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Theme:

Double Laplace Transform Decomposition Method for Solving Nonlinear Partial Differential Equations

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الملخص

تهدف هذه المذكرة الى دراسة طريقة فعالة لحل المعادلات التفاضلية الجزئية غير الخطية، من خلال الدمج بين طريقة ادوميان للتحلل وتحويل لابلاس المضاعف. يسمح هذا الدمج بتحويل المعادلات المعقدة الى معادلات جبرية يمكن التعامل معها بسهولة، مع الحفاظ على الطبيعة غير الخطية للمشكلة. تم تطبيق الطريقة على عدة امثلة، وأثبتت فعاليتها في إيجاد حلول تحليلية او شبه تحليلية دقيقة، مما يجعلها أداة واعدة لحل هذا النوع من المعادلات.

Résumé

Ce mémoire vise à étudier une méthode efficace pour résoudre des équations aux dérivées partielles non linéaires, en combinant la méthode de décomposition d'Adomian avec la transformée de Laplace double. Cette combinaison permet de transformer des équations complexes en formes algébriques plus faciles à manipuler, tout en conservant le caractère non linéaire du problème. La méthode a été appliquée à plusieurs exemples et a montré une grande efficacité pour obtenir des solutions analytiques ou semi-analytiques précises, ce qui en fait un outil prometteur dans ce domaine

Abstract

This thesis aims to study an effective method for solving nonlinear partial differential equations by combining the Adomian Decomposition Method with the Double Laplace Transform. This combination transforms complex differential equations into solvable algebraic forms while preserving the nonlinear nature of the problem. The method was applied to several examples and demonstrated high efficiency in obtaining accurate analytical or semi-analytical solutions, making it a promising tool for handling such equations.

اهداء

الحمد الله رب العالمين، حمدا يليق بجلاله، حمدا على عطائه و توفيقه و ستره، و الصلاة و السلام على سيدنا محمد، خير خلق الله، من بعثه رحمة للعالمين، وعلى اله وصحبه اجمعين.

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الى اخوتي الأحبة: **زبيدة، جيلاني، مروة، سارة، إسماعيل،** واخر العنقود **فاطمة**.

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Notations

$\mathcal{L}\{f(t)\}$	Laplace transform of a function with respect to time t.
$L_x L_t\{f(x,t)\}$	Double Laplace transform with respect to variables x and t.
$\mathcal{L}^{-1}\{F(s)\}$	Inverse Laplace transform.
$L_x^{-1} L_t^{-1}\{F(p,s)\}$	Inverse double Laplace transform.
$\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}$	Partial derivatives of u with respect to t and x.
$(f * g)(t)$	Convolution of two functions in one variable.
$(f * * g)(x,t)$	Double convolution of two functions in two variables
$U(p,s)$	Double Laplace transform of the function $u(x,t)$.
A_n	The n-th Adomian polynomial corresponding to the nonlinear operator N.
λ	Auxiliary parameter used to generate Adomian polynomials

CONTENTS

Introduction

Differential equations especially partial differential equations are fundamental tools in the mathematical modeling of physical and engineering systems. However, many real-world phenomena are described by **nonlinear differential equations**, whose solutions are often difficult or even impossible to obtain analytically using classical methods. This complexity arises from the nonlinear interaction between variables and their derivatives, making the analysis and solution of such equations a significant mathematical challenge.

To address this difficulty, **transform methods** have proven to be highly effective, particularly in converting differential equations into simpler algebraic equations. Among these, the **Laplace transform** is widely used to solve ordinary and partial differential equations involving a single independent variable. Yet, in many applications involving more than one independent variable such as time and space there arises the need for the **double Laplace transform**, which extends the classical Laplace method to two dimensions, thereby simplifying the treatment of multi-variable differential problems.

In parallel, the **Adomian decomposition method (ADM)** has emerged as a powerful analytical technique for solving both linear and nonlinear differential equations without linearization or discretization. This method expresses the solution as an infinite series and deals with the nonlinear part through the construction of **Adomian polynomials**, which are systematically computed to represent the nonlinear terms accurately.

By combining the strengths of both techniques, the **double Laplace-Adomian method** offers an efficient hybrid approach for solving nonlinear partial differential equations. This method involves applying the double Laplace transform to simplify the equation, followed by the decomposition process to handle nonlinearity, thus enabling the construction of accurate analytical or semi-analytical solutions.

This thesis consists of an introduction and three chapters. The first chapter is devoted to the presentation of the definition of the ordinary Laplace transform, as well as the conditions of existence and some basic properties. We also presented the definition and properties of the double Laplace transform, as well as the fundamental formulas of the double Laplace transform for ordinary partial derivatives.

In the second chapter we will brieáy present the Adomian decomposition method with the presentation of an efficient method to show how to calculate Adomian polynomials, then presented the basic theories of the convergence of this method and its proofs.

The third chapter is devoted to applications of this method (the double Laplace decomposition method) to solve partial differential equations such as the gas-dynamic equation, the wave-like equation, nonlinear system of PDE and the Klein-Gordon equation.

Chapter 1

Double Laplace Transform

In this chapter, we will review the ordinary Laplace transform, and then move on to the double Laplace transform, we will introduce the basics and important properties of this transform.

1.1 The Laplace transform

The Laplace transform method is a powerful tool used to solve differential and integral equations. The Laplace transform changes the differential and integral equations into algebraic equations that can be easily solved, and thus using the inverse Laplace transform gives the solution of the linear differential equation .

1.1.1 Definition of the Laplace transform

Definition 1.1.1 *[1]* Let $f(t)$ be a function of t specified for $t > 0$. Then the Laplace transform of $f(t)$, denoted by $\mathcal{L}\{f(t)\}$, is defined by

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt, \tag{1.1}$$

where we assume at present that the parameter s is real.

1.1.2 Conditions for existence of Laplace transform

Definition 1.1.2 *[1]* If real constants $M > 0$ and γ exist such that for all $t > N$

$$|e^{-\gamma t} f(t)| < M \quad \text{or} \quad |f(t)| < Me^{\gamma t}, \tag{1.2}$$

we say that $f(t)$ is a function of exponential order γ as $t \to \infty$, briefly, is of exponential order.

Example 1.1 Both $\sin at$ and $\cos at$ are of exponential order e^t since

$$|e^{-t} \sin at| < 1 \quad \text{for } t > 1. \tag{1.3}$$

Here we have taken γ, M, and N equal to 1. (Actually, the condition is satisfied for $t > 0$.)

Example 1.2 e^{3t} is of exponential order e^{at} for any $a > 3$, since

$$|e^{-at}e^{3t}| = e^{(3-a)t} < 1 \quad \text{for } t > 1. \tag{1.4}$$

Clearly, e^{t^2} is not of exponential order.

Theorem 1.1.3 *[1]* If $f(t)$ is sectionally continuous in every finite interval $0 \leq t \leq N$ and of exponential order γ for $t > N$, then its Laplace transform $F(s)$ exists for all s $> \gamma$.

proof.
We have for any positive number N,

$$\int_0^\infty e^{-st} f(t) dt = \int_0^N e^{-st} f(t) dt + \int_N^\infty e^{-st} f(t) dt.$$

Since $f(t)$ is sectionally continuous in every finite interval $0 \leq t \leq N$, the first integral on the right exists. Also the second integral on the right exists, since $f(t)$ is of exponential order γ for $t > N$. To see this we have only to observe that in such case

$$\begin{aligned} \left| \int_N^\infty e^{-st} f(t) dt \right| &\leq \int_N^\infty \left| e^{-st} f(t) \right| dt \\ &\leq \int_0^\infty e^{-st} \left| f(t) \right| dt \\ &\leq \int_0^\infty e^{-st} M e^{\gamma t} dt = \frac{M}{s - \gamma}, \end{aligned}$$

thus the Laplace transform exists for $s > \gamma$.

It must be emphasized that the stated conditions are sufficient to guarantee the existence of the Laplace transform. If the conditions are not satisfied, however, the Laplace transform may or may not exist .

1.1.3 Definition of the inverse Laplace transform

Definition 1.1.4 *[2]* If the Laplace transform of $f(t)$ is $F(s)$, then we say that the inverse Laplace transform of $F(s)$ is $f(t)$, given by:

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \exp(st) F(s) ds, \tag{1.5}$$

where $\mathcal{L}^{-1}$ is the inverse Laplace transform operator.

1.1.4 Properties of the Laplace transform

In this part, we will present some properties of this transform

a. Linearity property

If c_1 and c_2 are any constants and $f_1(t)$ and $f_2(t)$ are functions with Laplace transforms $F_1(s)$ and $F_2(s)$ respectively, then

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\} = c_1 F_1(s) + c_2 F_2(s). \tag{1.6}$$

The result is easily extended to more than two functions.

Proof.

The linearity of the Laplace transform is a direct consequence of the properties of the integral.

Example 1.3 The hyperbolic cosine function

$$\cosh \omega t = \frac{e^{\omega t} + e^{-\omega t}}{2}. \tag{1.7}$$

From formula (1.6), we obtain

$$\mathcal{L}\{\cosh \omega t\} = \frac{1}{2}[\mathcal{L}\{e^{\omega t}\} + \mathcal{L}\{e^{-\omega t}\}] \tag{1.8}$$

$$= \frac{1}{2}\left(\frac{1}{s-\omega} + \frac{1}{s+\omega}\right) \tag{1.9}$$

$$= \frac{s}{s^2 - \omega^2}. \tag{1.10}$$

Similarly,

$$\mathcal{L}\{\sinh \omega t\} = \frac{\omega}{s^2 - \omega^2}. \tag{1.11}$$

b. Change of scale property

If $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\{f(at)\} = \frac{1}{a} F(\frac{s}{a}). \tag{1.12}$$

Proof.

By definition of Laplace transform, we obtain

$$\mathcal{L}\{f(at)\} = \int_0^\infty e^{-st} f(at) dt,$$

unsing change of variable $t = u/a$

$$\begin{aligned}\mathcal{L}\{f(at)\} &= \int_0^x e^{-s(u/a)} f(u) d(u/a) \\ &= \frac{1}{a}\int_0^\infty e^{-s(u/a)} f(u) du \\ &= \frac{1}{a} F(\frac{s}{a}).\end{aligned}$$

c. First shifting property

If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > 0$, then for any constant $a \in \mathbb{R}$

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a), \quad (s > a). \tag{1.13}$$

proof.

From formula (1.1), we obtain

$$\begin{aligned}\mathcal{L}\{e^{at} f(t)\} &= \int_0^\infty e^{-st}\{e^{at} f(t)\} dt \\ &= \int_0^\infty e^{-(s-a)t} f(t) dt \\ &= F(s-a).\end{aligned}$$

Example 1.4 Consider

$$f(t) = e^{-t}\cos(2t). \tag{1.14}$$

Since

$$\mathcal{L}\{\cos(2t)\} = \frac{s}{s^2+4}, \tag{1.15}$$

we obtain

$$\mathcal{L}\{e^{-t}\cos(2t)\} = \frac{s+1}{(s+1)^2+4} = \frac{s+1}{s^2+2s+5}. \tag{1.16}$$

d. Laplace transform of integral

If $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\left\{\int_0^t f(u) du\right\} = \frac{F(s)}{s}. \tag{1.17}$$

proof.

Let $g(t) = \int_0^t f(u)du$, then $g'(t) = f(t)$ and $g(0) = 0$. Taking the Laplace transform of both sides, we have

$$\begin{aligned}\mathcal{L}\{g'(t)\} &= s\ \mathcal{L}\{g(t)\} - g(0) \\ &= s\ \mathcal{L}\{g(t)\},\end{aligned}$$

since $g(0) = 0$

$$\mathcal{L}\{g'(t)\} = F(s).$$

Thus

$$\mathcal{L}\{g(t)\} = \frac{F(s)}{s},$$

or

$$\mathcal{L}\{\int_0^t f(u)du\} = \frac{F(s)}{s}.$$

Example 1.5 Since

$$\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}, \tag{1.18}$$

we have

$$\mathcal{L}\left\{\int_0^t \sin 2u\, du\right\} = \frac{2}{s(s^2+4)}. \tag{1.19}$$

as can be verified directly.

e. Multiplication by t^n

If $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}(t^n f(t)) = (-1)^n \frac{d^n}{ds^n} F(s) = (-1)^n F^{(n)}(s). \tag{1.20}$$

Example 1.6 Since

$$\mathcal{L}\{e^{2t}\} = \frac{1}{s-2}, \tag{1.21}$$

we have

$$\mathcal{L}\{te^{2t}\} = -\frac{d}{ds}(\frac{1}{s-2}) = \frac{1}{(s-2)^2} \tag{1.22}$$

$$\mathcal{L}\{t^2e^{2t}\} = \frac{d^2}{ds^2}(\frac{1}{s-2}) = \frac{2}{(s-2)^3}. \tag{1.23}$$

g. Derivative property of the Laplace transform

The Laplace transform of the derivative of order $n \in \mathbb{N}^*$ of the function f is given by

$$\mathcal{L}(f^{(n)})(s) = s^n \mathcal{L}(f)(s) - \sum_{k=1}^{n} s^{k-1} f^{(n-k)}(0). \tag{1.24}$$

proof.

The demonstration is done by recurrence,
for $n = 1$,

$$\begin{aligned}\mathcal{L}(f^{(n)})(s) &= \mathcal{L}(f')(s)\\ &= \int_0^{+\infty} f'(t)e^{-st}dt.\end{aligned}$$

We perform integration by parts with $u = e^{-st}$ and $dv = f'(t)dt$.

$$\begin{aligned}\mathcal{L}(f')(s) &= \lim_{x\to\infty}[e^{-st}f(t)]_0^x + s\int_0^{+\infty} f(t)e^{-st}dt\\ &= s\mathcal{L}(f)(s) - f(0).\end{aligned}$$

And

$$s^n\mathcal{L}(f)(s) - \sum_{k=1}^{n} s^{k-1}f^{(n-k)}(0) = s\mathcal{L}(f)(s) - f(0).$$

Thus, equality (1.24), is true when $n = 1$.

Let us assume that formula (1.24), holds for a certain rank n, and let us show that it remains valid for rank $(n + 1)$.

So

$$\begin{aligned}\mathcal{L}(f^{(n+1)})(s) &= \mathcal{L}((f')^{(n)})(s)\\ &= s^n\mathcal{L}(f')(s) - \sum_{k=1}^{n} s^{k-1}(f')^{(n-k)}(0)\\ &= s^n(s\mathcal{L}(f)(s) - f(0)) - \sum_{k=1}^{n} s^{k-1}f^{(n+1-k)}(0)\\ &= s^{n+1}\mathcal{L}(f)(s) - s^n f(0) - \sum_{k=1}^{n} s^{k-1}f^{(n+1-k)}(0)\\ &= s^{n+1}\mathcal{L}(f)(s) - \sum_{k=1}^{n+1} s^{k-1}f^{(n+1-k)}(0).\end{aligned}$$

Finally, formula (1.24), is indeed at rank $(n + 1)$.

Hence by mathematical induction (1.24), is valid for all positive integers n.

Example 1.7 For $n = 2$ from the relation (1.24), we get

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0). \tag{1.25}$$

For $n = 3$ from the relation (1.24), we get

$$\mathcal{L}\{f^{(3)}(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0). \tag{1.26}$$

h. The transform of the convolution product

Definition 1.1.5 Let the functions $f, g : \mathbb{R}^+ \to \mathbb{C}$ vanish on the negative half-plane. We define the convolution product by:

$$(f * g)(t) = \int_0^t f(t-u)g(u)\,du. \tag{1.27}$$

Proposition 1.1.6 Let $f, g : \mathbb{R}^+ \to \mathbb{C}$ be functions vanishing on the negative half-line and having Laplace transforms $\mathcal{L}(f)(s)$ and $\mathcal{L}(g)(s)$, respectively. Then:

$$\mathcal{L}(f * g)(s) = \mathcal{L}(f)(s) \cdot \mathcal{L}(g)(s)$$

where the convolution is defined by

$$(f * g)(t) = \int_0^t f(t-u)g(u)\,du.$$

proof.
We start with the definition of the Laplace transform of the convolution

$$\mathcal{L}(f * g)(s) = \int_0^\infty (f * g)(t)e^{-st}\,dt = \int_0^\infty \left(\int_0^t f(t-u)g(u)\,du\right) e^{-st}\,dt.$$

By Fubini's theorem, we can switch the order of integration

$$\mathcal{L}(f * g)(s) = \int_0^\infty g(u) \left(\int_u^\infty f(t-u)e^{-st}\,dt\right) du.$$

We perform the change of variable $v = t - u$, so that $t = u + v$, and $dt = dv$. The inner integral becomes:

$$\int_u^\infty f(t-u)e^{-st}\,dt = \int_0^\infty f(v)e^{-s(v+u)}\,dv = e^{-su} \int_0^\infty f(v)e^{-sv}\,dv = e^{-su}\mathcal{L}(f)(s).$$

Thus, we have:

$$\mathcal{L}(f * g)(s) = \int_0^\infty g(u)e^{-su}\mathcal{L}(f)(s)\,du = \mathcal{L}(f)(s) \int_0^\infty g(u)e^{-su}\,du = \mathcal{L}(f)(s) \cdot \mathcal{L}(g)(s).$$

1.1.5 Table of Laplace transforms

The following table contains the Laplace transform of some elementary functions .

	$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
1.	1	$\frac{1}{s}, \quad s > 0$
2.	t	$\frac{1}{s^2}, \quad s > 0$
3.	$t^n, \quad n = 0, 1, 2, \ldots$	$\frac{n!}{s^{n+1}}, \quad s > 0$
4.	e^{at}	$\frac{1}{s-a}, \quad s > a$
5.	$\sin(at)$	$\frac{a}{s^2+a^2}, \quad s > 0$
6.	$\cos(at)$	$\frac{s}{s^2+a^2}, \quad s > 0$
7.	$\sinh(at)$	$\frac{a}{s^2-a^2}, \quad s > \lvert a\rvert$
8.	$\cosh(at)$	$\frac{s}{s^2-a^2}, \quad s > \lvert a\rvert$

1.1.6 Example

In this example, we aim to solve the following partial differential equation using the Laplace transform

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}, \quad x, t \geq 0, \tag{1.28}$$

with the initial and boundary conditions

$$u(x, 0) = x, \quad u(0, t) = t. \tag{1.29}$$

We apply the Laplace transform with respect to t

$$\mathcal{L}\left\{\frac{\partial u}{\partial t}\right\} = \mathcal{L}\left\{\frac{\partial u}{\partial x}\right\}. \tag{1.30}$$

Define

$$\mathcal{L}\{u(x, t)\} = U(x, s). \tag{1.31}$$

Using (1.24) the Laplace transform property for time derivatives

$$\mathcal{L}\left\{\frac{\partial u}{\partial t}\right\} = sU(x, s) - u(x, 0), \tag{1.32}$$

and

$$\mathcal{L}\left\{\frac{\partial u}{\partial x}\right\} = \frac{\partial U}{\partial x}. \tag{1.33}$$

Therefore,

$$sU(x, s) - u(x, 0) = \frac{\partial U}{\partial x}. \tag{1.34}$$

Substituting the initial condition $u(x, 0) = x$, we obtain

$$sU(x, s) - x = \frac{\partial U}{\partial x}, \tag{1.35}$$

which we rearrange as

$$\frac{\partial U}{\partial x} - sU(x, s) = -x. \tag{1.36}$$

This is a first-order linear ordinary differential equation. We solve it using an integrating factor

$$\mu(x) = e^{-sx}. \tag{1.37}$$

Multiplying both sides of equation (1.36) by the integrating factor

$$e^{-sx}\frac{\partial U}{\partial x} - se^{-sx}U = -xe^{-sx} \Rightarrow \frac{d}{dx}(e^{-sx}U) = -xe^{-sx}.$$

Integrating both sides

$$e^{-sx}U(x, s) = \int -xe^{-sx}dx. \tag{1.38}$$

Using integration by parts

$$\int xe^{-sx}dx = \frac{-x}{s}e^{-sx} - \frac{1}{s^2}e^{-sx}, \tag{1.39}$$

we get

$$e^{-sx}U(x, s) = \frac{x}{s}e^{-sx} + \frac{1}{s^2}e^{-sx} + C, \tag{1.40}$$

and therefore

$$U(x, s) = \frac{x}{s} + \frac{1}{s^2} + Ce^{sx}. \tag{1.41}$$

Now we apply the boundary condition $u(0, t) = t$. Taking the Laplace transform of both sides

$$U(0, s) = \mathcal{L}\{t\} = \frac{1}{s^2}. \tag{1.42}$$

Substitute into the solution

$$U(0, s) = \frac{0}{s} + \frac{1}{s^2} + C \Rightarrow C = 0. \tag{1.43}$$

Thus, the solution in the Laplace domain is

$$U(x, s) = \frac{x}{s} + \frac{1}{s^2}. \tag{1.44}$$

Taking the inverse Laplace transform

$$u(x, t) = \mathcal{L}^{-1}\left\{\frac{x}{s} + \frac{1}{s^2}\right\} = x + t. \tag{1.45}$$

Therefore, the solution to the partial differential equation is

$$u(x, t) = x + t. \tag{1.46}$$

1.2 Double Laplace Transform

1.2.1 Definition of double Laplace transform

Definition 1.2.1 *[8]* Let $f(x,t)$ be a function of two variables x and t, where x, $t > 0$. The double Laplace transform of $f(x,t)$ is defined as

$$L_x L_t\{f(x,t)\} = F(p,s) = \int_0^\infty e^{-px} \int_0^\infty e^{-st} f(x,t)\, dt\, dx. \tag{1.47}$$

whenever the improper integral converges. Here p, s are real numbers .

Example 1.8 Let $f(x,t) = 1$ be a continuous function. The Laplace transform is easily found to be as follows

$$\begin{aligned} L_x L_t\{1\} &= \int_0^\infty e^{-px} \int_0^\infty e^{-st} \cdot 1\, dt\, dx \\ &= \int_0^\infty e^{-px} \left(\int_0^\infty e^{-st}\, dt \right) dx \\ &= \int_0^\infty e^{-px} \cdot \frac{1}{s}\, dx \\ &= \frac{1}{s} \int_0^\infty e^{-px} dx \\ &= \frac{1}{s} \cdot \frac{1}{p} \\ &= \frac{1}{ps}. \end{aligned}$$

Then we get

$$L_x L_t\{1\} = \frac{1}{ps}. \tag{1.48}$$

1.2.2 Definition of the inverse double Laplace transform

Definition 1.2.2 *[8]* The inverse of the double Laplace transform, it is written in the form

$$L_x^{-1} L_t^{-1}\{F(p,s)\} = f(x,t), \tag{1.49}$$

is defined by

$$L_x^{-1} L_t^{-1}\{F(p,s)\} = f(x,t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{px} F(p,s) dp \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} F(p,s) ds. \tag{1.50}$$

1.2.3 Existence and uniqueness of the double Laplace transform

Theorem 1.2.3 *[3] If $f(x,t)$ is of exponential order, the double Laplace transform of f exists.*

Proof.

Suppose f is of exponential order, that is

$$\sup_{x,t\geq 0}\left|\frac{f(x,t)}{e^{ax+bt}}\right| = M,$$

for some $a, b, M \in \mathbb{R}$. Then

$$|f(x,t)| \leq Me^{(ax+bt)},$$

for all $x, t \geq 0$. Thus

$$\begin{aligned}
\left|\int_0^\infty e^{-st}\left(\int_0^\infty e^{-px} f(x,t)\,dx\right)dt\right| &\leq \int_0^\infty e^{-st}\left|\int_0^\infty e^{-px} f(x,t)\,dx\right| dt \\
&\leq \int_0^\infty e^{-st}\int_0^\infty e^{-px}|f(x,t)|\,dxdt \\
&\leq M\int_0^\infty e^{-st}\int_0^\infty e^{-px}e^{(ax+bt)}\,dxdt \\
&= M\int_0^\infty e^{-(s-b)t}\int_0^\infty e^{-(p-a)x}\,dxdt \\
&= \frac{M}{(s-b)(p-a)},
\end{aligned}$$

thus the integral in question converges for $p > a$ and $s > b$.

Theorem 1.2.4 Let $f(x,t)$ and $g(x,t)$ be continuous functions defined for $x, t \geq 0$ and having double Laplace transforms $F(p,s)$ and $G(p,s)$ respectively
if

$$F(p,s) = G(p,s), \tag{1.51}$$

then

$$f(x,t) = g(x,t). \tag{1.52}$$

Proof. [7] If α and β are sufficiently large, then the integral representation given by

$$f(x,t) = \frac{1}{(2\pi i)^2}\int_{\alpha-i\infty}^{\alpha+i\infty} e^{px}\left[\int_{\beta-i\infty}^{\beta+i\infty} e^{st}F(p,s)ds\right]dp,$$

for the inverse double Laplace transform, can be used to obtain

$$\begin{aligned}
f(x,t) &= \frac{1}{(2\pi i)^2}\int_{\alpha-i\infty}^{\alpha+i\infty}\int_{\beta-i\infty}^{\beta+i\infty} e^{px}e^{st}F(p,s)dsdp \\
&= \frac{1}{(2\pi i)^2}\int_{\alpha-i\infty}^{\alpha+i\infty}\int_{\beta-i\infty}^{\beta+i\infty} e^{px}e^{st}G(p,s)dsdp \\
&= g(x,t).
\end{aligned}$$

1.2.4 Properties of Double Laplace Transform

a. Linearity property

If $f(x,t)$ and $g(x,t)$ be two functions of x and t such that

$$L_xL_t\{f(x,t)\} = F(p,s), \tag{1.53}$$

and

$$L_xL_t\{g(x,t)\} = G(p,s), \tag{1.54}$$

then

$$L_xL_t\{\alpha f(x,t) + \beta g(x,t)\} = \alpha F(p,s) + \beta G(p,s), \tag{1.55}$$

where α and β are constants.

Proof.

This follows easily from the linearity of the integral [4].

b. Change of scale property

If $L_xL_t\{f(x,t)\} = F(p,s)$, then

$$L_xL_t\{f(ax,bt)\} = \frac{1}{ab}F(\frac{p}{a},\frac{s}{b}), \tag{1.56}$$

where a and b are real constants [5].

Proof.

From (1.47), we have

$$L_xL_t\{f(ax,bt)\} = \int_0^\infty e^{-px}\int_0^\infty e^{-st}f(ax,bt)\,dt\,dx. \tag{1.57}$$

We put $ax = u$ and $bt = v$ in the integral of (1.57), where u and v takes the limit from 0 to ∞. Hence, we get

$$\begin{aligned} L_tL_x\{f(ax,bt)\} &= \int_0^\infty e^{-p(\frac{u}{a})}\int_0^\infty e^{-s(\frac{v}{b})}f(u,v)\,\frac{dv}{b}\frac{du}{a} \\ &= \frac{1}{ab}\int_0^\infty e^{-p(\frac{u}{a})}\int_0^\infty e^{-s(\frac{v}{b})}f(u,v)\,dv\,du = \frac{1}{ab}F(\frac{p}{a},\frac{s}{b}), \end{aligned}$$

thus

$$L_xL_t\{f(ax,bt)\} = \frac{1}{ab}F(\frac{p}{a},\frac{s}{b}).$$

c. First shifting property

If $L_xL_t\{f(x,t)\} = F(p,s)$, then

$$L_xL_t\{e^{ax+bt}f(x,t)\} = F(p-a,s-b), \tag{1.58}$$

where a and b are constants [5].
Proof.

From (1.47), we have

$$\begin{aligned} L_x L_t\{e^{ax+bt} f(x,t)\} &= \int_0^\infty e^{-px} \int_0^\infty e^{-st} e^{ax+bt} f(x,t)\, dt\, dx \\ &= \int_0^\infty e^{-(p-a)x} \int_0^\infty e^{-(s-b)t} f(x,t)\, dt\, dx \\ &= F(p-a, s-b). \end{aligned}$$

Thus

$$L_x L_t\{e^{ax+bt} f(x,t)\} = F(p-a, s-b).$$

d. Double Laplace transform of integral

If $L_x L_t\{f(x,t)\} = F(p,s)$, then

$$L_x L_t\{\int_0^x \int_0^t f(u,v)\, du\, dv\} = \frac{F(p,s)}{ps}. \quad p > 0,\ s > 0 \tag{1.59}$$

e. Multiplication by $x^m t^n$

If $L_x L_t\{f(x,t)\} = F(p,s)$, then

$$L_x L_t\{x^m t^n f(x,t)\} = (-1)^{m+n} \frac{\partial^{m+n}}{\partial p^m \partial s^n} F(p,s). \tag{1.60}$$

1.2.5 The Double Laplace transform of derivatives.

In this section we present the double Laplace transform of the partial derivatives of two variables x and t functions.

Theorem 1.2.5 *[3] (the double Laplace transform of the first order partial derivatives)*

If $f(x,t)$ be a continuous function and its first order partial derivatives are of exponential order, then

$$L_x L_t \left\{ \frac{\partial f(x,t)}{\partial x} \right\} (p,s) = p L_x L_t\{f(x,t)\} - L_t\{f(0,t)\}. \tag{1.61}$$

$$L_x L_t \left\{ \frac{\partial f(x,t)}{\partial t} \right\} (p,s) = s L_x L_t\{f(x,t)\} - L_x\{f(x,0)\}. \tag{1.62}$$

respectively, where $x, t > 0$.
Proof.

We will prove the following relationship (1.61),
from (1.47), by using the definition of the double Laplace transform

$$L_xL_t\{f(x,t)\} = \int_0^\infty e^{-px}\int_0^\infty e^{-st}f(x,t)dtdx,$$

we now calculate the double Laplace transform of the partial derivative with respect to x

$$\begin{aligned}
L_xL_t\left\{\frac{\partial f(x,t)}{\partial x}\right\} &= \int_0^\infty e^{-st}\left\{\int_0^\infty e^{-px}\frac{\partial f(x,t)}{\partial x}dx\right\}dt \\
&= \int_0^\infty e^{-st}\left\{\left[e^{-px}f(x,t)\right]_0^\infty - \int_0^\infty (-p)e^{-px}f(x,t)dx\right\}dt \\
&= \int_0^\infty e^{-st}\left\{0 - f(0,t) + p\int_0^\infty e^{-px}f(x,t)dx\right\}dt \\
&= -\int_0^\infty e^{-st}f(0,t)dt + p\int_0^\infty e^{-st}\int_0^\infty e^{-px}f(x,t)dxdt \\
&= -L_t\{f(0,t)\} + pL_tL_x\{f(x,t)\} \\
&= pL_tL_x\{f(x,t)\} - L_t\{f(0,t)\}.
\end{aligned}$$

Thus

$$L_tL_x\left\{\frac{\partial f(x,t)}{\partial x}\right\} = pL_tL_x\{f(x,t)\} - L_t\{f(0,t).$$

Similarly, we can formula (1.62).

Theorem 1.2.6 *[3] (the double Laplace transform of the second order partial derivatives)*

Let $f(x,t)$ be a continuous function of exponential order such that its second partial derivatives are continuous functions of exponential order as well, then

$$L_xL_t\left\{\frac{\partial^2 f(x,t)}{\partial x^2}\right\} = p^2L_xL_t\{f(x,t)\} - pL_t\{f(0,t)\} - L_t\left\{\frac{\partial f(0,t)}{\partial x}\right\}. \tag{1.63}$$

$$L_xL_t\left\{\frac{\partial^2 f(x,t)}{\partial t^2}\right\} = s^2L_xL_t\{f(x,t)\} - sL_x\{f(x,0)\} - L_x\left\{\frac{\partial f(x,0)}{\partial t}\right\}. \tag{1.64}$$

Proof.

We will prove the following relationship (1.63)

From (1.61), we get

$$L_xL_t\left\{\frac{\partial f(x,t)}{\partial x}\right\} = pL_xL_t\{f(x,t)\} - L_t\{f(0,t)\},$$

using (1.61), we get

$$\begin{aligned}
L_xL_t\left\{\frac{\partial^2 f(x,t)}{\partial x^2}\right\} &= p[pL_xL_t\{f(x,t)\} - L_t\{f(0,t)\}] - L_t\{f_x(0,t)\} \\
&= p^2L_xL_t\{f(x,t)\} - pL_t\{f(0,t)\} - L_t\{f_x(0,t)\}.
\end{aligned}$$

Thus

$$L_xL_t\left\{\frac{\partial^2 f(x,t)}{\partial x^2}\right\} = p^2L_xL_t\{f(x,t)\} - pL_t\{f(0,t)\} - L_t\left\{\frac{\partial f(0,t)}{\partial x}\right\}.$$

Similarly, we can formula (1.64).

1.2.6 The double convolution

Definition 1.2.7 *[14]* The double convolution between two continuous functions $f(x,t)$ and $g(x,t)$ is defined by

$$(f**g)(x,t) = \int_0^x\int_0^t f(\nu,\tau)g(x-\nu,t-\tau)\,d\tau d\nu. \tag{1.65}$$

Theorem 1.2.8 *[14] **(the double Laplace transform of convolution)***
If $L_xL_t\{f(x,t)\} = F(p,s)$, and $L_xL_t\{g(x,t)\} = G(p,s)$

$$L_xL_t\{(f**g)(x,t)\} = L_xL_t\{f(x,t)\}L_xL_t\{g(x,t)\} \tag{1.66}$$
$$= F(p,s)G(p,s). \tag{1.67}$$

Proof.
From the definition of double Laplace transform and the definition of double convolution we get

$$\begin{aligned}L_xL_t\{(f**g)(x,t)\} &= \int_0^\infty\int_0^\infty e^{-px}e^{-st}\int_0^x\int_0^t f(\nu,\tau)g(x-\nu,t-\tau)d\tau d\nu dx dt\\ &= \int_0^\infty\int_0^\infty\int_0^x\int_0^t e^{-px}e^{-st}f(\nu,\tau)g(x-\nu,t-\tau)d\tau d\nu dx dt.\end{aligned}$$

If one uses the change of variable

$$\begin{cases}x = u+\nu\\ t = w+\tau\end{cases}$$

and

$$\begin{cases}\nu = \nu\\ \tau = \tau\end{cases}$$

we get

$$\begin{aligned}L_xL_t\{(f**g)(x,t)\} &= \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-p(u+\nu)}e^{-s(w+\tau)}f(\nu,\tau)g(u,w)d\tau d\nu du dw\\ &= \int_0^\infty\int_0^\infty e^{-pu}e^{-sw}g(u,w)du dw\int_0^\infty\int_0^\infty e^{-p\nu}e^{-s\tau}f(\nu,\tau)d\nu d\tau\\ &= F(p,s)G(p,s).\end{aligned}$$

1.2.7 Table of double Laplace transform

In this section, we provide a table of commonly used double Laplace transforms, which serve as useful references when solving partial differential equations involving two independent variables [13].

$f(x,t)$	$L_xL_t\{f(x,t)\}$
1	$\frac{1}{ps}$
$e^{(ax+bt)}$	$\frac{1}{(p-a)(s-b)}$
$e^{i(ax+bt)}$	$\frac{(ps-ab)+i(as+bp)}{(p^2+a^2)(s^2+b^2)}$
$\cos(ax+bt)$	$\frac{ps-ab}{(p^2+a^2)(s^2+b^2)}$
$\sin(ax+bt)$	$\frac{as-bp}{(p^2+a^2)(s^2+b^2)}$
$\cosh(ax+bt)$	$\frac{1}{2}\left[\frac{1}{(p-a)(s-b)}+\frac{1}{(p+a)(s+b)}\right]$
$\sinh(ax+bt)$	$\frac{1}{2}\left[\frac{1}{(p-a)(s-b)}-\frac{1}{(p+a)(s+b)}\right]$
$e^{-ax-bt}f(x,t)$	$F(p+a,s+b)$
$(xt)^n$	$\frac{(n!)^2}{(ps)^{n+1}}$

1.2.8 Example

In this example, we aim to solve the following partial differential equation using the double Laplace transform

$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial x}, \quad x,t\geq 0 \tag{1.68}$$

with initial and boundary conditions

$$u(x,0)=x, \quad u(0,t)=t. \tag{1.69}$$

We apply the double Laplace transform to (1.68), we obtain

$$\mathcal{L}_x\mathcal{L}_t\left\{\frac{\partial u}{\partial t}\right\}=\mathcal{L}_x\mathcal{L}_t\left\{\frac{\partial u}{\partial x}\right\}. \tag{1.70}$$

We define

$$\mathcal{L}_x\mathcal{L}_t\{u(x,t)\}=F(p,s). \tag{1.71}$$

According to (1.61) and (1.62), we get

$$sF(p,s) - \mathcal{L}_x\{u(x,0)\} = pF(p,s) - \mathcal{L}_t\{u(0,t)\}, \tag{1.72}$$

Substituting the initial and boundary conditions, $u(x,0) = x$ and $u(0,t) = t$, and their respective Laplace transforms

$$\mathcal{L}_x\{x\} = \frac{1}{p^2}, \quad \mathcal{L}_t\{t\} = \frac{1}{s^2}, \tag{1.73}$$

Now, equating the transformed PDE

$$sF(p,s) - \frac{1}{p^2} = pF(p,s) - \frac{1}{s^2}. \tag{1.74}$$

Rearranging to solve for $F(p,s)$

$$(s-p)F(p,s) = \frac{1}{p^2} - \frac{1}{s^2} = \frac{(s-p)(s+p)}{p^2 s^2}. \tag{1.75}$$

Dividing both sides by $s-p$

$$F(p,s) = \frac{s+p}{p^2 s^2} = \frac{1}{p^2 s} + \frac{1}{p s^2}. \tag{1.76}$$

We apply the double inverse Laplace transform to (1.76), we obtain

$$\mathcal{L}_x^{-1}\mathcal{L}_t^{-1}\{F(p,s)\} = \mathcal{L}_x^{-1}\mathcal{L}_t^{-1}\left\{\frac{1}{p^2 s} + \frac{1}{p s^2}\right\}, \tag{1.77}$$

therefore, the solution of the equation (1.68), is

$$u(x,t) = x + t. \tag{1.78}$$

Chapter 2

Adomian Decomposition Method (ADM)

The Adomian Decomposition Method (ADM) is an analytical technique developed by the American mathematician **George Adomian** in the 1980s for solving linear and nonlinear differential equations. The method gained wide recognition for its ability to produce accurate solutions without the need for linearization or approximation. ADM works by decomposing the solution into a series of terms and handling nonlinearities using specially constructed **Adomian polynomials**, making it efficient and easy to apply in various scientific and engineering fields.

In this chapter, we will explain the main principles of this method. We then propose the practical method of calculating Adomian polynomials and some examples, after we will study the convergence of the method

2.1 Principle of the ADM method

Consider the functional equation:

$$Ay = f. \tag{2.1}$$

Where A is a differential operator containing linear terms and nonlinear terms and f is a known function. The linear term of the operator A is decomposed into $L + R$ where L is invertible and R the remainder of (2.1). We denote by N the nonlinear term of A and therefore $A = L + R + N$, then (2.1) is written as

$$Ly + Ry + Ny = f, \tag{2.2}$$

since L is invertible, if L^{-1} is its inverse we have

$$y = \Phi + L^{-1}f - L^{-1}Ry - L^{-1}Ny, \tag{2.3}$$

where Φ is the constant of integration.

Adomian's method consists of searching for the solution in the form of a series

$$y = \sum_{n=0}^{+\infty} y_n, \tag{2.4}$$

and to decompose the non-linear term Ny in the form of a series:

$$Ny = F(y) = \sum_{n=0}^{+\infty} A_n. \tag{2.5}$$

The terms A_n are called Adomian polynomials and are obtained through the following relationship

$$A_n(y_0, y_1, \ldots, y_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n} N\left[(\sum_{n=0}^{+\infty} \lambda^n y_n)\right]_{\lambda=0}, \tag{2.6}$$

where λ is a real parameter.

Replacing relations (2.4) and (2.5) in (2.3), we obtain

$$\sum_{n=0}^{+\infty} y_n = \Phi + L^{-1}f - L^{-1}\sum_{n=0}^{+\infty} Ry_n - L^{-1}\sum_{n=0}^{+\infty} A_n. \tag{2.7}$$

Which leads to identification bet

$$\begin{cases} y_0 = \Phi + L^{-1}f, \\ y_1 = -L^{-1}Ry_0 - L^{-1}A_0, \\ \vdots \\ y_{n+1} = -L^{-1}Ry_n - L^{-1}A_n. \end{cases} \tag{2.8}$$

All terms of the series $\sum_{n=0}^{\infty} y_n$ cannot be calculated, we use the approximation

$$\varphi_n = \sum_{i=0}^{n-1} y_i, \quad n \geq 1, \quad \text{with} \quad \lim_{n\to\infty} \varphi_n = y \tag{2.9}$$

2.2 Adomian polynomials

There are several methods to calculate Adomian polynomials, we will present a simple method to calculate them.

Theorem 2.2.1 *[6]* Suppose nonlinear function $Ny = F(y)$, and the parameterized representation of y is $y(\lambda) = \sum_{k=0}^{\infty} \lambda^k y_k$, where λ is a parameter, then we have

$$\left.\frac{\partial^n F(y(\lambda))}{\partial \lambda^n}\right|_{\lambda=0} = \left.\frac{\partial^n F\left(\sum_{k=0}^{\infty} \lambda^k y_k\right)}{\partial \lambda^n}\right|_{\lambda=0} = \left.\frac{\partial^n F\left(\sum_{k=0}^{n} \lambda^k y_k\right)}{\partial \lambda^n}\right|_{\lambda=0}. \tag{2.10}$$

Proof. Adomian decomposition method Adomian decomposition method Since

$$y(\lambda) = \sum_{k=0}^{\infty} \lambda^k y_k = \sum_{k=0}^{n} \lambda^k y_k + \sum_{k=n+1}^{\infty} \lambda^k y_k,$$

we have such result as following

$$\begin{aligned}\left.\frac{\partial^n F(y(\lambda))}{\partial \lambda^n}\right|_{\lambda=0} &= \left.\frac{\partial^n F\left(\sum_{k=0}^{\infty} \lambda^k y_k\right)}{\partial \lambda^n}\right|_{\lambda=0} = \left.\frac{\partial^n F\left(\sum_{k=0}^{n} \lambda^k y_k + \sum_{k=n+1}^{\infty} \lambda^k y_k\right)}{\partial \lambda^n}\right|_{\lambda=0} \\ &= \left.\frac{\partial^n F\left(\sum_{k=0}^{n} \lambda^k y_k\right)}{\partial \lambda^n}\right|_{\lambda=0},\end{aligned}$$

Therefore,we obtain

$$\left.\frac{\partial^n F(y(\lambda))}{\partial \lambda^n}\right|_{\lambda=0} = \left.\frac{\partial^n F\left(\sum_{k=0}^{\infty} \lambda^k y_k\right)}{\partial \lambda^n}\right|_{\lambda=0} = \left.\frac{\partial^n F\left(\sum_{k=0}^{n} \lambda^k y_k\right)}{\partial \lambda^n}\right|_{\lambda=0}. \quad \square$$

As the representation introduced by Adomian,we assume the following form again

$$Ny(\lambda) = F(y(\lambda)) = \sum_{k=0}^{\infty} \lambda^k A_k, \tag{2.11}$$

So we have

$$F(y(\lambda)) = F\left(\sum_{k=0}^{\infty} \lambda^k y_k\right) = \sum_{k=0}^{\infty} \lambda^k A_k. \tag{2.12}$$

In order to obtain A_n, we give n-order derivative of both sides of (2.12) with respect to λ and let $\lambda = 0$, that is

$$\left.\frac{\partial^n F(y(\lambda))}{\partial \lambda^n}\right|_{\lambda=0} = \left.\frac{\partial^n \left(\sum_{k=0}^{\infty} \lambda^k A_k\right)}{\partial \lambda^n}\right|_{\lambda=0}. \tag{2.13}$$

According to theorem 2.2.1

$$\left.\frac{\partial^n F(y(\lambda))}{\partial \lambda^n}\right|_{\lambda=0} = \left.\frac{\partial^n F\left(\sum_{k=0}^{n} \lambda^k y_k\right)}{\partial \lambda^n}\right|_{\lambda=0}, \tag{2.14}$$

and

$$\left.\frac{\partial^n\left(\sum_{k=0}^{\infty}\lambda^k A_k\right)}{\partial\lambda^n}\right|_{\lambda=0} = \left.\frac{\partial^n\left(\sum_{k=0}^{n}\lambda^k A_k\right)}{\partial\lambda^n}\right|_{\lambda=0}. \tag{2.15}$$

So

$$\left.\frac{\partial^n F\left(\sum_{k=0}^{n}\lambda^k y_k\right)}{\partial\lambda^n}\right|_{\lambda=0} = \left.\frac{\partial^n\left(\sum_{k=0}^{n}\lambda^k A_k\right)}{\partial\lambda^n}\right|_{\lambda=0}. \tag{2.16}$$

For (2.16) when $n = 0$, we can get A_0; when $n = 1$, we can obtain A_1; go on this course, we will get $A_2, A_3, \dots, A_{n-1}, A_n$.

The following is the algorithm for calculating $A_0, A_1, A_2, \dots, A_{n-1}, A_n$:

Step 1: Input nonlinear term $Ny = F(y)$ and n that is the order of *Adomian* polynomials.

Step 2: Set $y = y_0 + \lambda y_1 + \lambda^2 y_2 + \cdots + \lambda^n y_n$.

Step 3: Let $\sum_{k=0}^{n}\lambda^k A_k = F(y_0 + \lambda y_1 + \lambda^2 y_2 + \cdots + \lambda^n y_n)$.

Step 4: For $i = 0, 1, \dots, n$ do

1. i-th order derivative of both sides of the above equality with respect to λ

$$\frac{\partial^i\left(\sum_{k=0}^{n}\lambda^k A_k\right)}{\partial\lambda^i} = \frac{\partial^i F(y_0 + \lambda y_1 + \lambda^2 y_2 + \cdots + \lambda^n y_n)}{\partial\lambda^i}. \tag{2.17}$$

2. For (2.17), let $\lambda = 0$ and determine A_i by solving the equation with respect to A_i.

Step 5: Output $A_0, A_1, \dots, A_n$.

Example 2.1 We will calculate the Adomian polynomials of $F(y) = y^2$.

$$y(\lambda) = \sum_{k=0}^{\infty}\lambda^k y_k, \tag{2.18}$$

we have

$$F(y(\lambda)) = F\left(\sum_{k=0}^{\infty}\lambda^k y_k\right) = \sum_{k=0}^{\infty}\lambda^k A_k, \tag{2.19}$$

according to (2.19), we obtain

$$\sum_{k=0}^{\infty}\lambda^k A_k = (y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3 + \lambda^4 y_4 + \dots)^2. \tag{2.20}$$

For $\lambda = 0$, we obtain

$$A_0 = y_0^2. \tag{2.21}$$

If we take the derivative on both sides of (2.20) with respect to λ, then we take $\lambda = 0$, we obtient

$$\left.\frac{\partial(A_0+\lambda A_1)}{\partial\lambda}\right|_{\lambda=0}=\left.\frac{\partial(y_0+\lambda y_1)^2}{\partial\lambda}\right|_{\lambda=0}, \tag{2.22}$$

solve this equation with respect to A_1, we obtain

$$A_1=2y_0y_1. \tag{2.23}$$

Similarly, we take the second-order derivative on both sides of (2.20), with respect to λ (using (2.16)), then we take $\lambda=0$, and by solving the equation with respect to A_2, we can obtain

$$\left.\frac{\partial^2(A_0+\lambda A_1+\lambda^2A_2)}{\partial\lambda^2}\right|_{\lambda=0}=\left.\frac{\partial^2(y_0+\lambda y_1+\lambda^2y_2)^2}{\partial\lambda^2}\right|_{\lambda=0}, \tag{2.24}$$

so

$$2A_2=2y_1^2+4y_0y_2, \tag{2.25}$$

from where

$$A_2=y_1^2+2y_0y_2, \tag{2.26}$$

following the same steps, we obtain

$$\left.\frac{\partial^3(A_0+\lambda A_1+\lambda^2A_2+\lambda^3A_3)}{\partial\lambda^3}\right|_{\lambda=0}=\left.\frac{\partial^3(y_0+\lambda y_1+\lambda^2y_2+\lambda^3y_3)^2}{\partial\lambda^3}\right|_{\lambda=0}, \tag{2.27}$$

so

$$3A_3=6y_0y_3+6y_1y_2, \tag{2.28}$$

from where

$$A_3=2y_0y_3+2y_1y_2. \tag{2.29}$$

If we take the derivative of order four (4) on both sides of (2.20), we obtain

$$\left.\frac{\partial^4(A_0+\lambda A_1+\lambda^2A_2+\lambda^3A_3+\lambda^4A_4)}{\partial\lambda^4}\right|_{\lambda=0}=\left.\frac{\partial^4(y_0+\lambda y_1+\lambda^2y_2+\lambda^3y_3+\lambda^4y_4)^2}{\partial\lambda^4}\right|_{\lambda=0},$$

$$24A_4=48y_0y_4+24y_2^2+48y_1y_3, \tag{2.30}$$

from where

$$A_4=2y_0y_4+y_2^2+2y_1y_3. \tag{2.31}$$

Let's continue the same steps, we get $A_5, A_6, \ldots$.

Example 2.2 We will calculate the Adomian polynomials of $F(y)=y^3$.

Let

$$y(\lambda)=\sum_{k=0}^{\infty}\lambda^ky_k, \tag{2.32}$$

we have

$$F(y(\lambda))=F\left(\sum_{k=0}^{\infty}\lambda^ky_k\right)=\sum_{k=0}^{\infty}\lambda^kA_k, \tag{2.33}$$

According to (2.33), we obtain

$$\sum_{k=0}^{\infty} \lambda^k A_k = (y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3 + ...)^3, \tag{2.34}$$

For $\lambda = 0$, we obtain

$$A_0 = y_0^3. \tag{2.35}$$

If we take the derivative on both sides of (2.34) with respect to λ, then set $\lambda = 0$, we obtain

$$\left.\frac{\partial(A_0 + \lambda A_1)}{\partial \lambda}\right|_{\lambda=0} = \left.\frac{\partial(y_0 + \lambda y_1)^3}{\partial \lambda}\right|_{\lambda=0}, \tag{2.36}$$

solve this equation with respect to A_1, we obtain

$$A_1 = 3y_1 y_0^2. \tag{2.37}$$

Similarly, if we take the second-order derivative on both sides of (2.34) with respect to λ (using (2.16)), then set $\lambda = 0$, and solve for A_2, we obtain

$$\left.\frac{\partial^2(A_0 + \lambda A_1 + \lambda^2 A_2)}{\partial \lambda^2}\right|_{\lambda=0} = \left.\frac{\partial^2(y_0 + \lambda y_1 + \lambda^2 y_2)^3}{\partial \lambda^2}\right|_{\lambda=0}, \tag{2.38}$$

so

$$A_2 = 3y_2 y_0^2 + 3y_1^2 y_0. \tag{2.39}$$

When the following words appear, we have options

$$\left.\frac{\partial^3(A_0 + \lambda A_1 + \lambda^2 A_2 + \lambda^3 A_3)}{\partial \lambda^3}\right|_{\lambda=0} = \left.\frac{\partial^3(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)^3}{\partial \lambda^3}\right|_{\lambda=0}, \tag{2.40}$$

from which

$$A_3 = 3y_3 y_0^2 + 6y_2 y_1 y_0 + y_1^3. \tag{2.41}$$

Let's continue the same steps, we get $A_4, A_5, \ldots$.

2.3 Convergence of the Adomian Method

We will study the convergence of the ADM method

Let the differential equation be

$$u - Nu = f \tag{2.42}$$

This is called the canonical form, where u is the sought solution and N is the nonlinear operator, which is defined by an infinite series, and f is a given function.

By substituting the decomposition series into (2.42), we find

$$\sum_{n=0}^{\infty} u_n - \sum_{n=0}^{\infty} A_n = f \tag{2.43}$$

Then we obtain the recurrence relation

$$\begin{cases} u_0 = f \\ u_{n+1} = A_n(u_0, u_1, \dots, u_n) \end{cases} \tag{2.44}$$

From this formula, we deduce that

$$\sum_{i=1}^{n} u_i = \sum_{i=1}^{n} A_i \tag{2.45}$$

Indeed, from relation (2.43), we deduce

Theorem 2.3.1 *[9]* If $\sum\limits_{n=0}^{\infty} A_n < \infty$, then $\sum\limits_{n=0}^{\infty} u_n < \infty$, and vice versa

The Adomian method consists in determining the sequence s_n given by

$$\begin{cases} s_0 = 0 \\ s_n = u_1 + \cdots + u_n \end{cases} \tag{2.46}$$

and verifying the following recurrence relation

$$\begin{cases} s_0 = 0, \quad u_0 = f \\ s_{n+1} = N(u_0 + s_n), \quad n = 0, 1, 2, \dots \end{cases} \tag{2.47}$$

Hence the following theorem

Theorem 2.3.2 *[9]* If the operator N is a contraction mapping ($\|N\| \leq \delta < 1$), then the sequence $(s_n)_n$ satisfying the recurrence relation

$$\begin{cases} s_0 = 0 \\ s_{n+1} = N(u_0 + s_n), \quad n \geq 0 \end{cases} \tag{2.48}$$

converges to s, where s is the solution of the equation $s_{n+1} = N(u_0 + s_n)$.

Proof.

from relation (2.48) we have

$$\begin{aligned}
s_{n+1} - s_n &= N(u_0 + s_n) - N(u_0 + s) \\
\|s_{n+1} - s_n\| &= \|N(u_0 + s_n) - N(u_0 + s)\| \\
&\leq \|N\| \|s_n - s\| \\
&< \delta \|s_n - s\| \\
&< \delta^2 \|s_{n-1} - s\| \\
&< \delta^3 \|s_{n-2} - s\| \\
&\vdots \\
&< \delta^n \|s_1 - s\|
\end{aligned}$$

Thus, the sequence $(s_n)_n$ converges.
Moreover, we have

$$\sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} A_n \tag{2.49}$$

And since $\sum\limits_{n=1}^{\infty} u_n$ is convergent according to Theorem (2.3.1)
Thus, we obtain the following result

Corollary 2.3.3 If N is a contraction, then the series of u_n and A_n are convergent.
Furthermore, $\sum\limits_{n=0}^{\infty} u_n$ is a solution of the canonical equation (2.42).

2.4 Solving algebraic equations using the ADM method

The Adomian decomposition method also allows us to solve algebraic equations. We consider the following equation:

$$x^2 - 3.26x + 2.376 = 0, \tag{2.50}$$

where we have $Lx = -3.26x$, $Nx = x^2$, $f(x) = -2.376$.
L^{-1} is the inverse of -3.26, thus: $L^{-1} = \frac{-1}{3.26}$.
We seek the solution in the form of a series

$$x = \sum_{n=0}^{+\infty} x_n, \tag{2.51}$$

and we decompose the nonlinear term Nx into a series

$$Nx = F(x) = \sum_{n=0}^{+\infty} \lambda^n A_n, \tag{2.52}$$

The first terms of the Adomian polynomials are given by (see example (2.1))

$$\begin{aligned} A_0 &= x_0^2, \\ A_1 &= 2x_1x_0, \\ A_2 &= 2x_0x_2 + x_1^2, \end{aligned}$$

From which we obtain

$$\begin{cases} x_0 = L^{-1}f = \frac{2.376}{3.26} = 0.7288, \\ x_1 = -L^{-1}A_0 = \frac{0.5311}{3.26} = 0.1629, \\ x_2 = -L^{-1}A_1 = \frac{0.2374}{3.26} = 0.0728, \\ x_3 = -L^{-1}A_2 = \frac{0.1326}{3.26} = 0.0406. \end{cases} \tag{2.53}$$

The approximate solution is

$$x = \sum_{i=0}^{3} x_i = 0.7288 + 0.1629 + 0.0728 + 0.0406 \simeq 1.0051. \tag{2.54}$$

And since

$$\begin{cases} x + x^* = 3.26, \\ xx^* = 2.376. \end{cases} \tag{2.55}$$

Thus, the roots of equation (2.53) are given by

$$x \simeq 1.0051, \tag{2.56}$$

$$x^* \simeq 2.2549. \tag{2.57}$$

2.5 Resolution of differential equations by the ADM method

We consider the following ordinary differential equation

$$y' - y^2 + 2y = 1, \tag{2.58}$$

with the initial condition

$$y(0) = 2. \tag{2.59}$$

The exact solution of this equation is given by

$$y = \frac{1}{1-t} + 1, \quad |t| < 1. \tag{2.60}$$

According to the steps of the ADM method, we have

$$L = \frac{d}{dt}, \quad \text{and} \quad L^{-1} = \int_0^t (.)ds, \quad Ry = 2y, \quad Ny = y^2, \quad f = 1. \tag{2.61}$$

and

$$\sum_{n=0}^{+\infty} y_n = y(0) + L^{-1}\Big(\sum_{n=0}^{+\infty} A_n - 2\sum_{n=0}^{+\infty} y_n + 1\Big), \tag{2.62}$$

With A_i being the Adomian polynomials of the nonlinear term y^2, by identification, we have

$$\begin{cases} y_0 = y(0) + L^{-1}(1), \\ y_1 = L^{-1}(A_0 - 2y_0), \\ y_2 = L^{-1}(A_1 - 2y_1), \\ y_3 = L^{-1}(A_2 - 2y_2), \\ \vdots \end{cases} \tag{2.63}$$

The first terms of the Adomian polynomials are given by (see example (2.1))

$$\begin{cases} A_0 = y_0^2, \\ A_1 = 2y_1 y_0, \\ A_2 = 2y_0 y_2 + y_1^2. \end{cases} \tag{2.64}$$

By substituting (2.64) into (2.63), we obtain

$$\begin{cases} y_0 = y(0) + L^{-1}(1), \\ y_1 = L^{-1}(y_0^2 - 2y_0), \\ y_2 = L^{-1}(2y_1 y_0 - 2y_1), \\ y_3 = L^{-1}((2y_0 y_2 + y_1^2) - 2y_2), \\ \vdots \end{cases} \tag{2.65}$$

the first terms of the approximate solution are given by

$$\begin{cases} y_0 = 2 + t, \\ y_1 = t^2 + \frac{1}{3}t^3, \\ y_2 = \frac{2}{3}t^3 + \frac{2}{3}t^4 + \frac{2}{15}t^5, \\ \vdots \end{cases} \tag{2.66}$$

According to the ADM method, the solution is given by

$$y = \lim_{n\to+\infty} \sum_{n=0}^{+\infty} y_n, \tag{2.67}$$

thus, we have

$$y(t) = 1 + \lim_{n\to+\infty} \left(1 + t + t^2 + t^3 + \dots\right), \tag{2.68}$$

therefore

$$y(t) = \frac{1}{1-t} + 1, \quad |t| < 1. \tag{2.69}$$

Chapter 3

Application

In this chapter, we explore the practical implementation of the Double Laplace Decomposition Method (DLDM) a powerful analytical approach for solving nonlinear partial differential equations (PDEs) under given initial conditions. To demonstrate the method's robustness and broad applicability, we present four representative examples: the gas dynamics equation, a nonlinear wave-like equation, a system of coupled nonlinear PDEs, and the nonlinear Klien-Gordon equation. These examples illustrate how DLDM effectively handles nonlinearity and complexity, providing exact or highly accurate series solutions with systematic computational procedures.

3.1 Outline of The Method

This method is described as in the following manner. Let us consider the nonlinear nonhomogeneous partial differential equation in operator form

$$Lu(x,t) + Ru(x,t) + Nu(x,t) = h(x,t), \tag{3.1}$$

with initial conditions $u(0,t) = f(t)$ and $u_x(0,t) = g(t)$.

Here L is a second order partial differential operator with respect to x, R is a remaining linear operator, N represents a general nonlinear differential operator, and $h(x,t)$ is a source term.

At the beginning of this method, the double Laplace transform is applied to both sides of the equation (3.1). Then we have

$$L_x L_t \left[Lu(x,t) + Ru(x,t) + Nu(x,t)\right] = L_x L_t[h(x,t)]. \tag{3.2}$$

Using the linearity and the differentiation properties of the double Laplace transform

$$U(p,s) = \frac{F(s)}{p} + \frac{G(s)}{p^2} + \frac{1}{p^2} L_x L_t[h(x,t)] - \frac{1}{p^2}\left[L_x L_t[Ru(x,t)] + L_x L_t[Nu(x,t)]\right], \tag{3.3}$$

where $U(p,s)$, $F(s)$, and $G(s)$ represent the double Laplace transforms of $u(x,t)$, $f(t)$, and $g(t)$, respectively.

After this step, we use the following decomposition series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \dots \tag{3.4}$$

for the linear terms. And also, the infinite series defined by

$$N(u(x,t)) = \sum_{n=0}^{\infty} A_n(u(x,t)), \tag{3.5}$$

is used for the nonlinear terms. Here A_n represents the Adomian polynomials, described by

$$A_n = \frac{1}{n!}\frac{d^n}{d\alpha^n}\left[N\left(\sum_{i=0}^{\infty}\alpha^i u_i\right)\right]\Bigg|_{\alpha=0}, \quad n = 0,1,2,\dots \tag{3.6}$$

From this definition, we get the first terms as below:

$$A_0 = N(u_0), \quad A_1 = u_1 N'(u_0), \quad A_2 = u_2 N'(u_0) + \frac{1}{2!}u_1^2 N''(u_0). \tag{3.7}$$

Now we substitute (3.4) and (3.5) into the equation (3.3), and afterwards we get

$$\begin{aligned} L_x L_t[\sum_{n=0}^{\infty} u_n(x,t)] =& \frac{F(s)}{p} + \frac{G(s)}{p^2} + \frac{1}{p^2}L_x L_t[h(x,t)] \\ & - \frac{1}{p^2}[L_x L_t[R(\sum_{n=0}^{\infty} u_n(x,t))] + L_x L_t(\sum_{n=0}^{\infty} A_n)]. \end{aligned} \tag{3.8}$$

The inverse double Laplace transform is applied to both sides of the equation (3.8), and by the linearity of the inverse transform, we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x,t) =& f(t) + xg(t) + L_x^{-1}L_t^{-1}[\frac{1}{p^2}L_x L_t[h(x,t)]] \\ & - L_x^{-1}L_t^{-1}[\frac{1}{p^2}(L_x L_t[R(\sum_{n=0}^{\infty} u_n(x,t))] \\ & + L_x L_t(\sum_{n=0}^{\infty} A_n))]. \end{aligned} \tag{3.9}$$

Comparing both sides of the equation (3.9) yields the following equalities:

$$u_0(x,t) = f(t) + xg(t) + L_x^{-1}L_t^{-1}\left[\frac{1}{p^2}L_x L_t[h(x,t)]\right], \tag{3.10}$$

$$u_1(x,t) = -L_x^{-1}L_t^{-1}\left[\frac{1}{p^2}\left(L_xL_t[R(u_0(x,t))] + L_xL_t[A_0]\right)\right], \tag{3.11}$$

$$u_2(x,t) = -L_x^{-1}L_t^{-1}\left[\frac{1}{p^2}\left(L_xL_t[R(u_1(x,t))] + L_xL_t[A_1]\right)\right]. \tag{3.12}$$

The general form of the recursive relation is given by

$$u_{n+1}(x,t) = -L_x^{-1}L_t^{-1}\left[\frac{1}{p^2}\left(L_xL_t[R(u_n(x,t))] + L_xL_t[A_n]\right)\right], \quad n \geq 0. \tag{3.13}$$

Obtaining the components u_0, u_1, u_2, ... from the above recursive relation and putting them into the expansion (3.4) provide us with the solution $u(x,t)$.

3.2 Example 1: Gas-Dynamic equation

The gas dynamic equation is a fundamental model in fluid mechanics, describing the behavior of compressible fluid flow such as gases. In this example, we consider a nonlinear and homogeneous gas dynamic equation with a given initial condition. We employ the Adomian Decomposition Method (ADM) in combination with the double Laplace transform technique to obtain the exact solution to the equation, demonstrating the effectiveness of this analytical method .

Consider the nonlinear gas dynamics equation [11]

$$u_t + uu_x - u(1-u) = 0, \tag{3.14}$$

with the initial condition

$$u(x,0) = e^{-x}. \tag{3.15}$$

Applying the double Laplace transform to both sides of (3.14), we obtain

$$L_xL_t\left(u_t + uu_x - u + u^2\right) = 0. \tag{3.16}$$

Using the property

$$L_xL_t(u_t) = sU(p,s) - U(p,0),$$

and applying the initial condition (3.15), we have

$$sU(p,s) - \frac{1}{p+1} = L_xL_t\left(-uu_x + u - u^2\right). \tag{3.17}$$

Solving for $U(p,s)$, we get

$$U(p,s) = \frac{1}{s(p+1)} - \frac{1}{s}L_xL_t\left(uu_x - u + u^2\right). \tag{3.18}$$

Applying the inverse double Laplace transform to both sides of (3.18), we obtain

$$u(x,t) = e^{-x} - L_x^{-1}L_t^{-1}\left(\frac{1}{s}L_xL_t\left(uu_x - u + u^2\right)\right). \tag{3.19}$$

We now apply the Adomian Decomposition Method (ADM)

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{3.20}$$

and the nonlinear terms as follows

$$uu_x = \sum_{n=0}^{\infty} A_n(x,t), \quad u^2 = \sum_{n=0}^{\infty} B_n(x,t), \tag{3.21}$$

where A_n and B_n are Adomian polynomials corresponding to uu_x and u^2, respectively, defined as

$$\begin{aligned} A_0 &= u_0 u_{0x}, \\ A_1 &= u_0 u_{1x} + u_1 u_{0x}, \\ A_2 &= u_0 u_{2x} + u_1 u_{1x} + u_2 u_{0x}, \quad \dots \end{aligned}$$

$$\begin{aligned} B_0 &= u_0^2, \\ B_1 &= 2u_0 u_1, \\ B_2 &= u_1^2 + 2u_0 u_2, \quad \dots \end{aligned}$$

We set $u_0(x,t) = e^{-x}$ from the initial condition, and compute the successive terms using the recurrence relation

$$\begin{aligned} u_{n+1}(x,t) = &- L_x^{-1} L_t^{-1} \left(\frac{1}{s} L_x L_t \left(\sum_{n=0}^{\infty} A_n \right) \right) \\ &+ L_x^{-1} L_t^{-1} \left(\frac{1}{s} L_x L_t \left(\sum_{n=0}^{\infty} u_n \right) \right) \\ &- L_x^{-1} L_t^{-1} \left(\frac{1}{s} L_x L_t \left(\sum_{n=0}^{\infty} B_n \right) \right), \end{aligned} \tag{3.22}$$

Now we compute the first few components

$$\begin{aligned} u_1(x,t) = &- L_x^{-1} L_t^{-1} \left(\frac{1}{s} L_x L_t(A_0) \right) + L_x^{-1} L_t^{-1} \left(\frac{1}{s} L_x L_t(u_0) \right) - L_x^{-1} L_t^{-1} \left(\frac{1}{s} L_x L_t(B_0) \right). \\ &= -L_x^{-1} L_t^{-1} \left(\frac{1}{s} L_x L_t \left(-e^{-2x} \right) \right) + L_x^{-1} L_t^{-1} \left(\frac{1}{s} L_x L_t(e^{-x}) \right) \\ &- L_x^{-1} L_t^{-1} \left(\frac{1}{s} L_x L_t(e^{-2x}) \right) \\ &= L_x^{-1} L_t^{-1} \left(\frac{1}{s} L_x L_t(e^{-x}) \right) = L_x^{-1} L_t^{-1} \left(\frac{1}{s^2(p+1)} \right) = te^{-x}, \end{aligned} \tag{3.23}$$

$$u_2(x,t) = -L_x^{-1}L_t^{-1}\left(\frac{1}{s}L_xL_t\left(A_1\right)\right) + L_x^{-1}L_t^{-1}\left(\frac{1}{s}L_xL_t(u_1)\right) - L_x^{-1}L_t^{-1}\left(\frac{1}{s}L_xL_t(B_1)\right)$$
$$= L_x^{-1}L_t^{-1}\left(\frac{1}{s}L_xL_t(te^{-x})\right) = L_x^{-1}L_t^{-1}\left(\frac{1}{s^3(p+1)}\right) = e^{-x}\frac{t^2}{2!}, \tag{3.24}$$

$$u_3(x,t) = e^{-x}\frac{t^3}{3!}. \tag{3.25}$$

Thus, the series solution becomes

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = e^{-x}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right) = e^{-x}e^t \tag{3.26}$$

Hence, the exact solution of (3.14) is

$$u(x,t) = e^{t-x}. \tag{3.27}$$

3.3 Example 2: Wave-like equation

Nonlinear wave-like equations appear in various physical contexts such as fluid dynamics, elasticity, and wave propagation in non-homogeneous media. In this example, we demonstrate how to apply the Adomian Decomposition Method (ADM) combined with the double Laplace transform to solve a nonlinear wave-like partial differential equation .

Consider one dimensional nonlinear wave-like equation [10]

$$u_{tt} = x^2\frac{\partial}{\partial x}(u_xu_{xx}) - x^2(u_{xx})^2 - u, \tag{3.28}$$

with the initial conditions

$$u(x,0) = 0, \quad u_t(x,0) = x^2. \tag{3.29}$$

Applying the double Laplace transform to both sides of (3.28), and using the linearity of the transform, we obtain

$$L_xL_t(u_{tt}) = L_xL_t\left(x^2\frac{\partial}{\partial x}(u_xu_{xx}) - x^2(u_{xx})^2 - u\right), \tag{3.30}$$

using the double Laplace transform property for second-order time derivatives

$$L_xL_t(u_{tt}) = s^2U(p,s) - sU(p,0) - U_t(p,0),$$

and applying the initial conditions from (3.29), where $u(x,0) = 0 \Rightarrow U(p,0) = 0$ and $u_t(x,0) = x^2 \Rightarrow U_t(p,0) = \frac{2}{p^3}$, we get

$$s^2U(p,s) - \frac{2}{p^3} = L_xL_t\left(x^2\frac{\partial}{\partial x}(u_xu_{xx}) - x^2(u_{xx})^2 - u\right). \tag{3.31}$$

Solving for $U(p,s)$, we have

$$U(p,s) = \frac{2}{p^3 s^2} + \frac{1}{s^2} L_x L_t \left(x^2 \frac{\partial}{\partial x}(u_x u_{xx}) - x^2 (u_{xx})^2 - u \right). \tag{3.32}$$

Appling the inverse double Laplace transform of both sides of (3.32), we have

$$u(x,t) = x^2 t + L_x^{-1} L_t^{-1} \left(\frac{1}{s^2} L_x L_t \left(x^2 \frac{\partial}{\partial x}(u_x u_{xx}) - x^2 (u_{xx})^2 - u \right) \right), \tag{3.33}$$

where we define the nonlinear terms as

$$u_x u_{xx} = \sum_{n=0}^{\infty} A_n, \quad (u_{xx})^2 = \sum_{n=0}^{\infty} B_n. \tag{3.34}$$

The Adomian polynomials for $u_x u_{xx}$ and u_{xx} are computed as follows

$$\begin{aligned}
A_0 &= (u_0)_x (u_0)_{xx}, \\
A_1 &= (u_0)_x (u_1)_{xx} + (u_1)_x (u_0)_{xx}, \\
A_2 &= (u_0)_x (u_2)_{xx} + (u_1)_x (u_1)_{xx} + (u_2)_x (u_0)_{xx},
\end{aligned}$$

and

$$B_0 = (u_0)^2_{xx}, \quad B_1 = 2(u_0)_{xx}(u_1)_{xx}, \quad B_2 = (u_1)^2_{xx} + 2(u_0)_{xx}(u_2)_{xx}.$$

Starting with $u_0(x,t) = x^2 t$ and using

$$\begin{aligned}
u_{n+1}(x,t) =& L_x^{-1} L_t^{-1} \left(\frac{1}{s^2} L_x L_t \left(x^2 \frac{\partial}{\partial x} \sum_{n=0}^{\infty} A_n \right) \right) \\
&- L_x^{-1} L_t^{-1} \left(\frac{1}{s^2} L_x L_t \left(x^2 \sum_{n=0}^{\infty} B_n \right) \right) \\
&- L_x^{-1} L_t^{-1} \left(\frac{1}{s^2} L_x L_t \left(\sum_{n=0}^{\infty} u_n \right) \right).
\end{aligned} \tag{3.35}$$

We compute the successive terms as

$$\begin{aligned}
u_1(x,t) =& L_x^{-1} L_t^{-1} \left(\frac{1}{s^2} L_x L_t \left(x^2 \frac{\partial}{\partial x} A_0 \right) \right) - L_x^{-1} L_t^{-1} \left(\frac{1}{s^2} L_x L_t (x^2 B_0) \right) - L_x^{-1} L_t^{-1} \left(\frac{1}{s^2} L_x L_t (u_0) \right). \\
&= L_x^{-1} L_t^{-1} \left(\frac{1}{s^2} L_x L_t \left(x^2 4t^2 \right) \right) - L_x^{-1} L_t^{-1} \left(\frac{1}{s^2} L_x L_t (x^2 4t^2) \right) - L_x^{-1} L_t^{-1} \left(\frac{1}{s^2} L_x L_t (x^2 t^2) \right) \\
&= -L_x^{-1} L_t^{-1} \left(\frac{1}{s^2} L_x L_t (x^2 t^2) \right) = -L_x^{-1} L_t^{-1} \left(\frac{2}{p^3 s^4} \right) = -\frac{x^2 t^3}{3!},
\end{aligned} \tag{3.36}$$

$$u_2(x,t) = L_x^{-1}L_t^{-1}\left(\frac{1}{s^2}L_xL_t\left(x^2\frac{\partial}{\partial x}A_1\right)\right) - L_x^{-1}L_t^{-1}\left(\frac{1}{s^2}L_xL_t(x^2B_1)\right) - L_x^{-1}L_t^{-1}\left(\frac{1}{s^2}L_xL_t(u_1)\right)$$
$$= -L_x^{-1}L_t^{-1}\left(\frac{1}{s^2}L_xL_t(-\frac{x^2t^3}{3!})\right) = L_x^{-1}L_t^{-1}\left(\frac{2}{p^3s^6}\right) = \frac{x^2t^5}{5!}, \tag{3.37}$$

$$u_3(x,t) = -\frac{x^2t^7}{7!}. \tag{3.38}$$

Hence, the solution series is

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = x^2\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \dots\right), \tag{3.39}$$

the exact solution of (3.28), is

$$u(x,t) = x^2 sint. \tag{3.40}$$

3.4 Example 3: Nonlinear System of P.D.E.

Consider the nonlinear system of inhomogeneous partial differential equations [12]

$$\begin{cases} u_t + u_x v + u = 1 \\ v_t - uv_x - v = 1 \end{cases} \tag{3.41}$$

with the initial conditions

$$u(x,0) = e^x, \quad v(x,0) = e^{-x}. \tag{3.42}$$

Taking the double Laplace transform,is applied to both sides of Eq (3.41), we obtain

$$\begin{cases} L_xL_t\{u_t + u_x v + u\} = L_xL_t\{1\} \\ L_xL_t\{v_t - uv_x - v\} = L_xL_t\{1\}. \end{cases} \tag{3.43}$$

Using the double Laplace transform of the first order time derivatives

$$L_xL_t(u_t) = sU(p,s) - U(p,0),$$

$$L_xL_t(v_t) = sV(p,s) - V(p,0),$$

and applying the initial conditions from (3.42), where $u(x,0) = e^x \Rightarrow U(p,0) = \frac{1}{p-1}$ and $v(x,0) = e^{-x} \Rightarrow V(p,0) = \frac{1}{p+1}$, we get

$$\begin{cases} sU(p,s) - \frac{1}{p-1} + L_xL_t\left(vu_x + u\right) = \frac{1}{ps} \\ sV(p,s) - \frac{1}{p+1} - L_xL_t\left(uv_x - v\right) = \frac{1}{ps}. \end{cases} \tag{3.44}$$

Solving for $U(p,s)$ and $V(p,s)$, we have

$$\begin{cases} U(p,s) = \frac{1}{s(p-1)} + \frac{1}{ps^2} - \frac{1}{s} L_x L_t (v u_x + u) \\ V(p,s) = \frac{1}{s(p+1)} + \frac{1}{ps^2} + \frac{1}{s} L_x L_t (u v_x + v) \end{cases}. \tag{3.45}$$

Applying inverse double Laplace transform to (3.45), we get

$$\begin{cases} u(x,t) = e^x + t - L_x^{-1} L_t^{-1} \left[\frac{1}{s} L_x L_t (v u_x + u) \right] \\ v(x,t) = e^{-x} + t + L_x^{-1} L_t^{-1} \left[\frac{1}{s} L_x L_t (u v_x + v) \right] \end{cases}, \tag{3.46}$$

where we define the nonlinear terms as

$$\begin{cases} v u_x &= \sum_{n=0}^{\infty} A_n \\ u v_x &= \sum_{n=0}^{\infty} B_n \end{cases}. \tag{3.47}$$

The Adomian polynomials for vu_x and uv_x are computed as follows

$$A_0 = v_0 u_{0x}, \quad A_1 = v_0 u_{1x} + v_1 u_{0x}, \quad A_2 = v_0 u_{2x} + v_1 u_{1x} + v_2 u_{0x}.$$

$$B_0 = u_0 v_{0x}, \quad B_1 = u_0 v_{1x} + u_1 v_{0x}, \quad B_2 = u_0 v_{2x} + u_1 v_{1x} + u_2 v_{0x}. \tag{3.48}$$

Starting with $u_0(x,t) = e^x$ and using

$$u_{n+1} = t - L_x^{-1} L_t^{-1} \left[\frac{1}{s} L_x L_t (\sum_{n=0}^{\infty} A_n + \sum_{n=0}^{\infty} u_n) \right] \tag{3.49}$$

and

$$v_{n+1} = t - L_x^{-1} L_t^{-1} \left[\frac{1}{s} L_x L_t (\sum_{n=0}^{\infty} B_n + \sum_{n=0}^{\infty} u_n) \right]. \tag{3.50}$$

We compute the successive terms as

$$u_1 = t - L_x^{-1} L_t^{-1} \left[\frac{1}{s} L_x L_t (A_0 + u_0) \right] \tag{3.51}$$

$$= t - L_x^{-1} L_t^{-1} \left[\frac{1}{s} L_x L_t (1 + e^x) \right] \tag{3.52}$$

$$= t - L_x^{-1} L_t^{-1} \left[\frac{1}{s} \left(\frac{1}{ps} + \frac{1}{s(p-1)} \right) \right] = -te^x \tag{3.53}$$

$$u_2 = -L_x^{-1} L_t^{-1} \left[\frac{1}{s} L_x L_t (A_1 + u_1) \right]$$

$$= -L_x^{-1} L_t^{-1} \left[\frac{1}{s} \left(-\frac{1}{s^2(p-1)} \right) \right] = \frac{t^2 e^x}{2}$$

Starting with $v_0(x,t) = e^{-x}$ and using

$$
\begin{aligned}
v_1 &= t - L_x^{-1}L_t^{-1}\left[\frac{1}{s}L_xL_t(B_0+v_0)\right]\\
&= t - L_x^{-1}L_t^{-1}\left[\frac{1}{s}L_xL_t(-1+e^{-x})\right]\\
&= t - L_x^{-1}L_t^{-1}\left[\frac{1}{s}\left(-\frac{1}{ps}+\frac{1}{s(p+1)}\right)\right] = te^{-x}
\end{aligned} \tag{3.54}
$$

$$
\begin{aligned}
v_2 &= L_x^{-1}L_t^{-1}\left[\frac{1}{s}L_xL_t(B_1+v_1)\right]\\
&= L_x^{-1}L_t^{-1}\left[\frac{1}{s}\left(\frac{1}{s^2(p+1)}\right)\right] = \frac{t^2e^{-x}}{2}.
\end{aligned} \tag{3.55}
$$

Hence, the series solution is

$$
\begin{cases} u(x,t) = e^{x}\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \dots\right) \\ v(x,t) = e^{-x}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots\right) \end{cases}, \tag{3.56}
$$

the exact solution of (3.41), is

$$
\begin{cases} u(x,t) = e^{x-t} \\ v(x,t) = e^{-x+t} \end{cases}. \tag{3.57}
$$

3.5 Example 4: Klein-Gordon equation

The Klein-Gordon equation is a relativistic version of the Schrodinger equation describing free particles, which was proposed by Oskar Klein and Walter Gordon in 1926. It has many applications in physics and engineering such as quantum field theory of relativistic physics, dispersive wave phenomena, plasma physics, and nonlinear optics. Various methods have been developed to get approximate and numerical solutions of linear Klein-Gordon (LKG) and nonlinear Klein-Gordon (NLKG) equations.

Consider the following nonlinear Klein-Gordon equation similar to [15]

$$
u_{tt}(x,t) - u_{xx}(x,t) + u^2(x,t) = 2x^2 - 2t^2 + x^4t^4, \tag{3.58}
$$

with initial conditions

$$
u(x,0) = 0, \quad u_t(x,0) = 0, \tag{3.59}
$$

and boundary conditions

$$
u(0,t) = 0, \quad u_x(0,t) = 0. \tag{3.60}
$$

Applying the double Laplace transform on both sides of (3.58), we get

$$s^2U(p,s) - sU(p,0) - U_t(p,0) - p^2U(p,s) + pU(0,s) + U_x(0,s) \\ + L_xL_t[u^2(x,t)] = 2\frac{2}{p^3s} - 2\frac{2}{ps^3} + L_xL_t[x^4t^4]. \quad (3.61)$$

Further, applying single Laplace transform to initial (3.59) and boundary conditions (3.60), we get

$$U(p,0) = 0, \quad U_t(p,0) = 0, \quad U(0,s) = 0, \quad U_x(0,s) = 0. \quad (3.62)$$

By substituting (3.62) in (3.61) and simplifying, we obtain

$$U(p,s) = \frac{4}{p^3s^3} + \frac{1}{(s^2-p^2)}L_xL_t\left[x^4t^4 - u^2(x,t)\right]. \quad (3.63)$$

Applying inverse double Laplace transform to (3.63), we get

$$u(x,t) = x^2t^2 + L_x^{-1}L_t^{-1}\left[\frac{1}{(s^2-p^2)}L_xL_t\left[x^4t^4 - u^2(x,t)\right]\right], \quad (3.64)$$

where we define the nonlinear terms as

$$u^2 = \sum_{n=0}^{\infty} A_n(x,t). \quad (3.65)$$

The Adomian polynomials for u^2 and are computed as follows

$$A_0 = u_0^2, \\ A_1 = 2u_0u_1, \\ A_2 = u_1^2 + 2u_0u_2, \quad \ldots$$

Starting with $u_0(x,t) = x^2t^2$,
We compute the successive terms as

$$u_1(x,t) = L_x^{-1}L_t^{-1}\left[\frac{1}{(s^2-p^2)}L_xL_t\left[x^4t^4 - A_0\right]\right] \\ = L_x^{-1}L_t^{-1}\left[\frac{1}{(s^2-p^2)}L_xL_t\left[x^4t^4 - (u_0)^2\right]\right] = 0, \quad (3.66)$$

$$u_2(x,t) = L_x^{-1}L_t^{-1}\left[\frac{1}{(s^2-p^2)}L_xL_t\left[x^4t^4 - A_1\right]\right] = 0. \quad (3.67)$$

$$u_3(x,t) = 0 \quad (3.68)$$

The exact solution of (3.58), is

$$u(x,t) = x^2t^2 \quad (3.69)$$

Conclusion

In this work, we have seen the application of the double Laplace-Adomian decomposition method to solve linear and nonlinear partial differential equations. This method results from the combination of two famous methods: the double Laplace transform method and the Adomian decomposition method.This method has provenits effectiveness and power in solving this type of equations, as it enabled us to obtain the exact solution in a faster way than the classical methods.

The question that can be asked is: Can this method be applied to solve the partial differential equations of fractional order? This question can be answered in a research project related to preparing for a doctoral degree.

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