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Some homogenization results for wave equation

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Dedication

To the one my heart beats with love for, To my dearest **mother** the source of tenderness, strength, and endless prayers... I dedicate to you the fruit of my effort and success, for without you, I would not have reached this point, dreamed this dream, or persevered on this path.

To my beloved **father**, may Allah have mercy on his soul, Though you are gone, your words and advice still live within me. I know that your spirit watches over me with pride from above. May Allah grant you the highest reward — this success is a humble gift to your blessed memory.

To my precious **brother**, Who has never failed to support and stand by me, And to all my dear **siblings**, each by name, You have always been my strength and support. You have my deepest love and gratitude.

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Notations

- ▶ Ω : Bounded open domain in $\mathbb{R}^n$.
- ▶ Y : Unit periodic cell, typically $Y = (0,1)^n$.
- ▶ ε : Small parameter representing the microstructure scale.
- ▶ x : Macroscopic (slow) spatial variable.
- ▶ $y = \dfrac{x}{\varepsilon}$: Microscopic (fast) spatial variable.
- ▶ $A(y)$: Periodic coefficient matrix defined on Y.
- ▶ $A^\varepsilon(x)$: Oscillating coefficient: $A\left(\dfrac{x}{\varepsilon}\right)$.
- ▶ A^H : Homogenized (effective) matrix obtained from the cell problem.
- ▶ u^ε : Solution of the original (heterogeneous) wave equation.
- ▶ u_0 : Homogenized solution, the limit of u^ε as $\varepsilon \to 0$.
- ▶ χ_k : Corrector function solving the k-th cell problem.
- ▶ w^ε : Oscillating test function: $x_k + \varepsilon \chi_k \left(\dfrac{x}{\varepsilon}\right)$.
- ▶ ∇_x : Gradient operator with respect to x.
- ▶ ∇_y : Gradient operator with respect to y.
- ▶ div_x : Divergence operator with respect to x.
- ▶ div_y : Divergence operator with respect to y.

- ▶ $L^2(\Omega)$: Space of square-integrable functions on Ω.
- ▶ $H_0^1(\Omega)$: Sobolev space with zero boundary conditions.
- ▶ $H^{-1}(\Omega)$: Dual space of $H_0^1(\Omega)$.
- ▶ $L^2(0, T; H_0^1(\Omega))$: Time-dependent functions valued in $H_0^1(\Omega)$.
- ▶ u_0^ε, u_1^ε : Initial displacement and velocity of u^ε.
- ▶ p_i^ε : Flux: $a_{ij}^\varepsilon \dfrac{\partial u^\varepsilon}{\partial x_j}$.
- ▶ $\rightharpoonup$: Weak convergence.
- ▶ $\overset{*}{\rightharpoonup}$: Weak-* (weak star) convergence.

Introduction

Several problems in Engineering Sciences, Physics or Chemistry can be modeled by using partial differential equations with a periodic structure or exhibiting small heterogeneities relative to the domain dimension[10]. This is particularly the case for composite materials, which are increasingly used in industry due to their superior characteristics compared to their components.

Modeling phenomena occurring in these composite materials, such as wave or heat propagation, can result in partial differential equations with highly oscillating coefficients. When too numerous, these oscillations can cause problems in the numerical resolution of these equations. Homogenization theory makes it possible to remedy this by replacing problems with highly oscillating coefficients with approximate problems whose coefficients are constant, and therefore much simpler to solve numerically.

The aim of homogenization theory is to establish the macroscopic behavior of a system which is 'microscopically' heterogeneous, in order to describe some characteristics of the heterogeneous medium. This means that the heterogeneous material is replaced by a homogeneous fictitious one (the 'homogenized' material), whose global characteristics are a good approximation of the initial ones.

This work aims to study the homogenization of the wave equation (see [6])

$$\frac{\partial^2 u^\epsilon}{\partial t} - \operatorname{div}\left(A^\epsilon \nabla u^\epsilon\right) = f\epsilon, \quad \text{in } \Omega\times]0,T[$$

with periodically oscillating coefficients, using two main approaches : the multiple scale method ([2], [5]) and the Oscillating Test Function Method ([14], [16]), in order to derive a simpler (homogenized) model that accurately approximates the behavior of the original equation in media with fine-scale oscillations.

The dissertation is composed of tree chapters:

The first chapter is devoted to the presentation of some functional analysis results that will be useful to us in the rest of this work. In the second chapter, we are interested in recalling the main homogenization methods, namely the multiple scale method and the oscillating test function method of L. Tartar. The last chapter is dedicated to studying the homogenization of the wave equation. First, we show the existence and uniqueness of the solution, then we give a homogenization results

Chapter 1

Some results of functional analysis

. In this chapiter,we recall the notions of functional analysis that will be useful to us in the rest of this work. These results can be consulted in [3],([6],[12],[13]).

1.1 Weak convergence

Let $(E, \|\cdot\|_E)$ be a Banach space, and E' its dual, which is a Banach space for the norm

$$\|f\|_{E'} = \sup \frac{|\langle f, u\rangle_{E',E}|}{\|u\|_E}. \tag{1.1}$$

Definition 1 (Strong convergence) : A sequence (u_n) in E is said to converge strongly to u if and only if

$$\lim_{n\to+\infty} \|u_n - u\|_E = 0. \tag{1.2}$$

This strong convergence is denoted

$$u_n \to u \quad \text{in } E. \tag{1.3}$$

Definition 2 (Weak convergence) : A sequence (u_n) in E is said to converge weakly to u if and only if

$$\forall f \in E', \quad \langle f, u_n\rangle_{E',E} \to \langle f, u\rangle_{E',E}. \tag{1.4}$$

This weak convergence is denoted by

$$u_n \rightharpoonup u \quad \text{in } E. \tag{1.5}$$

Proposition 3 (i) Strong convergence implies weak convergence.

(ii) If $\dim E = N < +\infty$, the strong and weak convergences are equivalent.

Proposition 4 Let (u_n) be a sequence weakly convergent to u in E. Then

(i) (u_n) is a bounded sequence in E, i.e., there exists a constant C independent of n such that

$$\forall n \in \mathbb{N}, \quad \|u_n\|_E \leq C. \tag{1.6}$$

(ii) The norm on E is lower semi-continuous with respect to the weak convergence, i.e.,

$$\|u\|_E \leq \liminf_{n\to\infty} \|u_n\|_E. \tag{1.7}$$

Theorem 5 (Eberlein–Šmuljan) Assume that E is reflexive and let (u_n) be a bounded sequence in E. Then

(i) there exists a subsequence (u_{n_k}) of (u_n) and $u \in E$ such that, as $k \to \infty$,

$$u_(n_k) \rightharpoonup u \quad \text{weakly in } E.$$

(ii) if each weakly convergent subsequence of (u_n) has the same limit u, then the whole sequence (u_n) weakly converges to u, i.e.

$$u_n \rightharpoonup u \quad \text{weakly in } E.$$

Definition 6 (Weak* convergence) : A sequence (f_n) in E' is said to converge weakly* to f iff

$$\forall u \in E, \quad \langle f_n, u\rangle_{E',E} \to \langle f, u\rangle_{E',E}. \tag{1.8}$$

This weak* convergence is denoted

$$f_n \rightharpoonup^* f \quad \text{in } E'. \tag{1.9}$$

Proposition 7 Every weakly convergent sequence in E' is weakly* convergent. However, the opposite is false unless E is reflexive.

Proposition 8 Let (f_n) be a sequence weakly* convergent to f in E'. Then

(i) (f_n) is a bounded sequence in E', i.e., there exists a constant C independent of n such that

$$\forall n \in \mathbb{N}, \quad \|f_n\|_{E'} \leq C. \tag{1.10}$$

(ii) The norm is lower semi-continuous with respect to the weak* convergence, i.e.,

$$\|f\|_{E'} \leq \liminf_{n\to\infty} \|f_n\|_{E'}. \tag{1.11}$$

Theorem 9 Let E be a separable Banach space and let (f_n) be a bounded sequence in E'. Then

(i) There exists a subsequence (f_{n_k}) of (f_n) and $f \in E'$ such that, as $k \to \infty$

$$f_{n_k} \rightharpoonup^* f \quad \text{in } E'. \tag{1.12}$$

(ii) If each weakly* convergent subsequence of (f_n) has the same limit f, then the whole sequence (f_n) weakly* converges to f, i.e.,

$$f_n \rightharpoonup f \quad \text{in } E'. \tag{1.13}$$

1.2 Weak and Weak* Convergence in L^p

Let Ω be a bounded open set in $\mathbb{R}^n$.

Definition 10 Let $p \in \mathbb{R}$ be with $1 \leq p < \infty$. Define

$$L^p(\Omega) = \{f : \Omega \to \mathbb{R}, f \text{ measurable and } \int_\Omega |f|^p dx < \infty\} \tag{1.14}$$

$$L^\infty(\Omega) = \{f : \Omega \to \mathbb{R}, f \text{ measurable and } \exists C > 0, |f(x)| \leq C \text{ a.e. on } \Omega\}. \tag{1.15}$$

Proposition 11 Let $p \in \mathbb{R}$ be with $1 \leq p \leq +\infty$. The set $L^p(\Omega)$ is a Banach space for the norm

$$\|f\|_{L^p(\Omega)} = \begin{cases} \left(\int_\Omega |f|^p dx\right)^{1/p} & \text{if } p < +\infty \\ \inf\{C; |f(x)| \leq C \text{ a.e on } \Omega\} & \text{if } p = +\infty \end{cases}$$

If $p = 2$, the space $L^2(\Omega)$ is a Hilbert space for the scalar product

$$(f, g)_{L^2(\Omega)} = \int_\Omega f(x)g(x)dx.$$

The space $L^p(\Omega)$ is separable for $1 \leq p < +\infty$ and is reflexive for $1 < p < +\infty$,
1. If $1 \leq p < \infty$, then $u_n \rightharpoonup u$ in $L^p(\Omega)$ if

$$\int_\Omega u_n \phi dx \to \int_\Omega u\phi dx, \quad \forall \phi \in L^q(\Omega)$$

with $1/p + 1/q = 1$.
2. If $p = +\infty$, then $f_n \overset{*}{\rightharpoonup} f$ in $L^\infty(\Omega)$ if

$$\int_\Omega f_n \phi dx \to \int_\Omega f\phi dx, \quad \forall \phi \in L^1(\Omega)$$

Lemma 1 Let p, q and r be three reals in $[1, +\infty]$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Let (u_n) be a sequence in $L^p(\Omega)$ and (v_n) be a sequence in $L^q(\Omega)$ such that:

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in } L^p(\Omega) \\ v_n \to v & \text{strongly in } L^q(\Omega) \end{cases}$$

Then

$$u_n v_n \rightharpoonup uv \quad \text{weakly in } L^r(\Omega)$$

1.3 Periodic functions

Definition 12 (Periodic functions) Let $Y =]0, l_1[\times \cdots \times]0, l_n[$ and let f be a function defined almost everywhere on $\mathbb{R}^n$. The function f is called Y-periodic if

$$f(x + kl_i e_i) = f(x) \quad \text{a.e. on } \mathbb{R}^n, \quad \forall k \in \mathbb{Z}, \forall i \in \{1, \ldots, n\}, \tag{1.16}$$

where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$. In the case $n = 1$, we simply say that f is l_1-periodic.

Definition 13 Let Ω be a bounded open set of $\mathbb{R}^n$ and let $f \in L^1(\Omega)$. The mean value of f over Ω is given by

$$M_\Omega(f) = \langle f \rangle = \frac{1}{|\Omega|} \int_\Omega f(y)dy. \tag{1.17}$$

Lemma 2 Let f be a Y-periodic function in $L^1(Y)$. Let y_0 be a fixed point in $\mathbb{R}^n$ and denote by Y_0 the translated set of Y, defined by $Y_0 = y_0 + Y$. Set

$$f_\epsilon(x) = f\left(\frac{x}{\epsilon}\right) \quad \text{a.e. on } \mathbb{R}^n. \tag{1.18}$$

Then:

1. $\int_{Y_0} f(y)dy = \int_Y f(y)dy$,
2. $\int_{Y_0} f(x)dx = \int_Y f(x)dx = \epsilon^n \int_Y f(y)dy$.

Theorem 14 Let $1 \leq p \leq +\infty$ and let f be a Y-periodic function in $L^p(Y)$. Set

$$f_\epsilon(x) = f\left(\frac{x}{\epsilon}\right) \quad \text{a.e. on } \mathbb{R}^n. \tag{1.19}$$

Then,

- If $p < +\infty$, as $\epsilon \to 0$, we have $f_\epsilon \rightharpoonup M_\Omega(f)$ weakly in $L^p(\Omega)$.
- If $p = +\infty$, we have $f_\epsilon \overset{*}{\rightharpoonup} M_\Omega(f)$ weakly* in $L^\infty(\Omega)$.

1.4 The space $H^1_{per}(Y)$

Definition 15 (The space $H^1_{per}(Y)$) Let $C^\infty_{per}(Y)$ be the subset of $C^\infty(\mathbb{R}^n)$ of Y-periodic functions. We denote by $H^1_{per}(Y)$ the closure of $C^\infty_{per}(Y)$ for the H^1-norm.

Definition 16 We introduce the space $L^2(0, T; H^1_0(\Omega))$, which is the space of functions $t \mapsto u(t)$ with values in $H^1_0(\Omega)$ and square integrable in t. It is provided with the norm

$$\|u\|_{L^2(0,T;H^1_0(\Omega))} = \left(\int_0^T \|u(t, \cdot)\|^2_{H^1_0(\Omega)} \, dt \right)^{1/2}.$$

1.5 Fredholm Alternative

Lemma 3 (Fredholm Alternative [1] [8]) for Periodic Elliptic Equations, Let $f \in L^2(Y)$ be a Y periodic function. Then the boundary value problem

$$-\operatorname{div}(A(y)\nabla v(y)) = f(y), \quad y \in Y, \tag{1.20}$$

with $A \in M^{n\times n}(\mathbb{R})$

admits a Y-periodic solution $v \in H^1_{\text{loc}}\mathbb{R}^n$, unique up to an additive constant, if and only if the right-hand side satisfies the compatibility condition

$$\int_Y f(y)\,dy = 0. \tag{1.21}$$

Chapter 2

Basic on homogenization theory

In this chapter , we present the main homogenization methods , namely the multiple scale method (see [2],[5]) and the oscillating test function method of L.Tartar ([11],[14])

We consider the simplest problem of temperature diffusion inside a body $\Omega \subset \mathbb{R}^n$. The equation reads

$$\begin{cases} -\text{div}(A(x)\nabla u(x)) = f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Here, f is the source of heat inside the material while the tensor $A \in M^{n\times n}(\mathbb{R})$ stands for the diffusion coefficients.

Ω designates the domain occupied by a composite material with periodic structure, its period ϵ (a positive number which is assumed to be very small in comparison with the size of the domain), and the unit periodic cell $Y = (0,1)^n$. The presence of different materials inside Y is modeled by a matrix $A(y)$ where $y = \frac{x}{\epsilon} \in Y$ is the fast periodic variable, while $x \in \Omega$ is the slow variable. Equivalently, x is also called the macroscopic variable, and y the microscopic variable. By periodicity, it is easy to extend $A \in \mathbb{R}^n$, and furthermore $A\left(\frac{x}{\epsilon}\right)$ will represent the diffusion coefficients inside the ϵ-periodic material. and it is also assumed to satisfy certain conditions

$$\begin{cases} a^{\epsilon}_{ij} \in L^{\infty}(\Omega), \\ \exists\, \alpha > 0 \text{ such that } \displaystyle\sum_{i,j=1}^{n} a_{ij}(y)\xi_i\xi_j \geq \alpha|\xi|^2, \quad \forall y \in Y,\, \forall \xi \in \mathbb{R}^n. \end{cases} \tag{2.2}$$

Denoting by u^{ϵ} the solution on the periodically microstructured material, we transform the preceding problem into

$$\begin{cases} -\text{div}(A\left(\frac{x}{\epsilon}\right)\nabla u^\epsilon(x)) = f(x), & \text{in } \Omega, \\ u^\epsilon = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

The aim of homogenization is to precisely give the macroscopic properties of the composite by taking into account the properties of the microscopic structure. Observe that making the heterogeneities smaller and smaller means that we 'homogenize' the mixture, and from the mathematical point of view this means that ϵ tends to zero. So, the homogenizing problem (2.1) consists of:

1. Study the behavior of u^ϵ when ϵ tends to 0.
2. Find its limit u^*, if it exists.
3. Give, if possible, a limit problem satisfied by u^* that will be called the homogenized problem.

2.1 The Multiple Scale Method

The principle of this method is to seek a solution of (2.3) in the form of an asymptotic expansion using multiple scales (i.e., slow and fast variables)

$$u^\epsilon(x) = u_0(x,y) + \epsilon u_1(x,y) + \epsilon^2 u_2(x,y) + \dots, \tag{2.4}$$

with

$$y = \frac{x}{\epsilon}$$

where $u_j(x,y)$ for $j = 1, 2, \dots,$ such that

$$\begin{cases} u_j(x,y) \text{ is defined for } x \in \Omega \text{ and } y \in Y, \\ u_j(x,y) \text{ is Y-periodic in } y. \end{cases} \tag{2.5}$$

Let $\Phi(x) = \Phi(x,y)$ be a function depending on two variables of $\mathbb{R}^n$ and denote by Φ^ϵ the following:

$$\Phi^\epsilon(x) = \Phi(x, \frac{x}{\epsilon}), \tag{2.6}$$

which depends only on one variable. Notice that

$$\frac{\partial \Phi^\epsilon}{\partial x_i} = \frac{\partial \Phi}{\partial x_i}(x,y) + \frac{1}{\epsilon}\frac{\partial \Phi}{\partial y_i}(x,y). \tag{2.7}$$

Then

$$\nabla \Phi^\epsilon(x) = \nabla_x \Phi(x,y) + \frac{1}{\epsilon}\nabla_y \Phi(x,y), \tag{2.8}$$

$$\text{div } \Phi^\epsilon(x) = \text{div}_x \Phi(x,y) + \frac{1}{\epsilon}\text{div}_y \Phi(x,y). \tag{2.9}$$

Defining the operator $\mathcal{A}^\epsilon$ by

$$\mathcal{A}^\epsilon \Phi^\epsilon = -\text{div}(A^\epsilon \nabla \Phi^\epsilon). \tag{2.10}$$

Consequently, substituting (2.8) and (2.9) into (3.27), write $\mathcal{A}^\epsilon\Phi^\epsilon$ as follows:

$$\mathcal{A}^\epsilon\Phi^\epsilon(x) = (\epsilon^{-2}\mathcal{A}_0 + \epsilon^{-1}\mathcal{A}_1 + \mathcal{A}_2)\Phi(x,y), \tag{2.11}$$

where

$$\begin{cases} \mathcal{A}_0 = -\text{div}_y(A(y)\nabla_y), \\ \mathcal{A}_1 = -\text{div}_x(A(y)\nabla_y) - \text{div}_y(A(y)\nabla_x), \\ \mathcal{A}_2 = -\text{div}_x(A(y)\nabla_x). \end{cases} \tag{2.12}$$

Now, substituting into (3.28) Φ^ϵ by u^ϵ (2.7) and grouping terms of the same orders in powers of ϵ gives a hierarchy of an infinite system of equations, the first three systems of which are:

Order ϵ^{-2}

$$\begin{cases} \mathcal{A}_0 u_0 = 0, \quad \text{in } Y, \\ u_0 \text{ Y-periodic in } y. \end{cases} \tag{2.13}$$

Order ϵ^{-1}

$$\begin{cases} \mathcal{A}_0 u_1 = -\mathcal{A}_1 u_0, \quad \text{in } Y, \\ u_1 \text{ Y-periodic in } y. \end{cases} \tag{2.14}$$

Order ϵ^{0}

$$\begin{cases} \mathcal{A}_0 u_2 = f - \mathcal{A}_1 u_1 - \mathcal{A}_2 u_0, \quad \text{in } Y, \\ u_2 \text{ Y-periodic in } y. \end{cases} \tag{2.15}$$

Solution of (2.13):

$$\begin{cases} -\text{div}_y\left(A(y)\nabla_y u_0\right) = 0 \\ u_0 \quad Y\text{-periodic} \end{cases} \tag{2.16}$$

Then, by the Fredholm alternative (lemma 3), equation (2.16) admits a unique solution (up to an additive constant that depends only on x).

Multiply (2.16) by u_0 and integrating over Y:

$$-\int_Y \text{div}_y\left(A(y)\nabla_y u_0\right) u_0 \, dy = 0 \tag{2.17}$$

Using Green's formula yields:

$$\int_Y A(y)\nabla_y u_0 \cdot \nabla_y u_0 \, dy - \int_{\partial Y} A(y)\nabla_y u_0 \, u_0 \cdot \vec{n} \, ds = 0 \tag{2.18}$$

Since u_0 is Y-periodic, the boundary term vanishes, hence:

$$\int_Y A(y)\nabla_y u_0 \cdot \nabla_y u_0 \, dy = 0 \tag{2.19}$$

In view of the ellipticity of A (2.2), we get

$$0 = \int_Y A(y)\nabla_y u_0 \cdot \nabla_y u_0 \, dy \;\geq\; \alpha\|\nabla_y u_0\|^2 \tag{2.20}$$

That necessitates

$$\nabla_y u_0 = 0 \tag{2.21}$$

and the fact that

$$u_0(x, y) = u_0(x) \tag{2.22}$$

Solution of (2.14):

$$\begin{cases} -\operatorname{div}_x \left(A(y)\nabla_y u_0\right) = \operatorname{div}_y \left(A(y)\nabla_x u_0\right) + \operatorname{div}_y \left(A(y)\nabla_y u_1\right), \\ u_1 \text{ is } Y\text{-periodic} \end{cases} \tag{2.23}$$

Since u_0 depends only on x then

$$\begin{cases} -\operatorname{div}_y \left(A(y)\nabla_y u_1\right) = \operatorname{div}_y \left(A(y)\nabla_x u_0\right), \\ u_1 \text{ is } Y\text{-periodic} \end{cases} \tag{2.24}$$

which is equivalent to

$$\begin{cases} -\operatorname{div}_y \left(A(y)\nabla_y u_1\right) = \operatorname{div}_y \left(A(y)\right).\nabla_x u_0, \\ u_1 \text{ is } Y\text{-periodic} \end{cases} \tag{2.25}$$

We have

$$\int_Y \operatorname{div}_y \left(A(y)\nabla_x u_0\right) dy = \int_Y \nabla_x u_0 \cdot \operatorname{div}_y A(y) dy \tag{2.26}$$

$$= \int_Y \nabla_x u_0 \cdot \operatorname{div}_y A(y)\, dy$$

$$= \int_{dy} A(y)\, dy\, \nabla_x u_0 = 0 \tag{2.27}$$

According to the Fredholm alternative (lemma 3), problem (2.23) admits a unique solution u_1, up to an additive constant. Owing to the linearity of the operator $\mathcal{A}_0$ and the separability of variables in the right-hand side of equation (2.25), the solution u_1 can be expressed in a simple form

$$u_1(x, y) = \chi \nabla_x u_0 + \bar{u}_1(x) \tag{2.28}$$

$$= \sum_{k=1}^{N} \chi_k \frac{\partial u_0}{\partial x_k} + \bar{u}_1(x) \tag{2.29}$$

where $\chi = (\chi_k)_{1 \leq k \leq N}$, denotes the solution of the cell problem

$$\begin{cases} \chi_k \in H^1_{\text{per}}(Y), \\ -\operatorname{div}_y \left(A(y)\nabla_y \chi_k\right) = \operatorname{div}_y \left(A(y)e_k\right) \\ \displaystyle\int_Y \chi_k(y)\, dy = 0. \end{cases} \tag{2.30}$$

Solution of (2.15):

$$\begin{cases} \mathcal{A}_0 u_2 = f - \mathcal{A}_2 u_0 - \mathcal{A}_1 u_1, \\ \quad u_2 \text{ is } Y\text{-periodic.} \end{cases} \tag{2.31}$$

Thanks to **Fredholm alternative**, problem (2.15) admits a unique solution (up to an additive constant if and only if

$$\int_Y (f - \mathcal{A}_2 u_0 - \mathcal{A}_1 u_1)\, dy = 0 \tag{2.32}$$

This is equivalent to

$$\int_Y (\mathcal{A}_2 u_0 + \mathcal{A}_1 u_1)\, dy = f\,|Y| \tag{2.33}$$

Where |Y| = measure of Y hence

$$\frac{1}{|Y|}\left[\int_Y \mathcal{A}_2 u_0\, dy + \int_Y \mathcal{A}_1 u_1\, dy\right] = f \tag{2.34}$$

We have

$$\begin{aligned} \int_Y \mathcal{A}_1\, u_1\, dy &= -\int_Y \mathrm{div}_y\left(A(y)\, \nabla_x u_1\right) dy - \int_Y \mathrm{div}_x\left(A(y)\, \nabla_y u_1\right) dy \\ &= I_1 + I_2 \end{aligned} \tag{2.35}$$

The first term $I_1 = 0$ by the divergence theorem and the periodicity of A and u_1. Now we consider I_2:

$$\begin{aligned} I_2 &= -\int_Y \mathrm{div}_x \Big(A(y)\, \nabla_y u_1\Big) dy && (2.36) \\ &= -\int_Y \mathrm{div}_x\left(A(y)\, \nabla_y\, \chi \nabla_x u_0\right) dy && (2.37) \\ &= -\,\mathrm{div}_x\left(\int_y A(y) \nabla_y \chi\, dy\right) .\nabla_x u_0 && (2.38) \end{aligned}$$

On the other hand

$$\begin{aligned} \int_Y A_2\, u_0\, dy &= -\int_Y \mathrm{div}_x\left(A(y)\, \nabla_x u_0\right) dy && (2.39) \\ &= -\,\mathrm{div}_x\left(\int_Y A(y)\, dy\right) \nabla_x u_0 && (2.40) \end{aligned}$$

Finally, we get the homogeneous problem

$$f(x) = \frac{1}{|Y|}\left[-\mathrm{div}_x\left(\nabla_x u_0 \int_Y A(y) dy\right) - \mathrm{div}_x\left(\int_Y (A(y) \nabla_y \chi) dy\, \nabla_x u_0\right)\right]$$

Then

$$-\operatorname{div}_x\left[\frac{1}{|Y|}\int_Y (A(y)+A(y)\nabla_x\chi(y))\,dy\,\nabla_x u_0\right] \tag{2.41}$$

which can be read

$$-\operatorname{div}_x\left(A^H\nabla_x u_0\right)=f(x) \tag{2.42}$$

Where the homogenized coefficient $A^H=(a^H_{ij})_{1\le i,j\le n}$ is explicitly given by

$$a^H_{ik}=\frac{1}{|Y|}\int_Y\left[a_{ik}(y)+\sum_{j=1}^n a_{ij}(y)\frac{\partial\chi_k}{\partial y_j}(y)\right]\,dy \tag{2.43}$$

It remains to find the boundary conditions that u_0 needs to fulfill. Here, since $u^\epsilon=0$ on $\partial\Omega$, the multiscale expansion gives at order ϵ^0:

$$u_0(x,y)=u_0(x)=0 \quad \text{on } \partial\Omega \tag{2.44}$$

Therefore, the homogenized problem is

$$\begin{cases}-\operatorname{div}_x\left(A^H\nabla_x u_0\right)=f(x) & \text{in } \Omega,\\ u_0=0 & \text{on } \partial\Omega.\end{cases} \tag{2.45}$$

2.2 The oscillating test function method

The Oscillating Test Function Method is a very elegant and highly efficient technique for rigorously homogenizing partial differential equations. It was developed by the mathematician Tartar in his works(see [15],[11]). This method is also called the energy method.

Theorem 17 The sequence u^ϵ of solutions of equation (2.45) converges weakly in $H^1_0(\Omega)$ to a limit u_0, which is the unique solution of the homogenized problem:

$$\begin{cases}-\operatorname{div}\left(A^H\nabla u_0\right)=f(x) & \text{in } \Omega,\\ u_0=0 & \text{on } \partial\Omega.\end{cases} \tag{2.46}$$

Proof : We know that u^ϵ is a solution of

$$\int_\Omega A(x)\nabla u^\epsilon(x)\cdot\nabla v(x)\,dx=\int_\Omega f(x)v(x)\,dx,\quad \forall v\in H^1_0(\Omega)$$

with $f\in L^2(\Omega)$. This problem can be written as

$$\int_\Omega a_{ij}(x)\frac{\partial u^\epsilon}{\partial x_j}\frac{\partial v}{\partial x_i}\,dx=\int_\Omega f(x)v(x)\,dx,\quad \forall v\in H^1_0(\Omega) \tag{2.47}$$

Taking $v=u^\epsilon$ and using the ellipticity of A^ϵ and Poincare inequality:

$$\alpha\|u^\epsilon\|^2_{H^1_0(\Omega)}\le\int_\Omega a_{ij}(x)\frac{\partial u^\epsilon}{\partial x_j}\frac{\partial u^\epsilon}{\partial x_i}\,dx=\int_\Omega f(x)u^\epsilon(x)\,dx\le C\|f\|_{L^2(\Omega)}\|u^\epsilon\|_{H^1_0(\Omega)}$$

which gives

$$\|u^\epsilon\|_{H_0^1(\Omega)} \leq C\|f\|_{L^2} \tag{2.48}$$

Then, we can extract a subsequence, still denoted by u^ϵ, such that

$$\begin{cases} u^\epsilon \rightharpoonup u^* & \text{in } H_0^1(\Omega), \\ u^\epsilon \to u^* & \text{in } L^2(\Omega) \end{cases}$$

We put

$$p_i^\epsilon = a_{ij}(x)\frac{\partial u^\epsilon}{\partial x_j}, \tag{2.49}$$

It follows from (2.48) that

$$\|p_i^\epsilon\|_{L^2(\Omega)} \leq C$$

Therefore we can extract a subsequence, still denoted by p_i^ϵ, such that

$$p_i^\epsilon \to p_i^* \quad \text{in } L^2(\Omega).$$

Equation (2.47) can be written

$$\int_\Omega p_i^\epsilon \frac{\partial v}{\partial x_i}\, dx = \int_\Omega f(x)v(x)\, dx, \quad \forall v \in H_0^1(\Omega) \tag{2.50}$$

which gives in the limit

$$\int_\Omega p_i^* \frac{\partial v}{\partial x_i}\, dx = \int_\Omega f(x)v(x)\, dx, \quad \forall v \in H_0^1(\Omega). \tag{2.51}$$

Thanks to Green's formula, it yields

$$-\int_\Omega \frac{\partial p_i^*}{\partial x_i} v\, dx = \int_\Omega f(x)v(x)\, dx. \tag{2.52}$$

Hence, in the sense of distributions

$$-\frac{\partial p_i^*}{\partial x_i} = f, \quad \text{a.e. in } \Omega. \tag{2.53}$$

It remains to find the relation between p_i^* and u^*. Note that in (2.49) a_{ij} and $\dfrac{\partial u^\epsilon}{\partial x_j}$ converge weakly so we cannot pass to the limit in $a_{ij}\frac{\partial u^\epsilon}{\partial x_j}$. To overcome this difficulty, we use Luc Tartar's method (the method of oscillating test functions). We define the function

$$w^\epsilon(x) = x_k + \epsilon\chi(x/\epsilon),$$

where χ_k is the solution of the cell problem. It is obvious that

$$w^\epsilon \to x_k \quad \text{in } L^2(\Omega).$$

On the other hand, we have

$$
\begin{aligned}
-\frac{\partial}{\partial x_i}\left(a_{ij}^{\epsilon}\frac{\partial w^{\epsilon}}{\partial x_j}\right) &= -\left(\frac{\partial}{\partial x_i}+\frac{1}{\epsilon}\frac{\partial}{\partial y_i}\right)\left[a_{ij}(y)\left(\frac{\partial}{\partial x_j}+\frac{1}{\epsilon}\frac{\partial}{\partial y_j}\right)(x_k+\epsilon\chi_k(y))\right] \\
&= -\left(\frac{\partial}{\partial x_i}+\frac{1}{\epsilon}\frac{\partial}{\partial y_i}\right)\left[a_{ij}(y)\left(\delta_{jk}+\frac{\partial\chi_k}{\partial y_j}\right)\right] \\
&= -\left(\frac{1}{\epsilon}\frac{\partial}{\partial y_i}\left[a_{ij}(y)\left(\delta_{jk}+\frac{\partial\chi_k}{\partial y_j}\right)\right]\right) \\
&= 0
\end{aligned}
$$

Then w^{ϵ} satisfies the equation

$$
-\frac{\partial}{\partial x_i}\left[a_{ij}^{\epsilon}\frac{\partial w^{\epsilon}}{\partial x_j}\right]=0 \quad \text{on } \Omega. \tag{2.54}
$$

The associated variational formulation is

$$
\int_{\Omega} a_{ij}^{\epsilon}(x)\frac{\partial w^{\epsilon}}{\partial x_j}\frac{\partial v}{\partial x_i}\,dx=0, \quad \forall v\in H_0^1(\Omega). \tag{2.55}
$$

We now take in (2.50), $v=\phi u^{\epsilon}$, with $\phi\in\mathcal{D}(\Omega)$,we obtain

$$
\int_{\Omega} a_{ij}^{\epsilon}(x)\frac{\partial w^{\epsilon}}{\partial x_j}\left(\frac{\partial\phi}{\partial x_i}u^{\epsilon}+\phi\frac{\partial u^{\epsilon}}{\partial x_i}\right)dx=0, \quad \forall\phi\in\mathcal{D}(\Omega). \tag{2.56}
$$

Similarly, taking in (2.47), $v=\phi w^{\epsilon}$ with $\phi\in\mathcal{D}(\Omega)$, we obtain

$$
\int_{\Omega} a_{ij}^{\epsilon}(x)\frac{\partial u^{\epsilon}}{\partial x_j}\left(\frac{\partial\phi}{\partial x_i}w^{\epsilon}+\phi\frac{\partial w^{\epsilon}}{\partial x_i}\right)dx=\int_{\Omega} f\phi w^{\epsilon}\,dx, \quad \forall\phi\in\mathcal{D}(\Omega). \tag{2.57}
$$

Therefore, by subtraction, it yields

$$
\int_{\Omega}\left[a_{ij}^{\epsilon}(x)\frac{\partial u^{\epsilon}}{\partial x_j}w^{\epsilon}-a_{ij}^{\epsilon}(x)\frac{\partial w^{\epsilon}}{\partial x_j}u^{\epsilon}\right]\frac{\partial\phi}{\partial x_i}\,dx=\int_{\Omega} f\phi w^{\epsilon}\,dx, \quad \forall\phi\in\mathcal{D}(\Omega). \tag{2.58}
$$

We pass to the limit in this identity and use the following convergences:

$$
\begin{cases}
u^{\epsilon}\to u^{*} \quad \text{in } L^2(\Omega), \\
a_{ij}^{\epsilon}(x)\dfrac{\partial u^{\epsilon}}{\partial x_j}=p_i^{\epsilon}\rightharpoonup p_i^{*}, \\
w^{\epsilon}\to x_k \quad \text{in } L^2(\Omega), \\
a_{ij}^{\epsilon}(x)\dfrac{\partial w^{\epsilon}}{\partial x_j}\rightharpoonup\left\langle a_{ij}^{\epsilon}(x)\dfrac{\partial w^{\epsilon}}{\partial x_j}\right\rangle=\left\langle a_{ij}(y)\left(\delta_{jk}+\dfrac{\partial\chi_k}{\partial y_j}\right)\right\rangle=a_{ik}^{H}.
\end{cases}
$$

We obtain

$$
\int_{\Omega}\left[p_i^{\epsilon}x_k-a_{ik}^{H}u^{*}\right]\frac{\partial\phi}{\partial x_i}\,dx=\int_{\Omega} f\phi x_k\,dx, \quad \forall\phi\in\mathcal{D}(\Omega). \tag{2.59}
$$

Using Green's formula, we get

$$\int_\Omega \frac{\partial}{\partial x_i}\left(p_i^\epsilon x_k - a_{ik}^H u^*\right)\phi\, dx = \int_\Omega f\phi x_k\, dx, \quad \forall\phi\in\mathcal{D}(\Omega). \tag{2.60}$$

which gives in the distributional sense

$$\frac{\partial}{\partial x_i}\left(p_i^\epsilon x_k - a_{ik}^H u^*\right) = f\phi x_k. \tag{2.61}$$

By using (2.53), we get

$$p_k^* = a_{ik}^H \frac{\partial u^*}{\partial x_i}. \tag{2.62}$$

Hence,

$$-\frac{\partial}{\partial x_i}\left(a_{ik}^H \frac{\partial u^*}{\partial x_i}\right) = f. \tag{2.63}$$

This relation shows that u^* and u_0 verify the same problem. Hence

$$u^* = u_0. \tag{2.64}$$

Chapter 3

Homogenization of the Wave Equation

In this chapter we are concerned with the asymptotic behaviour as $\epsilon \to 0$ of the solution $u_\epsilon = u_\epsilon(x,t)$ of the wave equation :

$$\begin{cases} \dfrac{\partial^2 u_\epsilon}{\partial t^2} - \operatorname{div}\left(A^\epsilon(x)\nabla u_\epsilon\right) = f_\epsilon, & \text{in } \Omega\times]0,T[\\ u_\epsilon = 0, & \text{on } \partial\Omega\times]0,T[\\ u_\epsilon(x,0) = u^0_\epsilon, & \text{in } \Omega \\ \frac{\partial u_\epsilon(x,0)}{\partial t} = u^1_\epsilon, & \text{in } \Omega \end{cases} \tag{3.1}$$

where, as in the previous chapter, the operators div and ∇ are taken with respect to the space variable $x \in \Omega$ and $'$ denotes the derivative with respect to the time variable $t \in]0,T[$ with $T > 0$. We suppose that we are given the source term f_ϵ and the initial states u^0_ϵ and u^1_ϵ. The matrix A^ϵ is Y periodic and is defined by

$$a^\epsilon_{ij}(x) = a_{ij}\left(\frac{x}{\epsilon}\right) \quad \text{a.e. on } \mathbb{R}^N, \quad \forall i,j = 1,\dots,N \tag{3.2}$$

and

$$A^\epsilon(x) = A\left(\frac{x}{\epsilon}\right) = \left(a_{ij}(x)\right)_{1\le i,j\le N} \quad \text{a.e. on } \mathbb{R}^N, \tag{3.3}$$

where

$$\begin{cases} a_{ij} = a_{ji}, & \forall i,j = 1,\dots,N \\ a_{ij} \text{ is } Y\text{-periodic}, & \forall i,j = 1,\dots,N \end{cases} \tag{3.4}$$

and

$$A = (a_{ij})_{1 \le i,j \le N} \in M(\alpha, \beta, Y).$$

3.1 Existence and uniqueness

Let Ω be a bounded open set in $\mathbb{R}^N$ and consider the following problem:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \operatorname{div}(A\nabla u) = f \text{ in } \Omega \times \Big] 0, T[\\ u = 0 \text{ on } \partial\Omega \times]0, T[\\ u(x,0) = u^0(x) \text{ in } \Omega \\ \dfrac{\partial u}{\partial t}(x,0) = u^1(x) \text{ in } \Omega \end{cases} \tag{3.5}$$

under the following assumptions:

$$\begin{cases} i) & A \text{ is symmetric and elliptic} \\ ii) & f \in L^2(\Omega \times]0, T[) \\ iii) & u^0 \in H_0^1(\Omega) \\ iv) & u^1 \in L^2(\Omega) \end{cases} \tag{3.6}$$

Let us introduce the space

$$\mathcal{W} = \left\{ v \mid v \in L^2\left(0, T : H_0^1(\Omega)\right), \frac{\partial v}{\partial t} \in L^2(\Omega \times]0, T[) \right\}$$

This is a Banach space with the graph norm defined by

$$\|v\|_{\mathcal{W}} = \|v\|_{L^2\left(0,T;H_0^1(\Omega)\right)} + \left\| \frac{\partial v}{\partial t} \right\|_{L^2(\Omega \times]0,T[)}$$

Then, the variational formulation of problem (3.5) is:

$$\begin{cases} \text{Find } u \in \mathcal{W} \text{ such that} \\ \left(\dfrac{\partial^2 u}{\partial t^2}, v \right)_{H^{-1}(\Omega).H_0^1(\Omega)} \quad + \int_\Omega A(x)\nabla u(x,t)\nabla u(x)dx \\ \qquad\qquad = \int_\Omega f(x,t)v(x)dx \quad \text{in } \mathcal{D}'(0,T). \quad \forall v \in H_0^1(\Omega) \\ u(x,0) = u^0(x) \quad \text{in } \Omega \\ \dfrac{\partial u}{\partial t}(x,0) = u^1(x) \quad \text{in } \Omega. \end{cases} \tag{3.7}$$

Theorem 18 Suppose that assumptions (3.6) are fulfilled. Then problem (3.7) has a unique solution $u \in \mathcal{W}$. Moreover.

$$u \in L^\infty\left(0, T; H_0^1(\Omega)\right), \quad \frac{\partial u}{\partial t} \in L^\infty\left(0, T; L^2(\Omega)\right), \quad \frac{\partial^2 u}{\partial t^2} \in L^2\left(0, T; H^{-1}(\Omega)\right)$$

and there exists a constant c depending on α, β, Ω, and T such that

$$\begin{cases} \|u\|_{L^\infty(0,T;H_0^1(\Omega))} & + \left\|\frac{\partial u}{\partial t}\right\|_{L^\infty(0,T;L^2(\Omega))} + \left\|\frac{\partial^2 u}{\partial t^2}\right\|_{L^2(0,T;H^{-1}(\Omega))} \\ & \leq c\left(\|f\|_{L^2(\Omega\times]0,T)} + \|u^0\|_{L^2(\Omega)} + \|u^1\|_{H_0^1(\Omega)}\right). \end{cases} \tag{3.8}$$

For the proof, we need the classical Gronwall lemma [9]:

Lemma 4 Let v a function in $C([0,T])$ and suppose that there exists a positive number γ such that

$$v(t) \leq \gamma + \int_0^t v(\tau)d\tau. \quad \forall t \in [0,T], \tag{3.9}$$

Then

$$v(t) \leq \gamma e^T, \quad \forall t \in [0,T].$$

Of theorem: In the proof,we will use the Faedo-Galerkin method[[4]]. It is known from the spectral theory on self-adjoint operator that there exists a complete orthonormal system $(\omega_j)_{j\in\mathbb{N}}$ of $L^2(\Omega)$ given by the eigenfunction of the operator $-\operatorname{div}(A\nabla)$ (see [12]). The corresponding eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ verify $\lambda_j \to +\infty$ as $j \to +\infty$. Furthermore, (ω_j) is also a complete orthogonal system in $H_0^1(\omega)$.Let Let $(\lambda_j)_{j\in\mathbb{N}}$ be the eigenvalues of the operator $-\operatorname{div}(A\nabla)$, consider the functions ω_j be the solutions of the problems

$$-\operatorname{div}(A\nabla\omega_j) = \lambda_j\omega_j \quad \text{in } \Omega \tag{3.10}$$

The functions $(\omega_j)_{j\in\mathbb{N}}$ constitute a system of eigenvalues of the operators $-\operatorname{div}(A\nabla)$and form a complete orthonormal system of $L^2(\Omega)$. Let V_m be the space spanned by $\{\omega_1, \ldots, \omega_m\}$ and introduce the projection operator $\mathcal{P}_m$ from $L^2(\Omega)$ to V_m defined by

$$\mathcal{P}_m v = \sum_{i=1}^m (v, \omega_i)\,\omega_i, \quad \forall v \in L^2(\Omega)$$

From classical results on Hilbert spaces, one has

(i) $P_m v \to v$ strongly in $L^2(\Omega)$, for all $v \in L^2(\Omega)$

(ii) $\|P_m\|_{\mathcal{L}(L^2(\Omega),L^2(\Omega))} \leq 1$

Next we introduce, for any $m \in \mathbb{N}^*$, the finite dimensional approximate problem

$$\begin{cases} \text{Find } u_m = \sum_{j=1}^m g_j^m(t)w_j \in V_m \text{ such that} \\ \int_\Omega \frac{\partial^2 u_m}{\partial t^2}(x,t)w_k dx & + \int_\Omega A(x)\nabla u_m(x,t)\nabla w_k dx \\ & = \int_\Omega f(x,t)w_k dx. \quad \text{in } \mathcal{D}'(0,T) \quad \forall k = 1,\ldots,m \\ u_m(x,0) & = u_m^0(x) \quad \text{in } \Omega \\ \frac{\partial u_m}{\partial t}(x,0) & = u_m^1(x) \quad \text{in } \Omega, \end{cases} \tag{3.11}$$

where, according to assumptions (3.6)(iii) and (3.6)(iv), we set

$$u_m^0 = P_m u^0. \quad u_m^1 = P_m u^1$$

From the properties of P_m, we have

$$\begin{cases} i) & u_m^0 \to u^0 \quad \text{strongly in } H_0^1(\Omega) \\ ii) & u_m^1 \to u^1 \quad \text{strongly in } L^2(\Omega) \end{cases} \tag{3.12}$$

Since $u_m = \sum_{j=1}^m g_j^m(t) w_j$ we obtain:

$$\int_\Omega \frac{\partial^2 u_m}{\partial t^2}(x,t) w_k dx + \int_\Omega A(x) \nabla u_m(x,t) \nabla w_k dx = \int_\Omega f(x,t) w_k$$

$$\Rightarrow \int_\Omega \frac{\partial^2}{\partial t^2}(\sum_{j=1}^m g_j^m(t) w_j)(x,t) w_k dx + \int_\Omega A(x) \nabla (\sum_{j=1}^m g_j^m(t) w_j) \nabla w_k dx = \int_\Omega f(x,t) w_k$$

$$\Rightarrow \sum_{j=1}^m \int_\Omega \frac{\partial^2}{\partial t^2} g_j^m(t) w_j \cdot w_k dx + \sum_{j=1}^m \int_\Omega A(x) \nabla g_j^m(t) w_j \nabla w_k dx = \int_\Omega f(x,t) w_k$$

$$\Rightarrow \sum_{j=1}^m \frac{\partial^2}{\partial t^2} g_j^m(t)(w_j, w_k) + \sum_{j=1}^m g_j^m(t) \int_\Omega A(x) \nabla w_j \nabla w_k dx = \int_\Omega f(x,t) w_k$$

$$\Rightarrow \frac{\partial^2}{\partial t^2} g_k^m(t) + \sum_{j=1}^m g_j^m(t) \int_\Omega A(x) \nabla w_j \nabla w_k dx = \int_\Omega f(x,t) w_k$$

Similarly, we also find:

$$\sum_{k=1}^m g_j^m(0) w_k = \sum_{k=1}^m (u^0, w_k) w_k$$

$$\sum_{k=1}^m \frac{\partial}{\partial t} g_j^m(0) w_k = \sum_{k=1}^m (u^1, w_k) w_k$$

From the previous equations and because $\{w_1, w_2 \ldots, w_m\}$ is a basis for V_m, problem (3.11) is equivalent to the following system of m linear ordinary differential equations of the second order with unknowns $g_1^m, \ldots, g_m^m$:

$$\begin{cases} \frac{d^2 g_k^m}{dt^2} + \sum_{j=1}^m g_j^m(t) \int_\Omega A \nabla w_j \nabla w_k dx = \int_\Omega f(x,t) w_k dx \\ g_k^m(0) = (u^0, w_k) \\ (g_k^m)'(0) = (u^1, w_k) \end{cases}$$

for any $k = 1, \ldots, m$. Classical results of ODEs (see [7]) give the existence and uniqueness in $C^1([0,T])$ of a solution $\{g_1^m, \ldots, g_m^m\}$ of this system on the interval $[0,T]$. Hence, u_m is determined and u_m and $\frac{\partial u_m}{\partial t}$ are in $C^1([0,T]; V_m)$.

A priori estimates We will now prove that u_m satisfies some a priori estimates. To do so, let us multiply the k th equation in (3.11) by $\frac{\partial}{\partial t} g_k^m$

$$\int_\Omega \frac{\partial^2 u_m}{\partial t^2}(x,t) \frac{\partial}{\partial t} g_k^m(t) w_k dx + \int_\Omega A(x) \nabla u_m(x,t) \nabla (\frac{\partial}{\partial t} g_k^m(t) w_k dx = \int_\Omega f(x,t) \frac{\partial}{\partial t} g_k^m(t) w_k dx.$$

and sum over k from 1 to m we obtain

$$\begin{aligned}\sum_{k=1}^{m}\int_{\Omega}\frac{\partial^2 u_m}{\partial t^2}(x,t)(\frac{\partial}{\partial t}(g_k^m)(t)w_k)dx+\sum_{k=1}^{m}\int_{\Omega}A(x)\nabla u_m(x,t)\nabla(\frac{\partial}{\partial t}(g_k^m)(t)w_k dx\\ =\sum_{k=1}^{m}\int_{\Omega}f(x,t)(\frac{\partial}{\partial t}g_k^m)(t)w_k dx\end{aligned}.$$

Then

$$\int_{\Omega}\frac{\partial^2 u_m}{\partial t^2}(x,t)\frac{\partial u_m}{\partial t}(x,t)dx+\int_{\Omega}A(x)\nabla u_m(x,t)\nabla\frac{\partial u_m}{\partial t}(x,t)dx=\int_{\Omega}f(x,t)\frac{\partial u_m}{\partial t}(x,t)dx \tag{3.13}$$

We have:

$$\int_{\Omega}\frac{\partial^2 u_m}{\partial t^2}(x,t)\frac{\partial u_m}{\partial t}(x,t)=\frac{1}{2}\frac{\partial}{\partial t}\int_{\Omega}(\frac{\partial u_m}{\partial t}(x,t))^2dx$$

Due to the symmetry of A we have

$$\int_{\Omega}A(x)\nabla u_m(x,t)\nabla\frac{\partial u_m}{\partial t}(x,t)dx=\frac{1}{2}\frac{\partial}{\partial t}\int_{\Omega}A(x)\nabla u_m(x,t)\nabla u_m(x,t)dx.$$

Hence, by using the inequality

$$a^2+b^2\geq 2ab,\quad \forall a,b\in\mathbb{R}_+$$

(3.13) can be rewritten as

$$\begin{aligned}\frac{\partial}{\partial t}\left(\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(\Omega)}+\int_{\Omega}A(x)\nabla u_m(x,t)\nabla u_m(x,t)dx\right)&\leq 2\|f\|_{L^2(\Omega)}\left\|\frac{\partial u_m}{\partial t}\right\|_{L^2(\Omega)}\\ &\leq\|f\|^2_{L^2(\Omega)}+\left\|\frac{\partial u_m}{\partial t}\right\|^2_{L^2(\Omega)}\end{aligned}$$

Integrating on $(0,t)$ with $t\leq T$ and using the ellipticity of A. we get

$$\begin{aligned}&\left\|\frac{\partial u_m}{\partial t}(t)\right\|^2_{L^2(\Omega)}+\alpha\|u_m(x,t)\|^2_{H_0^1(\Omega)}\\ &\leq\left\|u_m^1\right\|^2_{L^2(\Omega)}+\int_{\Omega}A\nabla u_m^0\nabla u_m^0dx+\int_0^T\|f\|^2_{L^2(\Omega)}dt+\int_0^t\left\|\frac{\partial u_m}{\partial t}(\tau)\right\|^2_{L^2(\Omega)}d\tau\\ &\leq\left\|u_m^1\right\|^2_{L^2(\Omega)}+\beta\left\|u_m^0\right\|^2_{H_0^2(\Omega)}+\|f\|^2_{L^2(\Omega\times(0,T))}+\int_0^t\left\|\frac{\partial u_m}{\partial t}(\tau)\right\|^2_{L^2(\Omega)}d\tau.\end{aligned}$$

Using properties of the projection P_m, we finally have

$$\begin{aligned}&\left\|\frac{\partial u_m}{\partial t}(t)\right\|^2_{L^2(\Omega)}+\alpha\|u_m(x,t)\|^2_{H_0^1(\Omega)}\\ &\leq\quad\left\|u^1\right\|^2_{L^2(\Omega)}+\sigma\left\|u^0\right\|^2_{H_0^1(\Omega)}+\|f\|^2_{L^2(\Omega\times(0,T))}\\ &+\int_0^t\left[\left\|\frac{\partial u_m}{\partial t}(\tau)\right\|^2_{L^2(\Omega)}+\alpha\|u_m(\tau)\|^2_{H_0^1(\Omega)}\right]d\tau.\end{aligned}$$

Applying now Gronwall's lemma with

$$\gamma = \left\|u^1\right\|^2_{L^2(\Omega)} + \alpha \left\|u^0\right\|^2_{H^1_0(\Omega)} + \|f\|^2_{L^2(\Omega\times(0,T))},$$

we deduce the a priori estimate

$$\begin{aligned}&\|u_m\|_{L^\infty\left(0,T;H^1_0(\Omega)\right)} + \left\|\frac{\partial u_m}{\partial t}\right\|_{L^\infty(0,T;L^2(\Omega))}\\ &\leq c_1\left(\|f\|_{L^2(\Omega\Omega\times 0,T)} + \left\|u^0\right\|_{L^2(\Omega)} + \left\|u^1\right\|_{H^1_0(\Omega)}\right),\end{aligned}$$

where c_1 depends only on α, β. Ω and T. It remains to obtain an a priori estimate for $\dfrac{\partial^2 u_m}{\partial t^2}$. Observe that the equation in (3.11) implies that

$$\left(\frac{\partial^2 u_m}{\partial t^2}(t), v\right)_{L^2(\Omega)} = (-\operatorname{div}(A\nabla u_m(t)) + f, v)_{L^2(\Omega)}, \quad \forall v \in V_m.$$

This signifies that

$$\frac{\partial^2 u_m}{\partial t^2}(t) = -\left[P_m\left(\mathcal{F}\left(u_m\right) + f\right)\right](t),$$

where $\mathcal{F} = -\operatorname{div}(A\nabla)$. We obtain

$$\left\|\frac{\partial^2 u_m}{\partial t^2}\right\|_{L^2(0,T;H^{-1}(\Omega))} \leq c_2\left(\|f\|_{L^2(\Omega\times|0,T)} + \left\|u^0\right\|_{L^2(\Omega)} + \left\|u^1\right\|_{H^1_0(\Omega)}\right) \leq c_3,$$

where c_2 and c_3 are constants independent of m. Consequently,

$$\begin{aligned}&\|u_m\|_{L^\infty\left(0,T;H^1_0(\Omega)\right)} + \left\|\frac{\partial u_m}{\partial t}\right\|_{L^\infty(0,T;L^2(\Omega))} + \left\|\frac{\partial^2 u_m}{\partial t^2}\right\|_{L^2(0,T;H^{-1}(\Omega))}\\ &\leq c\left(\|f\|_{L^2(\Omega\times]0,T[)} + \left\|u^0\right\|_{L^2(\Omega)} + \left\|u^1\right\|_{H^1_0(\Omega)}\right),\end{aligned} \tag{3.14}$$

where c depends only on α, β, Ω and T.

In this step we pass to the limit in the approximate problem. Estimate (3.14) implies (Theorem 5), up to a subsequence, the following convergences

$$\begin{cases} u_m \rightharpoonup u & \text{weakly}^* \text{ in } L^\infty\left(0,T;H^1_0(\Omega)\right), \\ \dfrac{\partial u_m}{\partial t} \rightharpoonup \dfrac{\partial u}{\partial t} & \text{weakly* in } L^\infty\left(0,T;L^2(\Omega)\right), \\ \dfrac{\partial^2 u_m}{\partial t^2} \rightharpoonup \dfrac{\partial^2 u}{\partial t^2} & \text{weakly in } L^2\left(0,T;H^{-1}(\Omega)\right). \end{cases} \tag{3.15}$$

Let us now pass to the limit in (3.11) for $m \to \infty$. let $\psi \in \mathcal{D}(0,T)$ and $v \in H^1_0(\Omega)$. Multiplying the equation in (3.11) by $(v, w_k)_{L^2(\Omega)}\,\psi$. Summing over k from 1 to m and integrating in t over $(0,T)$.

we get

$$
\begin{cases}
\displaystyle\int_0^T\int_\Omega \frac{\partial^2 u_m}{\partial t^2}(x,t)\psi(t)\left(P_m v\right)(x)dx\,dt\\
\displaystyle+\int_0^T\int_\Omega A(x)\nabla u_m(x,t)\psi(t)\nabla\left(P_m v\right)(x)dx\,dt\\
\displaystyle=\int_0^T\int_\Omega f(x,t)\psi(t)\left(P_m v\right)(x)dx\,dt.
\end{cases}
\tag{3.16}
$$

By integration by parts with respect to t . We get

$$
\begin{cases}
\displaystyle-\int_0^T\int_\Omega \frac{\partial u_m}{\partial t}(x,t)\frac{\partial\psi}{\partial t}(t)\left(P_m v\right)(x)dx\,dt\\
\displaystyle+\int_0^T\int_\Omega A(x)\nabla u_m(x,t)\psi(t)\nabla\left(P_m v\right)(x)dx\,dt\\
\displaystyle=\int_0^T\int_\Omega f(x,t)\psi(t)\left(P_m v\right)(x)dx\,dt.
\end{cases}
\tag{3.17}
$$

Here, all the terms pass to the limit thanks to convergences (3.15). We obtain

$$
\begin{aligned}
-\int_0^T\int_\Omega \frac{\partial u}{\partial t}(x,t)\frac{\partial\psi}{\partial t}(t)v(x)dx\,dt+\int_0^T\int_\Omega A(x)\nabla u(x,t)\psi(t)\nabla v(x)dx\,dt&\\
=\int_0^T\int_\Omega f(x,t)\psi(t)v(x)dx\,dt&
\end{aligned}
\tag{3.18}
$$

So we deduce

$$
\begin{aligned}
&\int_0^T\int_\Omega \frac{\partial u}{\partial t}(x,t)\frac{\partial\psi}{\partial t}(t)v(x)dx\\
&\quad=-\int_0^T\left\langle\frac{\partial^2 u}{\partial t^2}(t),\psi(t)v\right\rangle_{H^{-1}(\Omega)\cdot H_0^1(\Omega)}dt+\int_0^T\frac{\partial}{\partial t}\int_\Omega\frac{\partial u}{\partial t}(x,t)\psi(t)v(x)dx\,dt\\
&\quad=-\int_0^T\left\langle\frac{\partial^2 u}{\partial t^2}(t),\psi(t)v\right\rangle_{H^{-1}(\Omega)\cdot H_0^1(\Omega)}dt
\end{aligned}
$$

since $\psi(0)=\psi(T)=0$. This together with (3.18) shows that u satisfies

$$
\begin{aligned}
-\int_0^T\left\langle\frac{\partial^2 u}{\partial t^2}(t),\psi(t)v\right\rangle_{H^{-1}(\Omega)\cdot H_0^1(\Omega)}dt+\int_0^T\int_\Omega A(x)\nabla u(x,t)\psi(t)\nabla v(x)dx\,dt&\\
=\int_0^T\int_\Omega f(x,t)\psi(t)v(x)dx\,dt.&
\end{aligned}
\tag{3.19}
$$

It remains to check that the initial conditions $u(x,0)=u^0(x)$ and $\frac{\partial u}{\partial t}(x,0)=u^1(x)$ are satisfied. Choose in (3.17) (which is still valid if $\frac{\partial\psi}{\partial t}\in C^\infty(\{0,T\})$) a function $\psi\in C^\infty([0,T])$ such that

$\frac{\partial\psi}{\partial t}(0) = 1$ and $\frac{\partial\psi}{\partial t}(T) = 0$. Then from (3.17), we get

$$-\int_0^T \int_\Omega \frac{\partial u_m}{\partial t}(x,t)\frac{\partial\psi}{\partial t}(t)\left(P_m v\right)(x)\,dx\,dt + \int_0^T \int_\Omega A(x)\nabla u_m(x,t)\psi(t)\nabla\left(P_m v\right)(x)dx\,dt = \int_0^T \int_\Omega f(x,t)\psi(t)\left(P_m v\right)(x)dx\,dt + \int_\Omega u_m^1(x)\left(P_m v\right)(x)dx$$

where we pass to the limit to obtain

$$-\int_0^T \int_\Omega \frac{\partial u}{\partial t}(x,t)\frac{\partial\psi}{\partial t}(t)v(x)dx\,dt + \int_0^T \int_\Omega A(x)\nabla u(x,t)\psi(t)\nabla v(x)dx\,dt = \int_0^T \int_\Omega f(x,t)\psi(t)u(x)dx\,dt + \int_\Omega u^1(x)v(x)dx$$

As $u \in C\left([0,T]; L^2(\Omega)\right)$, we have

$$\int_0^T \left\langle \frac{\partial^2 u}{\partial t^2}(t), \psi(t)\frac{\partial v}{\partial t}\right\rangle_{H^{-1}(\Omega)\cdot H_0^1(\Omega)} + \int_\Omega \frac{\partial u}{\partial t}(x,0)v(x)dx \quad + \int_0^T \int_\Omega A(x)\nabla u(x,t)\psi(t)\nabla v(x)dx\,dt = \int_0^T \int_\Omega f(x,t)\frac{\partial\psi}{\partial t}(t)v(x)dx\,dt + \int_\Omega u^1(x)v(x)dx$$

Since (3.19) is still valid for $\psi \in C^\infty([0,T])$. we deduce that

$$\int_\Omega \frac{\partial u}{\partial t}(x,0)v(x)dx = \int_\Omega u^1(x)v(x)dx, \quad \forall v \in H_0^1(\Omega)$$

This implies that $\frac{\partial u}{\partial t}(x,0) = u^1(x)$. To obtain the other initial condition, let us choose in (3.17) a function $\psi \in C^\infty([0,T])$ such that $\psi(0) = 0$. $\frac{\partial\psi}{\partial t}(0) = 1$ and $\frac{\partial\psi}{\partial t}(T) = \frac{\partial\psi}{\partial t}(T) = 0$. We get, by integrating twice by parts with respect to t.

$$\int_0^T \int_\Omega u_m(x,t)\frac{\partial^2\psi}{\partial t^2}(t)\left(P_m v\right)(x)dx\,dt + \int_0^T \int_\Omega A(x)\nabla u_m(x,t)\psi(t)\nabla\left(P_m v\right)(x)dx\,dt = \int_0^T \int_\Omega f(x,t)\psi(t)\left(P_m v\right)(x)dx\,dt - \int_\Omega u_m^0(x)v(x)dx$$

where we pass to the limit and obtain

$$\int_0^T \int_\Omega u(x,t)\frac{\partial^2\psi}{\partial t^2}(t)v(x)\,dx\,dt + \int_0^T \int_\Omega A(x)\nabla u(x,t)\psi(t)\nabla v(x)\,dx\,dt = \int_0^T \int_\Omega f(x,t)\psi(t)v(x)\,dx\,dt - \int_\Omega u^0(x)v(x)\,dx$$

We integrate by parts with respect to t in the first term, and get

$$\int_0^T \left\langle \frac{\partial^2 u}{\partial t^2}(t), \psi(t)v \right\rangle_{H^{-1}(\Omega)\cdot H_0^1(\Omega)} dt - \int_\Omega u(x,0)v(x)dx$$
$$+ \int_0^T \int_\Omega A(x)\nabla u(x,t)\psi(t)\nabla v(x)dx\, dt = \int_0^T \int_\Omega f(x,t)\psi(t)v(x)dx\, dt$$
$$- \int_\Omega u^0(x)v(x)dx.$$

This implies $u(x,0) = u^0(x)$.
This concludes the existence of a solution u to the problem (3.7).

Uniqueness Let u_1 and u_2 be two solutions corresponding to the same data. Their difference $w = u_1 - u_2$ satisfies (3.5) with $f \equiv 0, u^0 \equiv 0$ and $u^1 \equiv 0$, namely

$$\begin{cases} \left\langle \frac{\partial^2 w}{\partial t^2}(t), v \right\rangle_{H^{-1}(\Omega),H^1_{10}(\Omega)} + \int_{\Omega 2} A(x)\nabla \frac{\partial u}{\partial t}(x,t)\nabla v(x)dx = 0 \quad \text{in } \mathcal{D}'(0,T). \quad \forall v \in H_0^1(\Omega) \\ w(x,0) = 0 \text{ in } \Omega \\ \frac{\partial w}{\partial t}(x,0) = 0 \text{ in } \Omega. \end{cases}$$

Taking into account the result of the previous step:

$$\|w\|_{L^\infty\left(0,T;H_0^1(\Omega)\right)} + \left\|\frac{\partial w}{\partial t}\right\|_{L^\infty(0,T;L^2(\Omega))} + \left\|\frac{\partial^2 w}{\partial t^2}\right\|_{L^2(0,T;H^{-1}(\Omega))} \leq 0.$$

Hence,

$$w = 0$$

Therefore:

$$u_1 = u_2$$

3.2 The homogenization result

$$\begin{cases} \frac{\partial^2 u^\epsilon}{\partial t^2} - \operatorname{div}\left(A^\epsilon(x)\nabla u^\epsilon\right) = f, & \text{in } \Omega\times]0,T[\\ u^\epsilon = 0, & \text{on } \partial\Omega\times]0,T[\\ u^\epsilon(x,0) = u_0^\epsilon, & \text{in } \Omega \\ \frac{\partial u^\epsilon(x,0)}{\partial t} = u_1^\epsilon, & \text{in } \Omega \end{cases} \tag{3.20}$$

We assume that the solution $u^\epsilon(x,t)$ to the wave equation has the following two-scale asymptotic expansion:

$$u^\epsilon(t,x) = u_0(t,x,y) + \epsilon u_1(t,x,y) + \epsilon^2 u_2(t,x,y) + \cdots, \tag{3.21}$$

With

$$y = \frac{x}{\epsilon}$$

and

$$a_{ij}^{\epsilon} = a_{ij}\left(\frac{x}{y}\right) \tag{3.22}$$

$$\begin{cases} a_{ij} = a_j, \text{ for } i, j = 1, \dots, N, \\ a_{ij} \text{ is } Y\text{-periodic}, \text{ for all } i, j = 1, \dots, N, \\ A = (a_{ij})_{1 \le i,j \le N} \in M(\alpha, \beta, Y) \end{cases} \tag{3.23}$$

where each $u^{\epsilon}(t, x, y)$ is periodic with respect to $y \in Y = [0, 1]^n$.

$$\frac{\partial^2 u^{\epsilon}}{\partial t^2} = \frac{\partial^2}{\partial t^2}\left(u_0 + \epsilon u_1 + \epsilon^2 u_2\right) \tag{3.24}$$

$$\text{div}\, u^{\epsilon} = \text{div}_x u + \frac{1}{\epsilon}\text{div}_y u \tag{3.25}$$

$$\nabla u^{\epsilon} = \nabla_x u + \frac{1}{\epsilon}\nabla_y u \tag{3.26}$$

Defining the operator $\mathcal{A}^{\epsilon}$ by

$$\mathcal{A}^{\epsilon}\Phi^{\epsilon} = -\text{div}(A^{\epsilon}\nabla\Phi^{\epsilon}). \tag{3.27}$$

Consequently, using (3.26) and (3.25) into (3.27), one can write $\mathcal{A}^{\epsilon}\Phi^{\epsilon}$ as follows:

$$\mathcal{A}^{\epsilon}\Phi^{\epsilon}(x) = (\epsilon^{-2}\mathcal{A}_0 + \epsilon^{-1}\mathcal{A}_1 + \mathcal{A}_2)\Phi(x, y), \tag{3.28}$$

where

$$\begin{cases} \mathcal{A}_0 = -\text{div}_y(A(y)\nabla_y), \\ \mathcal{A}_1 = -\text{div}_x(A(y)\nabla_y) - \text{div}_y(A(y)\nabla_x), \\ \mathcal{A}_2 = -\text{div}_x(A(y)\nabla_x). \end{cases} \tag{3.29}$$

By substituting equations (3.21), (3.24), (3.25), and (3.26) into equation (3.20), we obtain

$$\frac{\partial^2}{\partial t^2}(u_0 + \epsilon u_1 + \epsilon^2 u_2) - \left(\text{div}_x + \frac{1}{\epsilon}\text{div}_y\right)\left(A(y)\left(\nabla_x + \frac{1}{\epsilon}\nabla_y\right)(u_0 + \epsilon u_1 + \epsilon^2 u_2)\right) = f \tag{3.30}$$

Hence

$$\begin{aligned} &\epsilon^0\left[\frac{\partial^2 u_0}{\partial t^2} - \text{div}_x(A\nabla_x u_0) - \text{div}_x(A\nabla_y u_1) - \text{div}_y(A\nabla_x u_1) - \text{div}_y(A\nabla_y u_2)\right] \\ &- \frac{1}{\epsilon}\left[\text{div}_x\left(\nabla_y u_0\right) + \text{div}_y(A\nabla_x u_0) + \text{div}_y(A\nabla_x u_2) + \text{div}_y(A\nabla_Y u_1)\right] \\ &- \frac{1}{\epsilon^2}\,\text{div}_y(\nabla_Y u_0) = f \end{aligned}$$

Therefore,

On order ϵ^{-2}

$$\begin{cases} -\operatorname{div}_y(A\nabla_y u_0) = 0 \\ u_0 \text{ is } Y\text{-periodic} \end{cases} \tag{3.31}$$

On order ϵ^{-1}

$$\begin{cases} -\operatorname{div}_X(A\nabla_y u_0) - \operatorname{div}_y(A\nabla_x u_0) - \operatorname{div}_y(A\nabla_y u_1) = 0 \\ u_1 \text{ is } Y\text{-periodic} \end{cases} \tag{3.32}$$

On order ϵ^0

$$\begin{cases} \dfrac{\partial^2 u_0}{\partial t^2} - \operatorname{div}_x(A\nabla_x u_0) - \operatorname{div}_x(A\nabla_y u_1) - \operatorname{div}_y(A\nabla_x u_1) - \operatorname{div}_y(A\nabla_y u_2) = f \\ u_2 \text{ is } Y\text{-periodic} \end{cases} \tag{3.33}$$

Now, substituting into (3.28) Φ^ϵ by u^ϵ (2.7) and grouping terms of the same orders in powers of ϵ gives a hierarchy of an infinite system of equations, the first three systems of which are:

Order ϵ^{-2}

$$\begin{cases} \mathcal{A}_0 u_0 = 0, \quad \text{in } Y, \\ u_0 \text{ Y-periodic in } y. \end{cases} \tag{3.34}$$

Order ϵ^{-1}

$$\begin{cases} \mathcal{A}_0 u_1 = -\mathcal{A}_1 u_0, \quad \text{in } Y, \\ u_1 \text{ Y-periodic in } y. \end{cases} \tag{3.35}$$

Order ϵ^0

$$\begin{cases} A_0 u_2 = f - A_2 u_0 - A_1 u_1 - \dfrac{\partial^2 u_0}{\partial t^2}, \\ \quad u_2 \text{ is } Y\text{-periodic.} \end{cases} \tag{3.36}$$

Solution (3.34):

$$\begin{cases} -\operatorname{div}_y\left(A(y)\nabla_y u_0\right) = 0 \\ u_0 \quad Y\text{-periodic} \end{cases} \tag{3.37}$$

Then, by the **Fredholm alternative (lemma 3)** equation (2.26) admits a unique solution(up to an additive constant that depends on x only)

Multiply (2.26) by u_0 and integrating over Y

$$-\int_Y \operatorname{div}_y\left(A(y)\nabla_y u_0\right) u_0 \, dy = 0 \tag{3.38}$$

Using integration by parts**(Green's formula):**

$$\int_Y A(y)\nabla_y u_0 \cdot \nabla_y u_0 \, dy - \int_{\partial Y} A(y)\nabla_y u_0 \, u_0 \cdot \vec{n} \, ds = 0 \tag{3.39}$$

Since u_0 is Y-periodic, the boundary term vanishes, hence:

$$\int_Y A(y)\nabla_y u_0 \cdot \nabla_y u_0 \, dy = 0 \tag{3.40}$$

By the coercivity of A(2.2) we get

$$0 = \int_Y A(y)\nabla_y u_0 \cdot \nabla_y u_0 \, dy \; \geq \; \alpha \|\nabla_y u_0\|^2 \tag{3.41}$$

that necessitates

$$\nabla_y u_0 = 0 \tag{3.42}$$

and the fact that

$$u_0(x, y, t) = u_0(x, t) \tag{3.43}$$

Solution of (3.35):

$$\begin{cases} -\operatorname{div}_y \left(A(y)\nabla_y u_1\right) = -\operatorname{div}_x \left(A(y)\nabla_y u_0\right) - \operatorname{div}_y \left(A(y)\nabla_x u_0\right), \\ u_1 \text{ is } Y\text{-periodic} \end{cases} \tag{3.44}$$

Then

$$\begin{cases} -\operatorname{div}_y \left(A(y)\nabla_y u_1\right) = \operatorname{div}_y \left(A(y)\nabla_x u_0\right), \\ u_1 \text{ is } Y\text{-periodic} \end{cases} \tag{3.45}$$

We have

$$\int_Y \operatorname{div}_y \left(A(y)\nabla_x u_0\right) dy = \left(\int_Y \operatorname{div}_y A(y) \, dy\right) \cdot \nabla_x u_0 \tag{3.46}$$

$$= \int_Y \nabla_x u_0 \cdot \operatorname{div}_y A(y) \, dy$$

$$= \int_{dy} A(y) \, dy \, \nabla_x u_0 = 0 \tag{3.47}$$

Then, after Lemma A, problem (2.33) has a unique solution

$$u_1(x, t, y) = \chi \nabla_x u_0 + \tilde{u}_1(x, t) \tag{3.48}$$

$$= \sum_{k=1}^{N} \chi_k \frac{\partial u_0}{\partial x_k} + \tilde{u}_1(x) \tag{3.49}$$

where $\chi = (\chi_k)_{1 \leq k \leq N}$, denotes the solution of the cell problem

$$\begin{cases} \chi \in H^1_{\text{per}}(Y), \\ -\operatorname{div}_y \left(A(y)\nabla_y \chi\right) = \operatorname{div}_y \left(A(y)e\right) \\ \displaystyle\int_Y \chi(y) \, dy = 0. \end{cases} \tag{3.50}$$

Solution of (2.15)

$$
\begin{cases} \text{div}_y\left(A(y)\nabla_y u_0\right) = f - \dfrac{\partial^2 u_0}{\partial t^2} + \text{div}_x\left(A(y)\nabla_x u_0\right) + \text{div}_x\left(A(y)\nabla_y u_1\right) \quad + \text{div}_y\left(A(y)\nabla_x u_2\right) \\ u_2 \text{ is } Y\text{-periodic.} \end{cases} \tag{3.51}
$$

The solution to the problem (2.15) is as follows:

$$
\begin{cases} A_0 u_2 = f - A_2 u_0 - A_1 u_1 - \dfrac{\partial^2 u_0}{\partial t^2}, \\ \quad u_2 \text{ is } Y\text{-periodic.} \end{cases} \tag{3.52}
$$

We show that (2.44) has a unique solution using the Fredholm alternative

$$
\int_Y \left(f - A_2 u_0 - A_1 u_1 - \frac{\partial^2 u_0}{\partial t^2} \right) dy = 0 \tag{3.53}
$$

this is equivalent to

$$
\int_Y \left(A_2 u_0 + A_1 u_1 + \frac{\partial^2 u_0}{\partial t^2} \right) dy = f\,|Y| \tag{3.54}
$$

Where |Y| = measure of Y hence

$$
\frac{1}{|Y|} \int_Y \left(A_2 u_0 + A_1 u_1 + \frac{\partial^2 u_0}{\partial t^2} \right) dy = f \tag{3.55}
$$

Hence

$$
\frac{1}{|Y|} \left[\frac{\partial^2 u_0}{\partial t^2} + \int_Y A_2 u_0 \, dy + \int_Y A_1 u_1 \, dy \right] = f \tag{3.56}
$$

We have

$$
\begin{aligned} \int_Y A_1 \, u_1 \, dy &= -\int_Y \text{div}_y\left(A(y)\,\nabla_x u_1\right) dy - \int_Y \text{div}_x\left(A(y)\,\nabla_y u_1\right) dy \\ &= I_1 + I_2 \end{aligned} \tag{3.57}
$$

The first term $I_1 = 0$ by the divergence theorem and the periodicity of A and u_1. Now we consider I_2:

$$
I_2 = -\int_Y \text{div}_x \Big(A(y)\,\nabla_y u_1 \Big) dy \tag{3.58}
$$

$$
= -\int_Y \text{div}_x\left(A(y)\,\nabla_y\,\chi \nabla_x u_0\right) dy \tag{3.59}
$$

$$
= -\,\text{div}_x \left(\int_y A(y) \nabla_y \chi \, dy \right) . \nabla_x u_0 \tag{3.60}
$$

On the other hand

$$\int_Y A_2\, u_0\, dy = -\int_Y \operatorname{div}_x \left(A(y)\, \nabla_x u_0\right) dy \tag{3.61}$$

$$= -\operatorname{div}_x \left(\int_Y A(y)\, dy\right) \nabla_x u_0 \tag{3.62}$$

Finally, we get the homogeneous problem

$$f(x) = \frac{1}{|Y|}\left[-\operatorname{div}_x\left(\nabla_x u_0 \int_Y A(y)dy\right) - \operatorname{div}_x\left(\int_Y (A(y)\nabla_y \chi)dy\, \nabla_x u_0\right)\right]$$

Then

$$-\operatorname{div}_x\left[\frac{1}{|Y|}\int_Y \left(A(y) + A(y)\nabla_x \chi(y)\right) dy\, \nabla_x u_0\right] \tag{3.63}$$

which can be read

$$-\operatorname{div}_x\left(A^H \nabla_x u_0\right) + \frac{\partial^2 u_0}{\partial t^2} = f \tag{3.64}$$

Where the homogenized coefficient $A^H = (a_{ij}^H)_{1\leq i,j\leq n}$ is explicitly given by

$$a_{ik}^H = \frac{1}{|Y|}\int_Y \left[a_{ik}(y) + \sum_{j=1}^{n} a_{ij}(y)\frac{\partial \chi_k}{\partial y_j}(y)\right] dy \tag{3.65}$$

It remains to find the boundary conditions that u_0 needs to fulfill. Here, since $u^\epsilon = 0$ on $\partial\Omega$, the multiscale expansion gives at order ϵ^0:

$$u_0(x,y) = u_0(x) = 0 \quad \text{on } \partial\Omega \tag{3.66}$$

Therefore, the homogenized problem is

$$\begin{cases} -\operatorname{div}_x\left(A^H \nabla_x u_0\right) + \dfrac{\partial^2 u_0}{\partial t^2} = f & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.67}$$

Tartar's Method of Oscillating Test Functions:

Let us now consider problem (3.1) and suppose we are given $f_\epsilon \in L^2(\Omega\times]0,T[)$, $u_\epsilon^0 \in H_0^1(\Omega)$ and $u_\epsilon^1 \in L^2(\Omega)$. The variational formulation of problem (3.1) is

$$\begin{cases} \text{Find } u_\epsilon \in W_2 \text{ such that} \\ \left\langle \dfrac{\partial^2 u_\epsilon}{\partial t^2}(t), v \right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \displaystyle\int_\Omega A^\epsilon(x)\nabla u_\epsilon(x,t)\cdot\nabla v(x)\, dx \\ \quad = \displaystyle\int_\Omega f_\epsilon(x,t)v(x)\, dx \quad \text{in } \mathcal{D}'(0,T), \quad \forall v \in H_0^1(\Omega), \\ u_\epsilon(x,0) = u_\epsilon^0(x) \quad \text{in } \Omega, \\ \dfrac{\partial u_\epsilon}{\partial t}(x,0) = u_\epsilon^1(x) \quad \text{in } \Omega. \end{cases} \tag{3.68}$$

Theorem19 provides the existence and uniqueness of a solution u_ϵ such that

$$u_\epsilon \in L^\infty(0,T;H_0^1(\Omega)), \quad \frac{\partial u_\epsilon}{\partial t} \in L^\infty(0,T;L^2(\Omega)).$$

We now study the asymptotic behaviour of problem (3.68) as $\epsilon \to 0$.the oscillations in (3.68) are only due to the variable x, so that in the homogenization process, the variable t will play the role of a parameter. In fact, we have the following result:

Theorem 19 Suppose that $f_\epsilon \in L^2(\Omega\times]0,T[)$, and $u_\epsilon^0 \in H_0^1(\Omega)$, $u_\epsilon^1 \in L^2(\Omega)$. Let u_ϵ be the solution of (3.68) with A_ϵ defined by (3.22)–(3.23). Assume that

$$\begin{cases} \text{i)} \quad u_\epsilon^0 \rightharpoonup u^0 \text{ weakly in } H_0^1(\Omega), \\ \text{ii)} \quad u_\epsilon^1 \rightharpoonup u^1 \text{ weakly in } L^2(\Omega), \\ \text{iii)} \quad f_\epsilon \rightharpoonup f \text{ weakly in } L^2(\Omega\times]0,T[). \end{cases} \tag{3.69}$$

Then, one has the convergences

$$\begin{cases} \text{i)} \quad u_\epsilon \rightharpoonup u \text{ weakly* in } L^\infty(0,T;H_0^1(\Omega)), \\ \text{ii)} \quad \dfrac{\partial u_\epsilon}{\partial t} \rightharpoonup \dfrac{\partial u}{\partial t} \text{ weakly* in } L^\infty(0,T;L^2(\Omega)), \\ \text{iii)} \quad A_\epsilon \nabla u_\epsilon \rightharpoonup A_0 \nabla u \text{ weakly in } L^2(\Omega\times]0,T[)^N, \end{cases}$$

where u is the solution of the homogenized problem:

$$\begin{cases} \dfrac{\partial u^2}{\partial t} - \text{div}(A_0\nabla u) = f & \text{in } \Omega\times]0,T[, \\ u = 0 & \text{on } \partial\Omega\times]0,T[, \\ u(x,0) = u^0(x) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial t}(x,0) = u^1(x) & \text{in } \Omega. \end{cases} \tag{3.70}$$

and the homogenized matrix A_0 is given by:

$$A_{ij}^0 = \int_Y A(y)(\nabla_y \chi_j(y) + e_j)\cdot e_i \, dy \tag{3.71}$$

Proof: we apply Tartar's method of oscillating test functions.
Since $A^\epsilon \in M(\alpha,\beta,\Omega)$, from assumption (3.69) and estimate (3.8) we have

$$\begin{aligned} &\|u_\epsilon\|_{L^\infty(0,T;H_0^1(\Omega))} + \left\|\frac{\partial u_\epsilon}{\partial t}\right\|_{L^\infty(0,T;L^2(\Omega))} + \left\|\frac{\partial^2 u_\epsilon}{\partial t^2}\right\|_{L^2(0,T;H^{-1}(\Omega))} \\ &\qquad \le c\left(\|f_\epsilon\|_{L^2(\Omega\times]0,T[)} + \|u_\epsilon^0\|_{L^2(\Omega)} + \|u_\epsilon^1\|_{H_0^1(\Omega)}\right) \le c_1, \end{aligned} \tag{3.72}$$

where the constant c_1 is independent of ϵ.

Then, if ξ^ϵ is defined by

$$\xi^\epsilon(x,t) = (\xi_1^\epsilon(x,t),\dots,\xi_N^\epsilon(x,t)) = A^\epsilon(t)\nabla u_\epsilon(x,t), \tag{3.73}$$

from the assumptions on A^ϵ, one has in particular

$$\|\xi^\epsilon\|_{(L^2(\Omega\times]0,T[))^N} \leq \beta c_1.$$

These estimations, provide the existence of a subsequence, still denoted by ϵ, such that

$$\begin{cases} \text{i)} & u_\epsilon \rightharpoonup u & \text{weakly* in } L^\infty(0,T;H_0^1(\Omega)) \\ \text{ii)} & u_\epsilon \to u & \text{strongly in } L^2(\Omega\times]0,T[) \\ \text{iii)} & \dfrac{\partial u_\epsilon}{\partial t} \rightharpoonup \dfrac{\partial u}{\partial t} & \text{weakly* in } L^\infty(0,T;L^2(\Omega)) \\ \text{iv)} & \xi^\epsilon \rightharpoonup \xi^0 & \text{weakly in } (L^2(\Omega\times]0,T[))^N \end{cases} \tag{3.74}$$

From definition (3.73) and problem (3.68), one has that ξ^ϵ satisfies

$$\begin{aligned} \int_0^T\int_\Omega \xi^\epsilon(x,t)\cdot\nabla v(x)\,\varphi(t)\,dx\,dt &= \int_0^T\int_\Omega f_\epsilon(x,t)v(x)\varphi(t)\,dx\,dt \\ &\quad + \int_0^T\int_\Omega \frac{\partial u}{\partial t}(x,t)v(x)\frac{\partial\varphi}{\partial t}(t)\,dx\,dt, \end{aligned} \tag{3.75}$$

for any $v\in H_0^1(\Omega)$ and $\varphi\in\mathcal{D}(0,T)$, where we can pass to the limit due to convergences (3.69) and (3.74). we obtain that ξ^0 satisfies

$$\begin{aligned} \int_0^T\int_\Omega \xi^0(x,t)\cdot\nabla v(x)\,\varphi(t)\,dx\,dt &= \int_0^T\int_\Omega f(x,t)v(x)\varphi(t)\,dx\,dt \\ &\quad + \int_0^T\int_\Omega \frac{\partial u}{\partial t}(x,t)v(x)\frac{\partial\varphi}{\partial t}(t)\,dx\,dt, \end{aligned} \tag{3.76}$$

This implies for all $\varphi\in\mathcal{D}(0,T)$;

$$\begin{aligned} \int_0^T\Bigg[\int_\Omega \xi^0(x,t)\cdot\nabla v(x)\,dx - \int_\Omega f(x,t)v(x)\,dx \\ - \int_\Omega \frac{\partial^2 u}{\partial t^2}(x,t)v(x)\,dx\Bigg]\varphi(t)\,dt = 0 \end{aligned} \tag{3.77}$$

Hence

$$\begin{cases} \langle\dfrac{\partial^2 u}{\partial t}(t),v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \displaystyle\int_\Omega \xi^0(x,t)\cdot\nabla v(x)\,dx \\ \quad = \displaystyle\int_\Omega f(x,t)v(x)\,dx \quad \text{in } \mathcal{D}'(0,T), \quad \forall v\in H_0^1(\Omega) \end{cases} \tag{3.78}$$

Let us prove that

$$\xi^0 = A^0\nabla u \tag{3.79}$$

We will once again employ the oscillating test functions w_λ^ϵ , which are defined by $w_\lambda^\epsilon(x) = \epsilon w_\lambda\left(\frac{x}{\epsilon}\right) = \lambda \cdot x - \epsilon\chi_\lambda\left(\frac{x}{\epsilon}\right)$, where the function x_λ is solution of cell problem.

We recall the following convergences:

$$\begin{cases} \text{(i)} \quad w_\lambda^\epsilon \rightharpoonup \lambda \cdot x \quad \text{weakly in } H^1(\Omega), \\ \text{(ii)} \quad w_\lambda^\epsilon \to \lambda \cdot x \quad \text{strongly in } L^2(\Omega), \end{cases} \tag{3.80}$$

and set $\eta_\lambda^\epsilon = {}^tA^\epsilon \nabla w_\lambda^\epsilon$, which satisfies the convergence :

$$\eta_\lambda^\epsilon \rightharpoonup \mathcal{M}_Y({}^tA\nabla w_\lambda) = {}^tA^0\lambda \quad \text{weakly in } (L^2(\Omega))^N. \tag{3.81}$$

and the equation

$$\int_\Omega \eta_\lambda^\epsilon \cdot \nabla v \, dx = 0, \quad \forall v \in H_0^1(\Omega).$$

Let $\psi \in \mathcal{D}(\Omega)$. Choose here $v = \psi u_\epsilon \varphi$ and integrate on $]0, T[$. Then

$$\int_0^T \int_\Omega \eta_\lambda^\epsilon \cdot \nabla u_\epsilon(x,t)\,\psi(x)\varphi(t)\,dx\,dt + \int_0^T \int_\Omega \eta_\lambda^\epsilon \cdot \nabla\psi(x)\,u_\epsilon(x,t)\varphi(t)\,dx\,dt = 0. \tag{3.82}$$

Taking $v = \psi w_\lambda^\epsilon$ in (3.75) and subtracting from (3.82), we obtain

$$\int_0^T \int_\Omega \xi^\epsilon(x,t) \cdot \nabla\psi(x)\,w_\lambda^\epsilon(x)\varphi(t)\,dx\,dt - \int_0^T \int_\Omega \eta_\lambda^\epsilon \cdot \nabla\psi(x)\,u_\epsilon(x,t)\varphi(t)\,dx\,dt$$

$$= \int_0^T \int_\Omega f_\epsilon(x,t)\,\psi(x)w_\lambda^\epsilon(x)\varphi(t)\,dx\,dt + \int_0^T \int_\Omega \frac{\partial}{\partial t}u(x,t)\psi(x)w_\lambda^\epsilon(x)\frac{\partial\varphi}{\partial t}(t)\,dx\,dt.$$

where we pass to the limit by using convergences (3.69), (3.74), (3.80), and (3.81). We obtain

$$\int_0^T \int_\Omega \xi^0(x,t) \cdot \nabla\psi(x)(\lambda \cdot x)\varphi(t)\,dx\,dt - \int_0^T \int_\Omega {}^tA^0\lambda \cdot \nabla\psi(x)\,u(x,t)\varphi(t)\,dx\,dt$$

$$= \int_0^T \int_\Omega f(x,t)\,\psi(x)(\lambda \cdot x)\varphi(t)\,dx\,dt + \int_0^T \int_\Omega \frac{\partial u}{\partial t}(x,t)\psi(x)(\lambda \cdot x)\frac{\partial\varphi}{\partial t}(t)\,dx\,dt.$$

From equation (3.78), by the same computation as in Section 2.2, we deduce (3.79).

To show that u satisfies the initial conditions in (3.70), we first choose a function in (3.82) as follows:

$$\psi \in C^0([0,T])$$

such that

$$\frac{\partial\psi}{\partial t}(0) = 1 \quad \text{and} \quad \frac{\partial\psi}{\partial t}(T) = 0.$$

Then, when passing to the limit, one obtains

$$\frac{\partial u}{\partial t}(x,0) = u^1(x),$$

in view of the convergences (3.69) and (3.74) .Then, choosing in (3.75) a function

$$\psi \in C_0^0([0,T])$$

such that

$$\psi(0)=0, \quad \frac{\partial \psi}{\partial t}(0)=1, \quad \psi(T)=0, \quad \frac{\partial \psi}{\partial t}(T)=0,$$

and passing again to the limit, one obtains

$$u(x,0)=u^0(x).$$

Finally, observe that since A^0 is elliptic provides the uniqueness of the solution of problem (3.70). Consequently, the whole sequences in (3.74) converge. This concludes the proof.

Conclusion

In this work, we have presented a study on the homogenization of the wave equation. This study is based on the multiple scale and the oscillating test function methods. After we have proved the existence and the uniqueness of the solution, we have shown a homogenization results concerning wave equation.

As a perspective of this work we can look for a numerical results by applying a multiscale finite element method to demonstrate the effectiveness of the proposed approach in simplifying complex models.

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Abstract

Abstract

This research aims to study the homogenization of the wave equation with periodically oscillating coefficients, using two main approaches: the Two-Scale Asymptotic Expansions and the Oscillating Test Function Method. We begin by presenting the theoretical foundations of homogenization, then we study the existence and uniqueness of the solution along with the homogenization results, and finally we provide some homogenization results.

Keywords: Homogenization, Wave equation, Two-scale expansion, Oscillating test function.

Résumé

Cette recherche vise à étudier l'homogénéisation de l'équation d'onde à coefficients oscillants périodiquement, en utilisant deux approches principales : les développements asymptotiques à deux échelles et la méthode des fonctions tests oscillantes. Nous commençons par présenter les fondements théoriques de l'homogénéisation, puis étudions l'existence et l'unicité de la solution, et enfin fournissons quelques résultats d'homogénéisation.

Mots-clés : Homogénéisation, Équation des ondes, Développement asymptotique à deux échelles, fonctions tests oscillantes.

ملخص

تهدف هذه الدراسة إلى بحث تجانس معادلة الموجة ذات المعاملات المتذبذبة دوريًا، وذلك باستخدام منهجين رئيسيين: التوسعات غير المتجانسة ثنائية المقياس، وطريقة دالة الاختبار المتذبذبة. نبدأ بعرض الأسس النظرية لعملية التجانس، ثم ندرس وجود الحل ووحدانيته، وأخيرًا نعرض بعض نتائج التجانس.

الكلمات المفتاحية: التجانس، معادلة الموجة، التوسعات متعددة المقاييس، دالة الاختبار المتذبذبة