KASDI MERBAH UNIVERSITY OUARGLA

Faculty of Mathematics and Substance Sciences

N° d'ordre :
N° de série :

Department of:
Mathematics

Doctorate

Path: Mathematics

Speciality: Modeling, control and optimization

Present by: Nadjat HAFSI

Theme:

Solving some differential problems of non-integer order via operational matrix combined with orthogonal polynomials

Represented in : 04/12/2025

Jury members:

Djamel Ahmed CHACHA	Prof	University of Ouargla	President
Brahim TELLAB	Prof	University of Ouargla	Director of thesis
Mabrouk MEFLAH	Prof	University of Ouargla	Co-Director of thesis
Abdelkader AMARA	Prof	University of Ouargla	Examiner
Mouafak BENCHOHRA	Prof	University Sidi Bel Abbes	Examiner
Jamal Eddine LAZREG	Prof	University Sidi Bel Abbes	Examiner

ملخص

تهدف هذه الأطروحة إلى تقديم دراسة معمقة حول الحساب الكسري وتطبيقاته في حل المعادلات التفاضلية الكسرية باستخدام المصفوفات التشغيلية بالاشتراك مع كثيرات الحدود المتعامدة. تركز الدراسة على استخدام كثيرات حدود محددة، مثل كثيرحدود برنشتاين، كثيرحدود ليجيندر المعدلة، وكثيرحدود جاكوبي المعدلة، بالإضافة إلى المصفوفات التشغيلية الكسرية لتطوير طرق عددية فعالة. يتم توضيح النتائج من خلال أمثلة عددية، مما يبرز فعالية هذه الطرق .

الكلمات المفتاحية: المصفوفات التشغيلية الكسرية، كثيرات حدود برنشتاين، كثيرات حدود ليجيندر المزاحة، كثيرات حدود جاكوبي المزاحة.

Résumé

L'objectif de cette thèse est de présenter une étude approfondie du calcul fractionnaire et de ses applications à la résolution des équations différentielles d'ordre fractionnaire en utilisant les matrices opérationnelles en combinaison avec les polynômes orthogonaux. L'étude se concentre sur l'utilisation de polynômes spécifiques, tels que les polynômes de Bernstein, les polynômes de Legendre décalés et les polynômes de Jacobi décalés, ainsi que les matrices opérationnelles fractionnaires pour développer des méthodes numériques efficaces. Les résultats sont illustrés par des exemples numériques, mettant en évidence l'efficacité de ces méthodes .

Mots-clés : Matrices opérationnelles fractionnaires, polynômes de Bernstein, polynômes de Legendre décalés, polynômes de Jacobi décalés

Abstract

The objective of this thesis is to provide a comprehensive study of fractional calculus and its applications in solving fractional-order differential equations using operational matrix based on orthogonal polynomials. Our results are based on some orthogonal polynomials, such as Bernstein and shifted Legendre polynomials, shifted Jacobi polynomials and fractional operational matrix. Our results are illustrated by a numerical examples, The results highlighting the effectiveness of these methods .

Key words : Fractional operational Matrix, Bernstein polynomials, shifted Legendre polynomials, shifted Jacobi polynomials.

Acknowledgments

In the Name of Allah, the Most Gracious, the Most Merciful. All praise is due to Allah for granting me the strength and opportunity to complete this work.

I extend my sincere gratitude to my supervisor, Prof. **Brahim TELLAB**, for his valuable guidance and support.Their patience and encouragement have been a great source of motivation for me throughout this journey.

My heartfelt thanks to my **parents, brothers, and sisters** for their love, encouragement, and constant support.

I also appreciate the efforts of the**Department of Mathematics** staff for their assistance throughout this journey.

Finally, thank you to everyone who contributed to the success of this work. I hope this effort proves beneficial .

Dedicace

To the one who instilled in my heart a love for knowledge, to the one who was my role model and source of inspiration, to the soul of my **late father**—may he rest in peace—who taught me that ambition knows no bounds, and to my dear **mother**, who has always been my pillar of strength and unwavering support, I present to you the fruit of my efforts and your invaluable sacrifices.

To my brothers**(Nouh and Mouhidine)** and sisters**(Wassila, Fadila and Latifa)**, and to their children who bring joy and light into our lives—thank you for your constant love, encouragement, and belief in me.

To my friends, your camaraderie and belief in me kept me focused and motivated.

To everyone who offered me a word of encouragement, a moment of care, or a sincere prayer—this achievement is also yours.

Contents

Introduction **1**

1 Preliminaries **4**
1.1 Special functions 5
1.1.1 Gamma function 5
1.1.2 Beta function 5
1.2 Integrals and fractional derivatives 5
1.2.1 Fractional derivatives in the sense of Grünwald-Letnikov 5
1.2.2 Integral and derivative in the sense of Riemann-Liouville 7
1.2.3 Derivative in the sense of Caputo 13
1.2.4 Relation between fractional derivatives in the sense of Caputo and those of Riemann-Liouville 14
1.2.5 Properties of fractional derivatives 16
1.2.6 Integration by parts 17
1.3 Polynomials Bases in Approximation 19
1.3.1 Laguerre polynomials 20
1.3.2 Bernstein polynomials 22
1.3.3 Chebyshev polynomials 25

2 Solving fractional BVPs with a Hybrid Bernstein and Legendre Polynomial Collocation Method **30**
2.1 Introduction 31
2.2 Study of existence 32
2.3 A Bernstein Polynomial Approach to Approximate Solutions 39
2.3.1 Matrix Representation of Integration Fractional 41
2.3.2 Matrix Representation of Fractional Derivatives 42
2.4 Numerical Simulation Through Selected Example 43

3 Solving a non-integer order differential equations by using operational matrix and shifted Legendre polynomials **47**
3.1 Introduction 48
3.2 Legendre's polynomial 48
3.2.1 Rodrigue's formula of Legendre polynomials 48
3.2.2 Generated function of Legendre polynomials 49
3.3 Properties of Legendre polynomials 50
3.3.1 Orthogonality of Legendre polynomials 50

3.3.2 Recurrence relation . . . 52
3.4 Operational matrix of Legendre . . . 57
3.4.1 Shifted Legendre polynomiale . . . 57
3.4.2 Function approximation using shifted Legendre polynomials . . . 59
3.4.3 Operational matrix of $\mathscr{D}^q\Phi(t)$ based on shifted Legendre polynomials: . . . 59
3.5 Shifted Legendre operational matrix of multi-order fractional differential equations: 63
3.5.1 Case of a linear problem: . . . 63
3.5.2 Case of a nonlinear problem: . . . 65

4 Solving a fractional differential equations via operational matrix and shifted Jacobi polynomials 68
4.1 Introduction . . . 69
4.2 Properties of shifted Jacobi polynomials . . . 69
4.3 Operational matrix of fractional integration . . . 70
4.4 Fractional shift Jacobi operational matrix for solving linear multi-order fractional differential equations . . . 72
4.5 Illustrative examples . . . 74

Conclusion 80

List of Figures

1.1 Laguerre polynomials of degree 0 to 4 22
1.2 Bernstein polynomials of degree 0 to 4 24
1.3 Chebyshev Polynomials of the first kind 0 to 4 26
1.4 Chebyshev Polynomials of the second kind 0 to 4 27

2.1 Contrast between analytical and numerical results $\vartheta_5(t)$. 46

3.1 Legendre polynomials of degree 0 to 5. 56
3.2 Shifted Legendre polynomials of degree 0 to 4. 58

Notations

Symbole	Meaning
$\Gamma(.)$	Euler gamma function
$\beta(.,.)$	Beta function
$L^P([a,b],\mathbb{R})$	The set of all measurable functions f, defined for $1 \leq p < +\infty$ defined by $\|f\|_p = \left(\int_{[a,b]} |f|^p\right)^{\frac{1}{p}}$.
$L^1([a,b],\mathbb{R})$	The Lebesgue integrable function space on $[a,b]$
$[.]$	The integer part
$\mathbb{B}(s)$	Bernstein polynomials
$\mathbb{P}_n(\mathfrak{z})$	Legendre polynomials
$\mathbb{P}_n^*(\mathfrak{z})$	Shifted Legendre polynomials
BVP	Boundary value problem
$\mathscr{D}^{(\alpha)}$	Operational matrix of the fractional derivative of order α
$\lceil \alpha \rceil$	The smallest integer greater than or equal to α

Introduction

Fractional calculus occupies a significant position within various domains of mathematics. The genesis of fractional calculus can be traced back to an inquiry posed in the year 1695 by Marquis de L'Hôpital (1661-1704) directed towards Gottfried Wilhelm Leibniz (1646-1716). Leibniz formally introduced the notation $\dfrac{d^n y}{dx^n}$ to denote the derivative of order $n = \frac{1}{2}$. On September 30, 1695, Leibniz [21] articulated a response stating: "It is an apparent paradox from which, one day, useful consequences will be drawn." In 1819, Lacroix allocated two pages (pp. 409-410) to the discussion of fractional calculus within his comprehensive 700-page treatise, titled "Traité du Calcul Différentiel et du Calcul Intégral," ultimately demonstrating that

$$\frac{d^{\frac{1}{2}}}{dv^{\frac{1}{2}}} v = \frac{2\sqrt{v}}{\sqrt{\pi}}$$

Subsequently, in 1869, Laurent, in 1884, Nekrassov in 1888, Krug in 1890, and Weyl in 1917 further contributed to this field.
The concept of fractional derivatives was articulated by prominent mathematicians such as Euler in the year 1730, followed by Lagrange in 1772, Laplace in 1812, Lacroix in 1819, Fourier in 1822, Liouville in 1832, Riemann in 1847, Greer in 1859, Holmgren in 1865, Griinwald in 1867, Letnikov in 1868, Sonin in 1869, Laurent in 1884, Nekrassov in 1888, Krug in 1890, and Weyl in 1917.
The concept of generalizing integration and differentiation to encompass non-integer orders has been present since the foundational development of integral and differential calculus. Nonetheless, notwithstanding the extensive scope of potential applications, this domain has garnered minimal scholarly attention until recent times. For instance, fractional calculus finds application in the modeling of viscoelastic materials, continuous media possessing memory, the transformation of temperature and humidity across atmospheric strata, diffusion equations, among various other fields. At a macro level, it is acknowledged that the hereditary characteristics and memory inherent in numerous processes, phenomena, and materials can be effectively modeled utilizing fractional operators. In this context, differential equations that incorporate fractional derivatives have recently been validated as instrumental in the modeling of a diverse array of structures across several scientific disciplines. In light of the rapidly accelerating developments and inquiries within the realm of fractional calculus, numerous scholarly works have been undertaken; refer to [3] [4, 8, 11, 30, 37, 44, 46, 50, 63].

Given that theoretical advancements facilitate a more profound comprehension of fractional models, a considerable number of mathematicians have concentrated their efforts on examining the existence of solutions for various configurations of fractional differential equations through an array of methodologies and techniques. For examples, see [2, 5, 18, 22, 32, 36, 51, 57, 59].

The equations of fractional order serve as the foundational framework for the mathematical representation of processes occurring within fractal media. In the course of developing mathematical models for geophysical processes, The introduction of the notion of the effective rate of change pertaining to specific physical quantities that characterize the simulated processes results in the formulation of differential equations that incorporate a combination of fractional differentiation operators originating from various sources. Similar to numerous other integro-differential equations, fractional-integral and fractional-differential equations resist exact solutions. Consequently, it becomes imperative to develop approximate methodologies for their resolution. Recently, a plethora of scholarly works concerning the fractional Caputo-Fabrizio derivative have been disseminated by Rezapour et al. [16]- [50]. A considerable number of comprehensive articles and reviews exist, notably including the work by Stanˇek [56], which examined the existence, multiplicity, and uniqueness of solutions; Ibrahim et al. [25], who investigated the existence and uniqueness of solutions for the boundary value problem; and Agarwal et al. [39], who explored the singular fractional Cauchy problem. An essential domain within contemporary mathematics is the exploration of fractional derivatives, which facilitates the examination of a relatively extensive class of equations.

Fractional calculus has become an indispensable tool across numerous fields of applied science, including control theory, economics, optics, thermodynamics, aerodynamics, mechanical systems, image and signal processing, biology, and biochemistry. In these disciplines, it has demonstrated both efficiency and effectiveness in a wide array of applications. See [16, 17, 47]. Over recent years, a substantial body of scientific literature has emerged, examining various classes of boundary value problems and integro-differential initial value problems using different forms of fractional derivatives, alongside key properties such as stability and positivity. The growing scope and impact of these studies underscore the flexibility of fractional calculus in formulating mathematical models. Many of these theoretical investigations are grounded in fixed point theory , see [41, 48, 49].

In practice, theoretical analyses are often complemented by numerical methods to obtain approximate solutions, which are vital for handling real-world problems. The following sections highlight several notable studies in this context

In [33], the homotopy method was applied to derive approximate solutions for the SEIR epidemic model incorporating fractional parameters. Other numerical techniques for similar problems are discussed in [34, 46, 52]. For instance, Chadha et al. proposed the Faedo-Galerkin approximation method to address a neutral nonlocal fractional differential equation within a separable Hilbert space. Further approximation methods have been explored in works such as [13, 24, 28],Among these, Bernstein polynomials are particularly notable for their useful properties, including continuity and the partition of unity on the interval $[0, 1]$ [43].

In recent years, a variety of numerical approaches have been formulated for solving fractional differential equations, including the $\hat{q}$-homotopy analysis transform method (HATM) [53] and the collective cubic uniform B-spline method (CUBSA) [54].

The standard finite difference method (SFDA) [42] , along with Fourier spectral techniques [55], has been successfully applied to a wide range of challenges in fields such as physics, engineering, and population dynamics [29] . Moreover, advanced tools like the Elzaki transform and quadruple decomposition have been employed to address intricate problems, including the Navier-Stokes equations [62] and inverse source problems related to fractional diffusion mod-

els [45]. Furthermore, hyperbolic and exponential ansatz techniques have been proposed to improve analytical precision and to investigate solution behaviors of 'fifth-order Korteweg–de Vries equations [15] One notable application involves the mathematical modeling of a mechanical device known as a "thermostat." This problem was formulated as a second-order boundary value problem on the interval $[0, 1]$ and analyzed in [19].
This thesis is organized as follows:

Chapter 1: we covers the theoretical essential foundations for the research. It introduces key definitions of special functions such as the Gamma and Beta functions and explores fractional integrals and derivatives in the Riemann-Liouville and Caputo senses. Important results and properties, including the relationships between fractional integrals and derivatives, are also discussed to set the groundwork for subsequent chapters.

Chapter 2:we focuses on solving a fractional boundary value problem (BVP) for a thermostat model. Bernstein's collocation method combined with Legendre polynomials is employed to derive approximate solutions. The chapter establishes sufficient conditions for the existence and uniqueness of solutions and provides numerical examples to demonstrate the efficiency of the proposed method.

Chapter 3:This chapter presents advanced techniques for solving fractional-order differential equations using operational matrices and shifted Legendre polynomials. It details the fundamental properties of Legendre polynomials, such as orthogonality and recurrence relations, and utilizes these properties in constructing solutions. Several examples are provided to illustrate the applicability of the method.

Chapter 4:This chapter introduces a method based on shifted Jacobi polynomials for solving fractional-order differential equations. The operational matrix for fractional integration is developed and applied to obtain accurate numerical solutions. Illustrative examples are included to compare the results with alternative approaches.

Chapter one

Preliminaries

Contents

1.1 **Special functions** 5
1.1.1 Gamma function 5
1.1.2 Beta function 5
1.2 **Integrals and fractional derivatives** 5
1.2.1 Fractional derivatives in the sense of Grünwald-Letnikov 5
1.2.2 Integral and derivative in the sense of Riemann-Liouville 7
1.2.3 Derivative in the sense of Caputo 13
1.2.4 Relation between fractional derivatives in the sense of Caputo and those of Riemann-Liouville 14
1.2.5 Properties of fractional derivatives 16
1.2.6 Integration by parts 17
1.3 **Polynomials Bases in Approximation** 19
1.3.1 Laguerre polynomials 20
1.3.2 Bernstein polynomials 22
1.3.3 Chebyshev polynomials 25

1.1 Special functions

1.1.1 Gamma function

Definition 1.1.1: [26]

The Euler gamma function $\Gamma(z)$ is defined by the so-called Euler integral of the second kind:

$$\Gamma(z)=\int_0^{+\infty} t^{z-1}e^{-t}dt, \quad Re(z)>0. \tag{1.1}$$

For this function the reduction formula

$$\Gamma(z+1)=\int_0^{\infty} e^{-t}t^z dt=[-e^{-t}t^z]_{t=0}^{t=\infty}+z\int_0^{\infty} e^{-t}t^{z-1}dt=z\Gamma(z)$$

holds; it is obtained from (1.1) by integration by parts.

1.1.2 Beta function

Definition 1.1.2: [26]

The beta function is defined by the Euler integral of the first kind:

$$B(z,w)=\int_0^1 t^{z-1}(1-t)^{w-1}dt,(Re(z)>0,Re(w)>0). \tag{1.2}$$

This function is connected with the gamma functions by the relation

$$B(\theta,\varrho)=\frac{\Gamma(\theta)\Gamma(\varrho)}{\Gamma(\theta+\varrho)}, \quad \theta,\varrho>0$$

1.2 Integrals and fractional derivatives

1.2.1 Fractional derivatives in the sense of Grünwald-Letnikov

The main idea behind the Grünwald-Letnikov fractional derivative is to generalize the classical definition of integer-order differentiation of a function to arbitrary orders.

The classical derivative (of order 1) of a function f at a point t is defined as:

$$f'(t)=\lim_{h\to 0}\frac{f(t)-f(t-h)}{h}. \tag{1.3}$$

By successive differentiation of the function f, a generalization of equation (1.3) to order n

(where n is a non-negative integer) takes the form:

$$f^{(n)}(t)=\lim_{h\to 0}\frac{1}{h^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}f(t-kh), \tag{1.4}$$

where

$$\binom{n}{k}=\frac{n!}{k!(n-k)!}=\frac{n(n-1)\cdots(n-k+1)}{k!}. \tag{1.5}$$

If n is positive, equation (1.4) represents the derivative of order n, and if n is negative, it represents the repeated integral of order n.

Using the fundamental property $\Gamma(n+1)=n!$, for all $n\in\mathbb{N}$, we can write:

$$(-1)^k\binom{n}{k}=\frac{(-n)(-n+1)\cdots(-n+k-1)}{k!}. \tag{1.6}$$

Definition 1.2.1: (Grünwald-Letnikov fractional derivatives) [26]

If f is a continuous function on the interval $[a,t]$, the fractional derivatives of order θ and $-\theta$ in the sense of Grünwald-Letnikov are defined as follows:

$${}^{GL}D_t^{\theta}f(t)=\lim_{h\to 0}\frac{1}{h^{\theta}}\sum_{k=0}^{\infty}\frac{\Gamma(k-\theta)}{\Gamma(k+1)\Gamma(-\theta)}f(t-kh), \tag{1.7}$$

which is equivalent to the integral representation:

$${}^{GL}D_t^{\theta}f(t)=\frac{1}{\Gamma(-\theta)}\int_a^t(t-\zeta)^{-\theta-1}f(\zeta)d\zeta. \tag{1.8}$$

Similarly, the negative-order derivative is given by:

$${}^{GL}D_t^{-\theta}f(t)=\lim_{h\to 0}\frac{1}{h^{-\theta}}\sum_{k=0}^{\infty}\frac{\Gamma(k+\theta)}{\Gamma(k+1)\Gamma(\theta)}f(t-kh), \tag{1.9}$$

which corresponds to the integral form:

$${}^{GL}D_t^{-\theta}f(t)=\frac{1}{\Gamma(\theta)}\int_a^t(t-\zeta)^{\theta-1}f(\zeta)d\zeta. \tag{1.10}$$

Proposition 1.2.1: (Fractional derivative of $(t-a)^{\theta}$)

The fractional derivative in the sense of Grünwald-Letnikov of order q of the polynomial function $f(t)=(t-a)^{\theta}$ is given by:

$${}^{GL}D_t^{q}(t-a)^{\theta}=\frac{\Gamma(\theta+1)}{\Gamma(-q+\theta+1)}(t-a)^{\theta-q}, \tag{1.11}$$

where:

$$(q<0,\theta>-1)\quad\text{or}\quad(0\le m\le q<m+1,\theta>m). \tag{1.12}$$

Proposition 1.2.2: (Fractional derivative of a constant)

The Grünwald-Letnikov fractional derivative of order θ of a constant function $f(t) = C$ is defined by:

$$ {}^{GL}D_t^{\theta} C = C \frac{1}{\Gamma(1-\theta)} (t-a)^{-\theta}. \tag{1.13}$$

Proposition 1.2.3: (Composition with integer-order derivatives) [26]

Let m be a strictly positive integer and q be a non-integer. Then:

$$\frac{d^m}{dt^m}\left({}^{GL}D_t^q f(t)\right) = {}^{GL}D_t^{m+q} f(t), \tag{1.14}$$

$$ {}^{GL}D_t^q \left(\frac{d^m}{dt^m} f(t)\right) = {}^{GL}D_t^{m+q} f(t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(a)}{\Gamma(k-q-m+1)} (t-a)^{k-q-m}. \tag{1.15}$$

Proposition 1.2.4: Composition of fractional derivatives [26]

We have three distinct cases:

1. If $q < 0$ and for all $p \in \mathbb{R}$:

$$ {}^{GL}D_t^p ({}^{GL}D_t^q f(t)) = {}^{GL}D_t^{p+q} f(t). \tag{1.16}$$

2. If $0 \le m < q < m+1$, $p < 0$, and the function f satisfies:

$$f^{(k)}(a) = 0, \quad \forall k = 0, 1, \ldots, m-1, \tag{1.17}$$

then:

$$ {}^{GL}D_t^p ({}^{GL}D_t^q f(t)) = {}^{GL}D_t^{p+q} f(t). \tag{1.18}$$

3. If $0 \le m < q < m+1$, $0 \le n < p < n+1$ and the function f satisfies:

$$f^{(k)}(a) = 0, \quad \forall k = 0, 1, \ldots, \max(m, n) - 1, \tag{1.19}$$

then:

$$ {}^{GL}D_t^p ({}^{GL}D_t^q f(t)) = {}^{GL}D_t^q ({}^{GL}D_t^p f(t)) = {}^{GL}D_t^{p+q} f(t). \tag{1.20}$$

1.2.2 Integral and derivative in the sense of Riemann-Liouville

If f is locally integrable on $(c, +\infty)$, then the nth integral of f is given by

$$D_{c/t}^{-n}[f](t) = \int_c^t ds_1 \int_c^{s_1} ds_2 \cdots \int_c^{s_{n-1}} f(s_n)\, ds_n = \frac{1}{(n-1)!} \int_c^t (t-s)^{n-1} f(s)\, ds,$$

for almost every t with $-1 \le c < t < 1$ and $n \in \mathbb{N}$ (since $\Gamma(n) = (n-1)!$). There is an immediate generalization, namely the fractional integral of f of order $\theta > 0$.

Definition 1.2.2: Integrals of Arbitrary Order

The left and right Riemann-Liouville integrals of order $\theta > 0$ of the function $f \in L^1(a,b)$ are defined by:

$$I^{\theta}_{a/t}(f)(t) = \frac{1}{\Gamma(\theta)} \int_a^t (t-\zeta)^{\theta-1} f(\zeta)\, d\zeta, \quad t \in (a,b]; \tag{1.21}$$

and

$$I^{\theta}_{t/b}(f)(t) = \frac{1}{\Gamma(\theta)} \int_t^b (\zeta-t)^{\theta-1} f(\zeta)\, d\zeta, \quad t \in [a,b); \tag{1.22}$$

respectively.

Remark 1.2.1

When $\theta = n \in \mathbb{N}$, definitions (1.21) and (1.22) correspond respectively to the n-th integrals of the form:

$$I^{n}_{a/t}(f)(t) = \frac{1}{(n-1)!} \int_a^t (t-\zeta)^{n-1} f(\zeta)\, d\zeta,$$

and

$$I^{n}_{t/b}(f)(t) = \frac{1}{(n-1)!} \int_t^b (\zeta-t)^{n-1} f(\zeta)\, d\zeta.$$

Example 1.2.1

Consider the function $f(t) = (t-a)^{\varrho}$. Then,

$$I^{\theta}_{a/t}(t-a)^{\varrho} = \frac{1}{\Gamma(\theta)} \int_a^t (t-\zeta)^{\theta-1} (\zeta-a)^{\varrho}\, d\zeta$$

and, using the change of variable $\zeta = a + (t-a)s$, one obtains

$$\begin{aligned} I^{\theta}_{a/t}(t-a)^{\varrho} &= \frac{(t-a)^{\theta+\varrho}}{\Gamma(\theta)} \int_0^1 (1-s)^{\theta-1} s^{\varrho}\, ds \\ &= \frac{B(\theta,\varrho+1)}{\Gamma(\theta)} (t-a)^{\theta+\varrho} \\ &= \frac{\Gamma(\varrho+1)}{\Gamma(\varrho+\theta+1)} (t-a)^{\theta+\varrho}, \end{aligned} \tag{1.23}$$

where $B(\theta,\varrho+1)$ is Euler's Beta function.

Remark 1.2.2

- Notice that this is a generalization of the case $\theta = 1$ because:

$$I^1_{a/t}(t-a)^{\varrho} = \frac{\Gamma(\varrho+1)}{\Gamma(\varrho+2)}(t-a)^{\varrho+1} = \frac{1}{\varrho+1}(t-a)^{\varrho+1}.$$

- Relation (1.23) shows that the fractional integral in the Riemann-Liouville sense of order θ of a constant is given by:

$$I^{\theta}_{a/t}C = \frac{C}{\Gamma(\theta+1)}(t-a)^{\theta}. \tag{1.24}$$

Proposition 1.2.5

Let θ and ϱ be two complex numbers and f be continuous on $[a,b]$. Then:

1. $$I^{\theta}_{a\to t}\left(I^{\varrho}_{a\to t}f\right) = I^{\theta+\varrho}_{a\to t}f; \quad (\Re(\theta)>0,\, \Re(\varrho)>0)$$

2. $$\frac{d}{dt}\left(I^{\theta}_{a\to t}f\right)(t) = \left(I^{\theta-1}_{a\to t}f\right)(t); \quad \Re(\theta)>1$$

3. $$\lim_{\theta\to 0^+}\left(I^{\theta}_{a\to t}f\right)(t) = f(t); \quad \Re(\theta)>0.$$

Proof

1. Using Fubini's theorem, we have

$$I^{\theta}_{a\to t}\left(I^{\varrho}_{a\to t}f\right)(t) = \frac{1}{\Gamma(\theta)}\int_a^t (t-s)^{\theta-1}\left(I^{\varrho}_{a\to t}f\right)(s)ds,$$

which can be written as

$$= \frac{1}{\Gamma(\theta)\Gamma(\varrho)}\int_a^t\int_a^s (t-s)^{\theta-1}(s-\zeta)^{\varrho-1}f(\zeta)d\zeta\, ds.$$

Changing the order of integration yields

$$= \frac{1}{\Gamma(\theta)\Gamma(\varrho)}\int_a^t f(\zeta)\left[\int_\zeta^t (t-s)^{\theta-1}(s-\zeta)^{\varrho-1}ds\right]d\zeta.$$

With the substitution $s = \zeta + (t-\zeta)u$ and using Euler's Beta function, one obtains

$$\int_\zeta^t (t-s)^{\theta-1}(s-\zeta)^{\varrho-1}ds = (t-\zeta)^{\theta+\varrho-1}\frac{\Gamma(\theta)\Gamma(\varrho)}{\Gamma(\theta+\varrho)}.$$

Hence,

$$I_{a\to t}^{\theta}\left(I_{a\to t}^{\varrho}f\right)(t)=\frac{\Gamma(\theta+\varrho)}{\Gamma(\theta)\Gamma(\varrho)}\int_a^t f(\zeta)(t-\zeta)^{\theta+\varrho-1}d\zeta=\left(I_{a\to t}^{\theta+\varrho}f\right)(t).$$

2. This follows by applying classical differentiation theorems for integrals depending on a parameter and using the relation $\Gamma(\theta)=(\theta-1)\Gamma(\theta-1)$.

3. From (1.24)

$$\left(I_{a\to t}^{\theta}1\right)(t)=\frac{(t-a)^{\theta}}{\Gamma(\theta+1)}\to 1 \quad \text{as } \theta\to 0^{+}.$$

Thus, for any fixed t we have

$$\left(I_{a\to t}^{\theta}f\right)(t)-\frac{(t-a)^{\theta}}{\Gamma(\theta+1)}f(t)=\frac{1}{\Gamma(\theta)}\int_a^t (t-\zeta)^{\theta-1}\left[f(\zeta)-f(t)\right]d\zeta.$$

Splitting the integral and using the continuity of f on $[a,b]$ (so that for every $\epsilon>0$ there exists $\delta>0$ such that if $|\zeta-t|<\delta$ then $|f(\zeta)-f(t)|<\epsilon$), one obtains estimates which imply that

$$\lim_{\theta\to 0^{+}}\left(I_{a\to t}^{\theta}f\right)(t)=f(t).$$

Definition 1.2.3: Derivatives of arbitrary order [26]

The left and right Riemann–Liouville derivatives of order $\theta>0$ of the function $f\in L^1(0,T)$ are defined by:

$$_aD_t^{\theta}(f)(t)=\frac{d^n}{dt^n}\left[I_{a\to t}^{n-\theta}f\right](t)=\frac{1}{\Gamma(n-\theta)}\frac{d^n}{dt^n}\left(\int_a^t (t-\zeta)^{n-\theta-1}f(\zeta)d\zeta\right),\quad n=[\theta]+1, \tag{1.25}$$

and

$$_tD_b^{\theta}(f)(t)=(-1)^n\frac{d^n}{dt^n}\left[I_{t\to b}^{n-\theta}f\right](t)=\frac{(-1)^n}{\Gamma(n-\theta)}\frac{d^n}{dt^n}\left(\int_t^b (\zeta-t)^{n-\theta-1}f(\zeta)d\zeta\right),\quad n=[\theta]+1, \tag{1.26}$$

respectively.
In particular, when $\theta=n\in\mathbb{N}$, then

$$_aD_t^{0}f(t)={}_tD_b^{0}f(t)=f(t),$$

$$_aD_t^{n}f(t)=f^{(n)}(t) \quad \text{and} \quad {}_tD_b^{n}f(t)=(-1)^n f^{(n)}(t),$$

where $f^{(n)}(t)$ denotes the classical nth derivative of $f(t)$. If $0<\theta<1$ then

$$_aD_t^{\theta}(f)(t)=\frac{1}{\Gamma(1-\theta)}\frac{d}{dt}\left(\int_a^t (t-\zeta)^{-\theta}f(\zeta)d\zeta\right),\quad t>a,$$

and

$$ {}_tD_b^{\theta}(f)(t) = -\frac{1}{\Gamma(1-\theta)}\frac{d}{dt}\left(\int_t^b (\zeta - t)^{-\theta} f(\zeta)d\zeta\right), \quad t < b. $$

Remark 1.2.3

- The fractional integral and the fractional derivative (to the left, respectively to the right) in the Riemann-Liouville sense—are used for the past (respectively future) of f.
- The fractional integral in the Riemann–Liouville sense of order $\theta > 0$ is exactly the fractional derivative in the Riemann-Liouville sense of order $-\theta$.

Example 1.2.2

Consider the function $f(t) = (t-a)^{\varrho}$. Then the Riemann–Liouville derivative of order $\theta > 0$ of f is given by:

$$ \begin{aligned} {}_aD_t^{\theta}(t-a)^{\varrho} &= \frac{1}{\Gamma(n-\theta)}\frac{d^n}{dt^n}\left(\int_a^t (t-\zeta)^{n-\theta-1}(\zeta - a)^{\varrho} d\zeta\right) \\ &= \frac{d^n}{dt^n}\left[\frac{\Gamma(\varrho+1)}{\Gamma(\varrho+1+n-\theta)}(t-a)^{\varrho+n-\theta}\right], \end{aligned} \tag{1.27} $$

and since

$$ \frac{d^n}{dt^n}\left[(t-a)^{\varrho+n-\theta}\right] = (\varrho+n-\theta)(\varrho+n-\theta-1)\cdots(\varrho-\theta+1)(t-a)^{\varrho-\theta}, \tag{1.28} $$

replacing (1.28) into (1.27) yields:

$$ {}_aD_t^{\theta}(t-a)^{\varrho} = \frac{\Gamma(\varrho+1)}{\Gamma(\varrho-\theta+1)}(t-a)^{\varrho-\theta}. \tag{1.29} $$

Remark 1.2.4

- This clearly generalizes the case $\theta = 1$ since:

$$ {}_aD_t^{1}(t-a)^{\varrho} = \frac{\Gamma(\varrho+1)}{\Gamma(\varrho)}(t-a)^{\varrho-1} = \varrho(t-a)^{\varrho-1} = \frac{d}{dt}(t-a)^{\varrho}. $$

- Replacing $\varrho = 0$ in (1.29), we obtain

$$ {}_aD_t^{\theta}(1) = \frac{(t-a)^{-\theta}}{\Gamma(1-\theta)}, $$

that is, the Riemann–Liouville fractional derivative of order θ of a constant is given by:

$$ {}_aD_t^{\theta}(C) = \frac{C}{\Gamma(1-\theta)}(t-a)^{-\theta}. $$

Thus, the Riemann–Liouville fractional derivative of order θ of a constant is neither zero nor constant.

Lemma 1.2.1: [23]

Let $\varrho \geq \rho > 0$, for $f \in L^P([a,b],\mathbb{R}^N),(1 \leq p \leq +\infty)$, we have

$$ (\mathscr{D}^{\rho}_{0^+} I^{\varrho}_{0^+} f)(s) = I^{\varrho-\rho}_{0^+} f(s) $$

almost everywhere on $[a,b]$. In the particular case $\varrho = \rho$, we get

$$ (\mathscr{D}^{\varrho}_{0^+} I^{\varrho}_{0^+} f)(s) = f(s) $$

Lemma 1.2.2: [23]

For $\varrho > 0$, if $f \in L^1([a,b],\mathbb{R}^N)$ and $I^{n-\varrho}_{0^+} f \in AC^n([a,b],\mathbb{R}^N)$, then we have

$$ I^{\varrho}_{0^+}\left(\mathscr{D}^{\varrho}_{0^+} f(s)\right) = f(s) - \sum_{k=1}^{n} \frac{f^{(n-k)}_{n-\varrho}(0) s^{\varrho-k}}{\Gamma(\varrho-k+1)}, \tag{1.30} $$

almost everywhere on $[a,b]$, where $n-1 \leq \varrho < n$.

Lemma 1.2.3: [6]

If $\varrho > 0$ and $\rho > 0$, then at almost point $s \in [a,b]$, we have

$$ I^{\varrho}_{0^+}(I^{\rho}_{0^+} f(s)) = I^{\varrho+\rho}_{0^+} f(s), $$

for $f \in L^p([a,b],\mathbb{R}^N)$ where $1 \leq p \leq +\infty$. Si $\varrho + \rho > 1$, the last equality holds at any point of $[a,b]$.

Lemma 1.2.4: [6]

If $\varrho > 0$ and $\rho > 0$, then

$$ I^{\varrho}_{0^+} s^{\rho-1} = \frac{\Gamma(\rho) s^{\varrho+\rho-1}}{\Gamma(\varrho-\rho)} $$

Remark 1.2.5

We have the following useful axioms :

(H1) For $0 \leq \nu < \rho$, the equality $\mathscr{D}^{\nu}_{0^+}\mathscr{I}^{\rho}_{0^+} f(z) = \mathscr{I}^{\rho-\nu}_{0^+} f(z)$ holds ;

(H2) For $\rho > -1$ such that $\rho \neq \nu - j$ $(j = 1, 2, ..., n)$, we have for $z \geq 0$

$$\mathscr{D}^{\nu}_{0^+} z^{\rho} = \frac{\Gamma(1+\rho)}{\Gamma(\rho-\nu+1)} z^{\rho-\nu} \quad \text{and} \quad \mathscr{D}^{\nu}_{0^+} z^{\nu-j} = 0, \quad (j = 1, 2, ..., n). \tag{1.31}$$

Proposition 1.2.6: [64]

Let $\mu > 0$ and $k = [\mu] + 1$. The general solution to the homogeneous equation

$$\mathscr{D}^{\mu}_{0^+} f(t) = 0,$$

in $\mathscr{C}([0,1],\mathbb{R}) \cap \mathscr{L}^1([0,1],\mathbb{R})$ is

$$f(t) = c_1 t^{\mu-1} + c_2 t^{\mu-2} + ... + c_k t^{\mu-k}, \quad t \in [0,1],$$

where $c_1, c_2, \ldots, c_k \in \mathbb{R}$.

Proposition 1.2.7: [64]

Assume that $\mu > 0$, f is contained in the space $\mathscr{C}([0,1],\mathbb{R}) \cap L^1([0,1],\mathbb{R})$ and $k = [\mu] + 1$. Then

$$\mathscr{I}^{\mu}_{0^+}\mathscr{D}^{\mu}_{0^+}\xi(t) = f(t) - c_1 t^{\mu-1} - c_2 t^{\mu-2} - \cdots - c_k t^{\mu-k}, \tag{1.32}$$

with $c_1, c_2, \ldots, c_k \in \mathbb{R}$.

1.2.3 Derivative in the sense of Caputo

Definition 1.2.4: [26]

Let $k-1 < \mu < k$ with $k = [\mu] + 1$ and $f : (0, +\infty) \longrightarrow \mathbb{R}$ belongs to $AC^{(k)}((0,\infty),\mathbb{R})$. Then

$$\mathscr{D}^{\mu} f(z) = \frac{1}{\Gamma(k-\mu)} \int_0^z (z-s)^{k-\nu-1} f^{(k)}(s) ds, \tag{1.33}$$

is called the Caputo fractional derivative of order μ such that it exists.

Remark 1.2.6

We have the following :

(1) For $0 \leq \mu < \rho$ the equality $\mathscr{D}^{\mu}\mathscr{I}^{\rho}f(z) = \mathscr{I}^{\rho-\mu}f(z)$ holds;

(2) For $\rho > -1$ with $\rho \neq \mu - j$ $(j = 1, 2, ..., n)$ and for each $z \geq 0$, we have

$$\mathscr{D}^{\mu}z^{\rho} = \frac{\Gamma(1+\rho)}{\Gamma(\rho-\mu+1)}z^{\rho-\mu} \quad \text{and} \quad \mathscr{D}^{\mu}z^{\mu-j} = 0, \quad (j = 1, 2, ..., n). \tag{1.34}$$

Proposition 1.2.8: [64]

Suppose that is contained in the space $L(0,1) \cap \mathscr{C}(0,1)$ and $k = [\mu] + 1$. Then

$$\mathscr{I}^{\rho}\mathscr{D}^{\rho}f(z) = f(z) + c_0 + c_1 z + c_2 z^2 + \cdots + c_{k-1}z^{k-1} \tag{1.35}$$

such that $c_0, c_2, \ldots, c_{k-1} \in \mathbb{R}$.

1.2.4 Relation between fractional derivatives in the sense of Caputo and those of Riemann-Liouville

Proposition 1.2.9

If $\theta \in \mathbb{R}_+$ and f is a sufficiently regular function on the interval $[a, b]$, then the left (resp. right) Caputo fractional derivative is related to the left (resp. right) Riemann-Liouville fractional derivative by:

$${}^{C}D_{a}^{\theta}f(t) = {}^{R}D_{a}^{\theta}f(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{\Gamma(k-\theta+1)}(t-a)^{k-\theta}, \quad n = [\theta] + 1 \tag{1.36}$$

respectively,

$${}^{C}D_{t}^{\theta}f(t) = {}^{R}D_{t}^{\theta}f(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{\Gamma(k-\theta+1)}(b-t)^{k-\theta}, \quad n = [\theta] + 1 \tag{1.37}$$

In particular, when $0 < \theta < 1$, we have:

$${}^{C}D_{a}^{\theta}f(t) = {}^{R}D_{a}^{\theta}f(t) - \frac{f(a)}{\Gamma(1-\theta)}(t-a)^{-\theta} \tag{1.38}$$

and

$${}^{C}D_{t}^{\theta}f(t) = {}^{R}D_{t}^{\theta}f(t) - \frac{f(b)}{\Gamma(1-\theta)}(b-t)^{-\theta} \tag{1.39}$$

Proof

The Taylor expansion of f at a is given by:

$$f(t)=f(a)+(t-a)f'(a)+\frac{(t-a)^2}{2!}f''(a)+\cdots+\frac{(t-a)^{n-1}}{(n-1)!}f^{(n-1)}(a)+R_{n-1}, \tag{1.40}$$

where:

$$R_{n-1}=\frac{1}{(n-1)!}\int_a^t(t-\zeta)^{n-1}f^{(n)}(\zeta)d\zeta=\frac{1}{\Gamma(n)}\int_a^t(t-\zeta)^{n-1}f^{(n)}(\zeta)d\zeta=I_a^n f^{(n)}(t) \tag{1.41}$$

Using the linearity of the Riemann-Liouville fractional derivative operator, we obtain:

$${}^R D_a^\theta f(t)={}^R D_a^\theta\left[\sum_{k=0}^{n-1}\frac{(t-a)^k}{\Gamma(k+1)}f^{(k)}(a)+R_{n-1}\right] \tag{1.42}$$

which simplifies to:

$${}^R D_a^\theta f(t)=\sum_{k=0}^{n-1}{}^R D_a^\theta\left[\frac{(t-a)^k}{\Gamma(k+1)}f^{(k)}(a)\right]+{}^R D_a^\theta R_{n-1} \tag{1.43}$$

From known properties of the Riemann-Liouville derivative, we obtain:

$${}^R D_a^\theta f(t)=\sum_{k=0}^{n-1}\frac{(t-a)^{k-\theta}}{\Gamma(k-\theta+1)}f^{(k)}(a)+I_a^{n-\theta}f^{(n)}(t) \tag{1.44}$$

which gives:

$${}^R D_a^\theta f(t)=\sum_{k=0}^{n-1}\frac{(t-a)^{k-\theta}}{\Gamma(k-\theta+1)}f^{(k)}(a)+{}^C D_a^\theta f(t) \tag{1.45}$$

The same reasoning applies for equation (1.37).

Remark 1.2.7

The relations (1.36) and (1.37) can also be rewritten as:

$${}^C D_a^\theta f(t)={}^R D_a^\theta\left[f(t)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(t-a)^k\right],\quad n=[\theta]+1 \tag{1.46}$$

and

$${}^C D_t^\theta f(t)={}^R D_t^\theta\left[f(t)-\sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{k!}(b-t)^k\right],\quad n=[\theta]+1 \tag{1.47}$$

In particular, when $0<\theta<1$, we obtain:

$$ {}^{C}D_a^{\theta} f(t) = {}^{R}D_a^{\theta}(f(t) - f(a)) \tag{1.48} $$

and

$$ {}^{C}D_t^{\theta} f(t) = {}^{R}D_t^{\theta}(f(t) - f(b)). \tag{1.49} $$

1.2.5 Properties of fractional derivatives

Proposition 1.2.10: (Linearity)

If D^p is any fractional derivative operator of order $p > 0$, for all f and g such that $D^p f(t)$ and $D^p g(t)$ exist, we have:

$$ D^p(\lambda f(t) + \mu g(t)) = \lambda D^p f(t) + \mu D^p g(t), \quad \forall \lambda, \mu \in \mathbb{C} \tag{1.50} $$

Proof

For example, if D^p is the Grünwald-Letnikov operator, by definition we have:

$$ \begin{aligned} G^a D_t^p(\lambda f(t) + \mu g(t)) &= \lim_{h\to 0} h^{-p} \sum_{k=0}^{\infty} \frac{\Gamma(k-p)}{\Gamma(k+1)\Gamma(-p)} \big(\lambda f(t-kh) + \mu g(t-kh)\big) \\ &= \lambda \lim_{h\to 0} h^{-p} \sum_{k=0}^{\infty} \frac{\Gamma(k-p)}{\Gamma(k+1)\Gamma(-p)} f(t-kh) \\ &\quad + \mu \lim_{h\to 0} h^{-p} \sum_{k=0}^{\infty} \frac{\Gamma(k-p)}{\Gamma(k+1)\Gamma(-p)} g(t-kh) \\ &= \lambda G^a D_t^p f(t) + \mu G^a D_t^p g(t) \end{aligned} $$

For the Riemann-Liouville fractional derivative of order p (where $n-1 \le p < n$), based on definition (1.25) and the linearity of classical integral and derivative, we get:

$$ \begin{aligned} R^a D_t^p(\lambda f(t) + \mu g(t)) &= \frac{1}{\Gamma(n-p)} \frac{d^n}{dt^n} \left(\int_a^t (t-\zeta)^{n-p-1} (\lambda f(\zeta) + \mu g(\zeta)) d\zeta \right) \\ &= \lambda \frac{1}{\Gamma(n-p)} \frac{d^n}{dt^n} \left(\int_a^t (t-\zeta)^{n-p-1} f(\zeta) d\zeta \right) \\ &\quad + \mu \frac{1}{\Gamma(n-p)} \frac{d^n}{dt^n} \left(\int_a^t (t-\zeta)^{n-p-1} g(\zeta) d\zeta \right) \\ &= \lambda R^a D_t^p f(t) + \mu R^a D_t^p g(t) \end{aligned} $$

Proposition 1.2.11: (Leibniz rule)

We know that the Leibniz rule for computing the n-th derivative of the product of two functions $f(t)$ and $g(t)$ is given by:

$$\frac{d^n}{dt^n}(f(t)g(t)) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(t) g^{(n-k)}(t) \tag{1.51}$$

Replacing integer n with a real parameter p in the right-hand side of (1.51), we obtain the formula:

$$D^p(f(t)g(t)) = \sum_{k=0}^{n} \binom{p}{k} f^{(k)}(t) D^{(p-k)} g(t) - R_n^p, \quad n = [p]+1 \tag{1.52}$$

where:

$$R_n^p(t) = \frac{1}{n!\Gamma(-p)} \int_a^t (t-\zeta)^{-p-1} g(\zeta) d\zeta \int_\zeta^t f^{(n+1)}(\xi)(\zeta-\xi)^n d\xi \tag{1.53}$$

with:

$$\lim_{n\to+\infty} R_n^p(t) = 0 \tag{1.54}$$

Thus, we have a generalization of the Leibniz rule for fractional orders. If f and g and all their derivatives are continuous on $[a,t]$, the Leibniz rule for fractional differentiation in the sense of Grünwald-Letnikov or Riemann-Liouville is written as:

$$D^p(f(t)g(t)) = \sum_{k=0}^{\infty} \binom{p}{k} f^{(k)}(t) D^{(p-k)} g(t) \tag{1.55}$$

Remark 1.2.8

According to formula (1.36), the Leibniz rule in the case of the Caputo derivative is written as:

$$C^a D_t^p(f(t)g(t)) = \sum_{k=0}^{n} \binom{p}{k} f^{(k)}(t) R^a D_t^{p-k} g(t) - \sum_{k=0}^{n-1} \frac{(f(t)g(t))^{(k)}(a)}{\Gamma(k-p+1)} (t-a)^{k-p}. \tag{1.56}$$

1.2.6 Integration by parts

This subsection relates to the integration by parts formula for fractional derivatives of Riemann-Liouville and Caputo.

Theorem 1.2.1

Let $0<\theta<1$ and f,g be two sufficiently regular functions on $[a,b]$, for all $t\in(a,b)$. We have:

$$\int_a^t \left[{}_a^R D_t^\theta f(\zeta)\right] g(\zeta)d\zeta = \int_a^t f(\zeta)\left[{}_\zeta^R D_t^\theta g(\zeta)\right]d\zeta \tag{1.57}$$

$$\int_t^b \left[{}_\zeta^R D_b^\theta f(\zeta)\right] g(\zeta)d\zeta = \int_t^b f(\zeta)\left[{}_t^R D_\zeta^\theta g(\zeta)\right]d\zeta \tag{1.58}$$

In particular:

$$\int_a^b \left[{}_a^R D_t^\theta f(\zeta)\right] g(\zeta)d\zeta = \int_a^b f(\zeta)\left[{}_\zeta^R D_b^\theta g(\zeta)\right]d\zeta \tag{1.59}$$

Proof

$$\begin{aligned}
\int_a^t \left[{}_a^R D_t^\theta f(\zeta)\right] g(\zeta)d\zeta &= \Gamma(1-\theta)\int_a^t \frac{d}{d\zeta}\left(\int_a^\zeta (\zeta-s)^{-\theta} f(s)ds\right) g(\zeta)d\zeta \\
&= -\frac{1}{\Gamma(1-\theta)}\int_a^t \left(\int_a^\zeta (\zeta-s)^{-\theta} f(s)ds\right) g'(\zeta)d\zeta \\
&\quad + \left[\Gamma(1-\theta)g(\zeta)\int_a^\zeta (\zeta-s)^{-\theta} f(s)ds\right]_{\zeta=a}^{\zeta=t} \\
&= -\frac{1}{\Gamma(1-\theta)}\int_a^t \left(\int_s^t (\zeta-s)^{-\theta} g'(\zeta)d\zeta\right) f(s)ds \\
&\quad + g(t)\frac{1}{\Gamma(1-\theta)}\int_a^t (t-s)^{-\theta} f(s)ds \\
&= \int_a^t f(s)\left[{}_s^C D_t^\theta g(s)\right]ds \\
&\quad + g(t)\frac{1}{\Gamma(1-\theta)}\int_a^t (t-s)^{-\theta} f(s)ds \\
&= \int_a^t f(s)\left[{}_s^R D_t^\theta g(s) - g(t)(\Gamma(1-\theta)(t-s)^{-\theta})\right]ds \\
&\quad + g(t)\frac{1}{\Gamma(1-\theta)}\int_a^t (t-s)^{-\theta} f(s)ds \\
&= \int_a^t f(s)\left[{}_s^R D_t^\theta g(s)\right]ds \\
&= \int_a^t f(\zeta)\left[{}_\zeta^R D_t^\theta g(\zeta)\right]d\zeta
\end{aligned}$$

The formula (1.58) is proved in the same way.

Remark 1.2.9

In the case of the Riemann-Liouville fractional derivative, we see that no boundary term appears in this formula, but in the case of the Caputo derivative, we have:

Theorem 1.2.2

Let $0 < \theta < 1$ and f, g be two sufficiently regular functions on $[a, b]$, for all $t \in (a, b)$. We have:

$$\int_a^t \left[{}_a^C D_t^\theta f(\zeta)\right] g(\zeta) d\zeta = \int_a^t f(\zeta)\left[{}_\zeta^C D_t^\theta g(\zeta)\right] d\zeta + g(t) I_{a=t}^{1-\theta} f(t) - f(a) I_{a=t}^{1-\theta} g(a) \quad (1.60)$$

$$\int_t^b \left[{}_\zeta^C D_b^\theta f(\zeta)\right] g(\zeta) d\zeta = \int_t^b f(\zeta)\left[{}_t^C D_\zeta^\theta g(\zeta)\right] d\zeta + g(t) I_{t=b}^{1-\theta} f(t) - f(b) I_{t=b}^{1-\theta} g(b) \quad (1.61)$$

In particular:

$$\int_a^b \left[{}_a^C D_t^\theta f(\zeta)\right] g(\zeta) d\zeta = \int_a^b f(\zeta)\left[{}_\zeta^C D_b^\theta g(\zeta)\right] d\zeta + g(b) I_{a=b}^{1-\theta} f(b) - f(a) I_{a=b}^{1-\theta} g(a) \quad (1.62)$$

Proof

The proof follows similar steps as before by incorporating the boundary terms for Caputo derivatives.

1.3 Polynomials Bases in Approximation

Orthogonal polynomials play a fundamental role in mathematical analysis, numerical approximation, and mathematical physics. They arise naturally in solving differential equations, in approximation theory, and in quadrature rules. Given a weight function $w(x)$ over an interval $[a, b]$, a sequence of polynomials $\{p_n(x)\}$ is said to be orthogonal if it satisfies the relation:

$$\int_a^b p_n(x) p_m(x) w(x) dx = 0, \quad \text{for } n \neq m. \quad (1.63)$$

These polynomials serve as bases for function spaces, enabling expansions similar to Fourier series. Their significance extends to computational mathematics, where they contribute to spectral methods, Gaussian quadrature, and polynomial approximation [20, 27, 58, 60].

In this section, we focus on three important families of orthogonal polynomials: Laguerre polynomials, Bernstein polynomials, and Chebyshev polynomials. Each of these families has distinct properties and applications, making them essential tools in analysis and approximation theory.

1.3.1 Laguerre polynomials

Definition 1.3.1

The Laguerre polynomials $L_n(x)$ are a sequence of orthogonal polynomials that arise in solutions to certain differential equations, particularly in quantum mechanics (e.g., the hydrogen atom). They are defined using the Rodrigues' formula:

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}\left(x^n e^{-x}\right) \tag{1.64}$$

where n is a non-negative integer.

Example 1.3.1

The first few Laguerre polynomials are:

$$\begin{aligned}
L_0(x) &= 1,\\
L_1(x) &= 1-x,\\
L_2(x) &= 1-2x+\frac{x^2}{2},\\
L_3(x) &= 1-3x+\frac{3x^2}{2}-\frac{x^3}{6},\\
L_4(x) &= 1-4x+3x^2-\frac{2x^3}{3}+\frac{x^4}{24}.
\end{aligned}$$

Proposition 1.3.1: Properties of Laguerre polynomials

- **Recurrence relation**

$$(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x) \tag{1.65}$$

- **Differential equation** The Laguerre polynomials satisfy the differential equation:

$$xL_n''(x) + (1-x)L_n'(x) + nL_n(x) = 0. \tag{1.66}$$

- **Orthogonality** The Laguerre polynomials are orthogonal with respect to the weight function e^{-x} over $[0,\infty)$:

$$\int_0^\infty L_n(x)L_m(x)e^{-x}\,dx = \delta_{nm}\frac{n!}{(n+m+1)!}, \tag{1.67}$$

where δ_{nm} is the Kronecker delta:

$$\delta_{n,k} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

- **Generating function** The generating function for Laguerre polynomials is:

$$\sum_{n=0}^{\infty} L_n(x) t^n = \frac{e^{-xt/(1-t)}}{1-t}, \quad |t| < 1. \tag{1.68}$$

- **Explicit formula** An alternative explicit formula for Laguerre polynomials is:

$$L_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n}{k} x^k. \tag{1.69}$$

Applications of Laguerre polynomials

Laguerre polynomials have several important applications across different fields. In **quantum mechanics**, they appear in the radial solutions of the Schrödinger equation for the hydrogen atom, playing a crucial role in describing atomic orbitals. In **numerical analysis**, they are used in Gaussian quadrature methods, specifically Laguerre quadrature, which is effective for evaluating integrals with an exponential weight function. In **probability theory**, these polynomials are connected to the Poisson distribution and moment problems in statistics, providing valuable insights into probability distributions and their properties.

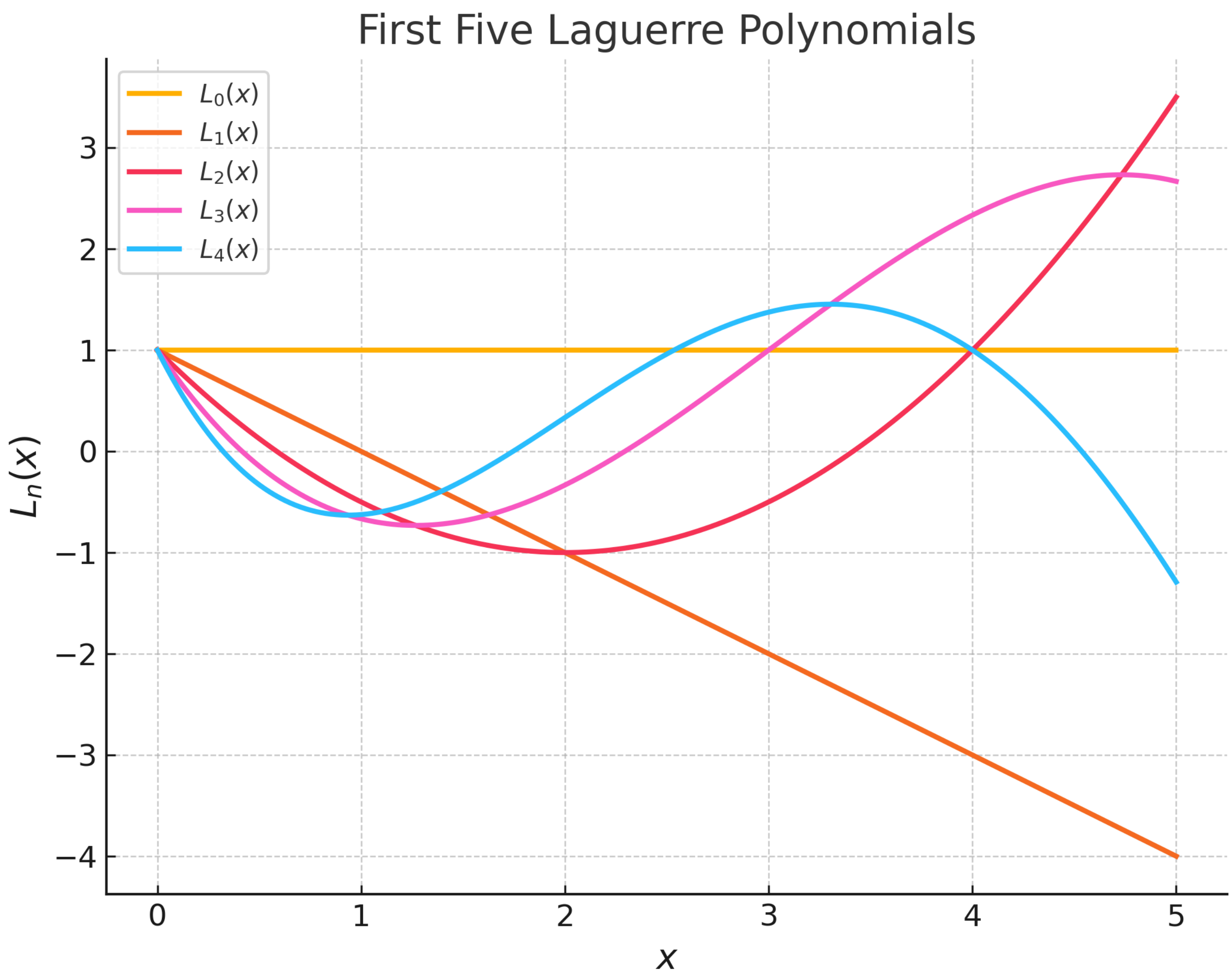

Figure 1.1: Laguerre polynomials of degree 0 to 4

1.3.2 Bernstein polynomials

Bernstein polynomials are a set of polynomials used mainly in approximation theory. The Bernstein polynomial of degree n for a function $f(x)$ is given by:

$$B_n(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}, \quad x \in [0,1]. \tag{1.70}$$

and

$$B_n, m(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad x \in [0,1].$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

is the binomial coefficient, and x is in the interval $[0,1]$.

A key property of Bernstein polynomials is that they provide uniform approximation to continuous functions on $[0,1]$, known as the Weierstrass approximation theorem:

$$\lim_{n\to\infty} B_n(f,x) = f(x), \quad \text{for all } x \in [0,1]. \tag{1.71}$$

These polynomials form a basis for the space of polynomials of degree at most n and are used in approximation theory and computer graphics.

Example 1.3.2

For $n=3$, the Bernstein polynomials are:

$$B_{0,3}(x) = \binom{3}{0} x^0 (1-x)^3 = (1-x)^3$$

$$B_{1,3}(x) = \binom{3}{1} x^1 (1-x)^2 = 3x(1-x)^2$$

$$B_{2,3}(x) = \binom{3}{2} x^2 (1-x)^1 = 3x^2(1-x)$$

$$B_{3,3}(x) = \binom{3}{3} x^3 (1-x)^0 = x^3$$

These polynomials sum to 1 for any x in $[0,1]$.

Properties

1. **Partition of unity**:

$$\sum_{k=0}^{n} B_{k,n}(x) = 1, \quad \forall x \in [0,1]$$

2. **Non-negativity**:

$$B_{k,n}(x) \geq 0, \quad \forall x \in [0,1]$$

3. **Symmetry**:

$$B_{k,n}(x) = B_{n-k,n}(1-x)$$

4. **Derivative formula**:

$$\frac{d}{dx} B_{k,n}(x) = n\left[B_{k-1,n-1}(x) - B_{k,n-1}(x)\right]$$

5. **Approximation Property (Weierstrass Theorem)**: The Bernstein polynomials approximate any continuous function $f(x)$ on $[0,1]$:

$$f_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}(x) \to f(x) \quad \text{as } n \to \infty$$

6. **Bézier Curves relation**: Bernstein polynomials are used in Bézier curves, where a curve is defined as:

$$C(x) = \sum_{k=0}^{n} P_k B_{k,n}(x)$$

where P_k are control points.

Application of Bernstein polynomials

Bernstein polynomials are fundamental in approximation theory, Bézier curves in computer graphics, and numerical analysis. Their properties make them useful for function approximation and interpolation.

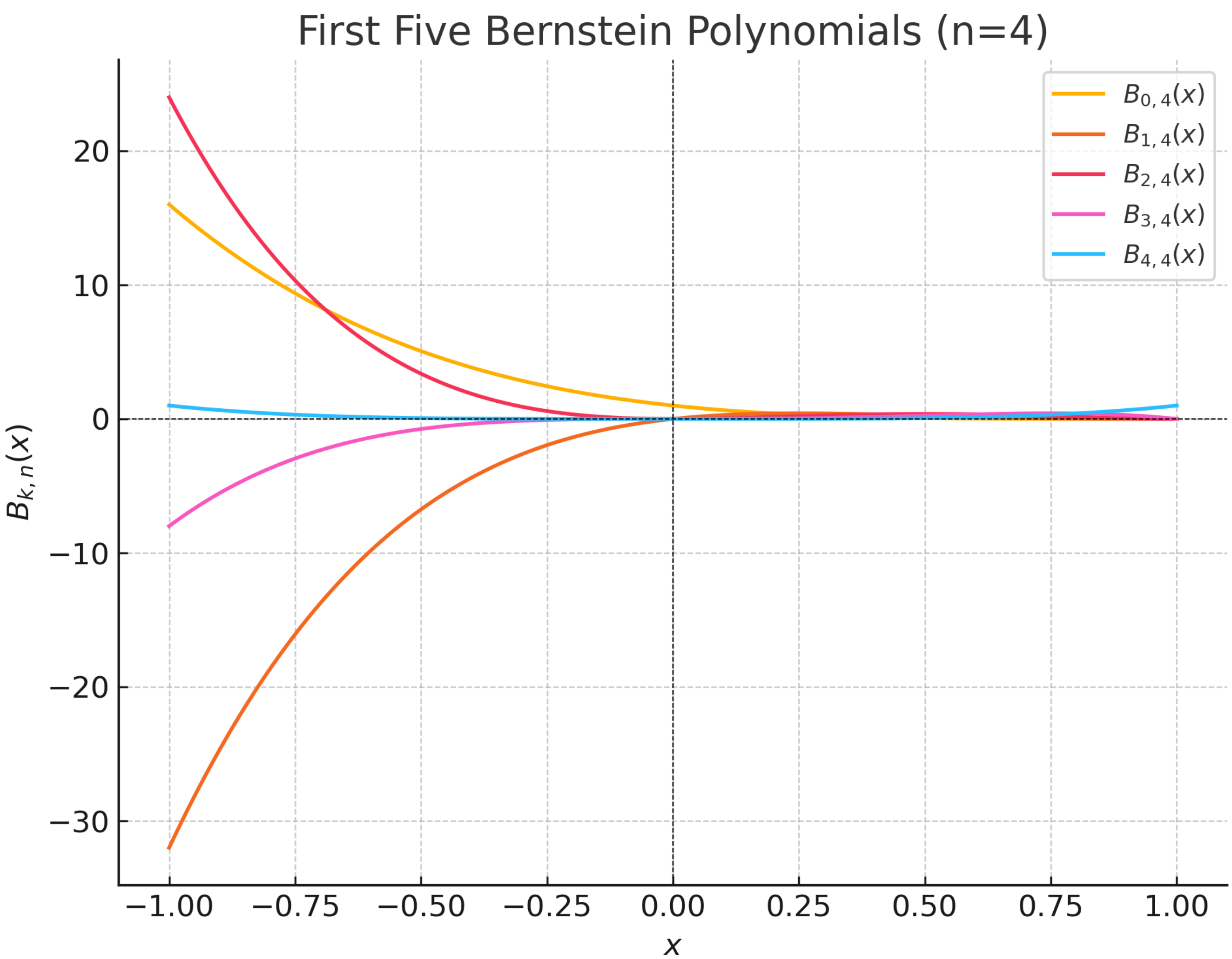

Figure 1.2: Bernstein polynomials of degree 0 to 4

1.3.3 Chebyshev polynomials

Definition 1.3.2: (Chebyshev polynomials of the first kind $T_n(x)$)

The Chebyshev polynomials of the first kind, $T_n(x)$, are defined recursively as:

$$\begin{aligned} T_0(x) &= 1, \\ T_1(x) &= x, \\ T_{n+1}(x) &= 2x\,T_n(x) - T_{n-1}(x), \quad \text{for } n \geq 1. \end{aligned}$$

Alternatively, they can be defined using the trigonometric identity:

$$T_n(x) = \cos(n\cos^{-1}(x)), \quad \text{for } x \in [-1, 1].$$

Example 1.3.3

$$\begin{aligned} T_0(x) &= 1, \\ T_1(x) &= x, \\ T_2(x) &= 2x^2 - 1, \\ T_3(x) &= 4x^3 - 3x, \\ T_4(x) &= 8x^4 - 8x^2 + 1. \end{aligned}$$

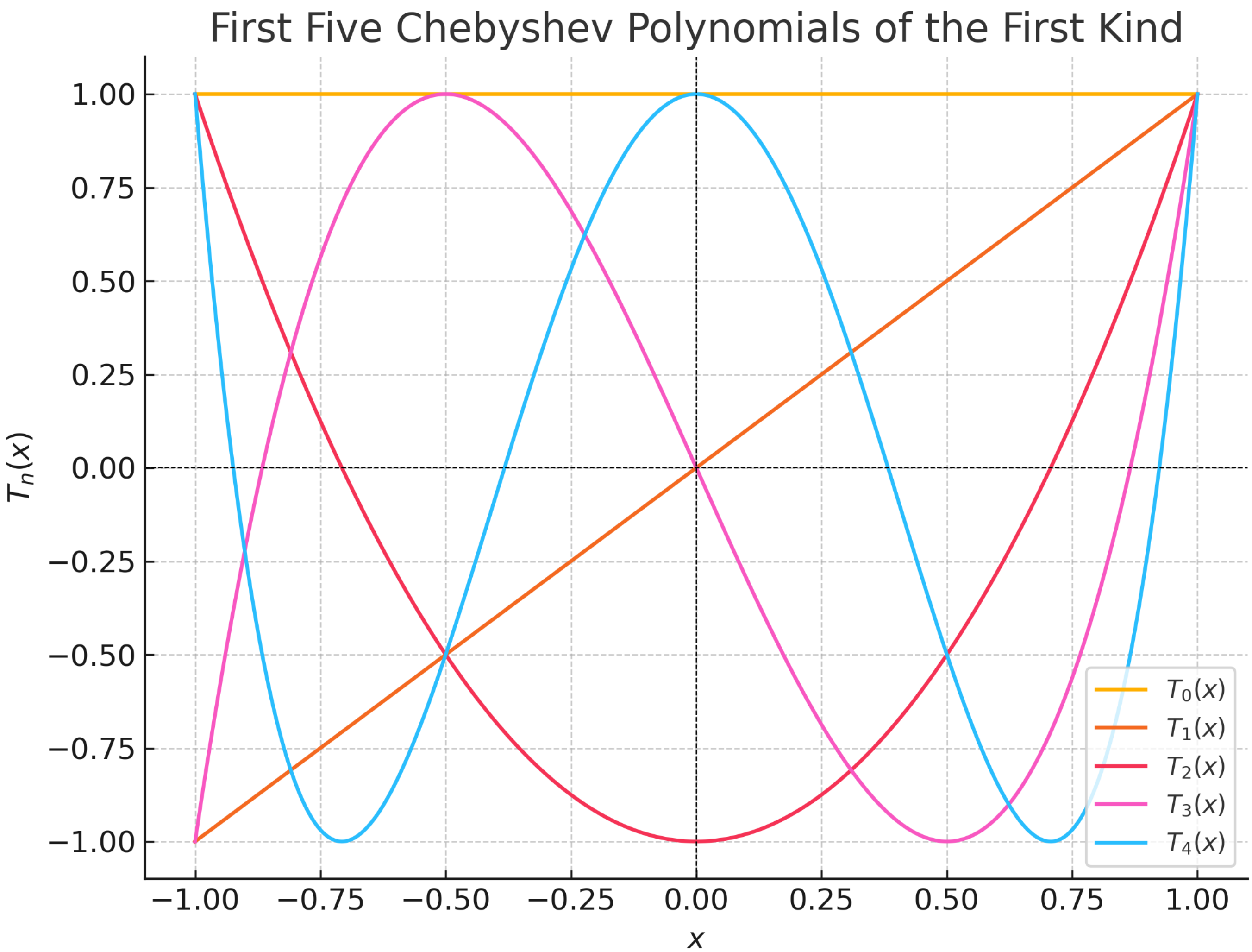

Figure 1.3: Chebyshev Polynomials of the first kind 0 to 4

Definition 1.3.3: (Chebyshev polynomials of the second kind $U_n(x)$)

The Chebyshev polynomials of the second kind, $U_n(x)$, are defined recursively as:

$$
\begin{aligned}
U_0(x) &= 1, \\
U_1(x) &= 2x, \\
U_{n+1}(x) &= 2xU_n(x) - U_{n-1}(x), \quad \text{for } n \geq 1.
\end{aligned}
$$

Alternatively, they can be expressed as:

$$U_n(x) = \frac{\sin((n+1)\cos^{-1}(x))}{\sin(\cos^{-1}(x))}, \quad \text{for } x \in [-1, 1].$$

Example 1.3.4

$$
\begin{aligned}
U_0(x) &= 1, \\
U_1(x) &= 2x, \\
U_2(x) &= 4x^2 - 1, \\
U_3(x) &= 8x^3 - 4x, \\
U_4(x) &= 16x^4 - 12x^2 + 1.
\end{aligned}
$$

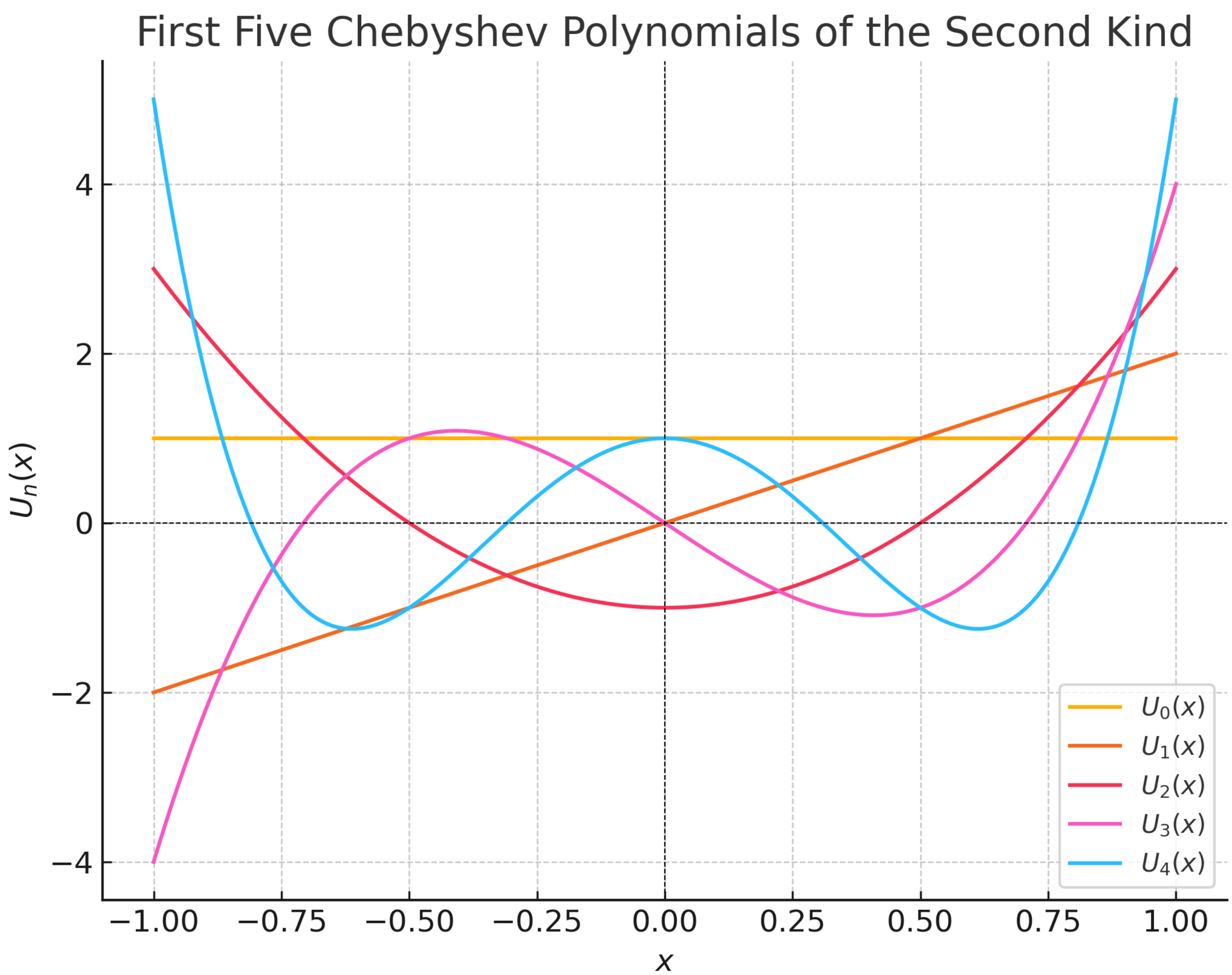

Figure 1.4: Chebyshev Polynomials of the second kind 0 to 4

Proposition 1.3.2: (Properties of Chebyshev polynomials)

•**Orthogonality** The polynomials $T_n(x)$ are orthogonal with respect to the weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$ on $x \in [-1,1]$:

$$\int_{-1}^{1} T_m(x)T_n(x)\frac{dx}{\sqrt{1-x^2}} = \begin{cases} 0, & \text{if } m \neq n, \\ \frac{\pi}{2}, & \text{if } m = n \neq 0, \\ \pi, & \text{if } m = n = 0. \end{cases}$$

Similarly, $U_n(x)$ are orthogonal with respect to $w(x) = \sqrt{1-x^2}$:

$$\int_{-1}^{1} U_m(x)U_n(x)\sqrt{1-x^2}\,dx = \frac{\pi}{2}\delta_{mn}.$$

•**Zeros** The zeros of $T_n(x)$ are:

$$x_k = \cos\left(\frac{(2k-1)\pi}{2n}\right), \quad k = 1, 2, \ldots, n.$$

The zeros of $U_n(x)$ are:

$$x_k = \cos\left(\frac{k\pi}{n+1}\right), \quad k = 1, 2, \ldots, n.$$

•**Relations between $T_n(x)$ and $U_n(x)$**

$$U_n(x) = \frac{T_{n+1}(x) - x\,T_n(x)}{1-x^2}.$$

•**Derivatives**

$$\frac{dT_n}{dx} = nU_{n-1}(x),$$
$$\frac{dU_n}{dx} = (n+1)T_{n+1}(x).$$

•**Chebyshev series expansion** Any function $f(x)$ on $[-1,1]$ can be approximated using Chebyshev series:

$$f(x) \approx \sum_{n=0}^{\infty} a_n T_n(x),$$

where the coefficients a_n are:

$$a_n = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)T_n(x)}{\sqrt{1-x^2}}\,dx.$$

Applications of Chebyshev polynomials

Chebyshev polynomials have numerous applications across various fields. In **approximation theory**, they are used in Chebyshev interpolation to minimize errors, making them valuable for function approximation. In **numerical analysis**, they play a key role in Chebyshev quadrature, which is useful for efficiently computing integrals. In **signal processing**, these polynomials are fundamental in designing digital and analog Chebyshev filters that provide an optimal trade-off between passband ripple and stopband attenuation. They also appear in **solving differential equations**, particularly in Sturm-Liouville problems, where they serve as basis functions. Furthermore, in **minimax polynomial approximation**, Chebyshev polynomials help minimize the maximum deviation from a given function, leading to highly efficient polynomial approximations.

Chapter two

Solving fractional BVPs with a Hybrid Bernstein and Legendre Polynomial Collocation Method

Contents

2.1 Introduction **31**

2.2 Study of existence **32**

2.3 A Bernstein Polynomial Approach to Approximate Solutions **39**

2.3.1 Matrix Representation of Integration Fractional 41

2.3.2 Matrix Representation of Fractional Derivatives 42

2.4 Numerical Simulation Through Selected Example **43**

2.1 Introduction

In this chapter, we investigate the conditions ensuring the existence and uniqueness of solutions. In addition, we propose an approach to approximate solutions of a generalized fractional boundary value problem using the Bernstein collocation method integrated with Legendre polynomials,see([1, 31, 38, 65]).

$$\begin{cases} {}^c\mathscr{D}^p\vartheta(\mathrm{t}) = \Upsilon(\mathrm{t}, \gamma\vartheta(\mathrm{t}), {}^c\mathscr{D}^\sigma\vartheta(\mathrm{t}), \mathscr{I}^\rho\vartheta(\mathrm{t})), & (\mathrm{t} \in \chi = [0,1]), \\ {}^c\mathscr{D}^1\vartheta(0) = \delta_1 \sum_{k=1}^{m}\vartheta(\eta_k), \quad \delta_2^c\mathscr{D}^{p-1}\vartheta(1) + \vartheta(\eta) = \delta_3\sum_{k=1}^{m}\vartheta(\eta_k), \end{cases} \tag{2.1}$$

Let $1 < p \leq 2$, $0 < \sigma < 1$, and $\gamma, \rho > 0$. We also assume $0 < \eta_1 < \eta_2 < \cdots < \eta_m < \eta < 1$, with $m \in \mathbb{N}$, and $\delta_1, \delta_2, \delta_3 \in \mathbb{R}$. The first-order Caputo derivative is denoted by ${}^c\mathscr{D}^1 = \frac{d}{d\mathrm{t}}$.

In addition, let the function $\Upsilon : \chi \times (\mathbb{R}^+)^3 \to \mathbb{R}^+$ be continuous. The operators ${}^c\mathscr{D}^\ell$ and $\mathscr{I}^\ell$ represent, respectively, the Caputo fractional derivative and the Riemann–Liouville fractional integral of order $\ell \in \{1, p, p-1, \sigma, \rho\}$.

To continue with our analysis, we consider a metric space $(\mathbb{E}, \mathrm{d})$, along with the mappings $\mathscr{K} : \mathbb{E} \to \mathbb{E}$, and $\nu : \mathbb{E}\times\mathbb{E} \to \mathbb{R}^+$. Furthermore, we define a family Ψ consisting of all non-decreasing functions $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that the infinite series $\sum_{n=1}^{\infty}\Phi^n(\mathrm{t})$ converges for every $\mathrm{t} > 0$.

Definition 2.1.1: [14]

The map $\mathscr{K}$ is called ν-Φ-contraction if

$$\nu(x,y)\mathbf{d}(\mathscr{K}x, \mathscr{K}y) \leq \Phi(\mathbf{d}(x,y)), \quad \text{for all} \quad (x,y) \in \mathbb{E}^2.$$

Definition 2.1.2: [14]

The map $\mathscr{K}$ is called ν-admissible if it satisfies the condition

$$\nu(x,y) \geqslant 1 \Rightarrow \nu(\mathscr{K}x, \mathscr{K}y) \geqslant 1, \quad \text{for all} \quad (x,y) \in \mathbb{E}^2.$$

Theorem 2.1.1: [14]

Assume that $(\mathbb{E}, \mathbf{d})$ is a complete metric space and
$\mathscr{K} : \mathbb{E} \longrightarrow \mathbb{E}$ is a ν-Φ-contraction, such that:
(i) $\mathscr{K}$ is ν-admissible,
(ii) There exists $\hat{x} \in \mathbb{E}$ satisfies the hypothesis: $\nu(\hat{x}, \mathscr{K}\hat{x}) \geqslant 1$.
(iii) For any sequence x_n in $\mathbb{E}$ which converges to $x \in \mathbb{E}$ where $\nu(x_n, x_{n+1}) \geq 1$ for each $n \in \mathbb{N}$, we have $\nu(x_n, x) \geq 1$, for any $n \in \mathbb{N}$.
Under these assumptions, there exists $\mathrm{t} \in \mathbb{E}$ such that $\mathscr{K}\mathrm{t} = \mathrm{t}$.

2.2 Study of existence

Consider the Banach space $\mathscr{B} = \{\vartheta : \vartheta, {}^c\mathscr{D}^{\sigma}\vartheta \in \mathscr{C}(\chi, \mathbb{R})\}$ endowed with the norm: $\|\vartheta\|_{\mathscr{B}} = \sup_{t\in\chi} |\vartheta(t)| + \sup_{t\in\chi} |{}^c\mathscr{D}^{\sigma}\vartheta(t)|$.Here, $\mathscr{C}(\chi, \mathbb{R})$ denotes the space of all continuous functions

To promote a clearer understanding of this study and to conduct a comprehensive analysis of problem (2.1), we introduce the following parameters, defined on χ and taking values in $\mathbb{R}$. .

$$\upsilon_1 = \delta_2, \qquad \upsilon_2 = 1 - m\delta_3, \qquad \upsilon_3 = \frac{1}{\delta_1} - \sum_{k=1}^{m} \eta_k$$
$$\upsilon_4 = \left(\frac{\delta_2}{\Gamma(3-p)} + \eta - \delta_3 \sum_{k=1}^{m} \eta_k \right), \qquad \upsilon = \upsilon_2\upsilon_3 + m\upsilon_4. \tag{2.2}$$

Lemma 2.2.1

Let $p \in (1,2], 0 < \eta_1 < \eta_2 < \cdots < \eta_m < \eta < 1, m \in \mathbb{N}, \delta_1, \delta_2, \delta_3 \in \mathbb{R}$, and $\hbar \in \mathscr{C}(\chi, \mathbb{R})$. Then, the solution of the generalized fractional boundary value problem (BVP) is given by the following expression:

$$\begin{cases} {}^c\mathscr{D}^p\vartheta(t) = \hbar(t), & (t \in \chi = [0,1]), \\ {}^c\mathscr{D}^1\vartheta(0) = \delta_1 \sum_{k=1}^{m} \vartheta(\eta_k), & \delta_2^c\mathscr{D}^{p-1}\vartheta(1) + \vartheta(\eta) = \delta_3 \sum_{k=1}^{m} \vartheta(\eta_k), \end{cases} \tag{2.3}$$

can be written in the form

$$\begin{aligned} \vartheta(t) \quad = \quad & \int_0^t \frac{(t-f)^{p-1}}{\Gamma(p)} \hbar(f)df + \frac{A(t)\upsilon_1}{\upsilon} \int_0^1 \hbar(f)df \\ & + \frac{A(t)}{\upsilon} \int_0^{\eta} \frac{(\eta-f)^{p-1}}{\Gamma(p)} \hbar(f)df + \frac{B(t)}{\upsilon} \sum_{k=1}^{m} \int_0^{\eta_k} \frac{(\eta_k-f)^{p-1}}{\Gamma(p)} \hbar(f)df, \end{aligned} \tag{2.4}$$

where $A, B \in \mathscr{C}(\chi, \mathbb{R})$ are given by

$$A(t) = -\upsilon_3 + mt, \quad B(t) = \upsilon_4 - \delta_3\upsilon_3 - (\upsilon_2 + \delta_3)t. \tag{2.5}$$

Proof

"We begin by assuming that ϑ^* fulfills the conditions of the generalized fractional boundary value problem 2.3. Consequently, the following relation holds:

$${}^c\mathscr{D}^p\vartheta^*(t) = \hbar(t).$$

Using the integral operator of order p on the preceding equation, we derive the following result:

$$\vartheta^*(t) = \frac{1}{\Gamma(p)} \int_0^t (t-f)^{p-1}\hbar(f)df + c_0 + c_1 t, \tag{2.6}$$

where c_0 and $c_1$denote real constants whose values are to be found. Furthermore, from Remark 1.2.3, we deduce that

$$ {}^{c}\mathscr{D}^{1}\vartheta^{*}(\mathrm{t})=\frac{1}{\Gamma(p-1)}\int_{0}^{\mathrm{t}}(\mathrm{t}-f)^{p-2}\hbar(f)df+c_1, \tag{2.7}$$

$$ {}^{c}\mathscr{D}^{p-1}\vartheta^{*}(\mathrm{t})=\int_{0}^{\mathrm{t}}\hbar(f)df+c_1\frac{\mathrm{t}^{2-p}}{\Gamma(3-p)}, \tag{2.8}$$

and from (2.6), we obtain

$$\sum_{\mathrm{k}=1}^{m}\vartheta(\eta_{\mathrm{k}})=\frac{1}{\Gamma(p)}\sum_{\mathrm{k}=1}^{m}\int_{0}^{\eta_{\mathrm{k}}}(\eta_{\mathrm{k}}-f)^{p-1}\hbar(f)df+mc_0+c_1\sum_{\mathrm{k}=1}^{m}\eta_{\mathrm{k}}. \tag{2.9}$$

Based on (2.2), along with the boundary conditions (2.3), and by utilizing (2.8) and (2.9), we ultimately obtain, after some calculations, the following result.

$$\begin{aligned} c_0 &= -\frac{\upsilon_3}{\upsilon}\int_{0}^{\eta}\frac{(\eta-f)^{p-1}}{\Gamma(p)}\hbar(f)df-\frac{\upsilon_1\upsilon_3}{\upsilon}\int_{0}^{1}\hbar(f)df \\ &\quad+\frac{\upsilon_4-\delta_3\upsilon_3}{\upsilon}\sum_{\mathrm{k}=1}^{m}\int_{0}^{\eta_{\mathrm{k}}}\frac{(\eta_{\mathrm{k}}-f)^{p-1}}{\Gamma(p)}\hbar(f)df, \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} c_1 &= \frac{m}{\upsilon}\int_{0}^{\eta}\frac{(\eta-f)^{p-1}}{\Gamma(p)}\hbar(f)df+\frac{m\upsilon_1}{\upsilon}\int_{0}^{1}\hbar(f)df \\ &\quad-\frac{(\upsilon_2+\delta_3)}{\upsilon}\sum_{\mathrm{k}=1}^{m}\int_{0}^{\eta_{\mathrm{k}}}\frac{(\eta_{\mathrm{k}}-f)^{p-1}}{\Gamma(p)}\hbar(f)df. \end{aligned} \tag{2.11}$$

As a result, by substituting the obtained values of c_0 and c_1 into (2.6), we derive the following desired outcome.

$$\begin{aligned} \vartheta^{*}(\mathrm{t}) &= \int_{0}^{\mathrm{t}}\frac{(\mathrm{t}-f)^{p-1}}{\Gamma(p)}\hbar(f)df+\frac{A(\mathrm{t})\upsilon_1}{\upsilon}\int_{0}^{1}\hbar(f)df \\ &\quad+\frac{A(\mathrm{t})}{\upsilon}\int_{0}^{\eta}\frac{(\eta-f)^{p-1}}{\Gamma(p)}\hbar(f)df+\frac{B(\mathrm{t})}{\upsilon}\sum_{\mathrm{k}=1}^{m}\int_{0}^{\eta_{\mathrm{k}}}\frac{(\eta_{\mathrm{k}}-f)^{p-1}}{\Gamma(p)}\hbar(f)df, \end{aligned}$$

This confirms that ϑ^* satisfies (2.4).
In the converse case, we suppose that v is a solution of (2.4) and we apply the operator ${}^{c}\mathscr{D}^{p}$ to both sides of (2.4) , we obtain immediately ${}^{c}\mathscr{D}^{p}v(\mathrm{t})=h(\mathrm{t})$, and the initial conditions of the problem (2.3), thereby completing the proof.

To proceed with our analysis, we define the operator $\mathfrak{J}:\mathscr{B}\longrightarrow\mathscr{B}$ related to the fractional problem BVP(2.1) as

$$\begin{aligned}(\mathfrak{J}\vartheta)(\mathfrak{t}) \;=\; & \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-f)^{p-1}}{\Gamma(p)}\Upsilon(f,\gamma\vartheta(f),{}^{c}\mathscr{D}^{\sigma}\vartheta(f),\mathscr{I}^{\rho}\vartheta(f))df \\ & +\frac{A(\mathfrak{t})\upsilon_1}{\upsilon}\int_0^1 \Upsilon(f,\gamma\vartheta(f),{}^{c}\mathscr{D}^{\sigma}\vartheta(f),\mathscr{I}^{\rho}\vartheta(f))df \\ & +\frac{A(\mathfrak{t})}{\upsilon}\int_0^{\eta} \frac{(\eta-f)^{p-1}}{\Gamma(p)}\Upsilon(f,\gamma\vartheta(f),{}^{c}\mathscr{D}^{\sigma}\vartheta(f),\mathscr{I}^{\rho}\vartheta(f))df \\ & +\frac{B(\mathfrak{t})}{\upsilon}\sum_{k=1}^{m}\int_0^{\eta_k} \frac{(\eta_k-f)^{p-1}}{\Gamma(p)}\Upsilon(f,\gamma\vartheta(f),{}^{c}\mathscr{D}^{\sigma}\vartheta(f),\mathscr{I}^{\rho}\vartheta(f))df\end{aligned} \tag{2.12}$$

where $A,B\in\mathscr{C}(\chi,\mathbb{R})$ are defined as in equation (2.5).

Before presenting our main results, we introduce a metric $\mathbf{d}$ defined in the Banach space $\mathscr{B}$ as follows:

$$\mathbf{d}(\mathfrak{t},\varpi)=\|\mathfrak{t}-\varpi\|_{\mathscr{B}}.$$

$(\mathscr{B},\mathbf{d})$ is a complete metric space.

Theorem 2.2.1:

Let $\Upsilon:\chi\times\mathbb{R}^3\longrightarrow\mathbb{R}$ be a continuous function, and assume that the following hypotheses hold:

($\mathbb{AY}$1) There exist two functions: $\Omega:\mathbb{R}^2\longrightarrow\mathbb{R}$ and $\Phi\in\Psi$ satisfying respectively $\Omega(x_1,x_2)\geq 0$ and

$$|\Upsilon(\mathfrak{s},x_1,x_2,x_3)-\Upsilon(\mathfrak{s},\tilde{x}_1,\tilde{x}_2,\tilde{x}_3)|\leq\frac{1}{\upsilon_1+\upsilon_2}\Phi\big(|x_1-\tilde{x}_1|+|x_2-\tilde{x}_2|+|x_3-\tilde{x}_3|\big),$$

for any $x_1,x_2,x_3,\tilde{x}_1,\tilde{x}_2,\tilde{x}_3\in\mathbb{R}$, where ℓ_1 and ℓ_2 are two positive constants satisfying:

$$\ell_1>\frac{1}{\Gamma(p+1)}+\frac{(|\upsilon_3|+m)|\upsilon_1|}{|\upsilon|}+\frac{|\upsilon_3|+m}{|\upsilon|\Gamma(p+1)}\eta^p+\frac{|\upsilon_4-\delta_3\upsilon_3|+|\upsilon_2+\delta_3|}{|\upsilon|\Gamma(p+1)}\sum_{\mathrm{k}=1}^{m}\eta_{\mathrm{k}}^{p},$$

and

$$\begin{aligned}\ell_2 \;>\; & \frac{1}{\Gamma(p-\sigma+1)}+\frac{(|\upsilon_3|+m)|\upsilon_1|}{|\upsilon|\Gamma(2-\sigma)} \\ & +\frac{|\upsilon_3|+m}{|\upsilon|\Gamma(2-\sigma)}\eta^{p-\sigma}+\frac{|\upsilon_4-\delta_3\upsilon_3|+|\upsilon_2+\delta_3|}{|\upsilon|\Gamma(p-\sigma+1)}\sum_{\mathrm{k}=1}^{m}\eta_{\mathrm{k}}^{p-\sigma}.\end{aligned}$$

($\mathbb{AY}$2) There exists $\tilde{x}\in\mathscr{B}$ so that $\Omega(\tilde{x}(\mathfrak{s}),\mathfrak{J}\tilde{x}(\mathfrak{s}))\geq 0$, for each $\mathfrak{s}\in\chi$.

($\mathbb{AY}$3) For any $x_1,x_2\in\mathscr{B}$,

if $\Omega(x_1(\mathfrak{s}),x_2(\mathfrak{s}))\geq 0$, then $\Omega(\mathfrak{J}x_1(\mathfrak{s}),\mathfrak{J}x_2(\mathfrak{s}))\geq 0$, for each $\mathfrak{s}\in\chi$.

($\mathbb{AY}4$)) For any sequence

$x_n \in \mathscr{B}$ converging to x_1 in $\mathscr{B}$ and suppose that $\Omega(x_n(\mathfrak{s}), x_{n+1}(\mathfrak{s})) \geq 0$, for all $\mathfrak{s} \in \chi$, and $n \in \mathbb{N}$, it follows that $\Omega(x_n(\mathfrak{s}), x_1(\mathfrak{s})) \geq 0$.

($\mathbb{AY}5$) The constant numbers γ and ρ satisfy the expression $\gamma + \dfrac{1}{\Gamma(\rho+1)} < 1$.

Then, the problem (2.1) has at least one solution.

Proof

Let us define a map $\nu : \mathscr{B} \times \mathscr{B} \longrightarrow \mathbb{R}^+$ as

$$\nu(x, y) = \begin{cases} 1, & if \quad \Omega(x(\mathrm{t}), y(\mathrm{t})) \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

For any $(x, y) \in \mathscr{B}^2$ where $\Omega(x(\mathrm{t}), y(\mathrm{t})) \geq 0$, and for each $\mathrm{t} \in \chi$, we obtain

$$\begin{aligned} & |\mathfrak{I}x(\mathrm{t}) - \mathfrak{I}y(\mathrm{t})| \\ \leq & \frac{1}{\Gamma(p)} \int_0^{\mathrm{t}} (\mathrm{t} - f)^{p-1} |\Upsilon(f, \gamma x(f), {}^c\mathscr{D}^{\sigma} x(f), \mathscr{I}^{\rho} x(f)) - \Upsilon(f, \gamma y(f), {}^c\mathscr{D}^{\sigma} y(f), \mathscr{I}^{\rho} y(f))| df \\ & + \frac{|A(\mathrm{t})||\upsilon_1|}{|\upsilon|} \int_0^1 |\Upsilon(f, \gamma x(f), {}^c\mathscr{D}^{\sigma} x(f), \mathscr{I}^{\rho} x(f)) - \Upsilon(f, \gamma y(f), {}^c\mathscr{D}^{\sigma} y(f), \mathscr{I}^{\rho} y(f))| df \\ & + \frac{|A(\mathrm{t})|}{|\upsilon|\Gamma(p)} \int_0^{\eta} (\eta - f)^{p-1} \\ & |\Upsilon(f, \gamma x(f), {}^c\mathscr{D}^{\sigma} x(f), \mathscr{I}^{\rho} x(f)) - \Upsilon(f, \gamma y(f), {}^c\mathscr{D}^{\sigma} y(f), \mathscr{I}^{\rho} y(f))| df \\ & + \frac{|B(\mathrm{t})|}{|\upsilon|\Gamma(p)} \sum_{\mathrm{k}=1}^{m} \int_0^{\eta_{\mathrm{k}}} (\eta_{\mathrm{k}} - f)^{p-1} \\ & |\Upsilon(f, \gamma x(f), {}^c\mathscr{D}^{\sigma} x(f), \mathscr{I}^{\rho} x(f)) - \Upsilon(f, \gamma y(f), {}^c\mathscr{D}^{\sigma} y(f), \mathscr{I}^{\rho} y(f))| df \\ \leq & \left[\frac{1}{\Gamma(p+1)} + \frac{(|\upsilon_3| + m)|\upsilon_1|}{|\upsilon|} + \frac{|\upsilon_3| + m}{|\upsilon|\Gamma(p+1)} \eta^p + \frac{|\upsilon_4 - \delta_3 \upsilon_3| + |\upsilon_2 + \delta_3|}{|\upsilon|\Gamma(p+1)} \sum_{\mathrm{k}=1}^{m} \eta_{\mathrm{k}}^p \right] \\ & \times \sup_{\mathrm{t} \in \chi} |\Upsilon(\mathrm{t}, \gamma x(\mathrm{t}), {}^c\mathscr{D}^{\sigma} x(\mathrm{t}), \mathscr{I}^{\rho} x(\mathrm{t})) - \Upsilon(\mathrm{t}, \gamma y(\mathrm{t}), {}^c\mathscr{D}^{\sigma} y(\mathrm{t}), \mathscr{I}^{\rho} y(\mathrm{t}))|, \end{aligned}$$

therefore,

$$|\mathfrak{I}x(\mathrm{t}) - \mathfrak{I}y(\mathrm{t})| \leq \ell_1 \sup_{\mathrm{t} \in \chi} |\Upsilon(\mathrm{t}, \gamma x(\mathrm{t}), {}^c\mathscr{D}^{\sigma} x(\mathrm{t}), \mathscr{I}^{\rho} x(\mathrm{t})) - \Upsilon(\mathrm{t}, \gamma y(\mathrm{t}), {}^c\mathscr{D}^{\sigma} y(\mathrm{t}), \mathscr{I}^{\rho} y(\mathrm{t}))|, \tag{2.13}$$

Similarly, we have

$$
\begin{aligned}
& |{}^{c}\mathscr{D}^{\sigma}\mathfrak{J}x(\mathrm{t}) - {}^{c}\mathscr{D}^{\sigma}\mathfrak{J}y(\mathrm{t})| \\
\leq\ & \frac{1}{\Gamma(p-\sigma)}\int_0^{\mathrm{t}}(\mathrm{t}-f)^{p-\sigma-1} \\
& |\Upsilon(f,\gamma x(f),{}^{c}\mathscr{D}^{\sigma}x(f),\mathscr{I}^{\rho}x(f)) - \Upsilon(f,\gamma y(f),{}^{c}\mathscr{D}^{\sigma}y(f),\mathscr{I}^{\rho}y(f))|df \\
& +\frac{|A(\mathrm{t})||\upsilon_1|}{|\upsilon|\Gamma(1-\sigma)}\int_0^1(1-f)^{-\sigma} \\
& |\Upsilon(f,\gamma x(f),{}^{c}\mathscr{D}^{\sigma}x(f),\mathscr{I}^{\rho}x(f)) - \Upsilon(f,\gamma y(f),{}^{c}\mathscr{D}^{\sigma}y(f),\mathscr{I}^{\rho}y(f))|df \\
& +\frac{|A(\mathrm{t})|}{|\upsilon|\Gamma(p-\sigma)}\int_0^{\eta}(\eta-f)^{p-\sigma-1} \\
& |\Upsilon(f,\gamma x(f),{}^{c}\mathscr{D}^{\sigma}x(f),\mathscr{I}^{\rho}x(f)) - \Upsilon(f,\gamma y(f),{}^{c}\mathscr{D}^{\sigma}y(f),\mathscr{I}^{\rho}y(f))|df \\
& +\frac{|B(\mathrm{t})|}{|\upsilon|\Gamma(p-\sigma)}\sum_{\mathrm{k}=1}^{m}\int_0^{\eta_{\mathrm{k}}}(\eta_{\mathrm{k}}-f)^{p-\sigma-1} \\
& |\Upsilon(f,\gamma x(f),{}^{c}\mathscr{D}^{\sigma}x(f),\mathscr{I}^{\rho}x(f)) - \Upsilon(f,\gamma y(f),{}^{c}\mathscr{D}^{\sigma}y(f),\mathscr{I}^{\rho}y(f))|df \\
\leq\ & \Bigg[\frac{1}{\Gamma(p-\sigma+1)}+\frac{(|\upsilon_3|+m)|\upsilon_1|}{|\upsilon|\Gamma(2-\sigma)}+\frac{|\upsilon_3|+m}{|\upsilon|\Gamma(2-\sigma)}\eta^{p-\sigma} \\
& +\frac{|\upsilon_4-\delta_3\upsilon_3|+|\upsilon_2+\delta_3|}{|\upsilon|\Gamma(p-\sigma+1)}\sum_{\mathrm{k}=1}^{m}\eta_{\mathrm{k}}^{p-\sigma}\Bigg] \\
& \times\sup_{\mathrm{t}\in\chi}|\Upsilon(\mathrm{t},\gamma x(\mathrm{t}),{}^{c}\mathscr{D}^{\sigma}x(\mathrm{t}),\mathscr{I}^{\rho}x(\mathrm{t})) - \Upsilon(\mathrm{t},\gamma y(\mathrm{t}),{}^{c}\mathscr{D}^{\sigma}y(\mathrm{t}),\mathscr{I}^{\rho}y(\mathrm{t}))|.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
|{}^{c}\mathscr{D}^{\sigma}\mathfrak{J}x(\mathrm{t}) - {}^{c}\mathscr{D}^{\sigma}\mathfrak{J}y(\mathrm{t})| \leq\ & \ell_2\sup_{\mathrm{t}\in\chi}|\Upsilon(\mathrm{t},\gamma x(\mathrm{t}),{}^{c}\mathscr{D}^{\sigma}x(\mathrm{t}),\mathscr{I}^{\rho}x(\mathrm{t})) \\
& -\Upsilon(\mathrm{t},\gamma y(\mathrm{t}),{}^{c}\mathscr{D}^{\sigma}y(\mathrm{t}),\mathscr{I}^{\rho}y(\mathrm{t}))| \qquad (2.14)
\end{aligned}
$$

consequently, (2.13) and (2.14) implies that:

$$
\begin{aligned}
\mathbf{d}(\mathfrak{J}x,\mathfrak{J}y) &= \sup_{t\in\chi}|\mathfrak{J}x(t)-\mathfrak{J}y(t)|+\sup_{t\in\chi}|^{c}\mathscr{D}^{\sigma}\mathfrak{J}x(t)-{}^{c}\mathscr{D}^{\sigma}\mathfrak{J}y(t)| \\
&\leq \ell_1\sup_{t\in\chi}\left|\Upsilon(t,\gamma x(t),{}^{c}\mathscr{D}^{\sigma}x(t),\mathscr{I}^{\rho}x(t))-\Upsilon(t,\gamma y(t),{}^{c}\mathscr{D}^{\sigma}y(t),\mathscr{I}^{\rho}y(t))\right| \\
&\quad +\ell_2\sup_{t\in\chi}\left|\Upsilon(t,\gamma x(t),{}^{c}\mathscr{D}^{\sigma}x(t),\mathscr{I}^{\rho}x(t))-\Upsilon(t,\gamma y(t),{}^{c}\mathscr{D}^{\sigma}y(t),\mathscr{I}^{\rho}y(t))\right| \\
&\leq (\ell_1+\ell_2)\sup_{t\in\chi}\Big|\Upsilon(t,\gamma x(t),{}^{c}\mathscr{D}^{\sigma}x(t),\mathscr{I}^{\rho}x(t)) \\
&\quad -\Upsilon(t,\gamma y(t),{}^{c}\mathscr{D}^{\sigma}y(t),\mathscr{I}^{\rho}y(t))\Big| \\
&\leq (\ell_1+\ell_2)\sup_{t\in\chi}\Bigg[\frac{1}{\ell_1+\ell_2}\Phi(\gamma|x(t)-y(t)|+|^{c}\mathscr{D}^{\sigma}x(t)-{}^{c}\mathscr{D}^{\sigma}y(t)| \\
&\quad +|\mathscr{I}^{\rho}x(t)-\mathscr{I}^{\rho}y(t)|\Bigg] \\
&\leq \Phi\Bigg(\gamma\sup_{t\in\chi}|x(t)-y(t)|+\sup_{t\in\chi}|^{c}\mathscr{D}^{\sigma}x(t)-{}^{c}\mathscr{D}^{\sigma}y(t)| \\
&\quad +\sup_{t\in\chi}|\mathscr{I}^{\rho}x(t)-\mathscr{I}^{\rho}y(t)|\Bigg) \\
&\leq \Phi\Bigg(\gamma\sup_{t\in\chi}|x(t)-y(t)|+\sup_{t\in\chi}|^{c}\mathscr{D}^{\sigma}x(t)-{}^{c}\mathscr{D}^{\sigma}y(t)| \\
&\quad +\frac{1}{\Gamma(\rho)}\sup_{t\in\chi}\Big[\int_0^t(t-r)^{\rho-1}|x(t)-y(t)|dr\Big]\Bigg) \\
&\leq \Phi\Bigg(\Big[\gamma+\frac{1}{\Gamma(\rho+1)}\Big]\sup_{t\in\chi}|x(t)-y(t)|+\sup_{t\in\chi}|^{c}\mathscr{D}^{\sigma}x(t)-{}^{c}\mathscr{D}^{\sigma}y(t)|\Bigg) \\
&\leq \Phi\Bigg(\sup_{t\in\chi}|x(t)-y(t)|+\sup_{t\in\chi}|^{c}\mathscr{D}^{\sigma}x(t)-{}^{c}\mathscr{D}^{\sigma}y(t)|\Bigg) \\
&\leq \Phi(\mathbf{d}(x,y)). \qquad (2.15)
\end{aligned}
$$

Hence, we have $\mathbf{d}(\mathfrak{J}x,\mathfrak{J}y)\leq\mathbf{d}(x,y)$. According to the definition of ν, we obtain

$$\nu(x,y)\mathbf{d}(\mathfrak{J}x,\mathfrak{J}y)\leq\Phi(\mathbf{d}(x,y)),\quad \forall x,y\in\mathscr{B}.$$

It is therefore established that $\mathfrak{J}$ is a ν-Φ-contraction. Furthermore, by invoking the definition of ν in conjunction with condition ($\mathbb{AY}3$), we confirm that $\mathfrak{J}$ is ν-admissible.
Next, let $\{x_n\}\subset\mathscr{B}$ be a sequence that converges to a point $x\in\mathscr{B}$, and assume that $\nu(x_n,x_{n+1})\geq 1$ for all $n\in\mathbb{N}$, with the additional condition $\Omega(x_n(t),x_{n+1}(t))\geq 0$ holding for each $t\in\chi$. By applying condition ($\mathbb{AY}5$) along with the definition of ν, we directly conclude that $\nu(x_n,x)\geq 1$ for all $n\in\mathbb{N}$.
Accordingly, all the hypotheses of Theorem 2.2 are satisfied. As a result, the mapping $\mathfrak{J}$ possesses at least one fixed point, which in turn corresponds to a solution of the considered fractional boundary value problem (2.1).

Theorem 2.2.2

Consider the continuous function $\Upsilon : \chi \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ such that ($\mathbb{A}\mathbb{Y}5$) with the following assumption hold:

($\mathbb{A}\mathbb{Y}6$) There exists $\kappa > 0$ such that for all $t \in \chi$, and $x, y, \mathfrak{t}, \tilde{x}, \tilde{y}, \tilde{\mathfrak{t}} \in \mathbb{R}$

$$|\Upsilon(\mathfrak{t}, x, y, \mathfrak{t}) - \Upsilon(\mathfrak{t}, \tilde{x}, \tilde{y}, \tilde{\mathfrak{t}}) \leq \kappa(|x-\tilde{x}| + |y-\tilde{y}| + |\mathfrak{t}-\tilde{\mathfrak{t}}|),$$

and

$$\begin{aligned}\Sigma &= \kappa\Bigg[\frac{1}{\Gamma(p+1)} + \frac{(|\upsilon_3|+m)|\upsilon_1|}{|\upsilon|} + \frac{|\upsilon_3|+m}{|\upsilon|\Gamma(p+1)}\eta^p \\ &\quad + \frac{|\upsilon_4-\delta_3\upsilon_3|+|\upsilon_2+\delta_3|}{|\upsilon|\Gamma(p+1)}\sum_{k=1}^{m}\eta_k^p + \frac{1}{\Gamma(p-\sigma+1)} \\ &\quad + \frac{(|\upsilon_3|+m)|\upsilon_1|}{|\upsilon|\Gamma(2-\sigma)} + \frac{|\upsilon_3|+m}{|\upsilon|\Gamma(2-\sigma)}\eta^{p-\sigma} + \frac{|\upsilon_4-\delta_3\upsilon_3|+|\upsilon_2+\delta_3|}{|\upsilon|\Gamma(p-\sigma+1)}\sum_{k=1}^{m}\eta_k^{p-\sigma}\Bigg] < 1.\end{aligned}$$

Consequently, the fractional BVP (2.1) has a unique solution.

Proof

Under the assumption that all the hypotheses of Theorem 2.2 hold, and by applying the same approach as in the proof of Theorem 2.2, we derive

$$\begin{aligned}&|\mathfrak{J}x(\mathfrak{t}) - \mathfrak{J}y(\mathfrak{t})| \\ \leq\ & \left[\frac{1}{\Gamma(p+1)} + \frac{(|\upsilon_3|+m)|\upsilon_1|}{|\upsilon|} + \frac{|\upsilon_3|+m}{|\upsilon|\Gamma(p+1)}\eta^p + \frac{|\upsilon_4-\delta_3\upsilon_3|+|\upsilon_2+\delta_3|}{|\upsilon|\Gamma(p+1)}\sum_{k=1}^{m}\eta_k^p\right] \\ & \times \sup_{\mathfrak{t}\in\chi}|\Upsilon(\mathfrak{t}, \gamma x(\mathfrak{t}), {}^c\mathscr{D}^\sigma x(\mathfrak{t}), \mathscr{I}^\rho x(\mathfrak{t})) - \Upsilon(\mathfrak{t}, \gamma y(\mathfrak{t}), {}^c\mathscr{D}^\sigma y(\mathfrak{t}), \mathscr{I}^\rho y(\mathfrak{t}))| \\ \leq\ & \kappa\left[\frac{1}{\Gamma(p+1)} + \frac{(|\upsilon_3|+m)|\upsilon_1|}{|\upsilon|} + \frac{|\upsilon_3|+m}{|\upsilon|\Gamma(p+1)}\eta^p + \frac{|\upsilon_4-\delta_3\upsilon_3|+|\upsilon_2+\delta_3|}{|\upsilon|\Gamma(p+1)}\sum_{k=1}^{m}\eta_k^p\right] \\ & \times \sup_{\mathfrak{t}\in\chi}\left(\gamma|x(\mathfrak{t}) - y(\mathfrak{t})| + |\mathscr{D}^\sigma x(\mathfrak{t}) - {}^c\mathscr{D}^\sigma y(\mathfrak{t})| + |\mathscr{I}^\rho x(\mathfrak{t}) - \mathscr{I}^\rho y(\mathfrak{t})|\right) \\ \leq\ & \kappa\left[\frac{1}{\Gamma(p+1)} + \frac{(|\upsilon_3|+m)|\upsilon_1|}{|\upsilon|} + \frac{|\upsilon_3|+m}{|\upsilon|\Gamma(p+1)}\eta^p + \frac{|\upsilon_4-\delta_3\upsilon_3|+|\upsilon_2+\delta_3|}{|\upsilon|\Gamma(p+1)}\sum_{k=1}^{m}\eta_k^p\right] \\ & \times \left(\left[\gamma + \frac{1}{\Gamma(\rho+1)}\right]\sup_{\mathfrak{t}\in\chi}|x(\mathfrak{t}) - y(\mathfrak{t})| + \sup_{\mathfrak{t}\in\chi}|{}^c\mathscr{D}^\sigma x(\mathfrak{t}) - {}^c\mathscr{D}^\sigma y(\mathfrak{t})|\right) \\ \leq\ & \kappa\left[\frac{1}{\Gamma(p+1)} + \frac{(|\upsilon_3|+m)|\upsilon_1|}{|\upsilon|} + \frac{|\upsilon_3|+m}{|\upsilon|\Gamma(p+1)}\eta^p + \frac{|\upsilon_4-\delta_3\upsilon_3|+|\upsilon_2+\delta_3|}{|\upsilon|\Gamma(p+1)}\sum_{k=1}^{m}\eta_k^p\right] \\ & \times \left(\sup_{\mathfrak{t}\in\chi}|x(\mathfrak{t}) - y(\mathfrak{t})| + \sup_{\mathfrak{t}\in\chi}|{}^c\mathscr{D}^\sigma x(\mathfrak{t}) - {}^c\mathscr{D}^\sigma y(\mathfrak{t})|\right).\end{aligned} \tag{2.16}$$

Similarly, we obtain

$$\begin{aligned}
&|{}^c\mathscr{D}^\sigma \mathfrak{J}x(\mathrm{t}) - {}^c\mathscr{D}^\sigma \mathfrak{J}y(\mathrm{t})| \\
\leq\ & \left[\frac{1}{\Gamma(p-\sigma+1)} + \frac{(|\upsilon_3|+m)|\upsilon_1|}{|\upsilon|\Gamma(2-\sigma)} + \frac{|\upsilon_3|+m}{|\upsilon|\Gamma(2-\sigma)}\eta^{p-\sigma} \right. \\
&\left. + \frac{|\upsilon_4-\delta_3\upsilon_3|+|\upsilon_2+\delta_3|}{|\upsilon|\Gamma(p-\sigma+1)}\sum_{\mathrm{k}=1}^{m}\eta_{\mathrm{k}}^{p-\sigma}\right] \times \sup_{\mathrm{t}\in\chi}|\Upsilon(\mathrm{t},\gamma x(\mathrm{t}), {}^c\mathscr{D}^\sigma x(\mathrm{t}), \mathscr{I}^\rho x(\mathrm{t})) \\
&-\Upsilon(\mathrm{t},\gamma y(\mathrm{t}), {}^c\mathscr{D}^\sigma y(\mathrm{t}), \mathscr{I}^\rho y(\mathrm{t}))| \\
\leq\ & \kappa\left[\frac{1}{\Gamma(p-\sigma+1)} + \frac{(|\upsilon_3|+m)|\upsilon_1|}{|\upsilon|\Gamma(2-\sigma)} + \frac{|\upsilon_3|+m}{|\upsilon|\Gamma(2-\sigma)}\eta^{p-\sigma} \right. \\
&\left. + \frac{|\upsilon_4-\delta_3\upsilon_3|+|\upsilon_2+\delta_3|}{|\upsilon|\Gamma(p-\sigma+1)}\sum_{\mathrm{k}=1}^{m}\eta_{\mathrm{k}}^{p-\sigma}\right] \\
&\times\left(\sup_{\mathrm{t}\in\chi}|x(\mathrm{t})-y(\mathrm{t})| + \sup_{\mathrm{t}\in\chi}|{}^c\mathscr{D}^\sigma x(\mathrm{t}) - {}^c\mathscr{D}^\sigma y(\mathrm{t})|\right).
\end{aligned} \tag{2.17}$$

Therefore, from (2.16) and (2.17)

$$\begin{aligned}
\|\mathfrak{J}x-\mathfrak{J}y\|_{\mathscr{B}} \leq\ & \kappa\left[\frac{1}{\Gamma(p+1)} + \frac{(|\upsilon_3|+m)|\upsilon_1|}{|\upsilon|} + \frac{|\upsilon_3|+m}{|\upsilon|\Gamma(p+1)}\eta^{p} \right. \\
& + \frac{|\upsilon_4-\delta_3\upsilon_3|+|\upsilon_2+\delta_3|}{|\upsilon|\Gamma(p+1)}\sum_{\mathrm{k}=1}^{m}\eta_{\mathrm{k}}^{p} + \frac{1}{\Gamma(p-\sigma+1)} + \frac{(|\upsilon_3|+m)|\upsilon_1|}{|\upsilon|\Gamma(2-\sigma)} \\
&\left. + \frac{|\upsilon_3|+m}{|\upsilon|\Gamma(2-\sigma)}\eta^{p-\sigma} + \frac{|\upsilon_4-\delta_3\upsilon_3|+|\upsilon_2+\delta_3|}{|\upsilon|\Gamma(p-\sigma+1)}\sum_{\mathrm{k}=1}^{m}\eta_{\mathrm{k}}^{p-\sigma}\right]\|x-y\|_{\mathscr{B}}.
\end{aligned} \tag{2.18}$$

The inequality (2.18) implies that

$$\|\mathfrak{J}x-\mathfrak{J}y\|_{\mathscr{B}} \leq \Sigma\|x-y\|_{\mathscr{B}},$$

for all $x, y \in \mathscr{B}$. Since $\Sigma < 1$, it follows from Banach's contraction principle that $\mathfrak{J}$ has a unique fixed point, which corresponds to the unique solution of our fractional BVP (2.1).

2.3 A Bernstein Polynomial Approach to Approximate Solutions

First, we provide a general overview of Bernstein basis polynomials, which serve as fundamental tools in deriving our desired results.

Definition 2.3.1: [43]

The $(n+1)$ basis polynomials of Bernstein of degree n are expressed on $[0,1]$ as

$$\mathfrak{b}_{j,n}(\mathfrak{s}) = \binom{n}{j}\mathfrak{s}^j(1-\mathfrak{s})^{n-j}, \quad 0 \le j \le n,$$

with

$$\binom{n}{j} = \frac{n!}{j!(n-j)!},$$

and can be expressed on $[0,1]$ using the following recursive relation:

$$\mathfrak{b}_{j,n}(\mathfrak{s}) = (1-\mathfrak{s})\mathfrak{b}_{j,n-1}(\mathfrak{s}) + \mathfrak{s}\mathfrak{b}_{j-1,n-1}(\mathfrak{s}).$$

It is straightforward to prove that each Bernstein basis polynomial is positive and satisfies the partition of unity property:

$$\sum_{j=0}^{n} \mathfrak{b}_{j,n}(\mathfrak{s}) = 1, \quad \text{for all } \mathfrak{s} \in [0,1].$$

Furthermore, any polynomial of degree n can be expressed as a linear combination of these basis functions.

$$\phi(\mathfrak{s}) = \sum_{j=0}^{n} c_j \mathfrak{b}_{j,n}(\mathfrak{s}) = \mathbb{C}^{\mathscr{T}}\mathbb{B}(\mathfrak{s}), \quad n \ge 1, \tag{2.19}$$

such that the Bernstein vector $\mathbb{B}(\mathfrak{s})$ and the coefficient vector $\mathbb{C}$ are given by:

$$\mathbb{B}(\mathfrak{s}) = \left[\mathfrak{b}_{0,n}(\mathfrak{s}), \mathfrak{b}_{1,n}(\mathfrak{s}), \ldots, \mathfrak{b}_{n,n}(\mathfrak{s})\right],$$

and $\mathbb{C}^{\mathscr{T}} = \left[c_0, c_1, \ldots, c_n\right]$ with

$$c_j = \int_0^1 \phi(\mathfrak{s})\mathrm{d}_{j,n}(\mathfrak{s})\mathrm{d}\mathfrak{s}, \quad 0 \le j \le n, \tag{2.20}$$

where $\mathrm{d}_{j,n}(\mathfrak{s})$ is expressed as

$$\mathrm{d}_{j,n}(\mathfrak{s}) = \sum_{\mathrm{k}=0}^{n} \zeta_{j,\mathrm{k}} \mathfrak{b}_{\mathrm{k},n}(\mathfrak{s}), \quad 0 \le j \le n, \tag{2.21}$$

see [12]. Also, for $0 \le j, \mathrm{k} \le n$, we have

$$\zeta_{j,\mathrm{k}} = \frac{(-1)^{j+\mathrm{k}}}{\binom{n}{j}\binom{n}{\mathrm{k}}} \sum_{i=0}^{\min(j,\mathrm{k})} (2i+1)\binom{n+i+1}{n-j}\binom{n-i}{n-j}\binom{n+i+1}{n-\mathrm{k}}\binom{n-i}{n-\mathrm{k}}. \tag{2.22}$$

In addition,

$$\mathscr{I}^f\mathbb{B}(\mathfrak{s}) \simeq \mathrm{I}^{(f)}\mathbb{B}(\mathfrak{s}), \quad f > 0, \tag{2.23}$$

where $\mathscr{I}^{f}$ represents the Riemann–Liouville fractional integral operator of order f, and $\mathrm{I}^{(f)}$ denotes its associated $(n+1)\times(n+1)$ operational matrix of the fractional R-L integral of order f.

By exploiting the binomial identity $(1-\mathfrak{s})^{n-j}=\sum_{\mathrm{k}=0}^{n-j}(-1)^{\mathrm{k}}\binom{n-j}{\mathrm{k}}\mathfrak{s}^{\mathrm{k}}$, we get

$$\begin{aligned}\mathfrak{b}_{j,n}(\mathfrak{s}) &= \binom{n}{j}\mathfrak{s}^{j}(1-\mathfrak{s})^{n-j}\\ &= \binom{n}{j}\mathfrak{s}^{j}\left(\sum_{\mathrm{k}=0}^{n-j}(-1)^{\mathrm{k}}\binom{n-j}{\mathrm{k}}\mathfrak{s}^{\mathrm{k}}\right)\\ &= \sum_{\mathrm{k}=0}^{n-j}(-1)^{\mathrm{k}}\binom{n}{j}\binom{n-j}{\mathrm{k}}\mathfrak{s}^{j+\mathrm{k}}\\ &= \sum_{i=j}^{n}(-1)^{i-j}\binom{n}{j}\binom{n-j}{i-j}\mathfrak{s}^{i}, \quad 0\le j\le n. \end{aligned} \tag{2.24}$$

Moreover, as stated in [12], we find that:

$$\int_0^1 \mathfrak{b}_{j,n}(\mathfrak{s})\mathrm{d}_{\mathrm{k},n}(\mathfrak{s})\mathrm{d}\mathfrak{s}=\delta_{j\mathrm{k}}, \quad 0\le j,\mathrm{k}\le n, \tag{2.25}$$

where $\{\mathrm{d}_{0,n}(\mathfrak{s}),\mathrm{d}_{1,n}(\mathfrak{s}),\ldots,\mathrm{d}_{n,n}(\mathfrak{s})\}$ represents the dual in basis of Bernstein on the interval $[0,1]$ and $\delta_{j\mathrm{k}}$ stands the symbol of Kronecker.

2.3.1 Matrix Representation of Integration Fractional

Theorem 2.3.1

Consider the Bernstein vector $\mathbb{B}(\mathfrak{s})$ given in (2.23) and the $(n+1)\times(n+1)$ operational matrix of fractional Riemann-Liouville integration $\mathrm{I}^{(f)}$ of order f, defined as follows:

$$\mathrm{I}^{(f)}=\begin{pmatrix} \Delta_{0,0} & \Delta_{0,1} & \cdots & \Delta_{0,j} & \cdots & \Delta_{0,n}\\ \Delta_{1,0} & \Delta_{1,1} & \cdots & \Delta_{1,j} & \cdots & \Delta_{1,n}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \Delta_{i,0} & \Delta_{i,1} & \cdots & \Delta_{i,j} & \cdots & \Delta_{i,n}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \Delta_{n,0} & \Delta_{n,1} & \cdots & \Delta_{n,j} & \cdots & \Delta_{n,n} \end{pmatrix}. \tag{2.26}$$

Hence, we have

$$\Delta_{i,j}=\sum_{q=i}^{n}\sum_{\mathrm{k}=0}^{n}(-1)^{q-i}\binom{n}{i}\binom{n-i}{q-i}\zeta_{j\mathrm{k}}\tau_{q\mathrm{k}}\frac{\Gamma(q+1)}{\Gamma(q+f+1)}, \tag{2.27}$$

with

$$\tau_{q\mathrm{k}} = \sum_{p=\mathrm{k}}^{n} \frac{(-1)^{p-\mathrm{k}}}{q+p+f+1} \binom{n}{\mathrm{k}} \binom{n-\mathrm{k}}{p-\mathrm{k}},$$

and $\zeta_{j\mathrm{k}}$ are defined by (2.22).

Proof

Taking into account the expressions (2.23) and (2.25), we obtain the following equality:

$$\mathrm{I}^{(f)} = \langle \mathscr{I}^f \mathbb{B}(\mathfrak{s}), D^{\mathscr{T}} \rangle. \tag{2.28}$$

In the expression (2.28), D represents the vector of dual polynomials corresponding to the Bernstein polynomials, and $\langle \mathscr{I}^f \mathbb{B}(\mathfrak{s}), D^{\mathscr{T}} \rangle$ denotes the $(n+1)\times(n+1)$-matrix expressed as

$$\langle \mathscr{I}^f \mathbb{B}(\mathfrak{s}), D^{\mathscr{T}} \rangle = \langle \mathscr{I}^f \mathfrak{b}_{i,n}(\mathfrak{s}), \mathrm{d}_{j,n}(\mathfrak{s}) \rangle, \quad 0 \leq i, j \leq n.$$

Here,

$$\varDelta_{ij} = \langle \mathscr{I}^f \mathfrak{b}_{i,n}(\mathfrak{s}), \mathrm{d}_{j,n}(\mathfrak{s}) \rangle = \int_0^1 \mathscr{I}^f \mathfrak{b}_{i,n}(\mathfrak{s}) \mathrm{d}_{j,n}(\mathfrak{s}) \mathrm{d}\mathfrak{s}.$$

By utilizing equation (2.24), we obtain:

$$\mathscr{I}^f \mathfrak{b}_{i,n}(\mathfrak{s}) = \sum_{q=i}^{n} (-1)^{q-i} \binom{n}{i} \binom{n-i}{q-i} \frac{\Gamma(q+1)}{\Gamma(q+f+1)}.$$

Therefore, from equation (2.21), we deduce that:

$$\varDelta_{ij} = \sum_{q=i}^{n} \sum_{\mathrm{k}=0}^{n} (-1)^{q-i} \binom{n}{i} \binom{n-i}{q-i} \zeta_{j\mathrm{k}} \frac{\Gamma(q+1)}{\Gamma(q+f+1)} \int_0^1 \mathfrak{s}^{q+f} \mathfrak{b}_{\mathrm{k},n}(\mathfrak{s}) \mathrm{d}\mathfrak{s}. \tag{2.29}$$

Since,

$$\int_0^1 \mathfrak{s}^{q+f} \mathfrak{b}_{\mathrm{k},n}(\mathfrak{s}) \mathrm{d}\mathfrak{s} = \sum_{p=\mathrm{k}}^{n} \frac{(-1)^{p-\mathrm{k}}}{q+p+f+1} \binom{n}{\mathrm{k}} \binom{n-\mathrm{k}}{p-\mathrm{k}} = \tau_{q\mathrm{k}}. \tag{2.30}$$

Thus, by combining equations (2.29) and (2.30), we conclude the proof of Theorem 2.3.1.

2.3.2 Matrix Representation of Fractional Derivatives

The derivative of $\mathbb{B}(\mathfrak{s})$ can be expressed as:

$$\frac{\mathrm{d}\mathbb{B}(\mathfrak{s})}{\mathrm{d}\mathfrak{s}} = \mathbb{D}^{(1)} \mathbb{B}(\mathfrak{s}), \tag{2.31}$$

where $\mathbb{D}^{(1)}$ represents the $(n+1)\times(n+1)$ derivative operational matrix defined as follows:

$$\mathbb{D}^{(1)} = \mathbb{A}\mathscr{W}\mathbb{B}^{\star}. \tag{2.32}$$

For further details, we refer to [9, 10, 35].

By utilizing equation (2.31), it is evident that for any given value: $\mathfrak{m}\in\mathbb{N}$

$$\frac{\mathrm{d}^{\mathfrak{m}}\mathbb{B}(\mathfrak{s})}{\mathrm{d}\mathfrak{s}^{\mathfrak{m}}} = \left(\mathbb{D}^{(1)}\right)^{\mathfrak{m}}\mathbb{B}(\mathfrak{s}).$$

Thus,

$$\mathbb{D}^{(\mathfrak{m})} = \left(\mathbb{D}^{(1)}\right)^{\mathfrak{m}}, \quad \mathfrak{m} = 1, 2, \ldots$$

To extend the derivative operational matrix to a non-integer order, we introduce the following formulas:

For $f > 0$, we have

$${}^{c}\mathscr{D}^{f}\mathbb{B}(\mathfrak{s}) \simeq {}^{c}\mathbb{D}^{(f)}\mathbb{B}(\mathfrak{s}) \tag{2.33}$$

where ${}^{c}\mathbb{D}^{(f)}$ represents the $(n+1)\times(n+1)$ operational matrix of the f^{th}-order Caputo derivative, defined as follows:

$${}^{c}\mathbb{D}^{(f)} = \begin{pmatrix} \sum_{j=\lceil f\rceil}^{n}\Delta_{0,j,0} & \sum_{j=\lceil f\rceil}^{n}\Delta_{0,j,1} & \cdots & \sum_{j=\lceil f\rceil}^{n}\Delta_{0,j,2} & \cdots & \sum_{j=\lceil f\rceil}^{n}\Delta_{0,j,n} \\ \sum_{j=\lceil f\rceil}^{n}\Delta_{1,j,0} & \sum_{j=\lceil f\rceil}^{n}\Delta_{1,j,1} & \cdots & \sum_{j=\lceil f\rceil}^{n}\Delta_{1,j,2} & \cdots & \sum_{j=\lceil f\rceil}^{n}\Delta_{1,j,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{j=\lceil f\rceil}^{n}\Delta_{i,j,0} & \sum_{j=\lceil f\rceil}^{n}\Delta_{i,j,1} & \cdots & \sum_{j=\lceil f\rceil}^{n}\Delta_{i,j,2} & \cdots & \sum_{j=\lceil f\rceil}^{n}\Delta_{i,j,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{j=\lceil f\rceil}^{n}\Delta_{n,j,0} & \sum_{j=\lceil f\rceil}^{n}\Delta_{n,j,1} & \cdots & \sum_{j=\lceil f\rceil}^{n}\Delta_{n,j,2} & \cdots & \sum_{j=\lceil f\rceil}^{n}\Delta_{n,j,n} \end{pmatrix}, \tag{2.34}$$

with

$$\Delta_{i,j,p} = (-1)^{j-i}\binom{n}{i}\binom{n-i}{j-i}\frac{\Gamma(j+1)}{\Gamma(j+1-f)}\sum_{\mathrm{k}=0}^{n}\zeta_{p\mathrm{k}}\tau_{\mathrm{k}j},$$

$\zeta_{p\mathrm{k}}$ is given by (2.22) and

$$\tau_{\mathrm{k}j} = \sum_{s=\mathrm{k}}^{n}\frac{(-1)^{s-\mathrm{k}}}{j-f+s+1}\binom{n-\mathrm{k}}{s-\mathrm{k}}.$$

2.4 Numerical Simulation Through Selected Example

First, we begin by introducing and applying Bernstein's collocation method to our fractional boundary value problem (BVP) (2.1) to obtain an accurate numerical solution. Next, we validate our theoretical findings by presenting numerical examples. To provide further clarification, we consider the following fractional BVP (2.35)-(2.36) for any $\mathfrak{t}\in\chi$.

$$
\begin{aligned}
{}^{c}\mathscr{D}^{p}\vartheta(t) &= \Upsilon(t,\gamma\vartheta(t),{}^{c}\mathscr{D}^{\sigma}\vartheta(t),\mathscr{I}^{\rho}\vartheta(t)), &(2.35)\\
{}^{c}\mathscr{D}^{1}\vartheta(0) &= \delta_1\sum_{k=1}^{m}\vartheta(\eta_k), \quad \delta_2^{c}\mathscr{D}^{p-1}\vartheta(1)+\vartheta(\eta)=\delta_3\sum_{k=1}^{m}\vartheta(\eta_k). &(2.36)
\end{aligned}
$$

To derive an approximate solution for the exact solution $\vartheta(t)$ using the Bernstein polynomials technique, we employ the fractional Riemann-Liouville integral of Bernstein polynomials in conjunction with the operational matrix of the Caputo derivative. For further details, refer to [9].

As shown in (2.19), the approximate solution of the exact solution $\vartheta(t)$ using Bernstein polynomials is given by:

$$\vartheta(t)\approx\phi(t)=\sum_{j=0}^{n}c_j\mathfrak{b}_{j,n}(t)=\mathbb{C}^{\mathscr{T}}\mathbb{B}(t), \quad n\geq 1, \tag{2.37}$$

where $\mathbb{C}$ represents an unknown vector. By substituting the approximate solution given by (2.37) into equations (2.35)-(2.36), we immediately obtain the following system of equations:

$$\mathbb{C}^{\mathscr{T}}\,{}^{c}\mathbb{D}^{(p)}\mathbb{B}(t)=\Upsilon\Big(t,\gamma\mathbb{C}^{\mathscr{T}}\mathbb{B}(t),\mathbb{C}^{\mathscr{T}}\,{}^{c}\mathbb{D}^{(\sigma)}\mathbb{B}(t),\mathbb{C}^{\mathscr{T}}\mathrm{I}^{(\rho)}\mathbb{B}(t)\Big), \tag{2.38}$$

$$-c_0\mathrm{m}+c_1\mathrm{m}=\mathbb{C}^{\mathscr{T}}\sum_{k=1}^{m}\mathbb{B}(\eta_k), \tag{2.39}$$

$$\delta_2\mathbb{C}^{\mathscr{T}}\,{}^{c}\mathbb{D}^{(p-1)}\mathbb{B}(1)+\mathbb{C}^{\mathscr{T}}\mathbb{B}(\eta)=\delta_3\mathbb{C}^{\mathscr{T}}\sum_{k=1}^{m}\mathbb{B}(\eta_k). \tag{2.40}$$

To simplify the calculations, we use the roots of the Legendre polynomials of degree $(m-1)$ within the interval $[0,1]$, along with equidistant points (nodes). Consequently, to compute an approximate solution for $\vartheta(t)$, we apply the collocation method by evaluating equation (2.38) at $(m-1)$ points while incorporating conditions (2.39) and (2.40). As a result, we obtain a system of $(m+1)$ equations with $(m+1)$ unknown coefficients. This system can then be solved efficiently to determine an approximate solution for $\vartheta(t)$.

Now, we introduce an illustrative example by implementing Bernstein's collocation method to approximate the solution of the given fractional boundary value problem. This approach enables us to assess the accuracy and efficiency of the method in solving such problems.

Example 2.4.1

Here, we are interested in the following FBVP

$$\begin{cases} {}^{c}\mathscr{D}^{\frac{5}{3}}\vartheta(t)=\xi(t)+\frac{1}{5}\vartheta(t)+\frac{3}{2}{}^{c}\mathscr{D}^{\frac{1}{3}}\vartheta(t)-\frac{1}{2}\mathscr{I}^{\frac{9}{4}}\vartheta(t), \quad t\in\chi,\\ {}^{c}\mathscr{D}^{1}\vartheta(0)=0, \quad 81\Gamma\Big(\dfrac{7}{3}\Big)^{c}\mathscr{D}^{\frac{2}{3}}\vartheta(1)+100\vartheta\Big(\dfrac{9}{10}\Big)=3\displaystyle\sum_{k=1}^{3}\vartheta\Big(\dfrac{k}{k+1}\Big),\end{cases} \tag{2.41}$$

where

$$\xi(\mathfrak{t}) = \frac{1}{\Gamma\left(\frac{21}{4}\right)}\mathfrak{t}^{\frac{17}{4}} - \frac{1}{5}\mathfrak{t}^2 - \frac{3}{\Gamma\left(\frac{8}{3}\right)}\mathfrak{t}^{\frac{5}{3}} + \frac{2}{\Gamma\left(\frac{4}{3}\right)}\mathfrak{t}^{\frac{1}{3}}.$$

In this example, we have $\delta_1 = 0,\ \delta_2 = \frac{81\Gamma(\frac{7}{3})}{100},\ \delta_3 = \frac{3}{100},\ \eta_k = \frac{k}{k+1},\ \ k \in \{1,2,3\},\ \eta = \frac{9}{10}$.

It is worth noting that the exact solution of the given problem in this example is $\vartheta(\mathfrak{t}) = \mathfrak{t}^2$. By setting $\Omega(x,y) = 1$ for all $(x,y) \in \mathscr{B}^2$ and $\Phi(\mathfrak{t}) = \mathfrak{t}$ for each $\mathfrak{t} \in \chi$, we can readily confirm that the fractional boundary value problem (BVP) (2.41) satisfies all the assumptions of Theorem 2.2.

If we substitute $\vartheta(\mathfrak{t})$ with $\mathbb{C}^{\mathscr{T}}\mathbb{B}(\mathfrak{t})$ in the fractional boundary value problem (BVP) (2.41), we derive the following system:

$$\begin{cases} \mathbb{C}^{\mathscr{T}}\, {}^{c}\mathbb{D}^{(\frac{5}{3})}\mathbb{B}(\mathfrak{t}) = \xi(\mathfrak{t}) + \frac{1}{5}\mathbb{C}^{\mathscr{T}}\mathbb{B}(\mathfrak{t}) + \frac{3}{2}\mathbb{C}^{\mathscr{T}}\, {}^{c}\mathbb{D}^{(\frac{1}{3})}\mathbb{B}(\mathfrak{t}) - \frac{1}{2}\mathbb{C}^{\mathscr{T}}\mathrm{I}^{(\frac{9}{4})}\mathbb{B}(\mathfrak{t}), \\ c_1 - c_0 = 0, \quad\ 81\Gamma\left(\dfrac{7}{3}\right)\mathbb{C}^{\mathscr{T}}\, {}^{c}\mathbb{D}^{(\frac{2}{3})}\mathbb{B}(1) + 100\mathbb{C}^{\mathscr{T}}\mathbb{B}\left(\dfrac{9}{4}\right) = 3\mathbb{C}^{\mathscr{T}}\displaystyle\sum_{k=1}^{3}\mathbb{B}\left(\dfrac{k}{k+1}\right). \end{cases}$$

At the conclusion of this example, we present in Table 1 the absolute errors $|\vartheta_5(\mathfrak{t}) - \vartheta(\mathfrak{t})|$, where the approximate solution $\vartheta_5(\mathfrak{t})$ is obtained using the roots of the shifted Legendre polynomials. Additionally, in Figure 2.1, we illustrate the graph of the exact solution of problem (2.41) alongside its approximate solution for visual comparison.

t	Approximate solution $\vartheta_5(t)$	Exact solution $\vartheta(t)$	Absolute error
0	0.000012	0.000000	0.0120×10^{-3}
0.1	0.013000	0.010000	3.0000×10^{-3}
0.2	0.036000	0.040000	4.0000×10^{-3}
0.3	0.094000	0.090000	4.0000×10^{-3}
0.4	0.158000	0.160000	2.0000×10^{-3}
0.5	0.247000	0.250000	3.0000×10^{-3}
0.6	0.358000	0.360000	2.0000×10^{-3}
0.7	0.486000	0.490000	4.0000×10^{-3}
0.8	0.643110	0.640000	3.1100×10^{-3}
0.9	0.813200	0.810000	3.2000×10^{-3}
1	1.000011	1.000000	0.0110×10^{-3}

Table 2.1: **Comparison of Approximate and Exact Solutions with Absolute Errors at Selected Points of t.**

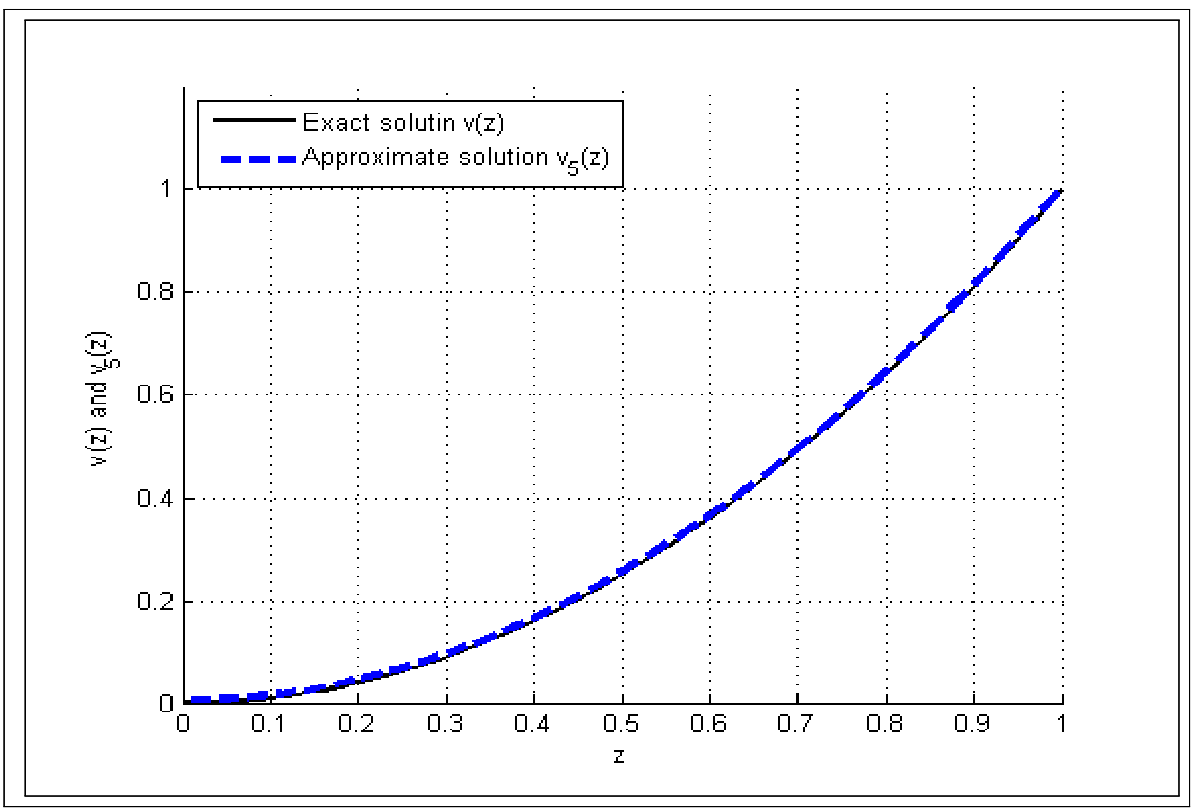

Figure 2.1: Contrast between analytical and numerical results $\vartheta_5(t)$.

Chapter three

Solving a non-integer order differential equations by using operational matrix and shifted Legendre polynomials

Contents

3.1 Introduction **48**

3.2 Legendre's polynomial **48**

3.2.1 Rodrigue's formula of Legendre polynomials 48

3.2.2 Generated function of Legendre polynomials 49

3.3 Properties of Legendre polynomials **50**

3.3.1 Orthogonality of Legendre polynomials 50

3.3.2 Recurrence relation 52

3.4 Operational matrix of Legendre **57**

3.4.1 Shifted Legendre polynomiale 57

3.4.2 Function approximation using shifted Legendre polynomials 59

3.4.3 Operational matrix of $\mathscr{D}^q\Phi(t)$ based on shifted Legendre polynomials: 59

3.5 Shifted Legendre operational matrix of multi-order fractional differential equations: **63**

3.5.1 Case of a linear problem: 63

3.5.2 Case of a nonlinear problem: 65

3.1 Introduction

This chapter introduces Solving of non-integer order differential equations by using operational matrix and shifted Legendre polynomials. It details the fundamental properties of Legendre polynomials, such as orthogonality and recurrence relations, and utilizes these properties in constructing solutions Finally, some illustrative examples are given.

3.2 Legendre's polynomial

The simplest orthogonal polynomials are the Legendre polynomials for which the orthogonality interval is $[-1,1]$ and the weight function is simply the constant function of value 1. For any natural number n, the Legendre polynomial $\mathbb{P}_n$ is defined by:

$$\mathbb{P}_n = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \frac{(2n-2k)!}{2^n n!(n-k)!(n-2k)!} t^{n-2k}$$

$\mathbb{P}_n$ is the n^{th} derivative of the polynomial $(x^2-1)^n$ which is a polynomial of degree $2n$ with coefficient dominant equal to 1, therefore,$\mathbb{P}_n$ is a polynomial of degree n with a dominant coefficient

$$2n(2n-1)...(2n-n+1) = \frac{(2n)!}{n}$$

Also Legendre polynomials are polynomial solutions over the interval $[-1,1]$, of the homogeneous linear differential equation of order 2 with variable coefficients called the Legendre differential equation. This equation is given by:

$$\left(1-t^2\right)\frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + n(n+1)y = 0.$$

3.2.1 Rodrigue's formula of Legendre polynomials

Proposition 3.2.1

The formula of Rodrigue is given by:

$$\mathbb{P}_n(t) = \frac{1}{n!2^n}\frac{d^n}{dt^n}\left(t^2-1\right)^n. \tag{3.1}$$

Proof

If $k \leq n$, we have

$$\frac{d^k}{dt^k}t^n = \frac{n!}{n-k}t^{n-k}$$

. According to the formula of Newtons binomial, we have

$$\left(t^2-1\right)^n=\sum_{k=0}^{n}\frac{n!}{(n-k)!k!}(-1)^k\frac{d^n}{dt^n}t^{2n-2k}$$

thus

$$\frac{1}{2^n n!}\frac{d^n}{dt^n}\left(t^2-1\right)^n=\frac{1}{2^n n!}\sum_{k=0}^{n}\frac{n!}{(n-k)!k!}(-1)^k\frac{d^n}{dt^n}t^{2n-2k}.$$

but

$$\frac{d^n}{dt^n}t^{2\left(n-k\right)}=0 \text{ if } 2\left(n-k\right)<n, \text{ i.e., if } k>\frac{n}{2}.$$

So we can replace $\sum_{k=0}^{n}$ by $\sum_{k=0}^{[\frac{n}{2}]}$.
if $k\leq\frac{n}{2}$ it was

$$\begin{aligned}\frac{d^n}{dt^n}t^{2(n-k))} &= \left[2(n-k)\right]\left[2(n-k)-1\right]\left[2(n-k)-2\right]...\left[2(n-k)-n+1\right]t^{2(n-k)-n}\\ &= \frac{(2n-2k)!}{(n-2k)!}t^{n-2k}.\end{aligned}$$

thus

$$\begin{aligned}\frac{1}{2^n n!}\frac{d^n}{dt^n}\left(t^2-1\right)^n &= \sum_{k=0}^{[\frac{n}{2}]}(-1)^k\frac{(2n-2k)!}{2^n n!(n-k)!(n-2k)!}t^{n-2k}\\ &= \mathbb{P}_n(t).\end{aligned}$$

3.2.2 Generated function of Legendre polynomials

Proposition 3.2.2

The generated function of Legendre polynomials is given by:

$$\left(1-2tf+f^2\right)^{-\frac{1}{2}}=\frac{1}{\sqrt{1-2tt+f^2}}=\sum_{n=0}^{+\infty}\mathbb{P}_n(t)f^n$$

3.3 Properties of Legendre polynomials

The Legendre polynomial have the following properties:

3.3.1 Orthogonality of Legendre polynomials

Theorem 3.3.1

Legendre polynomials are orthogonal to the interval $[-1,1]$ with a uniform weight. If $m, \mathrm{n} \in \mathbb{N}^*$,then

$$\int_{-1}^{1} \mathbb{P}_m(\mathrm{t})\mathbb{P}_{\mathrm{n}}(\mathrm{t})dt = \begin{cases} 0, & if \quad m \neq \mathrm{n}; \\ \dfrac{2}{2m+1}, & if \quad m = \mathrm{n}. \end{cases}$$

Proof

. It is shown that the Legendre polynomials are orthogonal to the scalar product $\langle \mathbb{P}_m(\mathrm{t}), \mathbb{P}_{\mathrm{n}}(\mathrm{t})\rangle = \int_{-1}^{1} \mathbb{P}_m(\mathrm{t})\mathbb{P}_{\mathrm{n}}(\mathrm{t})dt$
that is:

$$\int_{-1}^{1} \mathbb{P}_m(\mathrm{t})\mathbb{P}_n(\mathrm{t})dt = 0, \quad if \quad m \neq \mathrm{n} \tag{3.2}$$

To show (3.2) we write the Legendre equation

$$(1-\mathrm{t}^2)y" - 2\mathrm{t}y' + \mathrm{n}(\mathrm{n}+1)y = 0. \quad for \quad y = P_{\mathrm{n}}(\mathrm{t}),$$

in the form

$$\frac{d}{dt}\left[(1-\mathrm{t}^2)\mathbb{P}'_{\mathrm{n}}(\mathrm{t})\right] + \mathrm{n}(\mathrm{n}+1)\mathbb{P}_{\mathrm{n}}(\mathrm{t}) = 0 \tag{3.3}$$

Writing (3.3) for $\mathbb{P}_m(\mathrm{t})$:

$$\frac{d}{dt}\left[(1-\mathrm{t}^2)\mathbb{P}'_m(\mathrm{t})\right] + m(m+1)\mathbb{P}_m(\mathrm{t}) = 0 \tag{3.4}$$

By multiplying (3.3) by $\mathbb{P}_m(\mathrm{t})$ and (3.4) by $P_{\mathrm{n}}(\mathrm{t})$ and then subtracting we obtain:

$$\begin{aligned} &\frac{d}{dt}\left[(1-\mathrm{t}^2)\mathbb{P}'_{\mathrm{n}}(\mathrm{t})\right]\mathbb{P}_m(\mathrm{t}) + \mathrm{n}(\mathrm{n}+1)\mathbb{P}_{\mathrm{n}}(\mathrm{t})\mathbb{P}_m(\mathrm{t}) \\ &-\frac{d}{dt}\left[(1-\mathrm{t}^2)\mathbb{P}'_m(\mathrm{t})\right]\mathbb{P}_{\mathrm{n}}(\mathrm{t}) - m(m+1)\mathbb{P}_m(\mathrm{t})\mathbb{P}_{\mathrm{n}}(\mathrm{t}) = 0, \\ &\frac{d}{dt}\left[(1-\mathrm{t}^2)\mathbb{P}'_{\mathrm{n}}(\mathrm{t})\right]\mathbb{P}_m(\mathrm{t}) - \frac{d}{dt}\left[(1-\mathrm{t}^2)\mathbb{P}'_m(\mathrm{t})\right]\mathbb{P}_{\mathrm{n}}(\mathrm{t}) \\ &+\left[\mathrm{n}(\mathrm{n}+1) - m(m+1)\right]\mathbb{P}_{\mathrm{n}}(\mathrm{t})\mathbb{P}_m(\mathrm{t}) = 0. \end{aligned}$$

Like

$$\frac{d}{dt}\left((1-t^2)\Big[\mathbb{P}_m(t)\mathbb{P}'_n(t)-\mathbb{P}_n(t)\mathbb{P}'_m(t)\Big]\right)=\frac{d}{dt}\left(\mathbb{P}_m(t)(1-t^2)\mathbb{P}'_n-\mathbb{P}_n(t)(1-t^2)\mathbb{P}'_m\right)$$
$$=\mathbb{P}'_m(t)(1-t^2)\mathbb{P}'_n+\mathbb{P}_m(t)\frac{d}{dt}\left((1-t^2)\mathbb{P}'_n(t)\right)$$
$$-\mathbb{P}'_n(t)(1-t^2)\mathbb{P}'_m+\mathbb{P}_n(t)\frac{d}{dt}\left((1-t^2)\mathbb{P}'_m(t)\right),$$

$$\frac{d}{dt}(1-t^2)\Big[\mathbb{P}_m(t)\mathbb{P}'_n(t)-\mathbb{P}_n(t)\mathbb{P}'_m(t)\Big]=\Big[n(n+1)-m(m+1)\Big]\mathbb{P}_n(t)\mathbb{P}_m(t).$$

Integrating the two sides with respect to t between -1 and 1, we obtain

$$\Big[(1-t^2)\mathbb{P}_m(t)\mathbb{P}'_n(t)\Big]_{-1}^{1}=\Big[n(n+1)-m(m+1)\Big]\int_{-1}^{1}\mathbb{P}_n(t)\mathbb{P}_m(t)$$
$$0=\Big[n(n+1)-m(m+1)\Big]\int_{-1}^{1}\mathbb{P}_n(t)\mathbb{P}_m(t)$$

As $m\neq n$, then $\int_{-1}^{1}\mathbb{P}_n(t)\mathbb{P}_m(t)=0$.
And to demonstrate

$$\int_{-1}^{1}\mathbb{P}_n(t)\mathbb{P}_m(t)=\frac{2}{2n+1}$$

Using the generating function, we have

$$\left(1-2tf+t^2\right)^{-\frac{1}{2}}=\sum_{n=0}^{+\infty}\mathbb{P}_n(t)f^n, \tag{3.5}$$

$$\left(1-2tf+t^2\right)^{-\frac{1}{2}}=\sum_{m=0}^{+\infty}\mathbb{P}_m(t)f^m, \tag{3.6}$$

Multiplying equation (3.5) by equation (3.6) gives

$$\begin{aligned}\frac{1}{1-2tf+t^2}&=\sum_{n=0}^{+\infty}\mathbb{P}_n(t)f^n\sum_{m=0}^{+\infty}\mathbb{P}_m(t)f^m\\&=\sum_{n=0,m=0}^{+\infty}\mathbb{P}_n(t)\mathbb{P}_m(t)f^{n+m}.\end{aligned}\tag{3.7}$$

Integrating the two sides of equation (3.7) with respect to t from -1 to 1, we obtain

$$\sum_{n=0,m=0}^{+\infty}\int_{-1}^{1}\mathbb{P}_n(t)\mathbb{P}_m(t)f^{n+m}\,dt=\int_{-1}^{1}\frac{1}{1-2tf+t^2}\,dt.$$

We have $n = m$, then

$$\begin{aligned}
\sum_{n=0}^{+\infty}\int_{-1}^{1} \mathrm{f}^{2n}\left[\mathbb{P}_n(\mathrm{t})\right]^2 d\mathrm{t} &= \int_{-1}^{1}\frac{1}{1+\mathrm{t}^2-2\mathrm{tf}}d\mathrm{t}\\
&= \left[\frac{1}{-2\mathrm{f}} ln\left(1+\mathrm{t}^2-2\mathrm{tf}\right)\right]_{-1}^{1},\\
&= \frac{1}{-2\mathrm{f}} ln\left(1+\mathrm{t}^2-2\mathrm{f}\right) - \frac{1}{-2\mathrm{f}} ln\left(1+\mathrm{t}^2+2\mathrm{f}\right),\\
&= \frac{1}{-2\mathrm{f}}\left[ln(1-\mathrm{f})^2 - ln(1+\mathrm{f})^2\right],\\
&= \frac{1}{-2\mathrm{f}}2\left[ln\left(1-\mathrm{f}\right) - ln\left(1+\mathrm{f}\right)\right],\\
&= \frac{1}{\mathrm{f}}\left[ln(1+\mathrm{f}) - ln(1-\mathrm{f})\right].
\end{aligned}$$

Where from

$$\begin{aligned}
\sum_{\mathrm{n}=0}^{+\infty}\int_{-1}^{1} \mathrm{f}^{2\mathrm{n}}\left[\mathbb{P}_\mathrm{n}(\mathrm{t})\right]^2 d\mathrm{t} &= \frac{1}{\mathrm{f}}\left[\left(\mathrm{f}-\frac{\mathrm{f}^2}{2}+\frac{\mathrm{f}^3}{3}+\dots\right)-\left(-\mathrm{f}-\frac{\mathrm{f}^2}{2}-\frac{\mathrm{f}^3}{3}-\dots\right)\right]\\
&= \frac{2}{\mathrm{f}}\left[t+\frac{\mathrm{f}^3}{3}+\frac{\mathrm{f}^5}{5}+\dots\right]\\
&= 2\left[1+\frac{\mathrm{f}^2}{3}+\frac{\mathrm{f}^4}{5}+\dots+\frac{\mathrm{f}^{2\mathrm{n}}}{2\mathrm{n}+1}+\dots\dots\right]\\
&= 2\sum_{\mathrm{n}=0}^{+\infty}\frac{\mathrm{f}^{2\mathrm{n}}}{2\mathrm{n}+1}.
\end{aligned}$$

Therefore, $\|P_\mathrm{n}(\mathrm{t})\|_2^2 = \dfrac{2}{2\mathrm{n}+1}$.

3.3.2 Recurrence relation

Proposition 3.3.1

The Legendre polynomials satisfy the following recurrence relation:

$$\left(\mathrm{n}+1\right)\mathbb{P}_{\mathrm{n}+1}(\mathrm{t}) - \left(2\mathrm{n}+1\right)\mathrm{t}\mathbb{P}_\mathrm{n}(\mathrm{t}) + \mathrm{n}\mathbb{P}_{\mathrm{n}-1}(\mathrm{t}) = 0. \tag{3.8}$$

Proof

we have

$$(n+1)\mathbb{P}_{n+1}(t)-(2n+1)\mathbb{P}_n(t)+n\mathbb{P}_{n-1}(t)=0.$$

We will look for a recurrence relation between three consecutive polynomials $\mathbb{P}_{n+1}(t)$, $\mathbb{P}_n(t)$ and $\mathbb{P}_{n-1}(t)$. To do this, we use Rodrigues formula written for the degree $(n+1)$:

$$\mathbb{P}_{n+1}(t)=\frac{1}{2^{n+1}(n+1)!}\frac{d^{n+1}}{dt^{n+1}}\left(t^2-1\right)^{n+1}.$$

We derive the expression $\left(t^2-1\right)^{n+1}$ once, and we find:

$$\mathbb{P}_{n+1}(t)=\frac{1}{2^n n!}\frac{d^n}{dt^n}\left(t(t^2-1)^n\right).$$

That it can also be written:

$$\mathbb{P}_{n+1}(t)=\frac{1}{2^n n!}\frac{d^{n-1}}{dt^{n-1}}\left[\left(t^2-1\right)^n+2nt^2\left(t^2-1\right)^{n-1}\right].$$

Then we add and subtract $2n\left(t^2-1\right)^{n-1}$ in the derivative of the second term:

$$\begin{aligned}\mathbb{P}_{n+1}(t)&=\frac{1}{2^n n!}\frac{d^{n-1}}{dt^{n-1}}\left[\left(t^2-1\right)^n+2nt^2\left(t^2-1\right)^{n-1}-2n\left(t^2-1\right)^{n-1}+2n\left(t^2-1\right)^{n-1}\right]\\&=\frac{1}{2^n n!}\frac{d^{n-1}}{dt^{n-1}}\left[\left(t^2-1\right)^n+2n\left(t^2-1\right)^{n-1}\left(t^2-1\right)+2n\left(t^2-1\right)^{n-1}\right]\\&=\frac{1}{2^n n!}\frac{d^{n-1}}{dt^{n-1}}\left[(2n+1)\left(t^2-1\right)^n+2n\left(t^2-1\right)^{n-1}\right].\end{aligned}$$

Let us also :

$$\mathbb{P}_{n+1}(t)=\frac{1}{2^n n!}\frac{d^{n-1}}{dt^{n-1}}\left((2n+1)(t^2-1)^n\right)+\mathbb{P}_{n-1}(t).$$

Calculating the first term of the second member using the relation derived from Leibniz's rule, this quantity is equal to:

$$\begin{aligned}\frac{2n+1}{2^n n!}\frac{d^{n-1}}{dt^{n-1}}\left(t^2-1\right)^n&=\frac{2n+1}{n2^n n!}\frac{d^n}{dt^n}\left[t(t^2-1)^n\right]-\frac{(2n+1)t}{2^n n!}\frac{d^n}{dt^n}\left(t^2-1\right)^n\\&=\frac{2n+1}{n}\mathbb{P}_{n+1}(t)-\frac{2n+1}{n}t\mathbb{P}_n(t).\end{aligned}$$

Finally, it comes:

$$\mathbb{P}_{n+1}(t)=\frac{2n+1}{n}\mathbb{P}_{n+1}(t)-\frac{2n+1}{n}t\mathbb{P}_n(t)+\mathbb{P}_{n-1}(t),$$

we finally arrive at the desired recurrence relation:

$$-n\mathbb{P}_{n+1}(t)+(2n+1)\mathbb{P}_{n+1}(t)-(2n+1)t\mathbb{P}_n(t)+n\mathbb{P}_{n-1}(t)$$

$$(n+1)\mathbb{P}_{n+1}(t)-(2n+1)\mathbb{P}_n(t)+n\mathbb{P}_{n-1}(t)=0.$$

Lemma 3.3.1

Some recurrence relations in the presence of P_n derivatives:

$$\mathbb{P}'_n(t) = t\mathbb{P}'_{n-1}(t) + n\mathbb{P}_{n-1}(t) \tag{3.9}$$

$$n\mathbb{P}_n(t) = t\mathbb{P}'_n(t) - \mathbb{P}'_{n-1}(t), \quad n \neq 0. \tag{3.10}$$

Proof

We derive Rodrigues' formula with respect to t, we obtain:

$$\mathbb{P}'_n(t) = \frac{1}{2^n n!}\frac{d^n}{dt^n}\left(2nt(t^2-1)^{n-1}\right)$$

using Leibniz's relation once again, we arrive at:

$$\mathbb{P}'_n(t) = \frac{1}{2^{n-1}(n-1)!}\left(t\frac{d^n}{dt^n}\left(t^2-1\right)^{n-1} + n\frac{d^n}{dt^n}\left(t^2-1\right)^{n-1}\right)$$

hence, the first relation of recurrence

$$\mathbb{P}'_n(t) = t\mathbb{P}'_{n-1}(t) + n\mathbb{P}_{n-1}(t),$$

which can still be written:

$$\left(n+1\right)\mathbb{P}'_{n+1}(t) = \left(n+1\right)\left[t\mathbb{P}'_n(t) + n\mathbb{P}_n(t)\right], \tag{3.11}$$

The devivation of the relation (3.8)gives:

$$\left(n+1\right)\mathbb{P}'_{n+1}(t) = \left(2n+1\right)\mathbb{P}_n(t) + \left(2n+1\right)t\mathbb{P}'_n(t) - n\mathbb{P}'_{n-1}(t), \tag{3.12}$$

Subtracting expressions (3.11) and (3.12) leads to the second recurrence relation

$$n\mathbb{P}_n(t) = t\mathbb{P}'_n(t) - \mathbb{P}'_{n-1}(t), \quad n \neq 0.$$

These last two formulas are useful in the context of the study of the Gauss-Legendre method of integration.

Lemma 3.3.2

Legendre polynomials are solutions of the differential equation:

$$\left(1-t^2\right)y'' - 2ty' + n\left(n+1\right)y = 0.$$

Proof

Let us differentiate once with respect to t the relation (3.9):

$$\mathbb{P}"_{\mathrm{n}}(t)=\mathbb{P}'_{\mathrm{n-1}}(t)+t\mathbb{P}"_{\mathrm{n-1}}(t)+\mathrm{n}\mathbb{P}'_{\mathrm{n-1}}(t) \qquad =\left(\mathrm{n}+1\right)\mathbb{P}'_{\mathrm{n-1}}(t)+t\mathbb{P}"_{\mathrm{n-1}}(t).$$

Similarly, the differentiation of the relation (3.10) gives:

$$n\mathbb{P}'_{\mathrm{n}}(t)=t\mathbb{P}"_{n}(t)+\mathbb{P}'_{\mathrm{n}}(t)-\mathbb{P}"_{\mathrm{n-1}}(t).$$

From which we derive:

$$\mathbb{P}"_{\mathrm{n-1}}(t)=-\left(\mathrm{n}-1\right)\mathbb{P}'_{\mathrm{n}}(t)+t\mathbb{P}"_{\mathrm{n}}(t).$$

Eliminate $\mathbb{P}"_{n-1}(t)$ between these two relations, we have

$$\mathbb{P}"_{\mathrm{n}}(t)=\left(\mathrm{n}+1\right)\mathbb{P}'_{\mathrm{n-1}}(t)+t\left(t\mathbb{P}"_{\mathrm{n}}(t)-(\mathrm{n}-1)\mathbb{P}'_{\mathrm{n}}(t)\right).$$

It remains to eliminate $\mathbb{P}'_{\mathrm{n-1}}(t)$ thanks to the relation (3.10):

$$\mathbb{P}"_{\mathrm{n}}(t)=\left(\mathrm{n}+1\right)\left(t\mathbb{P}'_{\mathrm{n}}(t)-\mathrm{n}\mathbb{P}_{\mathrm{n}}(t)\right)+t^2\mathbb{P}"_{\mathrm{n}}(t)-\left(\mathrm{n}-1\right)t\mathbb{P}'_{\mathrm{n}}(t).$$

The last relation depends only on $\mathbb{P}_{\mathrm{n}}(t)$ and its first two derivatives:

$$\left(1-t^2\right)\mathbb{P}"_{\mathrm{n}}(t)-2t\mathbb{P}'_{\mathrm{n}}(t)+\mathrm{n}\left(\mathrm{n}+1\right)\mathbb{P}_{\mathrm{n}}(t)=0.$$

This last relation is true regardless of n, it is a linear differential equation of the second order called Legendre's differential equation.

Proposition 3.3.2

It is useful to have some particular values that follow, taken by the Legendre polynomials:

$$\mathbb{P}_{\mathrm{n}}(1)=(1)^{\mathrm{n}}=1, \forall \mathrm{n}\in\mathbb{N}$$

$$\mathbb{P}_{n}(-1)=(-1)^{n}=\begin{cases}1 & If\ m\ is\ even\\ -1 & If\ m\ is\ odd.\end{cases}$$

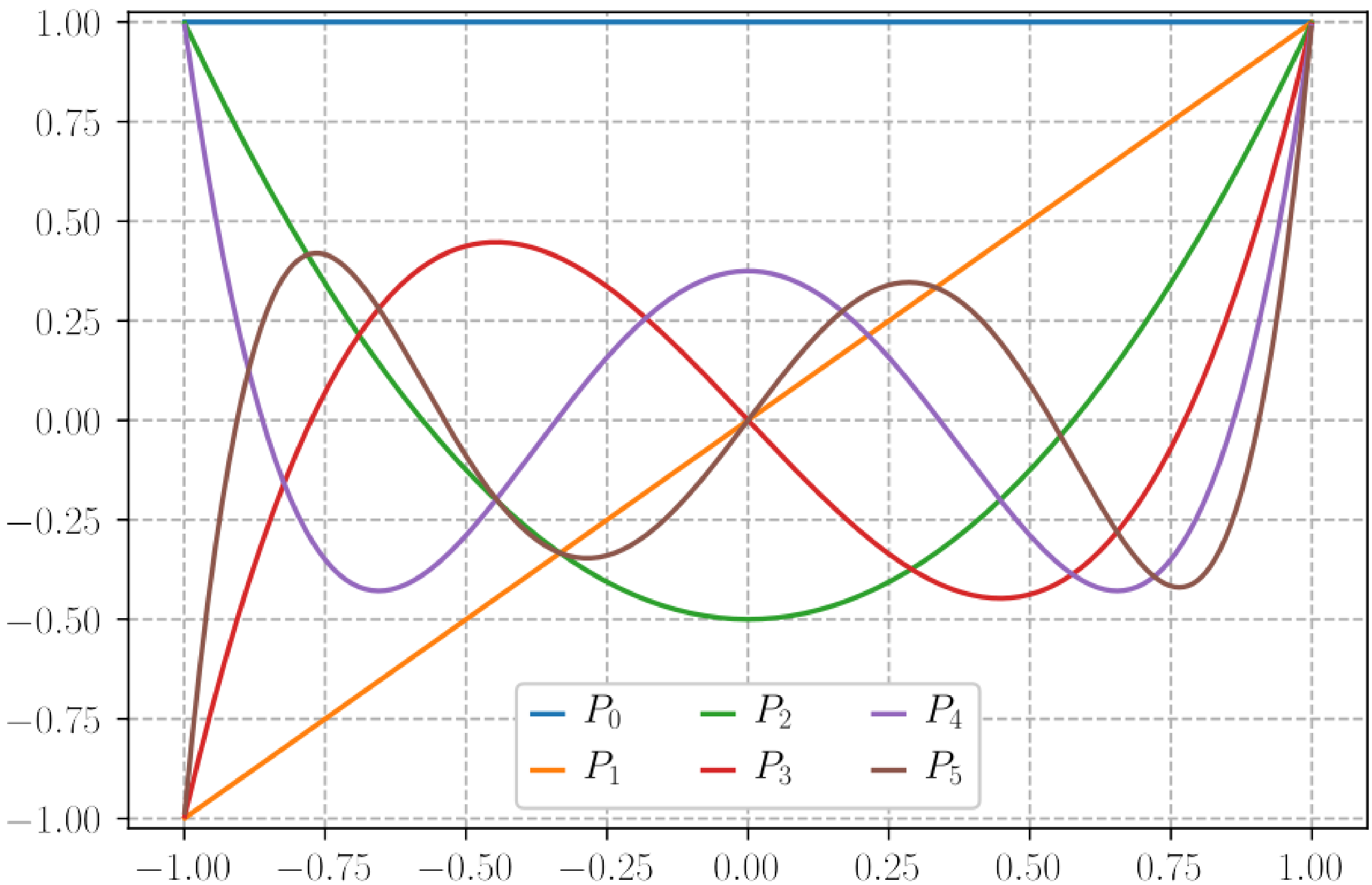

Figure 3.1: Legendre polynomials of degree 0 to 5.

Remark 3.3.1

The first six Legendre polynomials are:

$$\begin{cases} \mathbb{P}_0(t) = 1. \\ \mathbb{P}_1(t) = t. \\ \mathbb{P}_2(t) = \frac{1}{2}(3t^2 - 1). \\ \mathbb{P}_3(t) = \frac{1}{2}(5t^3 - 3t). \\ \mathbb{P}_4(t) = \frac{1}{8}(35t^4 - 30t^2 + 3). \\ \mathbb{P}_5(t) = \frac{1}{8}(63t^5 - 70t^3 + 15). \end{cases}$$

Remark 3.3.2

Roots of the first six Legendre polynomials:
The roots of the Legendre polynomial are solutions of the equation: $\mathbb{P}_n(t) = 0$

$$\begin{cases} n = 0, S = \varnothing \\ n = 1, S = \{0\} \\ n = 2, S = \left\{-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}\right\} \\ n = 3, S = \left\{-\sqrt{\frac{3}{5}}, 0, \sqrt{\frac{3}{5}}\right\} \\ n = 4, S = \left\{\pm\sqrt{\frac{15-2\sqrt{30}}{35}}, \pm\sqrt{\frac{15+2\sqrt{30}}{35}}\right\} \end{cases}$$

$n = 5$, we get the equation: $63t^5 - 70t^3 + 15 = 0$. Generally, there is no method to solve it algebraically, except by using numerical methods to find approximate solutions.

3.4 Operational matrix of Legendre

3.4.1 Shifted Legendre polynomiale

To use the Legendre polynomials on the interval $[0,1]$, we define the so-called shifted Legendre polynomials by introducing the following variable change:

$$s = 2t - 1$$

ou

$$t = \frac{1}{2}(s+1)$$

In this case, the shifted Legendre polynomials $\mathbb{P}^*_n(t)$ of order n in t are defined on $[0,1]$ by

$$\mathbb{P}^*_n(t) = \mathbb{P}_n(s) = \mathbb{P}_n(2t-1)$$

The first shifted Legendre polynomials are:

$$\begin{aligned} &\mathbb{P}^*_0(t) = 1, \\ &\mathbb{P}^*_1(t) = 2t - 1, \\ &\mathbb{P}^*_2(t) = 6t^2 - 6t + 1 \\ &\mathbb{P}^*_3(t) = 20t^3 - 30t^2 + 12t - 1, \\ &\mathbb{P}^*_4(t) = 70t^4 - 140t^3 + 90t^2 - 20t + 1, \\ &\mathbb{P}^*_5(t) = 252t^5 - 630t^4 + 560t^3 - 210t^2 + 30t - 1. \end{aligned}$$

- **Recurrence relation:**
  The shifted Legendre polynomials $\mathbb{P}^*_n(t)$ satisfying the following recurrence formula:

$$\mathbb{P}^*_{n+1}(t) = \frac{2n+1}{n+1}(2t-1)\mathbb{P}^*_n(t) - \frac{n}{n+1}\mathbb{P}^*_{n-1}(t), \ n \in \mathbb{N}^*.$$

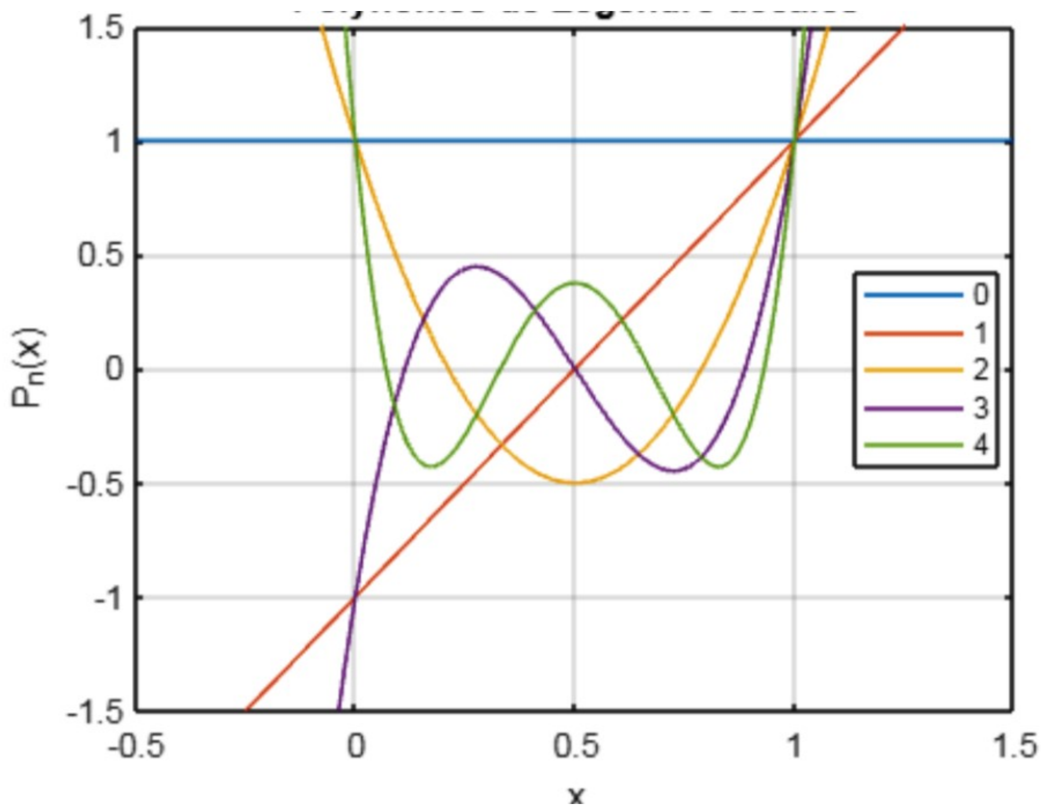

Figure 3.2: Shifted Legendre polynomials of degree 0 to 4.

- **Explicit form:**
An explicit expression for the shifted Legendre polynomials $\mathbb{P}_n^*$ of degree n in t is given by:

$$\mathbb{P}_n^*(t) = (-1)^n \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}(-t)^k. \tag{3.13}$$

$$\mathbb{P}_n^*(t) = \sum_{k=0}^{n} (-1)^{n+k} \frac{(n+k)!}{(n-k)!(k!)^2} t^k. \tag{3.14}$$

It is noted that

$$\begin{aligned} \mathbb{P}_n^*(0) &= (-1)^n, \\ \mathbb{P}_n^*(1) &= 1, \\ (\mathbb{P}_n^*)'(0) &= (-1)^{n-1} n(n+1), \\ (\mathbb{P}_n^*)'(1) &= n(n+1). \end{aligned}$$

- **Rodrigue's formula:**
The analogue of rodrigue's formula for shifted Legendre polynomials is:

$$\mathbb{P}_n^*(t) = \frac{1}{n!}\frac{d^n}{dt}\left(t^2 - t\right)^n.$$

- **Orthogonality:**
  The polynomials $\mathbb{P}_n^*(t)$ are orthogonal to the interval $[0,1]$, i.e:

$$\int_0^1 \mathbb{P}_{\mathrm{n}}^*(t)\mathbb{P}_m^*(t)dt = \begin{cases} \dfrac{1}{2\mathrm{n}+1}, & if \quad \mathrm{n}=m \\ 0, & if \quad m \neq n. \end{cases}$$

3.4.2 Function approximation using shifted Legendre polynomials

A function $u(t) \in \mathbb{L}^2(0,1)$ can be expressed in terms of the shifted Legendre polynomials basis. In practice only the first $(m+1)$ terms of shifted Legendre polynomials are considered. Hence:

$$u(t) = \sum_{i=0}^{\mathrm{n}} c_i \mathbb{P}_i^*(t) = C^T \Phi(t)$$

where

$$\mathbb{C}^T = \left[c_0, c_1, c_2, ..., c_m\right] \tag{3.15}$$

and

$$\Phi(t) = \left[\mathbb{P}_0^*, \mathbb{P}_1^*, \mathbb{P}_2^*, ..., \mathbb{P}_m^*\right]^T. \tag{3.16}$$

Then we have:

$$c_i = \left(2i+1\right)\int_0^1 u(t)\mathbb{P}_i^*(t)dt, \quad i = 1, 2, ..., m$$

3.4.3 Operational matrix of $\mathscr{D}^q\Phi(t)$ based on shifted Legendre polynomials:

The derivative of the vector $\Phi(t)$ can be expressed by:

$$\frac{d\Phi(t)}{dt} = \mathbb{D}^{(1)}\Phi(t), \tag{3.17}$$

where $\mathbb{D}^{(1)}$ is the $(m+1)\times(m+1)$ operational matrix of derivative given by:

$$\mathbb{D}^{(1)} = \left(d_{ij}\right) = \begin{cases} 2\left(2j+1\right), & j = i-k \\ 0 & otherwise, \end{cases}$$

$$\begin{cases} k = 1, 3, ...m, & if \ m \ odd, \\ k = 1, 3, ..., m-1, & if \ \mathrm{m} \ even, \end{cases}$$

for example for even m we have

$$\mathbb{D}^{(1)} = 2\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 5 & 0 & \cdots & 2m-3 & 0 & 0 \\ 0 & 3 & 0 & 7 & \cdots & 0 & 2m-1 & 0 \end{pmatrix}. \tag{3.18}$$

By using (3.17), it is clear that:

$$\frac{d^{n}\Phi(t)}{dt^{n}} = \left(\mathbb{D}^{(1)}\right)^{n}\Phi(t). \tag{3.19}$$

Where $n \in \mathbb{N}$ and $(\mathbb{D}^{(1)})^{n}$, represent the matrix powers. Therefore:

$$\mathbb{D}^{(n)} = \left(\mathbb{D}^{(1)}\right)^{n}, \quad n = 1, 2, \ldots \tag{3.20}$$

Example 3.4.1

Case 1: $m = 3$ (odd)
Then $k \in \{1,3\}$, so $\mathbb{D}^{(1)}$ of size 4×4 is :

$$\mathbb{D}^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2(2\cdot 0+1) & 0 & 0 & 0 \\ 0 & 2(2\cdot 1+1) & 0 & 0 \\ 2(2\cdot 0+1) & 0 & 2(2\cdot 2+1) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 2 & 0 & 10 & 0 \end{pmatrix}$$

Case 2 : $m = 4$ (even)
Then $k \in \{1,3\}$, so $\mathbb{D}^{(1)}$ of size 5×5 is :

$$\begin{aligned} \mathbb{D}^{(1)} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2(2\cdot 0+1) & 0 & 0 & 0 & 0 \\ 0 & 2(2\cdot 1+1) & 0 & 0 & 0 \\ 2(2\cdot 0+1) & 0 & 2(2\cdot 2+1) & 0 & 0 \\ 0 & 2(2\cdot 1+1) & 0 & 2(2\cdot 3+1) & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 2 & 0 & 10 & 0 & 0 \\ 0 & 6 & 0 & 14 & 0 \end{pmatrix} \end{aligned}$$

Lemma 3.4.1

Let $\mathbb{P}_{n}^{*}(t)$ be the shifted Legendre polynomial. Then:

$$\mathscr{D}^{q}\mathbb{P}_{n}^{*}(t) = 0, \quad n = 0, 1, \ldots, \lceil q \rceil - 1, \quad q > 0.$$

Theorem 3.4.1

Let $\Phi(t)$ be shifted Legendre vector defined in (3.16) and also suppose $q > 0$, then:

$$\mathscr{D}^{q}\Phi(t) \simeq \mathbb{D}^{(q)}\Phi(t), \tag{3.21}$$

where $\mathbb{D}^{(q)}$ is the$(m+1)\times(m+1)$ operational matrix of fractional derivative of order q in Caputo sense and is defined as follows:

$$\mathbb{D}^{(q)} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \sum_{k=\lceil q\rceil}^{\lceil q\rceil} \omega_{\lceil q\rceil,0,k} & \sum_{k=\lceil q\rceil}^{\lceil q\rceil} \omega_{\lceil q\rceil,1,k} & \cdots & \sum_{k=\lceil q\rceil}^{\lceil q\rceil} \omega_{\lceil q\rceil,m,k} \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{k=\lceil q\rceil}^{n} \omega_{n,0,k} & \sum_{k=\lceil q\rceil}^{n} \omega_{n,1,k} & \cdots & \sum_{k=\lceil q\rceil}^{n} \omega_{n,m,k} \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{k=\lceil q\rceil}^{m} \omega_{m,0,k} & \sum_{k=\lceil q\rceil}^{m} \omega_{m,1,k} & \cdots & \sum_{k=\lceil q\rceil}^{m} \omega_{m,m,k} \end{pmatrix}, \tag{3.22}$$

where $\omega_{n,j,k}$ is given by:

$$\omega_{n,j,k} = (2j+1)\sum_{l=0}^{j} \frac{(-1)^{n+j+k+l}(n+k)!(l+j)!}{(n-k)!k!\Gamma(k-q+1)(j-l)!(l!)^2(k+l-q+1)}. \tag{3.23}$$

Note that in $\mathbb{D}^{(q)}$, the first $\lceil q\rceil$ rows, are all zero.

Proof

Using the linearity property of $\mathbb{D}^{(q)}$

$$\mathbb{D}^{(q)}\big(\rho f_1(t) + \sigma f_2(t)\big) = \rho\mathbb{D}^{(q)} f_1(t) + \sigma\mathbb{D}^{(q)} f_2(t).$$

and the equalities (1.31) in (3.13)and (3.14), we have:

$$\begin{aligned}
\mathscr{D}^{\mathrm{q}}\mathbb{P}^{*}_{\mathrm{n}}(\mathrm{t}) &= \sum_{k=0}^{\mathrm{n}}(-1)^{\mathrm{n}+k}\binom{\mathrm{n}}{k}\binom{n+k}{k}\mathscr{D}^{\mathrm{q}}\mathrm{t}^{k},\\
&= \sum_{k=0}^{\lceil \mathrm{q}\rceil}(-1)^{\mathrm{n}+k}\binom{\mathrm{n}}{k}\binom{\mathrm{n}+k}{k}\frac{\Gamma(k+1)}{\Gamma(k-\mathrm{q}+1)}\mathrm{t}^{k-\mathrm{q}},\\
&= \sum_{k=\lceil \mathrm{q}\rceil}^{n}(-1)^{\mathrm{n}+k}\frac{(\mathrm{n}+k)!}{(\mathrm{n}-k)!(k!)\Gamma(k-\mathrm{q}+1)}\mathrm{t}^{k-\mathrm{q}}, \quad \mathrm{n}=\lceil \mathrm{q}\rceil,\dots,m.
\end{aligned} \tag{3.24}$$

Now, approximate $\mathrm{t}^{k-\mathrm{q}}$ by $(m+1)$ terms of shifted Legendre series, we have

$$\mathrm{t}^{k-\mathrm{q}} \simeq \sum_{j=0}^{m} d_{k,j}\mathbb{P}^{*}_{j}(\mathrm{t}), \tag{3.25}$$

where :

$$\begin{aligned}
d_{k,j} &= \left(2j+1\right)\int_{0}^{1}\mathrm{t}^{k-q}\mathbb{P}^{*}_{j}(\mathrm{t})d\mathrm{t},\\
&= \left(2j+1\right)\sum_{l=0}^{\lceil j\rceil}(-1)^{j+l}\binom{j}{l}\binom{j+l}{l}\int_{0}^{1}\mathrm{t}^{k+l-\mathrm{q}}d\mathrm{t},\\
&= \left(2j+1\right)\sum_{l=0}^{\lceil j\rceil}(-1)^{j+l}\binom{j}{l}\binom{j+l}{l}\frac{1}{k+l-\mathrm{q}+1},\\
&= \left(2j+1\right)\sum_{l=0}^{j}(-1)^{j+l}\frac{(j+l)!}{(j-l)!(l!)(k+l-\mathrm{q}+1)}.
\end{aligned} \tag{3.26}$$

Employing equations (3.24) and (3.26), we get:

$$\begin{aligned}
\mathscr{D}^{\mathrm{q}}\mathbb{P}^{*}_{\mathrm{n}}(\mathrm{t}) &= \sum_{k=\lceil \mathrm{q}\rceil}^{\mathrm{n}}\sum_{j=0}^{m}(-1)^{\mathrm{n}+k}\binom{\mathrm{n}}{k}\binom{\mathrm{n}+k}{k}\frac{\Gamma(k+1)}{\Gamma(k-\mathrm{q}+1)}d_{k,j}\mathbb{P}^{*}_{j}(\mathrm{t})\\
&= \sum_{k=\lceil \mathrm{q}\rceil}^{\mathrm{n}}\sum_{j=0}^{m}(-1)^{\mathrm{n}+k}\frac{(\mathrm{n}+k)!}{(\mathrm{n}-k)!(k!)\Gamma(k-\mathrm{q}+1)}d_{k,j}\mathbb{P}^{*}_{j}(\mathrm{t})\\
&= \sum_{j=0}^{m}\left(\sum_{k=\lceil \mathrm{q}\rceil}^{n}\omega_{\mathrm{n},j,k}\right)\mathbb{P}^{*}_{j}(\mathrm{t}), \quad \mathrm{n}=\lceil q\rceil,\dots,m
\end{aligned} \tag{3.27}$$

where $\omega_{\mathrm{n},j,k}$ is given in eq. (3.23). Rewrite eq. (3.27) as a vector form, we have for all $\mathrm{n}=\lceil \mathrm{q}\rceil,\dots,m$:

$$\mathscr{D}^{\mathrm{q}}\mathbb{P}^{*}_{\mathrm{n}}(\mathrm{t}) \simeq \left[\sum_{k=\lceil \mathrm{q}\rceil}^{\mathrm{n}}\omega_{\mathrm{n},0,k},\sum_{k=\lceil \mathrm{q}\rceil}^{\mathrm{n}}\omega_{\mathrm{n},1,k}\dots,\sum_{k=\lceil \mathrm{q}\rceil}^{\mathrm{n}}\omega_{\mathrm{n},m,k}\right]\Phi(\mathrm{t}) \tag{3.28}$$

Also according to Lemma 3.4.3, we can write

$$\mathscr{D}^{q}\mathbb{P}_{n}^{*}(t) \simeq \left[0,0,....,0\right]\Phi(t), n=0,1,...[q]-1. \tag{3.29}$$

A combination of eqs. (3.28) and (3.29) leads to the desired result.

Remark 3.4.1

If $q=n\in\mathbb{N}$, Then theorem 3.4.1 3.4.3 gives the same result as Eq. (3.20).

3.5 Shifted Legendre operational matrix of multi-order fractional differential equations:

In this section, in order to show the great importance of the fractional derivation operational matrix, we apply it to solve some types of fractional differential equations. The existence and uniqueness of solutions of these equations are discussed in two selected types of fractional differential equations.

3.5.1 Case of a linear problem:

Consider the linear multi-order fractional differential equation

$$\mathscr{D}^{\alpha}\varphi(t)=\lambda_1\mathscr{D}^{\beta}\varphi(t)+\lambda_2\varphi(t)+\lambda_3 f(t), \tag{3.30}$$

with initial conditions

$$\varphi(0)=\mu_0, \varphi'(0)=\mu_1. \tag{3.31}$$

where $\lambda_1,\lambda_2,\lambda_3$ are real constant coefficients,$n<\alpha\leq n+1, 0<\beta<\alpha$ and $\mathscr{D}^{\alpha},\mathscr{D}^{\beta}$ denote the fractional derivatives of order α and β Caputo respectively.
To solve problem (3.30)-(3.31) we approximate $u(x)$ and $g(x)$ by the modified Legendre polynomials as follows:

$$\varphi(t)\simeq\sum_{i=0}^{m}c_i\mathbb{P}_i^{*}(t)=\mathbb{C}^{T}\Phi(t) \tag{3.32}$$

$$f(t)\simeq\sum_{i=0}^{m}f_i\mathbb{P}_i^{*}(t)=F^{T}\Phi(t) \tag{3.33}$$

where the vector $F=\left[f_0,...,f_m\right]^{T}$ is known but $\mathbb{C}=\left[c_0,...,c_m\right]^{T}$ is an unknown vector. Using equations (3.32) and (3.31) we have:

$$\mathscr{D}^{\alpha}\varphi(t)\simeq\mathbb{C}^{T}\mathscr{D}^{\alpha}\Phi(t)\simeq\mathbb{C}^{T}\mathbb{D}^{(\alpha)}\Phi(t) \tag{3.34}$$

$$\mathscr{D}^{\beta}\varphi(t)\simeq\mathbb{C}^{T}\mathscr{D}^{\beta}\Phi(t)\simeq\mathbb{C}^{T}\mathbb{D}^{(\beta)}\Phi(t) \tag{3.35}$$

By exploiting equations (3.32) - (3.35). the residue $K_m(t)$ for equation (3.30) can be written:

$$K_m(t) \simeq \left(\mathbb{C}^T \mathbb{D}^{(\alpha)} - \mathbb{C}^T \lambda_1 \mathbb{D}^{(\beta)} - \lambda_2 C^T - \lambda_3 F^T\right)\Phi(t). \tag{3.36}$$

Now we generate $(m-1)$ linear equations by applying

$$\langle K_m(t), \mathbb{P}_j^* \rangle = \int_0^1 K_m(t)\mathbb{P}_j^* dt = 0 \quad j = 0, 1, ..., m-2. \tag{3.37}$$

Also, by substituting equations (3.20) and (3.32) in equation (3.31). we obtain

$$\varphi(0) = \mathbb{C}^T \Phi(0) = c_0, \quad \varphi'(0) = \mathbb{C}^T \mathbb{D}^{(1)} \Phi(0) = c_1 \tag{3.38}$$

Equations (3.37) and (3.38) generate $(m-n)$ and $(n+1)$ linear equations. These linear equations can be solved for unknown coefficients of the vector C . Therefore, $\Phi(t)$ given in equation (3.32) can be calculated.

Example 3.5.1

Consider the following initial value problem

$$\begin{cases} \mathscr{D}^2\Phi(t) = -\mathscr{D}^{\frac{3}{2}}\Phi(t) - \Phi(t) + 5 + t^3, \\ \Phi(0) = 5, \quad \Phi'(0) = 1, \end{cases} \tag{3.39}$$

It is easy to verify that the exact solution to this problem is $\Phi(t) = 5 + t^3$. Applying the technique described in subsection 3.5.1 with $m = 3$, we can express the approximate solution of the problem as follows:

$$\varphi(t) = \sum_{i=0}^{3} c_i \mathbb{P}_i^*(t) = \mathbb{C}^T \Phi(t),$$

that is:

$$\varphi(t) = c_0 \mathbb{P}_0^* + c_1 \mathbb{P}_1^* + c_2 \mathbb{P}_2^* = C^T \Phi(t),$$

with

$$\mathbb{P}_0^* = 1, \quad \mathbb{P}_1^* = 2t - 1, \quad \mathbb{P}_2^* = 1 - 6t + 6t^2, \quad \mathbb{P}_3^* = -1 + 12t - 30t^2 + 20t^3$$

According to theorem 3.4.3 we find:

$$\mathbb{D}^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 10 & 0 \end{pmatrix}; \mathbb{D}^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 12 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 \end{pmatrix} \mathbb{D}^{(\frac{3}{2})} = \left(\frac{16}{\sqrt{\pi}}\right) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -\frac{3}{5} & \frac{1}{7} & 0 \\ 3 & -\frac{18}{7} & \frac{6}{11} & -\frac{2}{13} \end{pmatrix} F = \begin{pmatrix} \frac{37}{6} \\ 2 \\ \frac{7}{6} \\ \frac{1}{4} \end{pmatrix}$$

So using equation (3.37) we get:

$$c_0 + \left(12 + \frac{16}{35\sqrt{\pi}}\right) c_2 - \frac{5}{4} = 0 \tag{3.40}$$

Now, applying using (3.38) we arrive at:

$$c_0 - c_1 + c_2 - c_3 - 5 = 0 \quad (3.41)$$
$$2c_1 - 6c_2 + 12c_3 - 1 = 0 \quad (3.42)$$
$$6c_2 - 20c_3 - 1 = 0 \quad (3.43)$$

Finally, the solution of equations (3.40) - (3.43) gives us:

$$c_0 = \frac{37}{6}, \quad c_1 = 2, \quad c_2 = \frac{7}{6}, \quad c_3 = \frac{1}{4}.$$

Thus we can write:

$$\varphi(t) = \left(\frac{37}{6}, 2, \frac{7}{6}, \frac{1}{4}\right)\begin{pmatrix} 1 \\ 2t-1 \\ 6t^2-6t+1 \\ 20t^3-30t^2+12t-1 \end{pmatrix} = 5 + t^3$$

which is the exact solution.

3.5.2 Case of a nonlinear problem:

Consider the following nonlinear fractional differential problem

$$\begin{cases} \mathscr{D}^{\alpha}\varphi(t) = F\big(t, \varphi(t), \mathscr{D}^{\beta}\varphi(t), \mathscr{D}^{\gamma}\varphi(t)\big), \\ \varphi(0) = c_0, \quad \varphi'(0) = c_1, \quad \varphi''(0) = c_2, \end{cases} \quad (3.44)$$

where $n \leqslant \alpha \leqslant n+1, 0 < \beta < \gamma < \alpha, \mathscr{D}^{\alpha}, \mathscr{D}^{\beta}, \mathscr{D}^{\gamma}$ respectively denote the fractional derivatives of order α, β and γ in the Caputo sense and F a nonlinear function.
In order to use modified Legendre polynomials for this problem, we first approximate $\varphi(t), \mathscr{D}^{\alpha}\varphi(t)$ and $, \mathscr{D}^{\beta}\varphi(t)$ and $\mathscr{D}^{\gamma}\varphi(t)$. By substituting equations (3.32), (3.34) and (3.35) respectively in equation (3.44). we obtain

$$\mathbb{C}^T\mathbb{D}^{(\alpha)}\varphi(t) \simeq F\big(t, \mathbb{C}^T\Phi(t), \mathbb{C}^T\mathbb{D}^{(\beta)}\Phi(t), \mathbb{C}^T\mathbb{D}^{(\gamma)}\Phi(t)\big) \quad (3.45)$$

Also, replacing equations (3.20) and (3.32) in (3.44) we find:

$$\varphi(0) = \mathbb{C}^T\Phi(0) = c_0, \quad (3.46)$$
$$\varphi'(0) = \mathbb{C}^T\mathbb{D}^{(1)}\Phi(0) = c_1, \quad (3.47)$$
$$\varphi''(0) = \mathbb{C}^T\mathbb{D}^{(2)}\Phi(0) = c_2. \quad (3.48)$$

To find the solution $\varphi(t)$, we first colocate equations (3.45) at $(m-2)$ points. For the appropriate collocation points, we use the first $(m-2)$ roots of the modified Legendre polynomials of $\mathbb{P}^*_{m+1}(t)$. These equations together with equation (3.46) generate $(m+1)$ nonlinear equations that can be solved by Newton's iterative method for example. Therefore $\varphi(t)$ given in equation (3.32) can be calculated.

Example 3.5.2

We consider the following nonlinear fractional differential problem with an initial value:

$$\begin{cases} \mathscr{D}^3\varphi(t) = t^4 - \mathscr{D}^{\frac{5}{2}}\varphi(t) - \varphi^2(t), \\ \varphi(0) = 0, \quad \varphi'(0) = 0, \quad \varphi''(0) = 2. \end{cases} \tag{3.49}$$

In this example we will solve the problem (3.49), applying the technique described in subSection 3.5.2 with $m = 3$, we approach the solution as follows:

$$\varphi(t) = c_0\mathbb{P}_0^* + c_1\mathbb{P}_1^* + c_2\mathbb{P}_2^* + c_3\mathbb{P}_3^* = C^T\Phi(t),$$

with

$$\mathbb{P}_0^* = 1, \quad \mathbb{P}_1^* = 2t - 1, \quad \mathbb{P}_2^* = 1 - 6t + 6t^2 . \mathbb{P}_3^* = -1 + 12t - 30t^2 + 20t^3.$$

Here, we have

$$\mathbb{D}^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 2 & 0 & 10 & 0 \end{pmatrix} \mathbb{D}^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 12 & 0 & 0 & 0 \\ 0 & 60 & 0 & 0 \end{pmatrix} \quad \mathbb{D}^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 120 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbb{D}^{(\frac{5}{2})} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{160}{\sqrt{\pi}} & \frac{96}{\sqrt{\pi}} & -\frac{160}{7\sqrt{\pi}} & \frac{32}{\sqrt{3\pi}} \end{pmatrix} \mathbb{C} = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

Using equation (3.45) we obtain:

$$\mathbb{C}^T\mathbb{D}^{(3)}\Phi(t) + \mathbb{C}^T\mathbb{D}^{(\frac{5}{2})}\Phi(t) + \left[\mathbb{C}^T\Phi(t)\right]^2 - t^4 = 0 \tag{3.50}$$

Now, we colocate equation (3.50) to the first root of $\mathbb{P}_4^*(t)$, i.e.:

$$t_0 = \frac{1}{2} + \frac{\sqrt{525 - 70\sqrt{3}}}{70} \simeq 0.069431844$$

Also, using equations (3.46) - (3.48) we find:

$$\begin{aligned} \mathbb{C}^T\Phi(0) &= c_0 - c_1 + c_2 - c_3 = 0, \\ \mathbb{C}^T\mathbb{D}^{(1)}\Phi(0) &= 2c_1 + 12c_3 - 6c_2 = 0, \\ \mathbb{C}^T\mathbb{D}^{(2)}\Phi(0) &= 12c_2 - 60c_3 = 2. \end{aligned} \tag{3.51}$$

Solving equations (3.50) and (3.51) we obtain:

$$c_0 = \frac{1}{3}, \quad c_1 = \frac{1}{2}, \quad c_2 = \frac{1}{6}, \quad c_3 = 0.$$

Consequently

$$\varphi(t) = \left(\frac{1}{3}, \frac{1}{2}, \frac{1}{6}, 0\right) \begin{pmatrix} 1 \\ 2t-1 \\ 6t^2-6t+1 \\ 20t^3-30t^2+12t-1 \end{pmatrix} = t^2$$

which represents the exact solution of the problem (3.49).

Chapter four

Solving a fractional differential equations via operational matrix and shifted Jacobi polynomials

Contents

4.1 Introduction 69

4.2 Properties of shifted Jacobi polynomials 69

4.3 Operational matrix of fractional integration 70

4.4 Fractional shift Jacobi operational matrix for solving linear multi-order fractional differential equations 72

4.5 Illustrative examples 74

4.1 Introduction

This study presents a novel shifted Jacobi operational matrix (SJOM) for fractional integration of arbitrary order, which is utilized in combination with the spectral tau method to solve linear fractional differential equations (FDEs). The fractional integration is defined in the Riemann-Liouville framework. The numerical approach is based on the shifted Jacobi tau technique, with a notable advantage being that it requires only a small number of shifted Jacobi polynomials to produce highly accurate results. Several example applications confirm the efficiency and effectiveness of this method in solving linear multi-term FDEs.

Jacobi polynomials have gained increasing importance in numerical analysis, both in theory and application. Recently, Doha et al. ([7]) developed the shifted Jacobi operational matrix for fractional derivatives, which was integrated with the spectral tau method to numerically solve general linear multi-term FDEs. In this chapter, we construct an operational matrix for fractional integration using shifted Jacobi polynomials, formulated in the Riemann-Liouville sense. This matrix is then employed to develop a direct solution method for FDEs. Notably, the shifted Chebyshev operational matrix of fractional integration, introduced by Bhrawy and Alofi [3], emerges as a specific case of the shifted Jacobi operational matrix of fractional integration. Finally, the accuracy and reliability of the proposed algorithm are validated through test problems.

4.2 Properties of shifted Jacobi polynomials

The well-known Jacobi polynomials $\mathscr{P}_m^{(\gamma,\delta)}(\mathfrak{z})$ are typically defined over the interval $(-1,1)$. To adapt these polynomials for use on the interval $\mathfrak{z} \in (0,t)$, we introduce the so-called shifted Jacobi polynomials through the variable transformation:

$$\mathfrak{z} = \frac{2x}{t} - 1.$$

This transformation maps the interval $(0,t)$ onto the standard domain $(-1,1)$, enabling the effective application of Jacobi polynomials in a modified range suitable for the given problem.

These shifted polynomials are denoted by $\mathscr{P}_{s,m}^{(\gamma,\delta)}(\mathfrak{z})$. The analytic form of the shifted Jacobi polynomials $\mathscr{P}_{s,m}^{(\gamma,\delta)}(\mathfrak{z})$ of degree m is:

$$\mathscr{P}_{s,m}^{(\gamma,\delta)}(\mathfrak{z}) = \sum_{k=0}^{m} \frac{(-1)^{m-k}\Gamma(m+\delta+1)\Gamma(m+k+\gamma+\delta+1)}{\Gamma(k+\delta+1)\Gamma(m+\gamma+\delta+1)(m-k)!k!s^k}\mathfrak{z}^k, \tag{4.1}$$

The value of the shift Jacobi polynomial at $\mathfrak{z}=0$ is:

$$\mathscr{P}_{s,m}^{(\gamma,\delta)}(0) = (-1)^m \frac{\Gamma(m+\delta+1)}{\Gamma(\delta+1)m!}$$

. The orthogonality condition for these polynomials is

$$\int_0^s \mathscr{P}_{s,n}^{(\gamma,\delta)}(\mathfrak{z})\mathscr{P}_{s,k}^{(\gamma,\delta)}(\mathfrak{z})w_s^{(\gamma,\delta)}(\mathfrak{z})d\mathfrak{z} = \hbar_{s,k}^{(\gamma,\delta)}\delta_{nk}$$

where the weight function is: $w_s^{(\gamma,\delta)}(\mathfrak{z}) = (t-\mathfrak{z})^\gamma \mathfrak{z}^\delta$ and the normalization constant is:

$$\hbar_{s,k}^{(\gamma,\delta)} = \frac{s^{\gamma+\delta+1}\Gamma(k+\gamma+1)\Gamma(k+\delta+1)}{(2k+\gamma+\delta+1)\Gamma(k+1)\Gamma(k+\gamma+\delta+1)}$$

. The special values of the derivative of the shifted Jacobi polynomials at $\mathfrak{z}=0$ are given by:

$$\mathscr{D}^q \mathscr{P}_{s,m}^{(\gamma,\delta)}(0) = \frac{(-1)^{m-q}\Gamma(m+\delta+1)(m+\gamma+\delta+1)_q}{s^q\Gamma(m-q+1)\Gamma(q+\delta+1)} \tag{4.2}$$

This will be crucial for later use. A function $v(\mathfrak{z})$, square integrable on the interval $(0,s)$, can be expressed as a series of shifted Jacobi polynomials:

$$v(\mathfrak{z}) = \sum_{n=0}^{\infty} c_n \mathscr{P}_{s,n}^{(\gamma,\delta)}(\mathfrak{z})$$

where the coefficients c_n are given by

$$c_n = \frac{1}{\hbar_{s,n}^{(\gamma,\delta)}} \int_0^s v(\mathfrak{z})\mathscr{P}_{s,n}^{(\gamma,\delta)}(\mathfrak{z}) w_s^{(\gamma,\delta)}(\mathfrak{z}) d\mathfrak{z}, \quad n = 0, 1, 2, \cdots.$$

In practice, only the first $(\mathrm{N}+1)$ terms of the shifted Jacobi polynomials are considered. Therefor, the function $v(\mathfrak{z})$ can be aproximated as:

$$v_{\mathrm{N}}(\mathfrak{z}) \simeq \sum_{n=0}^{\mathrm{N}} c_n \mathscr{P}_{s,n}^{(\gamma,\delta)}(\mathfrak{z}) = \mathbb{C}^T \varphi(\mathfrak{z})$$

where the shifted Jacobi coefficient vector $\mathbb{C}$ and the shifted Jacobi vector $\varphi(\mathfrak{z})$ are:

$$\mathbb{C}^T = [c_0, c_1, \cdots, c_{\mathrm{N}}], \tag{4.3}$$

$$\varphi(\mathfrak{z}) = \left[\mathscr{P}_{s,0}^{(\gamma,\delta)}(\mathfrak{z}), \mathscr{P}_{s,1}^{(\gamma,\delta)}(\mathfrak{z}), \cdots, \mathscr{P}_{s,\mathrm{N}}^{(\gamma,\delta)}(\mathfrak{z})\right]^T \tag{4.4}$$

Next, if we define the q times repeated integration of Jacobi vector $\varphi(\mathfrak{z})$ by $\mathscr{J}^q\varphi(\mathfrak{z})$. we have:

$$\mathscr{J}^q \varphi(\mathfrak{z}) \simeq \mathbb{P}^{(q)}\varphi(\mathfrak{z}), \tag{4.5}$$

where q is an integer and $\mathbb{P}^{(q)}$ is the operational matrix of integration for th vector $\varphi(\mathfrak{z})$.

4.3 Operational matrix of fractional integration

The goal of this section is to extend the Shift Jacobi Operational Matrix of integration to fractional calculus.

Theorem 4.3.1

Let $\varphi(\mathfrak{z})$ be the shifted Jacobi vector, and let $\rho > 0$. Then the fractional integration of order ρ is approximated as:

$$\mathscr{J}^{\rho}\varphi(\mathfrak{z}) \simeq \mathscr{P}^{(\rho)}\varphi(\mathfrak{z}) \tag{4.6}$$

where $\mathscr{P}^{(\nu)}$ is the $(\mathrm{N}+1)\times(\mathrm{N}+1)$ operational matrix of fractional integration of order ρ in the Riemann-Liouville sense, defined as :

$$\mathscr{P}^{(\rho)} = \begin{pmatrix} \Psi_\rho(0,0,\gamma,\delta) & \Psi_\rho(0,1,\gamma,\delta) & \cdots & \Psi_\rho(0,\mathrm{N},\gamma,\delta) \\ \Psi_\rho(1,0,\gamma,\delta) & \Psi_\rho(1,1,\gamma,\delta) & \cdots & \Psi_\rho(1,\mathrm{N},\gamma,\delta) \\ \vdots & \vdots & \cdots & \vdots \\ \Psi_\rho(m,0,\gamma,\delta) & \Psi_\rho(m,1,\gamma,\delta) & \cdots & \Psi_\rho(m,\mathrm{N},\gamma,\delta) \\ \vdots & \vdots & \cdots & \vdots \\ \Psi_\rho(\mathrm{N},0,\gamma,\delta) & \Psi_\rho(\mathrm{N},1,\gamma,\delta) & \cdots & \Psi_\rho(\mathrm{N},\mathrm{N},\gamma,\delta) \end{pmatrix} \tag{4.7}$$

Here, each element $\Psi_\rho(m,\mathrm{n},\gamma,\delta)$ is given by:

$$\Psi_\rho(m,\mathrm{n},\gamma,\delta) = \sum_{\mathrm{k}=0}^{m} \frac{(-1)^{m-\mathrm{k}}\Gamma(m+\delta+1)\Gamma(m+\mathrm{k}+\gamma+\delta+1)}{\Gamma(\mathrm{k}+\delta+1)\Gamma(m+\gamma+\delta+1)(m-\mathrm{k})!\Gamma(\mathrm{k}+\rho+1)} \tag{4.8}$$

$$\times \sum_{f=0}^{\mathrm{n}} \frac{(-1)^{\mathrm{n}-f}\Gamma(\mathrm{n}+f+\gamma+\delta+1)\Gamma(\gamma+1)\Gamma(f+\mathrm{k}+\rho+\delta+1)(2\mathrm{n}+\gamma+\delta+1)\mathrm{n}!s^{\rho}}{\Gamma(n+\gamma+1)\Gamma(f+\delta+1)(\mathrm{n}-f)!f!\Gamma(f+\mathrm{k}+\gamma+\delta+\nu+2)} \tag{4.9}$$

Proof

The analytic expression for the shifted Jacobi polynomials $\mathscr{P}^{(\gamma,\delta)}_{s,m}(\mathfrak{z})$ of degree m is given by equation (4.2). Using equations (4.1) and (4.2), and noying that the Riemann-Liouville's fractional integration is a linear operation, we obtain:

$$\mathscr{J}^{\rho}\mathscr{P}^{(\gamma,\delta)}_{s,m}(\mathfrak{z}) = \sum_{\mathrm{k}=0}^{m} \frac{(-1)^{m-\mathrm{k}}\Gamma(m+\delta+1)\Gamma(m+\mathrm{k}+\gamma+\delta+1)}{\Gamma(\mathrm{k}+\delta+1)\Gamma(m+\gamma+\delta+1)(m-\mathrm{k})!k!\sim^{k}} J^{\nu}\mathfrak{z}^{k} \tag{4.10}$$

$$= \sum_{k=0}^{m} \frac{(-1)^{m-\mathrm{k}}\Gamma(m+\delta+1)\Gamma(m+\mathrm{k}+\gamma+\delta+1)}{\Gamma(\mathrm{k}+\delta+1)\Gamma(m+\gamma+\delta+1)(m-\mathrm{k})!\Gamma(\mathrm{k}+\rho+1)s^{\mathrm{k}}}\mathfrak{z}^{\mathrm{k}+\rho}, \tag{4.11}$$

$$m = 0,1,\cdots,\mathrm{N}. \tag{4.12}$$

Next, applying the fractional integration $\mathfrak{z}^{\mathrm{k}+\rho}$, we get:

$$\mathfrak{z}^{k+\rho} = \sum_{n=0}^{\mathrm{N}} c_{k\mathrm{n}}\mathscr{P}^{(\gamma,\delta)}_{s,\mathrm{n}}(\mathfrak{z}), \tag{4.13}$$

where the coefficients c_{kn} are given by equation (4.6) with $v(\mathfrak{z}) = \mathfrak{z}^{\mathrm{k}+\rho}$, i.e,

$$c_{\mathrm{kn}} = \frac{(2\mathrm{n}+\gamma+\delta+1)\Gamma(\mathrm{n}+1)}{\Gamma(\mathrm{n}+\gamma+1)} s^{\mathrm{k}+\rho} \times \sum_{f=0}^{n} \frac{(-1)^{\mathrm{n}-f}\Gamma(\mathrm{n}+f+\gamma+\delta+1)\Gamma(\gamma+1)\Gamma(f+\mathrm{k}+\delta+\rho+1)}{\Gamma(f+\delta+1)(\mathrm{n}-f)!f!\Gamma(f+k+\gamma+\delta+\rho+2)}, \quad \mathrm{n} = 1,2,\cdots,\mathrm{N}.$$

Using equations (4.10) and (4.13), we get the following expression for $\mathscr{J}^{\rho}\mathscr{P}^{(\gamma,\delta)}_{s,m}(\mathfrak{z})$:

$$\mathscr{J}^{\rho}\mathscr{P}^{(\gamma,\delta)}_{s,m}(\mathfrak{z}) = \sum_{n=0}^{\mathrm{N}} \Psi_{\rho}(m,\mathrm{n})\mathscr{P}^{(\gamma,\delta)}_{s,\mathrm{n}}(\mathfrak{z}), \quad m = 0,1,\cdots,\mathrm{N} \tag{4.14}$$

where $\Psi_{\rho}(m,\mathrm{n}) = \sum_{\mathrm{k}=0}^{m} \zeta_{m\mathrm{nk}}$, and

$$\zeta_{m\mathrm{nk}} = \frac{(-1)^{m-\mathrm{k}}\Gamma(m+\delta+1)\Gamma(m+\mathrm{k}+\gamma+\delta+1)}{\Gamma(\mathrm{k}+\delta+1)\Gamma(m+\gamma+\delta+1)(m-\mathrm{k})!\Gamma(\mathrm{k}+\rho+1)} \times \sum_{f=0}^{\mathrm{n}} \frac{(-1)^{\mathrm{n}-f}\Gamma(\mathrm{n}+f+\gamma+\delta+1)\Gamma(\gamma+1)\Gamma(f+\mathrm{k}+\nu+\delta+1)(2\mathrm{n}+\gamma+\delta+1)\mathrm{n}!s^{\rho}}{\Gamma(\mathrm{n}+\gamma+1)\Gamma(f+\delta+1)(\mathrm{n}-f)!f!\Gamma(f+\mathrm{k}+\gamma+\delta+\rho+2)}, \quad \mathrm{n} = 1,2,\cdots\mathrm{N}.$$

Therefor , Equation (4.14) can be expressed in a vector form as:

$$\mathscr{J}^{\rho}\mathscr{P}^{(\gamma,\delta)}_{s,m}(\mathfrak{z}) \simeq \left[\Psi_{\rho}(m,0,\gamma,\delta),\Psi_{\rho}(m,1,\gamma,\delta),\Psi_{\rho}(m,2,\gamma,\delta),\cdots,\Psi_{\rho}(m,\mathrm{N},\gamma,\delta)\right]\varphi(\mathfrak{z}), \tag{4.15}$$

$$m = 0,1,\cdots,\mathrm{N}. \tag{4.16}$$

This leads to the desired result.

4.4 Fractional shift Jacobi operational matrix for solving linear multi-order fractional differential equations

In this section, the proposed multi-order fractional differential equation (FDE) is integrated ρ times, in the Riemann-Liouville sense, where ρ is the highest fractional-order. This is done by utilizing the formula that relates the expansion coefficients.

To highlight the importance of the Shifted Jacobi Operational Matrix (SJOM) for fractional integration, we first apply it to solve a multi-order fractional differential equation (FDE). The equation is given by:

$$\mathscr{D}^{\rho} v(\mathfrak{z}) = \sum_{m=0}^{k} \left(\eta_m \mathscr{D}^{\delta_m} v(\mathfrak{z}) + \lambda_m \mathscr{I}^{\sigma_m} v(\mathfrak{z})\right) + \mathbb{F}(\mathfrak{z}), \qquad \text{in } I = (0, s), \tag{4.17}$$

with initial conditions:

$$v^{(m)}(0) = l_m, \quad m = 0, \cdots, r-1, \tag{4.18}$$

where $\eta_m (m = 0, 1, \cdots, k)$ and $\lambda_m (m = 0, 1, \cdots, k)$ are real constants coefficients, and $r-1 < \rho \leq r, 0 < \delta_1 < \delta_2 < \cdots < \delta_k < \rho$. The term $\mathscr{D}^{\rho} v(\mathfrak{z}) \equiv v^{(\rho)}(\mathfrak{z})$ denotes the Riemann-Liouville fractional derivative of order ρ of $v(\mathfrak{z})$, $\mathscr{I}^{\sigma_m} v(\mathfrak{z})$ denotes the Riemann-Liouville fractional integral of order σ $(\sigma \in \mathbb{R}^+)$ of $v(\mathfrak{z})$ and $l_m (m = 0, \cdots, r-1)$ represents the initial values of $v(\mathfrak{z})$, while $\mathbb{G}(\mathfrak{z})$ is a given source function. For further details on existence, uniqueness and continuous dependence of the solution , refer to [8].

By applying the Riemann-Liouville integral of order ρ to equation (4.17) and using equation (1.30), we derive of the integrated form of (4.17), namely

$$\begin{aligned}
\mathscr{I}^{\rho} \mathscr{D}^{\rho} v(\mathfrak{z}) &= v(\mathfrak{z}) - \sum_{n=0}^{r-1} v^{(n)}\left(0^+\right) \frac{\mathfrak{z}^n}{n!} = \sum_{m=0}^{k} \mathscr{I}^{\rho} \left(\eta_m \mathscr{D}^{\delta_m} v(\mathfrak{z}) + \lambda_m \mathscr{I}^{\sigma_m} v(\mathfrak{z})\right) + \mathscr{I}^{\rho} \mathbb{F}(\mathfrak{z}), \\
&= \sum_{m=0}^{k} \eta_m \mathscr{I}^{\rho-\delta_m} \left[v(\mathfrak{z}) - \sum_{n=0}^{r-1} v^{(n)}\left(0^+\right) \frac{\mathfrak{z}^n}{n!} \right] + \lambda_m \mathscr{I}^{\rho} \mathscr{I}^{\sigma_m} v(\mathfrak{z}) + \mathscr{I}^{\rho} \mathbb{F}(\mathfrak{z}),
\end{aligned}$$

with the same initial conditions

$$v^{(m)}(0) = l_m, \quad m = 0, \cdots, r-1$$

where $r_m - 1 < \delta_m \leq r_m, r_m \in \mathrm{N}$ and $\sigma_m \in \mathbb{R}^+ (Re(\sigma_m) > 0)$ This equation can be simplified to:

$$v_{\mathrm{N}}(\mathfrak{z}) = \sum_{m=0}^{k} \left(\eta_m \mathscr{I}^{\rho-\delta_m} v(\mathfrak{z}) + \lambda_m \mathscr{I}^{\rho+\sigma_m} v(\mathfrak{z})\right) + \mathbb{G}(\mathfrak{z}), \qquad v^{(m)}(0) = l_m, \quad m = 0, \cdots, r-1, \tag{4.19}$$

where the term $\mathbb{G}(\mathfrak{z})$ is:

$$\mathbb{G}(\mathfrak{z}) = \mathscr{I}^{\rho} \mathbb{F}(\mathfrak{z}) + \sum_{n=0}^{r-1} l_n \frac{\mathfrak{z}^n}{n!} + \sum_{m=0}^{k} \eta_m \mathscr{I}^{\rho-\delta_m} \left(\sum_{n=0}^{r_m-1} l_n \frac{\mathfrak{z}^n}{n!} \right)$$

To solve the fully integrated problem (4.19) with the initial conditions (4.18) using the Tau method with SJOM, we approximate $v_{\mathrm{N}}(\mathfrak{z})$ and $\mathbb{G}(\mathfrak{z})$ using shifted Jacobi polynomials:

$$v_{\mathrm{N}}(\mathfrak{z}) \simeq \sum_{m=0}^{\mathrm{N}} c_m \mathscr{P}_{s,m}^{(\gamma,\delta)}(\mathfrak{z}) = \mathbb{C}^T \varphi(\mathfrak{z}) \tag{4.20}$$

$$\mathbb{G}(\mathfrak{z}) \simeq \sum_{m=0}^{\mathrm{N}} g_m \mathscr{P}_{s,m}^{(\gamma,\delta)}(\mathfrak{z}) = G^T \varphi(\mathfrak{z}) \tag{4.21}$$

where the vector $G=\left[g_0,g_1,\cdots,g_{\mathrm{N}}\right]^T$ is known, but $\mathbb{C}=[c_0,c_1,\cdots,c_{\mathrm{N}}]^T$ is an unknown vector.

Next, we apply the Riemann-Liouville integral of orders $\left(\rho-\delta_n\right)$ and $(\rho+\sigma_m)$ to the approximate solution (4.20), and using Theorem 4.3 (relation (4.6)), we express it as:

$$\mathscr{I}^{\rho+\sigma_m}v_{\mathrm{N}}(\mathfrak{z})\simeq\mathbb{C}^T\mathscr{I}^{\rho+\sigma_m}\varphi(\mathfrak{z})\simeq\mathbb{C}^T\mathscr{P}^{(\rho+\sigma_m)}\varphi(\mathfrak{z}) \tag{4.22}$$

where $\mathscr{P}^{(\rho)}$ is the operational matrix of fractional integration of order ρ. This step helps in applying the Tau method to approximate the solution of the multi-order fractional differential equation.

We begin with the equation:

$$\mathscr{I}^{\rho-\delta_n}v_{\mathrm{N}}(\mathfrak{z})\simeq\mathbb{C}^T\mathscr{I}^{\rho-\delta_n}\varphi(\mathfrak{z})\simeq\mathbb{C}^T\mathscr{P}^{(\rho-\delta_n)}\varphi(\mathfrak{z}),\quad n=1,\cdots,k, \tag{4.23}$$

Where $\mathscr{P}^{(\rho)}$ is the operational matrix of fractional integration of order ρ, and the vector $\mathbb{C}$ represents the unkown coefficients of the approximation.

Using equations (4.20) to (4.23) the residual $R_{\mathrm{N}}(\mathfrak{z})$ for Equation (4.19) can be written as:

$$R_{\mathrm{N}}(\mathfrak{z})=\left(\mathbb{C}^T\varphi(\mathfrak{z})-\mathbb{C}^T\sum_{n=0}^{k}\eta_n\mathscr{P}^{(\rho-\delta_n)}\varphi(\mathfrak{z})-\mathbb{C}^T\sum_{n=0}^{k}\lambda_n\mathscr{P}^{(\rho-\sigma_n)}\varphi(\mathfrak{z})-G^T\varphi(\mathfrak{z})\right)$$

$$R_{\mathrm{N}}(\mathfrak{z})=\left(\mathbb{C}^T-\mathbb{C}^T\left(\sum_{n=0}^{k}\eta_n\mathscr{P}^{(\rho-\delta_n)}-\lambda_n\mathscr{P}^{(\rho-\sigma_n)}\right)-G^T\right)\varphi(\mathfrak{z}) \tag{4.24}$$

To proceed with solving for the unknown coefficients, we apply the standard approach of the Tau method. We generate $\mathrm{N}-r+1$ linear algebraic equations by orthogonality, using the condition:

$$\left\langle R_{\mathrm{N}}(\mathfrak{z}),\mathscr{P}_{s,n}^{(\gamma,\delta)}(\mathfrak{z})\right\rangle=\int_0^s R_{\mathrm{N}}(\mathfrak{z})\mathscr{P}_{s,n}^{(\gamma,\delta)}(\mathfrak{z})d\mathfrak{z}=0,\quad n=0,1,\cdots,\mathrm{N}-r. \tag{4.25}$$

Moreover, by substituting equations (4.5) and (4.20) into equation (4.18), we get the initial conditions for the solution:

$$v^{(m)}(0)=\sum_{m=0}^{\mathrm{N}}c_m\mathscr{D}^{(m)}\mathscr{P}_{s,m}^{(\gamma,\delta)}(0)=l_m,\quad m=0,1,\cdots,r-1 \tag{4.26}$$

These generate $\mathrm{N}-r+1$ equations from the residual and r equations from the initial conditions. The system of equations can then be solved to obtain the unknown vector $\mathbb{C}$, providing an approximate solution to the multi-order fractional differential equation (4.17) with the given initial conditions (4.18).

4.5 Illustrative examples

To demonstrate the efficiency of the proposed method in this thesis, several test examples are presented in this section. The results obtained confirm that the proposed approach is highly effective and well-suited for solving linear fractional differential equations (FDEs).

Example 4.5.1

As an example of applying this method, consider the initial value problem:

$$\mathscr{D}^{\frac{3}{2}} v(\mathfrak{z}) + 3v(\mathfrak{z}) = 3\mathfrak{z}^3 + \frac{8}{\Gamma(0.5)}\mathfrak{z}^{1.5}, \quad v(0) = 0, v'(0) = 0, \quad \mathfrak{z} \in [0, s] \tag{4.27}$$

where the exact solution is given by $v(\mathfrak{z}) = \mathfrak{z}^3$.

Using the method outlined in Section 4 with $N = 3$, the approximate solution and the right-hand side are represented in terms of shifted Jacobi polynomials as:

$$v(\mathfrak{z}) = \sum_{m=0}^{3} c_m \mathscr{P}_{s,m}^{(\gamma,\delta)}(\mathfrak{z}) = \mathbb{C}^T \varphi(\mathfrak{z}), \quad \text{and} \quad \mathbb{G}(\mathfrak{z}) \simeq \sum_{m=0}^{3} g_m \mathscr{P}_{s,m}^{(\gamma,\delta)}(\mathfrak{z}) = G^T \varphi(\mathfrak{z}).$$

From equation (4.6), the matrix for the fractional integration is:

$$\mathscr{P}^{\left(\frac{3}{2}\right)} = \begin{pmatrix} \Psi_{\frac{3}{2}}(0,0,\gamma,\delta) & \Psi_{\frac{3}{2}}(0,1,\gamma,\delta)\Psi_{\frac{3}{2}}(0,2,\gamma,\delta) & \Psi_{\frac{3}{2}}(0,3,\gamma,\delta) \\ \Psi_{\frac{3}{2}}(1,0,\gamma,\delta) & \Psi_{\frac{3}{2}}(1,1,\gamma,\delta)\Psi_{\frac{3}{2}}(1,2,\gamma,\delta) & \Psi_{\frac{3}{2}}(1,3,\gamma,\delta) \\ \Psi_{\frac{3}{2}}(2,0,\gamma,\delta) & \Psi_{\frac{3}{2}}(2,1,\gamma,\delta)\Psi_{\frac{3}{2}}(2,2,\gamma,\delta) & \Psi_{\frac{3}{2}}(2,3,\gamma,\delta) \\ \Psi_{\frac{3}{2}}(3,0,\gamma,\delta) & \Psi_{\frac{3}{2}}(3,1,\gamma,\delta)\Psi_{\frac{3}{2}}(3,2,\gamma,\delta) & \Psi_{\frac{3}{2}}(3,3,\gamma,\delta) \end{pmatrix},$$

and the vector G is given as:

$$G = \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{pmatrix},$$

Thus, the method facilitates solving for $v(\mathfrak{z})$ and approximating the solution of fractional differential equations by leveraging the power of the SJOM and the Tau method.

The term $\Psi_{\frac{3}{2}}(m,n,\gamma,\delta)$ is defined in Equation (4.8), and $\mathbb{G}_n$ is given by:

$$\mathbb{G}_n = \frac{(2n+\gamma+\delta+1)n!}{s^{\gamma+\delta+1}\Gamma(n+\gamma+1)} \sum_{f=0}^{n} \frac{(-1)^{n-f}\Gamma(f+n+\gamma+\delta+1)}{s^f f!(n-f)!\Gamma(f+\delta+1)} \times \int_0^s \left(\frac{64\mathfrak{z}^{9/2}}{105\sqrt{\pi}} + \mathfrak{z}^3\right)\mathfrak{z}^{\delta+f}(t-\mathfrak{z})^{\gamma} d\mathfrak{z}$$

Using equations (4.23) and (4.25), we arrive at the following residual equations:

$$3\Psi_{\frac{3}{2}}(0,2,\gamma,\delta)c_0 + 3\Psi_{\frac{3}{2}}(1,2,\gamma,\delta)c_1 + 3\Psi_{\frac{3}{2}}(2,2,\gamma,\delta)c_2 + 3\Psi_{\frac{3}{2}}(3,2,\gamma,\delta)c_3 + c_2 - g_2 = 0, \tag{4.28}$$

$$3\Psi_{\frac{3}{2}}(0,3,\gamma,\delta)c_0+3\Psi_{\frac{3}{2}}(1,3,\gamma,\delta)c_1+3\Psi_{\frac{3}{2}}(2,3,\gamma,\delta)c_2+3\Psi_{\frac{3}{2}}(3,3,\gamma,\delta)c_3+c_3-g_3=0. \quad (4.29)$$

By applying equation (4.26) for the initial conditions, we get:

$$\mathbb{C}^T\varphi(0)=c_0-(\delta+1)c_1+\frac{(\delta+1)(\delta+2)}{2}c_2-\frac{(\delta+1)(\delta+2)(\delta+3)}{6}c_3=0, \quad (4.30)$$

$$\mathbb{C}^T D^{(1)}\varphi(0)=\frac{(\gamma+\delta+2)}{t}c_1-\frac{(\delta+2)(\gamma+\delta+3)}{s}c_2+\frac{(\delta+2)(\delta+3)(\gamma+\delta+4)}{2s}c_3=0 \quad (4.31)$$

Solving the system of equations (4.28) to (4.30) yields the approximate solution.

Special cases for the ultraspherical basis can be derived by setting $\gamma=\delta$ and replacing each with $\gamma-\frac{1}{2}$. Similarly, special cases for the Chebyshev basis (first, second, third, and fourth kinds) are obtained by setting $\gamma=\delta=\mp\frac{1}{2}$ and $\gamma=-\delta=\pm\frac{1}{2}$, respectively. For the Legendre basis, we set $\gamma=\delta=0$.
When $\gamma=\delta=0$, the coefficients are:

$$c_0=\frac{s^3}{4}, \quad c_1=\frac{9s^3}{20}, \quad c_2=\frac{s^3}{4}, \quad c_3=\frac{s^3}{20}$$

Thus, the approximate solution is:

$$v_{\mathrm{N}}(\mathfrak{z})=\sum_{m=0}^{3}c_m\mathscr{P}_{s,m}^{(0,0)}(\mathfrak{z})=\mathfrak{z}^3$$

which is the exact solution.
If $\gamma=-\frac{1}{2},\delta=\frac{1}{2}$, the coefficients become:

$$c_0=\frac{35s^3}{64}, \quad c_1=\frac{21s^3}{32}, \quad c_2=\frac{7s^3}{24}, \quad c_3=\frac{s^3}{20}$$

The approximate solution is:

$$v_{\mathrm{N}}(\mathfrak{z})=\sum_{m=0}^{3}c_m\mathscr{P}_{s,m}^{(-\frac{1}{2},\frac{1}{2})}(\mathfrak{z})=\mathfrak{z}^3$$

which is again the exact solution.

In the case where $\gamma=\frac{1}{2},\delta=-\frac{1}{2}$,the coefficients are:

$$c_0=\frac{5s^3}{64}, \quad c_1=\frac{9s^3}{32}, \quad c_2=\frac{5s^3}{24}, \quad c_3=\frac{s^3}{20}$$

The approximate solution is:

$$v_{\mathrm{N}}(\mathfrak{z})=\sum_{m=0}^{3}c_m\mathscr{P}_{s,m}^{(\frac{1}{2},-\frac{1}{2})}(\mathfrak{z})=\mathfrak{z}^3$$

which is also the exact solution.

Example 4.5.2

Consider the following equation:

$$\mathscr{D}^2 v(\mathfrak{z}) - 2\mathscr{D} v(\mathfrak{z}) + \mathscr{D}^{\frac{1}{2}} v(\mathfrak{z}) + v(\mathfrak{z}) = \mathfrak{z}^3 - 6\mathfrak{z}^2 + 6\mathfrak{z} + \frac{16}{5\sqrt{\pi}}\mathfrak{z}^{2.5}, \quad v(0)=0, v'(0)=0, \quad \mathfrak{z} \in [0, s] \tag{4.32}$$

Hhere the exact solution is $v(\mathfrak{z}) = \mathfrak{z}^3$.

We now implement the technique outlined in Example 1 with $\mathrm{N} = 3$.

The approximate solution obtained using the proposed method for different special cases of γ and δ is as folloxs:
Case 1. When $\gamma = \delta = 0$, the coefficient are:

$$c_0 = \frac{s^3}{4}, \quad c_1 = \frac{9s^3}{20}, \quad c_2 = \frac{s^3}{4}, \quad c_3 = \frac{s^3}{20}$$

The approximate solution is:

$$v_{\mathrm{N}}(\mathfrak{z}) = \sum_{m=0}^{3} c_m \mathscr{P}_{s,m}^{(0,0)}(\mathfrak{z}) = \mathfrak{z}^3$$

which represents the exact solution.

Case 2. When $\gamma = -\frac{1}{2}, \delta = \frac{1}{2}$, the coefficient are:

$$c_0 = \frac{35s^3}{64}, \quad c_1 = \frac{21s^3}{32}, \quad c_2 = \frac{7s^3}{24}, \quad c_3 = \frac{s^3}{20}$$

The approximate solution is:

$$v_{\mathrm{N}}(\mathfrak{z}) = \sum_{m=0}^{3} c_m \mathscr{P}_{s,m}^{(-\frac{1}{2},\frac{1}{2})}(\mathfrak{z}) = \mathfrak{z}^3$$

which represents the exact solution.

Case 3. When $\gamma = \frac{1}{2}, \delta = -\frac{1}{2}$, the coefficient are:

$$c_0 = \frac{5s^3}{64}, \quad c_1 = \frac{9s^3}{32}, \quad c_2 = \frac{5s^3}{24}, \quad c_3 = \frac{s^3}{20}$$

The approximate solution is:

$$v_{\mathrm{N}}(\mathfrak{z}) = \sum_{m=0}^{3} c_m \mathscr{P}_{s,m}^{(\frac{1}{2},-\frac{1}{2})}(\mathfrak{z}) = \mathfrak{z}^3,$$

which is the exact solution.
Case 4. When $\gamma = \delta = -\frac{1}{2}$, the coefficient are:

$$c_0 = \frac{5s^3}{16}, \quad c_1 = \frac{15s^3}{32}, \quad c_2 = \frac{3s^3}{16}, \quad c_3 = \frac{s^3}{32}$$

The approximate solution is:

$$v_N(\mathfrak{z}) = \sum_{m=0}^{3} c_m \mathscr{P}_{s,m}^{(-\frac{1}{2},-\frac{1}{2})}(\mathfrak{z}) = \mathfrak{z}^3$$

which is the exact solution.

Example 4.5.3

Consider the equation

$$\mathscr{D}^2 v(\mathfrak{z}) - 2\mathscr{D} v(\mathfrak{z}) + \mathscr{D}^{\frac{1}{2}} v(\mathfrak{z}) + v(\mathfrak{z}) = \mathfrak{z}^7 + \frac{2048}{429\sqrt{\pi}}\mathfrak{z}^{6.5} - 14\mathfrak{z}^6 + 42\mathfrak{z}^5 - \mathfrak{z}^2 - \frac{8}{3\sqrt{\pi}}\mathfrak{z}^{1.5} + 4\mathfrak{z} - 2, \tag{4.33}$$

$$v(0) = 0, \quad v'(0) = 0, \quad \mathfrak{z} \in [0, t] \tag{4.34}$$

with the exact solution $v(\mathfrak{z}) = \mathfrak{z}^7 - \mathfrak{z}^2$.

We now implement the method outlined in Example 4.5 with $N = 9$ for various specific values of γ and δ, resulting in the following cases:
Case 1: When $\gamma = \delta = 0$, the coefficient are:

$$c_0 = \frac{s^2}{24}\left(3s^5 - 8\right), \quad c_1 = \frac{s^2}{24}\left(7s^5 - 12\right), \quad c_2 = \frac{s^2}{24}\left(7s^5 - 4\right), \quad c_3 = \frac{49s^7}{264},$$
$$c_4 = \frac{7s^7}{88}, \quad c_5 = \frac{7s^7}{312}, \quad c_6 = \frac{s^7}{264}, \quad c_7 = \frac{s^7}{3432}, \quad c_8 = 0, \quad c_9 = 0,$$

The approximate solution is:

$$v_N(\mathfrak{z}) = \sum_{m=0}^{9} c_m \mathscr{P}_{s,m}^{(0,0)}(\mathfrak{z}) = \mathfrak{z}^7 - \mathfrak{z}^2,$$

which represents the exact solution.
Case 2: When $\gamma = \delta = \frac{1}{2}$, the coefficients are

$$c_0 = \frac{5s^2}{8192}\left(143s^5 - 512\right), \quad c_1 = \frac{s^2}{6144}\left(1001s^5 - 2048\right), \quad c_2 = \frac{s^2}{5120}\left(819s^5 - 512\right),$$
$$c_3 = \frac{13s^7}{128}, \quad c_4 = \frac{25s^7}{576}, \quad c_5 = \frac{15s^7}{1232}, \quad c_6 = \frac{7s^7}{3432}, \quad c_7 = \frac{s^7}{6435}, \quad c_8 = 0, \quad c_9 = 0,$$

The approximate solution is:

$$v_N(\mathfrak{z}) = \sum_{m=0}^{9} c_m \mathscr{P}_{s,m}^{(\frac{1}{2},\frac{1}{2})}(\mathfrak{z}) = \mathfrak{z}^7 - \mathfrak{z}^2$$

which is the exact solution.

Case 3: When $\gamma = -\frac{1}{2}, \delta = \frac{1}{2}$, the coefficients are

$$c_0 = \frac{s^2\left(6435s^5 - 10240\right)}{16384}, \quad c_1 = \frac{s^2\left(5005s^5 - 5120\right)}{8192}, \quad c_2 = \frac{s^2\left(3003s^5 - 1024\right)}{6144},$$
$$c_3 = \frac{273s^7}{1024}, \quad c_4 = \frac{13s^7}{128}, \quad c_5 = \frac{5s^7}{192}, \quad c_6 = \frac{5s^7}{1232}, \quad c_7 = \frac{s^7}{3432}, \quad c_8 = 0, \quad c_9 = 0,$$

The approximate solution is:

$$v_{\mathrm{N}}(\mathfrak{z}) = \sum_{m=0}^{9} c_m \mathscr{P}_{s,m}^{\left(-\frac{1}{2},\frac{1}{2}\right)}(\mathfrak{z}) = \mathfrak{z}^7 - \mathfrak{z}^2$$

which is the exact solution.

Case 4: When $\gamma = \frac{1}{2}, \delta = -\frac{1}{2}$, the coefficients are:

$$c_0 = \frac{s^2\left(429s^5 - 2048\right)}{16384}, \quad c_1 = \frac{s^2\left(1001s^5 - 3072\right)}{8192}, \quad c_2 = \frac{s^2\left(1001s^5 - 1024\right)}{6144},$$
$$c_3 = \frac{637s^7}{5120} \quad c_4 = \frac{39s^7}{640}, \quad c_5 = \frac{11s^7}{576}, \quad c_6 = \frac{13s^7}{3696}, \quad c_7 = \frac{s^7}{3432},$$

The approximate solution is:

$$v_{\mathrm{N}}(\mathfrak{z}) = \sum_{m=0}^{9} c_m \mathscr{P}_{s,m}^{\left(\frac{1}{2},-\frac{1}{2}\right)}(\mathfrak{z}) = \mathfrak{z}^7 - \mathfrak{z}^2,$$

which is the exact solution.

Conclusion

In this study, we conducted an in-depth investigation into fractional calculus and its application in solving fractional-order differential equations using operational matrices combined with orthogonal polynomials. Our approach leveraged Bernstein polynomials, modified Legendre polynomials, and shifted Jacobi polynomials to develop efficient numerical methods. The results demonstrated the robustness of these techniques in enhancing numerical accuracy and solution stability, as validated through multiple illustrative examples. The application of fractional-order differential equations spans numerous scientific and engineering fields, including control systems, viscoelastic material modeling, and signal processing. The use of operational matrices in conjunction with orthogonal polynomials provides a powerful tool for efficiently solving complex differential problems, reducing computational efforts while maintaining high precision. Moving forward, several avenues remain open for further exploration. One promising direction is the extension of these methods to higher-dimensional fractional differential equations, which could enhance their applicability in multi-dimensional physical and engineering systems. Additionally, the incorporation of machine learning techniques to optimize polynomial selection and parameter tuning could further improve computational efficiency and adaptability. Future research may also explore hybrid methods that integrate operational matrix approaches with finite element or spectral methods to enhance accuracy for highly nonlinear problems. Overall, this work contributes to the growing field of fractional calculus by offering effective numerical methods and highlighting potential areas for continued development and innovation. The combination of analytical rigor and computational techniques provides a solid foundation for tackling complex fractional-order systems in a wide range of scientific disciplines.

Bibliography

[1] A. Abbas, Bernstein operational matrix for solving variable order fractional differential equations with generalized Caputo-type, Wasit J. Pure Sci., 3(2) (2024), 35–42.

[2] A. Boulfoul, B. Tellab, N. Abdellouahab, Kh. Zennir, Existence and uniqueness results for initial value problem of nonlinear fractional integro-differential equation on an unbounded domain in a weighted Banach space, Math. Meth. Appl. Sci. (2020) DOI: 10.1002/mma.6957.

[3] A. Boutiara, K. Guerbati, M. Benbachir, Caputo-Hadamard fractional differential equation with three-point boundary conditions in Banach spaces, AIMS. Mathematics, **5** (2019) 259–272.

[4] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and applications of fractional differential equations, North Holland Mathematics Studies, 203, Amsterdam, (2006)

[5] A. Alsaedi, B. Ahmad, M. Alghanmi, S.K. Ntouyas, On a generalized Langevin type nonlocal fractional integral multivalued problem, Mathematics. **7**; (1015), 2019.

[6] A. Granas , J. Dugundji. Fixed point theory. New York: Springer-Verlag; (2003).

[7] A. H. BHRAWY, M. M. THARWAT AND M. A. ALGHAMDI, A New Operational Matrix of Fractional Integration for Shifted Jacobi Polynomials, Bull. Malays. Math. Sci. Soc. (2) 37(4) (2014), 983–995.

[8] A. Kheiryan, Sh. Rezapour, *A multi-singular fractional equation and the Hyers-Ulam stability*, Int. J. Appl. Comput. Math. **6**(155) (2020) doi:10.1007/s40819-020-00908-0

[9] A. Saadatmandi. Bernstein operational matrix of fractional derivatives and its applications. Appl. Math. Model. **38**(4), 1365-1372 (2014).
https://doi.org/10.1016/j.apm.2013.08.007

[10] A. Saadatmandi, M. Dehghan. A new operational matrix for solving fractional-order differential equations. Comp. Math. Appl. **59**(3), 1326–1336 (2010). https://-doi.org/10.1016/j.camwa.2009.07.006

[11] B. Ahmad, S.K. Ntouyas, J. Tariboon, On hybrid Caputo fractional integro-differential inclusions with nonlocal conditions, J. Nonl. Sci. Appl. **9** (2016) 4235–4246.

[12] B. Juttler.The dual basis functions for the Bernstein polynomials. Adv. Comput. Math. **8**, 345–352 (1998). https://doi.org/10.1023/A:1018912801267

[13] B.R. Sontakke, A. Shaikh. Approximate solutions of time fractional Kawahara and modified Kawahara equations by fractional complex transform. Commun. Numer. Anal. 2016(2), 218-229 (2016). https://doi.org/10.5899/2016/cna-00277

[14] B. Samet, C. Vetro, P. Vetro. Fixed point theorems for α-Ψ-contractive type mappings. Nonlinear Anal. Theory Methods Appl. 75(4), 2154-2165 (2012). https://-doi.org/10.1016/j.na.2011.10.014.

[15] C. Park, R.I. Nurudden, K.K. Ali, L. Muhammad, D. Baleanu. Novel hyperbolic and exponential ansatz methods to the fractional fifth-order Korteweg-de Vries equations. Adv. Differ. Equ. 2020, 627 (2020). https://doi.org/10.1186/s13662-020-03087-w

[16] D. Baleanu , S. Etemad , Sh. Rezapour . A hybrid Caputo fractional modeling for thermostat with hybrid boundary value conditions. Bound. Value Probl. 2020, 64 (2020). https://-doi.org/10.1186/s13661-020-01361-0

[17] D. Baleanu , S. Etemad , H. Mohammadi , Sh. Rezapour . A novel modeling of boundary value problems on the glucose graph. Commun. Nonlinear Sci. Numer. Simul. **100**, 105844 (2021). https://doi.org/10.1016/j.cnsns.2021.105844

[18] D. Baleanu, S. Etemad, Sh. Rezapour, On a fractional hybrid integro-differential equation with mixed hybrid integral boundary value conditions by using three operators, Alex. Eng. J. (2020) doi:10.1016/j.aej.2020.04.053

[19] G. Infante, J.R.L. Webb. Loss of positivity in a nonlinear scalar heat equation. Nonlinear Differ. Equ. Appl. **13**, 249-261 (2006). https://doi.org/10.1007/s00030-005-0039-y

[20] G. Szegő, *Orthogonal Polynomials*, American Mathematical Society, 1975.

[21] G. W. Leibniz, Letter from Hanover, Germany to G. F. A. L'Hôpital, September 30, 1695, in : Leibnizens Mathematische Schriften 1849. [reprinted in Hildesheim, Germany, Olns Verlag, 2, 1962, 301-302.].

[22] G. Wang, K. Pei, R.P. Agarwal, L. Zhang, B. Ahmad, Nonlocal Hadamard fractional boundary value problem with Hadamard integral and discrete boundary conditions on a half-line, J. Comput. Appl. Math. **343** (2018): 230–239.

[23] H. Brezis. Functional Analysis, Sobolev Spaces and Partial Differential Equations. New York Dordrecht Heidelberg London: Springer; (2010).

[24] H.K. Pathak, R. Rodriguez-Lopez. Existence and approximation of solutions to nonlinear hybrid ordinary differential equations. Appl. Math. Lett. 39, 101-106 (2015). https://-doi.org/10.1016/j.aml.2014.08.018

[25] I.B.H. Elbadawi, Q. Dong, Z. Fan. Existence for boundary value problems of two-term Caputo fractional differential equations. J. Nonlinear. Sci. Appl. **10**; (2017):511-520.

[26] I. Podlubny. Fractionl Differential Equations. New York: Acdemic Press; (1999).

[27] J. P. Boyd, *Chebyshev and Fourier Spectral Methods*, Dover Publications, 2001.

[28] K. Diethelm, N.J. Ford. Multi-order fractional differential equations and their numerical solution, Appl. Math. Comput. 154 (2004) 621-640. https://doi.org/10.1016/S0096-3003(03)00739-2

[29] K.M. Owolabi, R.P. Agarwal, E. Pindza, S. Bernstein, M.S. Osman. Complex Turing patterns in chaotic dynamics of autocatalytic reactions with the Caputo fractional derivative. Neural Comput & Applic 35, 11309-11335 (2023). https://doi.org/10.1007/s00521-023-08298-2

[30] L. Peng, Y. Zhou, *Bifurcation from interval and positive solutions of the three-point boundary value problem for fractional differential equations*, Appl. Math. Comput. **257** (2015) 458–466.

[31] L. S. Salih, S. S. Ahmed, A Bernstein operational matrix approach for solving nonlinear Volterra–Fredholm integro-differential equations involved by Caputo fractional derivative, J. Univ. Babylon Pure Appl. Sci., 33(3) (2025), 309–354.

[32] M. Ahmad, A. Zada, X. Wang, Existence, Uniqueness and stability of implicit switched coupled fractional differential equations of ψ-Hilfer type, Int. J. Nonlinear Sci. Numer. Simul. **21** (2020) 327–337.

[33] M.H. Al-Smadi , G. Gumah . On the homotopy analysis method for fractional SEIR epidemic model. Research J. Appl. Sci. Engrg. Technol. **7**(18), 3809–3820 (2014). https://-doi.org/10.19026/rjaset.7.738

[34] M. Muslim, R.P. Agarwal. Existence, uniqueness and convergence of approximate solutions of nonlocal functional differential equations. Carpathian J. Math. **27**(2), 249–259 (2011). https://doi.org/10.37193/CJM.2011.02.03

[35] M. Razzaghi, S. Yousefi. The Legendre wavelets operational matrix of integration. Int. J. Syst. Sci. **32**(4), 495–502 (2010). https://doi.org/10.1080/00207720120227

[36] M. Subramanian, A. Zada, Existence and uniqueness of solutions for coupled systems of Liouville-Caputo type fractional integro-differential equations with Erdelyi-Kober integral conditions, Int. J. Nonlinear. Sci. Numer. Simul. (2020):2019-0299 doi:10.1515/ijnsns

[37] N.D. Phuong, F.M. Sakar, S. Etemad, Sh. Rezapour, A novel fractional structure of a multi-order quantum multi-integro-differential problem, Adv. Diff. Equ. (2020) 2020:633.

[38] N. Hafsi, B. Tellab, M. Meflah . Approximate solutions for a fractional thermostat model boundary value problem via Bernstein's collocation method with Legendre polynomials. Mathematical Methods in the Applied Sciences, 47(3), 1234–1245(2023). https://-doi.org/10.1002/mma.9542

[39] R.P. Agarwal, D. O'Regan ,S. Staněk . Positive solutions for Dirichlet problems of singular nonlinear fractional differential equations. J. Math. Anal. Appl. **371**; (2010):57-68.

[40] R.T. Farouki. The bernstein polynomial basis: a centennial retrospective. Comput. Aided Geom. Des. **29**(6), 379-419 (2012). https://doi.org/10.1016/j.cagd.2012.03.001

[41] R.S. Adiguzel, U. Aksoy, E. Karapinar, I.M. Erhan. On the solution of a boundary value problem associated with a fractional differential equation. Math. Methods Appl. Sci. (2020). https://doi.org/10.1002/mma.6652

[42] R. Saima, K. Tul Kubra, S. Sobia, A. Praveen, M.S. Osman. An approximate analytical view of physical and biological models in the setting of Caputo operator via Elzaki transform decomposition method. Journal of Computational and Applied Mathematics, Vol. 413, No. 15, 114378 (2022). https://doi.org/10.1016/j.cam.2022.114378

[43] R.T. Farouki. The bernstein polynomial basis: a centennial retrospective. Comput. Aided Geom. Des. **29**(6), 379–419 (2012). https://doi.org/10.1016/j.cagd.2012.03.001

[44] S. Ben-Chikh, A. Amara, S. Etemad, Sh. Rezapour, On Hyers-Ulam stability of a multi-order boundary value problems via Riemann-Liouville derivatives and integrals, Adv. Diff. Equ. **2020** (2020) .1–20.

[45] S. Djennadi, N. Shawgfeh, M. Inc, MS. Osman , J.F. Gomez-Aguilar, O. Abu Arqub. The Tikhonov regularization method for the inverse source problem of time fractional heat equation in the view of ABC-fractional technique. Phys. Scr. 96 094006 (2021). https://-doi.org/10.1088/1402-4896/ac0867

[46] S. Etemad, B. Tellab, C.T. Deressa, J. Alzabut, Y. Li, Sh. Rezapour. On a generalized fractional boundary value problem based on the thermostat model and its numerical solutions via Bernstein polynomials. Advances in Difference Equations, (2021) 2021:458. https://-doi.org/10.1186/s13662-021-03610-7

[47] Sh. Rezapour, S. Etemad, H. Mohammadi. A mathematical analysis of a system of Caputo-Fabrizio fractional differential equations for the anthrax disease model in animals. Adv. Differ. Equ. **2020**, 481 (2020). https://doi.org/10.1186/s13662-020-02937-x

[48] Sh. Rezapour, A. Imran, A. Hussain, F. Martinez, S. Etemad, M.K.A. Kaabar. Condensing functions and approximate endpoint criterion for the existence analysis of quantum integro-difference FBVPs. Symmetry **13**(3), 469 (2021). https://doi.org/10.3390/sym13030469

[49] Sh. Rezapour, S.K. Ntouyas, M.Q. Iqbal, A. Hussain, S. Etemad, J. Tariboon. An analytical survey on the solutions of the generalized double-order ϕ-integrodifferential equation. J. Funct. Spaces **2021**, Article ID 6667757 (2021). https://doi.org/10.1155/2021/6667757

[50] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional integrals and derivatives: theory and applications, Amesterdam, Gordon and Breach, (1993).

[51] S. Kempfle, I. Schafer, H. Beyer, *Fractional calculus via functional calculus: theory and applications*, Nonl. Dyn. **29** (2002) 99–127.

[52] Sh. Rezapour, B. Azzaoui, B. Tellab, S. Etemad, H.P. Masiha. An analysis on the positives Solutions for a fractional configuration of the Caputo multiterm semilinear differential equation. J. Funct. Spaces 2021, Article ID 6022941 (2021). https://-doi.org/10.1155/2021/6022941

[53] S. Lei, R. Saima, S. Sobia, K. Aasma , A. Praveen, M.S. Osman. Semi-analytical view of time-fractional PDEs with proportional delays pertaining to index and Mittag-Leffler memory interacting with hybrid transforms. Fractals, Vol. 31, No. 4 2340071 (2023). https://-doi.org 10.1142/S0218348X23400716

[54] S. Lei, S. Tayebi, O. Abu Arqub, M.S. Osman, A. Praveen, W. Mahamoud, M.Abdel-Aty, M. Alhodaly . The novel cubic B-spline method for fractional Painlevé and Bagley-Trovik equations in the Caputo, Caputo-Fabrizio, and conformable fractional sense. Alexandria Engineering Journal. Vol. 65, 15, 413-426 (2023). https://doi.org/10.1016/j.aej.2022.09.039

[55] S.W. Yao, O.A. Arqub, S. Tayebi, M.S. Osman , W. Mahamoud, M. Inc, H. Alsulami. A novel collective algorithm using cubic uniform spline and finite difference approaches to solving fractional diffusion singular wave model through damping-reaction forces. Fractals, Vol. 31, No. 4, 2340069 (2023) . https://doi.org/10.1142/S0218348X23400698

[56] S. Staněk . Periodic problem for two-term fractinal differential equations. Fract Calc Appl Anal. **20**; (2017):662-678.

[57] S.T.M. Thabet, S. Etemad, Sh. Rezapour, On a new structure of the pantograph inclusion problem in the Caputo conformable setting, Bound. Value. Probl. (2020) 1–71.

[58] T. J. Rivlin, *Chebyshev Polynomials: From Approximation Theory to Algebra and Number Theory*, Wiley, 1990.

[59] U. Riaz, A. Zada, Analysis of (α,β)-order coupled implicit Caputo fractional differential equations using topological degree method, Int. J. Nonlinear Sci. Numer. Simul. (2021) doi:10.1515/ijnsns-897–915

[60] W. Gautschi, *Orthogonal Polynomials: Computation and Approximation*, Oxford University Press, 2004.

[61] X. Su. Boundary value problem for coupled system of nonlinear fractional differential equations. Appl. Math. Lett. 22(1), 64-69 (2009). https://doi.org/10.1016/j.aml.2008.03.001.

[62] Y. Chu, S. Rashid, K. Tul Kubra, M. Inc, Z. Hammouch, M.S. Osman. Analysis and Numerical Computations of the Multi-Dimensional, Time-Fractional Model of Navier-Stokes Equation with a New Integral Transformation. Computer Modeling in Engineering & Sciences 136(3), 3025-3060 (2023). https://doi.org/10.32604/cmes.2023.025470

[63] Y. Cui, W. Ma, Q. Sun, X. Su, New uniqueness results for boundary value problem of fractional differential equation, Nonlinear Anal. Model. Control. **23** (2018) 31–39.

[64] Y. Zhou. Basic theory of fractional differential equations, 6. Singapore:World Scientific; (2014).

[65] Z. Li, et al., Shifted Bernstein polynomial-based dynamic analysis for variable fractional order nonlinear viscoelastic bar, Fractals and Fractional, 9(3) (2025), 192