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Theme:

On Some Fuzzy Fractional Problems

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Dedication

To my beloved mother and father, whose prayers and encouragement have always been my strength.

.* * *

To my dear brothers and sisters, for their unwavering support and love.

.* * *

To all my friends and extended family, for being part of my journey.

.* * *

To my colleagues at the Department of Mathematics, Kasdi Merbah University of Ouargla, for their companionship and collaboration.

.* * *

I dedicate this humble work with deep appreciation and respect.

Brahim Ghrissi

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My sincere thanks also go to the chairman of the jury, ***Prof. Mabrouk Meflah****, for accepting this role and for his contribution to the defense process.*

الملخص

في هذه الأطروحة، نقدّم دراسة شاملة للنظرية الضبابية وحساب التفاضل والتكامل الكسري .وتتمثل المساهمة الرئيسة لهذا العمل في دراسة مسائل تفاضلية كسرية ضبابية ذات شروط ابتدائية، حيث تم إثبات وجود ووحدانية الحلول بالاعتماد على مبرهنة نقطة الثابت لباناخ .كما تم تحليل استقرار هذه الحلول وفق مفهومي استقرار يولام–هايرز واستقرار يولام–هايرز–راسياس .ومن جهة أخرى، تم الحصول على حلول تحليلية لمسألة تفاضلية كسرية ضبابية باستخدام تحويل لابلاس .تسهم هذه الأطروحة في تعميق الفهم النظري للمعادلات التفاضلية الكسرية الضبابية، كما توفّر أدوات رياضية فعّالة لدراستها وتحليلها.

الكلمات المفتاحية : المعادلات التفاضلية الكسرية الضبابية، مبرهنة نقطة الثابت لباناخ، الاستقرار، تحويل لابلاس.

Résumé

Dans cette thèse, nous présentons une étude approfondie de la théorie floue et du calcul différentiel et intégral fractionnaire. La contribution principale de ce travail consiste en l'étude de problèmes différentiels fractionnaires flous à conditions initiales, pour lesquels nous avons établi l'existence et l'unicité des solutions en utilisant le théorème du point fixe de Banach. Nous avons également analysé la stabilité des solutions selon les concepts de stabilité de Ulam–Hyers et de stabilité de Ulam–Hyers–Rassias. Par ailleurs, des solutions analytiques d'un problème différentiel fractionnaire flou ont été obtenues à l'aide de la transformée de Laplace. Cette thèse contribue à approfondir la compréhension théorique des équations différentielles fractionnaires floues et fournit des outils mathématiques efficaces pour leur étude et leur analyse.

Mots clés : Équations différentielles fractionnaires floues, théorème du point fixe de Banach, stabilité, transformée de Laplace.

Abstract

In this thesis, we present a comprehensive study of fuzzy theory and fractional calculus. The main contribution of this work lies in the investigation of fuzzy fractional differential problems with initial conditions, where the existence and uniqueness of solutions are established by means of Banach's fixed point theorem. The stability of the obtained solutions is also examined in the sense of Ulam–Hyers stability and Ulam–Hyers–Rassias stability. Furthermore, analytical solutions to a fuzzy fractional differential problem are derived using the Laplace transform. This thesis contributes to a deeper theoretical understanding of fuzzy fractional differential equations and provides effective mathematical tools for their analysis and study.

Keywords: Fuzzy fractional differential equations, Banach's fixed point theorem, stability, Laplace transform.

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Notations and conventions

- **crisp:** used to describe something that is precise, exact, or clearly defined.
- F^{crisp}: refers to a traditional, non-fuzzy function.
- $\ominus_H$: Hukuhara difference.
- $\tilde{A} \oplus \tilde{B}$: addition of two fuzzy sets.
- $\tilde{A} \ominus \tilde{B}$: subtraction of two fuzzy sets.
- $\tilde{A} \otimes \tilde{B}$: multiplication of two fuzzy sets.
- $\tilde{A} \div \tilde{B}$: division of two fuzzy sets.
- $A_{[\alpha]}$: the α-cut or α-level set of a fuzzy set A.
- $\mathbb{F}_{\mathbb{R}}$: the set of fuzzy numbers x defined on $\mathbb{R}$ with membership values in $[0, 1]$.
- $\mathcal{C}([a, b], \mathbb{F}_{\mathbb{R}})$: the space of all continuous fuzzy-number-valued functions defined on the closed interval $[a, b]$.
- **FFDEs**: Fuzzy Fractional Differential Equations.
- $\omega_{gH}^{(n)}$: the n-th order generalized Hukuhara derivative of a fuzzy-number-valued function ω.

- $\lfloor x \rfloor$: is the greatest integer less than or equal to x.
- $\lceil x \rceil$, the smallest integer greater than or equal to x.

Introduction

In many real-world problems, uncertainty and imprecision are inherent due to incomplete or vague information. Classical set theory and traditional mathematical models often fail to effectively capture such uncertainties. To address this limitation, fuzzy set theory, introduced by Zadeh in 1965, provides a mathematical framework for representing and processing imprecise data [46]. Fuzzy sets generalize classical sets by allowing elements to have degrees of membership rather than strict binary classification. This flexibility has led to widespread applications in artificial intelligence [36, 48], control systems [18, 30], decision-making [13, 48], engineering [25, 40], computing with words [47], electronic commerce [31], and medical supply chain management [35].

Building upon the foundation of fuzzy sets, fuzzy differential equations (FDEs) have been developed to model dynamic systems under uncertainty. However, many physical and engineering processes exhibit memory effects and hereditary properties that cannot be adequately described using integer-order derivatives. This has led to the emergence of fractional calculus, which extends classical differentiation and integration to non-integer orders, offering a more generalized and accurate description of such systems [17, 27, 33, 37, 42].

Combining fuzzy set theory with fractional calculus results in fuzzy fractional differential equations (FFDEs), which provide a robust mathematical tool for modeling complex phenomena characterized by both uncertainty and memory effects. These equations have gained significant attention in recent years due to their applicability in various scientific and engineering fields, including control theory, biological systems, signal processing, and finance [5, 2, 1, 4, 3].

This thesis comprises four chapters:

- **Chapter 1:**

  This chapter introduces the foundational concepts of fuzzy set theory, a generalization of classical set theory originally proposed by Lotfi Zadeh in 1965. Unlike classical sets, fuzzy sets allow for partial membership, enabling a more nuanced representation of uncertainty and imprecision. The chapter begins with a review of classical sets, then transitions into fuzzy set definitions, properties, and basic operations.

- **Chapter 2:** This chapter introduces the fundamental concepts of fractional calculus, extending classical differentiation and integration to non-integer orders. We begin with essential special functions like the Gamma, Beta, and Mittag-Leffler functions. Subsequently, we define the key fractional operators: the Riemann-Liouville fractional integral and derivative, followed by the widely used Caputo fractional derivative. The chapter concludes by outlining the crucial relationship between these two primary fractional derivative definitions, providing a solid foundation for understanding and applying fractional calculus.

- **Chapter 3:** In this chapter, we solve a fuzzy fractional differential equation using the proposed fuzzy fractional Laplace transform. The equation involves both Riemann-Liouville and Caputo fractional derivatives, as well as fractional initial conditions of Liouville and Caputo types:

$$\begin{cases} {}^{RL}D^{\beta}_{0+}x(t) + {}^{C}D^{\beta}_{0+}x(t) = f(t, x(t)), \quad t \in [0, T], \ T > 0 \quad \beta \in (0, 1), \\ x(0) = x_0, \quad I^{1-\beta}x(0) = x_1, \end{cases} \tag{1}$$

where $x \in AC^1[0,1]$, ${}^{RL}D^{\beta}_{0+}$ and ${}^{C}D^{\beta}_{0+}$ denote the Riemann-Liouville and Caputo fractional derivatives of order β, respectively. $I^{1-\beta}$ denotes the Riemann-Liouville fractional integral of order $1-\beta$, $x_0, x_1 \in \mathbb{F}_{\mathbb{R}}$, and $f : [0,T] \times \mathbb{F}_{\mathbb{R}} \to \mathbb{F}_{\mathbb{R}}$ is continuous.

- **Chapter 4:** In this chapter, we investigate the existence, uniqueness, and various types of stability, particularly Ulam-Hyers stability, of the solution to a nonlinear fuzzy fractional differential equation involving the Caputo derivative under generalized Hukuhara (gH) differentiability of order $\alpha \in (n-1, n)$:

$$ {}^{C}_{gH}\mathcal{D}^{\beta}\omega(\xi) = \varpi(\xi, \omega(\xi)), \quad \xi \in [a,b], \tag{2} $$

subject to the initial conditions:

$$ \omega^{(k)}_{gH}(a) = c_k, \quad 0 \leqslant k < n. \tag{3} $$

We establish explicit conditions that guarantee both the existence and uniqueness of the solution by employing the Banach fixed point theorem. Furthermore, to demonstrate the applicability of the theoretical results, we provide two illustrative examples.

Chapter 1

Foundations of Fuzzy Set Theory

In many real-world situations, information is vague, uncertain, or imprecise. Classical set theory, with its rigid boundaries and binary membership, is often insufficient to model such complexity. To address this limitation, fuzzy set theory was introduced as a powerful mathematical framework that allows elements to partially belong to a set, with degrees of membership ranging between 0 and 1. This chapter provides a foundational introduction to fuzzy set theory, presenting the basic concepts, operations, and structures necessary to understand and work with fuzzy sets. It lays the groundwork for more advanced topics in fuzzy mathematics, including fuzzy numbers, fuzzy functions, and the calculus of fuzzy systems. For detailed discussions, see References [5, 10, 11, 12, 19, 26, 43, 45, 46, 48], and the references therein.

1.1 Introduction to Fuzzy Sets

Classical set theory forms the cornerstone of mathematical logic and reasoning, offering a precise framework for grouping and analyzing collections of objects. In this section, we outline the core principles of classical set theory, including the notions of membership, subsets, and standard set operations such as union, intersection, and complement. These foundational concepts not only support a wide range of mathematical disciplines but also serve as a vital point of reference for the development of fuzzy set theory, where the boundaries of set membership become gradual rather than absolute.

Definition 1.1 *[5, 46][Classical Set] A classical set $A \subseteq X$ is defined via a characteristic function χ_A mapping each element of X to either 0 or 1:*

$$\chi_A : X \to \{0,1\}, \quad \chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Example 1.2 *Let $X = \mathbb{R}$, and define $A = [3,8]$. Then:*

$$\chi_A(x) = \begin{cases} 1, & \text{if } 3 \leq x \leq 8, \\ 0, & \text{otherwise.} \end{cases}$$

Fuzzy sets extend the concept of classical sets by allowing each element to have a degree of membership ranging between 0 and 1, rather than a strict binary status of belonging or not. This flexible framework makes it possible to model vagueness and uncertainty inherent in many real-world situations. Although technically referring to fuzzy sets, the term "fuzzy set" is widely used in the literature for simplicity and convenience.

Definition 1.3 *[46][fuzzy set] Let X be a classical set. A fuzzy set $\tilde{A} \subseteq X$ is defined by a membership function $\mu_{\tilde{A}} : X \to [0,1]$, where $\mu_{\tilde{A}}(x)$ represents the degree to which x belongs to $\tilde{A}$.*

Example 1.4 *Let $X = \{1,2,3,4,5\}$. A fuzzy set $\tilde{A}$ of X may be defined as:*

$$\tilde{A} = \{(1,0.1),(2,0.3),(3,0.5),(4,0.7),(5,1.0)\}$$

Example 1.5 *Define $\mu_{\tilde{A}} : \mathbb{R} \to [0,1]$ by:*

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \leq 2, \\ \dfrac{x-2}{98}, & 2 < x \leq 100, \\ 1, & x > 100. \end{cases}$$

This describes a fuzzy set with a smooth transition from non-membership to full membership.

Fuzzy Set Characteristics

Remark 1.6 *If a fuzzy set's membership function only takes values 0 and 1, it reduces to a classical subset. The universal fuzzy set has* $\mu(x) = 1$ *for all* x*, and the empty fuzzy set has* $\mu(x) = 0$ *for all* x*.*

Notation: Let X be the reference set:

- A fuzzy set $\tilde{A}$ is represented as $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$.
- If X is countable:
$$\tilde{A} = \sum_{x \in X} \frac{\mu_{\tilde{A}}(x)}{x}$$
- If X is uncountable:
$$\tilde{A} = \int_{x \in X} \frac{\mu_{\tilde{A}}(x)}{x}$$
- The collection of all fuzzy sets of X is denoted by $F(X)$.

Example 1.7 *Let* $Y = \{0, 10, 20, 30, 40, 50\}$*. Define fuzzy sets for "cold" and "hot" temperatures (e.g.,* C *and* H*) using appropriate membership values to represent linguistic concepts.*

1.2 Alpha-Cuts of a fuzzy set

Definition 1.8 *[19] Let* $\tilde{A}$ *be a fuzzy set of the real numbers* $\mathbb{R}$ *with membership function* $\mu_{\tilde{A}} : \mathbb{R} \to (0, 1]$*. For any* $\alpha \in [0, 1]$*, the* α***-cut*** *of* $\tilde{A}$ *is defined as:*

$$\tilde{A}_{[\alpha]} = \{x \in \mathbb{R} \mid \mu_{\tilde{A}}(x) \geq \alpha\} = \mu_{\tilde{A}}^{-1}([\alpha, 1]).$$

This set contains all real numbers whose degree of membership in $\tilde{A}$ is at least α.

- When $\alpha = 0$, the set $\tilde{A}_{[0]}$ corresponds to the **support** of the fuzzy set:

$$\text{supp}(\tilde{A}) = \{x \in \mathbb{R} \mid \mu_{\tilde{A}}(x) > 0\}.$$

- When $\alpha = 1$, $\tilde{A}[1]$ represents the **core** of the fuzzy set:

$$\text{core}(\tilde{A}) = \{x \in \mathbb{R} \mid \mu_{\tilde{A}}(x) = 1\}.$$

Example 1.9 *Consider a fuzzy set $\tilde{B}$:*

$$\tilde{B} = \{(p, 0.9), (q, 0.4), (r, 0.7)\} \tag{1.1}$$

The corresponding alpha-cuts are:

$$\tilde{B}_{0.5} = \{p, r\},$$

$$\tilde{B}_{0.8} = \{p\},$$

$$\tilde{B}_{0.3} = \{p, q, r\}$$

Properties. Given two fuzzy sets $\tilde{A}$ and $\tilde{B}$:

1. If $\tilde{A} \subset \tilde{B}$, then $\tilde{A}_\alpha \subset \tilde{B}_\alpha$.
2. $(\tilde{A} \cap \tilde{B})_\alpha = A_\alpha \cap B_\alpha$.
3. $(\tilde{A} \cup \tilde{B})_\alpha = \tilde{A}_\alpha \cup \tilde{B}_\alpha$.

1.3 Cartesian Product of fuzzy sets

When multiple reference sets $X_1, \dots, X_r$ are involved, we consider a global universe X formed as their Cartesian product:

$$X = X_1 \times X_2 \times \cdots \times X_r.$$

Elements of X are r-tuples $(x_1, \dots, x_r)$, where $x_i \in X_i$.

The Cartesian product of r fuzzy sets $\tilde{A}_1, \dots, \tilde{A}_r$, defined respectively on $X_1, \dots, X_r$, results in a fuzzy set $\tilde{A}$ of X with the membership function:

$$\forall x = (x_1, \dots, x_r), \quad \mu_{\tilde{A}}(x) = \min(\mu_{\tilde{A}_1}(x_1), \dots, \mu_{\tilde{A}_r}(x_r)).$$

Example 1.10 *Consider the reference sets $X_1 = \{x, y, z\}$ and $X_2 = \{\alpha, \beta\}$ with fuzzy sets:*

$$\tilde{A} = \{\langle x, 0.7\rangle, \langle y, 0.5\rangle, \langle z, 1.0\rangle\}, \quad \tilde{B} = \{\langle \alpha, 0.7\rangle, \langle \beta, 0.4\rangle\}.$$

Then, the Cartesian product $\tilde{A} \times \tilde{B}$ is:

$$\tilde{A} \times \tilde{B} = \begin{Bmatrix} \langle (x, \alpha), 0.7\rangle, \langle (x, \beta), 0.4\rangle, \langle (y, \alpha), 0.5\rangle, \\ \langle (y, \beta), 0.4\rangle, \langle (z, \alpha), 0.7\rangle, \langle (z, \beta), 0.4\rangle \end{Bmatrix}.$$

1.4 Convex Fuzzy Set

Definition 1.11 *[5] A fuzzy set $\tilde{A}$ defined on a vector space X is said to be* ***convex*** *if for all $x_1, x_2 \in X$ and all $\lambda \in [0,1]$, we have:*

$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\}.$$

Example 1.12 *Consider the fuzzy set $\tilde{A} : \mathbb{R} \to [0,1]$ defined by*

$$\mu_{\tilde{A}}(x) = \begin{cases} 1 - |x|, & \text{if } |x| \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let us verify the convexity condition for $x_1 = -0.5$, $x_2 = 0.5$, and $\lambda = 0.3$.

$$\begin{aligned} \mu_{\tilde{A}}(x_1) &= 1 - |-0.5| = 0.5, \\ \mu_{\tilde{A}}(x_2) &= 1 - |0.5| = 0.5, \\ \lambda x_1 + (1-\lambda)x_2 &= 0.3 \cdot (-0.5) + 0.7 \cdot 0.5 = -0.15 + 0.35 = 0.2, \\ \mu_{\tilde{A}}(0.2) &= 1 - |0.2| = 0.8 \geq \min\{0.5, 0.5\} = 0.5. \end{aligned}$$

Since the inequality holds, the fuzzy set $\tilde{A}$ satisfies the convexity condition.

Theorem 1.13 *If $\tilde{A}$ and $\tilde{B}$ are convex fuzzy sets of $\mathbb{R}$, their intersection is also convex.*

Proof. Let $\tilde{C} = \tilde{A} \cap \tilde{B}$. The membership function of $\tilde{C}$ is given by:

$$\mu_{\tilde{C}}(\lambda a + (1-\lambda)b) = \min(\mu_{\tilde{A}}(\lambda a + (1-\lambda)b), \mu_{\tilde{B}}(\lambda a + (1-\lambda)b)). \tag{1.2}$$

Since $\tilde{A}$ and $\tilde{B}$ are convex, we have:

$$\mu_{\tilde{A}}(\lambda a + (1-\lambda)b) \geq \min(\mu_{\tilde{A}}(a), \mu_{\tilde{A}}(b)),$$

$$\mu_{\tilde{B}}(\lambda a + (1-\lambda)b) \geq \min(\mu_{\tilde{B}}(a), \mu_{\tilde{B}}(b)).$$

Hence:

$$\mu_C(\lambda a + (1-\lambda)b) \geq \min(\mu_A(a), \mu_B(a), \mu_A(b), \mu_B(b)) = \min(\mu_C(a), \mu_C(b)). \tag{1.3}$$

Therefore, $\tilde{C}$ is also convex. ∎

1.5 Fuzzy Numbers

Fuzzy numbers extend the concept of real numbers to handle uncertainty and vagueness by allowing gradual transitions between membership and non-membership. They serve as foundational tools in fuzzy logic, decision-making, and systems modeling under uncertainty. Below are the key properties, types, and operations associated with fuzzy numbers.

1.5.1 Definition of a Fuzzy Number

Definition 1.14 *[11, 26] A fuzzy number $\tilde{A}$ is a fuzzy set of $\mathbb{R}$ satisfying:*

- ***Normality**: $\exists x \in \mathbb{R}$ such that $\mu_{\tilde{A}}(x) = 1$.*
- ***Convexity**: For all $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$,*

$$\mu_{\tilde{A}}(\lambda x + (1-\lambda)y) \geq \min\left(\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\right).$$

- ***upper semicontinuous**: $\mu_{\tilde{A}}(x)$ is upper semicontinuous for all $x \in \mathbb{R}$ if and only if $\limsup_{y\to x} \mu_{\tilde{A}}(y) \leq \mu_{\tilde{A}}(x)$*
- *The set $\overline{supp(\tilde{A})} = cl\{x \in \mathbb{R} \mid \mu_{\tilde{A}}(x) > 0\}$ is compact*

1.5.2 α-Cut (r-Level) Fuzzy Number

Definition 1.15 *[5] Let $\mu : \mathbb{R} \to [0,1]$ be the membership function of a fuzzy number $\tilde{A}$. For any level $\alpha \in [0,1]$, the α-**cut** (also called the r-**level set**) of $\tilde{A}$ is defined as:*

$$\tilde{A}_{[\alpha]} = \{x \in \mathbb{R} \mid \mu(x) \geq \alpha\}.$$

Equivalently, this can be written as:

$$\tilde{A}_{[\alpha]} = [\underline{\tilde{A}}(\alpha), \overline{\tilde{A}}(\alpha)], \quad \textit{for } \alpha \in [0,1],$$

where $\underline{\tilde{A}}(\alpha)$ and $\overline{\tilde{A}}(\alpha)$ are the lower and upper bounds of the interval at level α. or

$$\tilde{A}_{[\alpha]} = [\underline{\tilde{A}}(\alpha), \overline{\tilde{A}}(\alpha)],$$

where $\underline{\tilde{A}}(\alpha)$ is the left endpoint and $\overline{\tilde{A}}(\alpha)$ is the right endpoint of the fuzzy number at level α.

Example 1.16 *Consider thefuzzy number $\tilde{A}$. Its membership function $\mu_{\tilde{A}}(x)$ is defined as:*

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & \textit{if } x < 2, \\ \frac{x-2}{3} & \textit{if } 2 \leq x < 5, \\ 1 & \textit{if } x = 5, \\ \frac{8-x}{3} & \textit{if } 5 < x \leq 8, \\ 0 & \textit{if } x > 8. \end{cases}$$

Let us determine the α-cut set $\tilde{A}_{[\alpha]}$ for a given $\alpha \in [0,1]$. We solve the equations:

$$\mu_{\tilde{A}}(x) = \alpha \Rightarrow \quad \begin{cases} \frac{x-2}{3} = \alpha \Rightarrow x = 3\alpha + 2, & \textit{for } x \in [2,5], \\ \frac{8-x}{3} = \alpha \Rightarrow x = 8 - 3\alpha, & \textit{for } x \in [5,8]. \end{cases}$$

Thus, the α-cut set is the closed interval:

$$\tilde{A}_{[\alpha]} = [3r + 2,\ 8 - 3r], \quad \textit{for } 0 \leq \alpha \leq 1.$$

For example, when $\alpha = 0.5$, we have:

$$\tilde{A}_{[0.5]} = [3(0.5) + 2,\ 8 - 3(0.5)] = [3.5,\ 6.5].$$

1.5.3 Special Fuzzy Numbers

This subsection presents five widely used special fuzzy numbers, including their definitions, α-cuts, and characteristics.

Definition 1.17 *[43][Triangular Fuzzy Number] A triangular fuzzy number* $\tilde{A} = (a, b, c)$ *is defined by the membership function:*

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & x < a, \\ \frac{x-a}{b-a} & a \le x < b, \\ 1 & x = b, \\ \frac{c-x}{c-b} & b < x \le c, \\ 0 & x > c. \end{cases}$$

α-cut: $\tilde{A}_{[\alpha]} = [(b-a)\alpha + a, c - (c-b)\alpha]$, *for* $\alpha \in [0, 1]$.

Example 1.18 *Let* $a = 1$, $b = 3$, $c = 5$, *then:*

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & \text{if } x \le 1, \\ \frac{x-1}{2} & \text{if } 1 < x \le 3, \\ \frac{5-x}{2} & \text{if } 3 < x \le 5, \\ 0 & \text{if } x > 5. \end{cases}$$

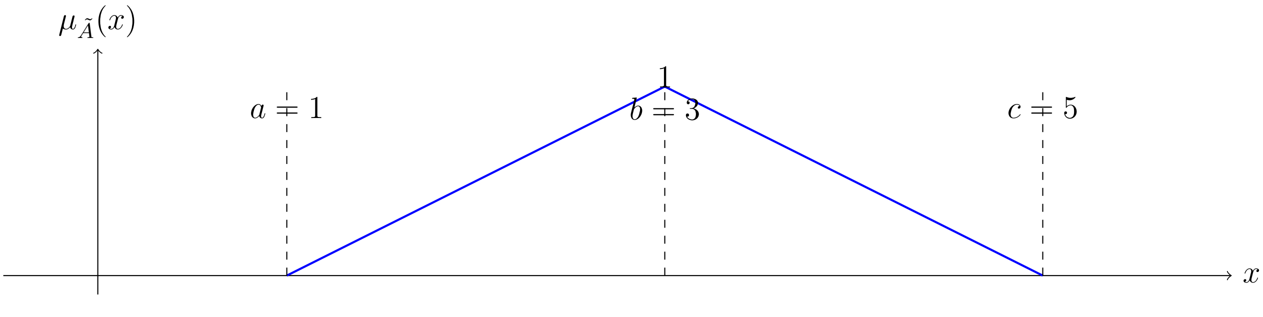

Figure 1.1: Triangular Membership Function of $\tilde{A}$

Definition 1.19 *[19][Trapezoidal Fuzzy Number] A trapezoidal fuzzy number $\tilde{A} = (a, b, c, d)$ is defined by the membership function:*

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & x < a, \\ \frac{x-a}{b-a} & a \leq x < b, \\ 1 & b \leq x \leq c, \\ \frac{d-x}{d-c} & c < x \leq d, \\ 0 & x > d. \end{cases}$$

α-cut: $\tilde{A}_{[\alpha]} = [(b-a)\alpha + a, d - (d-c)\alpha]$, *for* $\alpha \in [0, 1]$.

Example 1.20 *Let $a = 1$, $b = 2$, $c = 4$, $d = 5$, then:*

$$\mu_{\tilde{B}}(x) = \begin{cases} 0 & \text{if } x \leq 1, \\ x - 1 & \text{if } 1 < x \leq 2, \\ 1 & \text{if } 2 < x \leq 4, \\ 5 - x & \text{if } 4 < x \leq 5, \\ 0 & \text{if } x > 5. \end{cases}$$

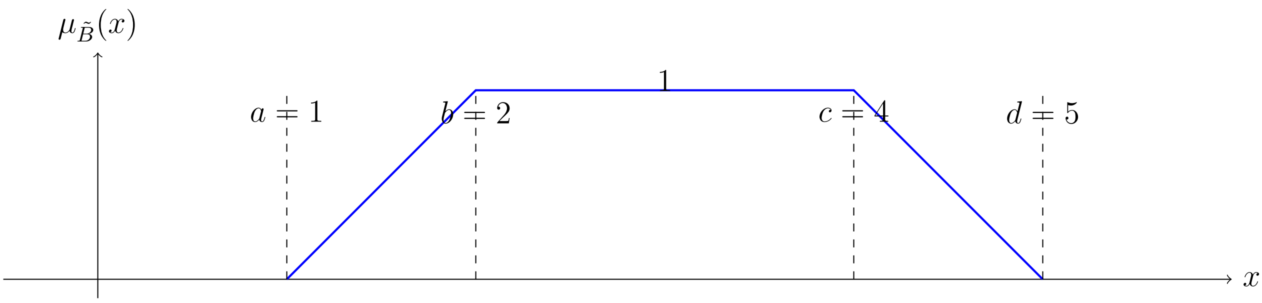

Figure 1.2: Trapezoidal Membership Function of $\tilde{B}$

Definition 1.21 *[48][Singleton Fuzzy Number] A singleton fuzzy number $\tilde{A} = a$ is defined by the membership function:*

$$\mu_{\tilde{A}}(x) = \begin{cases} 1 & x = a, \\ 0 & x \neq a. \end{cases}$$

α-cut: $\tilde{A}_{[\alpha]} = [a, a]$, *for all* $\alpha \in (0, 1]$.

Example 1.22 *A singleton fuzzy number $\tilde{A}$ at $x_0 = 3$ is defined by:*

$$\mu_{\tilde{A}}(x) = \begin{cases} 1 & \text{if } x = 3 \\ \\ 0 & \text{if } x \neq 3 \end{cases}$$

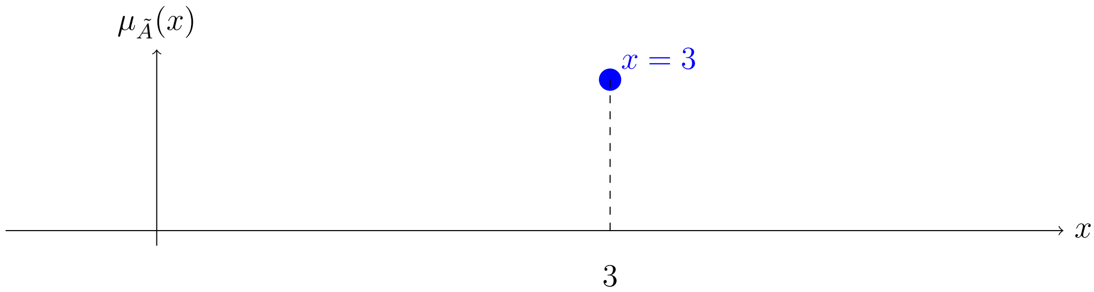

Figure 1.3: Singleton Fuzzy Number at $x = 3$

Definition 1.23 *[5][Gaussian Fuzzy Number] A Gaussian fuzzy number with mean m and standard deviation σ has the membership function:*

$$\mu_{\tilde{A}}(x) = \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right).$$

α-cut: $\tilde{A}_{[\alpha]} = [m - \sigma\sqrt{-2\ln\alpha}, m + \sigma\sqrt{-2\ln\alpha}]$, *for* $\alpha \in (0, 1]$.

Example 1.24 *A Gaussian fuzzy number $\tilde{A}$ centered at $m = 0$ with standard deviation $\sigma = 1$ is given by:*

$$\mu_{\tilde{A}}(x) = \exp\left(-\frac{(x-0)^2}{2 \cdot 1^2}\right) = \exp\left(-\frac{x^2}{2}\right)$$

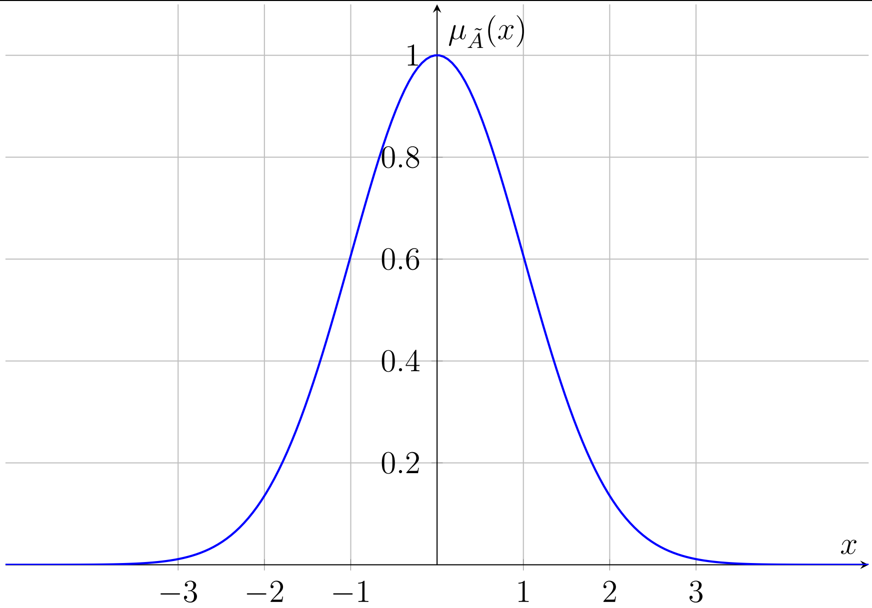

Figure 1.4: Gaussian Membership Function Centered at $x = 0$, $\sigma = 1$

Definition 1.25 *[48][Bell-Shaped Fuzzy Number] A generalized bell-shaped fuzzy number is given by:*

$$\mu_{\tilde{A}}(x) = \frac{1}{1 + \left|\frac{x-c}{a}\right|^{2b}},$$

where a, b, and c determine the width, slope, and center of the bell.

α-cut: *The α-cut must be computed numerically due to the transcendental nature of the inverse.*

Example 1.26 *For example, with $c = 0$, $a = 1$, and $b = 2$, we have:*

$$\mu_{\tilde{A}}(x) = \frac{1}{1 + |x|^4}$$

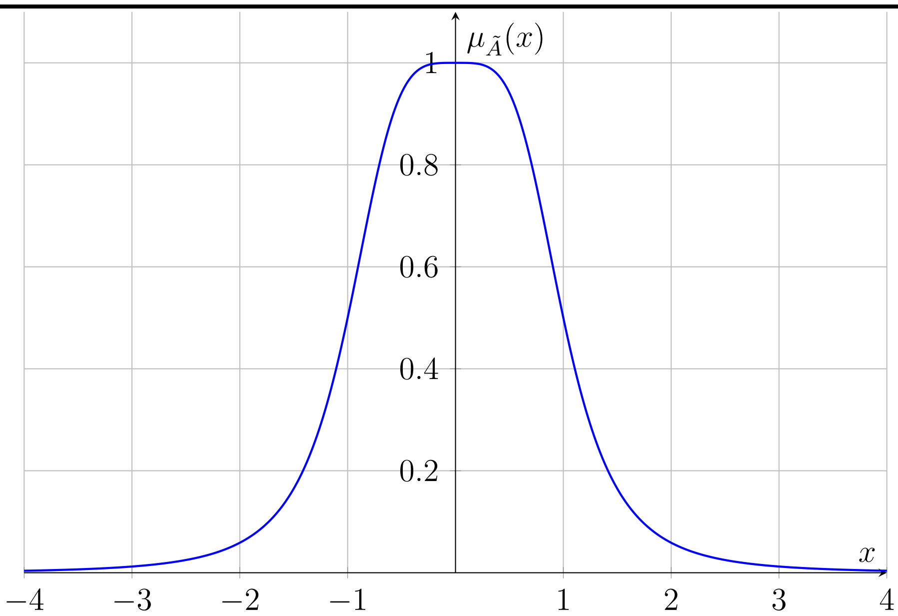

Figure 1.5: Bell-Shaped Membership Function with $c = 0$, $a = 1$, $b = 2$

1.6 Zadeh's Extension Principle for Fuzzy Arithmetic

Let $\tilde{A}, \tilde{B}$ be fuzzy sets on $\mathbb{R}$ with membership functions $\mu_{\tilde{A}}(x)$ and $\mu_{\tilde{B}}(y)$. According to Zadeh's extension principle for two fuzzy sets, the membership function of the fuzzy set $\tilde{C} = \tilde{A} \circ \tilde{B}$, denoted $\mu_{\tilde{C}}(z)$, is given by:

$$\mu_{\tilde{C}}(z) = \sup_{\substack{x,y\in\mathbb{R} \\ x\circ y=z}} \min\left(\mu_{\tilde{A}}(x),\ \mu_{\tilde{B}}(y)\right)$$

where $\circ \in \{+, -, \times, \div\}$.

Example 1.27 *Let:*

$$\tilde{A} = (1, 2, 3), \quad \tilde{B} = (2, 3, 4)$$

The membership functions are:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \leq 1 \\ x - 1, & 1 < x \leq 2 \\ 3 - x, & 2 < x < 3 \\ 0, & x \geq 3 \end{cases} \quad \mu_{\tilde{B}}(y) = \begin{cases} 0, & y \leq 2 \\ y - 2, & 2 < y \leq 3 \\ 4 - y, & 3 < y < 4 \\ 0, & y \geq 4 \end{cases}$$

We compute $\tilde{C} = \tilde{A} + \tilde{B}$. Then:

$$\mu_{\tilde{C}}(z) = \sup_{x+y=z} \min\left(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(z - x)\right)$$

Evaluate $\mu_{\tilde{C}}(5)$

Let $z = 5$. We look for all x such that $x + y = 5$, i.e., $y = 5 - x$. Then:

$$\mu_{\tilde{C}}(5) = \sup_{x} \min \left(\mu_{\tilde{A}}(x),\ \mu_{\tilde{B}}(5 - x)\right)$$

Choose some values of x:

x	$y = 5 - x$	$\mu_{\tilde{A}}(x)$	$\mu_{\tilde{B}}(y)$	$\min(\mu_{\tilde{A}}, \mu_{\tilde{B}})$
1.5	3.5	0.5	0.5	0.5
2	3	1	1	1
2.5	2.5	0.5	0.5	0.5

$$\Rightarrow \mu_{\tilde{C}}(5) = \sup\{0.5, 1, 0.5\} = 1$$

1.6.1 Addition $\tilde{A} \oplus \tilde{B}$

The addition of two fuzzy sets is defined by applying Zadeh's Extension Principle. However, a more practical and computationally efficient approach is to use the α-cut representation. Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy sets with well-defined α-cuts for each $\alpha \in [0, 1]$, given by:

$$\tilde{A}_{[\alpha]} = [\underline{\tilde{A}}(\alpha), \overline{\tilde{A}}(\alpha)], \quad \tilde{B}_{[\alpha]} = [\underline{\tilde{B}}(\alpha), \overline{\tilde{B}}(\alpha)].$$

Then, the α-cut of the fuzzy sum $\tilde{C} = \tilde{A} \oplus \tilde{B}$ is defined by:

$$\tilde{C}_{[\alpha]} = [\underline{\tilde{A}}(\alpha) + \underline{\tilde{B}}(\alpha),\ \overline{\tilde{A}}(\alpha) + \overline{\tilde{B}}(\alpha)].$$

This interval is obtained by summing the lower bounds and the upper bounds of the respective α-cuts.

Example 1.28 *Suppose the α-cuts are:*

$$\tilde{A}_{[\alpha]} = [1 + \alpha,\ 3 - \alpha], \quad \tilde{B}_{[\alpha]} = [2\alpha,\ 4 - 2\alpha].$$

Then the addition gives:

$$\tilde{C}_{[\alpha]} = [1 + \alpha + 2\alpha,\ 3 - \alpha + 4 - 2\alpha] = [1 + 3\alpha,\ 7 - 3\alpha].$$

1.6.2 Subtraction $\tilde{A} \ominus \tilde{B}$

The subtraction of two fuzzy sets is defined using the α-cut method. Let $\tilde{A}$ and $\tilde{B}$ be fuzzy sets with α-cuts given by:

$$\tilde{A}_{[\alpha]} = [\underline{\tilde{A}}(\alpha), \overline{\tilde{A}}(\alpha)], \quad \tilde{B}_{[\alpha]} = [\underline{\tilde{B}}(\alpha), \overline{\tilde{B}}(\alpha)].$$

Then, the α-cut of the fuzzy difference $\tilde{C} = \tilde{A} \ominus \tilde{B}$ is given by:

$$\tilde{C}_{[\alpha]} = [\underline{\tilde{A}}(\alpha) - \overline{\tilde{B}}(\alpha),\ \overline{\tilde{A}}(\alpha) - \underline{\tilde{B}}(\alpha)].$$

Example 1.29 *Suppose the α-cuts are:*

$$\tilde{A}_{[\alpha]} = [2 + \alpha,\ 5 - \alpha], \quad \tilde{B}_{[\alpha]} = [1 + \alpha,\ 3 - \alpha].$$

Then the subtraction gives:

$$\tilde{C}_{[\alpha]} = [(2 + \alpha) - (3 - \alpha),\ (5 - \alpha) - (1 + \alpha)] = [-1 + 2\alpha,\ 4 - 2\alpha].$$

So the resulting fuzzy set $\tilde{C}$ has α-cuts that describe a decreasing range as α increases.

1.6.3 Multiplication $\tilde{A} \otimes \tilde{B}$

The multiplication of two fuzzy sets is defined using the α-cut method. Let $\tilde{A}$ and $\tilde{B}$ be fuzzy sets with α-cuts:

$$\tilde{A}_{[\alpha]} = [\underline{\tilde{A}}(\alpha), \overline{\tilde{A}}(\alpha)], \quad \tilde{B}_{[\alpha]} = [\underline{\tilde{B}}(\alpha), \overline{\tilde{B}}(\alpha)].$$

Then, the α-cut of the fuzzy product $\tilde{C} = \tilde{A} \otimes \tilde{B}$ is:

$$[\tilde{C}]_{[\alpha]} = [\min S,\ \max S],$$

where

$$S = \left\{ \underline{\tilde{A}}(\alpha)\underline{\tilde{B}}(\alpha),\ \underline{\tilde{A}}(\alpha)\overline{\tilde{B}}(\alpha),\ \overline{\tilde{A}}(\alpha)\underline{\tilde{B}}(\alpha),\ \overline{\tilde{A}}(\alpha)\overline{\tilde{B}}(\alpha) \right\}.$$

Example 1.30 *Suppose:*

$$\tilde{A}_{[\alpha]} = [1 + \alpha,\ 2 - \alpha], \quad \tilde{B}_{[\alpha]} = [2,\ 3].$$

Then we compute the four products:

$$x_1 = (1+\alpha)(2),$$
$$x_2 = (1+\alpha)(3),$$
$$x_3 = (2-\alpha)(2),$$
$$x_4 = (2-\alpha)(3).$$

Thus:

$$S = \{2(1+\alpha),\ 3(1+\alpha),\ 2(2-\alpha),\ 3(2-\alpha)\} = \{2+2\alpha,\ 3+3\alpha,\ 4-2\alpha,\ 6-3\alpha\}.$$

Finally:

$$\tilde{C}_{[\alpha]} = [\min\{2+2\alpha,\ 3+3\alpha,\ 4-2\alpha,\ 6-3\alpha\},\ \max\{2+2\alpha,\ 3+3\alpha,\ 4-2\alpha,\ 6-3\alpha\}].$$

1.6.4 Division $\tilde{A} \div \tilde{B}$

The division of two fuzzy sets is defined using the α-cut method. Let $\tilde{A}$ and $\tilde{B}$ be fuzzy sets with α-cuts:

$$\tilde{A}_{[\alpha]} = [\underline{\tilde{A}}(\alpha),\ \overline{\tilde{A}}(\alpha)], \quad \tilde{B}_{[\alpha]} = [\underline{\tilde{B}}(\alpha),\ \overline{\tilde{B}}(\alpha)],$$

with the condition that $0 \notin [\underline{\tilde{B}}(\alpha),\ \overline{\tilde{B}}(\alpha)]$ for all $\alpha \in [0,1]$.

Then, the α-cut of the division $\tilde{C} = \tilde{A} \div \tilde{B}$ is:

$$\tilde{C}_{[\alpha]} = [\min D,\ \max D],$$

where

$$D = \left\{ \frac{\underline{\tilde{A}}(\alpha)}{\underline{\tilde{B}}(\alpha)}, \frac{\underline{\tilde{A}}(\alpha)}{\overline{\tilde{B}}(\alpha)}, \frac{\overline{\tilde{A}}(\alpha)}{\underline{\tilde{B}}(\alpha)}, \frac{\overline{\tilde{A}}(\alpha)}{\overline{\tilde{B}}(\alpha)} \right\}.$$

Example 1.31 *Suppose:*

$$\tilde{A}_{[\alpha]} = [1+\alpha,\ 2-\alpha], \quad \tilde{B}_{[\alpha]} = [2,\ 3] \quad (\textit{note: } 0 \notin [2,3]).$$

Then the values to compute are:

$$d_1 = \frac{1+\alpha}{2},$$
$$d_2 = \frac{1+\alpha}{3},$$
$$d_3 = \frac{2-\alpha}{2},$$
$$d_4 = \frac{2-\alpha}{3}.$$

So:

$$D = \left\{ \frac{1+\alpha}{2}, \frac{1+\alpha}{3}, \frac{2-\alpha}{2}, \frac{2-\alpha}{3} \right\}.$$

Thus, the α-cut for $\tilde{C} = \tilde{A} \div \tilde{B}$ is:

$$\tilde{C}_{[\alpha]} = [\min D,\ \max D].$$

1.7 Hukuhara Difference of Fuzzy Numbers

Definition 1.32 *[5] The* ***Hukuhara Difference*** *between fuzzy numbers $\tilde{A}$ and $\tilde{B}$, denoted $\tilde{C} = \tilde{A} \ominus_H \tilde{B}$, exists if there exists a fuzzy number $\tilde{C}$ satisfying:*

$$\tilde{A} = \tilde{B} + \tilde{C}$$

where + represents fuzzy number addition. The α-cuts of $\tilde{C}$ are given by:

$$\tilde{C}_{[\alpha]} = \tilde{A}_{[\alpha]} - \tilde{B}_{[\alpha]} = \left[\underline{\tilde{A}}(\alpha) - \underline{\tilde{B}}(\alpha),\ \overline{\tilde{A}}(\alpha) - \overline{\tilde{B}}(\alpha)\right]$$

For $\tilde{C}$ to be a valid fuzzy number:

1. $\underline{\tilde{C}}(\alpha) \leq \overline{\tilde{C}}(\alpha),\ \forall \alpha \in [0,1]$
2. $\underline{\tilde{C}}(\alpha)$ is non-decreasing
3. $\overline{\tilde{C}}(\alpha)$ is non-increasing

Existence Condition

The Hukuhara Difference exists **if and only if**:

$$\overline{\tilde{A}}(\alpha) - \underline{\tilde{A}}(\alpha) \geq \overline{\tilde{B}}(\alpha) - \underline{\tilde{B}}(\alpha), \quad \forall \alpha \in [0,1]$$

Example 1.33 :

1. ***Fuzzy Result***
   Let $\tilde{A} = (0,2,4)$, $\tilde{B} = (0,1,2)$:

$$\tilde{C} = \tilde{A} \ominus_H \tilde{B} = (0,1,2)$$

   Verification: $\tilde{B} + \tilde{C} = (0+0, 1+1, 2+2) = (0,2,4)$

2. ***Trapezoidal Fuzzy Numbers***
Let $\tilde{A} = (1, 3, 5, 7)$, $\tilde{B} = (0, 1, 2, 3)$*:*

$$\tilde{C} = \tilde{A} \ominus_H \tilde{B} = (1, 2, 3, 4)$$

Verification: $\tilde{B} + \tilde{C} = (0+1, 1+2, 2+3, 3+4) = (1, 3, 5, 7)$

3. ***Non-Existence Example***
For $\tilde{A} = (1, 2, 3)$ *and* $\tilde{B} = (0, 2, 4)$*:*

$$\overline{\tilde{A}}(\alpha) - \underline{\tilde{A}}(\alpha) = 2 < 4 = \overline{\tilde{B}}(\alpha) - \underline{\tilde{B}}(\alpha)$$

The Hukuhara difference $\tilde{A} \ominus_H \tilde{B}$ *does not exist.*

Definition 1.34 *[12] The **generalized Hukuhara Difference (gH-Difference)** between fuzzy numbers* $\tilde{A}$ *and* $\tilde{B}$*, denoted* $\tilde{C} = \tilde{A} \ominus_{gH} \tilde{B}$*, exists if there exists a fuzzy number* C *such that:*

$$\begin{cases} \tilde{A} = \tilde{B} + \tilde{C} & \text{(Case i: Standard H-difference)}, \\ \text{or} \\ \tilde{B} = \tilde{A} + (-\tilde{C}) & \text{(Case ii: Inverse H-difference)}. \end{cases} \tag{1.4}$$

For α*-cuts* $[\tilde{A}]^\alpha = [\underline{\tilde{A}}(\alpha), \overline{\tilde{A}}(\alpha)]$ *and* $[\tilde{B}]^\alpha = [\underline{\tilde{B}}(\alpha), \overline{\tilde{B}}(\alpha)]$*:*

$$[C]_\alpha = \begin{cases} [\underline{\tilde{A}}(\alpha) - \underline{\tilde{B}}(\alpha), \, \overline{\tilde{A}}(\alpha) - \overline{\tilde{B}}(\alpha)] & \text{(Case 1)}, \\ [\overline{\tilde{A}}(\alpha) - \overline{\tilde{B}}(\alpha), \, \underline{\tilde{A}}(\alpha) - \underline{\tilde{B}}(\alpha)] & \text{(Case 2)}. \end{cases} \tag{1.5}$$

For $\tilde{C}$ to be a valid fuzzy number:

1. $\underline{\tilde{C}}(\alpha) \leq \overline{\tilde{C}}(\alpha) \; \forall \alpha \in [0, 1]$
2. $\underline{\tilde{C}}(\alpha)$ is non-decreasing in α
3. $\overline{\tilde{C}}(\alpha)$ is non-increasing in α

The gH-Difference exists **if either**:

1. **Case 1**: $\overline{\tilde{A}}(\alpha) - \underline{\tilde{A}}(\alpha) \geq \overline{\tilde{B}}(\alpha) - \underline{\tilde{B}}(\alpha)$
2. **Case 2**: $\overline{\tilde{B}}(\alpha) - \underline{\tilde{B}}(\alpha) \geq \overline{\tilde{A}}(\alpha) - \underline{\tilde{A}}(\alpha)$

Example 1.35 *Let* $\tilde{A} = (1, 2, 3)$ *and* $\tilde{B} = (0, 2, 4)$*:*
Standard H-difference:

$$\tilde{A} \ominus_H \tilde{B} \text{ does not exist (spread}(\tilde{A}) = 2 < 4)$$

gH-Difference (Case ii):

$$-\tilde{C} = \tilde{B} - \tilde{A} = (0 - 1, 2 - 2, 4 - 3) = (-1, 0, 1)$$
$$\implies \tilde{C} = (1, 0, -1)$$

Invalid fuzzy number

1.8 Metric Space Framework

In this section, we present the basic metric structure used to study fuzzy numbers and fuzzy-valued functions.

Definition 1.36 *[16][Metric Space] A metric space is a pair* (X, d)*, where* X *is a non-empty set and* $d : X \times X \to \mathbb{R}$ *is a function called a metric that satisfies the following properties for all* $x, y, z \in X$*:*

1. *(Non-negativity)* $d(x, y) \geq 0$*,*
2. *(Identity of indiscernibles)* $d(x, y) = 0$ *if and only if* $x = y$*,*
3. *(Symmetry)* $d(x, y) = d(y, x)$*,*
4. *(Triangle inequality)* $d(x, z) \leq d(x, y) + d(y, z)$*.*

1.8.1 Hausdorff Distance

The Hausdorff distance is a fundamental concept used to measure the distance between two subsets of a metric space.

Definition 1.37 *[39][Hausdorff Distance Between Sets] Let* (X, d) *be a metric space, and let* $A, B \subset X$ *be two non-empty subsets. The Hausdorff distance* $H(A, B)$ *between* A *and* B *is defined as:*

$$H(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(b, a) \right\}.$$

Definition 1.38 *[21][Hausdorff Distance Between Intervals in $\mathbb{R}$] Let $A = [a_1, a_2]$ and $B = [b_1, b_2]$ be two compact intervals in $\mathbb{R}$. The Hausdorff distance between A and B is given by:*

$$H(A, B) = \max\{|a_1 - b_1|, |a_2 - b_2|\}.$$

Definition 1.39 *[16][Metric on the Space of Fuzzy Numbers] Let $\mathbb{F}_{\mathbb{R}}$ denote the set of all fuzzy numbers on $\mathbb{R}$. A commonly used metric d on $\mathbb{F}_{\mathbb{R}}$ is defined via the Hausdorff distance between the corresponding α-cuts:*

$$d(\tilde{A}, \tilde{B}) = \sup_{\alpha \in [0,1]} H([\tilde{A}]^\alpha, [\tilde{B}]^\alpha),$$

where H denotes the Hausdorff distance, and $[\tilde{A}]^\alpha, [\tilde{B}]^\alpha$ are the α-cuts of the fuzzy numbers $\tilde{A}$ and $\tilde{B}$, respectively.

1.9 Calculus of Fuzzy Functions

1.9.1 Fuzzy Functions

Fuzzy functions extend classical functions to domains or codomains involving fuzzy numbers. These functions are crucial in modeling and analyzing systems with uncertainty and imprecision. There are three principal types of fuzzy functions, depending on whether the domain and/or codomain is fuzzy.

From Real Numbers to Fuzzy Numbers: $f : \mathbb{R} \to \mathbb{F}_{\mathbb{R}}$

This type of function assigns to each crisp real number a fuzzy number. It is useful when the input is precise but the output is uncertain.

Example 1.40 *Let $f(x) = (x - 1, x, x + 1)$ is a triangular fuzzy number centered at x. Then*

$$f : \mathbb{R} \to \mathbb{F}_{\mathbb{R}}$$

.

From Fuzzy Numbers to Real Numbers: $f : \mathbb{F}_{\mathbb{R}} \to \mathbb{R}$

This function maps a fuzzy number to a crisp value and is typically used in defuzzification processes.

Example 1.41 *Let $f(\tilde{A})$ be the centroid (center of gravity) of the fuzzy number $\tilde{A}$:*

$$f(\tilde{A}) = \frac{\int x\mu_{\tilde{A}}(x)\,dx}{\int \mu_{\tilde{A}}(x)\,dx}.$$

From Fuzzy Numbers to Fuzzy Numbers: $f : \mathbb{F}_{\mathbb{R}} \to \mathbb{F}_{\mathbb{R}}$

This is the most general case, where both the input and output are fuzzy. The fuzzy extension principle or α-cut method is often used to define such functions.

Definition 1.42 *[5] Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued (crisp) function. Then the fuzzy function $\tilde{f} : \mathbb{F}_{\mathbb{R}} \to \mathbb{F}_{\mathbb{R}}$, defined via the extension principle, is given by:*

$$\tilde{f}(\tilde{A}_{[\alpha]}) = [\min_{x\in\tilde{A}_{[\alpha]}} f(x),\ \max_{x\in\tilde{A}_{[\alpha]}} f(x)], \quad \forall\alpha \in [0,1],$$

where $\tilde{A}_{[\alpha]}$ denotes the α-cut of the fuzzy number $\tilde{A}$.

Example 1.43 *Let $f(x) = x^2$ and consider a triangular fuzzy number $\tilde{A} = (1, 2, 3)$. The α-cut of $\tilde{A}$ is:*

$$\tilde{A}_{[\alpha]} = [1 + \alpha, 3 - \alpha], \quad \alpha \in [0,1].$$

Then the image of $\tilde{A}$ under f is:

$$\tilde{f}(\tilde{A}_{[\alpha]}) = \left[\min_{x\in[1+\alpha,3-\alpha]} x^2,\ \max_{x\in[1+\alpha,3-\alpha]} x^2\right].$$

Fuzzy functions generalize classical real-valued functions to operate on fuzzy numbers or fuzzy sets.

1.9.2 Continuity of Fuzzy Function

Definition 1.44 *[24][Continuity of Fuzzy Function] A fuzzy function $\tilde{f} : \mathbb{F}_{\mathbb{R}} \to \mathbb{F}_{\mathbb{R}}$ is said to be continuous at a fuzzy number $\tilde{A}$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that:*

$$d(\tilde{A}, \tilde{B}) < \delta \Rightarrow d(\tilde{f}(\tilde{A}), \tilde{f}(\tilde{B})) < \varepsilon,$$

where d is a metric, typically the supremum of the Hausdorff distance over $[0, 1]$:

$$d(\tilde{A}, \tilde{B}) = \sup_{\alpha\in[0,1]} H(\tilde{A}_{[\alpha]}, \tilde{B}_{[\alpha]}).$$

Definition 1.45 *[5] Let $\tilde{f} : \mathbb{R} \to \mathbb{F}_{\mathbb{R}}$ be a fuzzy function, where $\mathbb{F}_{\mathbb{R}}$ denotes the set of all fuzzy numbers on $\mathbb{R}$. The function $\tilde{f}$ assigns to each real number $x \in \mathbb{R}$ a fuzzy number $\tilde{f}(x) \in \mathbb{F}_{\mathbb{R}}$.*
For each $\alpha \in [0, 1]$, the α-cut set of $\tilde{f}(x)$, denoted by $\tilde{f}_{[\alpha]}(x)$, is an interval defined as:

$$\tilde{f}_{[\alpha]}(x) = [\underline{f}(\alpha, x), \overline{f}(\alpha, x)],$$

where $\underline{f}(\alpha, x)$ and $\overline{f}(\alpha, x)$ are the lower and upper bounds of the fuzzy function at level r.

Definition 1.46 *[19] A fuzzy function $\tilde{f} : D \subseteq \mathbb{R} \to \mathbb{F}_{\mathbb{R}}$ is said to be continuous at $x_0 \in D$ if, for every $\alpha \in [0, 1]$, the endpoint functions $\underline{f}(\alpha, x)$ and $\overline{f}(\alpha, x)$ are continuous at x_0. That is,*

$$\lim_{x\to x_0} \underline{f}(\alpha, x) = \underline{f}(\alpha, x_0), \quad \textit{and} \quad \lim_{x\to x_0} \overline{f}(\alpha, x) = \underline{f}(\alpha, x_0).$$

Example 1.47 *Let $\tilde{f}(x)$ be a fuzzy function defined by:*

$$\tilde{f}(x) = (x - 1, x, x + 1), \quad x \in \mathbb{R},$$

where the output at each point x is a triangular fuzzy number. Then for any $\alpha \in [0, 1]$, the α-cut set of $\tilde{f}(x)$ is:

$$\tilde{f}_{[\alpha]}(x) = [x - 1 + \alpha, x + 1 - \alpha].$$

Thus, $\underline{f}(\alpha, x) = x-1+\alpha$ and $\overline{f}(\alpha, x) = x+1-\alpha$, which are both continuous in x. Hence, $\tilde{f}(x)$ is a continuous fuzzy function.

1.9.3 Differentiation of Fuzzy Functions

Fuzzy-Valued Function

Let $\tilde{f} : \mathbb{R} \to \mathbb{F}_{\mathbb{R}}$ be a fuzzy-valued function, where $\mathbb{F}_{\mathbb{R}}$ denotes the set of fuzzy numbers on $\mathbb{R}$. For each real number x, $\tilde{f}(x)$ is a fuzzy number.

Differentiability via α-Cuts

A common approach to define the derivative of a fuzzy function is through its α-cuts.

Definition 1.48 *[5] Let $f : \mathbb{R} \to \mathbb{F}_{\mathbb{R}}$ be a fuzzy function. For every $\alpha \in [0, 1]$, the α-cut of $f(x)$ is given by:*

$$f_{[\alpha]}(x) = [\underline{f}(\alpha, x), \overline{f}(\alpha, x)],$$

where $\underline{f}(\alpha, x)$ and $\overline{f}(\alpha, x)$ are the lower and upper bounds of the α-cut of the fuzzy number $f(x)$.

*We say that f is **differentiable at** $x_0 \in \mathbb{R}$ if, for all $\alpha \in [0, 1]$, the functions $\underline{f}(\alpha, x)$ and $\overline{f}(\alpha, x)$ are differentiable at x_0, and the fuzzy derivative $f'(x_0)$ is defined by:*

$$f'_{[\alpha]}(x_0) = \left[\frac{d}{dx}\underline{f}(\alpha, x_0), \frac{d}{dx}\overline{f}(\alpha.x_0) \right].$$

Hukuhara and Generalized Hukuhara Differentiability

Definition 1.49 *[5]*
*Let $f : \mathbb{R} \to \mathbb{F}_{\mathbb{R}}$. We say that f is **Hukuhara differentiable** at $x_0 \in \mathbb{R}$ if there exists a fuzzy number $f'(x_0) \in \mathbb{F}_{\mathbb{R}}$ such that*

$$\lim_{h\to 0^+} \frac{f(x_0 + h) \ominus_H f(x_0)}{h} = \lim_{h\to 0^-} \frac{f(x_0 + h) \ominus_H f(x_0)}{h} = f'(x_0),$$

provided the Hukuhara difference exists.

Example 1.50 *Let $f(x) = (x - 1, x, x + 1)$ be a triangular fuzzy number-valued function. The α-cut is given by:*

$$f_{[\alpha]}(x) = [x - 1 + \alpha, x + 1 - \alpha].$$

Then, the lower and upper bounds are:

$$\underline{f}(\alpha, x) = x - 1 + \alpha, \quad \overline{f}(\alpha, x) = x + 1 - \alpha.$$

Differentiating both with respect to x, we obtain:

$$f'(x)_{[\alpha]}(x) = \left[\frac{d}{dx}\underline{f}(\alpha, x), \frac{d}{dx}\overline{f}(\alpha, x)\right] = [1, 1].$$

Thus, the derivative is the constant fuzzy number $f'(x) = (1, 1, 1)$.

Definition 1.51 *[5] A fuzzy-valued function f is said to be* ***generalized Hukuhara differentiable*** *(gH-differentiable) at x_0 if there exists a fuzzy number $f'(x_0)$ such that either*

$$\lim_{h\to 0} \frac{f(x_0 + h) \ominus_{gH} f(x_0)}{h} = f'(x_0), \quad or \quad \lim_{h\to 0} \frac{f(x_0) \ominus_{gH} f(x_0 - h)}{h} = f'(x_0)$$

exists in the sense of the generalized Hukuhara difference.

1.9.4 Integration of Fuzzy Functions

Definition 1.52 *[38] Let $f : [a, b] \to \mathbb{F}_{\mathbb{R}}$ be a fuzzy-valued function. For each $\alpha \in [0, 1]$, let*

$$f_{[\alpha]}(x) = [\underline{f}(\alpha, x), \overline{f}(\alpha, x)]$$

be the α-cut of $f(x)$, where $\underline{f}(\alpha, x)$ and $\overline{f}(\alpha, x)$ are real-valued functions representing the lower and upper bounds.

Then the definite integral of f over $[a, b]$ is the fuzzy number whose α-cut is defined by:

$$\left[\int_a^b f(x)\, dx\right]_\alpha = \left[\int_a^b \underline{f}(\alpha, x)\, dx, \int_a^b \overline{f}(\alpha, x)\, dx\right].$$

Note

This definition assumes that $\underline{f}(\alpha, x)$ and $\overline{f}(\alpha, x)$ are integrable over $[a, b]$ for every $\alpha \in [0, 1]$.

Example 1.53 *Let $f(x) = (x - 1, x, x + 1)$ be a triangular fuzzy number-valued function on $[0, 1]$. Its α-cut is given by:*

$$f_{[\alpha]}(x) = [x - 1 + \alpha, x + 1 - \alpha].$$

Then:

$$\underline{f}(\alpha, x) = x - 1 + \alpha, \quad \overline{f}(\alpha, x) = x + 1 - \alpha.$$

Compute the integral over $[0, 1]$*:*

$$\left[\int_0^1 f(x)\,dx\right]_\alpha = \left[\int_0^1 (x - 1 + \alpha)\,dx, \int_0^1 (x + 1 - \alpha)\,dx\right].$$

Compute each:

$$\int_0^1 (x - 1 + \alpha)\,dx = \left[\frac{x^2}{2} - x + \alpha x\right]_0^1 = \left(\frac{1}{2} - 1 + \alpha\right) = -\frac{1}{2} + \alpha,$$

$$\int_0^1 (x + 1 - \alpha)\,dx = \left[\frac{x^2}{2} + x - \alpha x\right]_0^1 = \left(\frac{1}{2} + 1 - \alpha\right) = \frac{3}{2} - \alpha.$$

So the integral is:

$$\left[\int_0^1 f(x)\,dx\right]_\alpha = \left[-\frac{1}{2} + \alpha, \frac{3}{2} - \alpha\right],$$

which corresponds to the fuzzy number $\left(-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\right)$.

Chapter 2

Notation and Preliminaries of Fractional Calculus

2.1 Introduction

Fractional calculus extends the classical notions of differentiation and integration to non-integer orders, offering a powerful framework for modeling systems with memory, hereditary properties, and complex dynamics. Unlike traditional calculus, which focuses on integer-order operators, fractional calculus provides tools that capture more intricate behaviors often observed in physical, biological, and engineering systems.

This chapter introduces the fundamental mathematical structures underlying fractional calculus. It begins with special functions that generalize classical mathematical concepts and are essential for defining and analyzing fractional operators. These include the Gamma and Beta functions, as well as the Mittag-Leffler function, which plays a central role in the solutions of fractional differential equations.

The chapter then develops the theory of Riemann-Liouville and Caputo fractional derivatives, highlighting their definitions and properties. Emphasis is placed on understanding the differences between these two formulations.

By the end of this chapter, readers will have a solid foundation in the mathematical tools and concepts required to study and apply fractional calculus effectively.

For further insights and comprehensive treatment of this subject, the reader may consult References [17, 22, 27, 28, 33], as well as the bibliographic sources they contain

2.2 Special Functions

2.2.1 The Gamma Function

Euler's Gamma function naturally extends the factorial to real numbers.

Definition 2.1 *[7] The Gamma function is defined by:*

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t}\, dt \quad ; \quad x \in \mathbb{C},\ \Re(x) > 0. \tag{2.1}$$

Some Properties

The Γ function satisfies the following properties:

- Recurrence relation:

$$\Gamma(x+1) = x\Gamma(x), \quad (\Re(x) > 0). \tag{2.2}$$

- $\Gamma(1) = \Gamma(2) = 1$, $\Gamma(0_+) = +\infty$, and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.
- For all $n \in \mathbb{N}$, $\Gamma(n+1) = n\Gamma(n) = n(n-1)! = n!$.
- Reflection formula:

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)} \quad (x \notin \mathbb{Z};\ 0 < \Re(x) < 1). \tag{2.3}$$

2.2.2 The Beta Function

The Beta function is also called Euler's integral of the first kind.

Definition 2.2 *[7] The Beta function is defined by:*

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad \Re(x) > 0, \ \Re(y) > 0. \tag{2.4}$$

Example 2.3

$$\begin{aligned} B(2,3) &= \int_0^1 t(1-t)^2 dt \\ &= \int_0^1 (t - 2t^2 + t^3)dt \\ &= \frac{1}{12}. \end{aligned}$$

Note

The relationship between the Gamma function and the Beta function is given by:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x,y \in \mathbb{C}, \ \Re(x), \Re(y) > 0. \tag{2.5}$$

.

Corollary 2.4 *The Beta function is symmetric:* $B(x,y) = B(y,x)$.

Proof.

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \frac{\Gamma(y)\Gamma(x)}{\Gamma(y+x)} = B(y,x).$$

■

2.2.3 The Mittag-Leffler Function

The Mittag-Leffler function plays an important role in the theory of fractional differential equations and has widespread applications. It was introduced by G. M. Mittag-Leffler and studied by A. Wiman.

Definition 2.5 *[7] The Mittag-Leffler function $E_\alpha(x)$ is defined by:*

$$E_\alpha(x) = \sum_{n=0}^{+\infty} \frac{x^n}{\Gamma(n\alpha+1)}, \quad x \in \mathbb{C}, \ \alpha > 0. \tag{2.6}$$

The two-parameter generalization of the Mittag-Leffler function is:

$$E_{\alpha,\beta}(x) = \sum_{n=0}^{+\infty} \frac{x^n}{\Gamma(n\alpha+\beta)}, \quad \alpha, \beta > 0. \tag{2.7}$$

Example 2.6 *For some special values:*

$$E_{1,1}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x,$$
$$E_{1,2}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+2)} = \sum_{k=0}^{\infty} \frac{x^k}{(k+1)!} = \frac{e^x-1}{x},$$
$$E_{1,3}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+3)} = \sum_{k=0}^{\infty} \frac{x^k}{(k+2)!} = \frac{e^x-1-x}{x^2}.$$

2.3 Riemann-Liouville Fractional Integrals and Derivatives

2.3.1 Fractional Integral

Definition 2.7 *[27] Let $f \in L^1[a,+\infty)$, $a \in \mathbb{R}$, and $\alpha \in \mathbb{R}_+^*$. The Riemann-Liouville fractional integral of order α of the function f, starting from the lower limit a, is defined by:*

$$\mathcal{I}_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-r)^{\alpha-1} f(r)\, dr, \quad -\infty \le a < t < +\infty. \tag{2.8}$$

Remark 2.8 *For $\alpha = 0$, equation (2.8) reduces to the identity operator:*

$$\mathcal{I}_a^0 f = f.$$

Example 2.9 *Let $f(x) = (x-a)^\beta$. Then,*

$$\mathcal{I}_a^{\alpha}(x-a)^\beta = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} (t-a)^\beta dt.$$

By changing the variable via $t = a + (x-a)r$, we get:

$$\mathcal{I}_a^\alpha (x-a)^\beta = \frac{(x-a)^{\alpha+\beta}}{\Gamma(\alpha)} \int_0^1 (1-r)^{\alpha-1} r^\beta dr.$$

Using the Beta function identity, we obtain:

$$\mathcal{I}_a^\alpha (x-a)^\beta = \frac{B(\alpha, \beta+1)}{\Gamma(\alpha)} (x-a)^{\alpha+\beta}.$$

Applying the identity $B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$, we arrive at:

$$\mathcal{I}_a^\alpha (x-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} (x-a)^{\beta+\alpha}. \tag{2.9}$$

Properties of the Fractional Integral

Theorem 2.10 *[15] If $f \in L^1([a,b])$ and $\alpha > 0$, then $\mathcal{I}_a^\alpha f(t)$ exists almost everywhere and:*

$$\mathcal{I}_a^\alpha f \in L^1([a,b]).$$

Theorem 2.11 *For $f \in L^1([a,b])$, the Riemann-Liouville fractional integral satisfies the semigroup property:*

$$\mathcal{I}_a^\alpha \left[\mathcal{I}_a^\beta f(t)\right] = \mathcal{I}_a^{\alpha+\beta} f(t) = \mathcal{I}_a^\beta \left[\mathcal{I}_a^\alpha f(t)\right], \quad \text{for } \alpha, \beta > 0, \tag{2.10}$$

for all $t \in [a,b]$.

Proof. We compute:

$$\mathcal{I}_{a^+}^\alpha \left[\mathcal{I}_{a^+}^\beta f(t)\right] = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t (t-r)^{\alpha-1} \int_a^r (r-\tau)^{\beta-1} f(\tau)\, d\tau\, dr.$$

By Fubini's theorem, we get:

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t f(\tau) \left(\int_\tau^t (t-r)^{\alpha-1} (r-\tau)^{\beta-1}\, dr \right) d\tau.$$

Using the change of variable $r = \tau + u(t-\tau)$, the inner integral becomes:

$$(t-\tau)^{\alpha+\beta-1} \int_0^1 (1-u)^{\alpha-1} u^{\beta-1}\, du = (t-\tau)^{\alpha+\beta-1} B(\alpha,\beta).$$

Therefore:

$$\mathcal{I}_a^{\alpha}\left[\mathcal{I}_a^{\beta} f(t)\right] = \frac{B(\alpha,\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t f(\tau)(t-\tau)^{\alpha+\beta-1}\, d\tau = \mathcal{I}_a^{\alpha+\beta} f(t).$$

The same result holds if α and β are swapped. ■

Lemma 2.12
For any function $f \in C([a,b])$, the fractional integral operator is linear:

$$\mathcal{I}_a^{\alpha}(c_1 f + c_2 g)(t) = c_1 \mathcal{I}_a^{\alpha} f(t) + c_2 \mathcal{I}_a^{\alpha} g(t),$$

for constants $c_1, c_2 \in \mathbb{R}$ and functions $f, g \in C([a,b])$.

2.3.2 Riemann-Liouville Fractional Derivative

Definition 2.13 *[27] Let $\alpha > 0$, and let $m = \lfloor \alpha \rfloor + 1$. The **Riemann-Liouville fractional derivative** of order α of a function f is defined by:*

$${}_a^R D_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t (t-\tau)^{m-\alpha-1} f(\tau)\, d\tau,$$

where $\Gamma(\cdot)$ is the Gamma function.

Example 2.14 *:*

Let us compute the Riemann-Liouville fractional derivative of the function

$$f(t) = (t-a)^{\beta}.$$

According to formula 2.9, we can write:

$$\begin{aligned} {}_a^R D_t^{\alpha} (t-a)^{\beta} &= \left(\frac{d}{dt}\right)^m \left[\frac{\Gamma(\beta+1)}{\Gamma(\beta+1+m-\alpha)} (t-a)^{\beta+m-\alpha}\right] \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+m-\alpha)} \left(\frac{d}{dt}\right)^m (t-a)^{\beta+m-\alpha}. \end{aligned} \tag{2.23}$$

We know that:

$$\left(\frac{d}{dt}\right)^m (t-a)^{\beta+m-\alpha} = (\beta+m-\alpha)(\beta+m-\alpha-1)\cdots(\beta-\alpha+1)(t-a)^{\beta-\alpha}. \tag{2.24}$$

Substituting (2.24) into (2.23), we get:

$$
\begin{aligned}
{}_a^R D_t^\alpha (t-a)^\beta &= \frac{\Gamma(\beta+1)(\beta+m-\alpha)(\beta+m-\alpha-1)\cdots(\beta-\alpha+1)}{\Gamma(\beta+1+m-\alpha)}(t-a)^{\beta-\alpha} \\
&= \frac{\Gamma(\beta+1)(\beta+m-\alpha)(\beta+m-\alpha-1)\cdots(\beta+1)\Gamma(\beta+1)}{(\beta+m-\alpha)(\beta+m-\alpha-1)\cdots(\beta+1)\Gamma(\beta+\alpha-\alpha+1)}(t-a)^{\beta-\alpha} \\
&= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}. \qquad (2.25)
\end{aligned}
$$

Remark

(i) For $\alpha = 1$, the formula (2.25) reduces to:

$$ {}_a^R D_t^1 (t-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta)}(t-a)^{\beta-1} = \beta(t-a)^{\beta-1} = \frac{d}{dt}(t-a)^\beta. \qquad (2.26) $$

(ii) If we take $\beta = 0$ in the previous example, we obtain the result:

$$ {}_a^R D_t^\alpha 1 = \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}, \qquad (2.11) $$

which means that the Riemann-Liouville fractional derivative of a constant is neither zero nor constant. However:

$$ {}_a^R D_t^\alpha C = \frac{C}{\Gamma(1-\alpha)}(t-a)^{-\alpha}. \qquad (2.12) $$

2.4 Caputo Fractional Derivatives

Definition 2.15 *[27] Let $n-1 < \beta < n$, where $n \in \mathbb{N}$, and let $f \in AC^n[a,b]$. The Caputo fractional derivative of order β of the function f, denoted by ${}^C D_{a+}^\beta f(t)$, is defined as*

$$ {}^C D_{a+}^\beta f(t) = \frac{1}{\Gamma(n-\beta)} \int_a^t (t-\tau)^{n-\beta-1} f^{(n)}(\tau)\, d\tau, \quad t \in (a,b]. $$

Remark 2.16 *From Definition 2.15, we can write:*

$$ {}^{C}\mathcal{D}_{a^+}^{\alpha} f(t) = \left(\mathcal{I}_{a^+}^{n-\alpha} D^n f\right)(t), \tag{2.13}$$

In particular, for $0 < \alpha < 1$, *we have:*

$$ {}^{C}\mathcal{D}_{a^+}^{\alpha} f(t) = \left(\mathcal{I}_{a^+}^{1-\alpha} D f\right)(t), \tag{2.14}$$

If $\alpha = n \in \mathbb{N}$ *and the classical derivative* $f^{(n)}(t)$ *exists, then:*

$$ {}^{C}\mathcal{D}_{a^+}^{n} f(t) = f^{(n)}(t), \tag{2.15}$$

where $\mathcal{D}^n = \dfrac{d^n}{dt^n}$.

Example 2.17 :

- ***Caputo derivative of a constant function:***

$$ {}^{C}\mathcal{D}^{\alpha} C = 0. \tag{2.16}$$

- ***Caputo derivative of*** $f(t) = (t-a)^{\beta}$***:***
  Let $0 \leq n-1 < \alpha < n$ *and* $\beta > n-1$. *We have:*

$$ f^{(n)}(t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-n+1)}(t-a)^{\beta-n}, \tag{2.17}$$

so that:

$$ {}^{C}\mathcal{D}^{\alpha}(t-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(n-\alpha)\Gamma(\beta-n+1)} \int_a^t (t-r)^{n-\alpha-1}(r-a)^{\beta-n}\, dr. \tag{2.18}$$

Making the substitution $r = a + u(t-a)$, *we get:*

$$ \begin{aligned} {}^{C}\mathcal{D}^{\alpha}(t-a)^{\beta} &= \frac{\Gamma(\beta+1)}{\Gamma(n-\alpha)\Gamma(\beta-n+1)}(t-a)^{\beta-\alpha} \int_0^1 (1-u)^{n-\alpha-1} u^{\beta-n}\, du \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}. \end{aligned}$$

Properties of the Caputo Fractional Derivative

Theorem 2.18 *Let $\alpha > 0$, $n = \lceil \alpha \rceil$, then the following properties hold for almost all $t \in [a, b]$:*

- $$ {}^{C}\mathcal{D}^{\alpha}\mathcal{I}_a^{\alpha} f = f. \tag{2.19}$$
- $$\mathcal{I}_a^{\alpha}\left({}^{C}\mathcal{D}^{\alpha} f(t)\right) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(t-a)^k. \tag{2.20}$$

Proof.

- From Remark 2.16 and using the semigroup property (Theorem 2.11):

$$ {}^{C}\mathcal{D}^{\alpha}\mathcal{I}_a^{\alpha} f(t) = \left(\mathcal{I}_a^{n-\alpha} D^n \mathcal{I}_a^{\alpha} f\right)(t) = \mathcal{I}_a^{0} f = f.$$

- We compute:

$$\mathcal{I}_a^{\alpha}\left({}^{C}\mathcal{D}^{\alpha} f(t)\right) = \mathcal{I}_a^{\alpha}\left(\mathcal{I}_a^{n-\alpha} D^n f\right)(t) = \mathcal{I}_a^{n} D^n f(t),$$

and by the known formula:

$$\mathcal{I}_a^{n} D^n f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(t-a)^k.$$

■

Theorem 2.19 *Let f and g be functions for which the Caputo derivative of order α exists. For all $\lambda, \mu \in \mathbb{R}$, the Caputo derivative is linear:*

$$ {}^{C}\mathcal{D}^{\alpha}(\lambda f(t) + \mu g(t)) = \lambda\, {}^{C}\mathcal{D}^{\alpha} f(t) + \mu\, {}^{C}\mathcal{D}^{\alpha} g(t). \tag{2.21}$$

2.5 Relation between the Caputo and Riemann-Liouville Fractional Derivative

1) Let $\alpha \in \mathbb{R}^+$, $m \in \mathbb{N}^*$ and $m = \lfloor \alpha \rfloor + 1$. If ${}_a^C D_t^{\alpha} f(t)$ and ${}_a^R D_t^{\alpha} f(t)$ exist, then:

i)

$$ {}_a^C D_t^\alpha f(t) = {}_a^R D_t^\alpha f(t) - \sum_{i=0}^{m-1} \frac{f^{(i)}(a)(t-a)^{i-\alpha}}{\Gamma(i-\alpha+1)}, $$

We deduce that if $f^{(i)}(a) = 0$ for all $i = 0, 1, \dots, m-1$, then

$$ {}_a^C D_t^\alpha f(t) = {}_a^R D_t^\alpha f(t). $$

ii)

$$ {}_a^C D_t^\alpha f(t) = {}_a^R D_t^\alpha \left(f(t) - \sum_{i=0}^{m-1} \frac{f^{(i)}(a)}{i!}(t-a)^i \right). $$

2) If $0 < \alpha < 1$, the Riemann-Liouville and Caputo fractional derivatives are defined respectively by:

$$ \begin{aligned} {}_a^R D_t^\alpha f(t) &= \frac{d}{dt}\left({}_a^R D_t^{1-\alpha} f(t)\right) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-\tau)^{-\alpha} f(\tau) d\tau, \\ {}_a^C D_t^\alpha f(t) &= {}_a^R D_t^{1-\alpha} \left(\frac{df(t)}{dt} \right) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\tau)^{-\alpha} f'(\tau) d\tau. \end{aligned} $$

and we have the following properties:

i)

$$ \begin{aligned} {}_a^R D_t^\alpha f(t) &= \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(a)}{(t-a)^\alpha} + \int_a^t (t-\tau)^{-\alpha} f'(\tau) d\tau \right) \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{f(a)}{(t-a)^\alpha} + {}_a^C D_t^\alpha f(t), \end{aligned} $$

ii)

$$ {}_a^C D_t^\alpha f(t) = {}_a^R D_t^\alpha \left(f(t) - f(a) \right). $$

Chapter 3

Solving Fuzzy Fractional Differential Equations via Laplace Transform

3.1 Introduction

In this chapter, we have addressed the solution of a fuzzy fractional differential equation by employing the Laplace transform method. This approach effectively handles the inherent uncertainty in the system by incorporating fuzzy logic with fractional calculus. The formulation includes both Riemann-Liouville and Caputo types of fractional derivatives, which are widely used in modeling memory and hereditary properties of various physical and engineering processes. Furthermore, the problem is equipped with fractional initial conditions of both Liouville and Caputo types.

Jafarian et al. in [23] generalized the fractional Laplace transformation to the fuzzy fractional Laplace transformation. They then applied the proposed fuzzy fractional Laplace transformation to solve a fuzzy fractional differential equation corresponding to the following problem

$$^{RL}D^{\beta}_{0+}\tilde{x}(t) = f(t,\tilde{x}), \quad t \in [0,T], \quad \lambda > 0, \quad \beta \in (0,1),$$

with fuzzy initial conditions:

$$\lim_{t\to o^+} I^{1-m}\tilde{x}(0) = \tilde{x}_0,$$

where $f : [0,T] \times \mathbb{F}_{\mathbb{R}} \to \mathbb{F}_{\mathbb{R}}$ is continuous.

With the previous foundation, we now proceed to solve the following problem using the Laplace transform method

$$\begin{cases} ^{RL}D^{\beta}_{0+}\tilde{x}(t) \oplus {}^{C}D^{\beta}_{0+}x(t) = f(t,\tilde{x}(t)) \oplus \lambda\tilde{x}(t), \quad t \in [0,T],\ T > 0 \quad \beta \in (0,1), \\ \tilde{x}(0) = \tilde{x}_0, \quad I^{1-\beta}\tilde{x}(0) = \tilde{x}_1, \end{cases} \tag{3.1}$$

where $x \in AC^1[0,1]$, $\lambda > 0$, $^{RL}D^{\beta}_{0+}$ and $^{C}D^{\beta}_{0+}$ denote the Riemann-Liouville and Caputo fractional derivatives of order $\beta \in (0,1)$, respectively. $I^{1-\beta}$ denotes the Riemann-Liouville fractional integral of order $1-\beta$. The initial values $\tilde{x}_0, \tilde{x}_1 \in \mathbb{F}_{\mathbb{R}}$, and the function $f : [0,T] \times \mathbb{F}_{\mathbb{R}} \to \mathbb{F}_{\mathbb{R}}$ is assumed to be continuous.

3.2 Fundamental Properties of the Laplace Transform

This section covers the Laplace transform and some of its key properties. For further details, please consult references [1] and [23]

We begin by reviewing key properties of the Laplace transform. For a complex variable s, the Laplace transform of a function $f(t)$ is defined as

$$F(s) = \mathcal{L}\{f(t); s\} = \int_0^{\infty} e^{-st} f(t)\, dt, \tag{3.2}$$

where $f(t)$ is called the original function and $F(s)$ its transform. To ensure the integral converges, $f(t)$ must be of exponential order α, meaning there exist positive constants M and T such that

$$|f(t)| \leq Me^{\alpha t}, \quad \text{for all } t > T.$$

The original function can be retrieved from its Laplace transform by means of the inverse Laplace transform.

3.2.1 Some results on Laplace transformation

1. The Laplace transformation formula for the Caputo frcational derivative:

$$\mathcal{L}\{{}_0^C D^{\beta} x(t)\} = s^{\beta} X(s) - \sum_{k=0}^{n-1} s^{\beta-k-1} x^{(k)}(0), \tag{3.3}$$

$n-1<\beta<n$

For $0<\beta<1$

$$\mathcal{L}\{{}_0^C D^{\beta} x(t)\} = s^{\beta} X(s) - s^{\beta-1} x(0), \tag{3.4}$$

2. The Laplace transformation formula for the Riemann-Liouville fractional derivative:

$$\mathcal{L}\{D^{\gamma} x(t); s\} = s^{\gamma} X(s) - \sum_{k=0}^{n-1} s^{k} \left[D^{\beta-k-1} x(t)\right]_{t=0}, \tag{3.5}$$

$n-1<\gamma<n$ **For** $0<\gamma<1$

$$\mathcal{L}\{D^{\beta} x(t); s\} = s^{\beta} X(s) - I^{1-\beta} x(0), \tag{3.6}$$

3.

$$\mathcal{L}\left(t^{rk+s-1} E_{r,s}^{(k)}(\pm a t^{r}); s\right) = \frac{k! s^{r-s}}{(s^{r} \mp a)^{k+1}}$$

4.

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-\beta}}{s^{\alpha}-a}\right\} = t^{\beta-1} E_{\alpha,\beta}(a t^{\alpha})$$

3.3 Solution to the Fuzzy Fractional Differential Equation

We aim to solve the following fuzzy fractional differential equation using the Laplace transform method. The equation is given by:

$$\begin{cases} {}^{RL}D_{0+}^{\beta} \tilde{x}(t) \oplus {}^{C}D_{0+}^{\beta} \tilde{x}(t) = f(t, \tilde{x}(t)) \oplus \lambda \tilde{x}(t), \quad t \in [0,T],\ T>0 \quad \beta \in (0,1), \lambda > 0, \\ \tilde{x}(0) = \tilde{x}_0, \quad I_{0+}^{1-\beta} \tilde{x}(0) = \tilde{x}_1, \end{cases} \tag{3.7}$$

where $x \in AC^1[0,1]$, ${}^{RL}D^{\beta}_{0+}$ denotes the Riemann-Liouville fractional derivative of order β, and ${}^{C}D^{\beta}_{0+}$ denotes the Caputo fractional derivative of order β. $I^{1-\beta}_{0+}$ represents the Riemann-Liouville fractional integral of order $1-\beta$. $\tilde{x}(t)$ is the unknown fuzzy function, $\tilde{x}_0$ and $\tilde{x}_1$ are fuzzy initial values, λ is a positive crisp constant, and $f(t, \tilde{x}(t))$ is a fuzzy nonlinear function.

Step 1: Consider the Crisp Linear Case

To apply the Laplace transform effectively, we first consider a simplified, crisp, and linear version of the problem where $x(t)$, x_0, x_1 are crisp, and $f(t, x(t))$ is a known crisp function. The equation becomes:

$$ {}^{RL}D^{\beta}_{0+}x(t) + {}^{C}D^{\beta}_{0+}x(t) - \lambda x(t) = f(t, x(t)), \quad t > 0, \quad \beta \in (0,1), \lambda > 0 \tag{3.8} $$

with initial conditions $x(0) = x_0$ and $I^{1-\beta}_{0+}x(0) = x_1$.

Step 2: Apply the Laplace Transform

Taking the Laplace transform of each term, we use the following properties for $0 < \beta < 1$:

- $\mathcal{L}\{{}^{RL}D^{\beta}_{0+}x(t); s\} = s^{\beta}X(s) - [I^{1-\beta}_{0+}x(t)]_{t=0} = s^{\beta}X(s) - x_1$
- $\mathcal{L}\{{}^{C}D^{\beta}_{0+}x(t); s\} = s^{\beta}X(s) - s^{\beta-1}x(0) = s^{\beta}X(s) - s^{\beta-1}x_0$
- $\mathcal{L}\{\lambda x(t); s\} = \lambda X(s)$
- $\mathcal{L}\{f(t, x(t)); s\} = F(s)$

Substituting these into the transformed equation yields:

$$ (s^{\beta}X(s) - x_1) + (s^{\beta}X(s) - s^{\beta-1}x_0) - \lambda X(s) = F(s) $$

Step 3: Solve for $X(s)$

Rearranging the terms to solve for $X(s)$:

$$ X(s)(2s^{\beta} - \lambda) = F(s) + x_1 + s^{\beta-1}x_0 $$

$$X(s) = \frac{F(s) + x_1 + s^{\beta-1}x_0}{2s^\beta - \lambda} = \frac{F(s)}{2s^\beta - \lambda} + \frac{x_1}{2s^\beta - \lambda} + \frac{s^{\beta-1}x_0}{2s^\beta - \lambda}$$

Step 5: Apply the Inverse Laplace Transform

We use the Mittag-Leffler function $E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}$ and its inverse Laplace transform properties. The inverse Laplace transform of each term is:

- $\mathcal{L}^{-1}\left\{\frac{x_1}{2s^\beta-\lambda}\right\} = \frac{x_1}{2}t^{\beta-1}E_{\beta,\beta}\left(\frac{\lambda}{2}t^\beta\right)$
- $\mathcal{L}^{-1}\left\{\frac{s^{\beta-1}x_0}{2s^\beta-\lambda}\right\} = \frac{x_0}{2}E_{\beta,1}\left(\frac{\lambda}{2}t^\beta\right)$
- $\mathcal{L}^{-1}\left\{\frac{F(s)}{2s^\beta-\lambda}\right\} = \int_0^t f(\tau, x(\tau))\frac{1}{2}(t-\tau)^{\beta-1}E_{\beta,\beta}\left(\frac{\lambda}{2}(t-\tau)^\beta\right)d\tau$ (using the convolution theorem)

Step 6: The Solution for the Crisp Linear Case

Combining these, the solution for the crisp linear fractional differential equation is: eq31

$$x(t) = \int_0^t f(\tau, x(\tau))\frac{1}{2}(t-\tau)^{\beta-1}E_{\beta,\beta}\left(\frac{\lambda}{2}(t-\tau)^\beta\right)d\tau + \frac{x_1}{2}t^{\beta-1}E_{\beta,\beta}\left(\frac{\lambda}{2}t^\beta\right) + \frac{x_0}{2}E_{\beta,1}\left(\frac{\lambda}{2}t^\beta\right) \tag{3.9}$$

Step 7: Solution as a Fuzzy-Valued Function

1- Solution via Zadeh's Extension Principle

According to Equation 3.9, the solutions to Problem 3.1 are given by the following expression

$$\tilde{x}(t) = \frac{\tilde{x}_1}{2} \odot t^{\beta-1}E_{\beta,\beta}\left(\frac{\lambda t^\beta}{2}\right) \oplus \frac{\tilde{x}_0}{2} \odot E_{\beta,1}\left(\frac{\lambda t^\beta}{2}\right) \oplus \frac{1}{2}\int_0^t (t-\tau)^{\beta-1}E_{\beta,\beta}\left(\frac{\lambda(t-\tau)^\beta}{2}\right) \odot f(\tau, \tilde{x}(\tau))\,d\tau,$$

2- Fuzzy Solution via α-Cut Method

Let the fuzzy-valued function $\tilde{x}(t)$ be expressed in terms of its α-cut:

$$\tilde{x}_{[\alpha]}(t) = [\underline{\tilde{x}}(\alpha, t), \overline{\tilde{x}}(\alpha, t)],$$

with fuzzy initial conditions:

$$\tilde{x_0}_{[\alpha]} = [\underline{\tilde{x}}_0(\alpha), \overline{\tilde{x}}_0(\alpha)], \quad \tilde{x_1}_{[\alpha]} = [\underline{\tilde{x}}_1(\alpha), \overline{\tilde{x}}_1(\alpha)],$$

and

$$\tilde{f}_{[\alpha]}(t, \tilde{x}(t)) = [\underline{f}(\alpha, t, \tilde{x}(t)), \overline{f}(\alpha, t, \tilde{x}(t))].$$

Lower Bound Solution

$$\begin{aligned}\underline{\tilde{x}}(\alpha, t) = & \frac{1}{2} \cdot \underline{\tilde{x}}_1(\alpha) \cdot t^{\beta-1} E_{\beta,\beta}\left(\frac{\lambda}{2} t^{\beta}\right) \\ & + \frac{1}{2} \cdot \underline{\tilde{x}}_0(\alpha) \cdot E_{\beta,1}\left(\frac{\lambda}{2} t^{\beta}\right) \\ & + \int_0^t \frac{1}{2}(t-\tau)^{\beta-1} E_{\beta,\beta}\left(\frac{\lambda}{2}(t-\tau)^{\beta}\right) \underline{f}(\alpha, \tau, \tilde{x}(\tau))\, d\tau\end{aligned}$$

Upper Bound Solution

$$\begin{aligned}\overline{x}(\alpha, t) = & \frac{1}{2} \cdot \overline{x}_1(\alpha) \cdot t^{\beta-1} E_{\beta,\beta}\left(\frac{\lambda}{2} t^{\beta}\right) \\ & + \frac{1}{2} \cdot \overline{x}_0(\alpha) \cdot E_{\beta,1}\left(\frac{\lambda}{2} t^{\beta}\right) \\ & + \int_0^t \frac{1}{2}(t-\tau)^{\beta-1} E_{\beta,\beta}\left(\frac{\lambda}{2}(t-\tau)^{\beta}\right) \overline{f}(\alpha, \tau, \tilde{x}(\tau))\, d\tau\end{aligned}$$

Final Fuzzy-Valued Solution

The fuzzy-valued solution at each α-level is given by:

$$\tilde{x}_{[\alpha]}(t) = [\underline{\tilde{x}}(\alpha, t), \overline{\tilde{x}}(\alpha, t)]$$

3.4 Examples

Example 3.1 *Consider the following fuzzy fractional differential equation:*

$${}^{RL}D^{\beta}_{0+}\tilde{x}(t) \oplus {}^{C}D^{\beta}_{0+}\tilde{x}(t) = \lambda\tilde{x}(t), \quad t > 0, \quad \beta \in (0, 1), \lambda > 0$$

with fuzzy initial conditions $\tilde{x}_0 = (1, 2, 3)$ and $\tilde{x}_1 = (2, 3, 4)$, where both are triangular fuzzy numbers.

Step 1: α-Cut Representation of Initial Conditions

- ***For** $\tilde{x}_0 = (1, 2, 3)$**:** Applying the definition with $a = 1, b = 2, c = 3$:*

$$\underline{x_0}(\alpha) = 1 + (2 - 1)\alpha = 1 + \alpha$$

$$\overline{x_0}(\alpha) = 3 - (3 - 2)\alpha = 3 - \alpha$$

  Thus,

$$\tilde{\boldsymbol{x}}_{\boldsymbol{0}[\alpha]} = [1 + \alpha, 3 - \alpha].$$

- ***For** $\tilde{x}_1 = (2, 3, 4)$**:** Applying the definition with $a = 2, b = 3, c = 4$:*

$$\underline{x_1}(\alpha) = 2 + (3 - 2)\alpha = 2 + \alpha$$

$$\overline{x_1}(\alpha) = 4 - (4 - 3)\alpha = 4 - \alpha$$

  Thus,

$$\tilde{x}_{1[\alpha]} = [2 + \alpha, 4 - \alpha].$$

Step 2: Determine Lower and Upper Solutions

To find the lower bound $\underline{x}(t, \alpha)$ and upper bound $\overline{x}(t, \alpha)$ of the fuzzy solution, we substitute the corresponding lower and upper bounds of the α-cuts of $\tilde{x}_0$ and $\tilde{x}_1$.

***Lower Solution ($\underline{x}(t, \alpha)$):** We use the lower bounds $\underline{x_0}(\alpha) = 1 + \alpha$ and $\underline{x_1}(\alpha) = 2 + \alpha$:*

$$\underline{x}(t, \alpha) = \frac{2 + \alpha}{2} t^{\beta - 1} E_{\beta,\beta}\left(\frac{\lambda}{2} t^{\beta}\right) + \frac{1 + \alpha}{2} E_{\beta,1}\left(\frac{\lambda}{2} t^{\beta}\right).$$

This can be factored as:

$$\underline{x}(t, \alpha) = \frac{1}{2}\left((2 + \alpha) t^{\beta - 1} E_{\beta,\beta}\left(\frac{\lambda}{2} t^{\beta}\right) + (1 + \alpha) E_{\beta,1}\left(\frac{\lambda}{2} t^{\beta}\right)\right).$$

***Upper Solution ($\overline{x}(t, \alpha)$):** We use the upper bounds $\overline{x_0}(\alpha) = 3 - \alpha$ and $\overline{x_1}(\alpha) = 4 - \alpha$:*

$$\overline{x}(t, \alpha) = \frac{4 - \alpha}{2} t^{\beta - 1} E_{\beta,\beta}\left(\frac{\lambda}{2} t^{\beta}\right) + \frac{3 - \alpha}{2} E_{\beta,1}\left(\frac{\lambda}{2} t^{\beta}\right)$$

This can be factored as:

$$\overline{x}(t,\alpha) = \frac{1}{2}\left((4-\alpha)t^{\beta-1}E_{\beta,\beta}\left(\frac{\lambda}{2}t^{\beta}\right) + (3-\alpha)E_{\beta,1}\left(\frac{\lambda}{2}t^{\beta}\right)\right)$$

Final Fuzzy-Valued Solution

The fuzzy-valued solution at each α-level is given by the interval:

$$\tilde{x}_{[\alpha]}(t) = [\underline{x}(\alpha,t), \overline{x}(\alpha,t)]$$

Thus, the lower and upper solutions of the fuzzy fractional differential equation for the given initial conditions are $\underline{x}(\alpha,t)$ and $\overline{x}(\alpha,t)$ as derived above.

In Figures 3.1, 3.2, 3.3, and 3.4 we have the solutions for the of fuzzy fractional differential equation

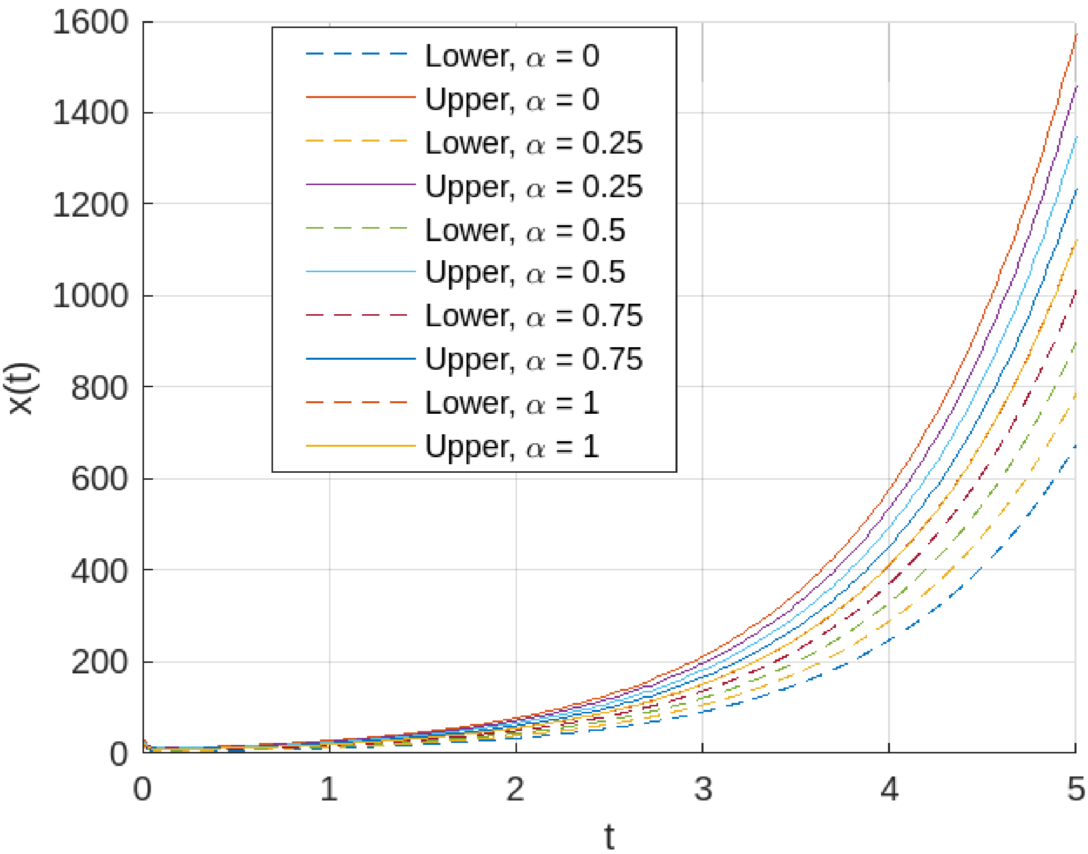

Figure 3.1: Fuzzy Solution Bounds for Different α Values and $\beta = 0.33$ and $\lambda = 2$

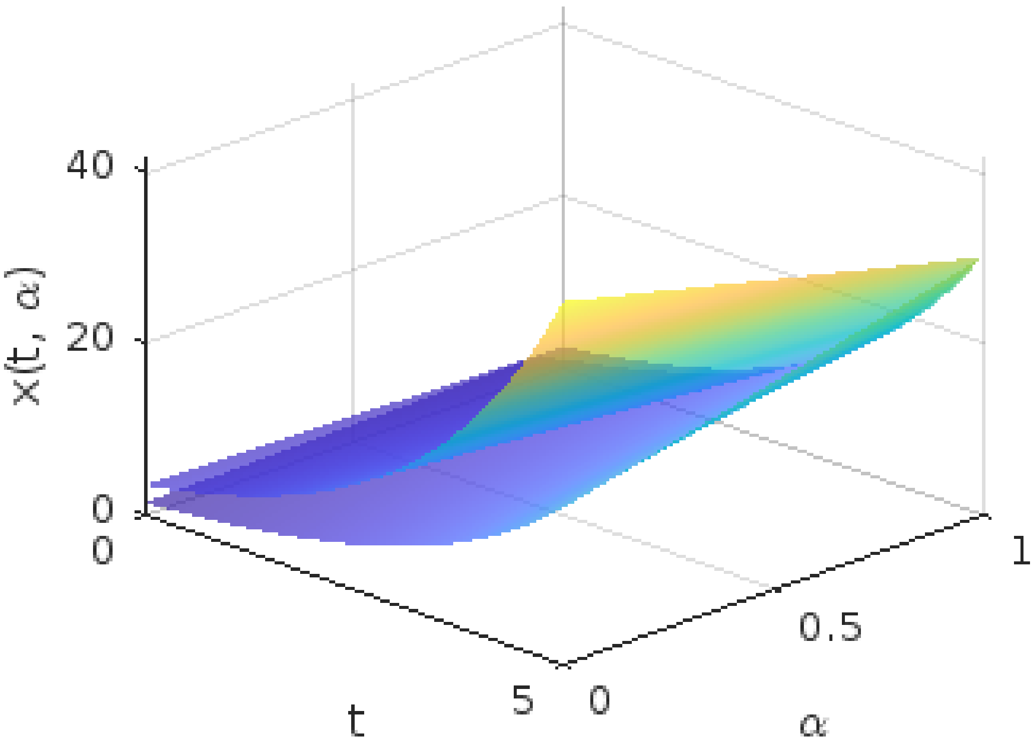

Figure 3.2: Lower and Upper Fuzzy Solutions for $\beta = 0.99$ and $\lambda = 1$

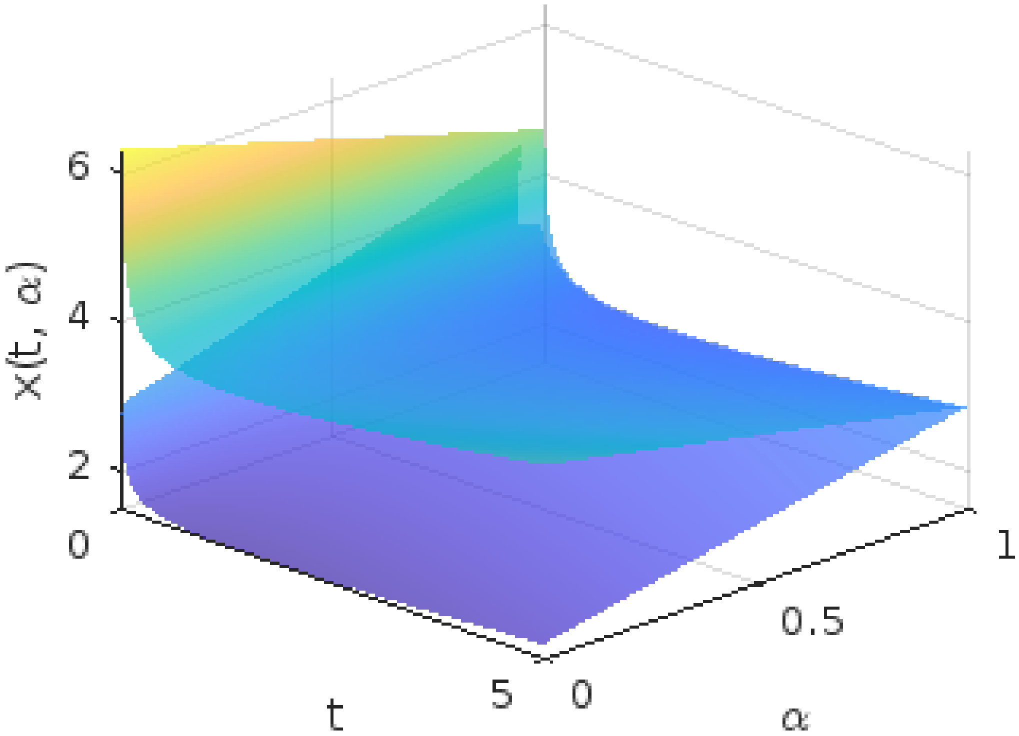

Figure 3.3: Lower and Upper Fuzzy Solutions for $\beta = 0.77$ and $\lambda = 0.2$

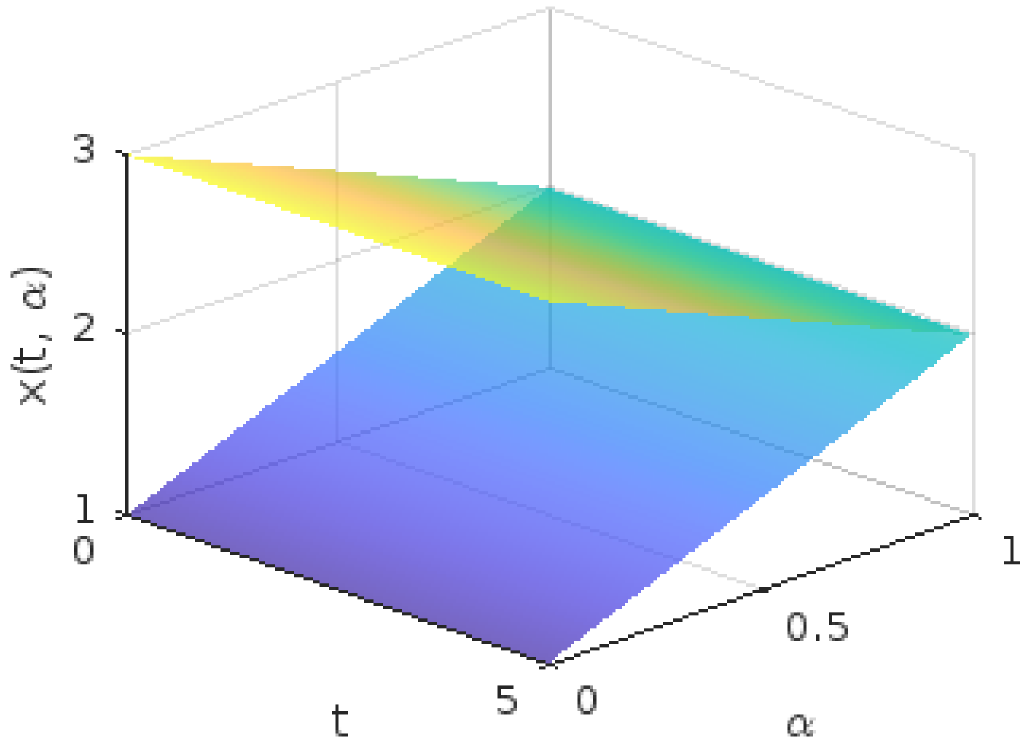

Figure 3.4: Lower and Upper Fuzzy Solutions for $\beta = 0.01$ and $\lambda = 1$

Example 3.2 *We are solving the fuzzy fractional differential equation:*

$$^{RL}D_{0+}^{\beta}\tilde{x}(t) \oplus {}^{C}D_{0+}^{\beta}\tilde{x}(t) = f(t, \tilde{x}(t)) \oplus \lambda\tilde{x}(t), \quad t > 0, \quad \beta \in (0, 1), \lambda > 0 \tag{3.10}$$

with the given fuzzy initial conditions:

- $\tilde{x}(0) = \tilde{x}_0 = (2, 3, 4)$ *(triangular fuzzy number)*
- $I_{0+}^{1-\beta}\tilde{x}(0) = \tilde{x}_1 = (3, 4, 5)$ *(triangular fuzzy number)*

and the right-hand side function being a constant fuzzy number:

- $\tilde{f}(t, \tilde{x}(t)) = F = (1, 2, 3)$ *(triangular fuzzy number)*

1. α-Cut Representation of Fuzzy Numbers

- ***For the constant fuzzy input*** $F = (1, 2, 3)$***:*** *Here,* $a = 1, b = 2, c = 3$.

$$\underline{F}(\alpha) = 1 + (2 - 1)\alpha = 1 + \alpha$$

$$\overline{F}(\alpha) = 3 - (3-2)\alpha = 3 - \alpha$$

Thus, the α-cut for the fuzzy input is $F_{[\alpha]} = [1+\alpha, 3-\alpha]$.

- ***For the initial condition $\tilde{x}_0 = (2,3,4)$:*** *Here, $a = 2, b = 3, c = 4$.*

$$\underline{\tilde{x}_0}(\alpha) = 2 + (3-2)\alpha = 2 + \alpha$$

$$\overline{\tilde{x}_0}(\alpha) = 4 - (4-3)\alpha = 4 - \alpha$$

Thus, the α-cut for $\tilde{x}_0$ is $\tilde{x}_{0[\alpha]} = [2+\alpha, 4-\alpha]$.

- ***For the initial condition $\tilde{x}_1 = (3,4,5)$:*** *Here, $a = 3, b = 4, c = 5$.*

$$\underline{x_1}(\alpha) = 3 + (4-3)\alpha = 3 + \alpha$$

$$\overline{\tilde{x}_1}(\alpha) = 5 - (5-4)\alpha = 5 - \alpha$$

Thus, the α-cut for $\tilde{x}_1$ is $\tilde{x}_{1[\alpha]} = [3+\alpha, 5-\alpha]$.

2. Derivation of the Crisp Solution Form

Applying the Laplace transform to equation (3.10) under the assumption of crisp (non-fuzzy) functions yields:

$$\left(s^\beta X(s) - I_{0+}^{1-\beta}x(0)\right) + \left(s^\beta X(s) - s^{\beta-1}x(0)\right) + \lambda X(s) = F(s)$$

Substitute the crisp initial conditions $x(0) = x_0^{crisp}$ and $I_{0+}^{1-\beta}x(0) = x_1^{crisp}$, and for a constant input $f(t) = F^{crisp}$, $F(s) = \frac{F^{crisp}}{s}$:

$$2s^\beta X(s) + \lambda X(s) = \frac{F^{crisp}}{s} + s^{\beta-1}x_0^{crisp} + x_1^{crisp}$$

$$(2s^\beta + \lambda)X(s) = \frac{F^{crisp}}{s} + s^{\beta-1}x_0^{crisp} + x_1^{crisp}$$

Solving for $X(s)$:

$$X(s) = \frac{F^{crisp}}{s(2s^\beta + \lambda)} + \frac{x_0^{crisp}s^{\beta-1}}{2s^\beta + \lambda} + \frac{x_1^{crisp}}{2s^\beta + \lambda}$$

To find the solution $x(t)$, we take the inverse Laplace transform of each term. Let $A = \lambda/2$. The denominator $2s^\beta + \lambda = 2(s^\beta + \lambda/2) = 2(s^\beta + A)$.

- ***Term from*** x_1^{crisp}***:***

$$\mathcal{L}^{-1}\left\{\frac{x_1^{crisp}}{2(s^\beta + A)}\right\} = \frac{x_1^{crisp}}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^\beta + A}\right\}$$
$$= \frac{x_1^{crisp}}{2}t^{\beta-1}E_{\beta,\beta}(-At^\beta)$$
$$= \frac{x_1^{crisp}}{2}t^{\beta-1}E_{\beta,\beta}\left(-\frac{\lambda}{2}t^\beta\right)$$

- ***Term from*** x_0^{crisp}***:***

$$\mathcal{L}^{-1}\left\{\frac{x_0^{crisp}s^{\beta-1}}{2(s^\beta + A)}\right\} = \frac{x_0^{crisp}}{2}\mathcal{L}^{-1}\left\{\frac{s^{\beta-1}}{s^\beta + A}\right\} = \frac{x_0^{crisp}}{2}E_{\beta,1}(-At^\beta) = \frac{x_0^{crisp}}{2}E_{\beta,1}\left(-\frac{\lambda}{2}t^\beta\right)$$

- ***Term from*** F^{crisp}***:***

$$\mathcal{L}^{-1}\left\{\frac{F^{crisp}}{s(2s^\beta + \lambda)}\right\} = \frac{F^{crisp}}{2}\mathcal{L}^{-1}\left\{\frac{1}{s(s^\beta + A)}\right\}$$

Using the identity $\mathcal{L}^{-1}\left\{\frac{1}{s(s^\beta+a)}\right\} = \frac{1}{a}(1 - E_{\alpha,1}(-at^\beta))$*, with* $a = A = \lambda/2$*:*

$$= \frac{F^{crisp}}{2}\cdot\frac{1}{\lambda/2}\left(1 - E_{\beta,1}\left(-\frac{\lambda}{2}t^\beta\right)\right) = \frac{F^{crisp}}{\lambda}\left(1 - E_{\beta,1}\left(-\frac{\lambda}{2}t^\beta\right)\right)$$

Combining these, the crisp solution $x(t)$ *is:*

$$x(t) = \frac{x_1^{crisp}}{2}t^{\beta-1}E_{\beta,\beta}\left(-\frac{\lambda}{2}t^\beta\right) + \frac{x_0^{crisp}}{2}E_{\beta,1}\left(-\frac{\lambda}{2}t^\beta\right) + \frac{F^{crisp}}{\lambda}\left(1 - E_{\beta,1}\left(-\frac{\lambda}{2}t^\beta\right)\right)$$

3. Lower and Upper Solutions using α-Cuts

The structure of the solution we will use is:

$$\underline{x}(t;\alpha) = \frac{\underline{x_1}(\alpha)}{2}t^{\beta-1}E_{\beta,\beta}\left(\frac{\lambda t^\beta}{2}\right) + \frac{\underline{x_0}(\alpha)}{2}E_{\beta,1}\left(\frac{\lambda t^\beta}{2}\right) + \frac{\underline{F}(\alpha)}{2\lambda}\left(1 - E_{\beta,1}\left(-\frac{\lambda t^\beta}{2}\right)\right),$$
$$\overline{x}(t;\alpha) = \frac{\overline{x_1}(\alpha)}{2}t^{\beta-1}E_{\beta,\beta}\left(\frac{\lambda t^\beta}{2}\right) + \frac{\overline{x_0}(\alpha)}{2}E_{\beta,1}\left(\frac{\lambda t^\beta}{2}\right) + \frac{\overline{F}(\alpha)}{2\lambda}\left(1 - E_{\beta,1}\left(-\frac{\lambda t^\beta}{2}\right)\right).$$

Lower Solution ($\underline{x}(t,\alpha)$***):*** *Substitute the lower bounds* $\underline{x_1}(\alpha) = 3+\alpha$*,* $\underline{x_0}(\alpha) = 2+\alpha$*, and* $\underline{F}(\alpha) = 1+\alpha$*:*

$$\underline{x}(t;\alpha) = \frac{3+\alpha}{2}t^{\beta-1}E_{\beta,\beta}\left(\frac{\lambda t^\beta}{2}\right) + \frac{2+\alpha}{2}E_{\beta,1}\left(\frac{\lambda t^\beta}{2}\right) + \frac{1+\alpha}{2\lambda}\left(1 - E_{\beta,1}\left(-\frac{\lambda t^\beta}{2}\right)\right)$$

Upper Solution ($\overline{x}(t,\alpha)$)***:*** *Substitute the upper bounds* $\overline{x_1}(\alpha) = 5-\alpha$, $\overline{x_0}(\alpha) = 4-\alpha$, *and* $\overline{F}(\alpha) = 3-\alpha$*:*

$$\overline{x}(t;\alpha) = \frac{5-\alpha}{2}t^{\beta-1}E_{\beta,\beta}\left(\frac{\lambda t^{\beta}}{2}\right) + \frac{4-\alpha}{2}E_{\beta,1}\left(\frac{\lambda t^{\beta}}{2}\right) + \frac{3-\alpha}{2\lambda}\left(1 - E_{\beta,1}\left(-\frac{\lambda t^{\beta}}{2}\right)\right)$$

4. Final Fuzzy-Valued Solution

The fuzzy-valued solution $\tilde{x}(t)$ *at any time* t *and for any* α*-level is given by the interval:*

$$[\tilde{x}(t)]_{\alpha} = [\underline{x}(t,\alpha), \overline{x}(t,\alpha)]$$

where $\underline{x}(t,\alpha)$ *and* $\overline{x}(t,\alpha)$ *are the detailed expressions derived above.*

In Figures 3.5, 3.6, and 3.7 we have the solutions for the of fuzzy fractional differential equation

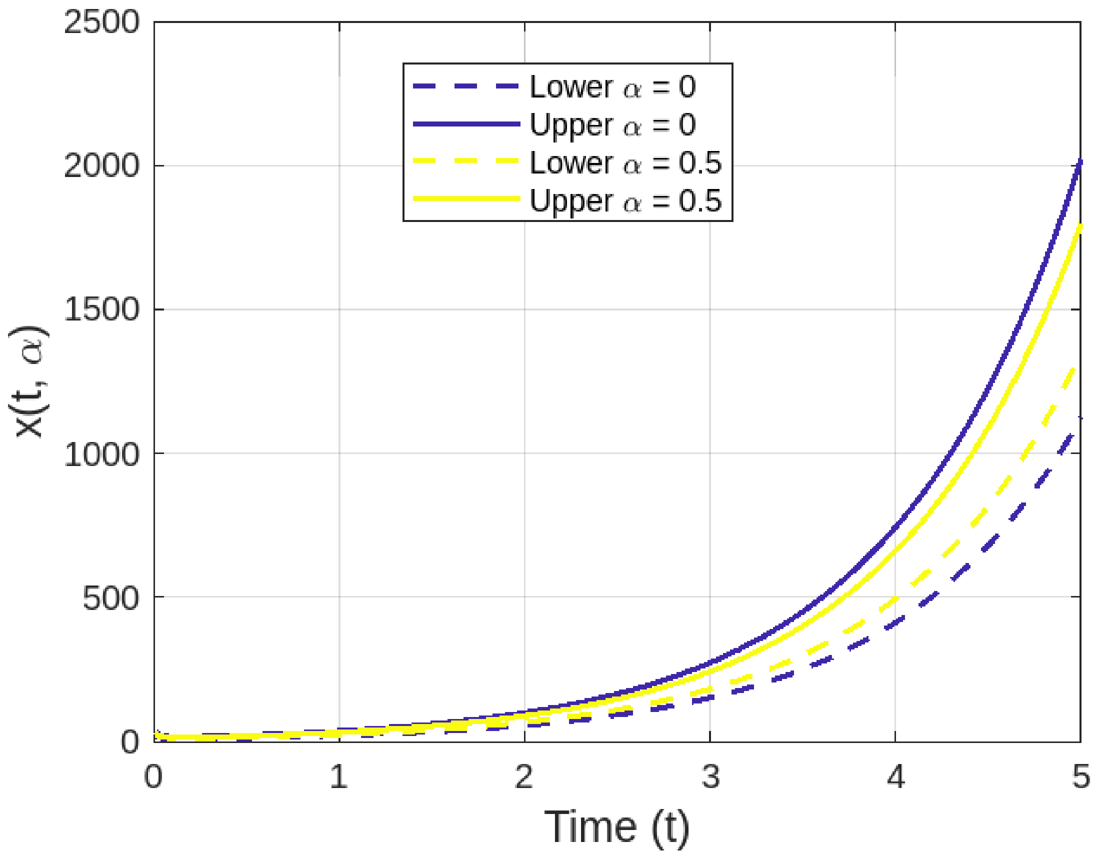

Figure 3.5: Fuzzy Solution Bounds for Different α Values and $\beta = 0.33$ and $\lambda = 2$

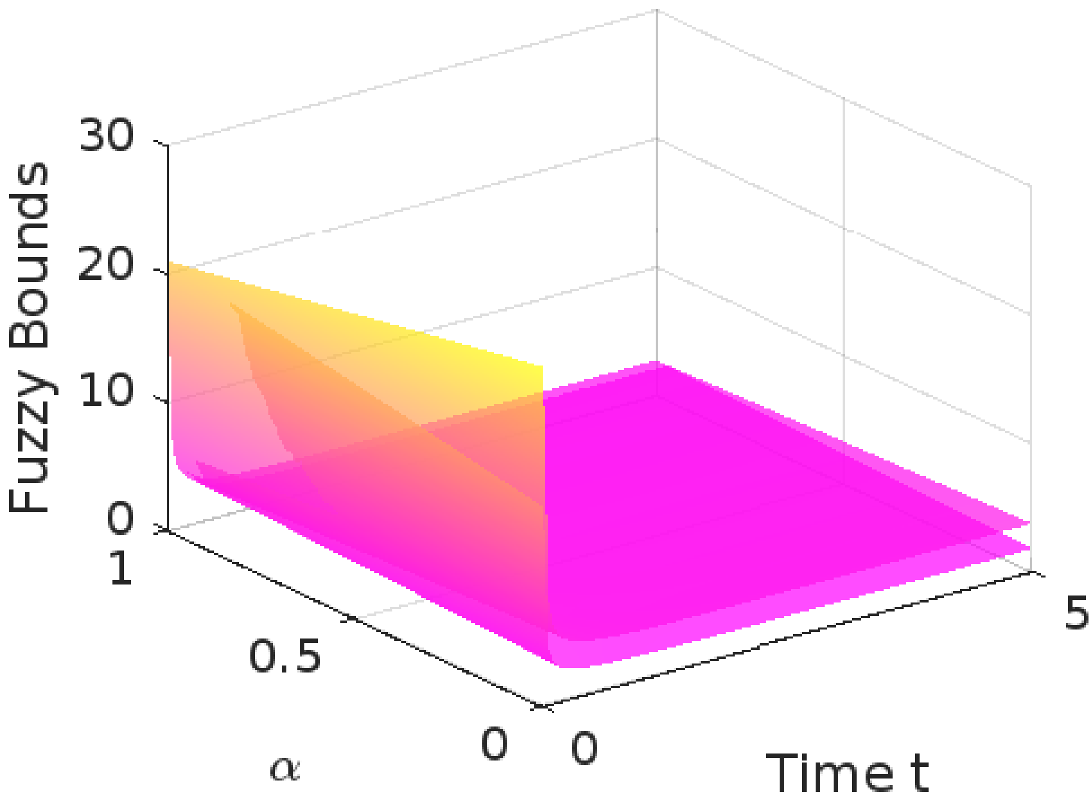

Figure 3.6: Lower and Upper Fuzzy Solutions for $\beta = 0.1$ and $\lambda = 0.5$

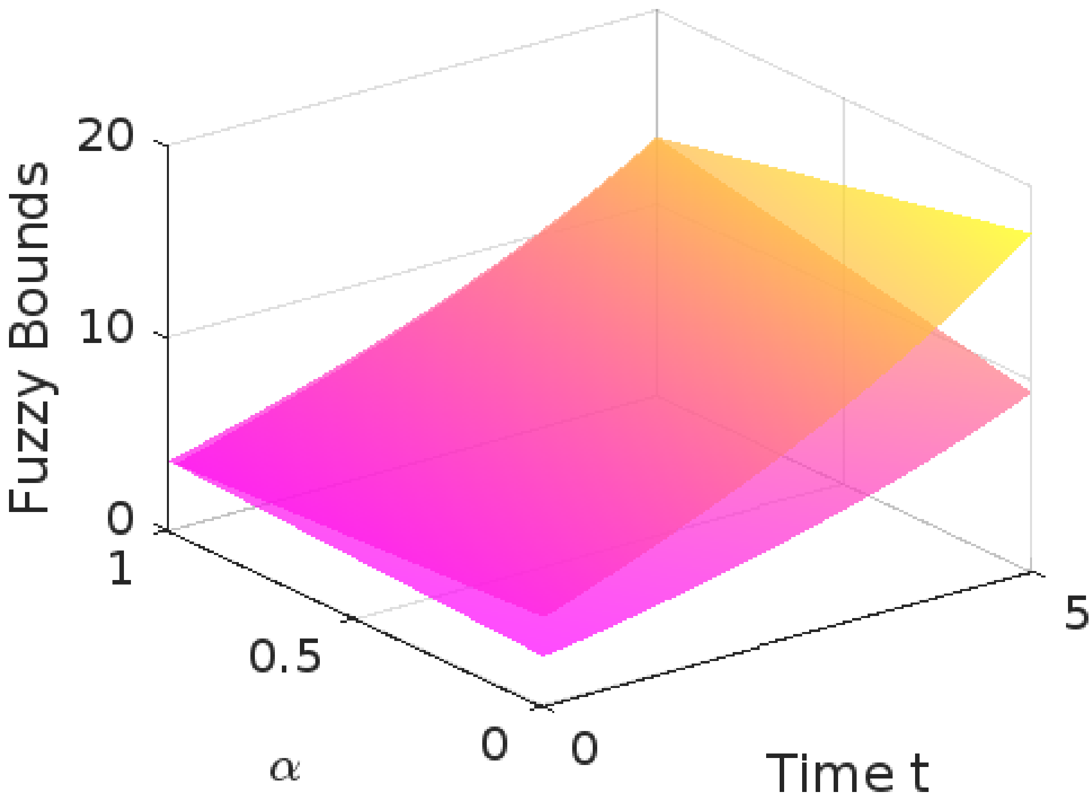

Figure 3.7: Lower and Upper Fuzzy Solutions for $\beta = 0.99$ and $\lambda = 0.5$

Example 3.3 *We are solving the fuzzy fractional differential equation:*

$$\begin{cases} {}^{RL}D^{\beta}_{0+}\tilde{x}(t) \oplus {}^{C}D^{\beta}_{0+}x(t) = f(t, \tilde{x}(t)) \oplus \lambda\tilde{x}(t), \quad t \in [0, T],\ T > 0 \quad \beta \in (0, 1), \lambda > 0, \\ \tilde{x}(0) = \tilde{x}_0, \quad I^{1-\beta}_{0+}\tilde{x}(0) = \tilde{x}_1, \end{cases} \tag{3.11}$$

with the specific conditions:

- $f(t, \tilde{x}(t)) = (1, 2, 3) \otimes \tilde{x}(t)$
- $\tilde{x}_0 = (2, 3, 4)$ *(triangular fuzzy number)*
- $\tilde{x}_1 = (3, 4, 5)$ *(triangular fuzzy number)*

Step 1: Reformulate the Equation with a Fuzzy Coefficient

Given $f(t, \tilde{x}(t)) = (1,2,3) \otimes \tilde{x}(t)$*, let* $F = (1,2,3)$ *be a constant fuzzy number. The equation becomes:*

$$^{RL}D^{\beta}_{0+}\tilde{x}(t) \oplus {}^{C}D^{\beta}_{0+}\tilde{x}(t) = F \otimes \tilde{x}(t) \oplus \lambda\tilde{x}(t)$$

The fuzzy expression

$$F \otimes \tilde{x}(t) \oplus \lambda\tilde{x}(t)$$

where:

- $F = (1,2,3)$ *is a triangular fuzzy number*
- $\tilde{x}(t)$ *has* α*-cut representation* $[\underline{x}(\alpha,t), \overline{x}(\alpha,t)]$
- $\lambda \in \mathbb{R}^+$ *is a positive crisp constant*

The α*-cut of* F *is:*

$$[\underline{F}(\alpha), \overline{F}(\alpha)] = [1+\alpha, 3-\alpha]$$

The α*-cut of the* $F \otimes \tilde{x}(t)$ *is:*

$$\Big[\min\Big\{(1+\alpha)\underline{x}(\alpha,t),\ (1+\alpha)\overline{x}(\alpha,t),\ (3-\alpha)\underline{x}(\alpha,t),\ (3-\alpha)\overline{x}(\alpha,t)\Big\} + \lambda\underline{x}(\alpha,t),$$
$$\max\Big\{(1+\alpha)\underline{x}(\alpha,t),\ (1+\alpha)\overline{x}(\alpha,t),\ (3-\alpha)\underline{x}(\alpha,t),\ (3-\alpha)\overline{x}(\alpha,t)\Big\} + \lambda\overline{x}(\alpha,t)\Big]$$

The α*-cut of the* $F \otimes \tilde{x}(t) \oplus \lambda\tilde{x}(t)$ *is:*

$$[\min(S) + \lambda\underline{x}(\alpha,t),\ \max(S) + \lambda\overline{x}(\alpha,t)]$$

where $S = \{(1+\alpha)\underline{x}(\alpha,t),\ (1+\alpha)\overline{x}(\alpha,t),\ (3-\alpha)\underline{x}(\alpha,t),\ (3-\alpha)\overline{x}(\alpha,t)\}$.
we can assume $\underline{x}(\alpha,t) > 0$ *and* $\overline{x}(\alpha,t) > 0$*, we have*
$\min(S) = (1+\alpha)\underline{x}(\alpha,t)$ *and* $\max(S) = (3-\alpha)\overline{x}(\alpha,t)$

The fuzzy fractional differential equation can now be written in terms of its α*-cuts:*

$$\begin{cases} ^{RL}D^{\beta}_{0+}[\underline{x}(\alpha,t), \overline{x}(\alpha,t)] + {}^{C}D^{\beta}_{0+}[\underline{x}(\alpha,t), \overline{x}(\alpha,t)] = [(\alpha+1)\underline{x}(\alpha,t) \quad , (3-\alpha)\overline{x}(\alpha,t)] \\ \qquad\qquad\qquad\qquad + [\lambda\underline{x}(\alpha,t), \lambda\overline{x}(\alpha,t)] \\ \tilde{x}_{[\alpha]}(0) = [\underline{\tilde{x}_0}(\alpha), \overline{\tilde{x}_0}(\alpha)] \\ I^{1-\beta}_{0+}x_{[\alpha]}(0) = [\underline{\tilde{x}_1}(\alpha), \overline{\tilde{x}_1}(\alpha)] \end{cases}$$

Step 2: α-Cut Representation of Initial Conditions

- ***For*** $\tilde{x}_0 = (2, 3, 4)$***:***

$$\underline{\tilde{x}_0}(\alpha) = 2 + (3 - 2)\alpha = 2 + \alpha$$

$$\overline{\tilde{x}_0}(\alpha) = 4 - (4 - 3)\alpha = 4 - \alpha$$

Thus, $\tilde{x}_{0[\alpha]} = [2 + \alpha, 4 - \alpha]$.

- ***For*** $\tilde{x}_1 = (3, 4, 5)$***:***

$$\underline{\tilde{x}_1}(\alpha) = 3 + (4 - 3)\alpha = 3 + \alpha$$

$$\overline{\tilde{x}_1}(\alpha) = 5 - (5 - 4)\alpha = 5 - \alpha$$

Thus, $\tilde{x}_{1[\alpha]} = [3 + \alpha, 5 - \alpha]$.

Step 3: Decouple the Fuzzy Equation into Crisp Equations

Lower Bound Equation:

$${}^{RL}D^{\beta}_{0+}\underline{\tilde{x}}(\alpha, t) + {}^{C}D^{\beta}_{0+}\underline{\tilde{x}}(\alpha, t) - (\lambda + 1 + \alpha)\underline{\tilde{x}}(\alpha, t) = 0$$

with initial conditions:

$$\underline{x}_0(\alpha) = 2 + \alpha$$

$$I^{1-\beta}_{0+}\underline{x}_0(\alpha) = 3 + \alpha$$

Upper Bound Equation:

$${}^{RL}D^{\beta}_{0+}\overline{\tilde{x}}(\alpha, t) + {}^{C}D^{\beta}_{0+}\overline{\tilde{x}}(\alpha, t) - (\lambda + 3 - \alpha)\overline{\tilde{x}}(\alpha, t) = 0$$

with initial conditions:

$$\overline{x}_0(\alpha) = 4 - \alpha$$

$$I^{1-\beta}_{0+}\overline{x}_0(\alpha) = 5 - \alpha$$

Step 4: Solve the Decoupled Crisp Equations

Both lower and upper bound equations are of the form:

Lower Bound Solution $\underline{x}_\alpha(t)$

Substitute $\underline{x_0}(\alpha) = 2 + \alpha$, *and* $\underline{x_1}(\alpha) = 3 + \alpha$*:*

$$\underline{\tilde{x}}(\alpha, t) = \frac{(3+\alpha)}{2} t^{\beta-1} E_{\beta,\beta}\left(\frac{\lambda + 1 + \alpha}{2} t^\beta\right) + \frac{(2+\alpha)}{2} E_{\beta,1}\left(\frac{\lambda + 1 + \alpha}{2} t^\beta\right)$$

Upper Bound Solution $\overline{x}_\alpha(t)$

Substitute , $\overline{x_0}(\alpha) = 4 - \alpha$, *and* $\overline{x_1}(\alpha) = 5 - \alpha$*:*

$$\overline{\tilde{x}}(\alpha, t) = \frac{(5-\alpha)}{2} t^{\beta-1} E_{\beta,\beta}\left(\frac{\lambda + 3 - \alpha}{2} t^\beta\right) + \frac{(4-\alpha)}{2} E_{\beta,1}\left(\frac{\lambda + 3 - \alpha}{2} t^\beta\right)$$

Step 5: Final Fuzzy-Valued Solution

The fuzzy-valued solution $\tilde{x}(t)$ *at each* α*-level is given by the interval:*

$$\tilde{x}_{[\alpha]}(t) = [\underline{\tilde{x}}(\alpha, t), \overline{\tilde{x}}(\alpha, t)]$$

where $\underline{\tilde{x}}(\alpha, t)$ *and* $\overline{\tilde{x}}(\alpha, t)$ *are the expressions derived above.*

In Figures 3.8 and 3.9 we have the solutions for the of fuzzy fractional differential equation

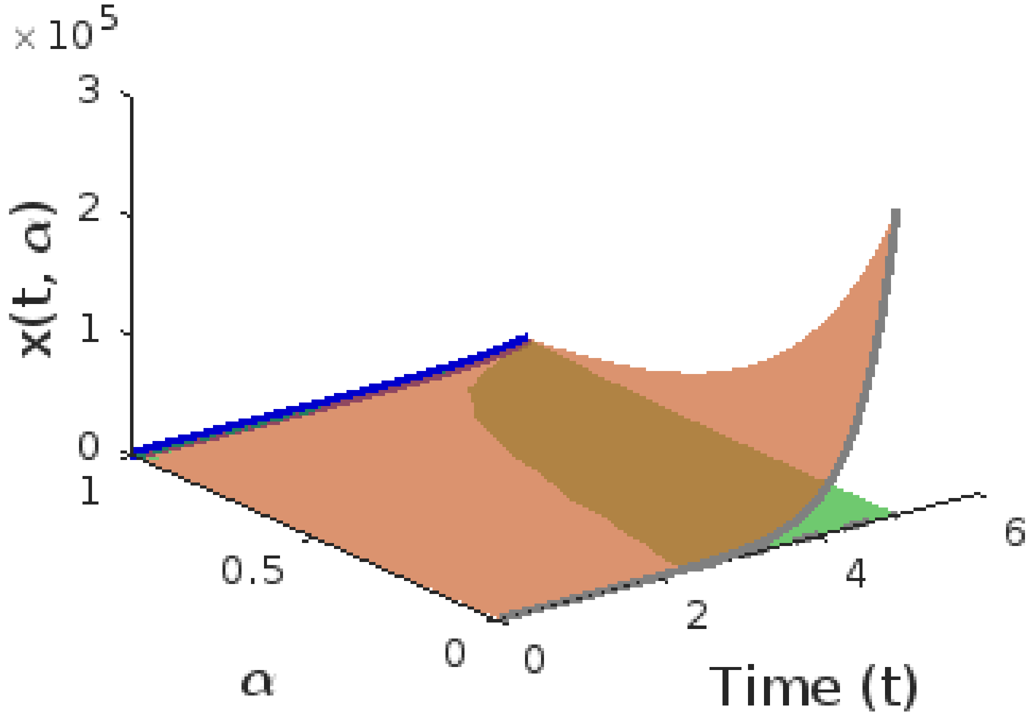

Figure 3.8: Lower and Upper Fuzzy Solutions for $\beta = 0.9$ and $\lambda = 1$

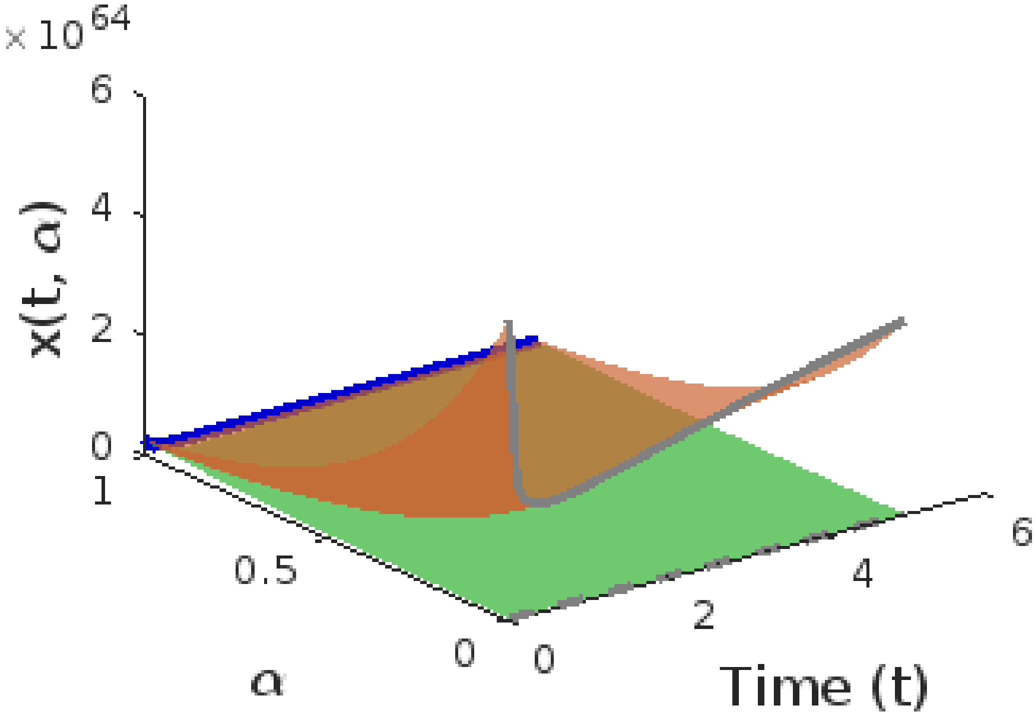

Figure 3.9: Lower and Upper Fuzzy Solutions for $\beta = 0.01$ and $\lambda = 20$

Chapter 4

Existence, Uniqueness and Stability of Fuzzy Fractional Differential Equations

4.1 Introduction

In this chapter, we explore the existence, uniqueness, and various stability properties of solutions to a nonlinear fuzzy fractional differential equation. Our focus is on Ulam-Hyers stability, considering the Caputo generalized Hukuhara differentiability of order $\beta \in (n-1, n)$. To ensure the existence and uniqueness of solutions, we establish explicit criteria based on the Banach fixed point theorem.
This body of literature underscores the contributions of researchers in establishing the existence of solutions for fuzzy fractional differential equations (FFDEs).

In 2012, Salahshour et al. [41] examined the existence, uniqueness, and approximations of solutions to FFDEs using Caputo's H-differentiability. Nieto et al. in [32] studied some results on boundary value problems for fuzzy differential equations with functional dependence. In 2013, Arshad [8] investigated these properties while focusing on Riemann-Liouville fractional derivatives.

In 2017, Rivaz et al. [39] proved that the problem

$$\begin{cases} {}_{gH}^{C}\mathcal{D}^{\beta}\omega(\xi) = \varpi(\xi, \omega(\xi)), \quad \beta \in [0,1], \quad \xi \in [\xi_0, a], \\ \omega(\xi_0) = \omega_0, \end{cases} \tag{4.1}$$

admits two unique solutions, where $\mathcal{D}_c^{\beta}$ represents the Caputo derivative, and ϖ is a fuzzy continuous function.

Building upon these findings, we explore the existence, uniqueness, and various stability types of higher-order Hyers-Ulam solutions to the problem

$$\begin{cases} {}_{gH}^{C}\mathcal{D}^{\beta}\omega(\xi) = \varpi(\xi, \omega(\xi)), \quad \xi \in [a, b], \\ \omega_{gH}^{(k)}(a) = c_k, \quad 0 \leqslant k < n-1, \end{cases} \tag{4.2}$$

where $n-1 < \beta < n$, c_k $(k = 0, ..., n-1)$ are constants, ϖ is a continuously fuzzy function, and ${}_{gH}^{C}\mathcal{D}_c^{\beta}$ denotes the Caputo generalized Hukuhara differentiability.

4.2 Existence and uniqueness results

4.2.1 Key Definitions and Theorems

Here, we address the existence and uniqueness results for our problem, considering the higher-order case where $n-1 < \beta < n$ with $n \in \mathbb{N}^*$, as defined in 4.2. We present and prove essential definitions and theorems that will be instrumental in establishing the existence and uniqueness of the solution to the problem formulated in 4.2.

Definition 4.1 *[5] Let $f_{gH}^{(m)} \in C[a,b] \cap L[a,b]$. The gH-fractional Caputo differentiability of fuzzy valued function f is defined as follows:*

$$ {}_{gH}^{C}D^{\alpha} f(x) = I^{m-\alpha}\left(f_{gH}^{(n)}\right)(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{(n-\alpha-1)} f_{gH}^{(n)}(t)\, dt \tag{8}$$

where $x > 0$ and $n = \lfloor \alpha \rfloor + 1$.

Definition 4.2 *[5] The Riemann-Liouville generalized Hukuhara differentiability of fuzzy-valued function ω, $n \in \mathbb{N}^*, \xi > a$, is defined as following:*

$$\mathcal{D}_{RL}^{\beta}\omega(\xi) = \mathcal{D}^n \mathcal{I}_{RL}^{n-\beta}\omega(\xi) = \left(\mathcal{I}_{RL}^{n-\beta}\omega(\xi)\right)_{gH}^{(n)}$$

$$= \frac{1}{\Gamma(n-\beta)} \left(\frac{d}{d\xi}\right)_{gH}^{n} \int_a^{\xi} (\xi - s)^{n-\beta-1}\omega(s)ds, \quad n-1 < \beta < n.$$

The Hausdorff distance, denoted as $d_\infty : \mathbb{F}_\mathbb{R} \times \mathbb{F}_\mathbb{R} \to \mathbb{R}^+ \cup \{0\}$, is given as follows

$$d_\infty(\omega, v) = H([\omega]^\flat, [v]^\flat) = \sup_{0 \le \flat \le 1} \max \left\{ \left|\underline{\omega}(\flat) - \underline{v}(\flat)\right|, \left|\bar{\omega}(\flat) - \bar{v}(\flat)\right| \right\}.$$

We have $(\mathbb{F}_\mathbb{R}, d_\infty)$ be a Banach space , with

(1) $d_\infty(\omega + s, v + s) = d_\infty(\omega, v), \quad \forall \omega, v, s \in \mathbb{F}_\mathbb{R}$;

(2) $d_\infty(\omega + s, h + o) \le d_\infty(\omega, h) + d_\infty(s, o) \quad \forall \omega^*, s, h, o \in \mathbb{F}_\mathbb{R}$;

(3) $d_\infty(\tau\omega, \tau v) = |\tau| d_\infty(\omega, v), \quad \forall \tau \in \mathbb{R}, \omega, v \in \mathbb{F}_\mathbb{R}$.

We have also $\mathcal{C}([a, b], \mathbb{F}_\mathbb{R})$ be fuzzy Banach spaces, where the distance function is defined as

$$d(w, v) = \sup_{a \le \xi \le b} d_\infty(w(\xi), v(\xi)). \tag{4.3}$$

Proposition 4.3 *[20] If $\omega \in \mathcal{C}([a, b], \mathbb{F}_\mathbb{R})$, then $\mathcal{D}_{RL}^{\beta} I_{RL}^{\beta}\omega(\xi) = \omega(\xi)$, for all $\beta > 0$.*

Proposition 4.4 *Assume $\beta > 0, \beta \notin \mathbb{N}, n = [\beta]+1$ and $\omega_{gH}^{(n)}(\xi)$ is a function whose values are continuous fuzzy numbers.. Based on Theorem 4.3 in [6], we choose the following cases,*

(i) If ω and its higher order derivatives are i-gH differentiable on$[a, b]$, then:

$$\omega(\xi) = \omega(a) \oplus \sum_{k=1}^{n-1} \frac{(\xi - a)^k}{k!} \omega_{gH}^{(k)}(a) \oplus I_{RLgH}^{\beta} {}^{C}\mathcal{D}^{\beta}\omega(\xi).$$

(ii) If $\omega \in \mathcal{C}^n([a, b], \mathbb{F}_{\mathbb{R}})$; $\omega$and its higher order derivatives are ii-gH differentiable on $[a, b]$, then

$$\omega(\xi) = \omega(a) \ominus (-1) \sum_{k=1}^{n-1} \frac{(\xi - a)^k}{k!} \omega_{gH}^{(k)}(a) \ominus_H (-1) I_{RLgH}^{\beta} {}^{C}\mathcal{D}^{\beta}\omega(\xi).$$

(iii) If $\omega \in \mathcal{C}^n([a, b], \mathbb{F}_{\mathbb{R}})$; ω is i-gH differentiable and its higher-order derivatives are changing i then

$$\omega(\xi) = \omega(a) \ominus (-1) \sum_{k=1,\ even}^{n-1} \frac{(\xi - a)^k}{k!} \omega_{gH}^{(k)}(a) \oplus \sum_{k=1, odd}^{m-1} \frac{(\xi - a)^k}{k!} \omega_{gH}^{(k)}(a) \oplus I_{RLgH}^{\beta} {}^{C}\mathcal{D}^{\beta}\omega(\xi).$$

(iv) If ω is ii-gH differentiable and its higher-order derivatives are changing in every other type then

$$\omega(\xi) = \omega(a) \ominus (-1) \sum_{k=1\ odd}^{n-1} \frac{(\xi - a)^k}{k!} \omega_{gH}^{(k)}(a) \oplus \sum_{k=1\ even}^{n-1} \frac{(\xi - a)^k}{k!} \omega_{gH}^{(k)}(a) \ominus (-1) I_{RLgH}^{\beta} {}^{C}\mathcal{D}^{\beta}\omega(\xi).$$

4.2.2 Study of the Existence and Uniqueness of the Solution

We are now in a position to study the existence and uniqueness of the solution to the problem defined in 4.2. To proceed, we begin by defining the following sets

$\mathcal{M}_1 = \{\omega \in \mathcal{C}^n([a,b], \mathbb{F}_\mathbb{R}); \omega$ and its higher order derivatives are i-gH differentiable on$[a,b]\}$.

$\mathcal{M}_2 = \{\omega \in \mathcal{C}^n([a,b], \mathbb{F}_\mathbb{R}); \omega$and its higher order derivatives are ii-gH differentiable on $[a,b]\}$.

$\mathcal{M}_3 = \{\omega \in \mathcal{C}^n([a,b], \mathbb{F}_\mathbb{R}); \omega$ is i-gH differentiable and its higher-order derivatives are changing in ev

$\mathcal{M}_4 = \{\omega \in \mathcal{C}^n([a,b], \mathbb{F}_\mathbb{R}); \omega$ is ii-gH differentiable and its higher-order derivatives are changing in ev

Theorem 4.5 *For $\beta > 0$, $n = [\beta] + 1$ and, $\omega_{gH}^{(n)} \in \mathcal{C}([a,b], \mathbb{F}_\mathbb{R})$, let $\varpi : [a,b] \times \mathbb{F}_\mathbb{R} \to \mathbb{F}_\mathbb{R}$ be continuous function.*

(i) if $\omega \in \mathcal{M}_1$, ω is solution of the problem (4.2) if and only if given by

$$\omega(\xi) = c_0 \oplus \sum_{k=1}^{n-1} \frac{(\xi - a)^k}{k!} c_k \oplus I_{RL}^{\beta} \varpi(\xi, \omega(\xi)). \tag{4.4}$$

(ii) If $\omega \in \mathcal{M}_2$, ω is solution of the problem (4.2) if and only if given by

$$\omega(\xi) = c_0 \ominus (-1) \sum_{k=1}^{n-1} \frac{(\xi - a)^k}{k!} c_k \ominus (-1) I_{RL}^{\beta} \varpi(\xi, \omega(\xi)). \tag{4.5}$$

(iii) If $\omega \in \mathcal{M}_3$, ω is solution of the problem (4.2) if and only if given by

$$\omega(\xi) = c_0 \ominus (-1) \sum_{k=1,even}^{n-1} \frac{(\xi - a)^k}{k!} c_k \oplus \sum_{k=1,odd}^{n-1} \frac{(\xi - a)^k}{k!} c_k \oplus I_{RL}^{\beta} \varpi(\xi, \omega(\xi)). \tag{4.6}$$

(iv) If $\omega \in \mathcal{M}_4$, ω is solution of the problem (4.2) if and only if given by

$$\omega(\xi) = c_0 \ominus (-1) \sum_{k=1\ ,odd}^{n-1} \frac{(\xi - a)^k}{k!} c_k \oplus \sum_{k=1,\ even}^{m-1} \frac{(\xi - a)^k}{k!} c_k \ominus (-1) I_{RL}^{\beta} \varpi(\xi, \omega(\xi)). \tag{4.7}$$

Proof. Only the first case is proven here; the proofs for the others are similar.

1- The first implication:

Let ω is a solution of (4.2), we have

$${}_{gH}^{C}\mathcal{D}^{\beta} \omega(\xi) = \varpi(\xi, \omega(\xi)), \quad \xi \in [a,b]. \tag{4.8}$$

And ω satisfies the following conditions

$$\omega_{gH}^{(k)}(a) = c_k, \quad 0 \leqslant k < n-1. \tag{4.9}$$

Since $\omega \in \mathcal{M}_1$, according to Proposition (4.4), it can be written as follows

$$\omega(\xi) = \omega(a) \oplus \sum_{k=1}^{n-1} \frac{(\xi-a)^k}{k!} \omega^{(k)}(a) \oplus I_{RL}^{\beta} {}_{gH}^{C}\mathcal{D}^{\beta} \omega(\xi). \tag{4.10}$$

From equation 4.8 and 4.9, equation 4.10 takes the following form

$$\omega(\xi) = c_0 \oplus \sum_{k=1}^{n} \frac{(\xi-a)^k}{k!} c_k \oplus I_{RL}^{\beta} \varpi(\xi, \omega(\xi)). \tag{4.11}$$

And this is the required result.

Now we prove the reverse implication:

Suppose that $\omega \in \mathcal{C}^n([a,b], \mathbb{F}_{\mathbb{R}})$ be a fuzzy function, it fulfills the integral equation shown below

$$\omega(\xi) = c_0 \oplus \sum_{k=1}^{n} \frac{(\xi-a)^k}{k!} c_k \oplus I_{RL}^{\beta} \varpi(\xi, \omega(\xi)). \tag{4.12}$$

Since $\omega \in \mathcal{M}_1$, according to Proposition (4.4), it can be written as follows

$$\omega(\xi) = \omega(a) \oplus \sum_{k=1}^{n-1} \frac{(\xi-a)^k}{k!} \omega^{(k)}(a) \oplus I_{RL}^{\beta} {}_{gH}^{C}\mathcal{D}^{\beta} \omega(\xi), \tag{4.13}$$

From Equations 4.12 and 4.13, we get

$$I_{RL}^{\beta} {}_{gH}^{C}\mathcal{D}^{\beta} \omega(\xi) = I_{RL}^{\beta} \varpi(\xi, \omega(\xi)), \tag{4.14}$$

and

$$\omega_{gH}^{(k)}(a) = c_k, \quad 0 \leqslant k < n-1. \tag{4.15}$$

Applying the Riemann-Liouville fractional derivative $\mathcal{D}_{RL}^{\beta}$ of order β to both sides of equations 4.14, and utilizing Proposition (4.3), we obtain (4.2)

$$ {}_{gH}^{C}\mathcal{D}^{\beta} \omega(\xi) = \varpi(\xi, \omega(\xi)), \quad \xi \in [a,b]. \tag{4.16}$$

The proof is completed using equations 4.16 and 4.15. ■

Theorem 4.6 *Assume that*

(H_1) $I^{\beta}\varpi(\xi,\omega(\xi)) \in \mathcal{M}_1 \cup \mathcal{M}_3$.

(H_2) $\exists L > 0, \forall \omega \in \mathcal{C}([a,b],\mathbb{F}_{\mathbb{R}}), \forall \xi \in [a,b]$,

$$d_{\infty}(\varpi(\xi,\omega(\xi)),\varpi(\xi,\omega(\xi))) < Ld_{\infty}(\omega(\xi),y(\xi)).$$

(H_3) *There is a positive number* γ *such that*

$$\gamma = \frac{L(b-a)^{\beta}}{\Gamma(\beta+1)} < 1.$$

If the assumptions H_1, H_2 *and* H_3 *hold, then problem (4.2) possesses two distinct solutions within the interval* $[a,b]$.

Proof. Let's define four operators from $\mathcal{C}([a,b],\mathbb{F}_{\mathbb{R}})$ in $\mathcal{C}([a,b],\mathbb{F}_{\mathbb{R}})$ as follows:

1- If $I^{\beta}\varpi(\xi,\omega(\xi)) \in \mathcal{M}_1$,

$$(F_1\omega)(\xi) = c_0 \oplus \sum_{k=1}^{n-1} \frac{(\xi-a)^k}{k!} c_k \oplus I^{\beta}_{RL}\varpi(\xi,\omega(\xi)).$$

$$(F_2\omega)(\xi) = c_0 \ominus (-1) \sum_{k=1}^{n-1} \frac{(\xi-a)^k}{k!} c_k \ominus (-1) I^{\beta}_{RL}\varpi(\xi,\omega(\xi)).$$

2- If $I^{\beta}_{RL}\varpi(\xi,\omega(\xi)) \in \mathcal{M}_3$,

$$(F_3\omega)(\xi) = c_0 \ominus (-1) \sum_{k=1,even}^{n-1} \frac{(\xi-a)^k}{k!} c_k \oplus \sum_{k=1,odd}^{n-1} \frac{(\xi-a)^k}{k!} c_k \oplus I^{\beta}_{RL}\varpi(\xi,\omega(\xi)).$$

$$(F_4\omega)(\xi) = c_0 \ominus (-1) \sum_{\text{k=1 ,odd}}^{n-1} \frac{(\xi-a)^k}{k!} c_k \oplus \sum_{\text{k=1, even}}^{n-1} \frac{(\xi-a)^k}{k!} c_k \ominus (-1) I^{\beta}_{RL}\varpi(\xi,\omega(\xi)).$$

For $\omega, y \in \mathcal{C}([a,b],\mathbb{F}_{\mathbb{R}}), \xi \in [a,b]$

$$
\begin{aligned}
d\left(F_i(\omega), F_i(y)\right) = \sup_{a\leq\xi\leq b} d_\infty\left(F_i(\omega)(\xi), F_i(y)(\xi)\right) &\leqslant \sup_{a\leq\xi\leq b} \frac{1}{\Gamma(\beta)}\int_a^\xi (\xi-s)^\beta d_\infty\left(\varpi(s,\omega(s)), \varpi(s,y(s))\right) ds \\
&\leqslant \sup_{a\leq\xi\leq b} \frac{L}{\Gamma(\beta)}\int_a^\xi (\xi-s)^\beta d_\infty(\omega(s),y(s))ds \\
&\leq \frac{L(b-a)^\beta}{\Gamma(\beta+1)} d(\omega,y).
\end{aligned}
\tag{4.17}
$$

Then for $i = 1, 2, 3, 4$

$$
d\left(F_i(\omega), F_i(y)\right) < \gamma d(\omega, y), \tag{4.18}
$$

by Banach fixed point theorem, F_i has a unique fixed point $\omega_i \in \mathcal{M}_i$ which is a solution of the integral equation (4.4), then ω_i is a solution for the differential equation (4.2) for $i = 1, 2, 3, 4$. ■

4.3 Stability Analysis

In this section, we explore several well-established stability concepts related to the solutions of problem 4.2, including Ulam-Hyers stability, generalized Ulam-Hyers stability, and Ulam-Hyers-Rassias stability. We begin by introducing the relevant definitions and theoretical foundations, drawing on the works presented in [9, 14, 29, 44].

Definition 4.7 *The problem 4.2 is Ulam-Hyers stability if $\exists K_\varpi \in \mathbb{R}_+^*$ such that for $\epsilon > 0$ and, for all $v \in \mathcal{M}_i$ $(i = 1, ..., 4)$ satisfying the inequality*

$$
d\Big[{}^C gH\mathcal{D}^\beta v(\xi), \varpi\left(\xi, v(\xi)\right)\Big] \leq \epsilon, \tag{4.19}
$$

there exists a solution, $\omega \in \mathcal{M}_i$ $(i = 1, ..., 4)$ to problem 4.2 with

$$
d[v(\xi), \omega(\xi)] \leq K_\varpi \epsilon,
$$

for all $\xi \in [a, b]$.

Definition 4.8 *The problem 4.2 is a generalized Ulam-Hyers stability if* $\exists \psi_{\varpi}(\xi) : \mathbb{R}^{+} \to \mathbb{R}^{+}$ *continuous such that* $\psi_{\varpi}(0) = 0$. *This is provided that for all solution* $v \in \mathcal{M}_i$ $(i = 1, ..., 4)$ *satisfies inequality (4.19), there exists another solution,* $\omega \in \mathcal{M}_i$ $(i = 1, ..., 4)$ *to problem 4.2, satisfying*

$$d[v(\xi), \omega(\xi)] \leq \psi(\epsilon),$$

for all $\xi \in [a, b]$.

Definition 4.9 *The problem 4.2 is a Ulam Hyers Rassias stability* $\exists K_{\varpi} \in \mathbb{R}_{+}^{*}$ *such that for every* $\varphi \in C\left([a, b], (0, +\infty)\right)$ *and for every* $v \in \mathcal{M}_i$ $(i = 1, ..., 4)$, *the following inequality holds:*

$$d\Big[{}_{gH}^{C}\mathcal{D}^{\beta} v(\xi), \varpi\left(\xi, v(\xi)\right)\Big] \leq \varphi(\xi), \tag{4.20}$$

there exists a solution, $\omega \in \mathcal{M}_i$ $(i = 1, ..., 4)$ *to the problem 4.2 with*

$$d[v(\xi), \omega(\xi)] \leq K_{\varpi}\varphi(\xi),$$

for all $\xi \in [a, b]$.

Remark 4.10 *A function* v *from* $\mathcal{M}i$ $(i = 1, ..., 4)$ *satisfies inequality (4.19) if and only if there exists a* $g \in \mathcal{C}([a, b], \mathbb{F}_{\mathbb{R}})$ *defined on* $[a, b]$ *such that:*

(i) $d\big[g(\xi), \widehat{0}\big] \leq \varepsilon$ *for any* $\xi \in [a, b]$;

(ii) $$\begin{cases} {}_{gH}^{C}\mathcal{D}^{\beta} v(\xi) = \varpi(\xi, v(\xi)) + g(\xi), & \xi \in [a, b], \\ v_{gH}^{(k)}(a) = c_k, & 0 \leqslant k < n - 1. \end{cases}$$

Remark 4.11 *A function* v *from* $\mathcal{M}i$ $(i = 1, ..., 4)$ *satisfies inequality (4.20) if and only if there exists a* $g \in \mathcal{C}([a, b], \mathbb{F}_{\mathbb{R}})$ *such that:*

(i) $d\big[g(\xi), \widehat{0}\big] \leq \varphi(\xi)$ *for any* $\xi \in [a, b]$;

(ii) $$\begin{cases} {}_{gH}^{C}\mathcal{D}^{\beta} v(\xi) = \varpi(\xi, v(\xi)) + g(\xi), & \xi \in [a, b], \\ v_{gH}^{(k)}(a) = c_k, & 0 \leqslant k < n - 1. \end{cases}$$

Theorem 4.12 *Under the Assumption H_2 and $\Gamma(\beta+1) > \mathcal{L}(b-a)^{\beta}$, the solution to problem 4.2 is Ulam Hers stable in $\mathcal{M}_i$ $(i = 1, ..., 4)$ And consequently, it exhibits generalized Ulam Hyers stability stable*

Proof. We will demonstrate the Hyers Ulam stability of our problem 4.2 for $\omega \in \mathcal{M}_1$, as well as for $\omega \in \mathcal{M}_i$ $(i = 2, ...4)$ in the same manner.

Since $v \in \mathcal{M}_1$ satisfies the inequality (4.19), by Remark 4.10 we obtain

$$v(\xi) = c_0 \oplus \sum_{k=1}^{n-1} \frac{(t-a)^k}{k!} c_k \oplus I_{RL}^{\beta} \varpi(\xi, v(\xi)) \oplus I_{RL}^{\beta} g(\xi),$$

if $\omega \in \mathcal{M}_1$ is a solution of 4.2, we have

$$d[v,\omega] = \sup_{a\leq\xi\leq b} d_{\infty}[v(\xi), \omega(\xi)] = \sup_{a\leq\xi\leq b} d_{\infty}\Bigg[c_0 \oplus \sum_{k=1}^{n-1} \frac{(\xi-a)^k}{k!} c_k \oplus I_{RL}^{\beta} \varpi(\xi, v(\xi)) \oplus I_{RL}^{\beta} g(\xi), c_0$$

$$\oplus \sum_{k=1}^{n-1} \frac{(\xi-a)^k}{k!} c_k \oplus I_{RL}^{\beta} \varpi(\xi, \omega(\xi))\Bigg]$$

$$\leq \sup_{a\leq\xi\leq b} \frac{1}{\Gamma(\beta)} \int_a^{\xi} (\xi-s)^{\beta-1} d_{\infty}\big[\varpi\left(s, v(s)\right), \varpi(s, \omega(s))\big] ds$$

$$+ \frac{1}{\Gamma(\beta)} \int_a^{\xi} (\xi-s)^{\beta-1} d_{\infty}\big[g(s), \widehat{0}\big] ds$$

$$\leq \sup_{a\leq\xi\leq b} L \frac{1}{\Gamma(\beta)} \int_a^{\xi} (\xi-s)^{\beta-1} d_{\infty}\big[v(s), \omega(s)\big] ds + \frac{\varepsilon}{\Gamma(\beta)} \int_0^{\xi} (\xi-s)^{\beta-1} ds$$

$$\leq \frac{L(b-a)^{\beta}}{\Gamma(\beta+1)} d\big[v,\omega\big] + \frac{\varepsilon(b-a)^{\beta}}{\Gamma(\beta+1)},$$

for any $\xi \in [a, b]$. Then

$$d[v,\omega] \leq \frac{(b-a)^{\beta}}{\Gamma(\beta+1) - L(b-a)^{\beta}} \varepsilon,$$

for any $\xi \in [a, b]$.

Let $K_{\varpi} = \frac{(b-a)^{\beta}}{\Gamma(\beta+1)-L(b-a)^{\beta}}$, we have

$$d[v,\omega] \leq K_{\varpi}\epsilon.$$

Thus it is deduced that the problem 4.2 is Ulam Hyers stable. Furthermore, assuming $\psi_{\varpi}(\epsilon) = K_{\varpi}\epsilon$. It is evident that $\psi_{\varpi}(0) = 0$. Thus, we conclude that the solution function of problem 4.2 exhibits generalized Ulam Hyers stable, thereby completing the proof. ■

Theorem 4.13 *According the assumptions $\mathcal{H}_2$ and $\Gamma(\beta+1) > \mathcal{L}(b-a)^{\beta}$. Let $\varphi : [a,b] \to (0,+\infty)$ be a continuous and increasing function, with the existence of a constant $\mathcal{W} > 0$ such that*

$$\frac{1}{\Gamma(\beta)} \int_0^{\xi} (\xi - s)^{\beta-1} \varphi(s) ds \leq \mathcal{W}\varphi(\xi), \qquad \text{for each} \qquad \xi \in [a,b].$$

The solution to problem 4.2 demonstrates Ulam Hyers Rassias stability within each $\mathcal{M}_i$ for $i = 1, ..., 4$

Proof. Since $v \in \mathcal{M}_1$ satisfies the inequality (4.20), by Remark 4.11, we obtain

$$v(t) = c_0 \oplus \sum_{k=1}^{n-1} \frac{(t-a)^k}{k!} c_k \oplus I_{RL}^{\beta} \varpi(\xi, v(\xi)) \oplus I_{RL}^{\beta} g(\xi),$$

and

$$d\big[g(\xi), \widehat{0}\big] \leq \varphi(\xi) \text{ for any } \xi \in [a,b].$$

If $\omega \in \mathcal{M}_1$ represents a solution to 4.2, then we have:

$$d[v,\omega] = \sup_{a\leq\xi\leq b} d_{\infty}[v(\xi), \omega(\xi)] = \sup_{a\leq\xi\leq b} d_{\infty}\Bigg[c_0 \oplus \sum_{k=1}^{n-1} \frac{(\xi-a)^k}{k!} c_k \oplus I_{RL}^{\beta} \varpi(\xi, v(\xi)) \oplus I_{RL}^{\beta} g(\xi), c_0$$

$$\oplus \sum_{k=1}^{n-1} \frac{(\xi-a)^k}{k!} c_k \oplus I_{RL}^{\beta} \varpi(\xi, \omega(\xi))\Bigg]$$

$$\leq \sup_{a\leq\xi\leq b} \frac{1}{\Gamma(\beta)} \int_a^{\xi} (\xi-s)^{\beta-1} d_{\infty}\big[\varpi\left(s, v(s)\right), \varpi(s, \omega(s))\big] ds + \frac{1}{\Gamma(\beta)} \int_a^{\xi} (\xi-s)^{\beta-1} \sup_{a\leq\xi\leq b} d_{\infty}\big[g(s)$$

$$\leq L \sup_{a\leq\xi\leq b} \frac{1}{\Gamma(\beta)} \int_a^{\xi} (\xi-s)^{\beta-1} d\big[v(s), \omega(s)\big] ds + \frac{1}{\Gamma(\beta)} \int_a^{t} (\xi-s)^{\beta-1} \varphi(s) ds$$

$$\leq L \sup_{a\leq\xi\leq b} \frac{1}{\Gamma(\beta)} \int_a^{\xi} (\xi-s)^{\beta-1} d[v(s), \omega(s)] ds + \mathcal{W}\varphi(\xi)$$

$$\leq \frac{L(b-a)^{\beta}}{\Gamma(\beta+1)} d[v,\omega] + \mathcal{W}\varphi(\xi),$$

for any $\xi \in [a,b]$. Then

$$d[v,\omega] \leq \frac{\mathcal{W}\Gamma(\beta+1)}{\Gamma(\beta+1)-L(b-a)^{\beta}}\varphi(\xi)$$

we put $K_{\varpi} = \frac{\mathcal{W}(b-a)^{\beta}}{\Gamma(\beta+1)-L(b-a)^{\beta}}$, we have

$$d[v,\omega] \leq K_{\varpi}\varphi(\xi).$$

Hence, it is concluded that problem 4.2 exhibits Ulam Hyers Rassias stability. ■

4.4 Examples

Several examples are provided in this part to demonstrate the practicality of our findings.

Example 4.14 *Assume that*

$$\begin{cases} \mathcal{D}_c^{3.33}\omega(\xi) = \varpi(\xi,\omega(\xi)), & \xi \in [0,1], \\ \omega_{gH}^{(k)}(0) = c_k, & 0 \leqslant k < 3, \end{cases} \tag{4.21}$$

Suppose that $\beta = 3.33 \in (3,4)$, *and*

$$\varpi(\xi,\omega(\xi)) = \begin{cases} \frac{1}{100\sqrt{\pi}}\omega(\xi) + (0,1,2)\xi, & \text{if} \quad \omega(\xi) \in \mathcal{M}_1, \\ 0, & \text{if} \quad \omega(\xi) \notin \mathcal{M}_1. \end{cases} \tag{4.22}$$

1- For $\omega(\xi) \in \mathcal{M}_1$ *we have* $\mathcal{I}^{\beta}\varpi(\xi,\omega(\xi)) \in \mathcal{M}_1$.

2- The function ϖ *fulfills the second condition (H2).*

$$\begin{aligned} d_{\infty}(\varpi(\xi,\omega_1(\xi)), f(t,\omega_2(\xi))) &= d_{\infty}\left(\frac{1}{100\sqrt{\pi}}\omega_1(\xi) + (0,1,2)\xi, \frac{1}{100\sqrt{\pi}}\omega_2(\xi) + (0,1,2)\xi\right) \\ &\leq d_{\infty}\left(\frac{1}{100\sqrt{\pi}}\omega_1(\xi), \frac{1}{100\sqrt{\pi}}\omega_2(\xi)\right) \\ &\leq \frac{1}{100\sqrt{\pi}}d_{\infty}(\omega_1(\xi),\omega_2(\xi)). \end{aligned}$$

This shows that the Lipschitz constant L for the function $\varpi(\xi, \omega(\xi)$ *is*

$$L = \frac{1}{100\sqrt{\pi}}.$$

3- A positive number, γ*, exists such that*

$$\gamma = \frac{L(b-a)^{\beta}}{\Gamma(\beta+1)} < 1.$$

From Theorem 4.6, it follows that the problem 4.21 possesses two unique solution.

We also have

$$\Gamma(\beta+1) > L(b-a)^{\beta}.$$

The previously discussed problem 4.21 exhibits both Ulam-Hyers stability and generalized Ulam-Hyers stability, based on the criteria provided in 4.12.

Example 4.15 *Suppose that* $n-1 < \beta < n$, c_k $(k = 0, ..., n-1)$ *is constant and*

$$\varpi(\xi, \omega(\xi)) = \big(-\exp(\lambda\xi), \exp(\lambda\xi)\big),$$

where $\lambda > 0$

$$\begin{cases} \mathcal{D}_c^{\beta}\omega(\xi) = \left(-\exp(\lambda\xi), \exp(\lambda\xi)\right), & \xi \in [0,1], \\ \omega_{gH}^{(k)}(0) = c_k, \quad 0 \leqslant k < n-1. \end{cases} \tag{4.23}$$

We have $\frac{\partial^n}{\partial t} I^{\beta} \exp(\lambda\xi) > 0$ *for all* $n \geq 0$*, then*

$$I^{\beta}\varpi(\xi, \omega(\xi)) \in \mathcal{M}_1.$$

The function ϖ *fulfills the second condition (H2) and* $\gamma = 0 < 1$.
According Theorem 4.6 our problem (4.23) admits two solutions one of them from $\mathcal{M}_1$ *and another solution from* $\mathcal{M}_2$.

Conclusion

The fuzzy theory, which gained considerable traction in the 1990s, has solidified its position as an indispensable tool across diverse scientific disciplines. Formulated by Professor Lotfi Zadeh, this theory's inherent appeal lies in its intuitive reasoning, effectively addressing subjectivity and vagueness without sacrificing mathematical rigor. Far from being a vague concept, fuzzy set theory is a precise mathematical framework adept at handling inherent uncertainty and imprecision in complex systems. This rigor has fueled significant contributions from researchers worldwide in fields spanning medical diagnosis, robotics, and fundamental mathematics. Indeed, fuzzy logic is now recognized as a foundational tool, with current research actively exploring its powerful combinations with genetic algorithms and neural systems. Our thesis stands as a testament to the enduring relevance and expanding utility of fuzzy theory. By integrating fuzzy logic with fractional calculus, we've developed and analyzed fuzzy fractional differential equations (FFDEs), offering a robust framework for modeling systems characterized by both uncertainty and memory effects. Our work demonstrates the precision with which FFDEs can be solved and their stability properties established, further solidifying the mathematical rigor of fuzzy theory in addressing real-world challenges. This research contributes directly to the ongoing advancements in harnessing fuzzy logic as a cornerstone for advanced modeling and problem-solving.

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