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On Some Fractional Problems with Variable Order

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Dedication

Would like to dedicate this work to my family, whose endless love, support, and encouragement have been the foundation of my success.
To my dear parents, my dear grandmother.
To my brothers Ahmed, Bachir, Sedik and my sisters Fatima Zohra and Khedidja.
To my dear uncles, aunts, and extended family members.
Thank you for believing in me even during the toughest times. This achievement is as much yours as it is mine.
To my friends Abir, Ahlem, Wafa, Radia, Fatna, Noussaiba, and Meriem whose constant companionship, wisdom, and laughter have made this journey unforgettable. Your encouragement and belief in me have been a source of strength.

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ملخص

في هذه الأطروحة، ندرس وجود ووحدانية الحلول لأنواع متعددة من المسائل الابتدائية والحدية غير الخطية التي تتضمن معادلات تفاضلية كسرية من نوع ريمان-ليوفيل ذات رتب متغيرة. يعتمد نهجنا على نظريات النقطة الثابتة (نظريات كراسنوسيلسكي، وشيفر، وشودر لإثبات الوجود ونظرية باناخ لإثبات الوحدانية) لتأسيس النتائج بشكل رياضي دقيق.

بالإضافة إلى ذلك، نقوم بتحليل استقرار الحلول التي تم الحصول عليها في سياق استقرار أولام-هايرز. ولتوضيح قابلية تطبيق نتائجنا، نقدم أمثلة عملية.

منهجيتنا مباشرة وتستخدم مؤثرًا كسريًا جديدًا يكون أكثر ملاءمة للمسألة قيد الدراسة، مما يتيح إمكانية الحل تحت افتراضات أقل تقييدًا. على عكس الطرق الموجودة في الأدبيات، التي تعتمد غالبًا على فترات معممة ودوال ثابتة على اجزاء، فإن منهجنا يوفر إطارًا أكثر مباشرة ومرونة.

الكلمات المفتاحية :
وجود ووحدانية الحلول، مسائل القيم الابتدائية والحدية، المعادلات التفاضلية الكسرية ذات الرتب المتغيرة غير المستقلة، نظرية النقطة الثابتة، المشتقات المختلطة، استقرار أولام-هايرز.

Abstract

In this thesis, we investigate the existence and uniqueness of solutions for several types of nonlinear initial and boundary value problems involving Riemann-Liouville fractional differential equations of variable order. Our approach relies on fixed point theorems (Krasnoselskii's, Schaefer's, and Schauder's theorems for existence and Banach's theorem for uniqueness) to rigorously establish the results.

Additionally, we analyze the stability of the obtained solutions in the context of Ulam-Hyers stability. To illustrate the applicability of our findings, we provide illustrative examples.

Our methodology is straightforward and employs a novel fractional operator that is more suitable for the considered problems, enabling solvability under less restrictive assumptions. Unlike existing methods in the literature, which often rely on generalized intervals and piecewise constant functions, our approach offers a more direct and flexible framework.

Keywords and phrases: Existence and uniqueness of solutions, initial and terminal value problems, fractional differential equations with non-autonomous variable order, fixed point theorem, mixed derivatives, Ulam-Hyers stability.

Résume

Dans cette thèse, nous étudions l'existence et l'unicité des solutions pour plusieurs types de problèmes non linéaires aux limites et aux valeurs initiales, impliquant des équations différentielles fractionnaires de Riemann-Liouville d'ordre variable. Notre approche s'appuie sur des théorèmes de point fixe (les théorèmes de Krasnoselskii, Schaefer et Schauder pour l'existence et le théorème de Banach pour l'unicité) afin d'établir rigoureusement les résultats.

De plus, nous analysons la stabilité des solutions obtenues dans le cadre de la stabilité d'Ulam-Hyers. Pour illustrer la pertinence de nos résultats, nous fournissons des exemples concrets.

Notre méthode est directe et repose sur un nouvel opérateur fractionnaire, plus adapté au problème considéré, permettant de démontrer la résolubilité sous des hypothèses moins restrictives. Contrairement aux techniques existantes dans la littérature, qui s'appuient souvent sur des intervalles généralisés et des fonctions constantes par morceaux, notre approche offre un cadre plus souple et plus intuitif.

Mots clés: Existence et unicité des solutions, problèmes aux valeurs initiales et terminales, équations différentielles fractionnaires avec ordre variable non autonome, théorème du point fixe , dérivées mixtes, stabilité d'Ulam-Hyers.

Publications and Communications of the thesis

International Publications

1. A. Benkerrouche, **S. Guedim,** S. N. Kottakkaran, A. Amara, M. S. Souid, H. Gunerhan, Fractional differential equations of non-autonomous variable-order. *Journal of Computational and Applied Mathematics*, **478**, 117235, (2025). https://doi.org/10.1016/j.cam.2025.117235.

2. M. Bensaid, M. S. Souid, A. Benkerrouche, **S. Guedim,** A. Amara, Initial and terminal conditions for differential equations of fractional derivative via non-autonomous variable order. *Palestine Journal of Mathematics*, **14(4)**, (2025), 187-205.

3. **S. Guedim,** A. Benkerrouche, S. G. Özyurt, M. S. Souid, S. Sabit, Initial and terminal value problem for fractional differential equations of variable order. *Filomat* **38:33**, (2024), 11805-11821. https://doi.org/10.2298/FIL2433805G.

4. **S. Guedim,** A. Benkerrouche, K. Sitthithakerngkiet, M. S. Souid, A. Amara, Initial value problem for mixed differential equations of variable order with finite delay. *Symmetry*, **17**, 295, (2025), 1-13. https:// doi.org/10.3390/sym17020295.

5. **S. Guedim,** A. Benkerrouche, M. S. Souid, A. Amara, J. Rashid, A. Imtiaz, Initial value problem for fractional differential equations of variable order. *Mathematical Modelling and Control*, **5(4)**, (2025), 379-389. https://doi.org/10.3934/mmc.2025026.

6. M. S. Souid, A. Benkerrouche, **S. Guedim,** S. Pinelas, A. Amara, Solvability of boundary value problems for differential equations combining ordinary and fractional derivatives of non-autonomous variable order. *Symmetry*, **17**, 184, (2025), 1-15. https://doi.org/10.3390/sym17020184.

7. M. S. Souid, **S. Guedim,** S. Boulaaras, A. Benkerrouche, A. Amara, T. Radwan, R. Jan, New approaches to solving initial value problems of mixed differential equations. *Filomat*, **40:2**, (2026), 649-659. https://doi.org/10.2298/FIL2602649S.

8. **S. Guedim,** A. Benkerrouche, M. S. Souid, A. Amara, H. Emadifar, Functional differential Equations with finite delay. (Submitted).

9. F. Li, A. Benkerrouche, **S. Guedim,** H. M. Baskonusc, M. S. Souid, A. Amara, Initial and terminal value problem for variable order differential equations with ordinary and fractional derivatives. (Submitted).

International Communications

1. Boundary value problems of Hadamard fractional differential equations of variable order. 1^{st} international conference on nonlinear mathematical analysis and its applications (IC-NMAA'24), 14-15 May, 2024, Bordj Bou Arréridj-Algeria.

2. On the boundary value problems of fractional differential equations of variable order. First international conference on applied mathematics and mathematical modeling $<< MAMM'2024 >>$, 12 november, 2024, El tarf-Algeria.

3. Uniqueness and stability of Hadamard variable-order boundary value problem. The 1st international conference Mohand Moussaoui on applied mathematics and modelling, 19-20 november, 2024, Guelma-Algeria.

4. New results for the integral equation with Hadamard derivatives of variable order. The international conference on mathematics and its applications in science and technology (ICMAST'2024), 15-16 december, 2024, Setif-Algeria.

National Communications

1. Existence and stability of Hadamard variable-ordre boundary value problem. National conference of applied sciences and engineering NCASE'24, 17-18 November, 2024 ENSTA-Algeria.

2. The existence and stability of fractional differential equation of variable-order. 1^{st} national conference on applied mathematics and artificial intelligence (NCAMAI

2025): fundamentals and modern applications, 4-5 May, 2025, Skikda-Algeria.

Contents

Introduction

The concept of fractional-order integration and differentiation dates back to the 16th century, though it wasn't until the 19th century that significant progress was made by Augustin-Louis Cauchy and Liouville, leading to the formal development of fractional derivatives and integrals. Since then, fractional calculus has found widespread applications across various fields.

Fractional calculus and fractional differential equations have gained considerable attention in recent years due to their broad applications in mathematical modeling. Unlike classical differential operators, fractional derivatives often provide more accurate representations of real-world phenomena, particularly in natural and biological systems. Below is a summary of key contributions in this field:

- **Ahmad et al.** [2] Studied Ulam-Hyersstability in nonlinear neutral stochastic fractional systems.

- **Aydoğan et al.** [3] Applied the Caputo–Fabrizio derivative to model rabies, solving the system via the Laplace-Adomian decomposition method.

- **Baleanu et al.** [4] Examined fractional boundary value problems in glucose molecule graph representations using novel vertex-labeling techniques.

- **Dehingia et al.** [17] Proposed a fractional-order epidemiological model for SARS-CoV-2 within-host dynamics, ensuring solution existence, non-negativity, and boundedness.
- **Hussain et al.** [23] Formulated stochastic models for COVID-19 spread under environmental noise, examining solution existence, disease persistence, and extinction.
- **Tuan et al.** [40] Investigated COVID-19 transmission using Caputo derivatives, deriving feasibility regions, equilibrium points, and reproduction numbers (R_0) while proving solution uniqueness via fixed-point theory.

Recent advances in fractional calculus have introduced variable order fractional operators, extending beyond conventional fixed-order approaches. Unlike their constant-order counterparts, these operators permit continuous variation in differentiation and integration orders based on either dependent or independent variables [18]. This enhanced flexibility has established them as powerful tools for modeling complex phenomena across diverse domains including biological systems, mechanical engineering, control theory, and transport processes.

In addition to variable order operators, many real phenomena are influenced by time delays. Delays arise naturally in engineering systems with feedback, in biological processes with incubation times, and in epidemiological models where infection depends on past exposure. The presence of delays can drastically change the stability and qualitative behavior of solutions [25, 42]. When combined with variable-order derivatives, delays lead to models that are both realistic and mathematically rich, but their analysis requires new theoretical tools. Despite their importance, the fractional differential equations of variable order with finite delays remain underexplored in the literature [41, 46]. Although some recent studies have addressed various aspects of variable-order equations [9, 10, 36, 37], the comprehensive analysis of variable order fractional systems with delays remains an open research area.

The ability of variable order operators to describe evolutionary dynamics has generated significant research interest. Numerous studies have successfully applied these methods to engineering and physical system modeling [19, 39]. However, investigations into nonlinear differential equations incorporating variable fractional orders remain comparatively limited [5, 8, 11, 13]. A key methodological aspect in this emerging field involves the use of piecewise constant functions (PWCFs) for problem formulation, as implemented in several recent studies [6, 14, 28, 37].

The standard PWCF approach partitions the existence interval $[0, t]$ as:

$$M := \{T1 = [0, t_1], T2 = (t_1, t_2], \ldots, T\sigma = t_{\sigma-1}, t]\},$$

where $\sigma \in \mathbb{N}$. The associated piecewise constant function $\alpha(t) : [0, t] \to (0, 1]$ is defined by:

$$\alpha(t) = \sum_{i=1}^{\sigma} \alpha_i l_i(t), \quad t \in [0, t],$$

with constants $0 < \alpha_i \leq 1$ $(i = 1, \ldots, \sigma)$. Here, $t_0 = 0$ and $t_\sigma = t$, where l_i serves as an indicator function:

$$l_i(t) = \begin{cases} 1 & \text{for } t \in [t_{i-1}, t_i], \\ 0 & \text{otherwise.} \end{cases}$$

This methodology enables the transformation of variable order fractional problems into equivalent constant-order formulations through domain partitioning and operator localization. The technique has become fundamental to current research in the field, providing a practical framework for analyzing complex fractional systems with varying orders [16, 28, 37].

In this thesis, we challenge the conventional piecewise constant approximation approach by asking: what if we could handle variable orders directly and in an integrated manner, without the need for partitioning or approximation?

Our core objective is to develop an entirely new practical framework that eliminates the step of piecewise functions and interval splitting at its root. To achieve this, we introduce an innovative method that directly addresses the variable order while recognizing that the classical properties of fractional calculus, designed for constant orders, cannot be used. The key element of our strategy lies in developing a more flexible mathematical operator?building upon recent advances in variable-order fractional calculus [28, 29, 35, 38], that naturally represents the smooth and dynamic variation of the order while preserving mathematical rigor and applicability. This work represents a methodological shift rather than merely an improvement: it simplifies modeling by removing an artificial layer of complexity, enhances accuracy by dealing directly with the variable core of the problem, and broadens horizons by opening doors to modeling systems that were previously difficult to handle with partitioned methods. The outcomes of this vision extend beyond new equations, offering a step toward more flexible mathematics for understanding an increasingly dynamic world. The thesis is structured into four chapters that collectively define and present these contributions.

The first chapter introduces the notations, definitions, and preliminary concepts that are utilized throughout the thesis.

In Chapter 2, we apply the existence and uniqueness properties on the following initial and terminal value problem (ITVP for short) of variable order

$$\begin{cases} \mathbb{D}_{0^+}^{\alpha(t)} x(t) = f(t, x(t)), \ \ t \in \mathbb{A} := [0, T], \\ x(0) = 0, \ \ x(T) = 0, \end{cases} \tag{1}$$

where $0 < T < +\infty$, $1 < \alpha(t) < 2$, $f : \mathbb{A} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $\mathbb{D}_{0^+}^{\alpha(t)}$ is the Riemann Liouville fractional derivative of variable order $\alpha(t)$.

In Chapter 3, we examine the existence and uniqueness, in addition to the stability of the solutions of the next non autonomous variable order initial value problem (NAVOIVP for short).

$$\begin{cases} \mathbb{D}_{0^+}^{\alpha(t,x(t))} x(t) + \varsigma x^{'}(t) = f(t, x(t)), \ \ t \in \mathbb{A} := [0, T], \\ x(0) = 0, \end{cases} \tag{2}$$

where $0 < T < +\infty$, $\varsigma > 0$ and $\alpha : \mathbb{A} \times \mathbb{R} \to (0, 1)$, $f : \mathbb{A} \times \mathbb{R} \to \mathbb{R}$ are a continuous functions and $\mathbb{D}_{0^+}^{\alpha(t,x(t))}$ is the Riemann Liouville fractional derivative of variable order $\alpha(t, x(t))$.

In Chapter 4, we study the existence, uniqueness and stability to the following finite delay of variable ordre initial value problem (FDVOIVP)

$$\begin{cases} \mathbb{D}^{\alpha(t)} x(t) + \varsigma x'(t) = f(t, x_t), \ \ t \in \mathbb{A} := [0, T], \\ x(t) = \phi(t), \ \ t \in [-\bar{z}, 0], \end{cases} \tag{3}$$

where $\alpha : \mathbb{A} \to (0,1)$, $f : \mathbb{A} \times C([-\bar{z}, 0], \mathbb{R}) \to \mathbb{R}$ is a specific function , $\phi \in C([-\bar{z}, 0], \mathbb{R})$ with $\phi(0) = 0$. We designate by x_t the element of $C([-\bar{z}, 0], \mathbb{R})$, defined by

$$x_t(\theta) = x(t + \theta), \theta \in [-\bar{z}, 0],$$

for any function x defined on $[-\bar{z}, T]$ and any t in $\mathbb{A}$. Here, $\mathbb{D}^{\alpha(t)}$ is the variable order $\alpha(t)$ fractional derivative of Riemann-Liouville, and $x_t(.)$ is the state's history from time $t - \bar{z}$ to the present time t.

Chapter 1

Preliminary

In this chapter, we present the notations, definitions, and preliminary results that will be utilized throughout the thesis.

1.1 Notations and definitions

Note that the set $\mathbb{E} = C(\mathbb{A}, \mathbb{R})$ is a Banach space of continuous functions x from $\mathbb{A}$ to $\mathbb{R}$, with the norm given by

$$\|x\| = sup\{|x(t)|/t \in \mathbb{A}\}.$$

Additionally, $\mathbb{E}_1 = C(\mathbb{A}, \mathbb{R})$ is a Banach space of continuous functions x from $\mathbb{A}$ to $\mathbb{R}$, where $x(0) = x(T) = 0$ with the norm defined as

$$\|x\| = \sup_{t\in\mathbb{A}} |x(t)|.$$

And, we have $\mathbb{E}_2 = C([-\bar{z}, T], \mathbb{R})$ is a Banach space of continuous functions x from $[-\bar{z}, T]$ into $\mathbb{R}$ with a norm defined as

$$\|x\| = sup\{|x(t)|/t \in [-\bar{z}, T]\}.$$

Definition 1.1 *Let X a Banach space and $Q : X \to X$ an operator.*

1. *Q is continuous if for every sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $(x_n)_{n\in\mathbb{N}}$ converge to x in X, the sequence $(Qx_n)_{n\in\mathbb{N}}$ converge to Qx.*

2. *Q is compact if for every bounded $K\subset X$, $Q(K)$ is relatively compact.*

3. *Q is completely continuous if Q is continuous and if the image for every bounded K in X, $Q(K)$ isrelatively compact.*

Definition 1.2 *Let M a subset of $C(\mathbb{A}, \mathbb{R})$.*

1. *M is called equicontinuous if and only if for all $\varepsilon > 0$, there exists $\delta > 0$, for all $t_1, t_2 \in \mathbb{A}$:*

$$|| t_2 - t_1 || \leq \delta \Rightarrow || f(t_2) - f(t_1) || \leq \varepsilon, \ for\, all\ f \in M.$$

2. *M is called uniformly bounded if and only if :*
   there exists $c > 0 : || f(t) || \leq c$ for all $t \in \mathbb{A}$ and for all $f \in M$.

Theorem 1.1 *[27](Ascoli Arzela) Let M be a subset of $C(\mathbb{A}, \mathbb{R})$, M is relatively compact if :*

- *M is uniformly bounded.*

- *M is equicontinuous.*

1.2 Fractional calculus

1.2.1 Fractional calculus of variable order

Definition 1.3 *([31]) Let $\alpha : \mathbb{A} \to (0, \infty)$ be a continuous function. The left Riemann Liouville fractional integral of variable order $\alpha(t)$ for the function $x(t)$ is defined by*

$$\mathbb{I}_{0^+}^{\alpha(t)} x(t) = \int_0^t \frac{(t-s)^{\alpha(s)-1}}{\Gamma(\alpha(s))} x(s) ds, \ \ t > 0, \tag{1.1}$$

where $\Gamma(.)$ *denotes the Gamma function defined as follow*

$$\Gamma(t) = \int_0^\infty s^{t-1}e^{-s}ds \quad (Re(t) > 0), \forall t \in \mathbb{A}.$$

Definition 1.4 *([31]) Let* $\alpha : \mathbb{A} \to (n-1, n)$ *be a continuous function. The left Riemann Liouville fractional derivative of variable order* $\alpha(t)$ *for the function* $x(t)$ *is defined by*

$$\mathbb{D}_{0^+}^{\alpha(t)}x(t) = \Big(\frac{d}{dt}\Big)^n \mathbb{I}_{0^+}^{n-\alpha(t)}x(t) = \Big(\frac{d}{dt}\Big)^n \int_0^t \frac{(t-s)^{n-\alpha(s)-1}}{\Gamma(n-\alpha(s))}x(s)ds, \quad t > 0. \tag{1.2}$$

Remark 1.1 *([12]) For general functions* $\alpha(t)$, $v(t)$, *we notice that the semi group property does not hold, i. e:*

$$\mathbb{I}_{a^+}^{\alpha(t)}\mathbb{I}_{a^+}^{v(t)}x(t) \neq \mathbb{I}_{a^+}^{\alpha(t)+v(t)}x(t).$$

Lemma 1.1 *([45]) Let* $\alpha : \mathbb{A} \to (0, \infty)$ *be a continuous function. Then for* $y \in C_\sigma(\mathbb{A}, \mathbb{R}) = \{y(t) \in C(\mathbb{A}, \mathbb{R}),\ t^\sigma y(t) \in C(\mathbb{A}, \mathbb{R}),\ (0 < \sigma < 1)\}$, *the variable order fractional integral* $\mathbb{I}_{0^+}^{\alpha(t)}y(t)$ *exists for* $t \in \mathbb{A}$.

Lemma 1.2 *([45]) Let* $\alpha \in C(\mathbb{A}, (0, \infty))$ *be a continuous function. Then* $\mathbb{I}_{0^+}^{\alpha(t)}y(t) \in C(\mathbb{A}, \mathbb{R})$ *for* $y \in C(\mathbb{A}, \mathbb{R})$.

1.2.2 Fractional calculus of non-autonomous variable order

Definition 1.5 *[31] Let* $\alpha : \mathbb{A} \times \mathbb{R} \to (0, 1)$ *be a continuous function, the left Riemann Liouville fractional integral of variable order* $\alpha(t, x(t))$ *for the function* $x(t)$ *is defined by*

$$\mathbb{I}_{0^+}^{\alpha(t,x(t))}x(t) = \int_0^t \frac{(t-s)^{\alpha(s,x(s))-1}}{\Gamma(\alpha(s, x(s)))}x(s)ds, \ \ t > 0, \tag{1.3}$$

where $\Gamma(.)$ *denotes the Gamma function.*

Definition 1.6 *[31] Let* $\alpha : \mathbb{A} \times \mathbb{R} \to (0, 1)$ *be a continuous function, the left Riemann Liouville fractional derivative of variable order* $\alpha(t, x(t))$ *for the function* $x(t)$ *is defined by*

$$\mathbb{D}_{0^+}^{\alpha(t,x(t))}x(t) = \Big(\frac{d}{dt}\Big)\mathbb{I}_{0^+}^{1-\alpha(t,x(t))}x(t) = \Big(\frac{d}{dt}\Big) \int_0^t \frac{(t-s)^{-\alpha(s,x(s))}}{\Gamma(1-\alpha(s, x(s)))}x(s)ds, \quad t > 0. \tag{1.4}$$

Remark 1.2 *[44] For general functions $\alpha(t, x(t))$, $v(t, x(t))$, we notice that the semi group property doesn't hold, i. e:*

$$\mathbb{I}_{a^+}^{\alpha(t,x(t))}\mathbb{I}_{a^+}^{v(t,x(t))}x(t) \neq \mathbb{I}_{a^+}^{\alpha(t,x(t))+v(t,x(t))}x(t).$$

Lemma 1.3 *[45] Let $\alpha : \mathbb{A} \times \mathbb{R} \to (0, 1)$ be a continuous function, then for $y \in C_\delta(\mathbb{A}, \mathbb{R}) = \{y(t) \in C(\mathbb{A}, \mathbb{R}),\ t^\delta y(t) \in C(\mathbb{A}, \mathbb{R}),\ (0 \leq \delta < \min \alpha(t, x(t)))\}$, the variable order fractional integral $\mathbb{I}_{0^+}^{\alpha(t,x(t))}y(t)$ exists $\forall t \in \mathbb{A}$.*

Lemma 1.4 *[45] Let $\alpha \in C(\mathbb{A} \times \mathbb{R}, (0, 1])$ be a continuous function, then $\mathbb{I}_{0^+}^{\alpha(t,x(t))}y(t) \in C(\mathbb{A}, \mathbb{R})$ for $y \in C(\mathbb{A}, \mathbb{R})$.*

1.3 Some fixed point theorems

Theorem 1.2 *([32]) Suppose X is a Banach space. If $Q : X \to X$ is a completely continuous operator and $\ominus = \{x \in X : x = \lambda Qx, 0 < \lambda < 1\}$ is bounded, then Q has a fixed pointin X.*

Theorem 1.3 *([24]) Let X be a Banach space and $Q : X \to X$ be a mapping such that, Q^n is a contraction, for some $n \in N$. Then Q has a unique fixed point in X.*

Theorem 1.4 *[22] Let (X, d) a complete metric space. the application $Q : X \to X$ is contraction with Lipschitz constant k. So Q accept only one fixed point $x \in X$.*

Theorem 1.5 *[22]*
Let X a Banach space, $K \subset X$ un a subset convex, closed, bounded and non empty, and let $Q : K \to K$ an operator completely continuous. So Q accept at least one fixed point.

Theorem 1.6 *[1]*
Let K be a closed, bounded and convex subset of a real Banach space X and let Q_1 and Q_2 be operators on K satisfying the following conditions:

1. $Q_1(K) + Q_2(K) \subset K$,

2. *Q_1 is continuous on K and $Q_1(K)$ isrelatively compact subset of X,*

3. *Q_2 is a strict contraction on K, i.e, there exists $p \in [0,1)$, such that*

$$\| Q_2(x) - Q_2(\tilde{y}) \| \leq p \| x - \tilde{y} \|,$$

for every $x, \tilde{y} \in K$.

Then, there exists $x \in K$ such that $Q_1(x) + Q_2(x) = x$.

1.4 The stability

Stability in math refers to a system's ability to resist change under small disturbances. It is crucial in differential equations, dynamical systems, and numerical methods, ensuring reliable solutions and accurate computations. Without stability, results become unpredictable, making it a key concept in modeling and analysis.

Definition 1.7 *[30] On account of NAVOIVP* (3.1)*, consider the inequality*

$$|D_{0^+}^{\alpha(t,r(t))} r(t) + \varsigma r^{'}(t) - f(t, r(t))| \leq \epsilon, \ t \in \mathbb{A}. \tag{1.5}$$

We say that NAVOIVP (3.1) *is Ulam-Hyers stable if there is $c_f > 0$ in a way that for any $\epsilon > 0$ and for any solution $r \in C(\mathbb{A}, \mathbb{R})$ of (1.5), there is a solution $x \in C(\mathbb{A}, \mathbb{R})$ of NAVOIVP* (3.1)*, such that*

$$|r(t) - x(t)| \leq c_f \epsilon, \ t \in \mathbb{A}.$$

Chapter 2

New Results to Solving Boundary Value Problems for Fractional Differential Equations with Variable Order [1]

2.1 Introduction

We apply existence and uniqueness criteria to the following variable order initial and terminal value problem (ITVP)

$$\begin{cases} \mathbb{D}_{0^+}^{\alpha(t)} x(t) = f(t, x(t)), \ \ t \in \mathbb{A} := [0, T], \\ x(0) = 0, \ \ x(T) = 0, \end{cases} \tag{2.1}$$

where $0 < T < +\infty$, $1 < \alpha(t) < 2$, $f : \mathbb{A} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $\mathbb{D}_{0^+}^{\alpha(t)}$ is the Riemann Liouville fractional derivative of variable order $\alpha(t)$.

[1] **S. Guedim,** A. Benkerrouche, S. G. Özyurt, M. S. Souid, S. Sabit, Initial and terminal value problem for fractional differential equations of variable order. *Filomat* **38:33**, (2024), 11805-11821. https://doi.org/10.2298/FIL2433805G.

2.2 Existence criteria

We begin by presenting the following assumptions.

(AS1) There exist constants $0 < \sigma < 1$, $\mathscr{P} > 0$, such that,

$t^{\sigma} \mid f\ (t, x(t))$ - $f\ (t, \mathrm{y}(t)) \mid \leq \mathscr{P} \mid x(t)$ - $\mathrm{y}(t)|, \quad \forall\ x, \mathrm{y} \in \mathbb{R}, t \in \mathbb{A}$.

(AS2) $\alpha : \mathbb{A} \to (1, \alpha^*]$ is a continuous function, such that, $1 < \alpha^* < 2$.

Remark 2.1 *1. The function* $\Gamma(2\text{-}\alpha(t))$ *is continuous as a composition of two continuous function. We set* $\mathcal{M}_\Gamma = \max_{t\in\mathbb{A}} \mid \frac{1}{\Gamma(2-\alpha(t))} \mid$.

2. By the continuity of the function $\alpha(t)$*, we let*

$T^{1-\alpha(t)} \leq 1$ *if* $1 \leq T < \infty$, $\quad T^{1-\alpha(t)} \leq T^{1-\alpha^*}$ *if* $0 \leq T \leq 1$.

We conclude that $T^{1-\alpha(t)} \leq \max\ (1, T^{1-\alpha^*}) = T^*$.

We will require the following lemma regarding the solution of the ITVP (2.1).

Lemma 2.1 *The ITVP (2.1) is equivalent to the following integral equation*

$$\int_0^t \frac{(t-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))} x(s) \ \ ds = \int_0^t (t-s) f(s, x(s)) \ \ ds$$

$$+ \frac{t}{T} \int_0^T \frac{(T-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))} x(s) \ \ ds - \frac{t}{T} \int_0^T (T-s) f(s, x(s)) \ \ ds, \tag{2.2}$$

such that, $x(0) = x(T) = 0$ *holds.*

proof

Using the definition of the fractional derivative of variable order provided in (1.2), the ITVP (2.1) can be expressed in the following form:

$$\frac{d^2}{dt^2} \int_0^t \frac{(t-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))} x(s) \ \ ds = f(t, x(t)).$$

Then,

$$\frac{d}{dt}\int_0^t \frac{(t-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))}x(s)\ ds = \int_0^t f(s,x(s))\ ds + c_1.$$

Thus,

$$\int_0^t \frac{(t-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))}x(s)\ \ ds = \int_0^t (t-s)f(s,x(s))\ ds + c_1 t + c_2. \tag{2.3}$$

Evaluating equation (2.3) at $t=0$ and $t=T$ gives us $c_2=0$ and

$$c_1 = \frac{1}{T}\Big[\int_0^T \frac{(T-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))}x(s)\ \ ds - \int_0^T (T-s)f(s,x(s))\ \ ds\Big].$$

Then,

$$\int_0^t \frac{(t-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))}x(s)\ \ ds = \int_0^t (t-s)f(s,x(s))\ ds + \frac{t}{T}\int_0^T \frac{(T-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))}x(s)\ \ ds$$
$$-\frac{t}{T}\int_0^T (T-s)f(s,x(s))\ \ ds.$$

On the other hand, by differentiating both sides of the equation (2.2), we obtain

$$\frac{d}{dt}\Big(\int_0^t \frac{(t-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))}x(s)ds\Big) = \int_0^t f(s,x(s))\ \ ds$$
$$+\frac{1}{T}\Big(\int_0^T \frac{(T-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))}x(s)\ \ ds - \int_0^T (T-s)f(s,x(s))\ \ ds\Big).$$

Differentiating once more, we obtain

$$\frac{d^2}{dt^2}\Big(\int_0^t \frac{(t-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))}x(s)ds\Big) = f(t,x(t)),$$

which results in the initial value problem (ITVP) (2.1). The first result follows from Theorem 1.2.

Theorem 2.1 *Assuming that the conditions (AS1) and (AS2) are satisfied, the initial value problem (ITVP) (2.1) has at least one solution on* $\mathbb{E}_1$.

proof We construct the following operator

$$C : \mathbb{E}_1 \to \mathbb{E}_1,$$

as follows,

$$Cx(t) = x(t) - \int_0^t \frac{(t-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))}x(s)\ \ ds + \int_0^t (t-s)f(s,x(s))\ \ ds$$

$$+\frac{t}{T}\int_0^T \frac{(T-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))}x(s)\ \ ds - \frac{t}{T}\int_0^T (T-s)f(s,x(s))\ \ ds.$$

Set

$$E_r = \{x \in \mathbb{E}_1, \| x \| < r, r > 0\}.$$

It is evident that E_r is a non empty, closed, and convex subset of $\mathbb{E}_1$.

Now, we will prove that the operator C satisfies the hypothesis of Theorem 1.2.

Step 1: C is continuous.

We assume that the sequence $(x_n)_{n\in\mathbb{N}}$ converges to x in $\mathbb{E}_1$. Then, we have

$$\mid Cx_n(t) - Cx(t) \mid$$

$$\begin{aligned}
\leq & \mid x_n(t) - x(t) \mid + \int_0^t \frac{(t-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))} \mid x(s) - x_n(s) \mid ds \\
+ & \int_0^t (t-s)s^{-\sigma}s^{\sigma} \mid f(s,x_n(s)) - f(s,x(s)) \mid ds \\
+ & \int_0^T \frac{(T-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))} \mid x_n(s) - x(s) \mid ds \\
+ & \int_0^T (T-s)s^{-\sigma}s^{\sigma} \mid f(s,x(s)) - f(s,x_n(s)) \mid ds \\
\leq & \| x_n - x \| + \mathcal{M}_\Gamma \| x_n - x \| \int_0^t (t-s)^{1-\alpha(s)} ds \\
+ & \mathscr{P} \| x_n - x \| \int_0^t (t-s)s^{-\sigma} ds + \mathcal{M}_\Gamma \| x_n - x \| \int_0^T (T-s)^{1-\alpha(s)} ds \\
+ & \mathscr{P} \| x_n - x \| \int_0^T (T-s)s^{-\sigma} ds \\
\leq & \| x_n - x \| + \mathcal{M}_\Gamma T^* \| x_n - x \| \int_0^t \left(\frac{t-s}{T}\right)^{1-\alpha^*} ds \\
+ & \mathscr{P} \| x_n - x \| \frac{t^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)} + \mathcal{M}_\Gamma T^* \| x_n - x \| \int_0^T \left(\frac{T-s}{T}\right)^{1-\alpha^*} ds
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \mathscr{P} \mid\mid x_n - x \mid\mid \frac{T^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)} \\
&\leq \quad \mid\mid x_n - x \mid\mid + \frac{\mathcal{M}_\Gamma T^*}{T^{1-\alpha^*}} \frac{(t)^{2-\alpha^*}}{(2-\alpha^*)} \mid\mid x_n - x \mid\mid + 2\mathscr{P} \mid\mid x_n - x \mid\mid \frac{T^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)} \\
&+ \quad \frac{\mathcal{M}_\Gamma T^*}{T^{1-\alpha^*}} \frac{(T)^{2-\alpha^*}}{2-\alpha^*} \\
&\leq \quad \mid\mid x_n - x \mid\mid + \frac{\mathcal{M}_\Gamma T T^*}{2-\alpha^*} \mid\mid x_n - x \mid\mid + 2\mathscr{P} \mid\mid x_n - x \mid\mid \frac{T^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)} \\
&+ \quad \frac{\mathcal{M}_\Gamma T T^*}{2-\alpha^*} \mid\mid x_n - x \mid\mid \\
&\leq \quad \mid\mid x_n - x \mid\mid + \frac{2\mathcal{M}_\Gamma T T^*}{2-\alpha^*} \mid\mid x_n - x \mid\mid + 2\mathscr{P} \mid\mid x_n - x \mid\mid \frac{T^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)} \\
&\leq \quad \left(1 + \frac{2\mathcal{M}_\Gamma T T^*}{2-\alpha^*} + 2\mathscr{P} \frac{T^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)}\right) \mid\mid x_n - x \mid\mid,
\end{aligned}
$$

which implies that,

$$
\mid\mid Cx_n(t) - Cx(t) \mid\mid \to 0, as \;\; n \to \infty.
$$

The above relation demonstrates that the operator C is continuous on $\mathbb{E}_1$.

Step 2: C maps bounded sets into bounded sets in $\mathbb{E}_1$.

Let $f^* = \sup_{t\in\mathbb{A}} \mid f(t,0) \mid$. Then, for $x \in E_r$, we have

$$
\mid Cx(t) \mid
$$

$$
\begin{aligned}
&\leq \quad \mid x(t) \mid + \int_0^t \frac{(t-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))} \mid x(s) \mid ds + \int_0^t (t-s) \mid f(s,x(s)) \mid ds \\
&+ \quad \frac{t}{T} \int_0^T \frac{(T-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))} \mid x(s) \mid ds + \frac{t}{T} \int_0^T (T-s) \mid f(s,x(s)) \mid ds \\
&\leq \quad \mid x(t) \mid + \int_0^t \frac{(t-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))} \mid x(s) \mid ds + \int_0^t (t-s) \mid f(s,x(s)) - f(s,0) \mid ds \\
&+ \quad \int_0^t (t-s) \mid f(s,0) \mid ds + \frac{t}{T} \int_0^T \frac{(T-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s)} \mid x(s) \mid ds \\
&+ \quad \frac{t}{T} \int_0^T (T-s) \mid f(s,x(s)) - f(s,0) \mid ds + \frac{t}{T} \int_0^T (T-s) \mid f(s,0) \mid ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq\ \mid x(t)\mid + \mathcal{M}_\Gamma T^* \int_0^t \Big(\frac{t-s}{T}\Big)^{1-\alpha^*} \mid x(s)\mid ds \\
&+\ \int_0^t (t-s)s^{-\sigma}s^{\sigma} \mid f(s,x(s)) - f(s,0)\mid ds \\
&+\ \int_0^t (t-s)\mid f(s,0)\mid ds + \mathcal{M}_\Gamma T^* \int_0^T \Big(\frac{T-s}{T}\Big)^{1-\alpha^*} \mid x(s)\mid ds \\
&+\ \int_0^T s^{-\sigma}s^{\sigma}(T-s)\mid f(s,x(s)) - f(s,0)\mid ds + \int_0^T (T-s)\mid f(s,0)\mid ds \\
&\leq\ \| x\| + \frac{\mathcal{M}_\Gamma T^*}{T^{1-\alpha^*}}\frac{T^{2-\alpha^*}}{2-\alpha^*}\| x\| + \int_0^t (t-s)s^{-\sigma}\mathscr{P}\mid x(s)\mid ds + \int_0^t (t-s)f^* ds \\
&+\ \frac{\mathcal{M}_\Gamma T^*}{T^{1-\alpha^*}}\frac{T^{2-\alpha^*}}{2-\alpha^*}\| x\| + \int_0^T (T-s)s^{-\sigma}\mathscr{P}\mid x(s)\mid ds + \int_0^T (T-s)f^* ds \\
&\leq\ \| x\| + \frac{2\mathcal{M}_\Gamma TT^*}{2-\alpha^*}\| x\| + \mathscr{P}\| x\| \int_0^t (t-s)s^{-\sigma}ds + f^*\frac{t^2}{2} \\
&+\ \mathscr{P}\| x\| \int_0^T (T-s)s^{-\sigma}ds + f^*\frac{T^2}{2} \\
&\leq\ \| x\| + \frac{2\mathcal{M}_\Gamma TT^*}{2-\alpha^*}\| x\| + \mathscr{P}\| x\| \frac{t^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)} + f^*T^2 \\
&+\ \mathscr{P}\| x\| \frac{T^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)} \\
&\leq\ \| x\| + \frac{2\mathcal{M}_\Gamma TT^*}{2-\alpha^*}\| x\| + 2\mathscr{P}\| x\| \frac{T^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)} + f^*T^2 \\
&\leq\ \Big[1 + \frac{2\mathcal{M}_\Gamma TT^*}{2-\alpha^*} + 2\mathscr{P}\frac{T^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)}\Big]\| x\| + f^*T^2,
\end{aligned}
$$

which implies that,

$$
\| Cx\| \leq \Big[1 + \frac{2\mathcal{M}_\Gamma TT^*}{2-\alpha^*} + 2\mathscr{P}\frac{T^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)}\Big] r + f^*T^2.
$$

Hence, $C(E_r)$ is uniformly bounded.

Step 3: C maps bounded sets into equicontinuous sets in $\mathbb{E}_1$.

Firstly, we can remark that the function $w_\alpha(s) = \left(\frac{t_1-s}{T}\right)^{1-\alpha(s)} - \left(\frac{t_2-s}{T}\right)^{1-\alpha(s)}$ is decreasing with respect to its exponent $1-\alpha(s)$, for $0 < \frac{t_1-s}{T} < \frac{t_2-s}{T} < 1$. Then, for $t_1, t_2 \in \mathbb{A}$, $t_1 < t_2$ and $x \in E_r$, we have

$$
\begin{aligned}
&| \, Cx(t_2) - Cx(t_1) \, | \\
\leq \; & | \, x(t_2) - x(t_1) \, | + | \int_0^{t_2} \frac{(t_2-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))} x(s) ds - \int_0^{t_1} \frac{(t_1-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))} x(s) ds \, | \\
+ \; & | \int_0^{t_2} (t_2-s) f(s,x(s)) ds - \int_0^{t_1} (t_1-s) f(s,x(s)) ds \, | \\
+ \; & | \, \frac{t_2}{T} \int_0^{T} \frac{(T-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))} x(s) ds - \frac{t_1}{T} \int_0^{T} \frac{(T-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))} x(s) ds \, | \\
+ \; & | \, \frac{t_2}{T} \int_0^{T} (T-s) f(s,x(s)) ds - \frac{t_1}{T} \int_0^{T} (T-s) f(s,x(s)) ds \, | \\
\leq \; & | \, x(t_2) - x(t_1) \, | + \int_0^{t_1} | \, \frac{1}{\Gamma(2-\alpha(s))} \, || \, (t_2-s)^{1-\alpha(s)} - (t_1-s)^{1-\alpha(s)} \, || \, x(s) \, | \, ds \\
+ \; & | \int_{t_1}^{t_2} \frac{(t_2-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))} x(s) ds \, | + | \int_0^{t_1} (t_2-s) f(s,x(s)) - (t_1-s) f(s,x(s)) ds \, | \\
+ \; & | \int_{t_1}^{t_2} (t_2-s) f(s,x(s)) ds \, | + | \, \frac{1}{T} \int_0^{T} \frac{(T-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))} x(s) ds \Big[t_2 - t_1 \Big] \, | \\
+ \; & | \, \frac{1}{T} \int_0^{T} (T-s) f(s,x(s)) ds \Big[t_2 - t_1 \Big] \, | \\
\leq \; & | \, x(t_2) - x(t_1) \, | + \mathcal{M}_\Gamma \, || \, x \, || \int_0^{t_1} \Big[(t_1-s)^{1-\alpha(s)} - (t_2-s)^{1-\alpha(s)} \Big] ds \\
+ \; & \int_{t_1}^{t_2} \frac{(t_2-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))} \, | \, x(s) \, | \, ds + \int_0^{t_1} \Big[(t_2-s) - (t_1-s) \Big] \, | \, f(s,x(s)) \, | \, ds \\
+ \; & \int_{t_1}^{t_2} (t_2-s) \, | \, f(s,x(s)) \, | \, ds + \frac{t_2-t_1}{T} \int_0^{T} \frac{(T-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))} \, | \, x(s) \, | \, ds \\
+ \; & \frac{t_2-t_1}{T} \int_0^{T} (T-s) \, | \, f(s,x(s)) \, | \, ds \\
\leq \; & | \, x(t_2) - x(t_1) \, | + \mathcal{M}_\Gamma \, || \, x \, || \int_0^{t_1} T^{1-\alpha(s)} \Big[\Big(\frac{t_1-s}{T} \Big)^{1-\alpha(s)} - \Big(\frac{t_2-s}{T} \Big)^{1-\alpha(s)} \Big] ds \\
+ \; & \mathcal{M}_\Gamma \, || \, x \, || \, T^* \int_{t_1}^{t_2} \Big(\frac{t_2-s}{T} \Big)^{1-\alpha^*} ds \\
+ \; & \int_0^{t_1} \Big[(t_2-s) - (t_1-s) \Big] \, | \, f(s,x(s)) - f(s,0) + f(s,0) \, | \, ds \\
+ \; & \int_{t_1}^{t_2} (t_2-s) \, | \, f(s,x(s)) - f(s,0) + f(s,0) \, | \, ds \\
+ \; & \frac{\mathcal{M}_\Gamma}{T} (t_2-t_1) \, || \, x \, || \, T^* \int_0^{T} \Big(\frac{T-s}{T} \Big)^{1-\alpha^*} ds \\
+ \; & \frac{t_2-t_1}{T} \int_0^{T} (T-s) \, | \, f(s,x(s)) - f(s,0) + f(s,0) \, | \, ds
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & \mid x(t_2)-x(t_1)\mid + \mathcal{M}_\Gamma \mid\mid x \mid\mid T^* \int_0^{t_1} \Big[\Big(\frac{t_1-s}{T}\Big)^{1-\alpha^*} - \Big(\frac{t_2-s}{T}\Big)^{1-\alpha^*}\Big] ds \\
+\ & \mathcal{M}_\Gamma \mid\mid x \mid\mid T^* \int_{t_1}^{t_2} \Big(\frac{t_2-s}{T}\Big)^{1-\alpha^*} ds \\
+\ & \int_0^{t_1} \Big[(t_2-s)-(t_1-s)\Big] s^{-\sigma} s^{\sigma} \mid f(s,x(s))-f(s,0)\mid ds \\
+\ & \int_0^{t_1} \Big[(t_2-s)-(t_1-s)\Big] \mid f(s,0)\mid ds \\
+\ & \int_{t_1}^{t_2} (t_2-s) s^{-\sigma} s^{\sigma} \mid f(s,x(s))-f(s,0)\mid ds + \int_{t_1}^{t_2} (t_2-s)\mid f(s,0)\mid ds \\
+\ & \frac{\mathcal{M}_\Gamma T^*}{2-\alpha^*} \mid\mid x \mid\mid (t_2-t_1) + \frac{t_2-t_1}{T} \int_0^T (T-s) s^{-\sigma} s^{\sigma} \mid f(s,x(s))-f(s,0)\mid ds \\
+\ & \frac{t_2-t_1}{T} \int_0^T (T-s)\mid f(s,0)\mid ds \\
\leq\ & \mid x(t_2)-x(t_1)\mid + \Big[\frac{\mathcal{M}_\Gamma \mid\mid x\mid\mid T^*}{T^{1-\alpha^*}(2-\alpha^*)}\Big]\Big[(t_1)^{2-\alpha^*} - (t_2)^{2-\alpha^*} + 2(t_2-t_1)^{2-\alpha^*}\Big] \\
+\ & \mathscr{P} \mid\mid x \mid\mid \int_0^{t_1} \Big[(t_2-s)-(t_1-s)\Big] s^{-\sigma} ds \\
+\ & f^*\Big[t_1 t_2 - t_1^2\Big] + \mathscr{P} \mid\mid x \mid\mid \int_{t_1}^{t_2} (t_2-s) s^{-\sigma} ds \\
+\ & f^*\Big[\frac{t_2^2}{2} + \frac{t_1^2}{2} - t_1 t_2\Big] + \Big[\frac{\mathcal{M}_\Gamma T^*}{2-\alpha^*} \mid\mid x \mid\mid \Big(t_2-t_1\Big)\Big] \\
+\ & \frac{t_2-t_1}{T} \mathscr{P} \mid\mid x \mid\mid \int_0^T (T-s) s^{-\sigma} ds + f^* \frac{t_2-t_1}{T}\frac{T^2}{2} \\
\leq\ & \mid x(t_2)-x(t_1)\mid + \Big[\frac{\mathcal{M}_\Gamma \mid\mid x\mid\mid T^*}{T^{1-\alpha^*}(2-\alpha^*)}\Big]\Big[(t_1)^{2-\alpha^*} - (t_2)^{2-\alpha^*} + 2(t_2-t_1)^{2-\alpha^*}\Big] \\
+\ & \mathscr{P} \mid\mid x \mid\mid \Big[(t_2-t_1)\frac{t_1^{-\sigma+1}}{-\sigma+1}\Big] + f^*\Big(\frac{-t_1^2}{2} + \frac{t_2^2}{2}\Big) \\
+\ & \mathscr{P} \mid\mid x \mid\mid \Big[\Big(-(t_2-t_1)\frac{t_1^{-\sigma+1}}{-\sigma+1}\Big) + \Big(\frac{t_2^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)} - \frac{t_1^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)}\Big)\Big] \\
+\ & \Big[\frac{\mathcal{M}_\Gamma T^*}{2-\alpha^*} \mid\mid x \mid\mid \Big(t_2-t_1\Big)\Big] + \frac{t_2-t_1}{T} \mathscr{P} \mid\mid x \mid\mid \frac{T^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)} + f^*(t_2-t_1)\frac{T}{2} \\
\leq\ & \mid x(t_2)-x(t_1)\mid + \Big[\frac{\mathcal{M}_\Gamma \mid\mid x\mid\mid T^*}{T^{1-\alpha^*}(2-\alpha^*)}\Big]\Big[(t_1)^{2-\alpha^*} - (t_2)^{2-\alpha^*} + 2(t_2-t_1)^{2-\alpha^*}\Big] \\
+\ & f^*\Big(\frac{t_2^2-t_1^2}{2}\Big) + \mathscr{P} \mid\mid x \mid\mid \Big(\frac{t_2^{-\sigma+2}-t_1^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)}\Big) \\
+\ & \Big[\frac{\mathcal{M}_\Gamma T^*}{2-\alpha^*} \mid\mid x \mid\mid + \frac{\mathscr{P} \mid\mid x \mid\mid}{T} \frac{T^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)} + f^*\frac{T}{2}\Big]\Big(t_2-t_1\Big).
\end{aligned}
$$

Hence, $\mid Cx(t_2) - Cx(t_1) \mid \to 0$ as $t_2 \to t_1$. It implies that $C(E_r)$ is equicontinuous. Consequently, the operator C is compact.

Step 4: The set $\ominus$ defined as

$$\ominus = \{x \in \mathbb{E}_1 : x = \gamma Cx, \;\; 0 < \gamma < 1\},$$

is bounded.

Let $x \in \ominus$. Then, for any $t \in \mathbb{A}$, we have

$$x(t) \;\; = \;\; \gamma Cx(t), \;\; 0 < \gamma < 1,$$

and

$$\begin{aligned}
\mid \gamma Cx(t) \mid \;\; &= \;\; \gamma \Big| x(t) - \int_0^t \frac{(t-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))} x(s) \;\; ds + \int_0^t (t-s) f(s, x(s)) ds \\
&+ \;\; \frac{t}{T} \int_0^T \frac{(T-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))} x(s) \;\; ds - \frac{t}{T} \int_0^T (T-s) f(s, x(s)) ds \Big| \\
&\leq \;\; \gamma \Big[\mid x(t) \mid + \int_0^t \frac{(t-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))} \mid x(s) \mid ds + \int_0^t (t-s) \mid f(s, x(s)) \mid ds \\
&+ \;\; \frac{t}{T} \int_0^T \frac{(T-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))} \mid x(s) \mid ds + \frac{t}{T} \int_0^T (T-s) \mid f(s, x(s)) \mid ds \Big] \\
&\leq \;\; \Big[\mid x(t) \mid + \int_0^t \frac{(t-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))} \mid x(s) \mid ds + \int_0^t (t-s) \mid f(s, x(s)) \mid ds \\
&+ \;\; \frac{t}{T} \int_0^T \frac{(T-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))} \mid x(s) \mid ds + \frac{t}{T} \int_0^T (T-s) \mid f(s, x(s)) \mid ds \Big] \\
&\leq \;\; \Big[1 + \frac{2\mathcal{M}_\Gamma T T^*}{2-\alpha^*} + 2\mathscr{P} \frac{T^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)} \Big] \mid\mid x \mid\mid + f^* T^2.
\end{aligned}$$

Now, for every $t \in \mathbb{A}$, we have

$$\mid\mid \gamma Cx \mid\mid \;\; \leq \;\; \Big[1 + \frac{2\mathcal{M}_\Gamma T T^*}{2-\alpha^*} + 2\mathscr{P} \frac{T^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)} \Big] \mid\mid x \mid\mid + f^* T^2 < \infty.$$

This implies that the set $\ominus$ is bounded independently of $\gamma \in (0, 1)$.

Then, all condition of Theorem 1.2 are satisfied and the ITVP (2.1) has at least one solution $x \in \mathbb{E}_1$.

2.3 Results of uniqueness

The next result is based on Theorem 1.3.

Theorem 2.2 *If the conditions (AS1) and (AS2) are satisfied, then the ITVP (2.1) has a unique solution on* $\mathbb{E}_1$.

proof: We consider the same operator

$$C : \mathbb{E}_1 \to \mathbb{E}_1,$$

defined as follows:

$$Cx(t) = x(t) - \int_0^t \frac{(t-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))} x(s) \quad ds + \int_0^t (t-s) f(s, x(s)) \quad ds$$
$$+ \frac{t}{T} \int_0^T \frac{(T-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))} x(s) \quad ds - \frac{t}{T} \int_0^T (T-s) f(s, x(s)) \quad ds.$$

For x, $x^* \in \mathbb{E}_1$, we may write

$$\mid Cx(t) - Cx^*(t) \mid$$

$$\begin{aligned}
&\leq \mid x(t) - x^*(t) \mid + \int_0^t \frac{(t-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))} \mid x^*(s) - x(s) \mid ds \\
&+ \int_0^t (t-s) s^{-\sigma} s^{\sigma} \mid f(s, x(s)) - f(s, x^*(s)) \mid ds \\
&+ \int_0^T \frac{(T-s)^{1-\alpha(s)}}{\Gamma(2-\alpha(s))} \mid x(s) - x^*(s) \mid ds \\
&+ \int_0^T (T-s) s^{-\sigma} s^{\sigma} \mid f(s, x^*(s)) - f(s, x(s)) \mid ds \\
&\leq \, || x - x^* || + \mathcal{M}_\Gamma || x - x^* || \int_0^t (t-s)^{1-\alpha(s)} ds
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \mathscr{P} \parallel x - x^* \parallel \int_0^t (t-s)s^{-\sigma} ds + \mathcal{M}_\Gamma \parallel x - x^* \parallel \int_0^T (T-s)^{1-\alpha(s)} ds \\
&+ \quad \mathscr{P} \parallel x - x^* \parallel \int_0^T (T-s)s^{-\sigma} ds \\
&\leq \quad \parallel x - x^* \parallel + \mathcal{M}_\Gamma T^* \parallel x - x^* \parallel \int_0^t \Big(\frac{t-s}{T}\Big)^{1-\alpha^*} ds \\
&+ \quad \mathscr{P} \parallel x - x^* \parallel \frac{t^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)} + \mathcal{M}_\Gamma T^* \parallel x - x^* \parallel \int_0^T \Big(\frac{T-s}{T}\Big)^{1-\alpha^*} ds \\
&+ \quad \mathscr{P} \parallel x - x^* \parallel \frac{T^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)} \\
&\leq \quad \parallel x - x^* \parallel + \frac{\mathcal{M}_\Gamma T^*}{T^{1-\alpha^*}} \frac{(t)^{2-\alpha^*}}{(2-\alpha^*)} \parallel x - x^* \parallel + 2\mathscr{P} \parallel x - x^* \parallel \frac{T^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)} \\
&+ \quad \frac{\mathcal{M}_\Gamma T^*}{T^{1-\alpha^*}} \frac{(T)^{2-\alpha^*}}{2-\alpha^*} \parallel x - x^* \parallel \\
&\leq \quad \parallel x - x^* \parallel + \frac{\mathcal{M}_\Gamma T T^*}{2-\alpha^*} \parallel x - x^* \parallel + 2\mathscr{P} \parallel x - x^* \parallel \frac{T^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)} \\
&+ \quad \frac{\mathcal{M}_\Gamma T T^*}{2-\alpha^*} \parallel x - x^* \parallel \\
&\leq \quad \parallel x - x^* \parallel + \frac{2\mathcal{M}_\Gamma T T^*}{2-\alpha^*} \parallel x - x^* \parallel + 2\mathscr{P} \parallel x - x^* \parallel \frac{T^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)} \\
&\leq \quad \Big(1 + \frac{2\mathcal{M}_\Gamma T T^*}{2-\alpha^*} + 2\mathscr{P} \frac{T^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)}\Big) \mid -x^* \parallel .
\end{aligned}
$$

We put $\sigma = 1 + \frac{2\mathcal{M}_\Gamma T T^*}{2-\alpha^*} + 2\mathscr{P} \frac{T^{-\sigma+2}}{(-\sigma+1)(-\sigma+2)}$ so that we have

$$\parallel Cx - Cx^* \parallel \leq \sigma \parallel x - x^* \parallel .$$

By induction, we can prove that

$$\parallel C^n x - C^n x^* \parallel \leq \frac{\sigma^n}{n!} \parallel x - x^* \parallel,$$

where $C^n = C \circ C \circ C \circ C \circ .... \circ C$ "n times".
We have $\lim_{n \to \infty} \dfrac{\sigma^n}{n!} = 0$. Then, for sufficiently large n, we get $\frac{\sigma^n}{n!} < 1$.
According to Theorem 1.3, the operator C has a unique fixed point which is the unique solution of the ITVP (2.1).

2.4 Numerical examples

Example 2.1 *Consider the following ITVP*

$$\begin{cases} D^{\alpha(t)}x(t) = f(t,x(t)), & t \in \mathbb{A} = [0,1], \\ x(0) = x(1) = 0, \end{cases} \tag{2.4}$$

where $\alpha(t) = 1 + \frac{t}{2}$, *and* $f(t,x) = t^2 + \frac{1}{3}x$.

Clearly $\alpha(t)$ *is a continuous function on* $[0,1]$ *and,* $1 < \alpha(t) < 1 + \frac{1}{2} = \frac{3}{2} = \alpha^* < 2$.

In addition, $f(t,x)$ *is a continuous function on* $\mathbb{A} \times R$, *and*

$$\begin{aligned} t^\sigma \mid f(t,x) - f(t,y) \mid &= t^\sigma \mid t^2 + \frac{1}{3}x - t^2 - \frac{1}{3}y \mid \\ &= t^\sigma \mid \frac{1}{3}(x-y) \mid \\ &\leq \frac{1}{3} \mid x - y \mid, \end{aligned}$$

so (AS1) satisfied for $\mathscr{P} = \frac{1}{3}$ *and* $\sigma \in (0,1)$.

By Theorem(2.2) the equation (2.4) has a unique solution.

Example 2.2 *Consider the following ITVP*

$$\begin{cases} D^{\alpha(t)}x(t) = f(t,x(t)), & t \in \mathbb{A} = [0,1], \\ x(0) = x(1) = 0, \end{cases} \tag{2.5}$$

where $\alpha(t) = \exp(t) - t$ *and* $f(t,x) = \frac{\exp(-t)}{(\exp(\exp(\frac{t2}{1+t}))+4\exp(2t)+1)(1+x)}$.

Clearly $\alpha(t)$ *is a continuous function on* $[0,1]$ *and,* $1 < \alpha(t) < \exp(1) - 1 = \alpha^* < 2$.

Also, $f(t,x)$ *is a continuous function on* $\mathbb{A} \times R$, *and*

$$\begin{aligned} t^\sigma \mid f(t,x) - f(t,y) \mid &= t^\sigma \mid \frac{\exp(-t)}{(\exp(\exp(\frac{t^2}{1+t})) + 4\exp(2t) + 1)} \Big(\frac{1}{1+x} - \frac{1}{1+y} \Big) \mid \\ &\leq t^\sigma \frac{\exp(-t) \mid x - y \mid}{(\exp(\exp(\frac{t^2}{1+t})) + 4\exp(2t) + 1)(1+x)(1+y)} \\ &\leq t^\sigma \frac{\exp(-t)}{(\exp(\exp(\frac{t^2}{1+t})) + 4\exp(2t) + 1)} \mid x - y \mid \\ &\leq \frac{\exp(-1)}{(\exp(\exp(\frac{1}{2})) + 4\exp(2) + 1)} \mid x - y \mid, \end{aligned}$$

so $(AS1)$ *satisfied for* $\mathscr{P} = \frac{\exp(-1)}{(\exp(\exp(\frac{1}{2}))+4\exp(2)+1)}$ *and* $\sigma \in (0,1)$.

By Theorem(2.2) the equation (2.5) has a unique solution.

Numerical results

Now, we present the numerical solution $x(t)$ for $\alpha(t) = \exp(t) - t$ with $t \in [0,1]$ and $x_i(t)$ for $\alpha(t_i) = \exp(t_i) - t_i$ where t_i is fixed.

In Figure (2.1), we plot the solution x depending on t.

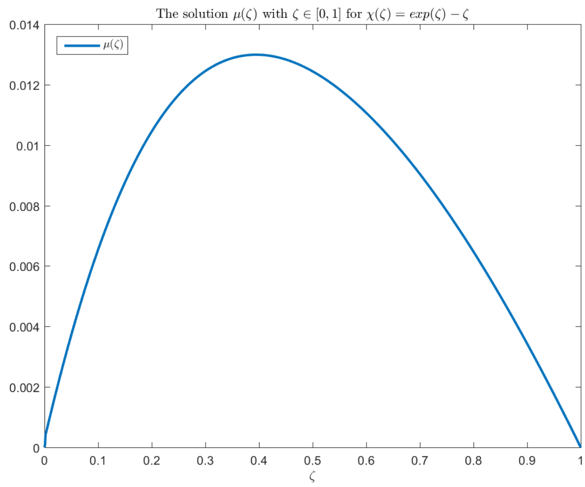
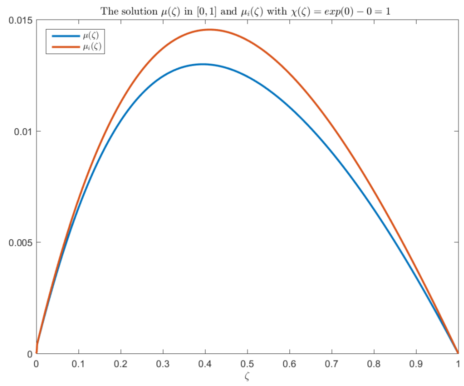
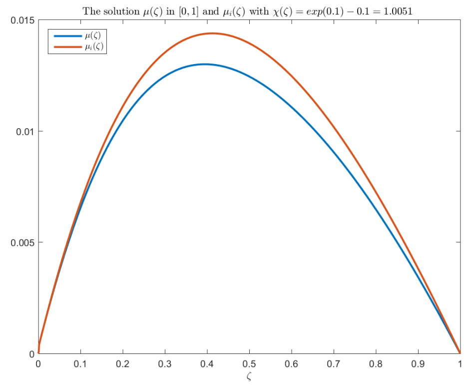
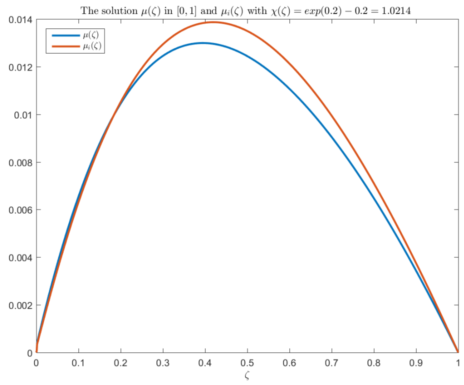


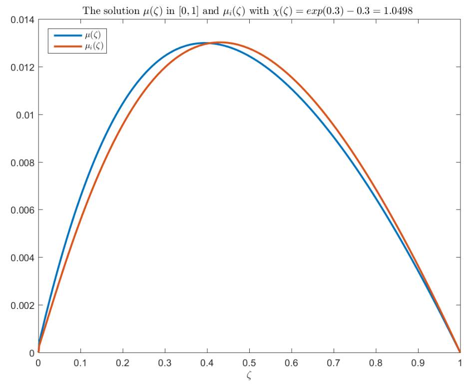
Figure 2.1: The solution $x(t)$ in $[0,1]$ with $\alpha(t) = \exp(t) - t$.

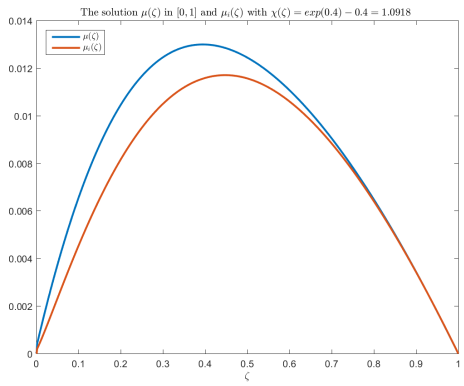
The following figures present a comparison between the solution x and the various solutions x_i, each with a different t.

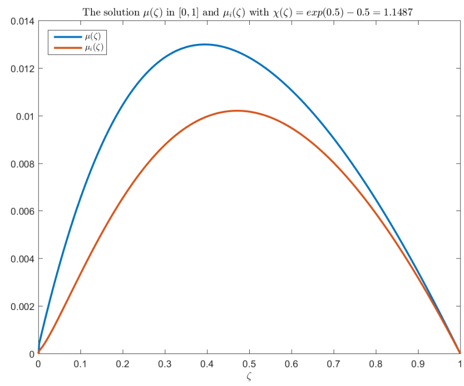

The solution $\mu(\zeta)$ in $[0,1]$ and $\mu_i(\zeta)$ with $\chi(\zeta) = exp(0) - 0 = 1$
$\mu(\zeta)$
$\mu_i(\zeta)$
ζ

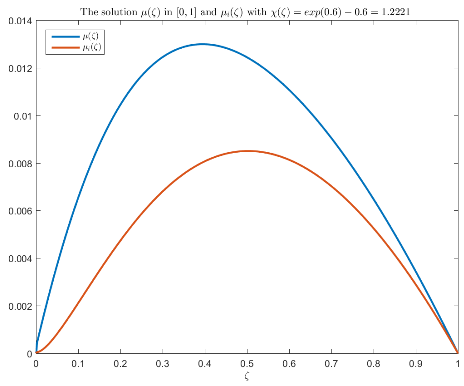

The solution $\mu(\zeta)$ in $[0,1]$ and $\mu_i(\zeta)$ with $\chi(\zeta) = exp(0.1) - 0.1 = 1.0051$
$\mu(\zeta)$
$\mu_i(\zeta)$
ζ

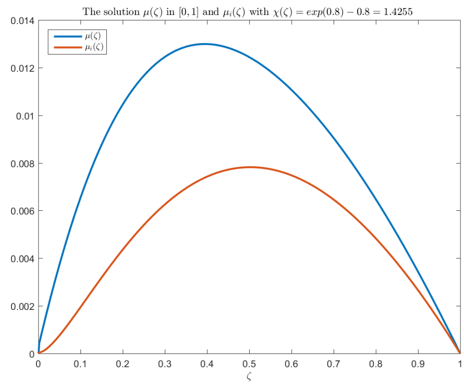

The solution $\mu(\zeta)$ in $[0,1]$ and $\mu_i(\zeta)$ with $\chi(\zeta) = exp(0.2) - 0.2 = 1.0214$
$\mu(\zeta)$
$\mu_i(\zeta)$
ζ


The solution $\mu(\zeta)$ in $[0,1]$ and $\mu_i(\zeta)$ with $\chi(\zeta) = exp(0.3) - 0.3 = 1.0498$
$\mu(\zeta)$
$\mu_i(\zeta)$
ζ


The solution $\mu(\zeta)$ in $[0,1]$ and $\mu_i(\zeta)$ with $\chi(\zeta) = exp(0.4) - 0.4 = 1.0918$
$\mu(\zeta)$
$\mu_i(\zeta)$
ζ


The solution $\mu(\zeta)$ in $[0,1]$ and $\mu_i(\zeta)$ with $\chi(\zeta) = exp(0.5) - 0.5 = 1.1487$
$\mu(\zeta)$
$\mu_i(\zeta)$
ζ


The solution $\mu(\zeta)$ in $[0,1]$ and $\mu_i(\zeta)$ with $\chi(\zeta) = exp(0.6) - 0.6 = 1.2221$
$\mu(\zeta)$
$\mu_i(\zeta)$
ζ


The solution $\mu(\zeta)$ in $[0,1]$ and $\mu_i(\zeta)$ with $\chi(\zeta) = exp(0.8) - 0.8 = 1.4255$
$\mu(\zeta)$
$\mu_i(\zeta)$
ζ

In this table, we present the $Norm_i = \max_{t \in [0,1]} \Big| x(t) - x_i(t) \Big|$ for $\alpha(t) \in [1, 2]$.

t	0	0.1	0.2	0.3	0.4
$\alpha(t)$	1	1.005	1.021	1.049	1.091
$Norm_i$	1.63×10^{-6}	1.49×10^{-6}	1.08×10^{-6}	1×10^{-6}	2.3×10^{-6}
t	0.5	0.6	0.8	0.9	1
$\alpha(t)$	1.148	1.222	1.425	1.559	1.718
$Norm_i$	3.92×10^{-6}	5.8×10^{-6}	6.26×10^{-6}	7.09×10^{-6}	7.44×10^{-6}

We observe that when t approaches to $0,4(\alpha = 1,091)$ the $Norm_i$ is decreasing and when t approaches to $1(\alpha = 1,718)$ is increasing.

Chapter 3

Initial Value Problem for Mixed Differential Equations with Non Autonomous Variable Order [2]

3.1 Introduction

We examine the existence and uniqueness, in addition to the Ulam-Hyersstability of the solutions of the next non autonomous variable order initial value problem (NAVOIVP for short).

$$\begin{cases} \mathbb{D}_{0^+}^{\alpha(t,x(t))}x(t) + \varsigma x^{'}(t) = f(t,x(t)), \ \ t \in \mathbb{A} := [0,T], \\ x(0) = 0, \end{cases} \tag{3.1}$$

where $0 < T < +\infty$, $\varsigma > 0$ and $\alpha : \mathbb{A} \times \mathbb{R} \to (0,1)$, $f : \mathbb{A} \times \mathbb{R} \to \mathbb{R}$ are a continuous functions and $\mathbb{D}_{0^+}^{\alpha(t,x(t))}$ is the Riemann Liouville fractional derivative of variable order $\alpha(t,x(t))$.

[2] M. S. Souid, **S. Guedim,** S. Boulaaras, A. Benkerrouche, A. Amara, T. Radwan, R. Jan, New approaches to solving initial value problems of mixed differential equations. *Filomat*, **40:2**, (2026), 649-659. https://doi.org/10.2298/FIL2602649S.

3.2 Existence criteria

Let us introduce the following assumptions:

(SY1) There exist constants $0 < \sigma < \min \alpha(t, x(t))$, $\lambda > 0$, such that, the function $t^{\sigma} f$ is a continuous function on $\mathbb{A} \times \mathbb{R}$ and:

$t^{\sigma} \mid f\ (t, x(t))$ - $f\ (t\ , \mathrm{y}(t)) \mid \leq \lambda \mid x(t)$ - $\mathrm{y}(t)|$, $\quad \forall\ x, \mathrm{y} \in \mathbb{R}, t \in \mathbb{A}$.

(SY2) $\alpha : \mathbb{A} \times \mathbb{R} \rightarrow (0, \alpha^*]$ is a continuous function, such that, $0 \leq \alpha(t, x(t)) \leq \alpha^* < 1$.

Remark 3.1 *[43]*

1. *The function* Γ*(1-*$\alpha(t, x(t))$*) is continuous as a composition of two continuous function, we can let*
   $\mathscr{M}_{\Gamma} = \max \mid \frac{1}{\Gamma(1-\alpha(t,x(t)))} \mid$.
2. *By the continuity of the function* $\alpha(t, x(t))$*, we let*
   $T^{-\alpha(t,x(t))} \leq$ *1 if* $1 \leq T < \infty$, $\quad T^{-\alpha(t,x(t))} \leq T^{-\alpha^*}$ *if* $0 \leq T \leq 1$.
   We conclude that $T^{-\alpha(t,x(t))} \leq \max$ *(1,*$T^{-\alpha^*}$*)*$= T^*$.

Remark 3.2 *[26] Assuming that* X *and* Y *are two real numbers, then*

$$\mid \varsigma_1 X -_{\beta} Y \mid \leq 2 \max(\varsigma, \beta) \mid X - Y \mid,$$

where ς_1 *and* β *are positive real numbers.*

Lemma 3.1 *[43] Let (SY 2) hold and Let* $x_n, x \in C[0, T]$*, assume that* $x_n(t) \rightarrow x(t)$*,* $t \in [0, T]$ *as* $n \rightarrow \infty$*, then*

$$\int_0^t \frac{(t-s)^{-\alpha(s,x_n(s))}}{\Gamma(1-\alpha(s, x_n(s)))} x_n(s)\ ds \rightarrow \int_0^t \frac{(t-s)^{-\alpha(s,x(s))}}{\Gamma(1-\alpha(s, x(s)))} x(s)\ ds, t \in [0, T],$$

as $n \rightarrow \infty$.

we will need the following lemma to solve NAVOIVP (3.1).

Lemma 3.2 *The function $x \in \mathbb{E}$ forms a solution of the NAVOIVP (3.1) if and only if x fulfills the integral equation*

$$x(t) = \frac{1}{\varsigma}\Big[\int_0^t f(s,x(s))\ \ ds - \int_0^t \frac{(t-s)^{-\alpha(s,x(s))}}{\Gamma(1-\alpha(s,x(s)))} x(s)\ \ ds\Big]. \tag{3.2}$$

Proof: By using the definition of fractional derivative of variable order provided by (1.4), it is possible to express the NAVOIVP (3.1) as follows:

$$(\frac{d}{dt})\int_0^t \frac{(t-s)^{-\alpha(s,x(s))}}{\Gamma(1-\alpha(s,x(s)))} x(s)\ \ ds + \varsigma x^{'}(t) = f(t,x(t)).$$

Then,

$$\int_0^t \frac{(t-s)^{-\alpha(s,x(s))}}{\Gamma(1-\alpha(s,x(s)))} x(s)\ \ ds + \varsigma x(t) = \int_0^t f(s,x(s))\ \ ds + c_3. \tag{3.3}$$

Evaluating Eq (3.3) at $t = 0$, gives us $c_3 = 0$. Thus,

$$\int_0^t \frac{(t-s)^{-\alpha(s,x(s))}}{\Gamma(1-\alpha(s,x(s)))} x(s)\ \ ds + \varsigma x(t) = \int_0^t f(s,x(s))\ \ ds.$$

So

$$x(t) = \frac{1}{\varsigma}\Big[\int_0^t f(s,x(s))\ \ ds - \int_0^t \frac{(t-s)^{-\alpha(s,x(s))}}{\Gamma(1-\alpha(s,x(s)))} x(s)\ \ ds\Big].$$

Conversely, by derivation both sides of the equation 3.2, we have

$$(\frac{d}{dt})\int_0^t \frac{(t-s)^{-\alpha(s,x(s))}}{\Gamma(1-\alpha(s,x(s)))} x(s)\ \ ds + \varsigma x^{'}(t) = f(t,x(t),$$

which means the NAVOIVP (3.1).

Now, we will prove the existence of solutions for the NAVOIVP (3.1). Theorem 1.6 forms the basis of the first finding.

Theorem 3.1 *Assume that conditions (SY1) and (SY2) hold. If*

$$\frac{\mathscr{M}_\Gamma T^* T}{1-\alpha^*} + \lambda\frac{T^{-\sigma+1}}{-\sigma+1} < \varsigma, \tag{3.4}$$

then the NAVOIVP (3.1) has at least one solution on $\mathbb{E}$.

Proof: We construct the following operators

$$Q_1,\ Q_2 : \mathbb{E} \to \mathbb{E},$$

as follows,

$$Q_1x(t) = \frac{1}{\varsigma}\Big[-\int_0^t \frac{(t-s)^{-\alpha(s,x(s))}}{\Gamma(1-\alpha(s,x(s)))}x(s)\ \ ds\Big],\ \ Q_2x(t) = \frac{1}{\varsigma}\int_0^t f(s,x(s))\ ds.$$

We consider the set

$$B_R = \{x \in \mathbb{E}, \| x \| \leq R\},$$

where

$$R \geq \frac{\frac{f^*T}{\varsigma}}{1-\frac{1}{\varsigma}\Big[\frac{\mathscr{M}_\Gamma T^*T}{1-\alpha^*}+\lambda\frac{T^{-\sigma+1}}{-\sigma+1}\Big]},$$

and

$$f^* = \sup_{t\in\mathbb{A}} | f(t,0) | .$$

Clearly, B_R is non empty, bounded, convexand closed.

We now prove that Q_1, Q_2 satisfy the conditions given by Theorem 1.6. The argument will be put into practice in a four phases.

Step 1:$Q_1(B_R) + Q_2(B_R) \subseteq B_R$.

For $x \in B_R$, we obtain

$$| Q_1x(t) + Q_2x(t) |$$

$$\begin{aligned}
&\leq \frac{1}{\varsigma}\Big[\Big| \int_0^t f(s,x(s))\ \ ds \Big| + \Big| \int_0^t \frac{(t-s)^{-\alpha(s,x(s))}}{\Gamma(1-\alpha(s,x(s)))}x(s)\ \ ds \Big| \Big] \\
&\leq \frac{1}{\varsigma}\Big[\int_0^t | f(s,x(s)) - f(s,0) + f(s,0) | ds + \int_0^t \frac{(t-s)^{-\alpha(s,x(s))}}{\Gamma(1-\alpha(s,x(s)))} | x(s) | \ \ ds\Big] \\
&\leq \frac{1}{\varsigma}\Big[\int_0^t s^{-\sigma}s^{\sigma} | f(s,x(s)) - f(s,0) | ds + \int_0^t | f(s,0) | ds \\
&+ \mathscr{M}_\Gamma \int_0^t T^{-\alpha(s,x(s))}\Big(\frac{t-s}{T}\Big)^{-\alpha(s,x(s))} | x(s) | ds\Big] \\
&\leq \frac{1}{\varsigma}\Big[\int_0^t \lambda | x(s) | s^{-\sigma} ds + \int_0^t f^* ds + \mathscr{M}_\Gamma T^* \int_0^t \Big(\frac{t-s}{T}\Big)^{-\alpha(s,x(s))} | x(s) | ds\Big]
\end{aligned}$$

$$\leq \frac{1}{\varsigma}\Big[\lambda \mid\mid x \mid\mid \frac{t^{-\sigma+1}}{-\sigma+1} + f^* t + \frac{\mathscr{M}_\Gamma T^*}{T^{-\alpha^*}} \mid\mid x \mid\mid \int_0^t (t-s)^{-\alpha^*} ds\Big]$$
$$\leq \frac{1}{\varsigma}\Big[\lambda \mid\mid x \mid\mid \frac{T^{-\sigma+1}}{-\sigma+1} + f^* T + \frac{\mathscr{M}_\Gamma T^*}{T^{-\alpha^*}} \frac{t^{1-\alpha^*}}{(1-\alpha^*)} \mid\mid x \mid\mid \Big]$$
$$\leq \frac{1}{\varsigma}\Big[\lambda \mid\mid x \mid\mid \frac{T^{-\sigma+1}}{-\sigma+1} + f^* T + \frac{\mathscr{M}_\Gamma T^* T}{(1-\alpha^*)} \mid\mid x \mid\mid \Big]$$
$$\leq \frac{1}{\varsigma}\Big[\frac{\mathscr{M}_\Gamma T^* T}{1-\alpha^*} + \lambda \frac{T^{-\sigma+1}}{-\sigma+1}\Big] \mid\mid x \mid\mid + \frac{f^* T}{\varsigma}$$
$$\leq \frac{1}{\varsigma}\Big[\frac{\mathscr{M}_\Gamma T^* T}{1-\alpha^*} + \lambda \frac{T^{-\sigma+1}}{-\sigma+1}\Big] R + \frac{f^* T}{\varsigma}$$
$$\leq R,$$

which means that $Q_1(B_R) + Q_2(B_R) \subseteq B_R$.

Step 2: Q_1 is continuous.

Let x_n be a sequence satisfying $x_n \to x$ in $\mathbb{E}$. For $t \in \mathbb{A}$, we estimate

$$\mid Q_1 x_n(t) - Q_1 x(t) \mid$$

$$\leq \frac{1}{\varsigma}\Big[\mid \int_0^t \frac{(t-s)^{-\alpha(s,x_n(s))}}{\Gamma(1-\alpha(s,x_n(s)))} x_n(s) ds - \int_0^t \frac{(t-s)^{-\alpha(s,x(s))}}{\Gamma(1-\alpha(s,x(s)))} x(s) ds \mid \Big].$$

By using lemma 3.1, we have

$$\mid\mid Q_1 x_n(t) - Q_1 x(t) \mid\mid \to 0, as \;\; n \to \infty.$$

The above relation shows that the operator Q_1 is continuous on $\mathbb{E}$.

Step 3: Q_1 is compact.

Here, we intend to prove the relative compactness of $Q_1(B_R)$ which means that Q_1 is compact. Evidently, $Q_1(B_R)$ is uniformly bounded, due to step1, we saw that

$$Q_1(B_R) = \{Q_1(x) : x \in B_R\} \subset Q_1(B_R) + Q_2(B_R) \subset B_R.$$

Hence, for every $x \in B_R$, we obtain $\mid\mid Q_1(x) \mid\mid \leq R$ meaning the uniform boundedness of $Q_1(B_R)$. Firstly, we can know that the function $w(t) = a^t - b^t, t \in (-1, 0), 0 < a < b < 1$, is decreasing. Indeed, since $\ln a < \ln b < 0$ and $a^t > b^t > 0$, we have that

$$w'(t) = a^t \ln a - b^t \ln b < b^t \ln a - b^t \ln b = b^t(\ln a - \ln b) < 0,$$

which implies that $w(t)$ is a decreasing. Thus, for $\tilde{K}(s) = \left(\frac{t_1-s}{T}\right)^{-\alpha(s,x(s))} - \left(\frac{t_2-s}{T}\right)^{-\alpha(s,x(s))}$ (where $0 < \frac{t_1-s}{T} < \frac{t_2-s}{T} < 1$), we may look $\tilde{K}(s)$ as the same type as $w(s)$, then $\tilde{K}(s)$ is decreasing with respect to its exponent $-\alpha(s, x(s))$. Then, for $t_1, t_2 \in \mathbb{A}$, $t_1 < t_2$ and $x \in B_R$, we have

$$
\begin{aligned}
& \mid Q_1 x(t_2) - Q_1 x(t_1) \mid \\
\leq\ & \frac{1}{\varsigma}\Big[\mid \int_0^{t_1} \frac{(t_1-s)^{-\alpha(s,x(s))}}{\Gamma(1-\alpha(s,x(s))))} x(s) - \int_0^{t_2} \frac{(t_2-s)^{-\alpha(s,x(s))}}{\Gamma(1-\alpha(s,x(s)))} x(s) ds \mid \Big] \\
\leq\ & \frac{1}{\varsigma}\Big[\mid \int_0^{t_1} \frac{(t_2-s)^{-\alpha(s,x(s))}}{\Gamma(1-\alpha(s,x(s)))} x(s) - \frac{(t_1-s)^{-\alpha(s,x(s))}}{\Gamma(1-\alpha(s,x(s)))} x(s) ds \mid \\
+\ & \mid \int_{t_1}^{t_2} \frac{(t_2-s)^{-\alpha(s,x(s))}}{\Gamma(1-\alpha(s,x(s)))} x(s) ds \mid \Big] \\
\leq\ & \frac{1}{\varsigma}\Big[\int_0^{t_1} \mid \frac{1}{\Gamma(1-\alpha(s,x(s)))} \mid\mid (t_2-s)^{-\alpha(s,x(s))} - (t_1-s)^{-\alpha(s,x(s))} \mid\mid x(s) \mid ds \\
+\ & \int_{t_1}^{t_2} \frac{(t_2-s)^{-\alpha(s,x(s))}}{\Gamma(1-\alpha(s,x(s)))} \mid x(s) \mid ds \Big] \\
\leq\ & \frac{1}{\varsigma}\Big[\mathscr{M}_\Gamma \mid\mid x \mid\mid \int_0^{t_1} (t_1-s)^{-\alpha(s,x(s))} - (t_2-s)^{-\alpha(s,x(s))} ds \\
+\ & \int_{t_1}^{t_2} \frac{(t_2-s)^{-\alpha(s,x(s))}}{\Gamma(1-\alpha(s,x(s)))} \mid x(s) \mid ds \Big] \\
\leq\ & \frac{1}{\varsigma}\Big[\mathscr{M}_\Gamma \mid\mid x \mid\mid \int_0^{t_1} T^{-\alpha(s,x(s))} \Big(\Big(\frac{t_1-s}{T}\Big)^{-\alpha(s,x(s))} - \Big(\frac{t_2-s}{T}\Big)^{-\alpha(s,x(s))} \Big) ds \\
+\ & \mathscr{M}_\Gamma \mid\mid x \mid\mid \int_{t_1}^{t_2} T^{-\alpha(s,x(s))} \Big(\frac{t_2-s}{T}\Big)^{-\alpha(s,x(s))} ds \Big] \\
\leq\ & \frac{1}{\varsigma}\Big[\mathscr{M}_\Gamma \mid\mid x \mid\mid T^* \int_0^{t_1} \Big(\Big(\frac{t_1-s}{T}\Big)^{-\alpha^*} - \Big(\frac{t_2-s}{T}\Big)^{-\alpha^*} \Big) ds \\
+\ & \mathscr{M}_\Gamma \mid\mid x \mid\mid T^* \int_{t_1}^{t_2} \Big(\frac{t_2-s}{T}\Big)^{-\alpha^*} ds \Big] \\
\leq\ & \frac{\mathscr{M}_\Gamma \mid\mid x \mid\mid T^*}{\varsigma[T^{-\alpha^*}(1-\alpha^*)]} \Big[(t_1)^{1-\alpha^*} - (t_2)^{1-\alpha^*} + 2(t_2-t_1)^{1-\alpha^*} \Big].
\end{aligned}
$$

Hence, $\mid Q_1 x(t_2) - Q_1 x(t_1) \mid \to 0$ as $t_2 \to t_1$. It implies that $Q_1(B_R)$ is equicontinuous.

Step 4: Q_2 is a strict contraction

For x, $\tilde{y} \in \mathbb{E}$ and $t \in \mathbb{A}$, we obtain

$$
\begin{aligned}
& \mid Q_2x(t) - Q_2\tilde{y}(t) \mid \\
\leq\ & \frac{1}{\varsigma}\Big[\int_0^t s^{-\sigma}s^{\sigma} \mid f(s,x(s)) - f(s,\tilde{y}(s)) \mid ds\Big] \\
\leq\ & \frac{1}{\varsigma}\Big[\lambda \mid\mid x-\tilde{y} \mid\mid \int_0^t s^{-\sigma}ds\Big] \\
\leq\ & \frac{1}{\varsigma}\Big[\lambda \mid\mid x-\tilde{y} \mid\mid \frac{t^{-\sigma+1}}{-\sigma+1}\Big] \\
\leq\ & \frac{\lambda}{\varsigma}\Big[\frac{T^{-\sigma+1}}{-\sigma+1}\Big] \mid\mid x-\tilde{y} \mid\mid .
\end{aligned}
$$

Consequently by (3.4), the operator Q_2 is a strict contraction.

Therefore, all condition of theorem 1.6 are fulfilled. We infer that the NAVOIVP(3.1) has at least one solution in $\mathbb{E}$.

3.3 Results of uniqueness

In the next result, we shall show the uniqueness of solutions for the NAVOIVP(3.1) based on the Banach contraction principle.

Theorem 3.2 *Let (SY1) and (SY2) be satisfied, if*

$$
\frac{1}{\varsigma}\Big[\lambda\frac{T^{-\sigma+1}}{-\sigma+1} + 4\mathscr{M}_{\Gamma}T^{*}T\Big] < 1, \tag{3.5}
$$

then, the NAVOIVP (3.1) has a unique solution on $\mathbb{E}$.

Proof

Consider the operator

$$
Q : \mathbb{E} \to \mathbb{E},
$$

as follows,

$$
Qx(t) = Q_1x(t) + Q_2x(t),\ \ for\ x \in \mathbb{E}.
$$

For x, $x^{*}\in \mathbb{E}$, we may write

$$| Qx(t) - Qx^*(t) |$$

$$\begin{aligned}
\leq\ & \frac{1}{\varsigma}\Big[\int_0^t s^{-\sigma}s^{\sigma} \mid f(s,x(s)) - f(s,x^*(s)) \mid ds \\
+\ & \mid \int_0^t \frac{(t-s)^{-\alpha(s,x(s))}}{\Gamma(1-\alpha(s,x(s)))}x(s)ds - \int_0^t \frac{(t-s)^{-\alpha(s,x^*(s))}}{\Gamma(1-\alpha(s,x^*(s)))}x^*(s)ds \mid \Big] \\
\leq\ & \frac{1}{\varsigma}\Big[\lambda \mid\mid x - x^* \mid\mid \int_0^t s^{-\sigma}ds \\
+\ & \mid \int_0^t \frac{(t-s)^{-\alpha(s,x(s))}}{\Gamma(1-\alpha(s,x(s)))}x(s)ds - \int_0^t \frac{(t-s)^{-\alpha(s,x^*(s))}}{\Gamma(1-\alpha(s,x^*(s)))}x^*(s)ds \mid \Big] \\
\leq\ & \frac{1}{\varsigma}\Big[\lambda \mid\mid x - x^* \mid\mid \frac{t^{-\sigma+1}}{-\sigma+1} \\
+\ & 2\int_0^t \Big(\sup_{t\in\mathbb{A}} \frac{(t-s)^{-\alpha(s,x(s))}}{\Gamma(1-\alpha(s,x(s)))} + \sup_{t\in\mathbb{A}} \frac{(t-s)^{-\alpha(s,x^*(s))}}{\Gamma(1-\alpha(s,x^*(s)))}\Big) \mid x - x^* \mid ds\Big] \\
\leq\ & \frac{1}{\varsigma}\Big[\lambda \frac{T^{-\sigma+1}}{-\sigma+1} \mid\mid x - x^* \mid\mid \\
+\ & 2 \mid\mid x - x^* \mid\mid \int_0^t (\mathscr{M}_\Gamma T^* + \mathscr{M}_\Gamma T^*)ds\Big] \\
\leq\ & \frac{1}{\varsigma}\Big[\lambda \frac{T^{-\sigma+1}}{-\sigma+1} \mid\mid x - x^* \mid\mid + 4\mathscr{M}_\Gamma T^* \mid\mid x - x^* \mid\mid t\Big] \\
\leq\ & \frac{1}{\varsigma}\Big[\lambda \frac{T^{-\sigma+1}}{-\sigma+1} + 4\mathscr{M}_\Gamma T^* T\Big] \mid\mid x - x^* \mid\mid .
\end{aligned}$$

Consequently by Equation (3.5), the operator Q will be a contraction. Accordingly to Banach contraction principle, the operator Q has a unique fixed point which is the unique solution of the NAVOIVP (3.1)on $\mathbb{E}$.

3.4 Ulam-Hyers stability

Theorem 3.3 *Let the conditions (SY1), (SY2) and inequality (3.5) be satisfied. Then, the NAVOIVP* (3.1) *is Ulam-Hyersstable.*

Proof: Let $\epsilon > 0$ be an arbitrary number and the function $r(t)$ from $C(\mathbb{A}, \mathbb{R})$ satisfy the following inequality

$$|D_{0^+}^{\alpha(t,r(t))}r(t) + \varsigma r'(t) - f(t, r(t))| \leq \epsilon,\ \ t \in \mathbb{A}. \tag{3.6}$$

We integrate both sides of the inequality (3.6), we obtain

$$\Big|r(t) \quad + \quad \frac{1}{\varsigma}\Big[\int_0^t \frac{(t-s)^{-\alpha(s,r(s))}}{\Gamma(1-\alpha(s,r(s)))}r(s) \ \ ds - \int_0^t f(s,r(s)) \ \ ds\Big]\Big| \\ \leq \quad \epsilon T$$

Let $t \in \mathbb{A}$, then

$$\begin{aligned}
& \mid r(t) - x(t) \mid \\
= \quad & \Big|r(t) + \frac{1}{\varsigma}\Big[\int_0^t \frac{(t-s)^{-\alpha(s,x(s))}}{\Gamma(1-\alpha(s,x(s)))}x(s) \ \ ds - \int_0^t f(s,x(s)) \ \ ds\Big]\Big| \\
= \quad & \Big|r(t) + \frac{1}{\varsigma}\Big[\int_0^t \frac{(t-s)^{-\alpha(s,x(s))}}{\Gamma(1-\alpha(s,x(s)))}x(s) \ \ ds - \int_0^t f(s,x(s)) \ \ ds\Big] \\
+ \quad & \frac{1}{\varsigma}\Big[\int_0^t \frac{(t-s)^{-\alpha(s,r(s))}}{\Gamma(1-\alpha(s,r(s)))}r(s) \ \ ds - \int_0^t \frac{(t-s)^{-\alpha(s,r(s))}}{\Gamma(1-\alpha(s,r(s)))}r(s) \ \ ds\Big] \\
+ \quad & \frac{1}{\varsigma}\Big[\int_0^t f(s,r(s)) \ \ ds - \int_0^t f(s,r(s)) \ \ ds\Big]\Big| \\
= \quad & \Big|r(t) + \frac{1}{\varsigma}\Big[\int_0^t \frac{(t-s)^{-\alpha(s,r(s))}}{\Gamma(1-\alpha(s,r(s)))}r(s) \ \ ds - \int_0^t f(s,r(s)) \ \ ds\Big]\Big| \\
+ \quad & \frac{1}{\varsigma}\Big[\Big|\int_0^t \frac{(t-s)^{-\alpha(s,x(s))}}{\Gamma(1-\alpha(s,x(s)))}x(s) \ \ ds - \int_0^t \frac{(t-s)^{-\alpha(s,r(s))}}{\Gamma(1-\alpha(s,r(s)))}r(s) \ \ ds\Big|\Big] \\
+ \quad & \frac{1}{\varsigma}\Big[\int_0^t \mid f(s,r(s)) - f(s,x(s)) \mid ds\Big] \\
\leq \quad & \epsilon T + \frac{1}{\varsigma}\Big[2\int_0^t \Big(\sup_{t\in\mathbb{A}} \frac{(t-s)^{-\alpha(s,x(s))}}{\Gamma(1-\alpha(s,x(s)))} + \sup_{t\in\mathbb{A}} \frac{(t-s)^{-\alpha(s,r(s))}}{\Gamma(1-\alpha(s,r(s)))}\Big) \mid x - r \mid ds \\
+ \quad & \int_0^t s^{-\sigma}s^{\sigma} \mid f(s,r(s)) - f(s,x(s)) \mid ds\Big] \\
\leq \quad & \epsilon T + \frac{1}{\varsigma}\Big[2 \mid r - x \mid\mid \int_0^t (\mathscr{M}_\Gamma T^* + \mathscr{M}_\Gamma T^*)ds + \lambda \mid\mid r - x \mid\mid \int_0^t s^{-\sigma}ds\Big] \\
\leq \quad & \epsilon T + \frac{1}{\varsigma}\Big[4\mathscr{M}_\Gamma T^* \mid\mid r - x \mid\mid \int_0^t ds + \lambda \mid\mid r - x \mid\mid \frac{t^{-\sigma+1}}{-\sigma+1}\Big] \\
\leq \quad & \epsilon T + \frac{1}{\varsigma}\Big[4\mathscr{M}_\Gamma T^* t \mid\mid r - x \mid\mid + \lambda\frac{T^{-\sigma+1}}{-\sigma+1} \mid\mid r - x \mid\mid \Big] \\
\leq \quad & \epsilon T + \frac{1}{\varsigma}\Big[4\mathscr{M}_\Gamma T^* T \mid\mid r - x \mid\mid + \lambda\frac{T^{-\sigma+1}}{-\sigma+1} \mid\mid r - x \mid\mid \Big] \\
\leq \quad & \epsilon T + \frac{1}{\varsigma}\Big[4\mathscr{M}_\Gamma T^* T + \lambda\frac{T^{-\sigma+1}}{-\sigma+1}\Big] \mid\mid r - x \mid\mid .
\end{aligned}$$

Then,

$$\mid\mid r - x \mid\mid \Big(1 - \frac{1}{\varsigma}\Big[4\mathscr{M}_\Gamma T^* T + \lambda\frac{T^{-\sigma+1}}{-\sigma+1}\Big]\Big) \leq \epsilon T.$$

We obtain, for each $t \in \mathbb{A}$

$$| r(t) - x(t) | \leq \| r - x \| \leq \frac{T}{1 - \frac{1}{\varsigma}\Big[4\mathscr{M}_\Gamma T^* T + \lambda \frac{T^{-\sigma+1}}{-\sigma+1}\Big]} \epsilon = c_f \epsilon.$$

Then, the NAVOIVP (3.1) is Ulam-Hyers stable.

3.5 Numerical examples

Example 3.1 *Let the following NAVOIVP*

$$\begin{cases} D^{\frac{t}{3}+\frac{1}{2}} x(t) + 3x'(t) = \frac{\exp(-t+4)}{\sqrt{2}\pi t + \frac{5}{6}}(t^2) + \frac{1}{3}x, \ t \in [0,1], \\ x(0) = 0, \end{cases} \tag{3.7}$$

Let $\varsigma = 3$ *and* $\alpha(t, x(t)) = \frac{t}{3} + \frac{1}{2}$. *Then, we have* α *is a continuous function with* $0 < \alpha(t, x(t)) < \frac{1}{3} + \frac{1}{2} = \frac{5}{6} = \alpha^* < 1$, *and* $\min_{t \in \mathbb{A}} | \alpha(t, x(t)) | = \frac{1}{2}$, *and we have*

$$\begin{aligned} t^{\frac{1}{6}} | f(t,x) - f(t,y) | &= t^{\frac{1}{6}} | \frac{\exp(-t+4)}{\sqrt{2}\pi t + \frac{5}{6}}(t^2) + \frac{1}{3}x - \frac{\exp(-t+4)}{\sqrt{2}\pi t + \frac{5}{6}}(t^2) - \frac{1}{3}y | \\ &= t^{\frac{1}{6}} | \frac{1}{3}x - \frac{1}{3}y | \\ &\leq t^{\frac{1}{6}} \frac{1}{3} | x - y | \\ &\leq \frac{1}{3} | x - y | . \end{aligned}$$

So (SY 1), (SY 2) satisfied with $\lambda = \frac{1}{3}$ *and* $\sigma = \frac{1}{6}$. *In addition to*

$$\begin{aligned} \frac{1}{\varsigma}\Big[\lambda \frac{T^{-\sigma+1}}{-\sigma+1} + 4\mathscr{M}_\Gamma T^* T\Big] &= \frac{1}{3}\Big[\frac{1}{3}\frac{1}{\frac{5}{6}} + 4\frac{1}{\sqrt{\pi}}\Big] = \frac{1}{3}\Big[\frac{1}{3}\frac{6}{5} + \frac{4}{\sqrt{\pi}}\Big] = \frac{1}{3}\Big[\frac{2}{5} + \frac{4}{\sqrt{\pi}}\Big] \\ &= \frac{1}{3}\Big[\frac{2\sqrt{\pi} + 20}{5\sqrt{\pi}}\Big] = \frac{23.54}{26.58} \approx 0.89 < 1. \end{aligned}$$

According to theorem (3.2), the NAVOIVP (3.7) has a unique solution, and by theorem (3.3), the NAVOIVP (3.7) is Ulam-Hyers stable.

Example 3.2 *Let the following NAVOIVP*

$$\begin{cases} D^{\frac{t}{2}+\frac{1}{4}} x(t) + 6x'(t) = \frac{8(t+1)}{4\sqrt{2}\pi} + (\exp(\sqrt{t+1})) + \frac{\pi}{4}x, \ t \in [0,1], \\ x(0) = 0 \end{cases} \tag{3.8}$$

Let $\varsigma = 6$ *and* $\alpha(t, x(t)) = \frac{t}{2} + \frac{1}{4}$. *Then, we have* α *is a continuous function with* $0 < \alpha(t, x(t)) < \frac{1}{2} + \frac{1}{4} = \frac{3}{4} = \alpha^* < 1$, *and* $\min_{t \in \mathbb{A}} \mid \alpha(t, x(t)) \mid = \frac{1}{4}$, *and we have*

$$
\begin{aligned}
t^{\frac{1}{16}} \mid f(t,x) - f(t,y) \mid &= t^{\frac{1}{16}} \mid \frac{8(t+1)}{4\sqrt{2}\pi} + (\exp(\sqrt{t+1})) + \frac{\pi}{4}x - \frac{8(t+1)}{4\sqrt{2}\pi} \\
&- (\exp(\sqrt{t+1})) - \frac{\pi}{4}y \mid \\
&= t^{\frac{1}{16}} \mid \frac{\pi}{4}x - \frac{\pi}{4}y \mid \\
&\leq t^{\frac{1}{16}} \frac{\pi}{4} \mid x - y \mid \\
&\leq \frac{\pi}{4} \mid x - y \mid .
\end{aligned}
$$

So (SY 1), (SY 2) satisfied with $\lambda = \frac{\pi}{4}$ *and* $\sigma = \frac{1}{16}$. *In addition to*

$$
\begin{aligned}
\frac{1}{6}\Big[\lambda \frac{T^{-\sigma+1}}{-\sigma+1} + 4\mathscr{M}_{\Gamma} T^* T\Big] &= \frac{1}{6}\Big[\frac{\pi}{4}\frac{1}{\frac{15}{16}} + 4 \times \frac{1}{1.2254}\Big] = \frac{1}{6}\Big[\frac{\pi}{4}\frac{16}{15} + 3.26\Big] \\
&= \frac{1}{6}\Big[\frac{4\pi}{15} + 3.26\Big] \approx 0.68 < 1.
\end{aligned}
$$

According to theorem (3.2), the NAVOIVP (3.8) has a unique solution. By theorem (3.3), the NAVOIVP (3.8) is Ulam-Hyers stable.

Chapter 4

Initial value problem for ordinary and fractional differential equations with finite delay [3]

4.1 Introduction

We apply the new technique to the following finite delay of variable ordre initial value problem (FDVOIVP)

$$\begin{cases} \mathbb{D}^{\alpha(t)}x(t) + \varsigma x'(t) = f(t, x_t), \ \ t \in \mathbb{A} := [0, T], \\ x(t) = \phi(t), \ t \in [-\bar{z}, 0], \end{cases} \tag{4.1}$$

where $\alpha : \mathbb{A} \to (0, 1)$, $f : \mathbb{A} \times C([-\bar{z}, 0], \mathbb{R}) \to \mathbb{R}$ is a specific function , $\phi \in C([-\bar{z}, 0], \mathbb{R})$ with $\phi(0) = 0$. We designate by x_t the element of $C([-\bar{z}, 0], \mathbb{R})$, defined by

$$x_t(\theta) = x(t + \theta), \theta \in [-\bar{z}, 0],$$

[3] **S. Guedim,** A. Benkerrouche, K. Sitthithakerngkiet, M. S. Souid, A. Amara, Initial value problem for mixed differential equations of variable order with finite delay. *Symmetry*, **17**, 295, (2025), 1-13. https:// doi.org/10.3390/sym17020295.

for any function x defined on $[-\bar{z}, T]$ and any t in $\mathbb{A}$. Here, $\mathbb{D}^{\alpha(t)}$ is the variable order $\alpha(t)$ fractional derivative of Riemann-Liouville, and $x_t(.)$ is the state's history from time $t-\bar{z}$ to the present time t.

4.2 Existence criteria

Remark 4.1 *Let's provide the following notes:*

1. *The function* Γ*(1-*α*(t)) is continuous as a composition of two continuous function, we can let*
   $M_\Gamma = \max \mid \frac{1}{\Gamma(1-\alpha(t))} \mid .$
2. *By the continuity of the function* $\alpha(t)$*, we let*
   $T^{-\alpha(t)} \leq 1$ *if* $1 \leq T < \infty, \quad T^{-\alpha(t)} \leq T^{-\alpha^*}$ *if* $0 \leq T \leq 1.$
   We conclude that $T^{-\alpha(t)} \leq \max\ (1, T^{-\alpha^*}) = T^*$.

Definition 4.1 *The function* $x \in \mathbb{E}_2$ *is a solution of FDVOIVP (4.1), if* x *satisfies the equation* $\mathbb{D}^{\alpha(t)}x(t) + \varsigma x'(t) = f(t, x_t)$ *on* $\mathbb{A}$*, and the condition* $x(t) = \phi(t)$ *on* $[-\bar{z}, 0]$.

We will need the following lemma to solve FDVOIVP (4.1).

Lemma 4.1 x *is a solution of the following initial value problem for the fractional differential equation*

$$\begin{cases} \mathbb{D}^{\alpha(t)}x(t) + \varsigma x'(t) = \nu(t), \quad t \in \mathbb{A}: = [0, T], \\ x(0) = 0, \end{cases} \tag{4.2}$$

where $0 < \alpha(t) < 1$*, and* $\nu\colon (0, T] \to \mathbb{R}$ *be continuous if and only if* x *is a solution of the fractional integral equation*

$$\int_0^t \frac{(t-s)^{-\alpha(s)}}{\Gamma(1-\alpha(s))} x(s)\ ds + \varsigma x(t) = \int_0^t \nu(s)\ ds. \tag{4.3}$$

Proof

By using the definition of fractional derivative of variable order provided by (1.2), it is possible to express the FDVOIVP (4.1) as follows:

$$(\frac{d}{dt})\int_0^t \frac{(t-s)^{-\alpha(s)}}{\Gamma(1-\alpha(s))}x(s)\ \ ds+\varsigma x^{'}(t)=\nu(t).$$

Then,

$$\int_0^t \frac{(t-s)^{-\alpha(s)}}{\Gamma(1-\alpha(s))}x(s)\ \ ds+\varsigma x(t)=\int_0^t \nu(s)\ \ ds+c_1. \tag{4.4}$$

Evaluating Eq (4.4) at $t=0$, gives us $c_1=0$.

Thus,

$$\int_0^t \frac{(t-s)^{-\alpha(s)}}{\Gamma(1-\alpha(s))}x(s)\ \ ds+\varsigma x(t)=\int_0^t \nu(s)\ \ ds.$$

Conversely, by derivation both sides of the equation 4.3, we have

$$(\frac{d}{dt})\int_0^t \frac{(t-s)^{-\alpha(s)}}{\Gamma(1-\alpha(s))}x(s)\ \ ds+\varsigma x^{'}(t)=\nu(t),$$

which means that x is a solution of FDVOIVP (4.2).

Forms the basis of the first finding.

We start by introducing the following assumptions.

(FY1) $f\colon \mathbb{A}\times C([-\bar{z},0],\mathbb{R})\to\mathbb{R}$ is a continuous function.

(FY2) There exist $\beta_1,\beta_2\in C(\mathbb{A},\mathbb{R}^+)$ such that $\mid f(t,x)\mid\leq \beta_1(t)+\beta_2(t)\mid\mid x\mid\mid_C$ for $t\in\mathbb{A}$ and each $x\in C([-\bar{z},0],\mathbb{R})$.

Theorem 4.1 *Assume that the condition (FY1), (FY2) be satisfied. Then, the FDVOIVP (4.1) has at least one solution on* $\mathbb{E}_2$.

Proof Convert the FDVOIVP (4.1) into a fixed point problem. Think about the operator

$$N:\mathbb{E}_2\to\mathbb{E}_2,$$

as follows,

$$N(x)(t)=\begin{cases}\phi(t), & if\ \ t\in[-\bar{z},0],\\ \frac{1}{\varsigma}\Big[\int_0^t f(s,x_s)\ \ ds-\int_0^t \frac{(t-s)^{-\alpha(s)}}{\Gamma(1-\alpha(s))}x(s)\ \ ds\Big], & if\ \ t\in[0,T].\end{cases}$$

Let the constant R such that:

$$R\geq\max\left\{\frac{\frac{||\beta_1||_\infty T}{\varsigma}}{1-\frac{1}{\varsigma}\Big[\frac{M_\Gamma T^* T}{1-\alpha^*}+\mid\mid\beta_2\mid\mid_\infty T\Big]},\mid\mid\phi\mid\mid_C\right\}.$$

We consider the set

$$B_R = \{x \in \mathbb{E}_2, \| x \| \leq R\}.$$

Clearly, B_R is non empty, bounded, convex and closed.

We now prove that N satisfies the conditions given by Theorem 1.5. The argument will be put into practice in a few stages .

Step 1: $N(B_R) \subseteq B_R$.

For $x \in B_R$, then for every $t \in [0, T]$, we have

$$
\begin{aligned}
&\mid N(x)(t) \mid \\
\leq\ & \frac{1}{\varsigma}\Big[\mid \int_0^t f(s, x_s) \ \ ds \mid + \mid \int_0^t \frac{(t-s)^{-\alpha(s)}}{\Gamma(1-\alpha(s))} x(s) \ \ ds \mid \Big] \\
\leq\ & \frac{1}{\varsigma}\Big[\int_0^t f(s, x_s) + \int_0^t \frac{(t-s)^{-\alpha(s)}}{\Gamma(1-\alpha(s))} \mid x(s) \mid \ \ ds \Big] \\
\leq\ & \frac{1}{\varsigma}\Big[\int_0^t \beta_1(s) + \beta_2(s) \| x_s \|_C \, ds + M_\Gamma \int_0^t T^{-\alpha(s)} \Big(\frac{t-s}{T}\Big)^{-\alpha(s)} \mid x(s) \mid ds \Big] \\
\leq\ & \frac{1}{\varsigma}\Big[\int_0^t \beta_1(s) ds + \int_0^t \beta_2(s) \| x_s \|_C \, ds + M_\Gamma T^* \int_0^t \Big(\frac{t-s}{T}\Big)^{-\alpha(s)} \mid x(s) \mid ds \Big] \\
\leq\ & \frac{1}{\varsigma}\Big[\| \beta_1 \|_\infty t + \| \beta_2 \|_\infty \| x \|_{\mathbb{E}_2} t + \frac{M_\Gamma T^*}{T^{-\alpha^*}} \| x \|_\infty \int_0^t (t-s)^{-\alpha^*} ds \Big] \\
\leq\ & \frac{1}{\varsigma}\Big[\| \beta_1 \|_\infty T + \| \beta_2 \|_\infty \| x \|_{\mathbb{E}_2} T + \frac{M_\Gamma T^*}{T^{-\alpha^*}} \frac{t^{1-\alpha^*}}{(1-\alpha^*)} \| x \|_{\mathbb{E}_2} \Big] \\
\leq\ & \frac{1}{\varsigma}\Big[\| \beta_1 \|_\infty T + \| \beta_2 \|_\infty \| x \|_{\mathbb{E}_2} T + \frac{M_\Gamma T^* T}{(1-\alpha^*)} \| x \|_{\mathbb{E}_2} \Big] \\
\leq\ & \frac{1}{\varsigma}\Big[\frac{M_\Gamma T^* T}{1-\alpha^*} + \| \beta_2 \|_\infty \| T \Big] \| x \|_{\mathbb{E}_2} + \frac{\| \beta_1 \|_\infty T}{\varsigma} \\
\leq\ & \frac{1}{\varsigma}\Big[\frac{M_\Gamma T^* T}{1-\alpha^*} + \| \beta_2 \|_\infty \| T \Big] R + \frac{\| \beta_1 \|_\infty T}{\varsigma} \\
\leq\ & R.
\end{aligned}
$$

If $t \in [-\bar{z}, 0]$, we have

$$\mid N(x)(t) \mid \leq \| \phi \|_C \leq R.$$

Thus, $N(B_R) \subseteq B_R$.

Step 2: N is continuous.

Let (x_n) be a sequence such that $x_n \to x$ in $\mathbb{E}_2$. If $t \in [-\bar{z}, 0]$, then

$$| N(x_n)(t) - N(x)(t) |= 0.$$

For $t \in \mathbb{A}$, we have

$$
\begin{aligned}
& | N(x_n)(t) - N(x)(t) | \\
\leq & \frac{1}{\varsigma}\Big[\int_0^t \sup_{t\in[0,T]} | f(s, x_{ns}) - f(s, x_s) | ds + \int_0^t \frac{(t-s)^{-\alpha(s)}}{\Gamma(1-\alpha(s))} | x(s) - x_n(s) | ds\Big] \\
\leq & \frac{1}{\varsigma}\Big[|| f(., x_n) - f(., x) ||_\infty \int_0^t ds + M_\Gamma || x_n - x ||_\infty \int_0^t (t-s)^{-\alpha(s)} ds\Big] \\
\leq & \frac{1}{\varsigma}\Big[t || f(., x_n) - f(., x) ||_\infty + M_\Gamma T^* || x_n - x ||_\infty \int_0^t \Big(\frac{t-s}{T}\Big)^{-\alpha^*} ds\Big] \\
\leq & \frac{1}{\varsigma}\Big[T || f(., x_n) - f(., x) ||_\infty + \frac{M_\Gamma T^*}{T^{-\alpha^*}} \frac{(t)^{1-\alpha^*}}{(1-\alpha^*)} || x_n - x ||_\infty \Big] \\
\leq & \frac{1}{\varsigma}\frac{M_\Gamma T^* T}{1-\alpha^*} | n - x ||_\infty + \frac{T}{\varsigma} || f(., x_n) - f(., x) ||_\infty .
\end{aligned}
$$

Since, $\left(\frac{1}{\varsigma}\frac{M_\Gamma T^* T}{1-\alpha^*}\right) || x_n - x ||_\infty \to 0$ as $n \to \infty$ and $\frac{T}{\varsigma} || f(., x_n) - f(., x) ||_\infty \to 0$ as $n \to \infty$, so

$$| N(x_n)(t) - N(x)(t) | \to 0 \;\; as \;\; n \to \infty.$$

Then,

$$|| N(x_n)(t) - N(x)(t) ||_{\mathbb{E}_2} \to 0 \;\; as \;\; n \to \infty.$$

The above relation shows that the operator N is continuous on $\mathbb{E}$.

Step 3: N is compact.

Here, we intend to prove the relative compactness of $N(B_R)$ which means that N is compact. Evidently, $N(B_R)$ is uniformly bounded, due to step1, we saw that

$$N(B_R) = \{N(x) : x \in B_R\} \subset B_R.$$

Hence, for every $x \in B_R$, we obtain $|| N(x) || \leq R$ meaning the uniform boundedness of $N(B_R)$. Firstly, we can know that the function $w(t) = a^t - b^t, t \in (-1, 0), 0 < a < b < 1$, is decreasing. Indeed, since $\ln a < \ln b < 0$ and $a^t > b^t > 0$, we have that

$$w'(t) = a^t \ln a - b^t \ln b < b^t \ln a - b^t \ln b = b^t(\ln a - \ln b) < 0,$$

which implies that $w(t)$ is a decreasing. Thus, for $K(s) = \left(\frac{t_1-s}{T}\right)^{-\alpha(s)} - \left(\frac{t_2-s}{T}\right)^{-\alpha(s)}$ (where $0 < \frac{t_1-s}{T} < \frac{t_2-s}{T} < 1$), we may look $K(s)$ as the same type as $w(s)$, then $K(s)$ is decreasing with respect to its exponent $-\alpha(s)$.

If $t_1, t_2 \in [-\bar{z}, 0]$

$$|Nx(t_2) - Nx(t_1)| = |\phi(t_2) - t(\jmath_1)| \to 0, \quad \text{as } t_2 \to t_1,$$

by the continuity of ϕ.

For $t_1, t_2 \in \mathbb{A}$, $t_1 < t_2$ and $x \in B_R$, we have

$$
\begin{aligned}
& \mid Nx(t_2) - Nx(t_1) \mid \\
\leq\ & \frac{1}{\varsigma}\Big[\mid \int_0^{t_2} f(s, x_s)\ ds - \int_0^{t_1} f(s, x_s)\ ds \mid \\
+\ & \mid \int_0^{t_1} \frac{(t_1 - s)^{-\alpha(s)}}{\Gamma(1-\alpha(s)))} x(s) - \int_0^{t_2} \frac{(t_2 - s)^{-\alpha(s)}}{\Gamma(1-\alpha(s))} x(s) ds \mid \Big] \\
\leq\ & \frac{1}{\varsigma}\Big[\mid \int_{t_1}^{t_2} f(s, x_s) \mid \\
+\ & \mid \int_0^{t_1} \frac{(t_2 - s)^{-\alpha(s)}}{\Gamma(1-\alpha(s))} x(s) - \frac{(t_1 - s)^{-\alpha(s)}}{\Gamma(1-\alpha(s))} x(s) ds \mid \\
+\ & \mid \int_{t_1}^{t_2} \frac{(t_2 - s)^{-\alpha(s)}}{\Gamma(1-\alpha(s))} x(s) ds \mid \Big] \\
\leq\ & \frac{1}{\varsigma}\Big[\int_{t_1}^{t_2} \mid f(s, x_s) \mid ds \\
+\ & \int_0^{t_1} \mid \frac{1}{\Gamma(1-\alpha(s))} \mid\mid (t_2 - s)^{-\alpha(s)} - (t_1 - s)^{-\alpha(s)} \mid\mid x(s) \mid ds \\
+\ & \int_{t_1}^{t_2} \frac{(t_2 - s)^{-\alpha(s)}}{\Gamma(1-\alpha(s))} \mid x(s) \mid ds \Big] \\
\leq\ & \frac{1}{\varsigma}\Big[\int_{t_1}^{t_2} \beta_1(s) + \beta_2(s) \mid\mid x \mid\mid_C + M_\Gamma \mid\mid x \mid\mid_{\mathbb{E}_2} \int_0^{t_1} (t_1 - s)^{-\alpha(s)} - (t_2 - s)^{-\alpha(s)} ds \\
+\ & \int_{t_1}^{t_2} \frac{(t_2 - s)^{-\alpha(s)}}{\Gamma(1-\alpha(s))} \mid x(s) \mid ds \Big] \\
\leq\ & \frac{1}{\varsigma}\Big[\mid\mid \beta_1 \mid\mid_\infty (t_2 - t_1) + \mid\mid \beta_2 \mid\mid_\infty \mid\mid x \mid\mid_{\mathbb{E}_2} (t_2 - t_1) \\
+\ & M_\Gamma \mid\mid x \mid\mid_{\mathbb{E}_2} \int_0^{t_1} T^{-\alpha(s)} \Big[\Big(\frac{t_1 - s}{T}\Big)^{-\alpha(s)} - \Big(\frac{t_2 - s}{T}\Big)^{-\alpha(s)} \Big] ds
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad M_\Gamma \mid\mid x \mid\mid_{\mathbb{E}_2} \int_{t_1}^{t_2} T^{-\alpha(s)} \Big(\frac{t_2-s}{T}\Big)^{-\alpha(s)} ds\Big] \\
&\leq \quad \frac{1}{\varsigma}\Big[\Big(\mid\mid \beta_1 \mid\mid_\infty + \mid\mid \beta_2 \mid\mid_\infty \mid\mid x \mid\mid_{\mathbb{E}_2} \Big)(t_2-t_1) \\
&+ \quad M_\Gamma \mid\mid x \mid\mid_{\mathbb{E}_2} T^* \int_0^{t_1} \Big[\Big(\frac{t_1-s}{T}\Big)^{-\alpha^*} - \Big(\frac{t_2-s}{T}\Big)^{-\alpha^*}\Big] ds \\
&+ \quad M_\Gamma \mid\mid x \mid\mid_{\mathbb{E}_2} T^* \int_{t_1}^{t_2} \Big(\frac{t_2-s}{T}\Big)^{-\alpha^*} ds\Big] \\
&\leq \quad \frac{1}{\varsigma}\Big[\Big(\mid\mid \beta_1 \mid\mid_\infty + \mid\mid \beta_2 \mid\mid_\infty \mid\mid x \mid\mid_{\mathbb{E}_2} \Big)(t_2-t_1) \\
&+ \quad \Big[(t_1)^{1-\alpha^*} - (t_2)^{1-\alpha^*} + 2(t_2-t_1)^{1-\alpha^*}\Big] \frac{M_\Gamma \mid\mid x \mid\mid_{\mathbb{E}_2} T^*}{T^{-\alpha^*}(1-\alpha^*)}\Big].
\end{aligned}
$$

Hence, $\mid Nx(t_2) - Nx(t_1) \mid \to 0$ as $t_2 \to t_1$. It implies that $N(B_R)$ is equicontinuous. Considering the first three phases and the Ascoli-Arzela theorem, we determine that N is completely continuous set. We infer that N has a fixed point x in B_R, which is the solution to FDVOIVP(4.1), as a result of theorem 1.5.

4.3 Results of uniqueness

The next result is based on Theorem (1.4).

We start by introducing the following assumption.

(FY3) Let $f : \mathbb{A} \times C([-\bar{z}, 0], \mathbb{R}) \to \mathbb{R}$ and there exist a constants $0 \leq \sigma < \min_{t\in\mathbb{A}|\alpha(t)|}$, $p > 0$, such that: $t^\sigma \mid f(t,x) - f(t,y) \mid \leq p \mid\mid x - y \mid\mid_C, \quad \forall x, y \in C([-\bar{z}, 0], \mathbb{R}), t \in \mathbb{A}$,

Theorem 4.2 *Assume that the condition (FY3) be satisfied. If*

$$
\frac{1}{\varsigma}\Big(\frac{M_\Gamma T^* T}{1-\alpha^*} + p\frac{T^{-\sigma+1}}{-\sigma+1}\Big) < 1, \tag{4.5}
$$

then, the FDVOIVP (4.1) has a unique solution on $\mathbb{E}_2$.

Proof

We consider the same operator

$$
N : \mathbb{E}_2 \to \mathbb{E}_2,
$$

defined as follows,

$$N(x)(t)=\begin{cases}\phi(t), & if\ \ t\in[-\bar{z},0],\\ \frac{1}{\varsigma}\Big[\int_0^t f(s,x_s)\ \ ds-\int_0^t \frac{(t-s)^{-\alpha(s)}}{\Gamma(1-\alpha(s))}x(s)\ \ ds\Big], & if\ \ t\in[0,T].\end{cases}$$

Let $x, x^* \in \mathbb{E}_2$. If $t \in [-\bar{z}, 0]$, then

$$\mid N(x)(t)-N(x^*)(t)\mid=0.$$

For $t \in \mathbb{A}$, we have

$$\begin{aligned}
&\mid N(x)(t)-N(x^*)(t)\mid\\
\leq\ & \frac{1}{\varsigma}\Big[\int_0^t s^{-\sigma}s^{\sigma}\mid f(s,x_s)-f(s,x_s^*)\mid ds\\
+\ & \int_0^t \frac{(t-s)^{-\alpha(s)}}{\Gamma(1-\alpha(s))}\mid x^*(s)-x(s)\mid ds\Big]\\
\leq\ & \frac{1}{\varsigma}\Big[p\int_0^t s^{-\sigma}\mid\mid x_s-x_s^*\mid\mid_C ds+M_\Gamma\mid\mid x-x^*\mid\mid_{\mathbb{E}_2}\int_0^t (t-s)^{-\alpha(s)}ds\Big]\\
\leq\ & \frac{1}{\varsigma}\Big[p\mid\mid x-x^*\mid\mid_{\mathbb{E}_2}\frac{t^{-\sigma+1}}{-\sigma+1}+M_\Gamma T^*\mid\mid x-x^*\mid\mid_{\mathbb{E}_2}\int_0^t\Big(\frac{t-s}{T}\Big)^{-\alpha^*}ds\Big]\\
\leq\ & \frac{1}{\varsigma}\Big[p\frac{T^{-\sigma+1}}{-\sigma+1}\mid\mid x-x^*\mid\mid_{\mathbb{E}_2}+\frac{M_\Gamma T^*}{T^{-\alpha^*}}\frac{(t)^{1-\alpha^*}}{(1-\alpha^*)}\mid\mid x-x^*\mid\mid_{\mathbb{E}_2}\Big]\\
\leq\ & \frac{1}{\varsigma}\Big[p\frac{T^{-\sigma+1}}{-\sigma+1}\mid\mid x-x^*\mid\mid_{\mathbb{E}_2}+\frac{M_\Gamma T^* T}{1-\alpha^*}\mid\mid x-x^*\mid\mid_{\mathbb{E}_2}\Big]\\
\leq\ & \frac{1}{\varsigma}\Big(\frac{M_\Gamma T^* T}{1-\alpha^*}+p\frac{T^{-\sigma+1}}{-\sigma+1}\Big)\mid\mid x-x^*\mid\mid_{\mathbb{E}_2}.
\end{aligned}$$

Consequently by Equation (4.5), the operator N is a contraction. Accordingly to Banach contraction principle, the operator N has a unique fixed point which is the unique solution of the FDVOIVP (4.1).

4.4 Ulam-Hyers stability

Definition 4.2 *[30] On account of FDVOIVP* (4.1)*, consider the inequality*

$$|D_{0^+}^{\alpha(t)}r(t)+\varsigma r'(t)-f(t,r_t)|\leq\epsilon,\ \ t\in\mathbb{A}. \tag{4.6}$$

We say that FDVOIVP (4.1) *is Ulam-Hyers stable if there is* $c_f > 0$ *in a way that for any* $\epsilon > 0$ *and for any solution* $r \in C(\mathbb{A}, \mathbb{R})$ *of (4.6), there is a solution* $x \in C(\mathbb{A}, \mathbb{R})$ *of*

FDVOIVP (4.1), *such that*

$$|r(t)-x(t)| \leq c_f\epsilon,\ t \in \mathbb{A}.$$

Theorem 4.3 *Let the condition (FY3) and inequality (4.5) be satisfied. Then, the FD-VOIVP* (4.1) *is Ulam-Hyers stable.*

Proof. Let $\epsilon > 0$ be an arbitrary number and the function $y(t)$ from $C(\mathbb{A},\mathbb{R})$ satisfy the following inequality

$$|D_{0^+}^{\alpha(t)}r(t)+\varsigma r^{'}(t)-f(t,r_t)| \leq \epsilon,\ t \in \mathbb{A}. \tag{4.7}$$

We integrate both sides of the inequality (4.7), we obtain

$$\begin{aligned}\Big|r(t) \ &+ \ \frac{1}{\varsigma}\Big[\int_0^t \frac{(t-s)^{-\alpha(s)}}{\Gamma(1-\alpha(s))}r(s)\ \ ds - \int_0^t f(s,r_s)\ \ ds\Big]\Big| \\ &\leq \ \epsilon T.\end{aligned}$$

If $t \in [-\bar{z},0)$

$$|r(t)-x(t)| = 0 \leq c_\eta\epsilon$$

.

Let $t \in \mathbb{A}$, then

$$|\ r(t)-x(t)\ |$$

$$
\begin{aligned}
&= \Big| r(t) + \frac{1}{\varsigma}\Big[\int_0^t \frac{(t-s)^{-\alpha(s)}}{\Gamma(1-\alpha(s))}x(s) \ \ ds - \int_0^t f(s,x_s) \ \ ds\Big]\Big| \\
&\leq \Big| r(t) + \frac{1}{\varsigma}\Big[\int_0^t \frac{(t-s)^{-\alpha(s)}}{\Gamma(1-\alpha(s))}r(s) \ \ ds - \int_0^t f(s,r_s) \ \ ds\Big]\Big| \\
&+ \frac{1}{\varsigma}\Big|\int_0^t \frac{(t-s)^{-\alpha(s)}}{\Gamma(1-\alpha(s))}x(s) \ \ ds - \int_0^t \frac{(t-s)^{-\alpha(s)}}{\Gamma(1-\alpha(s))}r(s) \ \ ds\Big| \\
&+ \frac{1}{\varsigma}\int_0^t \mid f(s,r_s) - f(s,x_s) \mid ds \\
&\leq \epsilon T + \frac{1}{\varsigma}\Big[\int_0^t \frac{(t-s)^{-\alpha(s)}}{\Gamma(1-\alpha(s))} \mid x(s) - r(s) \mid ds \\
&+ \int_0^t s^{-\sigma}s^{\sigma} \mid f(s,r_s) - f(s,x_s) \mid ds\Big] \\
&\leq \epsilon T + \frac{1}{\varsigma}\Big[M_\Gamma T^* \mid\mid y - x \mid\mid_{\mathbb{E}_2} \int_0^t \Big(\frac{t-s}{T}\Big)^{-\alpha^*} ds + p \mid\mid r - x \mid\mid_{\mathbb{E}_2} \frac{t^{-\sigma+1}}{-\sigma+1}\Big] \\
&\leq \epsilon T + \frac{1}{\varsigma}\Big[\frac{M_\Gamma T^*}{T^{-\alpha^*}} \frac{(t)^{1-\alpha^*}}{(1-\alpha^*)} \mid\mid r - x \mid\mid_{\mathbb{E}_2} + p\frac{T^{-\sigma+1}}{-\sigma+1} \mid\mid r - x \mid\mid_{\mathbb{E}_2} \Big] \\
&\leq \epsilon T + \frac{1}{\varsigma}\Big[\frac{M_\Gamma T^* T}{1-\alpha^*} \mid\mid r - x \mid\mid_{\mathbb{E}_2} + p\frac{T^{-\sigma+1}}{-\sigma+1} \mid\mid r - x \mid\mid_{\mathbb{E}_2} \Big] \\
&\leq \epsilon T + \frac{1}{\varsigma}\Big(\frac{M_\Gamma T^* T}{1-\alpha^*} + p\frac{T^{-\sigma+1}}{-\sigma+1}\Big) \mid\mid r - x \mid\mid_{\mathbb{E}_2} .
\end{aligned}
$$

Then,

$$
\mid\mid r - x \mid\mid_{\mathbb{E}_2} \Big(1 - \frac{1}{\varsigma}\Big[\frac{M_\Gamma T^* T}{1-\alpha^*} + p\frac{T^{-\sigma+1}}{-\sigma+1}\Big]\Big) \leq \epsilon T.
$$

We obtain, for each $t \in \mathbb{A}$

$$
\mid r(t) - x(t) \mid \leq \mid\mid r - x \mid\mid_{\mathbb{E}_2} \leq \frac{T}{1 - \frac{1}{\varsigma}\Big[\frac{M_\Gamma T^* T}{1-\alpha^*} + p\frac{T^{-\sigma+1}}{-\sigma+1}\Big]}\epsilon = c_f \epsilon.
$$

Then, the FDVOIVP (4.1) is Ulam-Hyers stable.

4.5 Numerical examples

Example 4.1 *Let the following FDVOIVP*

$$
\begin{cases}
D^{\frac{t}{3}+\frac{1}{2}}x(t) + 5x^{'}(t) = \log(t+1) + \pi^{\frac{\exp(-t+4)}{\sqrt{2}\pi t + \frac{5}{6}}}(t^2) + 4\cos(t) + \frac{\sqrt{\pi}}{3}x, \ \ t \in [0,T] = [0,1], \\
x(t) = \phi(t), \ \ t \in [-\bar{z}, 0].
\end{cases}
\tag{4.8}
$$

Let $\varsigma = 5$ *and* $\alpha(t, x(t)) = \frac{t}{3} + \frac{1}{2}$, *then we have* α *is a continuous function with* $0 < \alpha(t, x(t)) < \frac{1}{3} + \frac{1}{2} = \frac{5}{6} = \alpha^* < 1$, *and* $\min_{t\in\mathbb{A}} \mid \alpha(t, x(t)) \mid = \frac{1}{2}$, *and we have*

$$\begin{aligned} t^{\frac{1}{3}} \mid f(t,x) - f(t,y) \mid &= t^{\frac{1}{3}} \mid \log(t+1) + \pi\frac{\exp(-t+4)}{\sqrt{2}\pi t + \frac{5}{6}}(t^2) + 4\cos(t) + \frac{\sqrt{\pi}}{3}x \\ &- \log(t+1) - \pi\frac{\exp(-t+4)}{\sqrt{2}\pi t + \frac{5}{6}}(t^2) - 4\cos(t) - \frac{\sqrt{\pi}}{3}y \mid \\ &= t^{\frac{1}{3}} \mid \frac{\sqrt{\pi}}{3}x - \frac{\sqrt{\pi}}{3}y \mid \\ &\leq t^{\frac{1}{3}}\frac{\sqrt{\pi}}{3} \mid x - y \mid \\ &\leq \frac{\sqrt{\pi}}{3} \mid x - y \mid . \end{aligned}$$

So (FY 3) satisfied with $\sigma = \frac{1}{3}$ *and* $p = \frac{\sqrt{\pi}}{3}$. *In addition to*

$$\frac{1}{\varsigma}\Big[p\frac{T^{-\sigma+1}}{-\sigma+1} + 4M_\Gamma T^* T\Big] = \frac{1}{5}\Big[\frac{\sqrt{\pi}}{3}\frac{1}{\frac{2}{3}} + 4\frac{1}{\sqrt{\pi}}\Big] = \frac{1}{5}\Big[\frac{\sqrt{\pi}}{2} + \frac{4}{\sqrt{\pi}}\Big] = \frac{1}{5}\Big[\frac{\pi+8}{2\sqrt{\pi}}\Big] = \frac{11.14}{17.72} \approx 0.63 < 1.$$

According to theorem (4.2), the FDVOIVP (4.8) has a unique solution, and by theorem (4.3), the FDVOIVP (4.8) is Ulam-Hyers stable.

Example 4.2 *Let the following FDVOIVP*

$$\begin{cases} D^{\frac{t}{2}+\frac{1}{4}}x(t) + 8x'(t) = \frac{\log(t+1)}{t+1}\frac{8(t+1)}{4\sqrt{2}\pi} + (\exp(\sqrt{t+1})) + 3\sin(t) + \frac{\pi}{2}x, \ t \in [0, T] = [0, 1], \\ x(t) = \phi(t), \ \ t \in [-\bar{z}, 0]. \end{cases} \tag{4.9}$$

Let $\varsigma = 8$, $\alpha(t, x(t)) = \frac{t}{2} + \frac{1}{4}$, *then we have* α *is a continuous function with* $0 < \alpha(t, x(t)) < \frac{1}{2} + \frac{1}{4} = \frac{3}{4} = \alpha^* < 1$, *and* $\min_{t\in\mathbb{A}} \mid \alpha(t, x(t)) \mid = \frac{1}{4}$, *and we have*

$$\begin{aligned} t^{\frac{1}{8}} \mid f(t,x) - f(t,y) \mid &= t^{\frac{1}{8}} \mid \frac{\log(t+1)}{t+1}\frac{8(t+1)}{4\sqrt{2}\pi} + (\exp(\sqrt{t+1})) + 3\sin(t) + \frac{\pi}{2}x \\ &- \frac{\log(t+1)}{t+1}\frac{8(t+1)}{4\sqrt{2}\pi} - (\exp(\sqrt{t+1})) - 3\sin(t) - \frac{\pi}{2}y \mid \\ &= t^{\frac{1}{8}} \mid \frac{\pi}{2}x - \frac{\pi}{2}y \mid \\ &\leq t^{\frac{1}{8}}\frac{\pi}{2} \mid x - y \mid \\ &\leq \frac{\pi}{2} \mid x - y \mid . \end{aligned}$$

So (FY 3) satisfied with $\sigma = \frac{1}{8}$ *and* $p = \frac{\pi}{2}$. *In addition to*

$$\frac{1}{8}\Big[p\frac{T^{-\sigma+1}}{-\sigma+1} + 4M_\Gamma T^* T\Big] = \frac{1}{8}\Big[\frac{\pi}{2}\frac{1}{\frac{7}{8}} + 4 \times (\frac{1}{1.2254})\Big] = \frac{1}{8}\Big[\frac{4\pi}{7} + 3.26\Big] = \frac{5.05}{8} \approx 0.63 < 1.$$

According to theorem (4.2), the FDVOIVP (4.9) has a unique solution. By theorem (4.3), the FDVOIVP (4.9) is Ulam-Hyers stable.

Conclusion

In this thesis, we present novel results on the existence, uniqueness, and stability of solutions to variable order fractional differential equations. Chapter 2 addresses the initial-terminal value problem (ITVP) for nonlinear systems with order $1 < \alpha(t) < 2$, leveraging Schaefer's and Banach's fixed-point theorems. Chapter 3 extends this framework to non-autonomous systems (NAVOIVP) with mixed derivatives, where the order $0 < \alpha(t, x(t)) < 1$ introduces additional complexity; here, Krasnoselsky's theorem and the Banach contraction principle are key tools. In Chapter 4, we resolve finite-delay problems (FDVOIVP) of order $0 < \alpha(t) < 1$ via Schauder's theorem and Banach contraction . A central challenge arises from the loss of the semi-group property in variable order calculus, which precludes standard techniques and demands innovative analysis. We further establish Ulam-Hyers stability and validate our results numerically. We can expand the research in three main directions:

- Systems of equations to represent interactions between multiple components.
- General time delays (variable or distributed) to increase the realism of the models.

- More general boundary conditions (nonlocal).

These contributions open avenues for applications in transdisciplinary fields and lay the groundwork for future research, such as extending these models to incorporate stochasticity or multi-term fractional operators.

Bibliography

[1] R.P. Agarwal, Y. Zhou, Y. He, Existence of fractional neutral functional differential equations, *Computers and Mathematics with Applications*, **59(3)**, (2010), 1095–1100. https://doi.org/10.1016/j.camwa.2009.05.010.

[2] M. Ahmad, A. Zada, M. Ghaderi, R. George, S. Rezapour, On the existence and stability of a neutral stochastic fractional differential system. *Fractal and Fractional*, **6(4)**, 203, (2022). https://doi.org/10.3390/fractalfract6040203.

[3] S.M. Aydogan, D. Baleanu, H. Mohammadi et al, On the mathematical model of Rabies by using the fractional Caputo-Fabrizio derivative, *Advances in Difference Equations*, **2020:382**, (2020). https://doi.org/10.1186/s13662-020-02798-4.

[4] D. Baleanu, S. Etemad, H. Mohammadi, and S. Rezapour, A novel modeling of boundary value problems on the glucose graph, *Communications in Nonlinear Science and Numerical Simulation*, **100**, (2021), 105844. https://doi.org/10.1016/j.cnsns.2021.105844.

[5] A. Benkerrouche, D. Baleanu, M. S. Souid, A. Hakem, M. Inc, Boundary value problem for nonlinear fractional differential equations of variable order via Kura-

towski MNC technique, *Advances in Difference Equations*, **2021:365**, (2021), 1-19. https://doi.org/10.1186/s13662-021-03520-8.

[6] A. Benkerrouche, S. Etemad, M. S. Souid, S. Rezapour, H. Ahmad, T. Botmart, Fractional variable order differential equations with impulses: A study on the stability and existence properties, *AIMS Mathematics* , **8(1)**, 775-791, (2022).

[7] A. Benkerrouche, S. Guedim, S. N. Kottakkaran, A. Amara, M. S. Souid, H. Gunerhan, Fractional differential equations of non-autonomous variable-order. *Journal of Computational and Applied Mathematics*, **478**, 117235, (2025). https://doi.org/10.1016/j.cam.2025.117235.

[8] A. Benkerrouche, M.S. Souid, S. Chandok, A. Hakem, Existence and stability of a Caputo variable order boundary value problem, *Journal of Mathematics*, **2021**, Article ID 7967880, (2021), 1-16. https://doi.org/10.1155/2021/7967880.

[9] A. Benkerrouche, M.S. Souid, S. Etemad, A. Hakem, P. Agarwal, S. Rezapour, S. K. Ntouyas, J. Tariboon, Qualitative study on solutions of a Hadamard variable order boundary problem via the Ulam-Hyers-Rassias stability, *Fractal and Fractional*, **5(3), 108**, 1-20, (2021).

[10] A. Benkerrouche, M.S. Souid, F. Jarad, A. Hakem, On boundary value problems of Caputo fractional differential equation of variable order via Kuratowski MNC technique, *Advances in Continuous and Discrete Models: Theory and Applications*, **2022:43**, 1-19, (2022).

[11] A. Benkerrouche, M.S. Souid, E. Karapinar, A. Hakem, On the boundary value problems of Hadamard fractional differential equations of variable order, *Mathematical Methods in the Applied Sciences*, (2022), 1-17. https://doi.org/10.1002/mma.8306.

[12] A. Benkerrouche, M. S. Souid, K. Sitthithakerngkiet, A. Hakem, Implicit nonlinear fractional differential equations of variable order, *Boundary Value Problems*, **2021:64**, (2021), 1-16. https://doi.org/10.1186/s13661-021-01540-7.

[13] A. Benkerrouche, M. S. Souid, G. Stamov, I. Stamova, Multiterm impulsive Caputo-Hadamard type differential equations of fractional variable order, *Axioms*, **11, 634**, (2022), 1-14. https://doi.org/10.3390/axioms11110634.

[14] A. Benkerrouche, M. S. Souid, G. Stamov, I. Stamova, On the solutions for a quadratic integral equation of the Urysohn type of fractional variable order, *Entropy* , **24, 886**, 1-14, (2022).

[15] M. Bensaid, M. S. Souid, A. Benkerrouche, S. Guedim, A. Amara, Initial and terminal conditions for differential equations of fractional derivative via non-autonomous variable order, *Palestine Journal of Mathematics*, **14(4)**, (2025), 187-205.

[16] Z. Bouazza, M. S. Souid, A. Benkerrouche, A. Yakar, Solvability of quadratic integral equations of Urysohn type involving Hadamard variable-order operator, *Fundamental Journal of Mathematics and Applications*, **7(2)**, 108-117, (2024).

[17] K. Dehingia, A. A. Mohsen, S.A. Alharbi, R.D. Alsemiry, S. Rezapour, Dynamical behavior of a fractional order model for within-host SARS-CoV-2, *Mathematics*, **10(13)**, 2344, (2022). https://doi.org/10.3390/math10132344.

[18] G. Diaz, C. F. M. Coimbra, Nonlinear dynamics and control of a variable order oscillator with application to the van der Pol equation, *Nonlinear Dynamics*, **56**, (2009), 145-157. https://doi.org/10.1007/s11071-008-9385-8.

[19] C. Y. Gu, F. X. Zheng, B. Shiri, Mittag-Leffler stability analysis of tempered fractaional neural networks with short memory and variable order, *Fractals*, **29**, 2140029, (2021). https://doi.org/10.1142/S0218348X21400296.

[20] S. Guedim, A. Benkerrouche, S. G. Özyurt, M. S. Souid, S. Sabit, Initial and terminal value problem for fractional differential equations of variable order, *Filomat*, **38:33**, (2024), 11805-11821. https://doi.org/10.2298/FIL2433805G.

[21] S. Guedim, A. Benkerrouche, M. S. Souid, A. Amara, J. Rashid, A. Imtiaz, Initial value problem for fractional differential equations of variable order. *Mathematical Modelling and Control*, **5(4)**, (2025), 379-389. https://doi.org/10.3934/mmc.2025026.

[22] S. Guedim, A. Benkerrouche, K. Sitthithakerngkiet, M. S. Souid, A. Amara, Initial value problem for mixed differential equations of variable order with finite delay, *Symmetry*, **17**, 295, (2025), 1-13. https://doi.org/10.3390/sym17020295.

[23] S. Hussain, E. N. Madi, H. Khan, S. Etemad, S. Rezapour, T. Sitthiwirattham, N. Patanarapeelert. Investigation of the stochastic modeling of COVID-19 with environmental noise from the analytical and numerical point of view, *Mathematics*, **9(23)**, 3122, (2021). https://doi.org/10.3390/math9233122.

[24] S. Kesavan, *Functional Analysis, Hindustan Book Agency*, 1st ed, New Delhi, India, 2009.

[25] V. Lakshmikantham, S. Leela, J. V. Devi, *Theory of Fractional Dynamic Systems*, Cambridge scientific publishers, 2009.

[26] J. L. Lions, *Quelques méthodes de resolution des problèmes aux limites non linéaires*, Dunod, Paris, 1969.

[27] D. O'Regan, Y.Je Cho, Y.Q. Chen, *Topological Degree Theory and Applications*, Volume 10, Taylor et francis group, LLC, 2006.

[28] A. Refice, A. Benkerrouche, M. S. Souid, M. Bensaid, Existence and Ulam-Hyers Stability of Solutions for Caputo-Type Fractional Differential Equations with Variable Order, *Studies in Science of Science*, **43(5)**, 435-451, (2025).

[29] A. Refice, M. Bensaid, M.S. Souid, Innovative approaches to initial and terminal value problems of fractional differential equations with two different derivative orders, *Fixed Point Theory and Algorithms for Sciences and Engineering*, **32**, (2025).

[30] I. A. Rus, Ulam stabilities of ordinary differential equations in a Banach space, *Carpathian Journal of Mathematics*, **26**, (2010), 103-107.

[31] S. G. Samko, B. Boss, Integration and differentiation to a variable fractional order, *Integral Transforms and Special Functions*, **1**, (1993), 277-300. http://dx.doi.org/10.1080/10652469308819027.

[32] D. R. Smart, *Fixed Point Theorems*, Cambridge university press, 66, 1980.

[33] M.S. Souid, A. Benkerrouche, S. Guedim, S. Pinelas, A. Amara, Solvability of boundary value problems for differential equations combining ordinary and fractional derivatives of non-Autonomous variable order, *Symmetry*, **17**, 184, (2025), 1-15. https://doi.org/10.3390/sym17020184.

[34] M. S. Souid, S. Guedim, S. Boulaaras, A. Benkerrouche, A. Amara, T. Radwan, R. Jan, New approaches to solving initial value problems of mixed differential equations, *Filomat*, **40:2**, (2026), 649-659. https://doi.org/10.2298/FIL2602649S.

[35] M. S. Souid, Z. Bouazza, M. Bensaid, K. S. Mozhi, M. Mokhtari, J. K. K. Asamoah, Analytical Study of Variable-Order Fractional Differential Equations With Initial and Terminal Antiperiodic Boundary Conditions, *Journal of Applied Mathematics*, **2025(1)**, 1-13, (2025). https://doi.org/10.1155/jama/8863599

[36] M. S. Souid, Z. Bouazza, A. Yakar, Existence, Uniqueness, and stability of solutions to variable fractional order boundary value problems, *Journal of New Theory*, **41**, 82-93, (2022).

[37] M. S. Souid, A. Refice, K. Sitthithakerngkiet, Stability of p(·)-integrable solutions for fractional boundary value problem via piecewise constant functions, *Fractal and Fractional*, **7, 198** , 1-20, (2023).

[38] M. S. Souid, S. Sabit, Z. Bouazza, K. Sitthithakerngkiet, A study of Caputo fractional differential equations of variable order via Darbo's fixed point theorem and Kuratowski measure of noncompactness, *AIMS Mathematics*, **10(7)**, 15410-15432, (2025).

[39] H.G. Sun, W. Chen, Y.Q. Chen, Variable order fractional differential operators in anomalous diffusion modeling, *Physica A: Statistical Mechanics and its Applications*, **388**, (2009), 4586-4592. https://doi.org/10.1016/j.physa.2009.07.024.

[40] N. Tuan, H. Mohammadi, S. Rezapour. A mathematical model for COVID-19 transmission by using the Caputo fractional derivative, *Chaos Solitons Fractals*, **140**, (2020), 110107. https://doi.org/10.1016/j.chaos.2020.110107.

[41] J. Wang, Y. Zhou, Stability of variable order fractional differential equations with delays, *Fractional Calculus and Applied Analysis*, **22(5)**, (2019), 1231-1250.

[42] J. Wu, *Theory and Applications of Partial Functional Differential Equations*, Springer, 1996.

[43] S. Zhang, Existence and uniqueness result of solutions to initial value problems of fractional differential equations of variable order, *Journal of Fractional Calculus and Applications*, **4(1)**, (2013), 82–98.

[44] S. Zhang, L. Hu, Unique existence result of approximate solution to initial value problem for fractional differential equation of variable order involving the derivative arguments on the half-axis, *Mathematics*, **7(286)**, (2019), 1-23. https://doi.org/10.3390/math7030286.

[45] S. Zhang, S. Sun, L. Hu, Approximate solutions to initial value problem for differential equation of variable order, *Journal of Fractional Calculus and Applications*, **9(2)**, (2018), 93-112.

[46] Y. Zhang, X. Yang, R. P. Agarwal, Existence and stability of solutions for variable-order fractional systems with time delays, *Mathematical Methods in the Applied Sciences*, **44(18)**, (2021), 14107-14122.