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بسم الله الرحمن الرحيم

وقل رب زدني علماً

Dedication

To my beloved parents, Mohamed and Lamia, for their unconditional love, endless sacrifices, and unwavering belief in me.

To my dear husband, Mohamed Oussama, and to my beloved son, Rasim, for his constant support, patience, and encouragement throughout this journey.

To my wonderful siblings, Rihab, Lina, and Iyed, for their understanding and for always being my source of joy.

To my cherished friends, Souad, Oumaima, and Aya, for their motivation, companionship, and for always being there for me.

This work is dedicated to you all,
With all my love and gratitude.

✧✧✧ *Abir Mechaouf…* ✧✧✧

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Abir Mechaouf, September 2025

On the Optimal Control Problem Governed by Viscoelastic Marguerre–von Kármán Equations

A Theoretical study of Shallow Shells with Long- and Short-Memory Effects

Abstract

This thesis develops a rigorous mathematical framework for optimal control of viscoelastic shallow shells with geometric imperfections, governed by the Marguerre–von Kármán equations with memory effects. We establish the well-posedness of the state system and derive first-order optimality conditions using Gâteaux and Fréchet differentiability.

Keywords : *PDEs; non-linear elasticity; von Kármán equations; Marguerre-von Kármán equations; viscoelasticity; existence of solutions; optimal control; finite element methods.*

حول مشكلة التحكم الأمثل المحكوم بمعادلات مارغير-فون كارمان المرونة اللزجة

دراسة نظرية للهياكل الضحلة ذات تأثيرات الذاكرة الطويلة والقصيرة

ملـــخـــــص

تطور هذه الأطروحة إطاراً رياضياً متيناً للتحكم الأمثل للهياكل الضحلة المرنة اللزجة ذات العيوب الهندسية، المحكومة بمعادلات مارغير-فون كارمان بتأثيرات الذاكرة. نؤسس صلاحية نظام الحالة ونشتق شروط الأمثلة من الدرجة الأولى باستخدام قابلية الاشتقاق لغاتو وفريشيه.

الكلمات المفتاحية : المعادلات التفاضلية الجزئية؛ المرونة اللاخطية؛ معادلات فون كارمان؛ معادلات مارغير-فون كارمان؛ المرونة اللزجة؛ وجود الحلول؛ التحكم الأمثل؛ طرق العناصر المحددة.

Sur le Problème de Contrôle Optimal Gouverné par les Équations Viscoélastiques de Marguerre–von Kármán

Une Étude Théorique des Coques Peu Profondes avec Effets de Mémoire Longue et Courte

Résumé

Cette thèse développe un cadre mathématique rigoureux pour le contrôle optimal des coques viscoélastiques peu profondes présentant des imperfections géométriques, gouvernées par les équations de Marguerre–von Kármán avec effets de mémoire. Nous établissons la bien posé du système d'état et dérivons des conditions d'optimalité du premier ordre en utilisant les différentiabilités de Gâteaux et Fréchet.

Mots Clés : EDP ; élasticité non-linéaire ; équations de von Kármán ; équations de Marguerre-von Kármán ; viscoélasticité ; existence de solutions ; contrôle optimal ; méthodes des éléments finis.

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General Introduction

Historical Background

The study of thin plates and shallow shells represents a cornerstone in solid mechanics and structural analysis. The mathematical foundation of plate theory can be traced back to the 18th century with Euler's work on elastic curves . Still, it was Sophie Germain (1776–1831) who, in 1811, first proposed a differential equation for vibrating plates. Later, Gustav Kirchhoff (1824–1887) established the fundamental assumptions for thin plate theory in 1850, known as the Kirchhoff–Love hypotheses [13, 36], which form the basis for modern plate and shell theories.

The development of nonlinear plate and shell theories is strongly linked to the contributions of Theodore von Kármán (1881–1963) [56] and Hermann Marguerre (1880–1957) [11]. Von Kármán, a leading aerospace engineer and applied mathematician, derived in 1910 the celebrated **von Kármán equations** for thin elastic plates undergoing large deflections :

Figure 0.1: Theodore von Kármán (1881–1963)

$$\Delta^2 w - \frac{1}{E}[w, \phi] = q, \quad \Delta^2 \phi + \frac{E}{2}[w, w] = 0, \quad (1)$$

where w denotes the transverse displacement, ϕ the Airy stress function, q the applied transverse load, and $[\cdot, \cdot]$ is the von Kármán bracket operator defined as

$$[f, g] = f_{,xx}g_{,yy} + f_{,yy}g_{,xx} - 2f_{,xy}g_{,xy} \text{ [10, 13].}$$

Marguerre, a German engineer, extended von Kármán's work to shallow shells in the 1930s, providing a framework for analyzing curved thin structures [49]. The combination of their theories gives rise to the **Marguerre–von Kármán (Marguerre–von Kármán) equations**, which capture both geometric nonlinearities and shell curvature effects :

Figure 0.2: Stephen Timoshenko (1878–1972), refined plate theories including shear deformation.

$$\begin{cases} \Delta^2\zeta - [\zeta, \phi] + \text{viscoelastic terms} = f, \\ \Delta^2\phi + \frac{1}{2}[\zeta, \zeta] = 0, \end{cases} \tag{2}$$

Viscoelastic terms may be included to model material memory effects [4, 30, 31, 48].

In the mid-20th century, several researchers extended the Marguerre–von Kármán framework. Stephen Timoshenko (1878–1972) incorporated shear deformation effects [36], while Eric Reissner (1913–1996) developed refined theories accounting for transverse shear deformations [50]. The advent of computational mechanics enabled numerical solutions of these complex nonlinear equations using finite element methods [13, 35, 38, 39].

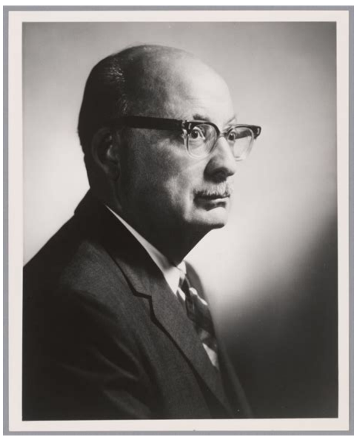

Figure 0.3: Eric Reissner (1913–1996), developed refined shell theories.

More recently, research has focused on **viscoelastic Marguerre–von Kármán shallow shells** with **memory effects** and their **optimal control**. Hwang [30–32] investigated control problems for von Kármán plates with extended memory, while Cavalcanti et al. [4] analyzed existence and decay rates for viscoelastic von Kármán systems. Chacha, Ghezal, and Bensayah [5, 20] extended these results to generalized Marguerre–von

Kármán shallow shells, providing well-posedness results and control characterizations. These modern contributions bridge the gap between **mathematical theory, numerical approximation, and engineering applications**, highlighting the relevance of Marguerre–von Kármán models in the design and control of advanced elastic structures.

Objectives and Contributions

The optimal control of nonlinear partial differential equations (PDEs) has become a pivotal area of research in applied mathematics and engineering, particularly for systems governed by viscoelastic Marguerre–von Kármán equations. Effective control of thin shell structures is essential to ensure structural integrity, suppress unwanted vibrations, and optimize performance under dynamic and uncertain loading conditions.

While the classical theory of von Kármán plates has been extensively studied [30], the extension to shallow shells, prevalent in modern engineering applications such as aircraft fuselages, architectural domes, and pressure vessels, remains largely unexplored. This thesis bridges this gap by developing a comprehensive theoretical framework for the mathematical analysis of viscoelastic shallow shell structures.

The primary objectives of this research are :

(1) **Mathematical Analysis :** To rigorously establish well-posedness results for the Marguerre–von Kármán system, including existence, uniqueness, and regularity of solutions for both short- and long-memory viscoelastic models [2, 13, 30, 49].

(2) **Optimal Control Formulation :** To define a precise mathematical framework for controlling Marguerre–von Kármán shallow shells, incorporating admissible control constraints, nonlinearities, and viscoelastic memory effects. The derivation of first-order necessary optimality conditions and the adjoint system are central to this objective [33, 36].

(3) **Analytical Investigations :** To examine the influence of geometric imperfections, boundary conditions, and material memory effects on the controlled system through rigorous mathematical analysis.

(4) **Theoretical Extensions :** To extend the classical von Kármán plate theory to the general geometry of Marguerre–von Kármán shallow shells, establishing a comprehensive mathematical foundation for future research in nonlinear viscoelastic structural control.

By achieving these objectives, this thesis provides a rigorous mathematical theory for optimal control problems in viscoelastic Marguerre–von Kármán shallow shells. The results address challenges that have persisted since the pioneering work of von Kármán and Marguerre, and laid a solid theoretical A foundation for future analytical studies on adaptive shell structures, with promising applications in aerospace, civil, and mechanical engineering.

Organization of the Thesis

The thesis is organized as follows :

- **Chapter 1 :** Preliminaries and theoretical framework, introducing the functional spaces, notation, and classical results on von Kármán and Marguerre–von Kármán equations.
- **Chapter 2 :** Analysis and optimal control of long-memory viscoelastic Marguerre–von Kármán shallow shells, including well-posedness results, optimal controls, and first-order necessary conditions.
- **Chapter 3 :** Short-memory Marguerre–von Kármán system : weak formulation, existence and uniqueness results, and optimal control problem formulation with associated adjoint system.
- **Chapter 4 :** Conclusion and perspectives, summarizing the main contributions and outlining potential directions for future research in controlling nonlinear viscoelastic shell structures.

Publications and Communications of the thesis

Internatioal Publications

- **A. Mechaouf,** . Ghezal, and R. Ghanem, Optimal control problems governed by marguerre–von kármán evolution equations with long memory. *Evol. Equ. Control Theory (EECT)* **14(4)**, (2025), 680–700.
http ://dx.doi.org/10.3934/eect.2025001.

International Communications

- The optimal control of a nonlinear model described by Marguerre von–Karman equations. 1^{st} international conference on nonlinear mathematical analysis and its applications (IC-NMAA'24), 14-15 May, 2024, Bordj Bou Arréridj-Algeria.

- On the optimal control problem governed by viscoelastic margurre von karman's equations. First international conference on applied mathematics and mathematical modeling $<< MAMM'2024 >>$, 12 november, 2024, El tarf-Algeria.

- Optimal Control of The Problems Evolution Equations. The 1st international conference Mohand Moussaoui on applied mathematics and modelling, 19-20 november, 2024, Guelma-Algeria.

- New results for the margurre von–karman equations with long memory. The international conference on mathematics and its applications in science and technology (ICMAST'2024), 15-16 december, 2024, Setif-Algeria.

National Communications

- The optimal control problems for a Marguerre–von Karman equations with long memory. National conference of applied sciences and engineering NCASE'24, 17-18 November, 2024 ENSTA-Algeria.

- The uniqueness of the optimal control for the quadratic cost. 1^{st} national conference on applied mathematics and artificial intelligence (NCAMAI 2025) : fundamentals and modern applications, 4-5 May, 2025, Skikda-Algeria.

Chapter 1

Preliminary

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This chapter establishes the mathematical foundation required for the analysis of viscoelastic Marguerre–von Kármán equations. We introduce the functional framework, Sobolev spaces, and key mathematical tools that will underpin the subsequent development of our optimal control theory. The results presented here provide the rigorous basis for the existence analysis, optimality conditions, and numerical simulations that follow in later chapters.

1.1 Sobolev Spaces and Functional Framework

1.1.1 Basic Sobolev Spaces and Notations

Lebesgue Spaces

Let $\Omega \subset R^2$ bounded domain with lipschitz boundary $\partial\Omega$, for $1 \leq p < \infty$, the space $L^p(\Omega)$ consists of all measurable functions $u : \Omega \to R$ such that

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p dx \right)^{1/p} < \infty.$$

For $p = \infty$, $L^\infty(\Omega)$ consists of all essentially bounded measurable functions with

$$\|u\|_{L^\infty(\Omega)} = \sup_{x\in\Omega} |u(x)| < \infty.$$

Sobolev Spaces

For $m \in N$ and $1 \leq p \leq \infty$, the Sobolev space $W^{m,p}(\Omega)$ consists of all functions $u \in L^p(\Omega)$ whose weak derivatives $D^\alpha u$ exist and belong to $L^p(\Omega)$ for all multi-indices α with $|\alpha| \leq m$. The norm is defined as :

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha|\leq m} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{1/p} \qquad \text{for } 1 \leq p < \infty,$$

and

$$\|u\|_{W^{m,\infty}(\Omega)} = \max_{|\alpha|\leq m} \|D^\alpha u\|_{L^\infty(\Omega)}.$$

Hilbert-Sobolev Spaces

When $p = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, which is a Hilbert space with the inner product :

$$(u, v)_{H^m(\Omega)} = \sum_{|\alpha|\leq m} \int_\Omega D^\alpha u(x) D^\alpha v(x) dx.$$

Sobolev Spaces with Boundary Conditions

The space $H_0^m(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $H^m(\Omega)$. For $m \geq 1$, this consists of functions satisfying homogeneous boundary conditions :

$$H_0^m(\Omega) = \{u \in H^m(\Omega) : u = \partial_\nu u = \cdots = \partial_\nu^{m-1} u = 0 \text{ on } \partial\Omega\}.$$

Dual Spaces

For $m \geq 1$, the dual space of $H_0^m(\Omega)$ is denoted by $H^{-m}(\Omega)$. Specifically :

- $H^{-1}(\Omega) = (H_0^1(\Omega))'$
- $H^{-2}(\Omega) = (H_0^2(\Omega))'$

1.1.2 Duality Pairings and Convergence Concepts

Definition 1.1 (Duality Pairing)**.** If Y is a Banach space and Y' its dual, the duality pairing $\langle\cdot,\cdot\rangle_{Y',Y}$ is defined as :

$$\langle f, u\rangle_{Y',Y} = f(u) \quad \text{for } f \in Y', u \in Y.$$

Definition 1.2 (Modes of Convergence)**.**

- **Strong convergence** : $u_n \to u$ in Y if $\|u_n - u\|_Y \to 0$
- **Weak convergence** : $u_n \rightharpoonup u$ in Y if $\langle f, u_n\rangle_{Y',Y} \to \langle f, u\rangle_{Y',Y}$ for all $f \in Y'$
- **Weak-* convergence** : $f_n \rightharpoonup^* f$ in Y' if $\langle f_n, u\rangle_{Y',Y} \to \langle f, u\rangle_{Y',Y}$ for all $u \in Y$

1.1.3 Green Operator and Biharmonic Theory

Definition 1.3 (Biharmonic Operator)**.** Let $u \in H^4(\Omega)$, the biharmonic operator Δ^2 is defined as :

$$\Delta^2 u = \frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4}.$$

Definition 1.4 (Green Operator)**.** Let $f \in L^2(\Omega)$ and $u \in H_0^2(\Omega)$. The Green operator $G = (\Delta^2)^{-1}$ associated with the Dirichlet problem for the biharmonic operator is defined as the solution operator for :

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases}$$

That is, $u = Gf$ is the unique solution of the above boundary value problem.

Theorem 1.5. [8, Remark 1.3.7][Regularity of Green Operator $G \equiv (\Delta^2)^{-1}$] For any integer $s \geq 2$, the biharmonic operator with Dirichlet boundary conditions is an isomorphism :

$$\Delta^2 : H^s(\Omega) \cap H_0^2(\Omega) \to H^{s-4}(\Omega).$$

Consequently, for $s \geq -2$, the Green operator satisfies :

$$G \in \mathcal{L}(H^s(\Omega), H^{s+4}(\Omega) \cap H_0^2(\Omega)).$$

1.1.4 Inner Products and Induced Norms

The scalar product on $L^2(\Omega)$ is denoted by $(\cdot, \cdot)_2$, which induces natural inner products on the subspaces $H_0^l(\Omega)$ for $l = 1, 2$:

Definition 1.6 (Inner Products on H_0^1 and H_0^2)**.** For all $u, v \in H_0^1(\Omega)$, we define :

$$((u, v))_{H_0^1} = (\nabla u, \nabla v)_2.$$

For all $u, v \in H_0^2(\Omega)$, we define :

$$((u, v))_{H_0^2} = (\Delta u, \Delta v)_2.$$

These inner products induce the following equivalent norms :

Proposition 1.7 (Equivalent Norms). The following norms are equivalent to the standard Sobolev norms :

$$\|u\|_{H_0^1} = \|\nabla u\|_{L^2}, \quad \forall u \in H_0^1(\Omega),$$
$$\|u\|_{H_0^2} = \|\Delta u\|_{L^2}, \quad \forall u \in H_0^2(\Omega).$$

Proof. The equivalence follows from Poincaré's inequality and elliptic regularity theory. For $u \in H_0^1(\Omega)$, Poincaré's inequality gives :

$$\|u\|_{L^2} \leq C\|\nabla u\|_{L^2},$$

which implies that $\|\nabla u\|_{L^2}$ is indeed a norm on $H_0^1(\Omega)$ equivalent to the standard H^1-norm. Similarly, for $u \in H_0^2(\Omega)$, elliptic regularity yields the equivalence with the standard H^2-norm. □

Theorem 1.8 (Sobolev Embedding Theorem). Let $\Omega \subset R^2$ be a bounded domain with Lipschitz boundary.

(1) If $mp > 2$, then $W^{m,p}(\Omega) \hookrightarrow C(\overline{\Omega})$

(2) If $mp = 2$, then $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $1 \leq q < \infty$

(3) If $mp < 2$, then $W^{m,p}(\Omega) \hookrightarrow L^{q^*}(\Omega)$ with $q^* = \frac{2p}{2-mp}$

Corollary 1.9. For the specific spaces used in this work :

- $H^1(\Omega) \hookrightarrow L^p(\Omega)$ for all $1 \leq p < \infty$
- $H^2(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega})$ for $0 < \alpha < 1$
- $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ continuously

Throughout this work, we employ the following simplified notation :

$$L^p \equiv L^p(\Omega), \quad W^{k,p} \equiv W^{k,p}(\Omega), \quad H^m \equiv W^{m,2}(\Omega),$$
$$\|.\|_p \equiv \|.\|_{L^p}, \quad \|.\| \equiv \|.\|_{L^2},$$
$$\langle .,. \rangle_{-2,2} \equiv \langle .,. \rangle_{H^{-2},H_0^2}, \quad \langle .,. \rangle_{2,-2} \equiv \langle .,. \rangle_{H_0^2,H^{-2}}.$$

The letter C will denote a generic positive constant that may vary from line to line.

1.2 Von Kármán and Marguerre–von Kármán Equations

1.2.1 Classical Von Kármán Equations

The von Kármán equations for a thin elastic plate subjected to transverse loads consist of the following coupled system :

$$\Delta^2\zeta = [\zeta, \phi] + f \quad \text{in } \Omega, \tag{1.1}$$

$$\Delta^2\phi = -\frac{1}{2}[\zeta, \zeta] \quad \text{in } \Omega, \tag{1.2}$$

with clamped boundary conditions :

$$\zeta = \partial_\nu\zeta = \phi = \partial_\nu\phi = 0 \quad \text{on } \partial\Omega,$$

where ζ represents the transverse displacement and ϕ is the Airy stress function [9, 10, 56].

Definition 1.10 (Von Kármán Bracket)**.** The von Kármán bracket $[\cdot, \cdot]$ is a nonlinear and symmetric operator defined for sufficiently smooth functions u, v by :

$$[u, v] = \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - 2\frac{\partial^2 u}{\partial x \partial y}\frac{\partial^2 v}{\partial x \partial y}.$$

This can be equivalently expressed as :

$$[u, v] = \det(D^2(u, v)) = u_{xx}v_{yy} + u_{yy}v_{xx} - 2u_{xy}v_{xy}.$$

Geometrically, the bracket represents the Gaussian curvature of the surface defined by the displacement field.

The first equation describes the equilibrium of forces, while the second represents the compatibility condition for the stresses.

Theorem 1.11 (Weak Formulation). The weak formulation of the von Kármán equations seeks $(u, v) \in H_0^2(\Omega) \times H_0^2(\Omega)$ such that for all $(\varphi, \psi) \in H_0^2(\Omega) \times$

$H_0^2(\Omega)$:

$$(\Delta u, \Delta\varphi)_2 = ([u, v], \varphi)_2 + (f, \varphi)_2, \tag{1.3}$$

$$(\Delta v, \Delta\psi)_2 = -\frac{1}{2}([u, u], \psi)_2. \tag{1.4}$$

This formulation is essential for the finite element approximation and existence analysis.

Remark (Physical Interpretation). The von Kármán equations can be derived from the following physical principles [9, 10] :

(1) The strain-displacement relationships for moderately large deformations

(2) The stress-strain constitutive relations for linear elasticity

(3) The equilibrium equations for forces and moments

(4) The compatibility conditions for strains

The nonlinearity arises from the geometric nonlinearities in the strain-displacement relations.

1.2.2 Marguerre–von Kármán with Geometric Imperfections

The Marguerre–von Kármán equations generalize the classical model to account for initial geometric imperfections $\Theta \in H_0^2(\Omega)$:

$$\Delta^2\zeta = [\zeta + \Theta, \phi] + f \quad \text{in } \Omega, \tag{1.5}$$

$$\Delta^2\phi = -\frac{1}{2}[\zeta, \zeta + 2\Theta] \quad \text{in } \Omega, \tag{1.6}$$

with boundary conditions :

$$\zeta = \partial_\nu\zeta = \phi = \partial_\nu\phi = 0 \quad \text{on } \partial\Omega.$$

The function Θ represents the initial deviation from a flat configuration [12, 23, 42, 49].

Definition 1.12 (Weak Formulation with Imperfections)**.** The weak formulation of the Marguerre–von Kármán equations seeks $(u, v) \in H_0^2(\Omega) \times H_0^2(\Omega)$ such that for all test functions $(\varphi, \psi) \in H_0^2(\Omega) \times H_0^2(\Omega)$:

$$(\Delta\zeta, \Delta\varphi)_2 = ([\zeta + \Theta, \phi], \varphi)_2 + (f, \varphi)_2, \tag{1.7}$$

$$(\Delta\phi, \Delta\psi)_2 = -\frac{1}{2}([\zeta, \zeta + 2\Theta], \psi)_2. \tag{1.8}$$

The presence of Θ introduces additional coupling terms that significantly affect the mathematical analysis.

Remark (Effect of Geometric Imperfections). The geometric imperfection Θ influences the system in several key ways [12, 23] :

(1) Reduces the critical buckling load of the structure

(2) Alters the post-buckling behavior and stability characteristics

(3) Introduces additional nonlinear coupling between in-plane and out-of-plane deformations

(4) Affects the sensitivity of the structure to external loads

The magnitude and spatial distribution of Θ determine the severity of these effects.

Remark (Special Cases). The Marguerre–von Kármán equations reduce to :

- Classical von Kármán equations when $\Theta = 0$
- Linear plate theory when both nonlinear terms and Θ are neglected
- Shallow shell equations for specific forms of Θ representing initial curvature

This generality makes the model suitable for analyzing various types of imperfect structures.

1.2.3 Mathematical Properties of the von Kármán Bracket

Lemma 1.13 (Continuity of Trilinear Form). The trilinear form $T : H^2(\Omega) \times H^2(\Omega) \times H^2(\Omega) \to R$ defined by :

$$T(u, v, w) = ([u, v], w)_2$$

is continuous. Moreover, T is symmetric if at least one of its three arguments belongs to H_0^2 or all arguments belong to $H^2 \cap H_0^1$.

Proof. See [8, Proposition 1.4.2] □

Lemma 1.14 (Bracket Estimates). The von Kármán bracket satisfies the following estimates :

(1) For $u, v \in H_0^2(\Omega)$ and any $\epsilon > 0$:

$$\|[u, v]\|_{H^{-1-\epsilon}} \leq C\|u\|_{H_0^2}\|v\|_{H_0^2}.$$

(2) For $u, v \in H^2(\Omega)$:

$$\|[u, v]\|_{H^{-1}} \leq C\|u\|_{H^2}\|v\|_{H^2}.$$

(3) For $u \in H^1(\Omega), v \in H^2(\Omega)$:

$$\|[u, v]\|_{H^{-2}} \leq C\|u\|_{H^1}\|v\|_{H^2}.$$

Proof. See [4, Lemma 3.2] for proof of part (i) and [8, Theorem 1.4.3] for proof of part (ii). □

Lemma 1.15 (Regularity of Bracket Compositions). The bilinear forms $(u, v) \mapsto G[u, v]$ from $H^2 \times H^2$ into $W^{2,\infty}$ and $(u, v) \mapsto [u, v]$ from $H^2 \times H^2$ into H^{-2} are continuous. We also have the following estimates :

$$\|G[u, v]\|_{W^{2,\infty}} \leq C\|u\|_{H^2}\|v\|_{H^2}, \quad u, v \in H^2, \tag{1.9}$$

$$\|[u, v]\|_{H^{-2}} \leq C\|u\|_{H^1}\|v\|_{H^2}, \quad u \in H^1, v \in H^2. \tag{1.10}$$

Consequently,

$$\|[w, G[u, v]]\| \le C\|w\|_{H^2}\|u\|_{H^2}\|v\|_{H^2}, \quad w, u, v \in H^2. \tag{1.11}$$

Proof. See [7, Lemma 1.2]. □

Another important lemma is the following.

Lemma 1.16. Let X, Y be two Banach spaces, $X \subset Y$ densely, and X be reflexive. Set

$$C_s\big([0,T];Y\big) := \Big\{u \in L^\infty(0,T;Y) | \,\forall\, v \in Y', t \mapsto \big\langle u(t), v\big\rangle_{Y,Y'} \quad \text{is continuous on } [0,T]\Big\}.$$

Then

$$L^\infty(0,T,X) \cap C_s\big([0,T],Y\big) = C_s\big([0,T],X\big). \tag{1.12}$$

Proof. See [40, Lemma 8.1]. □

Remark. The Marguerre–von Kármán equations reduce to the classical von Kármán equations when $\Theta = 0$. The presence of geometric imperfections significantly affects the buckling behavior and load-carrying capacity of thin shell structures [5, 19, 20].

1.3 Viscoelasticity with Memory Effects

1.3.1 Long-term and Short-term Memory Formulations

Viscoelastic materials exhibit time-dependent mechanical behavior that combines elastic solid and viscous fluid characteristics. The memory effects capture both short-term and long-term responses [46, 48, 57].

Definition 1.17 (Viscoelastic Constitutive Law)**.** The stress-strain relationship for a linear viscoelastic material is given by [43, 51] :

$$\sigma(t) = G_0\epsilon(t) + \int_0^t G(t-s)\dot{\epsilon}(s)ds,$$

where :

- $\sigma(t)$: Cauchy stress tensor
- $\epsilon(t)$: infinitesimal strain tensor
- G_0 : instantaneous elastic modulus
- $G(t)$: relaxation function characterizing memory effects

Alternative forms include the hereditary integral formulation [26] :

$$\sigma(t) = \int_{-\infty}^{t} G(t-s)\frac{d\epsilon(s)}{ds}ds.$$

Theorem 1.18 (Fading Memory Principle). If the relaxation function $G(t)$ satisfies :

(1) $G(t) \in L^1(0,\infty)$

(2) $G(t)$ is completely monotone

(3) $G(0+) < \infty$

then the viscoelastic system exhibits fading memory and is well-posed [15, 24, 47].

Proof. The well-posedness of Volterra integro-differential equations arising from linear viscoelasticity, which underpins the fading memory property, is a classical result. For a comprehensive analysis, including the existence, uniqueness, and stability of solutions, we refer the reader to the foundational texts on Volterra equations, such as the monographs by [15, 24], and the work of [47] on stability.

Definition 1.19 (Relaxation Function Classes)**.** The relaxation function $G(t)$ belongs to one of several mathematical classes [17, 37] :

(1) **Short-term memory** : $G(t) = G_\infty + (G_0 - G_\infty)e^{-t/\tau}$ (Maxwell model)

(2) **Long-term memory** : $G(t) = G_\infty + G_1 t^{-\alpha}$ (Power-law decay)

(3) **Multiple relaxation** : $G(t) = G_\infty + \sum_{k=1}^{N} G_k e^{-t/\tau_k}$ (Generalized Maxwell)

1.3.2 Viscoelastic Constitutive Relations

The viscoelastic response can be modeled through several equivalent formulations :

Definition 1.20 (Differential Formulation)**.** For a linear viscoelastic material, the stress-strain relation can be expressed as :

$$P(D)\sigma = Q(D)\epsilon,$$

where $P(D)$ and $Q(D)$ are linear differential operators with constant coefficients. This leads to classical models [46] :

$$\begin{aligned} \text{Maxwell :} \quad & \dot{\sigma} + \frac{\sigma}{\tau} = E\dot{\epsilon} \\ \text{Kelvin-Voigt :} \quad & \sigma = E\epsilon + \eta\dot{\epsilon} \\ \text{Standard Linear :} \quad & \sigma + \tau_\sigma\dot{\sigma} = E(\epsilon + \tau_\epsilon\dot{\epsilon}) \end{aligned}$$

Definition 1.21 (Integral Formulation)**.** The hereditary integral approach provides a more general framework [46] :

$$\sigma(t) = G[\epsilon](t) = G_0\epsilon(t) + \int_0^t G(t-s)\epsilon(s)ds,$$

where G is a Volterra integral operator.

Proposition 1.22 (Energy Dissipation). The viscoelastic system satisfies the energy inequality [43, 46] :

$$\frac{d}{dt}E(t) \leq -D(t) \leq 0,$$

where $E(t)$ is the total energy and $D(t)$ is the dissipation rate.

Proof. Using the second law of thermodynamics and the positivity of the relaxation function [46]. □

1.3.3 Mathematical Modeling of Memory Effects

Definition 1.23 (Viscoelastic Marguerre–von Kármán Equations)**.** The viscoelastic generalization incorporates memory effects through [4, 7, 8] :

$$\Delta^2\zeta + \mathcal{D}(\zeta_t) = [\zeta + \Theta, \phi] + f \quad \text{in } \Omega \times (0, T), \tag{1.13}$$

$$\Delta^2\phi = -\frac{1}{2}[\zeta, \zeta + 2\Theta] \quad \text{in } \Omega \times (0, T), \tag{1.14}$$

where $\mathcal{D}$ represents the viscoelastic damping operator.

Theorem 1.24 (Damping Operator Forms). The damping operator $\mathcal{D}$ can take several mathematically distinct forms [15, 26] :

(1) **Short-memory damping** : $\mathcal{D}(u_t) = \gamma u_t$ (viscous damping)

(2) **Long-memory damping** : $\mathcal{D}(u_t) = \int_0^t G(t-s)u_t(s)ds$ (hereditary damping)

(3) **Fractional damping** : $\mathcal{D}(u_t) = \partial_t^\alpha u$ (fractional derivative)

(4) **Kelvin-Voigt damping** : $\mathcal{D}(u_t) = -\eta\Delta^2 u_t$ (structural damping)

Theorem 1.25 (Existence and Uniqueness). Under appropriate conditions on $G(t)$, the viscoelastic von Kármán system admits a unique weak solution [24,47].

Proof. The proof uses Galerkin approximations and energy estimates ; see, e.g., [24, 47] for similar techniques. □

1.3.4 Well-posedness Analysis

The viscoelastic system exhibits several key analytical features [52] :

Lemma 1.26 (A Priori Estimates). The solutions satisfy the energy estimates [40] :

$$\|\zeta(t)\|_{H^2} + \|\phi(t)\|_{H^2} + \int_0^t \|\zeta_t(s)\|^2 ds \leq C(T).$$

Proof. The energy estimates are obtained by multiplying the equations by appropriate test functions, integrating by parts, and applying standard energy methods for viscoelastic evolution equations ; see, for example, Chapters 2–3 in [8] and the treatment of hereditary kernels in [24]. □

Theorem 1.27 (Long-time Behavior). As $t \to \infty$, solutions converge to equilibrium states [8, 21, 27].

Proof. Using Lyapunov functionals and LaSalle's invariance principle [8, Chapter 6]. □

1.4 Optimal Control Framework

1.4.1 Problem Formulation and Control Objectives

Definition 1.28 (Control Problem Setup)**.** [35] Let $\mathscr{U}_{ad}$ be the set of admissible controls, which is a closed convex subset of a Hilbert space U. For a given control $q \in \mathscr{U}_{ad}$, we consider the controlled viscoelastic Marguerre–von Kármán system :

$$\Delta^2\zeta + \mathcal{D}(\zeta_t) = [\zeta + \Theta, \phi] + f + Bq \quad \text{in } \Omega \times (0, T), \tag{1.15}$$

$$\Delta^2\phi = -\frac{1}{2}[\zeta, \zeta + 2\Theta] \quad \text{in } \Omega \times (0, T), \tag{1.16}$$

with boundary conditions :

$$\zeta = \partial_\nu\zeta = \phi = \partial_\nu\phi = 0 \quad \text{on } \partial\Omega \times (0, T),$$

and initial conditions :

$$\zeta(0) = \zeta_0, \quad \zeta_t(0) = \zeta_1 \quad \text{in } \Omega.$$

Here, $B : U \to L^2(\Omega)$ is the control operator [3, 18].

Definition 1.29 (Cost Functional)**.** [29,] The optimal control problem aims to minimize the cost functional :

$$J(q, \zeta) = \frac{1}{2}\int_0^T \|\zeta(t) - \zeta_d(t)\|_{L^2}^2 dt + \frac{\alpha}{2}\int_0^T \|q(t)\|_U^2 dt + \frac{\beta}{2}\|\zeta(T) - \zeta_T\|_{L^2}^2,$$

where :

- $\zeta_d \in L^2(0,T;L^2(\Omega))$ is the desired state trajectory
- $\zeta_T \in L^2(\Omega)$ is the desired terminal state
- $\alpha > 0, \beta \geq 0$ are regularization parameters
- $q \in L^2(0,T;U)$ is the control variable

Optimal Control Problem [8, 35] Find $q^* \in \mathscr{U}_{ad}$ such that :

$$J(q^*, \zeta(q^*)) = \min_{q\in\mathscr{U}_{ad}} J(q, \zeta(q)),$$

where $\zeta(q)$ is the solution of the controlled system corresponding to control q. problem

1.4.2 Existence Theory for Optimal Controls

Theorem 1.30 (Existence of Optimal Controls). Assume the following conditions hold :

(1) $\mathscr{U}_{ad}$ is closed, convex, and bounded in $L^2(0,T;U)$

(2) The control operator $B : \mathscr{U} \longrightarrow L^2(\Omega)$ is linear and bounded

(3) The relaxation function G satisfies the fading memory conditions

(4) The initial data $(\zeta_0, \zeta_1) \in H_0^2(\Omega) \times L^2(\Omega)$

(5) The imperfection function $\Theta \in H_0^2(\Omega)$

Then there exists at least one optimal control $q^* \in \mathscr{U}_{ad}$ minimizing the cost functional J.

Proof. [3, 8] The proof follows the direct method of calculus of variations :

(1) Let $\{q_n\} \subset \mathscr{U}_{ad}$ be a minimizing sequence : $J(q_n) \longrightarrow \inf_{q\in\mathscr{U}_{ad}} J(q)$

(2) By the boundedness of $\mathscr{U}_{ad}$, extract a weakly convergent subsequence $q_n \rightharpoonup q^*$ in $L^2(0,T;U)$

(3) Show that the corresponding states ζ_n converge to ζ^* in an appropriate topology

(4) Prove that ζ^* is the state corresponding to q^* (solution operator is weakly continuous)

(5) Use the weak lower semicontinuity of J to conclude :

$$J(q^*) \leq \liminf_{n\to\infty} J(q_n) = \inf_{q\in\mathcal{U}_{ad}} J(q)$$

The technical details require careful analysis of the nonlinear terms and compactness arguments [25, 54]. □

Lemma 1.31 (Continuous Dependence on Controls). [8] The solution mapping $q \mapsto \zeta(q)$ from $L^2(0,T;U)$ to $C([0,T];H_0^2(\Omega)) \cap C^1([0,T];L^2(\Omega))$ is sequentially weakly continuous.

Proof. [8] Let $q_n \rightharpoonup q$ in $L^2(0,T;U)$. The corresponding states ζ_n satisfy uniform energy estimates :

$$\sup_{t\in[0,T]} \left(\|\zeta_n(t)\|_{H^2}^2 + \|\zeta_{n,t}(t)\|_{L^2}^2 \right) \leq C,$$

which imply the existence of a convergent subsequence. The limit is identified as the solution corresponding to q using monotonicity arguments for the nonlinear terms [8, 25]. □

1.4.3 Optimality Conditions and Adjoint Systems

Theorem 1.32 (First-Order Necessary Conditions). If $q^* \in \mathcal{U}_{ad}$ is an optimal control with corresponding state ζ^*, then there exists an adjoint state $p \in L^2(0,T;H_0^2(\Omega))$ such that the following optimality system holds :

State Equations :

$$\Delta^2\zeta^* + \mathcal{D}(\zeta_t^*) = [\zeta^* + \Theta, \phi^*] + f + Bq^* \tag{1.17}$$

$$\Delta^2\phi^* = -\frac{1}{2}[\zeta^*, \zeta^* + 2\Theta] \tag{1.18}$$

Adjoint Equations :

$$\Delta^2 p - \mathcal{D}^*(p_t) = [p, \phi^*] + [\zeta^* + \Theta, \psi] + (\zeta^* - \zeta_d) \tag{1.19}$$

$$\Delta^2 \psi = [\zeta^* + \Theta, p] - \frac{1}{2}[\zeta^*, p] \tag{1.20}$$

with terminal conditions :

$$p(T) = \beta(\zeta^*(T) - \zeta_T), \quad p_t(T) = 0,$$

and boundary conditions for p, ψ identical to the state equations.

Variational Inequality :

$$\int_0^T (\alpha q^* + B^* p, q - q^*)_U dt \geq 0 \quad \forall q \in \mathcal{U}_{ad}.$$

Proof. The proof uses the Lagrangian approach :

(1) Define the Lagrangian functional incorporating state constraints.

(2) Compute Fréchet derivatives with respect to state and control variables.

(3) Derive the adjoint equations through integration by parts.

(4) Obtain the variational inequality from the stationarity condition.

The main technical challenge is handling the memory term and nonlinear bracket terms in the differentiation process ; see, e.g., [6, 41].

Corollary 1.33 (Unconstrained Case). [29] If $\mathcal{U}_{ad} = L^2(0, T; U)$, then the optimal control satisfies the pointwise relation :

$$q^*(t) = -\frac{1}{\alpha} B^* p(t) \quad \text{a.e. in } (0, T).$$

Adjoint Operator Properties

The adjoint damping operator $\mathcal{D}^*$ is defined by [6] we have :

$$\langle \mathcal{D}(u_t), v \rangle = \langle u_t, \mathcal{D}^*(v) \rangle + \text{boundary terms},$$

and satisfies the following properties :

(1) If $\mathcal{D}(u_t) = \gamma u_t$, then $\mathcal{D}^*(v_t) = \gamma v_t$

(2) If $\mathcal{D}(u_t) = \int_0^t G(t-s)u_t(s)ds$, then $\mathcal{D}^*(v_t) = \int_t^T G(s-t)v_t(s)ds$

(3) $\mathcal{D}^*$ preserves the dissipation properties of $\mathcal{D}$

Remark (Computational Implications). [28, 29, 58] The optimality system provides the basis for numerical algorithms :

- Gradient-based optimization methods
- SQP (Sequential Quadratic Programming) approaches
- Primal-dual active set strategies for constrained problems

Recent advances in computational methods for viscoelastic systems can be found in [45, 58].

Chapter 2

Long-Term Effects in Viscoelastic Marguerre–von Kármán Shallow Shells : Analysis and Optimal Control

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This chapter is based on the following published work : [44], "Optimal control of Marguerre–von Kármán evolution equations" Journal name, vol., pages, year.

2.1 Introduction

The canonical Marguerre–von Kármán system for a shallow shell subject to von Kármán-type boundary conditions [22, 49] is given by

$$\begin{cases} \Delta^2\zeta = [\zeta+\Theta,\phi]+f & \text{in } \Omega, \\ \Delta^2\phi = -[\zeta,\zeta+2\Theta] & \text{in } \Omega, \\ \zeta = \partial_\nu\zeta = 0 & \text{on } \Gamma, \\ \phi = \phi_0,\ \partial_\nu\phi = \phi_1 & \text{on } \Gamma, \end{cases} \tag{2.1}$$

where ζ denotes the transverse displacement, ϕ the Airy stress function, Θ the geometry of the shell, and f the vertical load.

When the shell is viscoelastic with long memory, the system includes an additional convolution term :

$$\zeta_{tt} - \Delta\zeta_{tt} + \Delta^2\zeta + \hbar\star\Delta^2\zeta = [\zeta+\Theta,\phi]+f \quad \text{in } Q, \tag{2.2}$$

Let us consider the domain $Q =]0,T[\times\Omega$ and its boundary portion $\Sigma =]0,T[\times\Gamma$, for a given $T > 0$. We define the convolution operator associated with the memory kernel $\hbar \in C^1([0,T])$ as

$$\hbar\star\Delta^2\zeta(t) := \int_0^t \hbar(t-s)\Delta^2\zeta(s)\,ds, \tag{2.3}$$

To recast problem (2.2) into a more convenient form, we introduce the transformations

$$\beth = \zeta+\Theta, \quad \Phi = \phi-\chi,$$

where χ is uniquely determined as the solution of the biharmonic boundary value problem

$$\begin{cases} \Delta^2\eta = 0 & \text{in } \Omega, \\ \eta = \phi_0, \quad \partial_\nu\eta = \phi_1 & \text{on } \Gamma. \end{cases} \tag{2.4}$$

Applying this change of variables, the original system (2.2) is equivalently rewritten as

$$\begin{cases} \beth_{tt} - \Delta\beth_{tt} + \Delta^2\beth + \hbar \star \Delta^2\beth = [\beth, \Phi] + [\beth, \chi] + \hbar \star \Delta^2\Theta + \Delta^2\Theta + f & \text{in } Q, \\ \Delta^2\Phi = [\Theta, \Theta] - [\beth, \beth] & \text{in } Q, \\ \beth = \partial_\nu\beth = \Phi = \partial_\nu\Phi = 0 & \text{on } \Sigma, \\ \beth(0, \cdot) = \beth_0(\cdot), \quad \beth_t(0, \cdot) = \beth_1(\cdot) & \text{in } \Omega. \end{cases} \tag{2.5}$$

The first contribution of this chapter is to prove the well-posedness of problem (2.5), extending existing results for von Kármán plates with memory [8]. Regularity results are also established by adapting techniques from [30, 32, 40].

The second and main contribution concerns the control problem governed by

$$\begin{cases} \beth_{tt}(v) - \Delta\beth_{tt}(v) + \Delta^2\beth(v) + \hbar \star \Delta^2\beth(v) = [\beth(v), \Phi(v)] + [\beth(v), \chi] \\ +\mathscr{B}v + \hbar \star \Delta^2\Theta + \Delta^2\Theta, \\ \Delta^2\Phi(v) = [\Theta, \Theta] - [\beth(v), \beth(v)] \quad \text{in } Q, \end{cases} \tag{2.6}$$

where $v \in \mathcal{V}$ is the control input. We prove the existence and uniqueness of an optimal control minimizing a quadratic cost functional, show the strong Gâteaux differentiability of the solution operator, and derive necessary optimality conditions for a class of observations.

Finally, the chapter ends with concluding remarks linking these results to the subsequent analysis of the thesis.

2.2 Mathematical Analysis and Well-Posedness for long memory Marguerre–von Kármán Shallow Shells

This section develops the well-posedness theory for memory-driven Marguerre–von Kármán shallow shells, extending the foundational results of Chueshov and Lasiecka [8] to our framework. In particular, we adapt their existence and uniqueness results to accommodate the short-memory effects introduced by the convolution term $\hbar \star \Delta^2\Theta$.

Theorem 2.1. Let

$$(\daleth_0, \daleth_1, \Theta, \chi) \in H_0^2(\Omega) \times H_0^1(\Omega) \times (H_0^2(\Omega) \cap H^4(\Omega)) \times H^2(\Omega)$$

and a forcing term $f \in L^2(0,T; L^2(\Omega))$, problem (2.5) admits a unique weak solution

$$\daleth \in C([0,T], H_0^2(\Omega)) \cap C^1([0,T], H_0^1(\Omega)).$$

Proof. The result is obtained by applying [8, Theorem 3.3.2] to our setting. The correspondence is established by setting :

$$\Gamma_1 = \emptyset, \quad \alpha = 1, \quad f = 0, \quad F_0 = \chi, \quad L = 0, \quad p = \hbar \star \Delta^2\Theta + \Delta^2\Theta + f,$$

$$G = G_0, \quad g = -\hbar.$$

The stress function Φ is defined as

$$\Phi = G\big([\Theta, \Theta] - [\daleth, \daleth]\big),$$

This ensures the nonlinear bracket term correctly captures the shell's memory effects. □

We introduce the solution space $\mathcal{W}(0,T)$ for problem (2.5) :

$$\mathcal{W}(0,T) := \Big\{\daleth \mid \daleth \in L^2(0,T; H_0^2(\Omega)),\ \daleth' \in L^2(0,T; H_0^1(\Omega)),\ \daleth'' \in L^2(0,T; L^2(\Omega))\Big\},$$

equipped with the norm

$$\|\daleth\|_{\mathcal{W}(0,T)} := \Big(\|\daleth\|^2_{L^2(0,T;H_0^2(\Omega))} + \|\daleth'\|^2_{L^2(0,T;H_0^1(\Omega))} + \|\daleth''\|^2_{L^2(0,T;L^2(\Omega))}\Big)^{1/2}.$$

Definition 2.2. A function $\daleth \in \mathcal{W}(0,T)$ is called a *weak solution* of problem (2.5) if, for all test functions $\eta \in H_0^2(\Omega)$ in the sense of $\mathcal{D}'(0,T)$, the following variational formulation holds :

$$\begin{cases} \langle \beth'' - \Delta \beth'', \eta \rangle_{-2,2} + (\Delta \beth + \hbar \star \Delta \beth, \Delta \eta)_2 \\ \quad = ([\beth, \Phi] + [\beth, \chi] + \hbar \star \Delta^2 \Theta + \Delta^2 \Theta + f, \eta)_2, \\ (\Delta \Phi, \Delta \eta)_2 = ([\Theta, \Theta], \eta)_2 - ([\beth, \beth], \eta)_2, \\ \beth(0) = \beth_0, \quad \beth'(0) = \beth_1. \end{cases} \tag{2.7}$$

A key difficulty in the analysis arises from the limited regularity of weak solutions, as $\beth'$ does not generally belong to $H_0^2(\Omega)$. This can be addressed using the double regularization technique introduced in [40, Lemmas 8.2 and 8.3].

Lemma 2.3. Let $\beth$ be a weak solution of problem (2.5). Then, after possible modification on a set of measure zero,

$$\beth \in C_s([0,T], H_0^2(\Omega)), \quad \beth' \in C_s([0,T], H_0^1(\Omega)).$$

Proof. The proof follows from [30, Lemma 3.2], employing regularity theory and properties of the convolution operator. □

Weak solutions satisfy fundamental energy equality, essential for establishing well-posedness and further regularity results.

Proposition 2.4 (Energy Equality). Let $\beth$ be a weak solution of problem (2.5). Then, for all $t \in [0,T]$, the energy identity

$$\begin{aligned} &\|\beth'(t)\|^2 + \|\nabla \beth'(t)\|^2 + \|\Delta \beth(t)\|^2 + \frac{1}{2}\|\Delta \Phi(t)\|^2 \\ &= -2(\hbar \star \Delta \beth(t), \Delta \beth(t))_2 + 2\int_0^t (\hbar' \star \Delta \beth(s), \Delta \beth(s))_2 ds \\ &\quad + 2\int_0^t \hbar(0)\|\Delta \beth(s)\|^2 ds + 2\int_0^t ([\beth(s), \chi] + \hbar \star \Delta^2 \Theta(s) + \Delta^2 \Theta + f(s), \beth'(s))_2 ds \\ &\quad + \|\beth_1\|^2 + \|\nabla \beth_1\|^2 + \|\Delta \beth_0\|^2 + \frac{1}{2}\|\Delta \Phi_0\|^2, \end{aligned} \tag{2.8}$$

holds, where $\Delta \Phi_0 = \Delta^{-1}([\Theta, \Theta] - [\beth_0, \beth_0])$.

Proof. Lemma 2.3 ensures the continuity required for justifying all terms in (2.8). The calculation follows [30, Proposition 3.1], with careful handling of the convolution terms and nonlinear brackets. □

Finally, we summarize the main well-posedness result, including existence, uniqueness, regularity, and continuous dependence on data.

Theorem 2.5. Let the initial data satisfy

$$(\daleth_0, \daleth_1, \Theta, \chi) \in H_0^2(\Omega) \times H_0^1(\Omega) \times (H_0^2(\Omega) \cap H^4(\Omega)) \times H^2(\Omega),$$

the source term $f \in L^2(0,T;L^2(\Omega))$, and the memory kernel $\hbar \in C^1([0,T])$. Then the system (2.5) admits a unique weak solution

$$\daleth \in \mathcal{S}(0,T) := \mathcal{W}(0,T) \cap C([0,T];H_0^2(\Omega)) \cap C^1([0,T];H_0^1(\Omega)). \quad (2.9)$$

Furthermore, the solution operator

$$\begin{aligned}\wp := H_0^2(\Omega) \times H_0^1(\Omega) \times L^2(0,T;L^2(\Omega)) &\to C([0,T];H_0^2(\Omega)) \times C([0,T];H_0^1(\Omega)) \\ &\times C([0,T];W^{2,\infty}(\Omega)), \\ p = (\daleth_0, \daleth_1, f_\Theta) &\mapsto (\daleth(p), \daleth_t(p), \Phi(p)),\end{aligned} \quad (2.10)$$

is locally Lipschitz continuous, where

$$f_\Theta(t) := \hbar \star \Delta^2\Theta(t) + \Delta^2\Theta(t) + f(t).$$

Proof. By the assumptions on the initial data and using Lemma 2.3, the second member of (2.8) depends continuously on time. Consequently, the mapping

$$t \mapsto \|\nabla\daleth'(t)\|^2 + \|\Delta\daleth(t)\|^2$$

It is continuous, which ensures

$$\daleth \in C([0,T];H_0^2(\Omega)) \cap C^1([0,T];H_0^1(\Omega)).$$

Applying Theorem 2.1 along with [32, Theorem 2.1] guarantees the existence and uniqueness of the weak solution $\beth \in \mathcal{S}(0,T)$, satisfying

$$\|\beth\|_{\mathcal{S}(0,T)} \le C\|p\|_{\wp}, \quad \text{with } \|p\|_{\wp} := \left(\|\beth_0\|^2_{H_0^2} + \|\beth_1\|^2_{H_0^1} + \|f_\Theta\|^2_{L^2(0,T;L^2)}\right)^{1/2}.$$

To prove the local Lipschitz property of the solution map, consider two sets of data $p_1 = (\beth_0^1, \beth_1^1, f_\Theta^1)$ and $p_2 = (\beth_0^2, \beth_1^2, f_\Theta^2)$ in $\wp$, and denote the corresponding solutions by $(\beth^1, \Phi^1)$ and $(\beth^2, \Phi^2)$. Defining the differences

$$\widetilde{\beth} := \beth^1 - \beth^2, \quad \widetilde{\Phi} := \Phi^1 - \Phi^2,$$

we see that $(\widetilde{\beth}, \widetilde{\Phi})$ satisfies

$$\begin{cases} \widetilde{\beth}_{tt} - \Delta\widetilde{\beth}_{tt} + \Delta^2\widetilde{\beth} + \hbar \star \Delta^2\widetilde{\beth} = [\widetilde{\beth}, \Phi^1] + [\beth^2, \widetilde{\Phi}] + [\widetilde{\beth}, \chi] + f^1 - f^2, \\ \Delta^2\widetilde{\Phi} = -[\widetilde{\beth}, \beth^1 + \beth^2], \\ \widetilde{\beth} = \partial_\nu\widetilde{\beth} = \widetilde{\Phi} = \partial_\nu\widetilde{\Phi} = 0 \quad \text{on } \Sigma, \\ \widetilde{\beth}(0) = \beth_0^1 - \beth_0^2, \quad \widetilde{\beth}_t(0) = \beth_1^1 - \beth_1^2. \end{cases} \tag{2.11}$$

Using an energy argument analogous to Proposition 2.4, the weak solution $\widetilde{\beth}$ satisfies

$$\begin{aligned} &\|\widetilde{\beth}'(t)\|^2 + \|\nabla\widetilde{\beth}'(t)\|^2 + \|\Delta\widetilde{\beth}(t)\|^2 \\ &= -2(\hbar \star \Delta\widetilde{\beth}(t), \Delta\widetilde{\beth}(t))_2 + 2\int_0^t (\hbar' \star \Delta\widetilde{\beth}(s), \Delta\widetilde{\beth}(s))_2 ds + 2\int_0^t \hbar(0)\|\Delta\widetilde{\beth}(s)\|^2 ds \\ &\quad + 2\int_0^t \left([\widetilde{\beth}(s), \Phi^1(s)] + [\beth^2(s), \widetilde{\Phi}(s)] + [\widetilde{\beth}(s), \chi] + f^1(s) - f^2(s), \widetilde{\beth}'(s)\right)_2 ds \\ &\quad + \|\widetilde{\beth}'(0)\|^2 + \|\nabla\widetilde{\beth}'(0)\|^2 + \|\Delta\widetilde{\beth}(0)\|^2. \end{aligned}$$

Applying standard estimates for the convolution and nonlinear bracket terms, together with Grönwall's inequality, yields

$$\|\nabla\widetilde{\beth}'(t)\|^2 + \|\Delta\widetilde{\beth}(t)\|^2 \le C(T, \hbar, p_1, p_2)\|p_1 - p_2\|^2_{\wp}. \tag{2.12}$$

Moreover, using the elliptic regularity for $\widetilde{\Phi}$, we obtain

$$\|\widetilde{\Phi}(t)\|^2_{W^{2,\infty}} \le C(T, p_1, p_2)\|p_1 - p_2\|^2_{\wp}. \tag{2.13}$$

Combining (2.12) and (2.13) gives the desired Lipschitz estimate

$$\|\nabla\widetilde{\beth}'(t)\| + \|\Delta\widetilde{\beth}(t)\| + \|\widetilde{\Phi}(t)\|_{W^{2,\infty}} \leq C\|p_1 - p_2\|_{\wp},$$

This proves the local Lipschitz continuity of the solution map. □

2.3 On the Existence of Optimal Controls for Memory-Driven Shallow Shell Equations

Consider a control operator $\mathscr{B}$ such that

$$\mathscr{B} \in \mathcal{L}(\mathcal{V}, L^2(0,T;L^2)).$$

By Theorem 2.5, for each $v \in \mathcal{V}$ there exists a unique state solution $\beth(v)$ via

$$v \mapsto \beth(v) \in \mathcal{S}(0,T).$$

The system observation is defined as (see, e.g., [35, Chapter 4])

$$\beth(v) := C\,\beth(v), \quad C \in \mathcal{L}(\mathcal{S}(0,T), \mathcal{M}),$$

where $\mathcal{M}$ is the observation Hilbert space.

The quadratic cost functional is

$$J(v) = \|\beth(v) - \beth_d\|^2_{\mathcal{M}} + (\mathcal{N}v, v)_{\mathcal{V}} \quad \forall v \in \mathcal{V}, \tag{2.14}$$

with $\beth_d \in \mathcal{M}$ the desired observation and $\mathcal{N} \in \mathcal{L}(\mathcal{V}, \mathcal{V})$ symmetric positive-definite :

$$(\mathcal{N}v, v)_{\mathcal{V}} = (v, \mathcal{N}v)_{\mathcal{V}} \geq \underline{C}\,\|v\|^2_{\mathcal{V}}. \tag{2.15}$$

Finally, let $\mathcal{V}_{ad} \subset \mathcal{V}$ be the closed convex set of admissible controls. An optimal control $v^* \in \mathcal{V}_{ad}$ satisfies

$$J(v^*) = \inf_{v \in \mathcal{V}_{ad}} J(v).$$

Theorem 2.6. If the assumptions of Theorem 2.5 are satisfied, then the cost functional (2.14) possesses at least one minimizing control v^*.

Proof. Define the optimal value $J^* = \inf_{v \in \mathcal{V}_{ad}} J(v)$. Since the set of admissible controls $\mathcal{V}_{\text{ad}}$ is nonempty, there exists a minimizing sequence $\{v_n\} \subset \mathcal{V}_{\text{ad}}$ such that

$$\inf_{v \in \mathcal{V}_{ad}} J(v) = \lim_{n \to \infty} J(v_n) = J^*.$$

The boundedness of the sequence $\{J(v_n)\}$ follows from its convergence. Consequently, by the coercivity condition (2.15), there exists a constant $\overline{C} > 0$ such that

$$\underline{C} \|v_n\|_{\mathcal{V}}^2 \leq (\mathcal{N} v, v)_{\mathcal{V}} \leq J(v_n) \leq \overline{C}. \tag{2.16}$$

This implies that $\{v_n\}$ is bounded in $\mathcal{V}$. Given that $\mathcal{V}_{\text{ad}}$ is closed and convex, we may extract a subsequence (still denoted by $\{v_n\}$) that converges weakly :

$$v_n \rightharpoonup v^* \text{ in } \mathcal{V} \quad \text{as } n \to \infty. \tag{2.17}$$

Let $\beth_n = \beth(v_n) \in \mathcal{W}(0, T)$ denote the state corresponding to the control v_n, which satisfies the system :

$$\begin{cases} \beth_{tt,n} - \Delta \beth_{tt,n} + \Delta^2 \beth_n + \hbar \star \Delta^2 \beth_n \\ = [\beth_n, \Phi_n] + [\beth_n, \chi] + \hbar \star \Delta^2 \Theta + \Delta^2 \Theta + \mathscr{B} v_n \quad \text{in } Q, \\ \Delta^2 \Phi_n = [\Theta, \Theta] - [\beth_n, \beth_n] \quad \text{in } Q, \\ \beth_n = \partial_\nu \beth_n = \Phi_n = \partial_\nu \Phi_n = 0 \quad \text{on } \Sigma, \\ \beth_n(0, .) = \beth_0(.) \text{ and } \beth_{t,n}(0, .) = \beth_1(.) \quad \text{in } \Omega. \end{cases} \tag{2.18}$$

From the boundedness estimate (2.16) and the continuity of the control operator $\mathscr{B}$, we obtain

$$\|\mathscr{B} v_n\|_{L^2(0,T;L^2)} \leq \|\mathscr{B}\|_{\mathcal{L}\left(\mathcal{V}, L^2(0,T;L^2)\right)} \left(\overline{C} \underline{C}^{-1}\right)^{\frac{1}{2}}.$$

By employing arguments analogous to those in the proof of Theorem 2.5, we derive the following uniform estimates :

$$\begin{aligned} \|\daleth_n\|_{\mathcal{S}(0,T)} &\leq C\|p\|_{\wp}, \\ \|\nabla\daleth_n'(t)\| + \|\Delta\daleth_n(t)\| &\leq C\|p\|_{\wp}, \\ \|\Phi_n(t)\|_{W^{2,\infty}} &\leq C\|p\|_{\wp}. \end{aligned}$$

Furthermore, the boundedness of the forcing term

$$\|\hbar\star\Delta^2\Theta+\Delta^2\Theta+\mathscr{B}v_n\|_{L^2(0,T;L^2)} \leq \|\hbar\star\Delta^2\Theta+\Delta^2\Theta\|_{L^2(0,T;L^2)}+\|\mathscr{B}v_n\|_{L^2(0,T;L^2)} = \overline{\overline{C}}.$$

implies, via standard energy estimates, the global bound

$$\|\daleth_n\|_{\mathcal{W}(0,T)}+\|\daleth_n(t)\|_{H_0^2}+\|\daleth_n'(t)\|_{H_0^1}+\|\Phi_n(t)\|_{W^{2,\infty}} \leq C\left(\|\daleth_0\|^2_{H_0^2}+\|\daleth_1\|^2_{H_0^1}+\overline{\overline{C}}^2\right)^{\frac{1}{2}}. \tag{2.19}$$

The estimate (2.19) guarantees the boundedness of the nonlinear term $[\daleth_n, \Phi_n]$ in $L^2(0, T; L^2)$. An application of Rellich's compactness theorem then permits the extraction of a subsequence (still denoted by $\{\daleth_n\}$) and the existence of limit functions

$$\begin{aligned} &\daleth \in \mathcal{W}(0,T) \cap L^\infty(0,T;H_0^2), \\ &\daleth' \in L^\infty(0,T;H_0^1), \\ &F \in L^2(0,T;L^2) \end{aligned}$$

such that the weak convergence statements (2.20)–(2.23) hold.

$$\daleth_n \rightharpoonup \daleth \text{ in } \mathcal{W}(0,T) \quad \text{as } n \to \infty, \tag{2.20}$$

$$\daleth_n \rightharpoonup^* \daleth \text{ in } L^\infty(0,T;H_0^2) \quad \text{as } n \to \infty, \tag{2.21}$$

$$\daleth_n' \rightharpoonup^* \daleth' \text{ in } L^\infty(0,T;H_0^1) \quad \text{as } n \to \infty, \tag{2.22}$$

$$[\daleth_n, \Phi_n] + [\daleth_n, \chi] \rightharpoonup F \text{ in } L^2(0,T;L^2) \quad \text{as } n \to \infty. \tag{2.23}$$

To identify the weak limit F with the expected nonlinear expression

$$F = [\daleth, G([\Theta,\Theta] - [\daleth,\daleth])] + [\daleth, \chi],$$

we follow the methodology of [30, Theorem 4.1] (originating from [14, pp. 561–566]). This analysis reveals that the weak limit $\beth$ satisfies the linearized problem

$$\begin{cases} \beth_{tt} - \Delta \beth_{tt} + \Delta^2 \beth + \hbar \star \Delta^2 \beth = F + \hbar \star \Delta^2 \Theta + \Delta^2 \Theta + \mathscr{B} v^* & \text{in } Q, \\ \beth = \partial_\nu \beth = 0 & \text{on } \Sigma, \\ \beth(0,.) = \beth_0(.) \text{ and } \beth_t(0,.) = \beth_1(.) & \text{in } \Omega. \end{cases} \tag{2.24}$$

We obtain the energy identity for solutions of (2.18) :

$$\begin{aligned} &\|\beth_n'(t)\|^2 + \|\nabla \beth_n'(t)\|^2 + \|\Delta \beth_n(t)\|^2 + 2\big(\hbar * \Delta \beth_n(t), \Delta \beth_n(t)\big)_2 \\ &= 2\int_0^t (\hbar' * \Delta \beth_n, \Delta \beth_n)_2 \, ds + 2\int_0^t \hbar(0) \|\Delta \beth_n\|^2 \, ds \\ &\quad + 2\int_0^t \big([\beth_n, \Phi_n] + [\beth_n, \chi] + \hbar \star \Delta^2 \Theta + \Delta^2 \Theta + \mathscr{B} v_n, \beth_n'\big)_2 \, ds \\ &\quad + \|\beth_1\|^2 + \|\nabla \beth_1\|^2 + \|\Delta \beth_0\|^2. \end{aligned} \tag{2.25}$$

A corresponding energy relation holds for weak solutions of the limiting problem (2.24).

$$\begin{aligned} &\|\beth'(t)\|^2 + \|\nabla \beth'(t)\|^2 + \|\Delta \beth(t)\|^2 + 2\big(\hbar * \Delta \beth(t), \Delta \beth(t)\big)_2 \\ &= 2\int_0^t (\hbar' * \Delta \beth, \Delta \beth)_2 \, ds + 2\int_0^t \hbar(0) \|\Delta \beth\|^2 \, ds \\ &\quad + 2\int_0^t (F + \hbar \star \Delta^2 \Theta + \Delta^2 \Theta + \mathscr{B} v^*, \beth')_2 \, ds + \|\beth_1\|^2 + \|\nabla \beth_1\|^2 + \|\Delta \beth_0\|^2. \end{aligned} \tag{2.26}$$

We first recall the following relations :

$$\begin{aligned} &\|\eta\|^2 + \|\psi\|^2 = \|\eta - \psi\|^2 + 2(\eta, \psi)_2, \quad \forall \eta, \psi \in L^2, \\ &(\eta_1, \eta_2)_2 + (\psi_1, \psi_2)_2 \\ &= (\eta_1 - \psi_1, \eta_2 - \psi_2)_2 + (\psi_1, \eta_2)_2 + (\eta_1, \psi_2)_2, \quad \forall \eta_i, \psi_i (i = 1, 2) \in L^2. \end{aligned}$$

By combining (2.26) with (2.25), it follows that the sequence $\varphi_n = \beth_n - \beth$

satisfies

$$\|\varphi_n'(t)\|^2 + \|\nabla\varphi_n'(t)\|^2 + \|\Delta\varphi_n(t)\|^2 + 2(\hbar * \Delta\varphi_n(t), \Delta\varphi_n(t))_2$$
$$= 2\int_0^t (\hbar' * \Delta\varphi_n, \Delta\varphi_n)_2\, ds + 2\int_0^t \hbar(0)\|\Delta\varphi_n\|^2\, ds + \varphi^0 + \sum_{i=1}^{5} \varphi_n^i,$$

where

$$\begin{aligned}
\varphi^0 &= 2\big(\|\daleth_1\|^2 + \|\nabla\daleth_1\|^2 + \|\Delta\daleth_0\|^2\big),\\
\varphi_n^1 &= -2\Big(\big(\daleth_n'(t), \daleth'(t)\big)_2 + \big(\nabla\daleth_n'(t), \nabla\daleth'(t)\big)_2 + \big(\Delta\daleth_n(t), \Delta\daleth(t)\big)_2\Big),\\
\varphi_n^2 &= -2\Big(\big(\hbar * \Delta\daleth(t), \Delta\daleth_n(t)\big)_2 + \big(\hbar * \Delta\daleth_n(t), \Delta\daleth(t)\big)_2\Big),\\
\varphi_n^3 &= 2\Big(\int_0^t \big(\hbar' * \Delta\daleth_n, \Delta\daleth\big)_2\, ds + \int_0^t \big(\hbar' * \Delta\daleth, \Delta\daleth_n\big)_2\, ds\Big),\\
\varphi_n^4 &= 4\int_0^t \hbar(0)(\Delta\daleth_n, \Delta\daleth)_2\, ds,\\
\varphi_n^5 &= 2\int_0^t \big([\daleth_n, \Phi_n] + [\daleth_n, \chi] + \hbar \star \Delta^2\Theta + \Delta^2\Theta + \mathscr{B}v_n, \daleth_n'\big)_2\, ds\\
&\quad + 2\int_0^t \big(F + \hbar \star \Delta^2\Theta + \Delta^2\Theta + \mathscr{B}v^*, \daleth'\big)_2\, ds.
\end{aligned}$$

Proceeding in the same spirit as in the proof of Theorem 2.5, one obtains

$$\|\varphi_n'(t)\|^2 + \|\nabla\varphi_n'(t)\|^2 + \|\Delta\varphi_n(t)\|^2 \le C(\hbar, T)\Big|\varphi^0 + \sum_{i=1}^{5}\varphi_n^i\Big|. \tag{2.27}$$

Consequently, there exists a subsequence $\{\daleth_{n_k}\} \subset \{\daleth_n\}$ such that

$$\begin{aligned}
\varphi_{n_k}^1 &\to -2\Big(\|\daleth'(t)\|^2 + \|\nabla\daleth'(t)|^2 + \|\Delta\daleth(t)\|^2\Big) \quad \text{as } k \to \infty,\\
\varphi_{n_k}^2 &\to -4\big(\hbar * \Delta\daleth(t), \Delta\daleth(t)\big)_2 \quad \text{as } k \to \infty,\\
\varphi_{n_k}^3 &\to 4\int_0^t (\hbar' * \Delta\daleth, \Delta\daleth)_2\, ds \quad \text{as } k \to \infty,\\
\varphi_{n_k}^4 &\to 4\int_0^t \hbar(0)\|\Delta\daleth\|^2\, ds \quad \text{as } k \to \infty.
\end{aligned}$$

Using the Aubin–Lions–Temam compactness theorem (see [53, pp. 271]), the compact embedding $H_0^1 \hookrightarrow L^2$ ensures that there exists a subsequence $\{\daleth_{n_k}\} \subset$

$\{\beth_n\}$ such that

$$\beth'_{n_k} \to \beth' \text{ in } L^2(0,T;L^2) \quad \text{as } k \to \infty. \tag{2.28}$$

Finally, combining (2.17), (2.23) and (2.28), we deduce that

$$\varphi^5_{n_k} \to 4\big(F + [\beth,\chi] + \hbar \star \Delta^2\Theta + \Delta^2\Theta + \mathscr{B}v^*, \beth'\big)_2 \quad \text{as } k \to \infty.$$

Taking into account (2.26), this yields

$$\varphi^0 + \sum_{i=1}^{5} \varphi^i_{n_k} \to 0 \quad \text{as } k \to \infty,$$

and thus, from (2.27), it follows that

$$\beth_{n_k} \to \beth \text{ in } H^2_0 \quad \text{as } k \to \infty, \quad \forall t \in [0,T]. \tag{2.29}$$

In view of Theorem 2.5 and Lemma 1.15, together with the strong convergence (2.29), we deduce that

$$\begin{aligned}
&\Big\|\big([\beth_{n_k},\Phi_{n_k}] + [\beth_{n_k},\chi]\big) - \big([\beth,\Phi] + [\beth,\chi]\big)\Big\|_{L^2(0,T;L^2)}\\
&= \big\|[\beth_{n_k},\Phi_{n_k} + \chi] - [\beth,\Phi + \chi]\big\|_{L^2(0,T;L^2)}\\
&= \big\|\,[\beth_{n_k} - \beth,\Phi_{n_k} + \chi] + [\beth,\Phi_{n_k} - \Phi]\big\|_{L^2(0,T;L^2)}\\
&\le \big\|[\beth_{n_k} - \beth,\Phi_{n_k} + \chi]\big\|_{L^2(0,T;L^2)} + \big\|[\beth, G[\beth,\beth] - G[\beth_{n_k},\beth_{n_k}]]\big\|_{L^2(0,T;L^2)}\\
&= \big\|[\beth_{n_k} - \beth,\Phi_{n_k} + \chi]\big\|_{L^2(0,T;L^2)} + \big\|[\beth, G[\beth - \beth_{n_k},\beth + \beth_{n_k}]\|_{L^2(0,T;L^2)}\\
&\le C\Big(\|\beth\|^2_{L^\infty(0,T;H^2_0)} + \|\beth_{n_k}\|^2_{L^\infty(0,T;H^2_0)} + \|\chi\|_{H^2}\Big)\|\beth_{n_k} - \beth\|_{L^2(0,T;H^2_0)}\\
&\le C\Big(\|p^*\|^2_{\wp} + \|p_{n_k}\|^2_{\wp} + \|\chi\|_{H^2}\Big)\|\beth_{n_k} - \beth\|_{L^2(0,T;L^2)} \to 0 \text{ as } k \to \infty,
\end{aligned} \tag{2.30}$$

where $p^* = (\beth_0, \beth_1, \mathscr{B}v^*)$ and $p_{n_k} = (\beth_0, \beth_1, \mathscr{B}v_{n_k})$.

Therefore, by combining (2.23) and (2.30), and invoking the uniqueness of weak limits, we obtain

$$F = [\beth,\Phi] + [\beth,\chi] = \big[\beth, G\big([\Theta,\Theta] - [\beth,\beth]\big)\big] + [\beth,\chi].$$

By replacing $\beth_n$ with $\beth_{n_k}$ in (2.18) and then passing to the limit (see, e.g., Dautray and Lions [14, pp 561–566]), we deduce that the limit $\daleth$ is a weak solution to the problem

$$\begin{cases} \daleth_{tt} - \Delta \daleth_{tt} + \Delta^2 \daleth + \hbar \star \Delta^2 \daleth = [\daleth, \Phi] + [\daleth, \chi] + \hbar \star \Delta^2 \Theta + \Delta^2 \Theta + \mathscr{B} v^* & \text{in } Q, \\ \Delta^2 \Phi = [\Theta, \Theta] - [\daleth, \daleth] \quad \text{in } Q, \\ \daleth = \partial_\nu \daleth = \Phi = \partial_\nu \Phi = 0 \quad \text{on } \Sigma, \\ \daleth(0, .) = \daleth_0(.) \text{ and } \daleth_t(0, .) = \daleth_1(.) \quad \text{in } \Omega. \end{cases} \tag{2.31}$$

According to Theorem 2.5, problem (2.31) admits a unique weak solution $\daleth \in \mathcal{S}(0,T)$. Consequently, we deduce that $\daleth = \daleth(v^*) \in \mathcal{S}(0,T)$ and that $\daleth(v_n) \rightharpoonup \daleth(v^*)$ weakly in $\mathcal{W}(0,T)$.

Since C is continuous on $\mathcal{S}(0,T) \subset \mathcal{W}(0,T)$ and $\|\cdot\|_{\mathcal{M}}$ is lower semi-continuous, it follows that

$$\|\daleth(v^*) - \daleth_d\|_{\mathcal{M}} \leq \liminf_{n\to\infty} \|\daleth(v_n) - \daleth_d\|_{\mathcal{M}}.$$

Moreover, we clearly have

$$(\mathcal{N} v^*, v^*)_{\mathcal{V}} \leq \liminf_{n\to\infty} (\mathcal{N} v_n, v_n)_{\mathcal{V}},$$

and hence

$$\|\mathcal{N}^{\frac{1}{2}} v^*\|_{\mathcal{V}} \leq \liminf_{n\to\infty} \|\mathcal{N}^{\frac{1}{2}} v_n\|_{\mathcal{V}}.$$

Therefore,

$$J(v^*) \leq J^* = \liminf_{n\to\infty} J(v_n).$$

Since by definition $J^* \leq J(v^*)$, we conclude that

$$J(v^*) = \inf_{v \in \mathcal{V}_{ad}} J(v).$$

□

2.4 Optimality Conditions and Control Characterization

This section analyzes the optimal control problem from two complementary perspectives. First, we establish the Gâteaux differentiability of the solution map, which provides the analytical foundation for the subsequent analysis. Then, we derive the necessary condition of optimal control, leading to the adjoint system and a precise characterization of the optimal controls.

2.4.1 Differentiability of the Control-to-State Map

The following definition of Gâteaux differentiability for the control-to-state mapping, adapted from [30], will be fundamental to our subsequent analysis :

Definition 2.7. Consider the solution operator

$$\begin{aligned} \mathcal{V} &\longrightarrow \mathcal{S}(0,T), \\ v &\longmapsto \daleth(v). \end{aligned} \tag{2.32}$$

is said to be Gâteaux differentiable at a point $v^* \in \mathcal{V}$ in the direction $w \in \mathcal{V}$ if there exists a bounded linear operator $D\daleth(v^*) \in \mathcal{L}(\mathcal{V}, \mathcal{S}(0,T))$ such that

$$\left\| \frac{\daleth(v^* + \lambda w) - \daleth(v^*)}{\lambda} - D\daleth(v^*)w \right\|_{\mathcal{S}(0,T)} \longrightarrow 0 \qquad \text{as } \lambda \to 0. \tag{2.33}$$

We emphasize that the optimal control v^* satisfies the following necessary optimality condition :

$$DJ(v^*)(v - v^*) \geq 0, \quad \forall v \in \mathcal{V}_{\text{ad}}, \tag{2.34}$$

where $DJ(v^*)$ denotes the Gâteaux derivative of the cost functional J evaluated at $v = v^*$. This inequality expresses that, at the optimal control, any admissible variation $v - v^*$ cannot decrease the value of the cost functional, thereby providing a first-order characterization of optimality.

By employing a similar reasoning as in the proof of Theorem 2.6, we can further establish the strong convergence of the corresponding variations, which will be instrumental in deriving the adjoint system and the precise structure of the optimal control.

Lemma 2.8. (Continuity of the Control-to-State Map) Let $w \in \mathcal{V}$ be any direction. Then the control-to-state map

$$\mathcal{S} : \mathcal{V} \longrightarrow \mathcal{S}(0,T), \quad v \longmapsto \beth(v),$$

It is continuous in the sense that.

$$\beth(v + \lambda w) \longrightarrow \beth(v) \quad \text{strongly in } \mathcal{S}(0,T) \quad \text{as } \lambda \to 0.$$

This property ensures that slight variations in the control produce small changes in the state, which is essential for differentiability analysis.

Theorem 2.9. (Gâteaux Differentiability and Linearization) The control-to-state map $\mathcal{S}$ is Gâteaux differentiable at $v^* \in \mathcal{V}$. For any admissible direction $h = v - v^*$, the derivative $z = D\mathcal{S}(v^*)h$ is the unique weak solution to the linearized system :

$$\begin{cases} z_{tt} - \Delta z_{tt} + \Delta^2 z + \hbar \star \Delta^2 z \quad = \left[z, G([\Theta,\Theta] - [\beth(v^*), \beth(v^*)])\right] & \\ +2[\beth(v^*), -G[z, \beth(v^*)]] + [z, \chi] + \mathscr{B}h & \text{in } Q, \\ z = \partial_\nu z = 0 & \text{on } \Sigma, \\ z(0,\cdot) = z_t(0,\cdot) = 0 & \text{in } \Omega. \end{cases} \tag{2.35}$$

This linearization captures the system's response to minor control variations and serves as a foundation for deriving the adjoint system and necessary optimality conditions.

Proof. Let $\lambda \in (-1,1)$ with $\lambda \neq 0$, and set

$$\beth_\lambda := \beth\big(v^* + \lambda(v - v^*)\big), \quad z_\lambda := \frac{\beth_\lambda - \beth(v^*)}{\lambda}.$$

Here, z_λ represents the scaled difference between the perturbed state $\beth_\lambda$ and the reference state $\beth(v^*)$, and it serves as an approximation to the Gâteaux derivative in the direction $v - v^*$.

From the state equation (2.6), $\daleth_\lambda$ satisfies

$$\begin{cases} \daleth_{\lambda,tt} - \Delta\daleth_{\lambda,tt} + \Delta^2\daleth_\lambda + \hbar\star\Delta^2\daleth_\lambda \\ = [\daleth_\lambda, \Phi_\lambda] + [\daleth_\lambda, \chi] + \hbar\star\Delta^2\Theta + \Delta^2\Theta + \mathscr{B}\big(v^* + \lambda(v - v^*)\big) \quad \text{in } Q, \\ \Delta^2\Phi_\lambda = [\Theta,\Theta] - [\daleth_\lambda, \daleth_\lambda] \quad \text{in } Q, \\ \daleth_\lambda = \partial_\nu\daleth_\lambda = \Phi_\lambda = \partial_\nu\Phi_\lambda = 0 \quad \text{on } \Sigma, \\ \daleth_\lambda\big(v^* + \lambda(v - v^*); 0, .\big) = \daleth_0(.) \text{ and } \daleth_{\lambda,t}\big(v^* + \lambda(v - v^*); 0, .\big) = \daleth_1(.) \quad \text{in } \Omega, \end{cases} \tag{2.36}$$

where $\Phi_\lambda = \Phi(v^* + \lambda(v - v^*))$. Similarly, the unperturbed state $\daleth(v^*)$ satisfies

$$\begin{cases} \daleth_{tt}(v^*) - \Delta\daleth_{tt}(v^*) + \Delta^2\daleth(v^*) + \hbar\star\Delta^2\daleth(v^*) \\ = \big[\daleth(v^*), \Phi(v^*)\big] + \big[\daleth(v^*), \chi\big] + \hbar\star\Delta^2\Theta + \Delta^2\Theta + \mathscr{B}v^* \quad \text{in } Q, \\ \Delta^2\Phi(v^*) = [\Theta,\Theta] - \big[\daleth(v^*), \daleth(v^*)\big] \quad \text{in } Q, \\ \daleth = \partial_\nu\daleth = \Phi(v^*) = \partial_\nu\Phi(v^*) = 0 \quad \text{on } \Sigma, \\ \daleth(v^*; 0, .) = \daleth_0(.) \text{ and } \daleth_t(v^*; 0, .) = \daleth_1(.) \quad \text{in } \Omega. \end{cases} \tag{2.37}$$

Subtracting (2.37) from (2.36) and dividing by λ, we obtain the evolution equation for z_λ :

$$\begin{cases} z_{\lambda,tt} - \Delta z_{\lambda,tt} + \Delta^2 z_\lambda + \hbar\star\Delta^2 z_\lambda = F_\lambda + \mathscr{B}(v - v^*) \quad \text{in } Q, \\ z_\lambda = \partial_\nu z_\lambda = 0 \quad \text{on } \Sigma, \\ z_\lambda(0, .) = z_{\lambda,t}(0, .) = 0 \quad \text{in } \Omega, \end{cases} \tag{2.38}$$

where

$$F_\lambda = \frac{1}{\lambda}\Big(\big[\daleth_\lambda, G\big([\Theta,\Theta] - [\daleth_\lambda, \daleth_\lambda]\big)\big] - \big[\daleth(v^*), G\big([\Theta,\Theta] - \big[\daleth(v^*), \daleth(v^*)\big]\big)\big]\Big)$$
$$+[z_\lambda, \chi] = \big[z_\lambda, G\big([\Theta,\Theta] - [\daleth_\lambda, \daleth_\lambda]\big)\big] + \big[\daleth(v^*), -G\big[z_\lambda, \daleth(v^*) + \daleth_\lambda\big]\big] + [z_\lambda, \chi]. \tag{2.39}$$

Next, by Theorem 2.5 and Lemmas 1.15, we derive the estimate

$$\begin{aligned}
&\|F_\lambda\|_{L^2(0,T;L^2)} \\
&\le C\|\Delta z_\lambda\|_{L^2(0,T;L^2)}\Big(\|\beth_\lambda\|^2_{L^\infty(0,T;H_0^2)} + \|\Theta\|^2_{H_0^2} \\
&\quad + \|\beth(v^*)\|_{L^\infty(0,T;H_0^2)}\big(\|\beth(v^*)\|_{L^\infty(0,T;H_0^2)} + \|\beth_\lambda\|_{L^\infty(0,T;H_0^2)}\big) + \|\chi\|_{H^2}\Big) \\
&\le C\|\Delta z_\lambda\|_{L^2(0,T;L^2)}\big(\|\beth_\lambda\|^2_{L^\infty(0,T;H_0^2)} + \|\beth(v^*)\|^2_{L^\infty(0,T;H_0^2)} + \|\Theta\|^2_{H_0^2} + \|\chi\|_{H^2}\big) \\
&\le C\|\Delta z_\lambda\|_{L^2(0,T;L^2)}\big(\|p_\lambda\|^2_{\wp} + \|p^*\|^2_{\wp} + \|\Theta\|^2_{H_0^2} + \|\chi\|_{H^2}\big),
\end{aligned} \tag{2.40}$$

where $p_\lambda = \Big(\beth_0, \beth_1, \hbar \star \Delta^2\Theta + \Delta^2\Theta + \mathscr{B}\big(v^* + \lambda(v - v^*)\big)\Big)$ and $p^* = (\beth_0, \beth_1, \hbar \star \Delta^2\Theta + \Delta^2\Theta + \mathscr{B}v^*)$.

Applying energy estimates to (2.38) as in Theorem 2.5, we obtain

$$\|z_\lambda\|_{\mathcal{S}(0,T)} \le C\|\mathscr{B}(v - v^*)\|_{L^2(0,T;L^2)}. \tag{2.41}$$

showing that the family $\{z_\lambda\}$ is uniformly bounded in $\mathcal{S}(0,T)$.

By weak compactness, there exists $z \in \mathcal{W}(0,T) \cap L^\infty(0,T;H_0^2)$ with $z' \in L^\infty(0,T;H_0^1)$, a subsequence $\{\lambda_k\} \to 0$, and $F \in L^2(0,T;L^2)$, as $k \to \infty$ such that

$$\begin{aligned}
&z_{\lambda_k} \rightharpoonup z \text{ in } \mathcal{W}(0,T), && z_{\lambda_k} \rightharpoonup^* z \text{ in } L^\infty(0,T;H_0^2), \\
&z'_{\lambda_k} \rightharpoonup^* z' \text{ in } L^\infty(0,T;H_0^1), && F_{\lambda_k} \rightharpoonup F \text{ in } L^2(0,T;L^2).
\end{aligned}$$

Passing to the limit in (2.38) along $\{\lambda_k\}$ and using continuity of the nonlinear operators (Lemma 2.8), we conclude that z satisfies

$$\begin{cases}
z_{tt} - \Delta z_{tt} + \Delta^2 z + \hbar \star \Delta^2 z = F + \mathscr{B}(v - v^*) & \text{in } Q, \\
z = \partial_\nu z = 0 \quad \text{on } \Sigma, \\
z(0,.) = z_t(0,.) = 0 \quad \text{in } \Omega,
\end{cases} \tag{2.42}$$

with

$$F = [z, G([\Theta,\Theta] - [\beth(v^*), \beth(v^*)])] + 2[\beth(v^*), -G[z, \beth(v^*)]] + [z,\chi] \in L^2(0,T;L^2).$$

From the convergence established in (2.43), we have that the sequence $\{z_{\lambda_k}\}$ converges strongly to z in the space $\mathcal{S}(0,T)$:

$$z_{\lambda_k} \to z \text{ in } \mathcal{S}(0,T) \quad \text{as } k \to \infty. \tag{2.43}$$

This strong convergence means that both the displacement z_{λ_k} and its time derivative $z_{\lambda_k,t}$ converge uniformly in the appropriate Sobolev norms, providing a solid basis to pass to the limit in nonlinear terms.

Next, we focus on the nonlinear operator G appearing in the linearized problem. Using Theorem 2.5 and Lemma 1.15, we can control the difference

$$\begin{aligned} &\|G[z_{\lambda_k}, \beth(v^*) + \beth_{\lambda_k}] - 2G[z, \beth(v^*)]\|_{C([0,T];W^{2,\infty})} \\ &\leq CT\Big((\|p^*\|_{\wp} + \|p_{\lambda_k}\|_{\wp}) \|z_{\lambda_k} - z\|_{C([0,T];H_0^2)} \\ &\quad + \|\mathscr{B}(v - v^*)\|_{L^2(0,T;L^2)} \|\beth_{\lambda_k} - \beth(v^*)\|_{C([0,T];H_0^2)} \Big), \end{aligned} \tag{2.44}$$

where

$$p_{\lambda_k} = \Big(\beth_0, \beth_1, \hbar \star \Delta^2\Theta + \Delta^2\Theta + \mathscr{B}(v^* + \lambda_k(v - v^*))\Big),$$
$$p^* = (\beth_0, \beth_1, \hbar \star \Delta^2\Theta + \Delta^2\Theta + \mathscr{B}v^*).$$

Here, the first term on the right-hand side arises from the difference in z_{λ_k} versus z, which multiplies the "size" of the data (p^*, p_{λ_k}), while the second term accounts for the variation in the state $\beth_{\lambda_k}$ relative to $\beth(v^*)$, weighted by the control difference $\mathscr{B}(v - v^*)$. Both terms vanish in the limit $k \to \infty$, thanks to the strong convergence of $z_{\lambda_k} \to z$ and $\beth_{\lambda_k} \to \beth(v^*)$ in $C([0,T]; H_0^2)$ (Lemma 2.8).

Consequently, we obtain the convergence of the nonlinear term :

$$G[z_{\lambda_k}, \beth(v^*) + \beth_{\lambda_k}] \to 2G[z, \beth(v^*)] \text{ in } C([0,T]; W^{2,\infty}) \quad \text{as } k \to \infty. \tag{2.45}$$

In the same way, for the other nonlinear component involving $[\beth_{\lambda_k}, \beth_{\lambda_k}]$, we have

$$G\big([\Theta,\Theta] - [\beth_{\lambda_k}, \beth_{\lambda_k}]\big) \to G\big([\Theta,\Theta] - [\beth(v^*), \beth(v^*)]\big) \text{ in } C([0,T]; W^{2,\infty}). \tag{2.46}$$

$$\text{as } k \to \infty$$

With these convergences, we can now consider the forcing term F_{λ_k} in (2.38). Using the linearity in the remaining terms and the convergences above, we deduce that

$$F_{\lambda_k} \to F = \left[z, G\left([\Theta,\Theta] - [\daleth(v^*), \daleth(v^*)]\right)\right] + 2\left[\daleth(v^*), -G[z, \daleth(v^*)]\right] + [z,\chi] \text{ in } L^2(0,T;L^2) \quad \text{as } k \to \infty. \tag{2.47}$$

Here, each term is interpreted carefully : the first term captures the variation in the “forcing” due to the linearized displacement z, the second term accounts for the interaction between the reference state $\daleth(v^*)$ and z, and the last term is the linear contribution from χ.

Finally, combining the strong convergence of z_{λ_k} in (2.43) with the convergence of F_{λ_k} in (2.47), we conclude that z is the unique weak solution of the linearized problem (2.35). Explicitly, we have

$$z_{\lambda_k} \to z = D\daleth(v^*)(v - v^*) \text{ in } \mathcal{S}(0,T) \quad \text{as } k \to \infty. \tag{2.48}$$

□

This completes the rigorous demonstration of the Gâteaux differentiability of the control-to-state map, providing a solid foundation for deriving the necessary optimality conditions through the adjoint system.

2.4.2 Adjoint System and Optimality Conditions

Theorem 2.9 establishes the Gâteaux differentiability of the quadratic cost functional $J(v)$ at the optimal control v^* along admissible directions $v - v^*$. Consequently, the first-order necessary optimality condition (2.34) takes the form :

$$\begin{aligned}&\Big(C\beth(v^*) - \beth_d, C\big(D\beth(v^*)(v-v^*)\big)\Big)_{\mathcal{M}} + \Big(\mathcal{N}v^*, v-v^*\Big)_{\mathcal{V}} \\ &= \big\langle C^*\Lambda_{\mathcal{M}}\big(C\beth(v^*) - \beth_d\big), D\beth(v^*)(v-v^*)\big\rangle_{\mathcal{W}(0,T)',\mathcal{W}(0,T)} \\ &\quad + \Big(\mathcal{N}v^*, v-v^*\Big)_{\mathcal{V}} \geq 0, \quad \forall v \in \mathcal{V}_{ad},\end{aligned} \tag{2.49}$$

Where $\Lambda_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}'$ denotes the canonical isomorphism induced by the Riesz representation theorem.

In many physical applications, the observation operator is given by $C\beth(v) = \beth(v;\cdot) \in L^2(0,T;L^2)$, which naturally leads to the choice $\mathcal{M} = L^2(0,T;L^2)$. With this specification, the cost functional (2.14) reduces to :

$$J(v) = \int_0^T \|\beth(v) - \beth_d\|^2 dt + (\mathcal{N}v, v)_{\mathcal{V}}, \quad \forall v \in \mathcal{V}_{ad} \subset \mathcal{V}, \tag{2.50}$$

where $\beth_d \in L^2(0,T;L^2)$ represents a target state trajectory. Let v^* be an optimal control for problem (2.6)–(2.50). Then the optimality condition (2.49) simplifies to :

$$\int_0^T (\beth(v^*) - \beth_d, z)_2 dt + (\mathcal{N}v^*, v-v^*)_{\mathcal{V}} \geq 0, \quad \forall v \in \mathcal{V}_{ad}, \tag{2.51}$$

where $z = D\beth(v^*)(v-v^*)$ is the solution to the linearized system (2.35).

To derive an explicit optimality system, we introduce the adjoint state $p(v^*)$ as the solution to the following backward-in-time problem :

$$\begin{cases} p_{tt}(v^*) - \Delta p_{tt}(v^*) + \Delta^2 p(v^*) + \int_t^T \hbar(\sigma - t)\Delta^2 p(v^*;\sigma)\, d\sigma \\ = \big[p(v^*), G\big([\Theta,\Theta] - [\beth(v^*),\beth(v^*)]\big)\big] + 2\big[\beth(v^*), -G[p(v^*),\beth(v^*)]\big] \\ \quad +[p(v^*),\chi] + \beth(v^*) - \beth_d \quad \text{ in } Q, \\ p(v^*) = \partial_\nu p(v^*) = 0 \quad \text{ on } \Sigma, \\ p(v^*;T) = p_t(v^*;T) = 0 \quad \text{ in } \Omega. \end{cases} \tag{2.52}$$

Proposition 2.10. Under the assumptions of Theorem 2.5, the adjoint system (2.52) admits a unique weak solution $p(v^*) \in \mathcal{S}(0,T)$.

The adjoint state $p(v^*)$ plays a crucial role in the optimality system. Specifically, it allows us to express the Gâteaux derivative of J in terms of $p(v^*)$, leading to the final form of the optimality condition :

$$(\mathscr{B}^* p(v^*) + \mathcal{N} v^*, v - v^*)_{\mathcal{V}} \geq 0, \quad \forall v \in \mathcal{V}_{ad},$$

This completes the optimal control v^* characterization through the coupled state, adjoint, and variational inequality system.

Proof. We start by observing that

$$\int_{T-t}^{T} \hbar(\sigma - T + t)\Delta^2 p(v^*;\sigma)\, d\sigma = \int_0^t \hbar(t-s)\Delta^2 p(u^*;T-s)\, ds,$$

which allows us, after the time reversal transformation $(t \mapsto T - t)$ applied to problem (2.52), to reformulate it in the equivalent form

$$\begin{cases} \psi_{tt} - \Delta\psi_{tt} + \Delta^2\psi + \int_0^t \hbar(t-s)\Delta^2\psi(s)\, ds \\ = \big[\psi, G\big([\Theta,\Theta] - [\beth(v^*),\beth(v^*)]\big)\big] + 2\big[\beth(v^*), -G[\psi,\beth(v^*)]\big] \\ \quad +[\psi,\chi] + \beth(v^*) - \beth_d \quad \text{in } Q, \\ \psi = \partial_\nu\psi = 0 \quad \text{on } \Sigma, \\ \psi(0,.) = \psi_t(0,.) = 0 \quad \text{in } \Omega, \end{cases} \tag{2.53}$$

where $\psi(t) = p(v^*;T-t)$.

where we set $\psi(t) = p(v^*;T-t)$.

Proceeding as in (2.40), we derive the following estimate :

$$\Big\|\big[\psi, G\big([\Theta,\Theta] - [\beth(v^*),\beth(v^*)]\big)\big] + 2\big[\beth(v^*), -G[\psi,\beth(v^*)]\big] + [\psi,\chi]\Big\|_{L^2(0,T;L^2)} \\ \leq C\|\Delta\psi\|_{L^2(0,T;L^2)}\big(\|p^*\|^2_{\wp} + \|\Theta\|^2_{H_0^2} + \|\chi\|_{H^2}\big). \tag{2.54}$$

Hence, invoking Theorem 2.5 together with the result of [30, Proposition 4.1], it follows that problem (2.52) admits a unique solution $\psi \in \mathcal{S}(0,T)$. □

To proceed, we multiply equation (2.52) by the test function z and integrate over the interval $]0,T[$, obtaining

$$\begin{aligned}
&\int_0^T \left\langle p''(v^*) - \Delta p''(v^*), z\right\rangle_{-2,2} dt + \int_0^T \Big(\Delta p(v^*) \\
&+ \int_t^T \hbar(\sigma - t)\Delta p(v^*;\sigma)\, d\sigma, \Delta z\Big)_2 dt - \int_0^T \Big(\big[p(v^*), G\big([\Theta,\Theta] - [\beth(v^*),\beth(v^*)]\big)\big] \\
&+ 2\big[\beth(v^*), -G[p(v^*),\beth(v^*)]\big] + [p(v^*),\chi], z\Big)_2 dt \\
&= \int_0^T \left(\beth(v^*) - \beth_d, z\right)_2 dt.
\end{aligned} \tag{2.55}$$

Applying Fubini's theorem to the convolution term, we arrive at

$$\begin{aligned}
&\int_0^T \Big(\int_t^T \hbar(\sigma - t)\Delta p(v^*;\sigma)\, d\sigma, \Delta z\Big)_2 dt \\
&= \int_0^T \Big(\int_0^t \hbar(t-s)\Delta z(s)\, ds, \Delta p(v^*)\Big)_2 dt \\
&= \int_0^T \Big(\int_0^t \hbar(t-s)\Delta^2 z(s) ds, p(v^*)\Big)_{-2,2} dt.
\end{aligned} \tag{2.56}$$

Moreover, Lemma 1.13 gives

$$\begin{aligned}
&\int_0^T \Big(\big[p(v^*), G\big([\Theta,\Theta] - [\beth(v^*),\beth(v^*)]\big)\big], z\Big)_2 dt \\
&= \int_0^T \Big(\big[z, G\big([\Theta,\Theta] - [\beth(v^*),\beth(v^*)]\big)\big], p(v^*)\Big)_2 dt,
\end{aligned} \tag{2.57}$$

and

$$\int_0^T \Big([p(v^*),\chi], z\Big)_2 dt = \int_0^T \Big([z,\chi], p(v^*)\Big)_2 dt. \tag{2.58}$$

Exploiting again Lemma 1.13, the continuous embedding $L^1 \hookrightarrow H^{-2}$, and the self-adjointness of G, we deduce

$$\begin{aligned}&\int_0^T \Big(2\big[\beth(v^*), -G[p(v^*), \beth(v^*)]\big], z\Big)_2\, dt \\ &= \int_0^T \Big\langle 2[z, \beth(v^*)], -G[p(v^*), \beth(v^*)]\Big\rangle_{-2,2}\, dt \\ &= \int_0^T \Big\langle -2G[z, \beth(v^*)], [p(v^*), \beth(v^*)]\Big\rangle_{2,-2}\, dt \\ &= \int_0^T \Big(2\big[\beth(v^*), -G[z, \beth(v^*)]\big], p(v^*)\Big)_2\, dt.\end{aligned} \tag{2.59}$$

Finally, gathering relations (2.56)–(2.59) together with (2.35) and (2.52), we integrate the left-hand side of (2.55) to obtain

$$\begin{aligned}&\int_0^T \Big\langle p(v^*), z'' - \Delta z'' + \Delta^2 z + \int_0^t \hbar(t-s)\Delta^2 z(s)\, ds\Big\rangle_{2,-2}\, dt \\ &- \int_0^T \Big(p(v^*), \big[z, G\big([\Theta,\Theta] - [\beth(v^*), \beth(v^*)]\big)\big] \\ &+ 2\big[\beth(v^*), -G[z, \beth(v^*)]\big] + [z,\chi]\Big)_2\, dt = \int_0^T \Big(p(v^*), \mathscr{B}(v - v^*)\Big)_2\, dt.\end{aligned} \tag{2.60}$$

We have thus established that the first-order necessary optimality condition (2.51) is equivalent to the variational inequality

$$\int_0^T \big(p(v^*), \mathscr{B}(v - v^*)\big)_2\, dt + \big(\mathcal{N} v^*, v - v^*\big)_{\mathcal{V}} \geq 0, \quad \forall v \in \mathcal{V}_{ad}. \tag{2.61}$$

where $\mathscr{B}^*$ denotes the adjoint of the control operator $\mathscr{B}$.

This fundamental result and the state and adjoint equations form the complete optimality system, which we summarize as follows.

Theorem 2.11. [Optimality System] A control $v^* \in \mathcal{V}_{ad}$ is optimal for problem (2.50) if and only if it satisfies, together with the corresponding state $\beth(v^*)$ and adjoint state $p(v^*)$, the following system :

(1) State equation : (2.37)

(2) Adjoint equation : (2.52)

(3) Variational inequality : (2.61)

Moreover, this system admits a unique solution.

2.5 Concluding Remarks

This research has made several significant contributions to the theory of optimal control for nonlinear shell structures :

- **Existence and Uniqueness** : We have established well-posedness results for optimal control problems governed by memory-dependent Marguerre-von Kármán equations.
- **Optimality Conditions** : A complete optimality system has been derived, comprising the state equations, adjoint equations, and variational inequalities.
- **Generalization** : Our results extend previous work on von Kármán plates to the more geometrically complex case of shallow shells.
- **Mathematical Framework** : The developed theory provides a foundation for numerical implementation and further theoretical investigations.

These achievements advance the mathematical understanding of control problems for structural systems with memory effects and open new avenues for theoretical and applied research in innovative structure technology.

Chapter 3

Optimal Control Problems and Analysis for a Marguerre–von Kármán System with Short Memory : Theory, Weak Formulation, and Existence Results

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This chapter presents original research intended for publication in a scientific journal.

3.1 Introduction

This chapter analyzes optimal control problems with short memory effects for the Marguerre–von Kármán system. Building on the classical model [42, 56], we formulate a well-posed initial-boundary value problem, derive existence results for both state and control solutions, and establish first-order optimality conditions. The analysis presented here focuses on the theoretical foundations of the control problem, while the numerical aspects and penalization methods are reserved for future research.

The structure of this chapter is as follows : Section 2 develops the initial-boundary value problem formulation for the system with short memory. Section 3 introduces the weak formulation and canonical operator form. Section 4 proves existence theorems for the canonical problem. Finally, Section 5 addresses the optimal control problem, including existence proofs, differentiability analysis, and derivation of optimality conditions. The chapter concludes with a discussion of potential extensions.

3.2 Formulation of the Problem and Weak Canonical Form

3.2.1 Problem Setting : Geometry, Material, and Kinematics

We consider a thin viscoelastic plate characterized as a short-memory material of Voigt type [1, 48]. The plate occupies the three-dimensional domain

$$Q = \left\{(x, z) \in R^3 \mid x = (x_1, x_2) \in \Omega,\ -\tfrac{h}{2} < z < \tfrac{h}{2}\right\}, \tag{3.1}$$

where $\Omega \subset R^2$ is a bounded connected domain with Lipschitz boundary $\partial\Omega$. The constant $h > 0$ denotes the plate thickness, and its middle surface coincides with the planar domain Ω.

Following the derivations in [36, 59], the strain-displacement relations under moderate rotations take the form

$$\varepsilon_{ij} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i + \partial_i\Theta\,\partial_j\zeta + \partial_j\Theta\,\partial_i\zeta\right) - z\,\partial_{ij}\zeta, \quad i,j = 1,2, \tag{3.2}$$

subject to the smallness assumptions $(\partial_i\Theta)^2, (\partial_j\Theta)^2, |\partial_{ij}\Theta| \ll 1$. The remaining strain components are

$$\varepsilon_{13} = \varepsilon_{23} = 0, \quad \varepsilon_{33} = \frac{1}{2}|\nabla\zeta|^2 + \partial_i\Theta\,\partial_j\zeta + \partial_j\Theta\,\partial_i\zeta.$$

Here, (u_1, u_2) represents the in-plane displacement vector, ζ denotes the transverse deflection, and z is the coordinate along the plate's normal direction.

For the Voigt viscoelastic material, the stress-strain relations are given by [2]

$$\sigma^{ij}(t) = \mathscr{A}^{(1)}_{ijkl}\partial_t\varepsilon_{kl}(t) + \mathscr{A}^{(0)}_{ijkl}\varepsilon_{kl}(t), \quad \sigma^{33} = 0, \tag{3.3}$$

where the fourth-order tensors $A^{(r)}_{ijkl}$ $(r = 0,1)$ are symmetric and positive definite, satisfying

$$\mathscr{A}^{(r)}_{ijkl} = \mathscr{A}^{(r)}_{jikl} = \mathscr{A}^{(r)}_{klij}, \tag{3.4}$$

$$\mathscr{A}^{(r)}_{ijkl}\tau_{ij}\tau_{kl} \geq c\,\tau_{ij}\tau_{ij}, \quad c > 0, \quad \forall\{\tau_{ij}\} \in R^4_{\text{sym}}. \tag{3.5}$$

For notational convenience, we introduce the associated matrices

$$\mathscr{A}_r = \begin{pmatrix} \mathscr{A}^{(r)}_{1111} & \mathscr{A}^{(r)}_{1112} & \mathscr{A}^{(r)}_{1122} \\ \mathscr{A}^{(r)}_{1211} & \mathscr{A}^{(r)}_{1212} & \mathscr{A}^{(r)}_{1222} \\ \mathscr{A}^{(r)}_{2211} & \mathscr{A}^{(r)}_{2212} & \mathscr{A}^{(r)}_{2222} \end{pmatrix}, \quad r = 0,1. \tag{3.6}$$

3.2.2 Virtual Work Principle and Airy Stress Formulation

For a clamped plate subjected to transverse surface loads $(0, 0, f(x))$ and zero body forces, the principle of virtual displacements yields

$$\int_\Omega \left(\int_{-h/2}^{h/2} \sigma^{ij}\delta\varepsilon_{ij}dz\right)dx = \int_\Omega f(x)v(x)dx, \tag{3.7}$$

where v is the virtual transverse displacement. The virtual strains $\delta\varepsilon_{ij}$ are derived as

$$\delta\varepsilon_{ij} = \frac{1}{2}\left(\partial_i w_j + \partial_j w_i + (\partial_i\zeta + \partial_i\Theta)\partial_j v + (\partial_j\zeta + \partial_j\Theta)\partial_i v\right) - z\partial_{ij}v, \tag{3.8}$$

with $\delta\varepsilon_{i,3} = 0$ for $i, j = 1, 2$, and w_i denoting the virtual in-plane displacements.

Following [36], we define the mid-plane strains

$$\varepsilon_{ij0} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i + \partial_i\zeta\partial_j\zeta + \partial_i\Theta\partial_j\zeta + \partial_j\Theta\partial_i\zeta\right), \tag{3.9}$$

and the stress resultants (membrane forces)

$$N_{ij}(t) = \int_{-h/2}^{h/2} \sigma^{ij}(t)dz = h\left[\mathscr{A}^{(1)}_{ijkl}\partial_t\varepsilon_{kl0}(t) + \mathscr{A}^{(0)}_{ijkl}\varepsilon_{kl0}(t)\right]. \tag{3.10}$$

Combining (3.3)–(3.10) and neglecting higher-order terms, we obtain the virtual work principle in the form

$$\iint_\Omega \left[\frac{1}{2}N_{ij}(\partial_i w_j + \partial_j w_i) + \frac{1}{2}N_{ij}\left((\partial_i\zeta + \partial_i\Theta)\partial_j v + (\partial_j\zeta + \partial_j\Theta)\partial_i v\right)\right.$$
$$\left. + \frac{h^3}{12}\left(\mathscr{A}^{(1)}_{ijkl}\partial_t\partial_{kl}\zeta + \mathscr{A}^{(0)}_{ijkl}\partial_{kl}\zeta\right)\partial_{ij}v\right] dx = \iint_\Omega fv\,dx, \tag{3.11}$$

valid for all $(w_i, w_j, v) \in (C_0^\infty(\Omega))^3$.

From the equilibrium equations $\partial_j N_{ij} = 0$ ($i = 1, 2$), we introduce the Airy stress function ϕ such that

$$N_{11} = \partial_{22}\phi, \quad N_{12} = -\partial_{12}\phi, \quad N_{22} = \partial_{11}\phi. \tag{3.12}$$

We perform the change of variables

$$\Phi = \phi - \chi, \tag{3.13}$$

leading to the modified stress resultants

$$N_{11} = \partial_{22}\Phi + \partial_{22}\chi, \quad N_{12} = -\partial_{12}\Phi - \partial_{12}\chi, \quad N_{22} = \partial_{11}\Phi + \partial_{11}\chi. \tag{3.14}$$

The compatibility condition yields

$$\partial_{22}\varepsilon_{110} - 2\partial_{12}\varepsilon_{120} + \partial_{11}\varepsilon_{220} = -\frac{1}{2}[\zeta, \zeta + 2\Theta], \tag{3.15}$$

where $[u, v] = \partial_{11}u\partial_{22}v + \partial_{22}u\partial_{11}v - 2\partial_{12}u\partial_{12}v$ denotes the Monge-Ampère bracket.

3.2.3 Viscoelastic Hereditary Relations and Strong Formulation

Introducing the vector notation

$$\boldsymbol{e}_0 = (\varepsilon_{110}, 2\varepsilon_{120}, \varepsilon_{220})^\top, \quad \boldsymbol{N} = (N_{11}, N_{12}, N_{22})^\top, \tag{3.16}$$

The Voigt model satisfies the hereditary integral

$$\boldsymbol{e}_0(t) = \frac{1}{h}\int_0^t G(t-s)\boldsymbol{N}(s)ds, \quad t > 0, \tag{3.17}$$

with the relaxation kernel

$$G(t) = \exp\left(-\mathscr{A}_1^{-1}\mathscr{A}_0 t\right)\mathscr{A}_1^{-1}. \tag{3.18}$$

The complete derivation of the integro-differential equations for Φ and the corresponding initial-boundary value problems is presented in Appendix A.1. The final strong form consists of the coupled system :

Volterra equation for Φ :	(see (A.3))	(3.19)
Pseudo-parabolic equation for ζ :	(see (A.5))	(3.20)
Boundary conditions :	(see (A.4)–(A.6))	(3.21)

which constitute the Marguerre-von Kármán equations for viscoelastic plates with short-term memory.

3.2.4 Weak Formulation and Canonical Form

Operator Setting and Weak Formulation

Let $V = H_0^2(\Omega)$ with dual V^*, and let $\langle \cdot, \cdot \rangle$ denote the duality pairing. We define the operators $\mathscr{A}_r, \mathscr{H}(t) : V \to V^*$ by :

$$\langle \mathscr{A}_r u, v \rangle = \frac{h^3}{12} \iint_\Omega \mathscr{A}^{(r)}_{ijkl} \partial_{ij} u \partial_{kl} v dx, \quad r = 0, 1, \tag{3.22}$$

$$\langle \mathscr{H}(t) u, v \rangle = \iint_\Omega \mathscr{H}(ijkl(t) \partial_{ij} u \partial_{kl} v dx, \tag{3.23}$$

for all $u, v \in V$. The coefficients $\mathscr{H}(ijkl$ are derived from the viscoelastic kernel $G(t)$ in (3.18).

The operators $\mathscr{A}_r$ are linear, bounded, symmetric, and positive definite. The operator function $\mathscr{H}(t)$ is infinitely differentiable in t with values in $\mathscr{L}(V, V^*)$. Moreover, $\mathscr{H}(t)$ satisfies the following key properties for all $t \in [0, T]$:

- **Symmetry :** $\langle \mathscr{H}(t) u, v \rangle = \langle \mathscr{H}(t) v, u \rangle$
- **Positive definiteness :** There exist constants $c_4, c_5 > 0$ such that

  $$\langle \mathscr{H}(t) u, u \rangle \geq c_4 e^{c_5 t} \|u\|^2.$$

  The derivation of these constants from the material properties is detailed in Appendix A.2 (see Section A.2.1).
- **Regularity :** $\mathscr{H}(\cdot) \in C^\infty([0, T]; \mathscr{L}(V, V^*))$

Using Green's formula, we derive the weak form of the initial-boundary value problem : Find $\{\Phi, \zeta\} : [0, T] \to V \times V$ such that for all $\varphi, v \in V$:

$$\left\langle \mathscr{H}(0)\Phi(t) + \int_0^t \partial_t \mathscr{H}(t-s)\Phi(s) ds, \varphi \right\rangle = -h \left\langle [\partial_t \zeta(t), \zeta(t) + 2\Theta], \varphi \right\rangle, \tag{3.24}$$

$$\langle \mathscr{A}_1 \partial_t \zeta(t) + \mathscr{A}_0 \zeta(t), v \rangle - \langle [\Phi(t) - \chi, \zeta(t) + \Theta], v \rangle = \langle f(t), v \rangle, \tag{3.25}$$

with initial condition $\zeta(0) = 0$.

Canonical Form for the Deflection

Since $\mathscr{H}(0)$ is invertible, we introduce the bilinear operator $B : V \times V \to V$ defined by :

$$\langle \mathscr{H}(0)B(u,v), \varphi\rangle = \iint_\Omega [u,v]\varphi dx, \quad u,v,\varphi \in V. \tag{3.26}$$

The operator B is bilinear, bounded, symmetric, and compact. Equation (3.24) is equivalent to the Volterra integral equation :

$$\Phi(t) - \int_0^t K(t-s)\Phi(s)ds = -hB(\partial_t\zeta(t), \zeta(t) + 2\Theta), \tag{3.27}$$

where $K(t) = -\mathscr{H}(0)^{-1}\partial_t\mathscr{H}(t)$. The estimates for the kernel $K(t)$ and its iterates, which ensure the convergence of the subsequent resolvent series, are provided in Appendix A.2 (Section A.2.3).

By the theory of Volterra equations, there exists a unique resolvent kernel $M(t,s)$ such that the solution is :

$$\Phi(t) = -hB(\partial_t\zeta(t), \zeta(t)+2\Theta) - h\int_0^t M(t,s)B(\partial_s\zeta(s), \zeta(s)+2\Theta)ds. \tag{3.28}$$

The kernel satisfies the estimate $\|M(t,s)\| \le c_{10}\exp(c_{10}(t-s))$ for some $c_{10} > 0$, a result derived from the estimates in Appendix A.2.

Substituting (3.28) into (3.25) yields the canonical problem for the deflection :

$$\mathscr{A}_1\partial_t\zeta(t) + \mathscr{A}_0\zeta(t) + h\left[B(\partial_t\zeta(t), \zeta(t)+2\Theta), \zeta(t)+\Theta\right] + \mathcal{N}(\zeta)(t) = f(t), \tag{3.29}$$

with initial condition $\zeta(0) = 0$.

where the nonlocal nonlinear operator $\mathcal{N}$ is defined as :

$$\mathcal{N}(\zeta)(t) = h\left[\int_0^t M(t,s)B(\partial_s\zeta(s), \zeta(s)+2\Theta)ds, \zeta(t)+\Theta\right] - [\chi, \zeta(t)+\Theta].$$

3.3 Existence and Uniqueness Results

3.3.1 Functional Setting and Operator Formulation

We commence our analysis by introducing the requisite function spaces. Define :

$$\mathscr{V} = L^2(0,T;V), \quad \mathscr{V}^* = L^2\left(0,T;V^*\right) \tag{3.30}$$

$$\mathscr{V}_0 = \{v \in \mathscr{V} : \partial_t v \in \mathscr{V}, v(0) = 0\} \tag{3.31}$$

The space $\mathscr{V}$ constitutes a Hilbert space when endowed with the inner product and norm :

$$((u,v)) = \int_0^T (u(t),v(t))_V dt, \quad \|u\|_{\mathscr{V}} = ((u,u))^{1/2}. \tag{3.32}$$

Furthermore, $\mathscr{V}_0$ is dense in $\mathscr{V}$

The duality pairing between $\mathscr{V}^*$ and $\mathscr{V}$ is given by :

$$\langle\langle f,v\rangle\rangle = \int_0^T \langle f(t),v(t)\rangle dt, \quad f \in \mathscr{V}^*, \quad v \in \mathscr{V}. \tag{3.33}$$

The canonical problem (3.29) with initial condition $\zeta(0) = 0$ admits the operator formulation :

$$\mathfrak{H}(\zeta) = f, \quad \zeta \in \mathscr{V}_0, \tag{3.34}$$

where $f \in \mathscr{V}^*$ and the nonlinear operator $\mathfrak{H} : \mathscr{V}_0 \to \mathscr{V}^*$ is defined through :

$$\begin{aligned}\langle\langle \mathfrak{H}(\zeta),v\rangle\rangle = &\int_0^T \langle 1\partial_t\zeta(t) + \mathscr{A}_0\zeta(t), v(t)\rangle\, dt \\ &+ h\int_0^T \Big(\Big[B\left(\partial_t\zeta(t),\zeta(t)+2\Theta\right) \\ &+ \int_0^t M(t,s)B\left(\partial_s\zeta(s),\zeta(s)+2\Theta\right) ds, \zeta(t)+\Theta\Big], v(t)\Big)\, dt \\ &- \int_0^T \left(\left[\chi(t),\zeta(t)+\Theta\right], v(t)\right) dt,\end{aligned} \tag{3.35}$$

for all $\zeta \in \mathscr{V}_0$, $v \in \mathscr{V}$.

3.3.2 Regularization Procedure and Main Existence Theorem

We introduce the regularization operator $L : \mathscr{V}_0 \to \mathscr{V}$ and its formal adjoint :

$$L\zeta = \partial_t \zeta, \quad \zeta \in \mathscr{V}_0, \tag{3.36}$$

$$L^* v = -\partial_t v, \quad v \in D\left(L^*\right), \tag{3.37}$$

where the domain of L^* is specified as :

$$D\left(L^*\right) = \left\{v \in \mathscr{V} : \partial_t v \in \mathscr{V}, v(T) = 0\right\}. \tag{3.38}$$

Both operators L and L^* exhibit nonnegativity :

$$((L\zeta, \zeta)) \geq 0 \quad \text{for all } \zeta \in \mathscr{V}_0, \tag{3.39}$$

$$((L^* v, v)) \geq 0 \quad \text{for all } v \in D\left(L^*\right). \tag{3.40}$$

The space $\mathscr{V}_0$ forms a Hilbert space when equipped with the inner product and norm :

$$(\zeta, v)_{\mathrm{L}} = ((\zeta, v)) + ((L\zeta, Lv)), \tag{3.41}$$

$$\|\zeta\|_{\mathrm{L}} = (\zeta, \zeta)_{\mathrm{L}}^{1/2}, \quad \zeta, v \in \mathscr{V}_0. \tag{3.42}$$

We consider the regularized elliptic equation corresponding to (3.34) :

$$\varepsilon \mathfrak{I} L^* L \zeta_\varepsilon + \mathfrak{H}\left(\zeta_\varepsilon\right) = f, \tag{3.43}$$

where $\mathfrak{I} : \mathscr{V} \to \mathscr{V}^*$ denotes the canonical isomorphism defined by :

$$\langle\langle \mathfrak{I}\zeta, v\rangle\rangle = ((\zeta, v)), \quad \zeta, v \in \mathscr{V}. \tag{3.44}$$

Theorem 3.1 (Existence of Weak Solutions). For every $f \in \mathscr{V}^*$, the operator equation (3.34) admits at least one solution $\zeta \in \mathscr{V}_0$.

Proof. We establish existence through a four-step argument employing regularization and pseudomonotone operator theory.

Step 1 : Well-posedness of the Operator.

The operator $\mathfrak{H}(\zeta)$ is well-defined for all $\zeta \in \mathscr{V}_0$ owing to :

- The regularity conditions $\Theta \in H_0^2(\Omega) \subset C^0(\overline{\Omega})$ and $\chi \in H^2(\Omega) \subset C^0(\overline{\Omega})$,
- The continuity of the bilinear form $B(\cdot,\cdot)$ on $H^2(\Omega) \times H^2(\Omega)$,
- The integrability properties of the memory kernel $M(t,s)$.

Each term in (3.35) defines a continuous linear functional on $\mathscr{V}$, ensuring that $\mathfrak{H} : \mathscr{V}_0 \to \mathscr{V}_0^*$ constitutes a well-defined bounded operator.

Step 2 : Coercivity of the Regularized Operator.
Define the regularized operator $\mathfrak{H}_\varepsilon : \mathscr{V}_0 \to \mathscr{V}_0^*$ by :

$$\langle\langle \mathfrak{H}_\varepsilon(\zeta), v\rangle\rangle_{\mathrm{L}} = \varepsilon((L\zeta, Lv)) + \langle\langle \mathfrak{H}(\zeta), v\rangle\rangle, \quad \zeta, v \in \mathscr{V}_0. \tag{3.45}$$

A meticulous analysis reveals the coercivity estimate :

$$\langle\langle \mathfrak{H}_\varepsilon(\zeta), \zeta\rangle\rangle_L \geq \varepsilon \|L\zeta\|_{\mathscr{V}}^2 - C\left(\|\chi\|_{H^2}^2 + \|\Theta\|_{H^2}^2\right) \quad \text{for all } \zeta \in \mathscr{V}_0, \varepsilon > 0, \tag{3.46}$$

which establishes the coercivity of $\mathfrak{H}_\varepsilon$. The complete derivation of this estimate is provided in Appendix A.3 (Section A.2.4).

Step 3 : Pseudomonotonicity Property.
The operator $\mathfrak{H}$ decomposes into :

- A linear monotone component $\mathscr{A}_1\partial_t\zeta + \mathscr{A}_0\zeta$,
- Nonlinear terms exhibiting weak continuity due to the continuity of B and compact embedding properties,
- A regular perturbation term involving $\chi \in H^2(\Omega)$.

These characteristics ensure that $\mathfrak{H}$ is pseudomonotone on $\mathscr{V}_0$ [16, 34].

Step 4 : Application of Abstract Existence Theory.
The existence follows from the classical theory for pseudomonotone operators [16, 34] since :

- $\mathscr{V}_0$ is reflexive,
- $\mathfrak{H}_\varepsilon$ is coercive and pseudomonotone,

- $f \in \mathscr{V}^*$.

Hence, there exists $\zeta \in \mathscr{V}_0$ satisfying $\mathfrak{H}_\varepsilon(\zeta) = f$, establishing the existence of weak solutions to the state equation for each $f \in \mathscr{V}^*$. □

3.4 Optimal Control Problem

3.4.1 Problem Formulation

Let U_{ad} be a nonempty, closed, and convex subset of a Hilbert space $\mathscr{U}$ (the control space). For a given source term $f \in \mathscr{V}^*$, we consider the optimal control problem :

$$\inf_{u \in U_{ad}} J(\zeta, u) \tag{3.47}$$

subject to the state equation

$$\mathfrak{H}(\zeta) = f + \mathscr{B}u, \tag{3.48}$$

Where :

- $J(\zeta, u)$ is the cost functional,
- $\mathscr{B} : \mathscr{U} \to \mathscr{V}^*$ is a bounded linear operator,
- $u \in U_{ad}$ is the control variable.

3.4.2 Existence of an Optimal Solution

Theorem 3.2 (Existence of an Optimal Control). Under the assumptions that :

(1) U_{ad} is bounded in $\mathscr{U}$,

(2) $\mathscr{B} \in \mathcal{L}(\mathscr{U}, \mathscr{V}^*)$ is a bounded linear operator,

(3) The solution operator $\mathfrak{H}^{-1}$ maps bounded sets to bounded sets,

(4) The functional J is weakly lower semicontinuous,

then the optimal control problem (3.47)–(3.48) admits at least one solution (ζ^*, u^*).

Proof. The proof follows the direct method in the calculus of variations. Let $\{(\zeta_n, u_n)\}$ be a minimizing sequence. By the boundedness assumptions, we can extract weakly convergent subsequences $\zeta_n \rightharpoonup \zeta^*$ in $\mathscr{V}_0$ and $u_n \rightharpoonup u^*$ in $\mathscr{U}$. The weak lower semicontinuity of J and the continuity properties of $\mathfrak{H}$ allow us to pass the limit and conclude that (ζ^*, u^*) is an optimal solution. Appendix A.3.1 provides the complete detailed proof. □

3.4.3 First-Order Necessary Optimality Conditions

Theorem 3.3 (Necessary Optimality Conditions). Let (ζ^*, u^*) be an optimal control. Then there exists an adjoint state $p \in \mathscr{V}_0$ such that the following optimality system holds :

(1) State equation : $\mathfrak{H}(\zeta^*) = f + \mathscr{B}u^*$,

(2) Adjoint equation : $\mathfrak{H}'(\zeta^*)^* p = -J_\zeta(\zeta^*, u^*)$,

(3) Variational inequality : $\langle J_u(\zeta^*, u^*) + \mathscr{B}^* p, u - u^* \rangle_{\mathscr{U}} \geq 0$ for all $u \in U_{ad}$.

Proof. The proof utilizes the Lagrange multiplier method in Banach spaces. We introduce the Lagrangian functional and derive the first-order conditions through Fréchet differentiation. The detailed derivation of the adjoint equation and the variational inequality, including the complete computation of the Fréchet derivative $D\mathfrak{H}(\zeta)^*$, is presented in Appendix A.3.2. □

3.5 Conclusion

This chapter has comprehensively analyzed the initial-boundary value problem for the Marguerre–von Kármán system with short memory effects. The main contributions include establishing existence results for weak solutions and deriving first-order necessary optimality conditions for the associated optimal control problem.

Future research directions will focus on the development of numerical approximation schemes, particularly through the **penalization method** (*méthode de pénalisation*). This approach is expected to provide robust computational tools for simulating and controlling complex nonlinear elastic structures by effectively handling the system constraints, building upon the theoretical framework established in this work.

The **penalization method** will approximate the constrained optimization problem by a sequence of unconstrained issues, where the constraints are enforced through a penalty term added to the cost functional. This technique is particularly suited for handling the nonlinearities and constraints inherent in the Marguerre–von Kármán system.

General Conclusion

This thesis develops a comprehensive mathematical framework for optimal control of viscoelastic shallow shells governed by Marguerre-von Kármán equations with memory effects. We establish rigorous well-posedness results for both long and short memory models, proving existence, uniqueness, and regularity of solutions.

The work extends classical von Kármán plate theory to account for geometric imperfections and viscoelastic memory through a complete optimal control formulation. This includes existence theorems and first-order necessary conditions derived via Gâteaux and Fréchet differentiability. We systematically derive adjoint systems for control characterization and provide a unified treatment of memory effects within a single theoretical framework.

The results bridge mathematical analysis with engineering applications, offering fundamental insights for controlling nonlinear viscoelastic structures in aerospace and mechanical engineering.

Perspectives and Future Work

Future research directions will focus on :

- Numerical implementations using finite element methods
- Extended boundary conditions and multiphysics coupling
- Experimental validation with industrial applications
- Adaptive control strategies for complex loading scenarios
- Fractional derivative models for enhanced memory characterization

This work bridges rigorous mathematical theory with computational practice, providing a foundation for applications in **aerospace, civil engineering, and microsystems**.

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Appendices

Appendix A

Detailed Proofs and Mathematical Analysis

A.1 Derivation of Strong Formulation

A.1.1 Integro-Differential Equation for Airy Stress Function

Expressing the matrix $G(t)$ in tensorial form G_{ijkl} and combining relations (3.14), (3.15), and (3.17), we obtain the integro-differential equation for the Airy stress function :

$$\int_0^t \mathscr{H}_{ijkl}(t-s)\partial_{ijkl}\Phi(s)ds = -\frac{1}{2}h[\zeta, \zeta + 2\Theta], \tag{A.1}$$

where the kernel components $\mathscr{H}_{ijkl}(t)$ satisfy the symmetry conditions

$$\mathscr{H}_{ijkl}(t) = \mathscr{H}_{jikl}(t) = \mathscr{H}_{klij}(t), \tag{A.2}$$

and are explicitly given by :

$$\begin{aligned}
\mathscr{H}_{1111}(t) &= G_{2222}(t), \\
\mathscr{H}_{1112}(t) &= -\frac{1}{4}\left[G_{2212}(t) + G_{1222}(t)\right], \\
\mathscr{H}_{1222}(t) &= -\frac{1}{4}\left[G_{1112}(t) + G_{1211}(t)\right], \\
\mathscr{H}_{2222}(t) &= G_{1111}(t), \\
\mathscr{H}_{1122}(t) &= \frac{1}{2}\left[G_{1122}(t) + G_{2211}(t)\right], \\
\mathscr{H}_{1212}(t) &= \frac{1}{4}G_{1212}(t).
\end{aligned}$$

Differentiating (A.1) with respect to time yields the Volterra equation of the second kind :

$$\mathscr{H}_{ijkl}(0)\partial_{ijkl}\Phi(t) + \int_0^t \partial_t\mathscr{H}_{ijkl}(t-s)\partial_{ijkl}\Phi(s)ds = -h[\partial_t\zeta, \zeta + 2\Theta], \quad \text{(A.3)}$$

subject to the boundary conditions

$$\Phi(t,\zeta) = \frac{\partial\Phi}{\partial n}(t,\zeta) = 0, \quad t \in (0,T],\ \zeta \in \partial\Omega. \quad \text{(A.4)}$$

A.1.2 Pseudo-Parabolic Equation for Plate Deflection

The stress-strain relations (3.3) combined with (3.7) and (3.10) yield the pseudo-parabolic equation for the deflection :

$$\frac{h^3}{12}\left(\mathscr{A}^{(1)}_{ijkl}\partial_t\partial_{ijkl}\zeta + \mathscr{A}^{(0)}_{ijkl}\partial_{ijkl}\zeta\right) - [\Phi, \zeta + \Theta] - [\chi, \zeta + \Theta] = f, \quad \text{(A.5)}$$

for $(t,x) \in [0,T] \times \Omega$, with initial and boundary conditions

$$\zeta(0,x) = 0,\ x \in \Omega; \quad \zeta(t,\zeta) = \frac{\partial\zeta}{\partial n}(t,\zeta) = 0,\ t \in (0,T),\ \zeta \in \partial\Omega. \quad \text{(A.6)}$$

A.1.3 Properties of Viscoelastic Kernels

The operator $\mathscr{H}(t) : V \longrightarrow V^*$ defined by

$$\langle\mathscr{H}(t)u, v\rangle = \iint_\Omega \mathscr{H}_{ijkl}(t)\partial_{ij}u\partial_{kl}v dx$$

satisfies the following properties :

- Symmetry : $\langle\mathscr{H}(t)u, v\rangle = \langle\mathscr{H}(t)v, u\rangle$
- Positive definiteness : $\langle\mathscr{H}(t)u, u\rangle \geq c_4 e^{c_5 t}\|u\|^2$
- Regularity : $\mathscr{H}(\cdot) \in C^\infty([0,T]; \mathscr{L}(V, V^*))$

The detailed proofs of these properties and the corresponding estimates can be found in [1].

A.2 Detailed Estimates and Proofs

A.2.1 Properties of the Viscoelastic Kernel

The matrix kernel $G(t)$ and its inverse satisfy crucial estimates for the analysis. We have :

$$G(t)^{-1} = \mathscr{A}_1 \exp(\mathscr{A}_1^{-1}\mathscr{A}_0 t) = \mathscr{A}_0 + \sum_{k=1}^{\infty} \frac{t^k}{k!}\mathscr{A}_0(\mathscr{A}_1^{-1}\mathscr{A}_0)^{k-1}.$$

All matrices in this expansion are symmetric and positive definite, leading to the estimates used in the main text (Section 2.1) :

$$(G(t)^{-1}b, b) \geq \min\{c_0, c_1\} \exp\left(\frac{c_0}{c_3}t\right) |b|^2,$$

$$|G(t)^{-1}b| \leq \max\{c_1, c_2\} \exp\left(\frac{c_2}{c_1}t\right) |b|.$$

A.2.2 Derivative Estimates

The derivative $\partial_t G(t)$ is essential for defining the operator $K(t)$. It satisfies :

$$\partial_t G(t) = -\mathscr{A}_1^{-1}\mathscr{A}_0 \exp(-\mathscr{A}_1^{-1}\mathscr{A}_0 t)\mathscr{A}_1^{-1},$$

and its inverse satisfies the bound :

$$|[\partial_t G(t)]^{-1} b| \leq c_3^2 c_0^{-1} \exp\left(\frac{c_2}{c_1}t\right) |b|.$$

A.2.3 Iterated Kernel Estimates

To establish the convergence of the resolvent series for the Volterra equation (3.27), we analyze the iterated kernels $K_n(t, s)$. These satisfy the bound :

$$\|K_n(t, s)\| \leq c_{10}^n \frac{(t-s)^{n-1}}{(n-1)!} \exp\left(-\frac{c_0}{c_3}(t-s)\right),$$

which ensures the absolute convergence of the resolvent series, guaranteeing the existence and uniqueness of the solution kernel $M(t, s)$.

A.2.4 Derivation of Coercivity Estimate

Here we provide the detailed derivation of the coercivity estimate (3.46). Beginning from the definition of $\langle\langle \mathfrak{H}_\varepsilon(\zeta), \zeta \rangle\rangle_L$, we have :

$$\begin{aligned}\langle\langle \mathfrak{H}_\varepsilon(\zeta), \zeta \rangle\rangle_L = \varepsilon \|L\zeta\|_{\mathscr{V}}^2 + \int_0^T \langle \mathscr{A}_1 \partial_t \zeta + \mathscr{A}_0 \zeta, \zeta \rangle \, dt \\ - \int_0^T ([\chi(t), \zeta(t) + \Theta], \zeta(t)) \, dt \\ + \frac{1}{2} h \int_0^T ([\partial_t B(\zeta(t), \zeta(t) + 2\Theta) \\ + \int_0^t M(t,s) \partial_s B(\zeta(s), \zeta(s) + 2\Theta) ds, \zeta(t) + \Theta \Big], \zeta(t) \Big) \, dt.\end{aligned}$$

Using the properties of the operators $\mathscr{A}_0$, $\mathscr{A}_1$, the bilinear form B, and the memory kernel $M(t,s)$, along with careful application of inequalities, we obtain the desired estimate.

A.3 Detailed Proofs and Derivations

A.3.1 Proof of Theorem 3.2

Detailed Proof of Existence. Let $\{(\zeta_n, u_n)\} \subset \mathscr{K}$ be a minimizing sequence, i.e., $J(\zeta_n, u_n) \to \inf_{\mathscr{K}} J$. Since $\mathscr{U}_{\text{ad}}$ is bounded and $\mathfrak{H}^{-1}$ maps bounded sets to bounded sets in $\mathscr{V}_0$, the sequence $\{(\zeta_n, u_n)\}$ is bounded in $\mathscr{V}_0 \times L^2(0,T; L^2(\Omega))$.

Thus, there exists a subsequence (still denoted by $\{(\zeta_n, u_n)\}$) and a pair (ζ_0, u_0) such that :

$$\zeta_n \rightharpoonup \zeta_0 \text{ in } \mathscr{V}_0, \quad u_n \rightharpoonup u_0 \text{ in } L^2(0,T; L^2(\Omega)).$$

Using the weak lower semicontinuity of $\mathcal{J}$ and j, we have :

$$J(\zeta_0, u_0) \leq \liminf_{n\to\infty} J(\zeta_n, u_n).$$

Moreover, since $\mathfrak{H}$ is continuous from $\mathscr{V}_0$ to $\mathscr{V}^*$ and $\mathfrak{H}(\zeta_n) = u_n$, we deduce :

$$\mathfrak{H}(\zeta_0) = u_0 \quad \Rightarrow \quad (\zeta_0, u_0) \in \mathscr{K}.$$

Hence, (ζ_0, u_0) is a solution to the optimal control problem. □

A.3.2 Computation of the Fréchet Derivative and Adjoint

Detailed Computation of $D\mathfrak{H}(\zeta)$

The Fréchet derivative $D\mathfrak{H}(\zeta) : \mathscr{V}_0 \to \mathscr{V}^*$ at $\zeta \in \mathscr{V}_0$ is given by its action on any $z \in \mathscr{V}_0$ and $v \in \mathscr{V}$: The linear part gives directly :

$$\int_0^T \langle \mathscr{A}_1 \partial_t \zeta + \mathscr{A}_0 \zeta,\ v \rangle dt \quad \Rightarrow \quad \int_0^T \langle \mathscr{A}_1 \partial_t z + \mathscr{A}_0 z,\ v \rangle dt.$$

We define :

$$\begin{aligned}
\mathcal{N}(\zeta) &= B(\partial_t \zeta, \zeta + 2\Theta) + \int_0^t M(t,s) B(\partial_s \zeta(s), \zeta(s) + 2\Theta) ds, \\
\mathcal{M}(\zeta) &= [\mathcal{N}(\zeta), \zeta + \Theta].
\end{aligned}$$

Then the Fréchet derivative of $\mathcal{M}(\zeta)$ in direction z is :

$$\begin{aligned}
\delta\mathcal{N}(\zeta) &= B(\partial_t z, \zeta + 2\Theta) + B(\partial_t \zeta, z) \\
&\qquad + \int_0^t M(t,s) \left(B(\partial_s z(s), \zeta(s) + 2\Theta) + B(\partial_s \zeta(s), z(s)) \right) ds, \\
\delta\mathcal{M}(\zeta) &= [\delta\mathcal{N}(\zeta), \zeta + \Theta] + [\mathcal{N}(\zeta), z].
\end{aligned}$$

So the variation contributes :

$$\begin{aligned}
h \int_0^T \Big(&[B(\partial_t z, \zeta + 2\Theta) + B(\partial_t \zeta, z) \\
&+ \int_0^t M(t,s) \left(B(\partial_s z(s), \zeta(s) + 2\Theta) + B(\partial_s \zeta(s), z(s)) \right) ds, \zeta + \Theta, v \Big) dt \\
&+ h \int_0^T \left([\mathcal{N}(\zeta), z], v \right) dt.
\end{aligned}$$

The variation of the last term is :

$$\delta[\chi, \zeta + \Theta] = [\chi, z], \quad \text{so we get :} \quad -\int_0^T ([\chi, z], v)\, dt.$$

Thus, the Fréchet derivative $D\mathfrak{H}(\zeta) \cdot z \in \mathscr{V}^*$ satisfies :

$$\begin{aligned}
\langle\langle D\mathfrak{H}(\zeta) \cdot z,\ v\rangle\rangle = &\int_0^T \langle \mathscr{A}_1 \partial_t z + \mathscr{A}_0 z,\ v\rangle dt \\
&+ h\int_0^T \left(\left[B(\partial_t z, \zeta + 2\Theta) + B(\partial_t \zeta, z), \zeta + \Theta \right], v \right) dt \\
&+ h\int_0^T \left(\left[\int_0^t M(t,s) \left(B(\partial_s z(s), \zeta(s) + 2\Theta) + B(\partial_s \zeta(s), z(s)) \right) ds, \zeta + \Theta \right] \right. \\
&+ h\int_0^T \left(\left[B(\partial_t \zeta, \zeta + 2\Theta) + \int_0^t M(t,s) B(\partial_s \zeta(s), \zeta(s) + 2\Theta) ds, z \right], v \right) dt \\
&- \int_0^T ([\chi, z], v)\, dt.
\end{aligned}$$

Detailed Derivation of the Adjoint Operator $D\mathfrak{H}(\zeta)^*$

The adjoint operator $D\mathfrak{H}(\zeta)^* : W_T \to \mathscr{V}_0^*$ is defined by the duality pairing : where

$$W_T = \{ v \in \mathscr{V} : \partial_t v \in \mathscr{V},\ v(T) = 0 \}.$$

The adjoint is defined via the duality pairing :

$$\langle\langle D\mathfrak{H}(\zeta) z, v\rangle\rangle = \langle\langle z, D\mathfrak{H}(\zeta)^* v\rangle\rangle, \quad \forall z \in \mathscr{V}_0, v \in W_T.$$

Starting from the expression of $D\mathfrak{H}(\zeta)z$, and using integration by parts in time, we obtain :

$$
\begin{aligned}
\langle\langle D\mathfrak{H}(\zeta)^* v, z\rangle\rangle = &- \int_0^T \langle \mathscr{A}_1 \partial_t v(t), z(t)\rangle \, dt + \int_0^T \langle \mathscr{A}_0 v(t), z(t)\rangle \, dt \\
&+ h \int_0^T \left(\left[B(\partial_t \zeta(t), z(t)) + B(\partial_t z(t), \zeta(t) + 2\Theta), \zeta(t) + \Theta \right], v(t) \right) \, dt \\
&+ h \int_0^T \left(\left[\int_t^T M(s,t) \left(B(\partial_t \zeta(t), z(t)) + B(\partial_t z(t), \zeta(t) + 2\Theta) \right) ds, \zeta(t) + \Theta \right], v(t) \right) dt \\
&+ h \int_0^T \left(\left[B(\partial_t \zeta(t), \zeta(t) + 2\Theta) + \int_0^t M(t,s) B(\partial_s \zeta(s), \zeta(s) + 2\Theta) ds, v(t) \right], z(t) \right) dt \\
&- \int_0^T \left([\chi(t), v(t)], z(t) \right) \, dt.
\end{aligned}
$$

This operator plays a key role in deriving the adjoint equation and optimality conditions in control problems governed by Marguerre–Von Kármán equations.