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Perturbation of certain semigroups and application to heat echange equations.

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Dedication

I dedicate this effort and my success to: Dear dad, **"Kamel"**, Allah Save and take care of him. Dear mother, **"Naoual"**, may Allah prolong her life. I hope that one day I will be able to give them back a little of what they have done for me, may Allah grant them happiness and long life. My sister **"Choukrame", "Amina", "seouar","Ranime","Tesnim"**, My brother is **"Issem", "Remsi", "Mouhamed".**, my friends and colleagues. My big family is all in his name. And have mercy on him. All my teachers and those who supported me in this work financially or morally. And to everyone who has forgotten my pen and has not forgotten my heart... I ask Allah to accept it from me and benefit those who are after me.

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Contents

Abstract

This work aims to study the perturbation of some semigroups and their application to the heat equation. Using semigroup theory, we proved the existence and uniqueness of the solution. Additionally, we provided a practical example for studying a semilinear heat evolution equation.

Furthermore, we applied the results of the previous chapters to the heat equation by examining the effect of perturbations on the behavior of the solution, along with presenting applied examples that highlight the importance of this theory.

Keywords: Semigroup of operators, heat equation, semi-linear evolution equation.

الملخص

يهدف هذا العمل الى دراسة اضطراب بعض أشباه الزمر وتطبيقاتها على معادلة الحرارة باستخدام نظرية semigroup ,قمنا بإثبات وجود ووحدانية الحل ,بالإضافة لي ذلك قدمنا مثالا عمليا لدراسة معادلة التطور شبه خطي للحرارة .

كما قمنا بتطبيق نتائج الفصول السابقة على معادلة الحرارة من خلال تأثير الاضطرابات على سلوك الحل مع تقديم الأمثلة التطبيقية التي تبرز أهمية هذه النظرية.

كلمات مفتاحية: معادلة الحرارة، معادلة التطور الشبه خطي.

Notations

- A : Operator.
- X : Banach space.
- $| \, . \, |_X$: The norm equipped with the Banach space X.
- $\mathcal{L}(X, Y)$: The space of continuous linear operator.
- $\|.\|_{\mathcal{L}(X,Y)}$: The norm of the operators.
- $D(A)$: The domain of the linear operator A.
- $\rho(A)$: The resolvent set of A.
- $\sigma(A)$: The spectrum of A.
- $G(A)$: Graph of A.
- $R(\lambda, A)$: The resolvent of A.
- A_ω : The set $\{\lambda\}$.
- $\{A_\lambda\}_{\lambda \in A_\omega}$: The generalized Yosida approximation of the operator A.
- $\langle ., . \rangle$: canonical form .

- ► A^* : adjoint of A.
- ► $\widehat{F}$; Fourier transform of f.
- ► $\dfrac{\partial^k}{\partial x_i^k}$: The k^{th} partial derivative.
- ► $|\alpha| = \alpha_1 + \cdots + \alpha_n, \quad \alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n$.
- ► I : Identity operator.
- ► ∇ : The gradient operator.
- ► Δ : The Laplace operator.
- ► $\Delta^2 = \Delta\Delta$: The biharmonic operator.
- ► $\partial_n = n \cdot \nabla$: The derivative according to the normal.
- ► $\partial\Omega$: The boundary of Ω.
- ► $\bar{\Omega}$: The closure of Ω.
- ► $\mathring{\Omega}$: The interior of Ω.
- ► $\lim\limits_{t \uparrow s}$:The limit of the function as t approaches S from below" or "from the left.
- ► $h \downarrow 0$:h tends to 0 from above.

Introduction

Partial differential equations (PDEs) are among the most important mathematical tools used to model various physical phenomena, such as heat conduction, material diffusion, and the evolution of dynamical systems. Among these equations, the **heat equation** holds a central place as a fundamental model for studying thermal diffusion and dissipative processes.

In this context, **semigroup theory** provides a powerful mathematical framework for solving first-order time-evolution equations (see**[10]**). The general solution of an abstract evolution equation of the form:

$$\frac{du(t)}{dt} = Au(t), \quad u(0) = u_0$$

can be expressed using a one-parameter semigroup of linear operators $\{S(t)\}_{t\geq 0}$ generated by the operator A. However, in practical applications, mathematical models are rarely perfect; they often undergo slight modifications or *perturbations* due to external influences or changes in boundary conditions or the physical environment.

This brings us to the importance of studying **perturbations of semigroups**, which essentially means analyzing how the semigroup $S(t)$ generated by an operator A is affected when A is modified by adding another operator B, resulting in a new operator $A + B$ **(see[26])**.

This naturally raises several key questions, such as:

- Does the perturbed operator $A + B$ still generate a semigroup?
- What are the properties of the new semigroup?
- How do the physical solutions, such as those to the heat equation, evolve under such perturbations?

The aim of this work is to provide a systematic study of semigroup perturbation theory, with a focus on fundamental results. These theoretical concepts will then be applied to the heat equation in an appropriate Hilbert space setting. Practical examples will also be explored to illustrate how perturbations influence the time evolution of solutions. This research is divided into three chapters.

Chapter 1: This chapter introduces the general concepts that we will use later, such as Sobolev space (**see [3]**) and semigroup theory **(see [24])**.

Chapter 2: We studied the existence and uniqueness of the solution to non-homogeneous evolution problem. We also studied the behavior, existence and uniqueness of the different solutions to the semi-linear problem (generalized global solutions, classical global solutions, local solutions and maximal solutions) **(see [5])**.

Chapter 3: We applied the results of the previous chapters to the heat equation by studying the effect of perturbation on the solution's behavior **(see [13])**.

Chapter 1

Preliminaries

1.1 Banach Space

Definition 1.1.1 *A normed vector spaces (X,$\|.\|$) is called a Banach space if every Cauchy sequence in X converges in X.*

1.2 Hilbert Spaces

Definition 1.2.1 *see([6]) let H be a vector space. A scalar product (u, v) is a bilinear form on $H \times H$ with values in $\mathbb{R}$ (i.e., a map from $H \times H$ to $\mathbb{R}$ that is linear in both variables) such that*

$$(u,v) = (v,u) \quad \forall\ u,v \in H \qquad (\text{ symmety}),$$
$$(u,u) \geq 0 \quad \forall\ u \in H \qquad (\text{ positive}),$$
$$(u,u) \neq 0 \quad \forall\ u \neq 0 \qquad (\text{ definite}),$$

Let us recall that a scalar product satisfies the Cauchy–Schwartz inequality

$$|(u,v)| \leq (u,u)^{\frac{1}{2}}(u,v)^{\frac{1}{2}} \quad \forall\ u,v \in H\ .$$

1.3 The spaces $L^p(\Omega)$ for p $\in [1, \infty]$

Throughout this work, Ω will generally be an open subset of $\mathbb{R}^n$ for some arbitrary $n \in \mathbb{N}$ and I an interval in $\mathbb{R}$, not necessarily bounded.

Definition 1.3.1 *(Measurable Function)*
A function $f : \Omega \to \mathbb{R}$ is said to be measurable if for all $\alpha \in \mathbb{R}$, the set

$$E_\alpha = \{x \in \Omega \mid f(x) \geq \alpha\}.$$

is measurable in the sense of Lebesgue.

Definition 1.3.2 *(Integrable Function)*
A measurable function $f : \Omega \to \mathbb{R}$ is daid to be integrable in the sense of Lebesgue if:

$$\int_\Omega \mid f \mid < \infty.$$

Definition 1.3.3 *see([6]) (Lebesgue Spaces)*
let $p \in \mathbb{R}$, $1 \leq p < \infty$. We denote by the Lebesgue space $L^p(\Omega)$ the set measurable function

$$L^p(\Omega) = \{f : \Omega \to \mathbb{R} \mid f \textit{ is measurable and} \mid f \mid^p \textit{ is integrable}\} .$$

Moreover, for every function $f \in L^p(\Omega)$, we define:

$$\|f\|_{L^p} = \left(\int_\Omega \mid f(x) \mid^p dx\right)^{\frac{1}{p}}$$

if $p = \infty$ and $f : \Omega \to \mathbb{R}$ is measurable, then we define $\|.\|_{L^\infty}$:

$$\|f\|_{L^\infty} = \sup(f) = \inf\{\alpha :\mid f(x) \mid \leq \alpha \quad e.a\}.$$

Theorem 1.3.4 *The following properties hold*

- *L^p is a vector space and $\|.\|_{L^p}$ is a norm for all $1 \leq p \leq \infty$.*
- *L^p is a Banach space for all $1 \leq p \leq \infty$.*
- *$L^p$is separable for all $1 \leq p < \infty$.*

1.4 $L^1((0,T),D(A))$ Space

Definition 1.4.1 *The space $L^1((0,T),D(A))$ consists of all* ***Bochner-integrable*** *functions $f\colon (0,T) \to D(A)$ such that:*

$$\int_0^T \|f(t)\|_{D(A)}\, dt < \infty, \tag{1.1}$$

where:

$$\|f\|_{L^1(D(A))} = \int_0^T \|f(t)\|_{D(A)}\, dt \tag{1.2}$$

1.5 Fourier transform

Definition 1.5.1 *See([3]) consider the Fourier transform*

$$(Fu)(\sigma) = \tfrac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\sigma x} u(x) dx, \quad \sigma \in \mathbb{R},$$

and the inverse of the Fourier transform

$$(F^{-1}u)(x) = \tfrac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\sigma x} u(\sigma) d\sigma, \quad \sigma \in \mathbb{R}.$$

the Fourier transform has a number of properties, noted as follows:

$$(Fu^{(k)})(x) = (i\sigma)^k (Fu)(\sigma), \quad K \in \mathbb{N}.$$

1.6 Sobolev Spaces and Embedding Theorems

Let $\Omega \subset \mathbb{R}^n$ be an open set and let $\partial\Omega$ denote its boundary .

Definition 1.6.1 *See([3]) Let Ω be an open subset of $\mathbb{R}^n$. For $m \in \mathbb{N}$ and $1 \leq p \leq +\infty$, the Sobolev space denoted by $W^{m,p}(\Omega)$ consists of the functions in $L^p(\Omega)$ whose partial derivatives up to order m, in the sense of distributions, can be identifed with function in $L^p(\Omega)$.*
For these derivatives, we set $\alpha = (\alpha_1, ..., \alpha_N)$ and $\mid \alpha \mid = \sum_1^N \alpha_i$. Moreover, we use the notation

$D^\alpha u = \frac{\partial^{|\alpha|}u}{\partial^{\alpha_1}x_1......\partial^{\alpha_n}x_n}$ *the definition above can now be written as*

$W^{m,p}(\Omega) = \{u \in L^p(\Omega) \mid \forall \alpha \in \mathbb{N}^n, \mid \alpha \mid \leq m \Rightarrow D^\alpha u \in L^p(\Omega)\}$.

Remark 1.6.2 • *For $p = 2$, the notation $W^{m,2}(\Omega)$ is generally replaced by $H^m(\Omega)$.*
• *When $\Omega = \mathbb{R}^N$,we can use the Fourier transform $\xi \longmapsto \widehat{u}(\xi)$ of a function u in $L^2(\mathbb{R}^N)$ to give the following equivalent definition:*
$W^{m,2}(\mathbb{R}^N) = H^m(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) \mid \xi \longmapsto (1+ \mid \xi \mid^2)^{\frac{m}{2}} \widehat{u}(\xi) \in \mathcal{L}^2(\mathbb{R}^N)\}$

1.7 $W^{1,1}((0,T),X)$ Space

Definition 1.7.1 *The **Sobolev space** $W^{1,1}((0,T),X)$ contains all functions $f \in L^1((0,T),X)$ whose **weak derivative** f' exists and satisfies $f' \in L^1((0,T),X)$. Formally:*

$$W^{1,1}((0,T),X) := \left\{ f \in L^1((0,T),X) \;\middle|\; \exists\, g \in L^1((0,T),X) \text{ s.t. } \int_0^T f(t)\phi'(t)\,dt = -\int_0^T g(t)\phi(t)\,dt \right. \tag{1.3}$$

where g is identified as the weak derivative f'.
*The **norm** on the Sobolev space $W^{1,1}((0,T),X)$ is defined as:*

$$\|f\|_{W^{1,1}} = \|f\|_{L^1(X)} + \|f'\|_{L^1(X)},$$

1.8 The resolvent and the spectrum

Definition 1.8.1 *See([9]) The set*

$$\rho(A) = \{\lambda \in \mathbb{C}, \lambda I - A : D(A) \to X \textit{is a bejective operator }\}$$

is called the resolvent set of A.

Definition 1.8.2 *See([9]) The application*

$$R(.,A) : \rho(A) \to \mathcal{L}(X)$$

$$\lambda \to R(\lambda, A) = (\lambda I - A)^{-1}, \quad \forall \lambda \in \rho(A)$$

is called the resolvent of A.

Definition 1.8.3 *The set*

$$\sigma(A) = \mathbb{C} - \rho(A)$$

is called the spectrum of A.

Definition 1.8.4 *Let $(K, +, \cdot)$ be a field and $A \in \mathcal{M}_{n\times m}(K)$. A scalar $\lambda \in K$ is called an eigenvalue of A if there exists a vector $x \in K^n, x \neq 0$, such that:*

$$Ax = \lambda x$$

the vector x is called an eigenvector of A associated with the eigenvalue λ.

Definition 1.8.5 *Let V be a vector space over K and $F \in L(V, V)$. A scalar $\lambda \in K$ is called an eigenvalue of E it there exists a vector $v \in V$, $v \neq 0$. such that $f(v) = \lambda v$.*
the vector v is called an eigenvector of F associated with the eigenvalue λ.

1.9 Semigroups of linear operators.

1.9.1 Uniformly continuous Semigroups

Let X be a Banach space and let $\mathcal{L}(X)$ be the set of all linear bounded operators from X to X. Endowed with the operator norm $\|.\|_{\mathcal{L}(X)}$, defined by

$$\|\mathrm{U}\|_{\mathcal{L}(x)} = \sup_{\|x\|\leq 1} \|Ux\|$$

for each U$\in \mathcal{L}(x)$, $\mathcal{L}(x)$ is a Banach space.

Definition 1.9.1 *A family $\{S(t)\}_{t\geq 0}$ in $\mathcal{L}(X)$ is a semigroup of linear operators on X, or simply semigroup if:*
(i) S(0)=I (I The unit element of algabrs $\mathcal{L}(X)$).
(ii) $S(t + s) = S(t).S(s)$ for each t,s $\geq$ 0.
If, in addition, it satisfies the continuity condition at t=0

$$\lim_{t \downarrow 0} S(t) = I,$$

in the norm topology of $\mathcal{L}(X)$, the semigroup is called uniformly continuous.

Definition 1.9.2 *A semigroup $(S(t))_{t \geq 0} \subset \mathcal{L}(X)$ is called uniformly continuous on X if it satisfies the following property:*

$$\lim_{t \mapsto 0^+} \|S(t) - I\| = 0. \tag{1.4}$$

Definition 1.9.3 *The infinitesimal generator or generator of the semigroup of linear operators $\{S(t)\}_{t \geq 0}$ is the operator $A : D(A) \subseteq X \to X$, defined by*

$$D(A) = \left\{ x \in X; \lim_{t \to 0} \frac{1}{t}(S(t)x - x) \quad exists \right\}. \tag{1.5}$$

and

$$Ax = \lim_{t \to 0^+} \frac{1}{t}(S(t)x - x) = \frac{dS(t)x}{dt} \mid_{t=0} \quad \text{for all} \quad x \in D(A). \tag{1.6}$$

Equivalently .we say that A generator $\{S(t)\}_{t \geq 0}$.

Remark 1.9.4 *Uniformly continuous semigpops are C_0–semigroups sence :*

$$\|S(t)x - x\| \leq \|S(t) - I\|.x$$

But there exist C_0-semigrous that are not uniformly continuous.

Remark 1.9.5 *The semigroup $(S(t))_{t \geq 0} \subset \mathcal{L}(X)$ is strongly continuous if and only if :*

$$\forall f \in X, \quad S(t)f \to f \quad as \quad t \to 0.$$

1.9.2 Elementary properties of uniformly continuous semigroups

In the following,we study some problems concerning uniformly continuous semigroups of elementary of the algebra $\mathcal{L}(X)$, and X is a Banach space.

Lemma 1.9.6 *Let $A \in \mathcal{L}(X)$ then $(e^{tA})_{t\geq 0}$ is a uniformly continuous semigroup of element whose infinitesimal is A.*

Lemma 1.9.7 *Let $A \in \mathcal{L}(X)$ be a bounded operator. There exists a unique uniformly continuous semigroup $(S(t))_{t\geq 0}$ such that*

$$S(t) = e^{At}, \forall t \geq 0.$$

Theorem 1.9.8 *An operator $A : X \to X$ is the infinitesimal generator of a uniformly continuous semigroup if and only if A is a bounded linear operator .*

Corollary 1.9.9 *Let $(S(t))_{t\geq 0}$ to be a uniformly continuous semigroup and A its infinitesimal generator, then*
i) *there exists $w \geq 0$ such that $\|S(t)\| \leq e^{w}t \quad \forall t \geq 0$.*
ii) *the application $t \in [0, \infty] \mapsto S(t) \in \mathcal{L}(X)$ is differentiable for the norm*

$$\tfrac{dT(t)}{dt} = AS(t) = S(t)A \quad t \geq 0.$$

1.10 C_0–SEMIGROUPS DEFINITIONS AND GENERALS PROPERTIES

In this section we introduce a class of semigroups of linear operator, strictly larger that of uniformly continuous semigroups, class which prowes very useful in the study of many partial differntail of parabolic or hyperbolic type A semigroup of linear operators $\{S(t)\}_{t\geq 0}$ is called a semigroup of class C_0, or C_0 − semigroup if for each x ∈ X we have

$$\lim_{t\downarrow 0} S(t)x = x.$$

Theorem 1.10.1 *If $\{S(t)\}_{t\geq 0}$ is a C_0–semigroup then there exist $M \geq 1$, and $\omega \in \mathbb{R}$ such*

$$\|S(t)\|_{\mathcal{L}(X)} \leq e^{t\omega}$$

for each $t \geq 0$.

Theorem 1.10.2 *Let* A:$D(A) \subseteq X \longrightarrow X$ *be the infinitesimal generator of a* C_0 *semigroup* $\{S(t)\}_{t\geq 0}$. *Then*

(i) for each $x \in$ *and each* $t \geq 0$, *we have*

$$\lim_{h\downarrow 0} \frac{1}{h} \int_t^{t+h} S(\tau)xd\tau = S(t)x.$$

(ii)for each $x \in X$ *and each* $t \geq 0$, *we have*

$$\int_0^t S(\tau)xd\tau \in D(A) \text{ and } A(\int_0^t S(\tau)xd\tau) = S(t)x - x.$$

(iii) for each $x \in D(A)$ *and each* $t \geq 0$, *we have* $S(t)x \in D(A)$. *In addition, the mapping* $t \longrightarrow S(t)x$ *is of class* C^1 *on* $[0,+\infty)$, *and satisfies*

$$\frac{d}{dt}(S(t)x) = AS(t)x = S(t)Ax.$$

(iv)for each $x \in D(A)$ *and each* $0 \leq s \leq t \leq +\infty$

$$\int_0^t AS(\tau)xd\tau = \int_s^t S(\tau)Axd\tau = S(t)x - S(s)x.$$

1.11 m-dissipativs operators in a Banach spaces

Definition 1.11.1 *An unbounded linear operator* $(A, D(A))$ *in* X *is dissipative if:*

$$\forall x \in D(A), \quad \exists\lambda > 0, \quad \|\lambda x - Ax\| \geq \lambda\|x\|.$$

Definition 1.11.2 *An unbounded linear operator* $(A, D(A))$ *,in* X *is m-dissipative if*

- *A is dissipative ,*
- $\forall f \in X, \quad \forall\lambda \geq 0, \quad \exists x \in D(A)$ *such that* $\lambda x - Ax = f$.

Theorem 1.11.3 *If* A *is m-dissipative then, for all* $\lambda > 0$, *the operator* $(\lambda I - A)$ *has an inverse,* $(\lambda I - A)^{-1}$ *belongs to* $D(A)$ *for all* $f \in X$, *and* $(\lambda I - A)^{-1}$ *is a bounded linear operator on* X *satisfying:*

$$\|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda}$$

Theorem 1.11.4 *Let $(A, D(A))$ be an unbouded linear dissipative operator in X The operator A is m-dissipative if and only*

$$\exists\lambda_0 \geq 0 \quad \textit{such that} \quad \forall\, f \in X, \quad \exists x \in D(A) \quad \textit{satisfying} \quad \lambda_0 x - Ax = f$$

Theorem 1.11.5 *Let $(A, D(A))$ an unbounded operator in X if there exists $\lambda_0 \geq 0$ for which the operator $\lambda_0 I - A$ is a bijection from $D(A)$ to X, and if $(\ \lambda_0 I - A)^{-1}$. is a bounded on X, then A is closed. In particular, if A is m–dissipative, then A is closed.*

Definition 1.11.6 *Let A be m-dessipative operator in X. The family of operators $R(\lambda; A)$, $\lambda > 0$, define by $R(\lambda, A) = (\lambda I - A)^{-1}$ is called the resolvent of A.*
The operator A_λ is called the Yosida approximation of A.

Theorem 1.11.7 *Let A be an m-dissipative operator with a dense domain in X Then*

$$\lim_{\lambda \to \infty} \|\lambda R(\lambda, A)x - x\| = 0, \quad \textit{for all} x \in X.$$

Moreover

$$\lim_{\lambda \to \infty} \|A_\lambda x - Ax\| = 0. \quad \textit{for all } x \in D(A).$$

Theorem 1.11.8 *Let $(A, D(A))$ be a dissipative operator with a dense domain in X .if A is closed and A^* is, Then A is m-dissipative.*

1.12 m–dissipative Operator in a Hilbert space

In this section we assume that H is a Hilbert space .

Theorem 1.12.1 *An unbounded linear operator $(A, D(A))$, in H is dissipative if and only if*

$$\forall x \in D(A), \quad (Ax, x) \leq 0.$$

In the case of a complex Hilbert space,the previous condition is repaced by

$$\forall x \in D(A), \quad Re(Ax, x) \leq 0.$$

Theorem 1.12.2 *if A is m–dissipative then $D(A)$ is dense in H.*

Theorem 1.12.3 *Let A be a dissipative operator with a dense domain in H. Then A is m–dissipative if and only if A closed and A^* is dissipative .*

Definition 1.12.4 *An unbounded linear operator $(A, D(A))$, with a dense domain in H is said to be self- adjoint if $A = A^*$. it is said to be anti–adjoint if $A = -A^*$.*

Definition 1.12.5 ***(C_0 semigroup of contraction)***
Let $S(t)_{t\geq 0}$ be a C_0-semigroup on X.
1. *$S(t)_{t\geq 0}$ to be uniformly bounded on X if there exists $M \geq 1$ such that $\|S(t)\| \leq M$,$\forall t \geq 0$.*
2. *$S(t)_{t\geq 0}$ is called a C_0-semigroup of contraction if $\|S(t)\| \leq 1, \quad \forall t \geq 0$.*

1.13 The Hille–Yosida Theorem

Theorem 1.13.1 ***(Hille–Yosida)*** *A linear operator*

$$A\colon D(A) \subset X \to X$$

is the infinitesmal generator of a semigroup $\{S(t)\}_{t\geq 0} \in SG(M, \omega)$ if and anly if :
(i) *A is closed operator and $\overline{D(A)} = X$.*
(ii) *There exist $\omega \geq 0$ and $M \geq 1$ such that $A_\omega \subset \rho(A)$ and for $\lambda \in A_\omega$ one has:*

$$\|R(\lambda; A)^n\| \leq \frac{M}{(Re\lambda - \omega)^n} \quad \text{for all} \quad n \in \mathbb{N}^*$$

1.14 The Lumer– Phillips Theorem

Theorem 1.14.1 (The Lumer– Phillips) *let A be a liner operator defined on a linear subspace $D(A)$ of the X then A generates a contraction semigroup if and only if:*

- *D(A) is dense in X.*
- *A is dissipative .*
- $A - \lambda_0 I$ *is surjective for some* $\lambda_0 > 0$.

Remark 1.14.2 *Let A be a densely defined linear operator on Banach space X .If both A and its adjoint are dissipative,then A is the infinitesimal generator of a* C_0 *semigroup of contractions.*

Chapter 2

Inhomogeneous equation and abstract semilinear problems

Throughout this chapter, we assume that X is a Banach space and that A is an m-dissipative operator with dense domain. We denote by $(T(t))_{t\geq 0.}$ the contraction semigroup generated by A.

2.1 Inhomogeneous equations

Let $T > 0$. Given $x \in X$ and $f : [0.T] \to X$, our aim is to solve the problem :

$$\begin{cases} u \in C([0,T], D(A)) \cap C^1([0,T], X); & (2.1) \\ u'(t) = Au(t) + f(t), \quad \forall t \in [0,T]; & (2.2) \\ u(0) = x. & (2.3) \end{cases}$$

As in the case of ordinary differential equation, we have the following result (the variation of parameters formula, or Duhamel´s formula).

Lemma 2.1.1 *Let $x \in D(A)$ and let $f \in C([0;T];X)$. We consider a solution $u \in C([0,T], D(A)) \cap C^1([0,T], X)$ of problem (2.1)-(2.3).Then,we have*

$$u(t) = S(t)x + \int_0^t S(t-s)f(s)ds, \tag{2.4}$$

for all $t \in [0;T]$.

Proof. Let $t \in [0;T]$ Set

$$\omega(s) = S(t-s)u(s),$$

for $s \in [0,t]$.Let $s \in [0,t]$ and $h \in (0,t-s]$.We have

$$\begin{gathered} \tfrac{\omega(s+t)-\omega(s)}{h} = S(t-s-h)\left\{\tfrac{\omega(s+h)-\omega(s)}{h} - \tfrac{S(h)-I}{h}u(s)\right\}. \\ \longrightarrow S(t-s)\{u'(s) - Au(s)\} = S(t-s)f(s). \end{gathered}$$

as $h \downarrow 0$. Since $S(t-\cdot)f(\cdot) \in C([0,t],X)$ we deduce that $\omega \in C^1([0.t],X)$ and that

$$\omega'(s) = S(t-s)f(s). \tag{2.5}$$

for all $s \in [0,t]$. Integratimg (2.5) between 0 and $\tau < t$, and letting $\tau < t$, and letting $\tau \uparrow t$, we obtain (2.4) $\blacksquare$

Corollary 2.1.2 *For all* $x \in D(A)$ *and* $f \in C([0,T],X)$ *problem(2.1)-(2.3) has at most one solution.*

Remark 2.1.3 *For all* $x \in X$*and* $f \in C([0,T],X)$ *formula (2.4) defines a function* $u \in C([0,T],X)$. *Now we looking for sufficient condition for* u *given by (2.4) to be the solution of (2.1)-(2.3).*

Remark 2.1.4 *It is clear that if* u *is a solution of (2.1)-(2.3),then* $x \in D(A)$.*However, this condition is not sufficient .Indeed, assume that* $(S(t))_{t\in\mathbb{R}}$ *is an isometry group, and let*
$y \in X \setminus D(A)$, $S(t)y \notin D(A)$,*for all* $t \in \mathbb{R}$. *Take* $f(t) = S(t)y$,*and* $x = 0 \in D(A)$. *It follows easily that (2.4) given* $u(t) = S(t)y \notin D(A)$. *for* $t \neq 0$. .

Lemma 2.1.5 *for all* $x \in X$ *and* $f \in L^1([0,T],X)$, *formula (2.4) defines a function* $u \in C([0,T],X)$. *In addition,we have*

$$\|u\|_{C([0,T],X)} \leq \|x\| + \|f\|_{L^1(0,T,X)}$$

Proof. The result is clear if $f \in C([0,T],X)$ and follows by density in the general case .

■

Proposition 2.1.6 *Let $x \in D(A)$ and let $f \in C([0,T],X)$. Assume that at least one of the following condition is satisfied:*
(i) $f \in L^1((0,T),D(A))$.
(ii) $f \in W^{1,1}((0,T),X)$.
Then u given by (2.4) is the solution of (2.1)-(2.3).

Proof. We proceed in four steps.set

$$v(t) = \int_0^t S(t-s)f(s)ds = \int_0^t S(s)f(t-s)ds,$$

for $t \in [0,T]$.
step 1. We have $v \in C^1([0,T],X)$. Indeed, if $f \in L^1((0,T),D(A))$,for $t \in [0,t]$ and $h \in [0,t-s]$, write

$$\frac{v(t+h)-v(t)}{h} = \int_0^t S(t-s)\frac{S(h)-I}{h}f(s)ds + \frac{1}{h}\int_0^{t+h} S(t+h-s)f(s)ds.$$

and let $h \downarrow 0$.Note that

$$\frac{S(h)-I}{h}f \longrightarrow Af$$

in $L^1((0,T),X)$ as $h \downarrow 0$,and apply Lemma 2.1.5 It follows that

$$\frac{d^+v}{dt}(t) = \int_0^t S(t-s)Af(s)ds + f(t)ds,$$

for all $t \in [0,T]$. If $f \in W^{1,1}([0,T],X)$ for $t \in [0,T]$ and $h \in [0,T-t]$, we write

$$\frac{v(t+s)-v(t)}{h} = \int_0^t S(s)\frac{f(t+h-s)-f(t-s)}{h}ds + \frac{S(t)}{h}\int_0^t S(t-s)f(s)ds,$$

and we let $h \downarrow 0$.

$$\frac{f(t+h-\cdot)-f(t-\cdot)}{h} \to f'(t-\cdot), \text{ as } h \downarrow 0.$$

for all $t \in [0.T]$ In both cases ,we have $\frac{d^+v}{dt} \in C([0,T],X)$ and $h \downarrow 0.$ in $L^1((0,t),X)$ It follows that in $L^1([0,t],X)$ and apply Lemma 2.1.5.It follows that

$$\frac{d^+v}{dt}(t) = \int_0^t S(s)f'(t-s)ds + S(t)f(0),$$

for all $t \in [0.T]$ In both cases ,we have $\frac{d^+v}{dt} \in C([0,T],X)$ and $h \downarrow 0.$ in $L^1((0,t),X)$ and so $v \in C^1([0,T],X)$

stap 2. similarly,we show that $(\frac{d^+v}{dt}(S))$ makes sense and is equal to $\lim_{t\uparrow S} v'(t)$; and so $v \in C^1([0,T],X)$

step 3. Let $t \in [0,T]$ and let $h \in [0,T-t]$, we have

$$\begin{aligned}\frac{T(h)-I}{h}v(t) &= \frac{1}{h}\int_0^t S(t+h-s)f(s)ds - \frac{1}{h}\int_0^t S(t-s)f(s)ds \\ &= \frac{v(t+h)-v(t)}{h} - \frac{1}{h}\int_0^{t+h} S(t+h-s)f(s)ds.\end{aligned}$$

Letting $h \downarrow 0$,we deduce $u(t) \in D(A)$,and $Au = v'(t) - f(t)$,This is still true for $t = S$, since the graph of A is closed ,It follows that $v \in C([0.T],D(A))$ and that v satisfies (2.2)

stap 4. We have

$$u'(t) = AS(t)x + Av(t) + f(t) = Au(t) + f(t),$$

for all $t \in [0,T]$, Hence, we have (2.2) and (2.3) is immediate. ∎

Corollary 2.1.7 *Let $x \in X, f \in C([0,T]X)$ and let u be given by (2.4) .*
u is the unique solution of the problem

$$\begin{cases} u \in C([0,T],X) \cap C^1([0,T],\overline{X}); \\ u' = \overline{A}u(t) + f(t), \quad \forall t \in [0,T]; \\ u(0) = x. \end{cases}$$

Corollary 2.1.8 *Let $x \in X, f \in C([0,T],X)$ and let u be given by (2.4).*
Asume that at least one of the following condition is satisfied:

(i) $u \in C([0;T],D(A))$;

(ii) $u \in C^1([0,T],X)$.

Then u is the solution of (2.1)-(2.3).

Proof. Assume that (i) holds.By Corollary(2.1.7) ,we have $u' \in C([0,T],X)$; and so $u \in C^1([0,T],X)$, hence the result.

Now assume that (ii) holds, By Corollary(2.1.7), we have $\overline{A}u \in C([0,T],X)$ and $u \in C([0.T],D(A))$; hence the result.

■ we have supposed that $f \in C([0,T],X)$. But in order to give a sense to (2.4),it suffices that $f \in L^1([0,T],X)$ Lemma 2.1.5 .In this case ,we have the following result

Proposition 2.1.9 *Let $x \in X, f \in L^1((0,T),X)$:*
Assume further that $u \in L^1((0,T),D(A))$ or that $u \in W^{1.1}((0,T),X)$, Then u verifies (2.4) if and only if u

$$\begin{cases} u \in L^1((0,T),D(A)) \cap W^{1,1}([0,T],X) \\ u' = Au(t) + f(t), \quad \textit{for almost every} \quad t \in [0,T]; \\ u(0) = x. \end{cases}$$

Proof. First note that if $u \in W^{1,1}([0,T],X)$ then $u \in C([0,T],X)$ so the condition $u(0) = x$ makes sense. Let us first show that the assumptions of the theorem are sufficient to have (2.4) To see this, we argue us .
We consider $t \in [0,T]$ and we set $w(s) = S(t-s)u(s)$, for almost every $s \in [0,t]0$, For all $h \in [0,t]$, and for almost every $s \in [0,t-h]$ we have

$$\tfrac{w(s+h)-w(s)}{h} = S(t-s-h)\left\{\tfrac{w(s+h)-w(s)}{h} - \tfrac{S(h)-I}{h}u(s)\right\}$$

It follows that w is absolutely continuous from [0,T] to Y. Moreover, the righthand member for almost every $s \in [0,t]$ to $S(t-s)u(s) - \overline{A}u(s) = S(t-s)f(s)$, as $h \downarrow 0$. Therefore w is right differentiable almost everywhere on [0,t] and

$$\tfrac{d^+w}{ds}(s).$$

Similarty, we show that w is left differentiable almost everywhere on [0,t] and that

$$\tfrac{d^-w}{ds}(s) = S(t-s)f(s).$$

Consequently $w \in W^{1,1}([0,T],Y)$ and $w' = T(t-s)f(s)$ almost everywhere. We deduce (2.4)

Conversely, assume that u satisfies (2.4). Let $(f_n)_{n\geq 0}$ be a sequence of $C([0,T],X)$ such that $f_n \longrightarrow f$ in $L^1([0,T],X)$ as $n \to \infty$, and let u_n be the corresponding solution of (2.4). Invoking Corollary2.1.7 ,we have

$$u_n'(t) = \overline{A}u_n(t) + f_n(t),$$

for all $t \in [0,T]$; and so

$$u_n = (t) = x + \int_0^t (\overline{A}u_n(s) + f_n(s))ds,$$

for all $t \in [0,T]$, Letting $n \to 0$we obtain(Lemma 2.1.5)

$$u(t) = x + \int_0^t (\overline{A}u(s) + f(s))ds$$

for all $t \in [0,T]$, It following that $u \in W^{1.1}([0,T],Y)$ and that

$$u'(t) = \overline{A}u(t) + f(t),$$

for almost every $t \in [0,T]$, If $u \in W^{1.1}([0,T])$. we have $\overline{A}u \in L^1([0,T],D(A))$ we have $u' \in L^1([0,T],X)$ This completes the proof ■

2.2 Semilinear Problems

Lemma 2.2.1 ***(Gronwall)*** *Let $T > 0, \lambda \in L^1([0;T]), \lambda \geq 0 \quad a.e$ and $\quad C_1C_2 \geq 0$. Let $\varphi \in L^1([0;T]), \varphi \geq 0 \quad a.e$ such that as $\lambda\varphi \in L^1([0;T])$ and*

$$\varphi(t) \leq C_1 + C_2 \int_0^t \lambda(s)\varphi(s)ds,$$

for almost every $t \in (0,T)$, Them we have

$$\varphi(t) \leq C_1 exp\left(C_2 \int_0^t \lambda(s)ds\right). \quad a.e \quad t \in [0;T].$$

for almost every $t \in (0,T)$.

Lemma 2.2.2 ***(fixed point)*** *Let E be a compact metric space, non empty . We note d the distance on E and we consider fan application of E in itself . We suppose F contracting , that is to say: there exists a positive constant k , strictly less than 1, such that : $d \mid F(x), F(y) \mid \leq kd(x, y)$for every one $x, y \in X$ Then: there is a unique point $a \in X$ such as $F(a) = a$ moreover, this point can be obtained as a limit of the sequance $(X_n)_{n \in N}$ iterates , defined by induction starting from any point x_0 of E according to $x_{n+1} = F(x_n)$ .We also have :*

$$\forall n \geq 1 : d(x_n, \alpha) \leq \tfrac{K^n}{1-K} d(x_0, x). \quad \textit{(method of Picard)}$$

We have already defined during the study of the non-homogeneous problem the term of classical solution. We will subsequently in an analogous way highlight different types of solutions of a give semilinear problem.

Definition 2.2.3 *A function $F : X \to X$ is lipschitz continuous function on bounded subsets of X. provided that for all $M > 0$, there exist a constant $L(M)$ such that:*

$$\|F(y) - F(x)\| \leq L(M)\|y - x\| \quad \forall x, y \in B_M,$$

where B_M is the ball of center 0 and of radius M.

Definition 2.2.4 ***(Generalized global solution)***

We call a generalized global solution of problam (P') any function $u \in C([0, \infty[; X)$ such that :

$$\forall t \geq 0, u(t) = S(t)u_0 + \textstyle\int_0^t S(t - s)F(u(s))ds$$

Where $S(t)$ denotes the semigroup associated with the operator A

Theorem 2.2.5 *If $u_0 \in X$. if F is lipschitzian wiht lipschitz constant $M > 0$,then problem (P') admits a unique generalized global solution denoted u.*
if us $D(A)$, u is locally lipschitzian.

Proof. First part

• Existence

Let

$$\psi(u)(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))ds$$

The map $\psi \in C([0, +\infty[; X)$.

We will apply Banach fixed point theorem to ψ in the space to be determined, defined by:

$$X_\alpha = \{u \in C([0, +\infty[; X) : \sup_{t>0} e^{-\alpha t} \mid u(t) \mid_X < +\infty\}$$

equipped with the norm :$\mid u \mid_{X_\alpha} = \sup_s (e^{-\alpha s} \mid u(s) \mid_X)$. This space is complete because it is closed in the complete space $C([0, +\infty]; X)$.

Let us determine under what condition the application ψ is a contraction from X_α into X_α.

The application ψ sends X_α out of X_α for everything $\alpha > 0$.

$$\begin{aligned} \forall \geq 0, \quad \mid \psi(u)(t) \mid_X \quad &\leq \quad \mid S(t)u_0 \mid_X + \int_0^t \|S(t-s)\|_{\mathcal{L}(X)} \mid \mathbf{F\ (u(s))} \mid_{\mathbf{x}} \mathbf{ds} \\ &\leq \quad \mid u_0 \mid_X + \int_0^t \mid F(u(s)) \mid_X ds, \quad \text{because} \quad \|S(t)\|_{\mathcal{L}(\mathcal{X})} \leq 1, t \geq 0. \end{aligned}$$

Gold,

$$\begin{aligned} \forall s \in [0.t], \quad \mid F(u(s)) \mid_X \quad &\leq \mid F(u(s)) \mid_X \quad \mid F(u(s)) - F(0) \mid_X + \mid F(0) \mid_X \\ &\leq M \mid u(s) \mid_X + C_1. \end{aligned}$$

from where :

$$\mid \psi(u)(t) \mid_X \quad \leq \quad \mid u_0 \mid_X + \int_0^t (M \mid u(s) \mid_X + C)ds.$$

Multiply this last inequality by $e^{-\alpha t}$ for $t > 0$ $\quad (e^{-\alpha t} < 1 \quad$ for all $\quad t > 0.$) We obtain .

$$\begin{aligned} \forall t > 0, \quad e^{-\alpha t} \mid \psi(u)(t) \mid_X &\leq e^{-\alpha t} \mid u_0 \mid_X + \int_0^t e^{-\alpha(t-s)} e^{-\alpha s}(M \mid u(u) \mid_X + C)ds \\ &\leq e^{-\alpha t} \mid u_0 \mid_X + M \sup_s (e^{-\alpha s} \mid u(s) \mid_X) \int_0^t e^{-\alpha(t-s)} ds + C(\int_0^t ds)e^{-\alpha t} \\ &\leq e^{-\alpha t} \mid u_0 \mid_X + M \mid u \mid_{X_\alpha} \int_0^t e^{-\alpha(t-s)} ds + Cte^{-\alpha t}. \end{aligned}$$

What´s more,

$$\forall t > 0, \int_0^t e^{-\alpha(t-s)} ds = e^{-\alpha t} \int_0^t t e^{\alpha s} ds$$
$$= \frac{1-e^{-\alpha t}}{\alpha} \leq \frac{1}{\alpha}.$$

so

$$\forall t > 0, e^{-\alpha t} \mid \psi(u)(t) \mid_X \leq \mid u_0 \mid_X + \frac{1}{\alpha} M \mid u \mid_{X_\alpha} + C \sup_s (t e^{-\alpha t})$$

and

$$\sup_{t>0} e^{-\alpha t} \mid \psi(u)(t) \mid_X < +\infty,$$

that is to say for all $\alpha > 0, \psi : X_\alpha \to X_\alpha$

the application ψ is a contraction if $\alpha > M$ Indeed, let $u, v \in X_\alpha$. we have, for all $t \geq 0$:

$$\mid \psi(u)(t) - \psi(v)(t) \mid_X = \mid \int_0^t S(t-s)(F(u(s)) - F(u(s)))ds \mid_X$$

and

$$\mid \psi(u)(t) - \psi(u)(t) \mid_X \leq \int_0^t \|S(t-s)\|_{\mathcal{L}(X)} \mid (F(u(s)) - F(u(s))) \mid_X ds$$
$$\leq M \int_0^t \mid u(s) - \psi(s) \mid_X ds$$

$$\forall t > 0, \quad e^{-\alpha t} \mid \psi(u)(t) - \psi(v)(t) \mid_X \leq M \int_0^t e^{-\alpha(t-s) e^{-\alpha s}} \mid u(s) - v(s) \mid_X ds$$

$$\leq \frac{M}{\alpha s} \mid u(s) - v(s) \mid_{X_\alpha}$$

Therefore ψ is acontraction on X_α if $\frac{M}{\alpha} < 1$.

For $\alpha > M$ if F is Lipschitzian, there exists a unique fixed poit for ψ on $X_{\alpha.}$ thus the cauchy problame (P') has a solution in $X_\alpha \subset C([0, +\infty[; X)$ if F is Lipschitz.

• uniqussite

Let U and V be two generalized solution of (P') We have :

$$\forall t \geq 0, \quad u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))ds$$
$$\forall t \geq \quad v(t) = S(t)v_0 + \int_0^t S(t-s)F(v(s))ds$$

hence, by difference:

$$\forall t \geq 0, u(t) - v(t) = S(t)(u_0 - v_0) + \int 0^t S(t-s)(F(u(s)) - F(v(t)))ds.$$

them

$$\forall t \geq 0, \mid u(t) - v(t) \mid_X \geq \mid u_0 - v_0 \mid_X ds.$$

By Gronwall lemma, we therefore obtain :

$$\begin{aligned} \forall t \geq 0, \quad \mid u(t) - v(t) \mid_X \quad &\leq \quad \mid u_0 - v_0 \mid_X exp\left(M \int_0^t ds \right) \\ &\leq \mid u_0 - v_0 \mid_X e^{Mt} \end{aligned}$$

Now, $u_0 = v_0$ if u and v are generalized soluctions of $(P\prime)$ .therefore $u = v$.

• Second part:

Let $u_0 \in D(A)$. **Thene u is locally lipschitiz** .

Indeed, let $h > 0$ and $t \in [0, T]$.lrt us study $u(t+h) - u(t)$.

To do this, consider $u(t+h)$ asthe solution at time t of :

$$\begin{cases} \frac{dv}{dt} + Av = F(v), \\ v(0) = v(h). \end{cases}$$

(this is possible given the shape of u and du the fact that $\{S(t)\}_{t\geq 0}$ defines a semigrop)

We have :

$$\forall t \geq 0, \quad \mid u(t+h) - u(t) \mid_X \leq e^{Mt} \mid u(h) - u_0(t) \mid_X,$$

$$\mid u(h) - u(0) \mid_X \quad \leq \mid S(h)u_0 - u_0 + \mid \int_0^h \mid F(u(s)) \mid_X ds.$$

what´s more .

$$\|S(h)x_0 - u_0\|_X \leq Ch$$

because, according to steps 4 and 5 of the demonstraction of the theorem of HILLE–Yosida (case Banach), the application $t \to u(t) = S(t)u_0 \in C^1([0,+\infty];X) \cap C([0,+\infty];D(A)$,hence in particular:

$$\int_0^t \mid F(u(s)) \mid ds \quad \leq \quad \int_0^t (M \mid u(s) \mid_X + C) ds$$

Hance, finally

$$\mid u(h) - u(0) \mid_X \quad \leq \quad C^1 h + \int_0^t (M \mid u(s) \mid_X + C) ds.$$

Like $u \in C([0,+\infty];X)$, it exists h_0 such as ;

$$\forall h \leq h_0. \mid u(s) \mid_X \leq 2 \mid u_0 \mid_X$$

and

$$\mid u(h) - u(0) \mid_X \leq C'' h$$

so

$$\forall t \geq 0, \mid u(t+h) - u(t) \mid_X \leq C'' h e^{Mt}$$

and

$$\forall t \in [0,T], \mid u(t+h) - u(t) \mid_X \leq C_T h.$$

■

Definition 2.2.6 *(Classic global solution)*
Let $u_0 \in D(A)$ We call ***global classical solution*** *any function $u \in C([0,+\infty];D(A)) \cap C^1([0;+\infty];X)$ telle que :*

$$\begin{cases} \frac{du}{dt} + Au = F(u), \quad on \quad [0;+\infty[\\ u(0) = u_0 \end{cases}$$

Moreover, in the same way as for generalized global solution ,we have an existence resulta of a classical global solution to the Cauchy problame (P').

Theorem 2.2.7 *if F is lipschitzian and C^1 (that is ,the application :)*

$$u \in X \to F(u) \in l(E)$$

is is continuous and $|F'(u)| \leq M$), so for everything $u_0 \in D(A)$, it exists u classic global solution of (P'). the proof of this theorem rests on the following essential fact : in the inhomogeneous case, if $u_0 \in D(A)$ and $f \in C^1([0;T];X)$ then any generalized solution is a classical solution . Moreover, if the space X is reflexive, we notice that we can specify the regularity of the generalized solution and show that it is a classical solution

Definition 2.2.8 ***(Local solution)***

1.We call ***local generalized solution*** *of (p') any function u such that*

$\forall u_0 \in X, \quad \exists T > 0, \quad \exists u \in C([0,T[: D(A)]);$

$\forall t > T, u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))ds.$

2.We say that (P') has a ***local classical solution*** *if:*

$\forall u_0 \in D(A), \quad \exists T > 0, \exists u \in C([0:T[: D(A)) \cap C^1([0;T[;X)$ *such as :*

$$\begin{cases} \frac{du}{dt} + Au = F(u), & on \quad [0;+\infty[\\ u(0) = u_0 \end{cases}$$

Definition 2.2.9 *We say that a function $F : X \to X$* ***is lipschitz on the bounds*** *of X if, for all $r > 0$, there is a constant noted M, such that as :*

$$\forall u, v \in B(0,r), \quad | F(u) - F(v) |_X \leq M_T \mid u - v \mid_X .$$

We have an existanc result concerning the local solution of the Cauchy problam (P').

$$\begin{cases} \frac{du}{dt} + Au = F(u), & on \quad [0;+\infty[\\ u(0) = u_0 \end{cases}$$

Theorem 2.2.10 *Let $F : X \to X$ lipschitzian on the bounded ones .Then:*

$\forall u_0 \in X, \exists T > 0, \quad \exists u \in C([0,T[;X)$ *local generalized solution of (P').*

Proof. Let's applycation Banach fixed point theorem to space :

$K_T = \{u \in C([0,T[: X) :\mid u(t) - u_0 \mid_X \leq 2 \mid u_0 \mid +1 \quad \forall t \leq T\}.$

and to the function :

$\phi : K_t \to K_T$ such that $\phi(u)(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))ds.$

Not that the space K_T is a Banach space because it is closed in is closed in the complet $(\mathcal{C}([0,T];X), |\ .\ |_X)$. **the app$\phi$ send K_T out K_T if**

$$K \leq \frac{1}{M_K(2|u_0|+1)+|F(u_0)|}$$

Inedeed ,we have :

$\forall t, |\ \phi(u(t)) - \phi(u_0)\ |_X \leq |\ S(t)u_0\ |_X + |\ u_0\ | + \int_0^t |\ S(t-s)\ |_X |\ F(u(s))\ |_X\ ds.$

Let M_K denote the lipschitz constant of on $B(0, 3\ |\ u_0\ |_X +1)$. par hypotase si $u \in K_T$.

$$\forall s \leq T, \quad |\ u(s)\ |_X \leq 2\ |\ u_0\ | +1$$

hence ;

$$\forall s \leq T, \quad u(t) \in B(0, 3\ |\ u_0\ |_X +1),$$

and

$\forall t \leq T, \forall s \leq T, \quad |\ F(u(s))\ |_X \leq |\ F(u(s)) - F(u_0)\ |_X + |\ F(u(s))\ |_X\ .$

them:

$$\forall t \geq T, |\ F(u(s))\ |_X \leq M_k\ |\ u(s) - u_0\ |_X + |\ F(u_0)\ |_X\ .$$

$\Longrightarrow \forall t \leq T, |\ \phi(u(t)) - u_0\ |_X \leq 2\ |\ u_0\ |_X + T[M_k(2\ |\ u_0\ |_X +1) + |\ F(u_0)\ |_X].$

We therefore deduce that ϕ has values in $K_T$if:

$T[M_K(2\ |\ u_0\ |_X +1) + |\ u_0\ | +1 + |\ F(u_0)\ |_X] \leq 1$

the application ϕ is contracting if $T < \frac{1}{M_K}$.

Indeed, let $u, v \in K_T$ we have ;

$$\phi(u)-\phi(v)=\int_0^t S(t-s)\mid F(u(s))-F(v(s))\mid ds.$$

Hence

$$\mid \phi(u)-\phi\mid_X \leq \int_0^t \mid F(u(s))-F(v(s))\mid_X ds$$

$$\leq M_X\int_0^t \mid u(s)v(s)\mid_X ds$$

$$\leq M_X\int_0^t \sup_{s\leq T}\mid u(s)-v(s)\mid_X ds$$

$$\leq M_K\int_0^t T\sup_{s\leq T}\mid u(s)-v(s)\mid_X$$

We therefore deduce that ϕ is a contraction for $M_K T<1$ By Banach fixed point theorem , there exists a unique$u\in K_T$ such that $u=\phi(u)$. For all $u\in X$, we have shown a time T and a solution $u\in K_t$ of $u\in\phi(u)$. ■

Proposition 2.2.11 • *If u_i is a local solution defined on $u\in[0;T_i]$ $for\, T_i<T_j$, $\quad u_j=u_i$, if $\sup\{T_i, i\in I\}=T_{max}\leq+\infty$, $\quad$ then we can define $u\in C([0;T_{max}[,X)$ such that :*

$$\forall t<T_{max}, u(t)=S(t)u_0+\int_0^t S(t-s)F(u(s))ds$$

*We call u **the generalized maximal solution***

• *Moreover, if $u\in C([0,T(u_0)];X)$ as for everything $t<T(u_0), u$ called **classical maximal solution.***

Theorem 2.2.12 *There is a function $T: X\longrightarrow[0,+\infty]$ with the following proprties: for all $u_0\in X$, there exist $u\in C([0:T(u_0)[;X)$ as for everything $t<T(u_0), u$ is the unique global generalized solution in $C([0;T];X)$ what`s more ,*

$$\forall t>\in[0;T(u_0)[, 2K(\mid F(0)\mid_X+2\mid u(t)\mid_X)\geq\frac{1}{T(u_0)-t}-2.$$

In particutar, we have the alhernation :

Let $T_{max}=T(u_0)=+\infty$ $\quad$ and the solution is global .

Let $T_{max}=T(u_0)<+\infty$ and $\mid u(t)\mid_X\longrightarrow_{t\to T(u_0)}+\infty$ and the solution explades in finite time .

This last result is the analogue of the explosin theorem for ordinary differential equations.

Example 2.1 ***(Heat Equation)***

Let us consider the heat equation with a source term*:*

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + g(x,t), & x \in \mathbb{R},\ t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}, \end{cases} \tag{2.6}$$

where f and g are continuous bounded functions from $\mathbb{R}$ to $\mathbb{R}$.

Partial Fourier Transform. *For $u \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, the Fourier transform in x is defined by:*

$$\widehat{u}(\xi,t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} u(x,t)\, dx, \qquad \xi \in \mathbb{R}.$$

We obtain the relations:

$$\widehat{\partial_x u} = i\xi \hat{u}, \qquad \widehat{\partial_{xx} u} = -\xi^2 \hat{u}, \qquad \widehat{\partial_t u} = \partial_t \hat{u}.$$

Ordinary Differential Equation. *Applying the Fourier transform to equation gives:*

$$\begin{cases} \frac{\partial \hat{u}}{\partial t}(\xi,t) = -\xi^2 \hat{u}(\xi,t) + \hat{g}(\xi,t), & t > 0,\ \xi \in \mathbb{R}, \\ \hat{u}(\xi,0) = \hat{f}(\xi), & \xi \in \mathbb{R}, \end{cases} \tag{2.7}$$

which is a first-order ODE in t whose solution is given by:

$$\hat{u}(\xi,t) = e^{-\xi^2 t} \hat{f}(\xi) + \int_0^t e^{-\xi^2(t-s)} \hat{g}(\xi,s)\, ds, \qquad t \geq 0,\ \xi \in \mathbb{R}.$$

Inverse Fourier Transform. *Using the identity:*

$$\mathcal{F}^{-1}\big[e^{-\xi^2 t}\big] = \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{x^2}{4t}}, \qquad t > 0,$$

we obtain by the convolution formula:

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} f(y)\, dy + \int_0^t \frac{1}{\sqrt{4\pi(t-s)}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4(t-s)}} g(y,s)\, dy\, ds, \quad t > 0.$$

Semigroup formulation. *Define the Gauss–Weierstrass operator:*

$$(T(t)h)(x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} h(y)\, dy.$$

Then the solution can be written compactly as:

$$u(\cdot,t) = T(t) f + \int_0^t T(t-s)\, g(\cdot,s)\, ds.$$

The first term represents the homogeneous solution, and the second term is the forced response (Duhamel's principle).

Chapter 3

Perturbation of the heat equation

3.1 introdution

In this chapter, we present a study of the perturbed heat equation,analyzing how adding a new term affects the stability of its solutions.

we define the following problems:

Nonhomogeneous heat equation:

$$\begin{cases} u_t = \frac{\partial^2 u}{\partial x^2}(x,t) + F(x,t) \quad 0 < x < 1, \quad \text{t} > 0. \\ u(x,0) = f(x) \\ u(1,t) = u(0,t) = 0. \end{cases} \tag{3.1}$$

Perturbed heat equation:

$$\begin{cases} u_t = \frac{\partial^2 u}{\partial x^2}(x,t) + q(x)u + F(x,t) \quad \text{q} \in L^\infty[0,1], \\ u(x,0) = f(x), \\ u(1,t) = u(0,t) = 0. \end{cases} \tag{3.2}$$

- u_t: partial derivative with respect to time t.
- $\frac{\partial^2 u}{\partial x^2}$:second partial derivative with respect to space x.
- $F(x,t)$: external source term (forcing function).

- $q(x)$:a bounded function that represents the perturbation.

Boundary and Initial Conditions

The function $u(x,t)$ must satisfy:

• Boundary conditions:

$$u(0,t) = u(1,t) = 0 \quad t > 0, \tag{3.3}$$

(i.e,Dirichlet boundary coditions)

• Initial condition:

$$\lim_{t \to 0^+} u(x,t) = f(x), \tag{3.4}$$

The aim and the purpose is to study the effect of **perturbation** on the solution,in terms of :

• How it changes the solution method

• Its effect on the operator (infinitesimal generator and differential generators) semigroup.

• Comparing the perturbed solution $(3.2)-(3.4)$ with the unperturbed solution $(3.1)-(3.3)-(3.4)$ when t is small.

The analysis will be done using functional analysis tools:

Banach spaces:

- $C_0[0,1]$ spaces of continuous function on [0,1] that vanish at the endpoints (i.e;f(0)=0=f(1)),with supremum norm $\|.\|_\infty$.

Let A be the linear differential operator defined by:

• The space is $C_0[0,1]$

$$Au = u^{(2)} + q(x)u. \tag{3.5}$$

the domain of A is define by:

$$D(A) = \{u \in C^2[0,1] : u(0) = u(1) = 0\},$$

that is,the set of twice continuously differentiable function on the interval [0,1] that vanish at the endpoints.This operator A is unbounded ,closed and densely defined in the appropriate function space .

Objective:

solve the following inhomogeneous equation

$$u_t = Au + u(x,0), \quad f(x) = F(x,t)$$

by representing the solution in terms of the operators $U(t)$ and the eigenfunction rule of the operator.

Assume that A has a set eigenvalues and corresponding eigenfunction such that forms an orthonormal rule in space,and

we presented the framework of spectral theory.

we begin by studying the homogeneous version of the differential equation:

$$Au = u_t$$

the goal is to find the solution to this equation when:

$$f(x) = u(x,0)$$

suppose that operator A has:

- Eigenfunctions u_k
- Eigenvalues λ_k such that

Let λ_k and $u_k(x), k = 1, 2, ...$ be the eigenvalues and eigenfunctions of an operator A.

such that $v_k(x) = \overline{u_k}(x)$(complex conjugate).

and the eigenfunctions are orthonormal bassis in the space $C_0[0,1]$:

$$\int_0^1 u_k \overline{v_j} = \delta_k j$$

$$\langle v_k, u_j \rangle = \begin{cases} 0 & k \neq j \\ 1 & k = j \end{cases} = \delta_{kj}$$

we write,

$$f(x) = \sum_{k=1}^{\infty} \langle v_k, f \rangle u_k(x)$$

where $\langle f, v_k \rangle$ is an inner product between f and v_k (the orthogonal pair of eigenfunction u_k.)

Hence,we write the general solution:

$$u(x,t) = \sum_{k=1}^{\infty} e^{\lambda_k t} \langle f, v_k \rangle u_k(x)$$

This is called the Fourier spectral evolution of the initial slate.

$$U(t)f(x) = \sum_{1}^{\infty} e^{\lambda_k t} \langle f, v_k \rangle u_k(x), \tag{3.6}$$

It is the solution of the homogeneous equation.

To solve the nonhomogeneous heat equation:

$$u_t = Au + F(x,t), \quad u(x,0) = f(x)$$

by expressing the solution in terms of the operator $U(t)$ and the spectral decomposition of the operator A.

Assume that the operator A has a countable set of eigenvalues λ_k and corresponding eigenfunctions $u_k(x)$, where $\{u_k\}$ forms an orthonormal basis in the space X, and:

$$Au_k = \lambda_k u_k, \quad k = 1, 2, \ldots$$

The function f is expanded in terms of the basis:

$$f(x) = \sum_{k=1}^{\infty} (f, v_k)\, u_k(x)$$

Therefore, the solution of the homogeneous problem $u_t = Au$ is given by:

$$U(t)f(x) = \sum_{k=1}^{\infty} e^{\lambda_k t} (f, v_k)\, u_k(x) \quad \text{(see equation 1.6)}$$

For each moment $s \in [0, t]$, the function $F(x, s)$ represents an external input.

To determine its effect on the solution at time t, we apply the operator:

$$U(t-s)F(\cdot, s)(x)$$

This expresses how the input at time s evolves to affect the system at time t.

We expand the function $F(\cdot, s)$ in the orthonormal basis:

$$F(\cdot, s) = \sum_{k=1}^{\infty} (F(\cdot, s), v_k)\, u_k(x)$$

Applying $U(t-s)$ to this expansion yields:

$$U(t-s)F(\cdot, s)(x) = \sum_{k=1}^{\infty} e^{\lambda_k (t-s)} (F(\cdot, s), v_k)\, u_k(x) \tag{3.7}$$

from the previous results (3.6) and (3.7), we find that the solution of the nonhomogeneous equation is given by the relation....

$$u(x, t) = U(t)f(t) + \int_0^t U(t-s)F(\cdot, s)(x)dx \tag{3.8}$$

given (3.8) the general solution to the inhomogeneous (trubulent) partial differential equation such that:

- $U(t)f(t)$ this represents the solution to the homogeneous equation (without F) $Au = f(x), u(x, 0) = u_t$.
- $U(t-s)F(\cdot, s)(x)$ this integral represents the cumulative effect of the term over the

time interval from s=0 to s=t.

The solution $u(x,t)$ must satisfy the following properties.
(i) u is a continuous function in time on the space X.
(ii) u is a differentiable in time beyond time zero.
(iii) that it takes values in the domain in which the operator A is defined.

we wants to express equation (3.2).which is the perturbation equation,but instead of writing it in this traditional from ,he wants to write it in an abstract mathematical from using what is called the differential operator A. This is common in the analysis of advanced partial equation.
instead of writing (3.2):

$$u_t = u_{xx} + q(x)u + F(x,t)$$

we write it in this more abstract way:

$$u_t = Au + F(x,t) \tag{3.9}$$

such that

$$Au = u_{xx} + q(x)u$$

u satisfies (3.4) in the sense that

$$\lim_{t\to 0^+} \|u(\cdot,t) - f\| = 0 \tag{3.10}$$

Assume the function $F(\cdot,t)$ is defined on the interval $[0,t_0]$ and is uniformly Holder continuous in t. that is there exist constants $C > 0$ and $0 < \beta \leq 1$ such that:

$$\|F(\cdot,t) - F(\cdot,t')\| \leq C \mid t - t' \mid^{\beta}, \quad \forall t,t' \in [0,t_0].$$

∘ The detailed proof and framework for this assumption are given in the literature,especially
∘ In particular,Theorem 3.2 establishes key properties of the operator and its semigroup under these conditions

the operator A corresponding to the unperturbed problem, has eigenvalues λ_K with asymptotic behavior :

$$\lambda_k = -(k\pi)^2 + O(1) \quad \text{as } K \to \infty.$$

but adding a real constant to function $q(x)$, the spectrum of A can be shifted such that :

$$Re(\lambda_k) < 0, \quad \forall K.$$

the resolvent operator is defined as:

$$R(A, \lambda) = (\lambda - A)^{-1}$$

and satisfies the estimate:

$$\|R(A, \lambda)\| \leq \tfrac{C}{|\lambda|+1}, \quad \text{for} \quad \mathrm{Re}(\lambda) > 0.$$

this is key analytic property that allows us to control the behavior of solutions in terms of spectral parameters

the operator A generates an analytic semigroup $\{U(t)\}_{t\geq 0}$ of bounded linear operators on the X
the semigroup satisfies an estimate of the from:

$$\|AU(t)\| \leq Ct^{-1}, \quad t > 0, \tag{3.11}$$

for some constant C and all $t > 0$
$t = 0$ this estimate shows the smoothing effect of the semigroup and controls its behavior near
the additions of a constant to $q(x)$ corresponds to a spectral shits,which translates to modifying $U(t)$ by multiplying it by an exponential factor

$$e^{\lambda t}U(t) \to U(t)$$

The operator A_0 is associated with the unperturbed problem where $q(x) \equiv 0$ (i.e., no perturbation term).

• Its eigenvalues are given by:

$$\lambda_k^0 = -(k\pi)^2 \quad \text{for} \quad k = 1, 2, \ldots$$

• The corresponding eigenfunctions are:

$$s_k(x) = \sqrt{2}\sin(k\pi x), \quad k = 1, 2, \ldots \tag{3.12}$$

The semigroup $\{U_0(t)\}$ is given by:

$$U_0(t)f(x) = \sum_{k=1}^{\infty} e^{\lambda_k^0 t}(f, s_k)s_k(x) \tag{3.13}$$

This study compares two semigroups:

- $\{U(t)\}$: The semigroup associated with the operator A (with perturbation or non-trivial conditions).
- $\{U_0(t)\}$: The semigroup of the unperturbed operator A_0 (where $q(x) \equiv 0$).

The goal is to analyze the behavioral differences between the two systems by studying their eigenvalues, eigenfunctions, and operator kernels. The kernel is given by:

$$K_N(x, t; y) = \sum_{k=1}^{N} \left[e^{\lambda_k t}u_k(x)\overline{v}_k(y) - e^{\lambda_k^0 t}s_k(x)s_k(y)\right] \tag{3.14}$$

such that:

- λ_k: Eigenvalues of the perturbed operator A.
- $u_k(x)$: Eigenfunctions of A.
- $\overline{v}_k(y)$: Adjoint eigenfunctions of A (for non-self-adjoint operators).
- $\lambda_k^0 = -(k\pi)^2$: Eigenvalues of the unperturbed operator A_0.

- $s_k(x) = \sqrt{2}\sin(k\pi x)$: Known eigenfunctions of A_0.

The difference is defined as:

$$\delta_k = \lambda_k - \lambda_k^0$$

where:

- λ_k: Eigenvalues of the perturbed operator
- λ_k^0: Eigenvalues of the unperturbed operator

These differences δ_k define a multiplier transform $M : X \to X$ of the form:

$$Mf = \sum_{k=1}^{\infty} \delta_k (f, s_k) s_k \tag{3.15}$$

The following equation appears between the perturbed generating operator and the unperturbed generating operator:

$$A = B[A_0 + M]B^{-1} \tag{3.16}$$

where:

- A is the perturbed generator
- A_0 is the unperturbed generator
- B is a continuous linear transformation (isomorphism)
- M is the multiplier operator representing the perturbation
- B^{-1} is the inverse transformation of B

3.2 Asymptotic estimates for the eigenvalues and eigenfunctions of A

The aim of this part is to study the effect of the perturbation $q(x)$ on the eigenvalues and eigenfunctions of the differential operators :

$$Au = -\frac{\partial^2 u}{\partial x^2}(x,t) + q(x)u$$

our goal is to obtain asymptotic estimates for the eigenvalues λ_K and eigenfunctions $u_k(x)$ as the order $K \to \infty$.this analysis is essential for understanding how the spectral properties of the operator change under perturbation and how they deviate from the unperturbed case

Let

$$Q(x) = \frac{1}{2}\int_0^x q(x)dy$$

this equation defines a function $Q(x)$ as a **normalized integral** of the function $q(y)$ from 0to x.
clearly $Q(0) = 0$

$$Q(1) = 0 \tag{3.17}$$

- A is a differential operator that is given by

$$A = -\frac{\partial^2}{\partial x^2} - q(x)$$

$$Au = \lambda u \tag{3.18}$$

equation $Au = \lambda u$ know the problem of the intrinsic value of the A. we consider the following eigenvalue problem with Dirichlet boundary condition:

$$u''(x) + q(x)u(x) = \lambda u(x) \tag{3.19}$$

$$u(0) = 0 = u(1). \tag{3.20}$$

$$Au \Leftrightarrow u''(x) + q(x)u(x)$$

suppose que $\lambda = -p^2$
◦ represents the eigenvalues of the operator A, by redefining it as $-p^2$, we shift our analysis to studying p in the complex plane.

∘ the half–plane lm>-k, defines the region in which we examine solution

For sufficiently large $\mid \rho \mid$, two linearly independent solutions of (3.19) are:

$$y_1(x,\rho) = e^{i\rho x}\left[1 - \frac{1}{i\rho}Q(x) - \frac{1}{2i\rho}\int_x^1 e^{-2i\rho(x-y)}q(y)\,dy + O(\rho^{-2};x)\right],$$
$$y_2(x,\rho) = e^{-i\rho x}\left[1 + \frac{1}{i\rho}Q(x) - \frac{1}{2i\rho}\int_0^x e^{2i\rho(x-y)}q(y)\,dy + O(\rho^{-2};x)\right],$$

where $O(\rho^{-2};x)$ denotes a function bounded by $|\rho^{-2}|$ as $|\rho| \to \infty$ in the half-plane, uniformly in $x \in [0,1]$.

First Derivatives of Solutions

$$y_1'(x,\rho) = i\rho e^{i\rho x}\left[1 - \frac{1}{i\rho}Q(x) + \frac{1}{2i\rho}\int_x^1 e^{-2i\rho(x-y)}q(y)\,dy + O(\rho^{-2};x)\right],$$
$$y_2'(x,\rho) = -i\rho e^{-i\rho x}\left[1 - \frac{1}{i\rho}Q(x) - \frac{1}{2i\rho}\int_0^x e^{2i\rho(x-y)}q(y)\,dy + O(\rho^{-2};x)\right].$$

Theorem 3.2.1 *Assume is valid, and let $\lambda = -p^2$.The set of λ are eigenvalues of A satisfies*

$$\rho_k = k\pi + \frac{1}{2k\pi}\int_0^1 q(y)\cos(2k\pi y)\,dy + O(k^{-2}), \quad k \to +\infty \tag{3.21}$$

Remark 3.2.2 *In[16], alightly weaker asymptotic formula is derived for a large class of differential operators.*

we shall find the following notation useful:

$$s_k = 2^{1\backslash 2} sink\pi x, \quad \sigma_k(x) = (1 \backslash 2)\int_0^x q(y)sin2k\pi ydy,$$
$$c_k = 2^{1\backslash 2} cosk\pi x, \quad \gamma_k(x) = (1 \backslash 2)\int_0^x q(y)cos2k\pi ydy.$$

Note that (3.21) becomes

$$\rho_k = k\pi + \gamma_k(1)/k\pi + O(k^{-2}) \tag{3.22}$$

Theorem 3.2.3 *The eigenfunctions $u_k(x)$ of A satisfy*

$$u_k(x) = s_k(x) + \frac{1}{k\pi}\Big[Q(x)c_k(x) + \big[x\gamma_k(1) - \gamma_k(x)\big]c_k(x) - \sigma_k(x)s_k(x)\Big] + O(k^{-2};x),$$
$$u_k^{(1)}(x) = \rho_k c_k(x) + \frac{\rho_k}{k\pi}\Big[-Q(x)s_k(x) - \big[x\gamma_k(1) - \gamma_k(x)\big]s_k(x) - \sigma_k(x)c_k(x)\Big]$$
$$+ \rho_k O(k^{-2};x), \quad as\ k \to +\infty.$$

This proof analyzes the eigenfunctions of the differential operator A, focusing on asymptotic estimates for A_k and B_k as $k \to +\infty$

Proof. • The eigenfunction $y(x, \rho_k)$ is given by:

$$y(x, \rho_k) = A_k y_1(x, \rho_k) + B_k y_2(x, \rho_k), \tag{3.23}$$

where:

$$A_k = y_2(0, \rho_k),$$
$$B_k = -y_1(0, \rho_k).$$

(a) Estimate for A_k

$$A_k = 1 + O(k^{-2}).$$

The dominant term is 1, with an error term decaying as k^{-2}

(b) Estimate for B_k

$$B_k = -1 + \frac{1}{2i\rho_k} \int_0^1 e^{2i\rho_k y} q(y)\, dy + O(k^{-2}).$$

The integral term captures oscillations of $q(y)$ at frequency $2\rho_k$.

- Simplification Using $\rho_k \approx k\pi$ Approximating $\rho_k \approx k\pi$:

$$B_k = -1 + \frac{1}{2k\pi i} \int_0^1 e^{2k\pi i y} q(y)\, dy + O(k^{-2}).$$

Here, the integral becomes a Fourier transform at $2k\pi$.Simplification Using $\rho_k \approx k\pi$ Approximating $\rho_k \approx k\pi$:

$$B_k = -1 + \frac{1}{2k\pi i} \int_0^1 e^{2k\pi i y} q(y)\, dy + O(k^{-2}).$$

Here, the integral becomes a Fourier transform at $2k\pi$.

Transformation from Equation (3.23) to (3.24)

$$y(x, \rho_k) = y_1(x, \rho_k) + y_2(x, \rho_k) \left[-1 + \frac{1}{2k\pi i} \int_0^1 e^{2k\pi i y} q(y)\, dy \right] + O(k^{-2}; x). \tag{3.24}$$

• Original Equation (3.23):

$$y(x, \rho_k) = A_k y_1(x, \rho_k) + B_k y_2(x, \rho_k),$$

• Substituting A_k and B_k :

$$y(x,\rho_k) = y_1(x,\rho_k) + y_2(x,\rho_k)\left[-1 + \frac{1}{2k\pi i}\int_0^1 e^{2k\pi i y} q(y)\,dy\right] + O(k^{-2};x). \tag{3.25}$$

Here A_k is replaced by 1, and B_k by its asymptotic expansion, while retaining the error term $O(k^{-2})$

$$\begin{aligned} y_1(x,\rho_k) &= e^{k\pi i x}\left[1 - \frac{x\gamma_k(1)}{k\pi i} + O(k^{-2};x)\right] \\ &\quad \times \left[1 - \frac{Q(x)}{k\pi i} - \frac{1}{2k\pi i}\int_x^1 e^{-2k\pi i(x-y)} q(y)\,dy + O(k^{-2};x)\right] \\ &= e^{k\pi i x}\left[1 - \frac{Q(x) + x\gamma_k(1)}{k\pi i} - \frac{1}{2k\pi i}\int_x^1 e^{-2k\pi i(x-y)} q(y)\,dy + O(k^{-2};x)\right]. \end{aligned} \tag{3.26}$$

using approximation of of (3.22), we find

The asymptotic expansion of the exponential term is given by:

$$e^{i\rho_k x} = e^{k\pi i x}\left[1 - \frac{x\gamma_k(1)}{k\pi i} + O(k^{-2};x)\right].$$

This approximation follows from the asymptotic relation $\rho_k \approx k\pi$ (assuming ρ_k is close to $k\pi$).

• Applying to $y_1(x,\rho)$ equation (3.21), we obtain :

$$\begin{aligned} y_1(x,\rho_k) &= e^{k\pi i x}\left[1 - \frac{x\gamma_k(1)}{k\pi i} + O(k^{-2};x)\right] \\ &\quad \times \left[1 - \frac{Q(x)}{k\pi i} - \frac{1}{2k\pi i}\int_x^1 e^{-2k\pi i(x-y)} q(y)\,dy + O(k^{-2};x)\right] \end{aligned} \tag{3.27}$$

■

3.3 Semigroup behavoir

The goal of this passage is to **study the behavior of semigroups of linear operators** and understand their convergence as time approaches zero from the right ($t \to 0^+$), with a particular focus on direct consequences of the Hille-Yosida Theorem.

The first equation:

$$\lim_{t\to 0^+} \|U(t)f - f\| = 0$$

Significance Confirms that the operator semigroup $U(t)$ approaches the identity operator when $(t \to 0^+)$

• we compare between two operator semigroups.

$$\lim_{t\to 0^+} \|U(t)f - U_0(t)f\| = 0 \tag{3.28}$$

- ∘ $U(t)$ The semigroup under study,
- ∘ $U_0(t)$ A reference semigroup (could be an approximation or analytical solution).

Theorem 3.3.1 *If f is in $L^1(0,1)$, then*

$$\lim_{t\to 0^+} [U(t) - U_0(t)]\, f(x) = 0 \tag{3.29}$$

uniformly in $x, 0 \le x \le 1$.

Proof. The given proof examines the difference between two operators, $U(t)$, and $U_0(t)$ and demonstrates that this difference converges to zero under certain conditions. Below is a detailed breakdown of the steps

The goal is to prove that $U(t)$ and $U_0(t)$ converge uniformly as $t \to 0^+$ for any function f in $L^1(0,1)$.

This is achieved by approximating f with a smoother function g and then controlling the difference.

• The difference between the two operators is expressed as a series:

$$[U(t) - U_0(t)]f(x) = \textstyle\sum_1^\infty \left[e^{\lambda_k t}(f, v_k)u_k(x) - e^{\lambda_k^0 t}(f, s_k)s_k(x)\right].$$

∘ If g is a function in $C^2[0,1]$ that satisfies the boundary conditions $(g(0) = g(1) = 0)$, then its coefficients (g, v_k) decay rapidly $(O(k^{-2}))$, ensuring uniform convergence of the series.

• For any $f \in L^1(0,1)$ and $\epsilon > 0$, there exists such a function g with $\|f - g\| < \epsilon$.

The difference between the operators is rewritten in terms of g:

$$[U(t) - U_0(t)]f(x) = \int_0^1 K(x,t;y)[f(y) - g(y)]\,dy + [U(t) - U_0(t)]g(x).$$

Here, $K(x,t;y)$ is a bounded integral kernel ($|K| \leq M$), which helps control the first term.

• The first term is estimated as $M\epsilon$ due to the approximation of f by g.

• The second term $[U(t) - U_0(t)]g(x)$ becomes smaller than ϵ for sufficiently small t, because g is smooth and $U(t)g$ converges to $U_0(t)g$.

Thus, for small t:

$$\|[U(t) - U_0(t)]f(x)\| < \epsilon(M + 1).$$

Since ϵ is arbitrary, $\|[U(t) - U_0(t)]f\| \to 0$ as $t \to 0$.The first term is estimated as $M\epsilon$ due to the approximation of f by g.

The second term $[U(t) - U_0(t)]g(x)$ becomes smaller than ϵ for sufficiently small t, because g is smooth and $U(t)g$ converges to $U_0(t)g$.

Thus, for small t:

$$\|[U(t) - U_0(t)]f(x)\| < \epsilon(M + 1).$$

Since ϵ is arbitrary, $\|[U(t) - U_0(t)]f\| \to 0$ as $t \to 0$. ■

Lemma 3.3.2 *Let g be in X,$\{\sigma_k\}$ in ℓ^2,and let $\{\beta_k(t)\}$ be a sequence of function on $0 \leq t \leq t_0$ such that $\mid \beta_k(t) \mid \leq M$ there.Let*

$$h(x,t) = \sum_{t}^{\infty} \sigma_k \beta_k(t)(g, s_k)s_k(x).$$

Then $h(\cdot,t)$ is in X, and

$$\|h(\cdot,t)\| \leq M\|g\|.$$

Proof. The proof examines the properties of a perturbed heat operator $U(t)$ in various functional spaces (C_0, L^p) and establishes precise convergence estimates as $t \to 0^+$.

suppose que we have $g \in L^q$, where q is the conjugate exponent of p satisfying the relation:

$$\frac{1}{p} + \frac{1}{q} = 1.$$

The series $h(x,t)$ converges absolutely since $\langle g, s_k \rangle \in \ell^2$, with estimate:

$$|h(x,t)|^2 \leq M^2 \sum_{k=1}^{\infty} |\delta_k|^2 \|g\|_2^2 \leq M^2 \|\delta_k\|^2 \|g\|_X^2.$$

Boundedness of δ_k and $\beta_k(t)$ ($|\beta_k(t)| \leq M$) ensures convergence.

- **case** $X = L^p$ **(for 1< p <2)**

The series may not converge absolutely, but it converges in the L^2 norm for each fixed t.

The map $X \to L^2$ is closed and bounded, ensuring that

$$\|h(\cdot,t)\|_2 \leq M\|g\|_X.$$

- Using the embedding $L^2 \subset X$, it follows that

$$\|h(\cdot,t)\|_X \leq M\|g\|_X.$$

■

Theorem 3.3.3 *For f in X,*

$$\|U(t)f - f\|_X \leq M\|U_0(t)B^{-1}f - B^{-1}f\|_X + O(t\|f\|_X) \tag{3.30}$$

as $t \to 0^+$.

Proof. Since the linear operator B is continuous in the norm of X,

$$U(t)f = \sum_1^{\infty} e^{\lambda_s t}(f,\, v_u)\, u_b(\mathbf{x}) = B \sum_1^{\infty} e^{\lambda_s^0 t} e^{\beta_s t}(B^{-1}f,\, s_b)\, s_b(\mathbf{x})$$

$$= B \sum_1^{\infty} e^{\lambda_s^0 t}(B^{-1}f,\, s_b)\, s_b(\mathbf{x}) + tB \sum_1^{\infty} e^{\lambda_s^0 t} \delta_b \beta_b(t)(B^{-1}f,\, s_b)\, s_b(\mathbf{x})$$

$$= BU_0(t)\, B^{-1}f + tB\sum_{1}^{\infty} e^{\lambda_s^0 t}\delta_b\beta_b(t)(g,\, s_b)\, s_b(\mathbf{x}), \tag{3.31}$$

where $g = B^{-1}f$. Note that $|\beta_b(t)| \leqslant M$ if $0 \leqslant t \leqslant t_0$. Thus by the previous Lemma,

$$\left\| B\sum_{1}^{\infty} e^{\lambda_s^0 t}\delta_b\beta_b(t)(g,\, s_b)s_b \right\| \leqslant M\|f\|. \tag{3.32}$$

Then

$$U(t)f(\mathbf{x}) - f(\mathbf{x}) = B[U_0(t)\, B^{-1}f - B^{-1}f] + O(t\|f\|). \quad \text{Q.E.D.} \tag{3.33}$$

If $\|[U(t) - I]f\| = o(t)$ as $t \to 0$, then f is in $D(A)$ and $Af = 0$. Consequently growth no better than $O(t)$ can be expected, and this is always realized if f is in $D(A)$. If X is reflexive, then $O(t)$ growth occurs if and only if f is in $D(A)$. Thus in (3.30), the first term on the right cannot be smaller than the second, as $t \to 0$, unless $f = 0$. This situation is now examined in greater detail. Let $\phi(s)$ be a positive, nonincreasing function on $(0, \infty)$, such that $\phi(s) \to 0$ as $s \to \infty$. Let Λ_s^0 denote the collection of all functions f in X such that

(i) $\|[U_0(t) - I]f\|_X = O(\phi(t^{-1}))$, $t \to 0$,

(ii) $\limsup_{t\to 0} \|[U_0(t) - I]f\|_X/\phi(t^{-1}) > 0$.

Let A_s denote the similar class for $U(t)$. ■

Theorem 3.3.4 *If* $s\phi(s) \to \infty$,*as*,*then* $\Lambda_\phi = B\Lambda_\phi^0$.

Proof. The condition $s\phi(s) \to \infty$ as $s \to \infty$ means that Λ_ϕ^0 is not the saturation class for $U_0(t)$. From (3.30), we see that if f is in Λ_ϕ^0, then

$$\|(U(t) - I)Bf\| = O(\phi(t^{-1})).$$

To verify condition (ii) for $B\Lambda_\phi^0$, we note from (3.33) that if f is in λ_ϕ^0, then

$$[U(t) - I]Bf \mid \phi(t^{-1}) = B[U_0(t) - I]f \mid \phi(t^{-1}) + [t \mid \phi(t^{-1})]O(1).$$

Since $s\phi(s) \to \infty$ as $s \to \infty$, we see that $t \,|\, \phi(t^{-1}) \to 0$ as $t \to 0$, yielding (ii). The converse follows from the symmetric roles of $U(t)$ and $U_0(t)$. ■

Theorem 3.3.5 *If $\phi(s) = s^{-1}$ and $\lambda = 0$ is not an eigenvalue of A, then $\Lambda_\phi = B\Lambda_\phi^0$.*

Proof. If $f \in \Lambda_\phi^0$, then the two terms on the right of (3.33) are both $O(t)$. If condition (ii) is not satisfied for $U(t)$ and Bf, then Bf lies in the null space of A, so f and Bf are zero and, in particular, $f \notin A_\phi^0$. The converse holds since $\lambda = 0$ is not an eigenvalue of Λ_0. ■

Conclusion

In this work, we studied and analyzed the perturbation of one-parameter linear semigroups, due to its central importance in solving evolutionary differential equations—particularly the heat equation. We began with the fundamental definitions and the theory of semigroups in Hilbert spaces, then moved on to explore how the addition of a perturbing operator to the original generator affects the existence and properties of the resulting semigroup.

We showed that certain types of perturbations, such as bounded or relatively bounded operators, do not prevent the perturbed operator $A + B$ from generating a semigroup, provided some technical conditions are satisfied. This was illustrated through fundamental results such as the *Miyadera–Voigt* and *Kato* theorems, which demonstrated the robustness of semigroups in accommodating structural changes to their generators.

In the application section, we focused on the heat equation as a canonical model, showing how the evolution of its solutions can be interpreted via perturbed semigroups.

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