+n+N)\right)dx > 0.$$

Thus,

$$I_1 = \sum_{n=0}^{+\infty} \int_n^{n+1} h(x)f(x+N)dx = \sum_{n=0}^{+\infty} u_n > 0.$$

For $r=0$ where $h(x)=-B_1(x)$, so $h(x)$ satisfies the following properties:

$$\begin{cases} h(x) \text{ is periodic with period 1.} \\ h\left(x+\frac{1}{2}\right)=-h(x), \quad (\forall x\in\mathbb{R}). \\ h(x)<0 \quad \text{for} \quad \frac{1}{2}<x<1 \quad \text{and} \quad h(x)>0 \quad \text{for} \quad 0<x<\frac{1}{2}. \end{cases}$$

Following the same steps as before, in the case of $r\geqslant 1$, it can be proven that $I_1>0$.

2. To simplify the writing, we set: $I_2=\int_N^{+\infty}|B_{2r}-B_{2r}(x)|\, f(x+N)dx$.

   From theorem (1.2.1), we have:

$$(-1)^{r-1}(B_{2r}-B_{2r}(x))\geq 0, \quad (-1)^{r-1}B_{2r}>0, \quad (r\in\mathbb{N}^*), \quad (x\in\mathbb{R}).$$

   These two inequalities imply:

$$\begin{aligned} |(B_{2r}-B_{2r}(x))| &= |(-1)^{r-1}(B_{2r}-B_{2r}(x))| \\ &= (-1)^{r-1}(B_{2r}-B_{2r}(x)). \end{aligned}$$

   Likewise,

$$(-1)^{r-1}B_{2r}=|(-1)^{r-1}B_{2r}|=|B_{2r}|.$$

   Thus, we get,

$$\begin{aligned} I_2 &= \int_N^{+\infty}(-1)^{r-1}(B_{2r})f(x)dx-(-1)^{r-1}\int_N^{+\infty}B_{2r}(x)f(x)dx. \\ &= |B_{2r}|\int_N^{+\infty}f(x)dx-(-1)^{r-1}\int_N^{+\infty}B_{2r}(x)f(x)dx. \end{aligned}$$

   Using integration by parts of $\int_N^{+\infty}B_{2r}(x)f(x)dx$, take into acount that: $B_{2r+1}(N)=0$, we obtain,

$$(-1)^{r-1}\int_N^{+\infty}B_{2r}(x)f(x)dx=\frac{(-1)^{r-1}}{2r+1}\int_N^{+\infty}B_{2r+1}(x)f'(x)dx>0.$$

   This confirms the expected result in the integral I_2.

□

Chapter 3

Computing values of the Riemann Zeta Functions ζ

In this chapter, we aim to explore the Riemann zeta function and its derivative, and study their convergence. We will also compute the values of the zeta function at positive even integers and examine its analytic continuation. Additionally, we will address the zeros of this function. Let us begin with the following definition:

3.1 Definition

Definition 3.1.1. ***(Zeta function)*** *The Zeta function was first defined by Euler in 1737 for every $s \in \mathbb{R}^+$, then Riemann extended the definition in 1859 for every $s \in \mathbb{C} - \{1\}$. For this reason, the function is called the Riemann Zeta function or simply the Zeta function of Riemann, defined as following:*

$$\zeta(s) := \sum_{n\geqslant 1} \frac{1}{n^s}, \quad (Re(s) > 1),$$

it is an analytic function in the set: $\quad E = \{s \in \mathbb{C} : Re(s) > 1\}.$

Definition 3.1.2. ***(Derivative of the zeta function)*** *The function ζ is defined and differentiable in the previously defined domain E, and its derivative of order k is given by the following expression:*

$$\zeta^{(k)}(s) := (-1)^k \sum_{n\geqslant 2} \frac{\ln^k(n)}{n^s}, \quad (Re(s) > 1), \quad (k \in \mathbb{N}^*)$$

Lemma 3.1.1. *[1]*
The series $\sum\limits_{n\geqslant 1} \frac{1}{n^s}$ and $\sum\limits_{n\geqslant 2} \frac{\ln^\alpha(n)}{n^s}$, with $(\alpha \in \mathbb{R})$, are absolutely convergent in the open half-plane $(Re(s) > 1)$, and uniformly convergent in every closed half-plane E, where

$$E = \{s \in \mathbb{C} : Re(s) \geqslant 1 + \delta, \quad \delta > 0\}.$$

Proof.
Study of the convergence of series $S_1 = \sum\limits_{n\geqslant 1} \frac{1}{n^s}$

- Absolute convergence.

  Let $s \in \mathbb{C}$, where $s = \sigma + it$. Then

$$0 \leqslant |S_1| = \left| \sum_{n=1}^{+\infty} \frac{1}{n^s} \right| \leqslant \sum_{n=1}^{+\infty} \frac{1}{|n^s|} = \sum_{n=1}^{+\infty} \frac{1}{n^\sigma}.$$

It is the Riemann series, which converges for all ($\sigma > 1$). Therefore S_1 convergence absolutely in the open half-plane ($Re(s) > 1$).

- Uniform convergence.
We consider $\left(\sum\limits_{n=1}^{N} \frac{1}{n^s}\right)_{N\geqslant 1}$ to be the sequence of partial sums of S_1.
We say that S_1 convergence uniformly on E if:

$$\sup_{x\in E}\left|S_1 - \sum_{n=1}^{N} \frac{1}{n^s}\right| \overset{n\to+\infty}{\longrightarrow} 0.$$

We have

$$\left|S_1 - \sum_{n=1}^{N} \frac{1}{n^s}\right| \leqslant \sum_{n>N} \frac{1}{n^{\delta+1}} \quad (s \in E).$$

On the other hand, we have

$$\sum_{N<n<M} \frac{1}{n^{\delta+1}} \leqslant \int_N^M \frac{dx}{x^{\delta+1}} = \frac{1}{\delta}\left(\frac{1}{N^\delta} - \frac{1}{M^\delta}\right) \quad (\forall N \geqslant 1).$$

From which,

$$\sum_{n>N} \frac{1}{n^{\delta+1}} \leqslant \int_N^{+\infty} \frac{dx}{x^{\delta+1}} = \frac{1}{\delta N^\delta}.$$

Thus,

$$\left|S_1 - \sum_{n=1}^{N} \frac{1}{n^s}\right| \leqslant \frac{1}{\delta N^\delta} \overset{N\to+\infty}{\longrightarrow} 0.$$

Therefore,

$$\sup_{x\in E}\left|S_1 - \sum_{n=1}^{N} \frac{1}{n^s}\right| \overset{N\to+\infty}{\longrightarrow} 0.$$

This proves the Uniform convergence of S_1.

Study of the convergence of series $S_2 = \sum\limits_{n\geqslant 2} \frac{\ln^\alpha(n)}{n^s}$.

- Absolute convergence.
We consider $\left(\sum\limits_{n=2}^{N} \frac{\ln^\alpha(n)}{n^s}\right)_{N\geqslant 2}$ to be the sequence of partial sums of S_2. Let $s \in \mathbb{C}$, where $s = \sigma + it$ ($\sigma > 1$).
We set $\sigma = 1 + 2\lambda$, where $\lambda = \frac{\sigma-1}{2} > 0$.
Since $\lim\limits_{n\to+\infty} \frac{\ln^\alpha(n)}{n^\lambda} = 0$, there exists $n_0 = n_0(\sigma)$ such that: $(\forall n > n_0) \quad \lim\limits_{n\to+\infty} \frac{\ln^\alpha(n)}{n^\lambda} < 1$.
Therefore,

$$\left|S_2 - \sum_{n=2}^{N} \frac{\ln^\alpha(n)}{n^s}\right| \leqslant \sum_{n>N} \frac{\ln^\alpha(n)}{n^\sigma} = \sum_{n>N} \frac{1}{n^{\lambda+1}}\left(\frac{\ln^\alpha(n)}{n^\lambda}\right) \leqslant \sum_{n>N} \frac{1}{n^{\lambda+1}}.$$

It is known that the function $x \to \frac{1}{x^{\lambda+1}}$ decreasing for all $(x \geqslant 1)$.

Thus,

$$\frac{1}{(n+1)^{\lambda+1}} \leqslant \int_{n}^{n+1} \frac{dx}{x^{\lambda+1}} \leqslant \frac{1}{n^{\lambda+1}}.$$

This requires that

$$\sum_{n>N} \frac{1}{n^{\lambda+1}} \leqslant \int_{N}^{+\infty} \frac{dx}{x^{\lambda+1}} = \frac{1}{\lambda N^{\lambda}} \quad (\forall N \geqslant 1).$$

Therefore,

$$\left| S_2 - \sum_{n=2}^{N} \frac{\ln^{\alpha}(n)}{n^s} \right| \leqslant \frac{1}{\lambda N^{\lambda}} \quad \overset{N \to +\infty}{\longrightarrow} 0.$$

This proves the absolute convergence of S_2.

- Uniform convergence

  Let $s = \sigma + it$, where $s \in E$. Accordingly $\sigma > \delta + 1$, that is $\lambda > \frac{\delta}{2}$.

  Thus,

$$\left| S_2 - \sum_{n=2}^{N} \frac{\ln^{\alpha}(n)}{n^s} \right| \leqslant \frac{1}{\lambda N^{\lambda}} \leqslant \frac{2}{\delta N^{\frac{\delta}{2}}} \quad \overset{N \to +\infty}{\longrightarrow} 0.$$

  Therefore,

$$\sup_{x \in E} \left| S_2 - \sum_{n=2}^{N} \frac{\ln^{\alpha}(n)}{n^s} \right| \quad \overset{N \to +\infty}{\longrightarrow} 0.$$

  This proves that S_2 convergence uniformly on E.

□

3.2 Determining Some Values of the Zeta Function ζ

Theorem 3.2.1. *[1, 3]*

Let B_{2r} be the Bernoulli number of index $2r$. Then for every $r \in \mathbb{N}^*$, we have the following identity:

$$\zeta(2r) = \frac{(2\pi)^{2r} B_{2r}}{2(2r)!}(-1)^{r+1}.$$

Proof.

From Theorem (2.2.1), we have for every real number $x \in \mathbb{R}$, and every natural number $r \in \mathbb{N}^*$, the following relation:

$$B_{2r}(x) = (-1)^{r-1} 2 \frac{(2r)!}{(2\pi)^{2r}} \sum_{n=1}^{+\infty} \frac{\cos(2\pi n x)}{n^{2r}}.$$

By substituting $x = 0$ into the previous relation, we get:

$$B_{2r}(0) = B_{2r} = (-1)^{r-1} 2 \frac{(2r)!}{(2\pi)^{2r}} \sum_{n=1}^{+\infty} \frac{1}{n^{2r}}.$$

Therefore,

$$\frac{(2\pi)^{2r} B_{2r}}{2(-1)^{r-1}(2r)!} = \sum_{n=1}^{+\infty} \frac{1}{n^{2r}}.$$

Thus,

$$\begin{aligned} \zeta(2r) &= \frac{(2\pi)^{2r} B_{2r}}{2(-1)^{r-1}(2r)!} \\ &= (-1)^{r+1} \frac{(2\pi)^{2r} B_{2r}}{2(2r)!}. \end{aligned}$$

□

Using this theorem, some values of the Zeta function can be determined.

Example:

If $r = 1$, then

$$\zeta(2) = \sum_{n \geq 1} \frac{1}{n^2} = \frac{(-1)^{1+1}(2\pi)^2 B_2}{2 \times (2!)} = \frac{\pi^2}{6}.$$

If $r = 2$, then

$$\zeta(4) = \sum_{n \geq 1} \frac{1}{n^4} = \frac{(-1)^{2+1}(2\pi)^4 B_4}{2 \times (4!)} = \frac{\pi^4}{90}.$$

In the same way, we can obtain the following values:

$$\zeta(6) = \frac{\pi^6}{945}, \quad \zeta(8) = \frac{\pi^8}{9450}, \quad \zeta(10) = \frac{\pi^{10}}{93555},$$

$$\zeta(12) = \frac{69\pi^{12}}{638512875}, \ \zeta(14) = \frac{2\pi^{14}}{18243225}, \ldots.$$

3.3 The Analytic Continuation of the Zeta Function

In this part, we aim to extend the Zeta function and its derivative. Let us explore the following theorem:

Theorem 3.3.1. *[1]*

For every two integers a, b such that $a < b$, and for every function f defined on the interval $[a, b]$ to $\mathbb{C}$, continuous and differentiable, we have:

$$\sum_{a<n\leq b} f(n) = \int_a^b f(t)\,dt + \frac{f(b)-f(a)}{2} + \int_a^b B_1(t)f'(t)\,dt.$$

***Proof*.**

We rely in the proof of theorem on computing the values of the two integrals:

$$\int_{n-1}^{n} f(t)\,dt \quad \text{and} \quad \int_{n-1}^{n} [t]f'(t)\,dt, \quad \text{for } a < n \leq b.$$

Using integration by parts, we compute their values and obtain the following:

$$\begin{aligned}\int_{n-1}^{n} f(t)\,dt &= [tf(t)]_{n-1}^{n} - \int_{n-1}^{n} tf'(t)\,dt \qquad\qquad (u = f(t), dv = dt)\\ &= nf(n) - (n-1)f(n-1) - \int_{n-1}^{n} tf'(t)\,dt.\end{aligned}$$

Likewise:

$$\begin{aligned}\int_{n-1}^{n} [t]f'(t)\,dt &= (n-1)\int_{n-1}^{n} f'(t)\,dt\\ &= (n-1)(f(n) - f(n-1))\\ &= [nf(n) - (n-1)f(n-1)] - f(n).\end{aligned}$$

From the last equation, we find that:

$$\begin{aligned}f(n) &= [nf(n) - (n-1)f(n-1)] - \int_{n-1}^{n} [t]f'(t)\,dt\\ &= [nf(n) - (n-1)f(n-1)] - \int_{n-1}^{n} (t - \{t\})f'(t)\,dt, \quad ([t] = t - \{t\})\\ &= [nf(n) - (n-1)f(n-1) - \int_{n-1}^{n} tf'(t)\,dt] + \int_{n-1}^{n} \{t\}f'(t)\,dt.\end{aligned}$$

Thus,

$$f(n) = \int_{n-1}^{n} f(t)\,dt + \int_{n-1}^{n} \{t\}f'(t)\,dt.$$

By summing with respect to n from $a+1$ to b on both sides of the last equation, we obtain,

$$\sum_{a<n\leq b} f(n) = \sum_{n=a+1}^{b} \int_{n-1}^{n} f(t)\,dt + \sum_{n=a+1}^{b} \int_{n-1}^{n} \{t\}f'(t)\,dt$$

$$\sum_{a<n\leq b} f(n) = \int_a^b f(t)\,dt + \int_a^b \{t\} f'(t)\,dt.$$

We now write $\{t\}$ in the following form:

$$\{t\} = t - [t] = t - [t] - \frac{1}{2} + \frac{1}{2} = B_1(t) + \frac{1}{2}.$$

We obtain,

$$\begin{aligned}\sum_{a<n\leq b} f(n) &= \int_a^b f(t)\,dt + \int_a^b \left(B_1(t) + \frac{1}{2}\right) f'(t)\,dt \\ &= \int_a^b f(t)\,dt + \frac{1}{2}\int_a^b f'(t)\,dt + \int_a^b B_1(t) f'(t)\,dt \\ &= \int_a^b f(t)\,dt + \frac{1}{2}[f(b) - f(a)] + \int_a^b B_1(t) f'(t)\,dt.\end{aligned}$$

□

Theorem 3.3.2. *[1]*

1. *The Riemann zeta function is analytically continued in the half-plane E where $E = \{s \in \mathbb{C} : Re(s) > 1\}$ as an analytic function except at the point $s = 1$, which is a simple pole of the zeta function, and its residue at that point is:*

   $$Res(\zeta(s), 1) = 1.$$

   The extended formula is known as the canonical form of the function $\zeta(s)$, and it is expressed by the following equality, where $(N \geqslant 1)$ is an arbitrary positive integer:

   $$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \frac{1}{(s-1)N^{s-1}} - \frac{1}{2N^s} - s\int_N^{+\infty} \frac{B_1(x)}{x^{s+1}} dx. \tag{3.1}$$

2. *The function $\zeta'(s)$ extends analytically in the half-plane E where $E = \{s \in \mathbb{C} : Re(s) > 0\}$ by an analytic function except at the point $s = 1$, which is a pole of order two of the function ζ', and its residue at that point is:*

   $$Res(\zeta'(s), 1) = -1.$$

The extended formula is known as the canonical form of the function $\zeta'(s)$, and it is expressed by the following equality:

$$\zeta'(s) = -\sum_{n=2}^{+\infty} \frac{\ln(n)}{n^s} - \frac{\ln N}{(s-1)N^{s-1}} - \frac{1}{(s-1)^2 N^{s-1}} + \frac{\ln N}{2N^s} - \int_N^{+\infty} \frac{B_1(x)(1-s\ln x)}{x^{s+1}} dx, \quad (3.2)$$

where $N \geq 1$ is an arbitrary integer.

Proof.

1. For every $s \in E$ and for every natural number $N \geq 1$, we have

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \sum_{n>N} \frac{1}{n^s} = \sum_{n=1}^{N} \frac{1}{n^s} + \lim_{M\to+\infty} \left(\sum_{n=N+1}^{M} \frac{1}{n^s} \right).$$

By applying **Theorem 3.3.1** to the sum $\sum\limits_{N<n\leq M} \frac{1}{n^s}$, where we take,

$$a = N, \quad b = M, \quad f(n) = \frac{1}{n^s}, \quad f'(n) = -\frac{s}{n^{s+1}},$$

we obtain,

$$\begin{aligned}
\sum_{N<n\leq M} \frac{1}{n^s} &= \int_N^M f(t)dt + \frac{f(M)-f(N)}{2} + \int_N^M B_1(t) f'(t) dt \\
&= \int_N^M \frac{1}{t^s} dt + \frac{1}{2}\left(\frac{1}{M^s} - \frac{1}{N^s}\right) + \int_N^M B_1(t) f'(t) dt \\
&= \int_N^M \frac{1}{t^s} dt + \frac{1}{2}\left(\frac{1}{M^s} - \frac{1}{N^s}\right) - s\int_N^M \frac{B_1(t)}{t^{s+1}} dt \\
&= \frac{1}{1-s}\left(\frac{1}{M^{s-1}} - \frac{1}{N^{s-1}}\right) + \frac{1}{2}\left(\frac{1}{M^s} - \frac{1}{N^s}\right) - s\int_N^M \frac{B_1(t)}{t^{s+1}} dt.
\end{aligned}$$

By proceeding to the end, we find

$$\lim_{M\to+\infty}\left(\sum_{N<n\leq M} \frac{1}{n^s}\right) = \lim_{M\to+\infty}\left[\frac{1}{1-s}\left(\frac{1}{M^{s-1}} - \frac{1}{N^{s-1}}\right) + \frac{1}{2}\left(\frac{1}{M^s} - \frac{1}{N^s}\right) - s\int_N^M \frac{B_1(t)}{t^{s+1}} dt\right].$$

But

$$\lim_{M\to+\infty} \int_N^M \frac{B_1(t)}{t^{s+1}} dt = \int_N^{+\infty} \frac{B_1(t)}{t^{s+1}} dt < +\infty, \quad (s \in E),$$

and also,

$$\lim_{M\to+\infty} \frac{1}{M^{s-1}} = 0, \text{and} \quad \lim_{M\to+\infty} \frac{1}{M^s} = 0, \quad (s \in E).$$

Therefore,

$$\lim_{M\to+\infty}\left(\sum_{N<n\leq M}\frac{1}{n^s}\right)=\frac{1}{(s-1)N^{s-1}}-\frac{1}{2N^s}-s\int_N^{+\infty}\frac{B_1(t)}{t^{s+1}}dt.$$

Thus,

$$\zeta(s)=\sum_{n=1}^{N}\frac{1}{n^s}+\frac{1}{(s-1)N^{s-1}}-\frac{1}{2N^s}-s\int_N^{+\infty}\frac{B_1(t)}{t^{s+1}}dt.$$

The right-hand side of this last equality has a finite value for all $s\in C-\{1\}$, such that $\mathrm{Re}(s)>0$.

The point $s=1$ is a simple pole of the function $\zeta(s)$, and the residue at that point is $\mathrm{Res}(\zeta(s),1)=1$.

In the same way, we continue integrating by parts on the right-hand side of this equation, such that with each integration by parts, the domain becomes wider.

Thus, we have obtained the expression for the analytic continuation of the Riemann zeta function, which allows us to determine its value at every point in the half-plane ($\mathrm{Re}(s)>0$) where $s\neq 1$.

2. To prove Identity (3.2), we follow the same method that was used to prove Identity (3.1). For every $s\in E$, and for every natural number $N\geq 1$, we have the following:

$$\begin{aligned}\zeta'(s)&=-\sum_{n\geq 2}\frac{\ln(n)}{n^s}\\&=-\sum_{n=2}^{N}\frac{\ln(n)}{n^s}-\lim_{M\to+\infty}\left(\sum_{N<n\leq M}\frac{\ln(n)}{n^s}\right).\end{aligned}$$

By applying **Theorem 3.3.1** to the sum $\sum\limits_{N<n\leq M}\frac{\ln(n)}{n^s}$, where we take,

$$a=N,\quad b=M,\quad f(n)=\frac{\ln(n)}{n^s},\quad f'(n)=-\frac{1-\ln(n)s}{n^{s+1}},$$

we obtain,

$$\sum_{N<n\leq M}\frac{\ln(n)}{n^s}=\int_N^M\frac{\ln(t)}{t^s}dt+\frac{1}{2}\left(\frac{\ln(M)}{M^s}-\frac{\ln(N)}{N^s}\right)+\int_N^M\frac{B_1(t)(1-s\ln(t))}{t^{s+1}}dt.$$

We have,

$$\int_N^M\frac{\ln(t)}{t^s}dt=\frac{\ln(N)}{(s-1)N^{s-1}}-\frac{\ln(M)}{(s-1)M^{s-1}}+\frac{1}{(s-1)^2N^{s-1}}-\frac{1}{(s-1)^2M^{s-1}}.$$

Therefore,

$$\lim_{M\to+\infty}\left(\sum_{N<n\leq M}\frac{\ln(n)}{n^s}\right)=\frac{\ln(N)}{(s-1)N^{s-1}}+\frac{1}{(s-1)^2N^{s-1}}-\frac{\ln(N)}{2N^s}+\int_N^{+\infty}\frac{B_1(t)(1-s\ln(t))}{t^{s+1}}dt.$$

Since

$$\lim_{M\to+\infty}\frac{\ln(M)}{2M^s}=0,$$

$$\lim_{M\to+\infty}\frac{\ln(M)}{(s-1)M^{s-1}}=0,$$

$$\lim_{M\to+\infty}\frac{1}{(s-1)^2M^{s-1}}=0,$$

and since

$$\int_N^{+\infty}\frac{B_1(t)(1-s\ln(t))}{t^{s+1}}dt<+\infty \quad \text{for all } (s\in E),$$

we have,

$$\sum_{n\geq 2}\frac{\ln(n)}{n^s}=-\sum_{n=2}^{N}\frac{\ln(n)}{n^s}-\frac{\ln(N)}{(s-1)N^{s-1}}-\frac{1}{(s-1)^2N^{s-1}}+\frac{\ln(N)}{2N^s}-\int_N^{+\infty}\frac{B_1(t)(1-s\ln(t))}{t^{s+1}}dt.$$

The right-hand side of this last equality has a finite value for all $s\in C-\{1\}$, such that $\text{Re}(s)>0$.

The point $s=1$ is a pole of order two for $\zeta'(s)$, and its residue is $\text{Res}(\zeta'(s),1)=-1$.

In the same way, we continue integrating by parts on the right-hand side of this equation, such that with each integration by parts, the domain becomes wider.

Thus, we have obtained the expression for the analytic continuation of the function $\zeta'(s)$.

□

3.4 The Zeros of the Zeta Function $\zeta(s)$

The Riemann zeta function, is analytic throughout the entire complex plane except for $s=1$, and it has a set of isolated zeros (roots) that can be identified based on the following relation (see, e.g. [1]):

$$\zeta(s)=\frac{(2\pi)^s}{\pi}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s) \quad (Re(s)<0). \tag{3.3}$$

The proof of this latter relation relies on methods that go beyond the scope of our dissertation, as it involves the Gamma function $\Gamma(s)$, which is defined as follows (see, e.g. [1]):

$$\Gamma(s) := \lim_{n\to+\infty}\left[\frac{n^s n!}{s(s+1)\dots(s+n)}\right].$$

Equation (3.3) can be proven in more than five different ways, and from it we derive the negative even roots.

Example:

If we take $s = -2k$ for $(k \in \mathbb{N}^*)$ in Equation (3.3), then we find

$$\zeta(-2) = \zeta(-4) = \zeta(-6) = \cdots = \zeta(-2k) = 0.$$

The complex zeros of zeta function lie in the region $0 \leqslant Re(s) < 1$ and are all symmetric with respect to the line $x = \frac{1}{2}$. However, Riemann made a conjecture about these zeros through his famous hypothesis.

The Riemann Hypothesis: There are many hypotheses concerning the distribution of the roots of the zeta function, the most famous of which is called the Riemann Hypothesis. It is stated as follows: All non-rel (nontrivial) zeros of the zeta function whose real part is positive lie on the line $Re(s) = \frac{1}{2}$.

Bibliography

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, New York, (1976).

[2] T. Arakawa, T. Ibukiyama, and M. Kaneko, *Bernoulli numbers and zeta functions*. Springer, (2014).

[3] H. Cohen, *Number theory: Volume II: Analytic and modern tools. Vol. 240*, Springer Science & Business Media, (2007).

[4] A. Derbal, *Quelques Outils d'Analyse Réelle en Théorie Analytique des Nombres*. Cours de post-graduation a l'école supérieure de Kouba, Algeria, (2012–2013).

[5] H. L. Montgomery and R.C. Vaughan, *Multiplicative Number Theory I. Classical Theory*. Cambridge University Press, New York, (2006).

[6] P. Prapanpong. *Analytic Number Theory for Beginners. Vol. 103*. American Mathematical Society, (2023).

[7] H. K. Rosen, *Elementary Number Theory and Its Applications*, Library of Congress Cataloging in Publication Data, United States of America, (1986).