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Dedication

In the name of God, the Most Gracious, the Most Merciful.
I dedicate this work to:
My dear father.
My beloved mother.
My brother and sisters, my greatest supporters.
Everyone who helped me during my Master's studies.
Thank you all.

Malki Ouissal

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Contents

NOTATIONS

- $\partial_t, \partial_{tt}$ First and second order partial derivatives with respect to time.
- Δ The Laplace operator (Laplacian)
- $\text{div}(\cdot)$ The divergence operator.
- ∇ The gradient operator with respect to the spatial variables.
- $\widehat{f}$ or $\mathcal{F}(f)$ Fourier transform
- $\mathcal{U}$ Floquet-Bloch transform
- ω Angular frequency
- k Wave number.
- β Longitudinal wavenumber.
- $\rho(x)$ Density or periodic coefficient.
- $\lambda^{\pm}$ Neumann-to-Dirichlet (NtD) coefficients.
- γ_n Dirichlet-to-Neumann (DtN) coefficients.
- $H^1(\mathbb{R})$, $H^2(\mathbb{R})$ Sobolev spaces.
- $L^2(\mathbb{R})$ Lebesgue space.
- $D(A)$ Domain of operator A
- $\hookrightarrow$ Continuous embedding.
- $\overset{c}{\hookrightarrow}$ Compact embedding

- ► $\langle \cdot, \cdot \rangle$ Duality pairing.
- ► $\Re(u)$, $\Im(u)$ Real and imaginary parts of u.
- ► I Identity operator.
- ► δ_{ij} Kronecker delta.
- ► $\sigma(A)$ The spectrum of A.
- ► $Ker(A)$ Kernel of operator A.
- ► $D(A)$ Domain of operator A.

INTRODUCTION

The study of wave propagation and structural dynamics in elastic media and unbounded domains constitutes a cornerstone of applied mathematics and engineering physics. Modeling and numerically analyzing these phenomena carry profound significance due to their widespread applications, ranging from civil and mechanical engineering to acoustic engineering and telecommunications via waveguides.

The mathematical foundation of this field traces back to the 18th and 19th centuries, during which the fundamentals of continuum mechanics were established. In structural elasticity, Leonhard Euler and Daniel Bernoulli formulated the classical "Euler-Bernoulli beam theory," which marked a major breakthrough in computing structural deformations and natural frequencies [10]. As engineering structures grew more advanced, simulating their dynamic response under time-dependent forces became essential. In the context of time-harmonic wave propagation, Hermann von Helmholtz introduced his famous equation in the late 19th century, laying the groundwork for acoustic and electromagnetic scattering [9].

By the early 20th century, modeling waves in open domains and waveguides introduced challenges regarding solution uniqueness, prompting Arnold Sommerfeld to establish his renowned radiation condition. This was later augmented by foundational concepts such as the "Limiting Absorption Principle" developed by mathematicians like Tosio Kato [12, 11]. With the advent of modern computing in the mid-20th century, the focus shifted toward "Numerical Analysis." The Finite Element Method (FEM) and time-integration schemes like the "Newmark Method" revolutionized the field. Nevertheless, structural challenges persist today, most notably the "Pollution Error" in high-frequency Helmholtz regimes [1, 16], driving continuous research into accurate artificial boundary conditions (such as NtD maps) [7, 8] and advanced numerical solvers.

Our work begins with a chapter that introduces mathematical concepts that will be used in subsequent chapters,and numerical challenges (such as computational efficiency and the pollution error).

In Chapter Two, we shift our focus to the structural dynamics, examining the analytical and numerical calculation of the natural frequencies of trusses and beams using the Finite

Element Method (FEM) and Newmark's schemes.

Chapter Three extends this analysis to wave mechanics, dealing with the general mathematical formulation of wave propagation in both closed and open waveguides, alongside a detailed study of guided modes.

Finally, Chapter Four investigates wave propagation within an infinite waveguide containing a local perturbation, incorporating the construction of artificial Neumann-to-Dirichlet (NtD) boundary conditions to ensure the numerical accuracy and physical validity of the open-domain solution."

Chapter 1

Challenges in Numerical Modeling

The Helmholtz equation, also known as the reduced wave equation or the time-harmonic wave equation, serves as the fundamental framework for describing the propagation of monochromatic (single-frequency) waves in various media. This equation is derived from the general wave equation

$$\frac{\partial^2 U}{\partial t^2} - c^2(x)\Delta U = F(x,t) \tag{1.1}$$

by assuming that the source oscillates harmonically $F(x,t) = f(x)e^{-i\omega t}$ (mono-chromatic) ,we can assume $U(x,t) = u(x)e^{-i\omega t}$with an angular frequency ω, resulting in the following form:

$$-\Delta u - \frac{\omega^2}{c(x)^2}u = f \tag{1.2}$$

where:

- $c(x)$: the local wave speed in the medium.
- ω: the angular frequency.
- $n(x) = \dfrac{1}{c(x)}$: the refractive index.
- $k^2 = n^2\omega^2$: the **wave number**, which is the reciprocal of the wavelength.

Remark 1.0.1 *When the wave number k is small, the Helmholtz equation is merely a "perturbation" of the familiar Laplace problem ($-\Delta u = f$). However, when k is large (i.e., high frequency), the solution u becomes highly oscillatory, and this is the source of immense mathematical and computational difficulties.*

1.1 Why is the "Computational Efficiency" Problem Hard?

After using a numerical method (such as finite differences or finite elements) to approximate the differential equation, we ultimately obtain a very large linear system to solve:

$$Au = b \tag{1.3}$$

Here, A is the system matrix, u is the unknown solution vector, and b is the source vector. The challenges related to solving this system efficiently are:

1.1.1 The Properties of Matrix A are Inconvenient:

- **Symmetric but Non-Hermitian:** In the presence of damped equations or Robin boundary conditions, matrix A loses the Hermitian property (for complex matrices). This complicates the process of finding fast solutions and disables some algorithms designed for symmetric positive definite matrices.

1.1.2 The Explosive Growth of the Problem Size:

- The size of the matrix n (the number of degrees of freedom) increases catastrophically with frequency ω. To resolve a wavelength, we need a grid with spacing h that is sufficiently small. The basic condition is $h \sim 1/\omega$.
- In a space of dimension d, the number of points in each direction $N \sim 1/h \sim \omega$. Consequently, the total degrees of freedom $n = N^d \sim \omega^d$.
- However! Due to the “Pollution Effect” (explained later), the condition becomes stricter: $N \sim \omega^{1+1/p}$, where p is the polynomial degree used in the finite element method [2].
- Therefore, the total size becomes:

  $$n = N^d \sim \omega^{(1+1/p)d} \tag{1.4}$$

  This means the problem size grows with an exponent larger than one relative to the frequency, making the problem unsolvable by traditional methods.

1.1.3 Arbitrarily Ill-conditioned:

- The condition number of matrix A becomes extremely large without bound as ω increases. This means the linear system is highly sensitive to any minor error in calculations or data.

1.1.4 Failure of Conventional Iterative Methods:

- All these factors combined (huge size, ill-conditioning, matrix properties) cause standard iterative methods (like conjugate gradients) to fail to converge to a solution within a reasonable time.

1.2 Why is the "Accuracy" Problem Hard?

The goal here is to maximize accuracy while minimizing the number of degrees of freedom (#DoF).

1.2.1 Pollution Error:

- One might mistakenly think that to maintain constant accuracy as frequency ω increases, it suffices to keep the number of points per wavelength (*ppw* or G) constant. That is, keeping the product $h\omega$ constant.
- **This is incorrect!** prove it in [1] that if $h\omega$ is kept constant, the error increases with ω. This additional part of the error is called the **Pollution Error** .A detailed finite element analysis of this phenomenon for acoustic scattering problems is given by Ihlenburg[9] and [15]

1.2.2 The Quasi-optimality Condition:

- To ensure the numerical solution is as close to the true solution as the approximation space allows (i.e., to achieve quasi-optimality), the mesh size h must satisfy a stricter condition [16]:
$$h^p \omega^{p+1} \lesssim 1 \tag{1.5}$$
- This leads to an even more precise condition for a bounded error[5]:
$$h \sim \omega^{-1-\frac{1}{2p}} \tag{1.6}$$
- Hence:

  The intuitive condition (to represent oscillation) $h \sim \omega^{-1}$
  The precise condition (to ensure accuracy)$h \sim \omega^{-1-1/p}$.

  The extra factor $\omega^{-1/p}$ is the price we pay to eliminate the pollution error.

1.2.3 Practical Consequences:

- The high-frequency solution u oscillates on a scale of $1/\omega$, requiring $h \sim 1/\omega$, which leads to a **very large number of degrees of freedom**.
- The pollution effect forces us to have $h \ll 1/\omega$, leading to an **even larger number of degrees of freedom**.
- There is a trade-off between:
  - The number of grid points per wavelength: $G = \dfrac{\lambda}{h} = \dfrac{2\pi}{\omega h}$.
  - The polynomial degree p used.

  This trade-off is subject to **Dispersion Analysis**, which measures the difference between the numerical and physical wave speeds. The higher the method's accuracy, the smaller this difference.

In this chapter we discuss some mathematical concepts that we should know them for use in our theme.

1.3 Some functional analysis theorems

Theorem 1.3.1 (Lax-Milgram) *Let H be a Hilbert space equipped with the norm $\|\cdot\|_H$. Consider the variational problem:*

$$\begin{cases} \text{Find } u \in H \text{ such that} \\ a(u,v) = \ell(v), \quad \forall v \in H \end{cases} \tag{1.7}$$

Assume that:

- *The bilinear form $a(\cdot,\cdot)$ is continuous, i.e.,*
$$\exists \beta > 0,\ \forall u, v \in H, \quad |a(u,v)| \le \beta \|u\|_H \|v\|_H.$$
- *The bilinear form $a(\cdot,\cdot)$ is coercive, i.e.,*
$$\exists \alpha > 0,\ \forall u \in H, \quad a(u,u) \ge \alpha \|u\|_H^2.$$
- *The linear form ℓ is continuous, i.e.,*
$$\exists \gamma > 0,\ \forall v \in H, \quad |\ell(v)| \le \gamma \|v\|_H.$$

Then there exists a unique solution $u \in H$. Moreover,

$$\|u\|_H \le \frac{\|\ell\|_{H'}}{\alpha}.$$

If in addition $a(\cdot,\cdot)$ is symmetric, then u is the unique minimizer of

$$J(v) = \frac{1}{2} a(v,v) - \ell(v), \quad \text{over } H.$$

Proof. For a proof ,see [3],p.140 ■

Theorem 1.3.2 (Rellich-Kondrachov) *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $m \in \mathbb{N}$ and $1 \leq p < \infty$.*

- *If $p < n$, then*
$$W^{1,p}(\Omega) \overset{c}{\hookrightarrow} L^q(\Omega), \quad \forall q \in [1, p^*),$$
where
$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}.$$
- *If $p = n$, then*
$$W^{1,p}(\Omega) \overset{c}{\hookrightarrow} L^q(\Omega), \quad \forall q \in [p, \infty).$$
- *If $p > n$, then*
$$W^{1,p}(\Omega) \overset{c}{\hookrightarrow} C(\overline{\Omega}).$$

In particular,
$$W^{1,p}(\Omega) \overset{c}{\hookrightarrow} L^p(\Omega), \quad \forall p.$$

Proof. See[3],p.285 ■

Definition 1.3.3 (self-adjoint operator) *[3] Let $A : D(A) \subset H \to H$ be an unbounded linear operator.*
*The operator A is said to be **self-adjoint** if*

1. *$D(A)$ is dense in H,*
2. *$A^* = A$ (which means that $D(A^*) = D(A)$ and $A^*u = Au$ for all $u \in D(A)$).*

1.4 Some functional transformations

1.4.1 The Fourier Transform

The Fourier transform is one of the most important tools in modern analysis and partial differential equations. Its fundamental idea is that a function can be represented as a superposition of elementary oscillatory waves of the form

$$e^{i\xi \cdot x}.$$

In this way, the Fourier transform converts a function from the physical space into a frequency space representation.

For a sufficiently regular function

$$f \in L^1(\mathbb{R}^n),$$

the Fourier transform is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\,dx, \qquad \xi \in \mathbb{R}^n.$$

The inverse Fourier transform is formally given by

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{f}(\xi)e^{ix\cdot\xi}\,d\xi.$$

One of the main reasons for the importance of the Fourier transform in PDE theory is that differentiation becomes multiplication in the frequency domain. Indeed,

$$\widehat{\partial_{x_j} f}(\xi) = i\xi_j \widehat{f}(\xi),$$

and therefore differential operators with constant coefficients become algebraic operators after transformation.

For example, the Laplace operator satisfies

$$-\Delta\widehat{f}(\xi) = |\xi|^2 \widehat{f}(\xi).$$

Consequently, many partial differential equations can be transformed into simpler algebraic equations in Fourier space. This makes the Fourier transform a fundamental tool for studying existence, uniqueness, regularity, and spectral properties of solutions.

In periodic settings, where coefficients are periodic rather than constant, the Fourier transform is replaced by the Floquet–Bloch transform, which plays an analogous role for periodic operators.

1.4.2 The Floquet–Bloch Transform

It is well known that the Fourier transform is an essential tool for the analysis of partial differential equations with constant coefficients. For differential operators with periodic coefficients, the appropriate analogue is the *Floquet–Bloch transform* [14]. This transform allows one to decompose functions on $\mathbb{R}^n$ into quasiperiodic components and reduces the analysis of periodic operators on the whole space to a family of problems posed on a single periodicity cell.

Periodic setting

Let

$$\Gamma \subset \mathbb{R}^n$$

be a lattice generated by linearly independent vectors

$$a_1, \dots, a_n,$$

that is,

$$\Gamma = \left\{ \sum_{j=1}^{n} m_j a_j \; ; \; m_j \in \mathbb{Z} \right\}.$$

A function or operator is called Γ-periodic if it is invariant under translations by vectors of Γ.

Let W denote a fundamental cell of the lattice Γ.

The dual lattice is defined by

$$\Gamma^* = \{ k \in \mathbb{R}^n \; ; \; k \cdot \gamma \in 2\pi\mathbb{Z}, \quad \forall \gamma \in \Gamma \}.$$

A fundamental domain of Γ^* is called the *Brillouin zone* and is denoted by B.

Bloch waves

The natural elementary solutions in a periodic medium are not plane waves, but rather *Bloch waves* of the form

$$u(x) = e^{ik\cdot x} p(x),$$

where

$$k \in B$$

is the *quasi-momentum* and

$$p(x+\gamma) = p(x), \qquad \forall \gamma \in \Gamma.$$

Equivalently,

$$u(x+\gamma) = e^{ik\cdot\gamma} u(x).$$

Such functions are called *k-quasiperiodic* or *Floquet functions.*

Floquet–Bloch transform

For a sufficiently regular function

$$f \in C_0^\infty(\mathbb{R}^n),$$

the Floquet–Bloch transform is defined by

$$(\mathcal{U}f)(x,k) = \sum_{\gamma\in\Gamma} f(x+\gamma) e^{-ik\cdot\gamma},$$

where

$$x \in W, \qquad k \in B.$$

For fixed k, the transformed function satisfies the quasi periodicity relation

$$(\mathcal{U}f)(x+\gamma,k) = e^{ik\cdot\gamma} (\mathcal{U}f)(x,k).$$

Thus, the transform decomposes functions into quasiperiodic components indexed by the quasi-momentum k.

Relation with periodic operators

Let

$$L(x, D)$$

be a differential operator with periodic coefficients. Under the Floquet–Bloch transform, the operator decomposes into a direct family of operators

$$L(k), \qquad k \in B,$$

acting on quasiperiodic functions defined on the single cell W.

Formally, one replaces

$$D_j \longmapsto D_j + k_j,$$

so that

$$L(x, D) \rightsquigarrow L(x, D + k).$$

Hence, instead of solving one problem on the whole space $\mathbb{R}^n$, one obtains a family of problems on the compact cell W:

$$L(k)u_k = f_k, \qquad k \in B.$$

1.4.3 Spectral decomposition

For each fixed $k \in B$, the operator $L(k)$ has discrete spectrum

$$\lambda_1(k) \le \lambda_2(k) \le \cdots .$$

The functions

$$\lambda_j(k)$$

are called the *band functions*. The spectrum of the original periodic operator is then given by

$$\sigma(L) = \bigcup_j \lambda_j(B).$$

This leads naturally to the band-gap structure characteristic of periodic media and solid-state physics.

Inverse transform

Up to normalization constants depending on conventions, the inverse Floquet–Bloch transform is formally written as

$$f(x) = \int_B (\mathcal{U}f)(x, k)\, dk.$$

Therefore, every function can be reconstructed as a superposition of quasiperiodic modes.

The Floquet–Bloch transform plays for periodic differential operators the same role that the Fourier transform plays for constant coefficient operators. It reduces global periodic problems on $\mathbb{R}^n$ to a family of quasiperiodic problems posed on a single periodicity cell, making it a fundamental tool in the spectral theory of periodic partial differential equations.

Chapter 2

Natural frequencies for trusses and beams

2.1 Introduction

This chapter presents the analytical and numerical methods for determining the natural frequencies of continuous systems, specifically strings-rods (1D wave equation) and Euler-Bernoulli beams. The finite element method (FEM) and Newmark time integration scheme are introduced for numerical approximation.

2.2 Natural Frequencies of the 1D Wave Equation and Finite Element Approximation

2.2.1 Exact Natural Frequency

In this part,we will start from the wave equation in search of the analytical expression $w(x,t)$ We consider the 1D wave equation:

$$\begin{cases} \dfrac{\partial^2 w}{\partial t^2} = c^2 \dfrac{\partial^2 w}{\partial x^2}, \quad x \in (0,L), \quad t > 0 \\ w(0,t) = \quad w(L,t) = 0 \end{cases} \tag{2.1}$$

First, it is assumed that the displacement $w(x,t)$ can be written as the product of two functions, one depending only on x and the other depending only on t (hence separation of variables). Thus

$$w(x,t) = X(x)T(t) \tag{2.2}$$

Substitution of this separated form into the string equation (2.1) yields:

$$c^2 X''(x)T(t) = X(x)\ddot{T}(t) \tag{2.3}$$

A simple rearrangement of equation (2.3) yields:

$$\frac{X''(x)}{X(x)} = \frac{\ddot{T}(t)}{c^2 T(t)} \tag{2.4}$$

Since each side of the equation is a function of a different variable, it is argued that each side must be constant. To see this, differentiate with respect to x. This yields

$$\frac{d}{dx}\left(\frac{\ddot{X}}{X}\right) = 0$$

which becomes, upon integration:

$$\frac{\ddot{X}}{X} = cte \tag{2.5}$$

The constant admits three cases ,requiring careful analysis:

(a) $cte > 0$ in this case ,the solution is written in the form :

$$X(x) = A\exp^{\sigma x} + B\exp^{-\sigma x}$$

,but the exponent does not yield a bounded oscillatory solution. (**Rejected**)

(b) $cte = 0$ in this case ,the solution is written in the form :

$$X(x) = Ax + b$$

(**Rejected**,because it does not satisfy the boundary condition)

(c) $cte < 0$ in this case ,the solution is written in the form :

$$X(x) = Asin(\sigma x) + Bcos(\sigma x)$$

Trigonometric solutions represent periodic oscillations and are satisfy the boundary conditions (**physically acceptable**).

then we obtain :

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\sigma^2. \tag{2.6}$$

1. $\ddot{X}(x) + \sigma^2 X(x) = 0$ The solution of this equation is:

$$\begin{cases} X(x) = A\cos(\sigma x) + B\sin(\sigma x) \\ X(0) = X(L) = 0 \end{cases} \tag{2.7}$$

- $X(0) = 0$we have:

$$A\cos(\sigma x) + B\sin(\sigma x) = 0 \implies A = 0$$

  then

$$X(x) = B\sin(\sigma x)$$

- $X(L) = 0$

$$A\sin(\sigma L) = 0 \quad ,\forall A \neq 0$$

  we have :

$$\sin(\sigma L) = 0 \Rightarrow \sigma L = \pi n \quad ,\forall n = 1,2,3$$
$$\Rightarrow \sigma_n = \frac{n\pi}{L}$$

  Since there are an infinite number of values of σ_n, the solution (2.7) then becomes the infinite number of solutions:

$$X_n(x) = A_n \sin(\frac{n\pi}{L}x) \quad ,\forall n = 1,2,.. \tag{2.8}$$

2.

$$\ddot{T}(t) + \sigma_n^2 c^2 T(t) = 0 \tag{2.9}$$

where $T(t)$ is now indexed because there is one solution for each value of σ_n. The coefficient of $T_n(t)$ in the temporal equation defines the natural frequency by noting that $\omega_n = c\sigma_n$ and hence

$$\omega_n = c\sigma_n = c\frac{n\pi}{L} \tag{2.10}$$

The general form of the solution of is given (2.9) as:

$$T_n(t) = A_n\cos(\omega_n t) + B_n\sin(\omega_n t), \quad \omega_n = c\frac{n\pi}{L}. \tag{2.11}$$

where A_n and B_n are constants of integration. Since both of the functions $X_n(x)$ and $T_n(t)$ are found to be dependent on n, the solution $w(x,t) = X_n(x)T_n(t)$must also be a function of n, so that

$$W_n(x,t) = c_n\sin(\frac{n\pi x}{L})\sin(\frac{n\pi c}{L}t) + d_n\sin(\frac{n\pi x}{L})\cos(\frac{n\pi c}{L}t) \tag{2.12}$$

where c_n and d_n are new constants to be determined. Hence the general solution is of the form

$$W(x,t) = \sum_{n=1}^{\infty}\sin(\frac{n\pi}{L}x)\left[c_n\sin(\frac{n\pi c}{L}t) + d_n\cos(\frac{n\pi c}{l}t)\right] \tag{2.13}$$

The set of constants c_n and d_n can be determined by applying the initial conditions on (x,t) and the orthogonality of the set of functions $\sin(\frac{m\pi}{L}x)$

(a) When

$$W(x,0) = W_0(x)$$

we obtain :

$$W(x,0) = \sum_{n=1}^{\infty} d_n \sin(\frac{n\pi}{L}x) = W_0(x) \tag{2.14}$$

To use the orthogonality of the trigonometric function ,we must multiply the sides by a function $\sin(\frac{m\pi}{L}x)$

$$W_0(x)\sin(\frac{m\pi}{L}x) = \sum_{n=1}^{\infty} d_n \sin(\frac{n\pi}{L}x)\sin(\frac{m\pi}{L}x) \tag{2.15}$$

The orthogonality of the set of functions $\sin(\frac{m\pi}{l}x)$ states that:

$$\int_0^L \sin(\frac{n\pi}{L}x)\sin(\frac{m\pi}{L}x)dx = \begin{cases} \frac{l}{2} & if \quad m = n \\ 0 & if \quad m \neq n \end{cases} \tag{2.16}$$

Then for $n = m$:

$$\int_0^L W_0(x)\sin(\frac{m\pi}{L}x)dx = d_m \cdot \frac{L}{2} \tag{2.17}$$

$$d_m = \frac{2}{L}\int_0^L W_0(x)\sin(\frac{m\pi}{L}x)dx \tag{2.18}$$

(b)

$$\dot{W}(x,0) = \dot{W}_0(x) \tag{2.19}$$

$$\frac{\partial W}{\partial t}(x,0) = \sum_{n=1}^{\infty} c_n(\frac{n\pi}{L}c)\sin(\frac{n\pi}{L}x) = \dot{W}_0(x)$$

Again, multiplying by $\sin(\frac{m\pi}{L}x)$, integrating over the length of the string, and applying the orthogonality condition of (2.16) yields

$$c_n = \frac{2}{m\pi c}\int_0^L \dot{W}_0(x)\sin(\frac{m\pi}{L}x)dx \tag{2.20}$$

2.3 Finite Element Approximation of Natural Frequencies

The Finite element method discretizes the spacial domain into a finite number of elements.

1. Approximate $u(x,t)$ using basis functions $\phi_j(x)$:

$$u(x,t) = \sum_j q_j(t)\phi_j(x). \tag{2.21}$$

2. Compute the mass matrix M and the stiffness matrix K

3. Solve the Generalized Eigenvalue Problem:

$$K\Phi = \omega^2 M\Phi. \tag{2.22}$$

4. The solution of each modal equation is given by:

$$q_j(t) = A_j \cos(\omega_j t) + B_j \sin(\omega_j t), \tag{2.23}$$

   where A_j and B_j are determined by initial conditions.

Remarks:

- The eigenvalues ω^2 approximate the squared natural frequencies.
- The relation between q_j and ω_j is explicitly given by:

$$q_j(t) = q_j(0)\cos(\omega_j t) + \frac{\dot{q}_j(0)}{\omega_j}\sin(\omega_j t).$$

- As N increases, the FEM approximation converges to the exact frequencies.

2.4 Newmark Method

Consider the undamped second-order system:

$$\begin{cases} M\ddot{q} + Kq &= 0, \\ q(0) &= q_0, \\ \dot{q}(0) &= q_1 \end{cases} \tag{2.24}$$

where:

- M:Mass Matrix
- K:Stiffness matrix

The Newmark method is defined by two parameters, β and γ, which control the accuracy and stability of the method [13]. Common choices include:

- **Average acceleration method**: $\beta = \frac{1}{4}$, $\gamma = \frac{1}{2}$ (unconditionally stable).
- **Linear acceleration method**: $\beta = \frac{1}{6}$, $\gamma = \frac{1}{2}$ (conditionally stable).

Initialize Time and Displacement Variables

- Choose a time step Δt and total simulation time T.
- Compute the total number of time steps $N_t = \frac{T}{\Delta t}$.
- Initialize displacement q_0 and velocity q_1.

Compute Effective System Matrices

Define the effective stiffness matrix:

$$K_{\text{eff}} = K + \frac{\gamma}{\beta \Delta t} M. \tag{2.25}$$

Define constants:

$$a_0 = \frac{1}{\beta \Delta t^2}, \qquad a_1 = \frac{\gamma}{\beta \Delta t}, \qquad a_2 = \frac{1}{\beta \Delta t}, \tag{2.26}$$

$$a_3 = \frac{1}{2\beta} - 1, \qquad a_4 = \frac{\gamma}{\beta} - 1, \qquad a_5 = \frac{\Delta t}{2}\left(\frac{\gamma}{\beta} - 2\right). \tag{2.27}$$

Time Stepping Loop

For each time step n:

- Compute the effective force:

$$F_{\text{eff}} = -M\left(a_0 q_n + a_2 \dot{q}_n + a_3 \ddot{q}_n\right). \tag{2.28}$$

- Solve for q_{n+1}:

$$K_{\text{eff}} q_{n+1} = F_{\text{eff}}. \tag{2.29}$$

- Compute velocity and acceleration:

$$\dot{q}_{n+1} = a_4(q_{n+1} - q_n) + a_5 \dot{q}_n + a_2 \ddot{q}_n, \tag{2.30}$$

$$\ddot{q}_{n+1} = a_0(q_{n+1} - q_n) - a_2 \dot{q}_n - a_3 \ddot{q}_n. \tag{2.31}$$

Update Values

Set:

$$q_n \to q_{n+1}, \quad \dot{q}_n \to \dot{q}_{n+1}, \quad \ddot{q}_n \to \ddot{q}_{n+1}, \tag{2.32}$$

and repeat.

2.5 Newmark Scheme for Damped Second-Order System

We want to solve the second-order ODE system:

$$M\ddot{q} + C\dot{q} + Kq = 0 \tag{2.33}$$
$$q(0) = q_0, \tag{2.34}$$
$$\dot{q}(0) = q_1. \tag{2.35}$$

using the Newmark method, which is a time-stepping integration scheme. The Newmark method introduces approximations for displacement q and velocity $\dot{q}$ at each time step:

$$q_{n+1} = q_n + \Delta t\dot{q}_n + \frac{\Delta t^2}{2}\left[(1-2\beta)\ddot{q}_n + 2\beta\ddot{q}_{n+1}\right] \tag{2.36}$$

$$\dot{q}_{n+1} = \dot{q}_n + \Delta t\left[(1-\gamma)\ddot{q}_n + \gamma\ddot{q}_{n+1}\right] \tag{2.37}$$

where:

- β and γ are Newmark parameters that control stability and accuracy.
- Common choices:
  - Average acceleration method: $\beta = \frac{1}{4}, \gamma = \frac{1}{2}$ (unconditionally stable).
  - Linear acceleration method: $\beta = \frac{1}{6}, \gamma = \frac{1}{2}$ (conditionally stable).

2.6 Algorithm

2.6.1 Step 1: Initialize Time and State Variables

- Choose a time step Δt and total simulation time T.
- Compute the number of time steps:

$$N_t = \frac{T}{\Delta t} \tag{2.38}$$

- Set initial displacement q_0 and velocity q_1.

2.6.2 Step 2: Compute Effective System Matrices

The effective stiffness matrix:

$$K_{\text{eff}} = K + \frac{\gamma}{\beta\Delta t}C + \frac{1}{\beta\Delta t^2}M \tag{2.39}$$

Define Newmark constants:

$$a_0 = \frac{1}{\beta \Delta t^2}, \quad a_1 = \frac{\gamma}{\beta \Delta t}, \quad a_2 = \frac{1}{\beta \Delta t} \tag{2.40}$$

$$a_3 = \frac{1}{2\beta} - 1, \quad a_4 = \frac{\gamma}{\beta} - 1, \quad a_5 = \frac{\Delta t}{2} \left(\frac{\gamma}{\beta} - 2 \right). \tag{2.41}$$

2.6.3 Step 3: Time Stepping Loop

For each time step n:

1. Compute Effective Force:

$$F_{\text{eff}} = -M \left(a_0 q_n + a_2 \dot{q}_n + a_3 \ddot{q}_n \right) - C \left(a_1 q_n + a_4 \dot{q}_n + a_5 \ddot{q}_n \right) \tag{2.42}$$

2. Solve for q_{n+1}:

$$K_{\text{eff}} q_{n+1} = F_{\text{eff}} \tag{2.43}$$

3. Compute Velocity and Acceleration:

$$\dot{q}_{n+1} = a_4 \frac{q_{n+1} - q_n}{\Delta t} + a_5 \dot{q}_n + a_2 \ddot{q}_n \tag{2.44}$$

$$\ddot{q}_{n+1} = a_0 (q_{n+1} - q_n) - a_2 \dot{q}_n - a_3 \ddot{q}_n \tag{2.45}$$

4. Update Values:

   - $q_n \to q_{n+1}$,
   - $\dot{q}_n \to \dot{q}_{n+1}$,
   - $\ddot{q}_n \to \ddot{q}_{n+1}$.

Newmark Scheme in Matrix Form

The Newmark scheme for the second-order system:

$$\mathcal{M}\frac{d^2U}{dt^2}(t) + \mathcal{C}\frac{dU}{dt}(t) + \mathcal{K}U(t) = b(t), \tag{2.46}$$

is discretized using the approximations:

$$\dot{U}^{n+1} = \dot{U}^n + \tau\left(\delta\ddot{U}^{n+1} + (1-\delta)\ddot{U}^n\right), \tag{2.47}$$

$$U^{n+1} = U^n + \tau\dot{U}^n + \frac{\tau^2}{2}\left(2\theta\ddot{U}^{n+1} + (1-2\theta)\ddot{U}^n\right). \tag{2.48}$$

Rearranging for $\ddot{U}^{n+1}$:

$$\ddot{U}^{n+1} = \frac{1}{2\theta}\left(\frac{2}{\tau^2}(U^{n+1} - U^n - \tau\dot{U}^n) - (1-2\theta)\ddot{U}^n\right). \tag{2.49}$$

Substituting this into the velocity equation:

$$\dot{U}^{n+1} = \frac{\delta}{\theta}\frac{1}{\tau}(U^{n+1} - U^n) + \left(1 - \frac{\delta}{\theta}\right)\dot{U}^n + \tau\left((1-\delta) + \frac{\delta}{\theta}\left(\frac{1}{2} - \theta\right)\right)\ddot{U}^n. \tag{2.50}$$

Substituting these expressions into the governing equation:

$$\mathcal{M}\ddot{U}^{n+1} + \mathcal{C}\dot{U}^{n+1} + \mathcal{K}U^{n+1} = b(t_{n+1}). \tag{2.51}$$

Grouping terms involving U^{n+1}, we obtain the system:

$$K_{\text{eff}}U^{n+1} = F_{\text{eff}}, \tag{2.52}$$

where the effective stiffness matrix is:

$$K_{\text{eff}} = \mathcal{K} + \frac{1}{\theta\tau^2}\mathcal{M} + \frac{\delta}{\theta\tau}\mathcal{C}, \tag{2.53}$$

and the effective force vector is:

$$\begin{aligned} F_{\text{eff}} = b(t_{n+1}) + \mathcal{M}\left[\frac{1}{\theta\tau^2}U^n + \frac{1}{\theta\tau}\dot{U}^n + \left(\frac{1}{2\theta} - 1\right)\ddot{U}^n\right] \\ + \mathcal{C}\left[\left(1 - \frac{\delta}{\theta}\right)\dot{U}^n + \tau\left((1-\delta) + \frac{\delta}{\theta}\left(\frac{1}{2} - \theta\right)\right)\ddot{U}^n\right]. \end{aligned} \tag{2.54}$$

$$\begin{aligned} F_{\text{eff}} = -\frac{\mathcal{M}}{\tau^2}(2U^n - U^{n-1}) - \mathcal{K}\left(\left(\frac{1}{2} + \delta - 2\theta\right)U^n + \left(\frac{1}{2} - \delta + \theta\right)U^{n-1}\right) + \theta b(t_{n+1}) + \\ \left(\frac{1}{2} + \delta - 2\theta\right)b(t_n) + \left(\frac{1}{2} - \delta + \theta\right)b(t_{n-1}). \end{aligned} \tag{2.55}$$

This formulation allows the system to be solved for U^{n+1} at each time step based on the known values from the previous step.

2.7 Natural frequencies for beams

2.7.1 Natural frequencies for the Euler-Bernoulli beam

Following the systematic approach presented by Inman [10], the governing differential equation for transverse vibration of a uniform beam is derived by applying Newton's second law to an infinitesimal beam element of length dx. From the vertical force equilibrium, we obtain:

$$\frac{\partial V(x,t)}{\partial x} + f(x,t) = \rho A(x)\frac{\partial^2 w(x,t)}{\partial t^2} \tag{2.56}$$

We then relate the shear force V to the bending moment M using the Euler-Bernoulli hypothesis (neglecting rotary inertia):

$$V = -\frac{\partial M}{\partial x} \tag{2.57}$$

From the mechanics of materials, we introduce the moment-curvature relation:

$$M = EI\frac{\partial^2 w}{\partial x^2} \tag{2.58}$$

Substituting (2.58) into (2.57) and then into (2.56) yields the well-known Euler-Bernoulli beam equation:

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) + \rho A\frac{\partial^2 w}{\partial t^2} = f(x,t) \tag{2.59}$$

where:

- E Young's elastic modulus
- I Second moment of area
- EI Flexural rigidity
- ρ Mass density
- A Cross-sectional area

Assuming the absence of external forces ($f(x,t) = 0$) and a uniform beam (EI and ρA are constant),equation (2.59) simplifies to the free vibration :

$$\frac{\partial^2 w(x,t)}{\partial t^2} + c^2\frac{\partial^4 w(x,t)}{\partial x^4} = 0, \quad c = \sqrt{\frac{EI}{\rho A}} \tag{2.60}$$

Note that unlike the previous equations, the free vibration equation (2.60) contains four spatial derivatives and hence requires four (instead of two) boundary conditions

A separation-of-variables solution of the form $w(x,t) = X(x)T(t)$ is assumed. This is substituted into the equation of motion, equation (2.60), to yield (after rearrangement)

$$c^2 \frac{X''''(x)}{X(x)} = -\frac{\ddot{T}(t)}{T(t)} = \omega^2 \tag{2.61}$$

where the partial derivatives have been replaced with total derivatives as before (note: $X'''' = d^4X/dx^4$, $\ddot{T} = d^2T/dt^2$). Here the choice of separation constant, ω^2, where a positive value is chosen in accordance with the physical nature of the problem,that the natural frequency comes from the temporal equation

$$\ddot{T}(t) + \omega^2 T(t) = 0 \tag{2.62}$$

which is the right side of equation (2.61). This temporal equation has a solution of the form

$$T(t) = A \sin \omega t + B \cos \omega t \tag{2.63}$$

where the constants A and B will eventually be determined by the specified initial conditions after being combined with the spatial solution.

The spatial equation comes from rearranging equation (2.61), which yields

$$X''''(x) - \left(\frac{\omega}{c}\right)^2 X(x) = 0 \tag{2.64}$$

By defining [recall equation (2.60)]

$$\beta^4 = \frac{\omega^2}{c^2} = \frac{\rho A \omega^2}{EI} \left(\text{so that } \omega = \beta^2 \sqrt{\frac{EI}{\rho A}} \text{ rad/s} \right) \tag{2.65}$$

and assuming a solution to equation (2.64) of the form $Ae^{\sigma x}$, the general solution of equation (2.64) can be calculated to be of the form

$$X(x) = a_1 \sin \beta x + a_2 \cos \beta x + a_3 \sinh \beta x + a_4 \cosh \beta x \tag{2.66}$$

Here the value for β and three of the four constants of integration a_1, a_2, a_3, and a_4 will be determined from the four boundary conditions. The fourth constant becomes combined with the constants A and B from the temporal equation, which are then determined from the initial conditions. The following example illustrates the solution procedure for a beam fixed at one end and simply supported at the other end.

2.8 Natural Frequencies of a Clamped-Pinned Beam

2.8.1 Example

We illustrate the procedure for calculating natural frequencies and mode shapes for the transverse vibration of a uniform beam of length l that is fixed at one end ($x = 0$) and pinned (simply supported) at the other end ($x = l$)[1]

The boundary conditions at the fixed end are zero displacement and zero slope:

$$w(0,t) = 0, \qquad \frac{\partial w}{\partial x}(0,t) = 0$$

At the pinned end, the boundary conditions are zero displacement and zero bending moment:

$$w(l,t) = 0, \qquad EI\frac{\partial^2 w}{\partial x^2}(l,t) = 0$$

Using separation of variables, we assume $w(x,t) = X(x)T(t)$. Substituting into the Euler-Bernoulli beam equation yields the spatial equation:

$$X''''(x) - \beta^4 X(x) = 0, \quad \text{where} \quad \beta^4 = \frac{\rho A \omega^2}{EI}$$

The general solution is (2.66)

Applying the boundary conditions at $x = 0$:

$$X(0) = 0 \Rightarrow a_2 + a_4 = 0 \tag{a}$$
$$X'(0) = 0 \Rightarrow \beta\,(a_1 + a_3) = 0 \tag{b}$$

Similarly, at $x = l$ the boundary conditions result in

$$X(l) = 0 \Rightarrow a_1 \sin\beta l + a_2 \cos\beta l + a_3 \sinh\beta l + a_4 \cosh\beta l = 0 \tag{c}$$
$$EIX''(l) = 0 \Rightarrow \beta^2\,(-a_1 \sin\beta l - a_2 \cos\beta l + a_3 \sinh\beta l + a_4 \cosh\beta l) = 0 \tag{d}$$

These four boundary conditions thus yield four equations [(a) through (d)] in the four unknown coefficients a_1, a_2, a_3, and a_4. These can be written as the single vector equation

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ \beta & 0 & \beta & 0 \\ \sin\beta l & \cos\beta l & \sinh\beta l & \cosh\beta l \\ -\beta^2 \sin\beta l & -\beta^2 \cos\beta l & \beta^2 \sin\beta l & \beta^2 \cosh\beta l \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

From linear algebra, a homogeneous system has a nonzero solution for the vector $\mathbf{a} = [a_1\ a_2\ a_3\ a_4]^T$ only if the determinant of the coefficient matrix vanishes (i.e., the

[1]This example is taken from Inman [10], Example 6.5.1.

matrix is singular).
Consequently, not all components of **a** can be uniquely determined.

Setting the determinant above equal to zero yields the characteristic equation

$$\tan\beta l = \tanh\beta l$$

This equality is satisfied for an infinite number of choices for β, denoted β_n ($n = 1, 2, 3, \ldots$). The first five solutions are:

$$\begin{array}{lll} \beta_1 l = 3.926602 & \beta_2 l = 7.068583 & \beta_3 l = 10.210176 \\ \beta_4 l = 13.351768 & \beta_5 l = 16.49336143 & \end{array}$$

For the rest of the modes (i.e., for values of the index $n > 5$), the solutions to the characteristic equation are well approximated by

$$\beta_n l = \frac{(4n+1)\pi}{4} \tag{2.67}$$

The weighted frequencies determine the system natural frequencies by

$$\omega_n = \beta_n^2\sqrt{\frac{EI}{\rho A}}\text{rad/s}, \quad \text{and} \quad f_n = \frac{\beta_n^2}{2\pi}\sqrt{\frac{EI}{\rho A}}\ \text{Hz}$$

With these values of the weighted frequencies $\beta_n l$, the individual modes of vibration can be calculated. Solving the preceding matrix equation for the individual coefficients a_i yields $a_1 = -a_3, a_2 = -a_4$, and

$$(\sinh\beta_n l - \sin\beta_n l)\, a_3 + (\cosh\beta_n l - \cos\beta_n l)\, a_4 = 0$$

Thus

$$a_3 = -\frac{\cosh\beta_n l - \cos\beta_n l}{\sinh\beta_n l - \sin\beta_n l} a_4$$

for each n. The coefficient a_4 remains undetermined due to the singularity of the coefficient matrix; it becomes the arbitrary magnitude of the eigenfunction. Denoting it by $(a_4)_n$ and substituting into the expression for $X(x)$ yields the mode shapes:

$$X_n(x) = (a_4)_n \left[\frac{\cosh\beta_n l - \cos\beta_n l}{\sinh\beta_n l - \sin\beta_n l}(\sinh\beta_n x - \sin\beta_n x) - \cosh\beta_n x + \cos\beta_n x\right]$$

These mode shapes are orthogonal, satisfying

$$\int_0^l X_n(x) X_m(x)\, dx = 0 \quad \text{for} \quad n \neq m$$

The general solution for the transverse displacement is then

$$w(x,t) = \sum_{n=1}^{\infty}(A_n \sin\omega_n t + B_n \cos\omega_n t) X_n(x)$$

where the constants A_n and B_n are determined from the initial conditions.

Chapter 3

A general mathematical wave propagation model

3.1 A General Wave Propagation Model

3.1.1 Wave Equation Formulation

We consider wave propagation phenomena described by a second-order hyperbolic equation of the form:

$$\alpha(\mathbf{x})\partial_t^2\mathbf{U} + \mathcal{L}(\mathbf{x}, \nabla, \partial_z)\mathbf{U} = \mathbf{F}(\mathbf{x}, t) \tag{3.1}$$

where:

- $\mathbf{U}(\mathbf{x}, t) : \Omega \times \mathbb{R}^+ \to \mathbb{R}^m$, $\Omega \subset \mathbb{R}^3$ is a vector field of dimension m
- $\alpha(\mathbf{x})$ is a positive coefficient
- $\mathcal{L}(\mathbf{x}, \nabla, \partial_z)$ is a linear differential operator of second order
- $\mathbf{F}(\mathbf{x}, t)$ represents a source term

Moreover $\mathcal{L}(\mathbf{x}, \nabla, \partial_z)$ is **homogeneous** in the sense that it contains only second order derivatives.

We assume that $\mathcal{L}(\mathbf{x}, \nabla, \partial_z)$ is **formally self-adjoint** and **positive**, i.e. that, for $(\mathbf{U}, \mathbf{V})$ compactly supported with adapted regularity:

$$(\mathbf{U}, \mathbf{V}) \longmapsto \int_\Omega \mathcal{L}(\mathbf{x}, \nabla, \partial_z)\mathbf{U} \cdot \mathbf{V}\, d\mathbf{x} \tag{3.2}$$

is symmetric and positive. This fundamental assumption ensures the conservation of energy and the reality of eigenvalues, as will become evident throughout this chapter.

The self-adjointness condition can be expressed explicitly as:

$$\int_\Omega \mathcal{L}(\mathbf{x}, \nabla, \partial_z)\mathbf{U} \cdot \mathbf{V}\, d\mathbf{x} = \int_\Omega \mathbf{U} \cdot \mathcal{L}(\mathbf{x}, \nabla, \partial_z)\mathbf{V}\, d\mathbf{x} \tag{3.3}$$

for all admissible $\mathbf{U}, \mathbf{V}$. The positivity condition requires:

$$\int_\Omega \mathcal{L}(\mathbf{x}, \nabla, \partial_z)\mathbf{U} \cdot \mathbf{U}\, d\mathbf{x} \geq 0 \tag{3.4}$$

with equality if and only if $\mathbf{U} = 0$.

3.2 Model problems

3.2.1 Acoustic Waves

For acoustic waves propagating in a fluid, we have $m = 1$ and $\mathbf{U} = p$, where p denotes the acoustic pressure. The coefficients are:

$$\alpha(\mathbf{x}) = \lambda^{-1} \tag{3.5}$$

where λ is Lamé's coefficient and:

$$\mathcal{L}(\mathbf{x}, \nabla, \partial_z) = -\operatorname{div}(\rho^{-1}\nabla) \tag{3.6}$$

where ρ is the fluid density. and $\nabla = (\nabla, \partial_z)$ denotes the full three-dimensional gradient, and $\operatorname{div} = \nabla^t$.

3.2.2 Electromagnetic Waves

For electromagnetic waves, we have $m = 3$ and $\mathbf{U} = \mathbf{E}$, the electric field. The coefficients are:

$$\alpha(\mathbf{x}) = \varepsilon$$

where ε is the electric permittivity, and:

$$\mathcal{L}(\mathbf{x}, \nabla, \partial_z) = \operatorname{curl}(\mu^{-1} \operatorname{curl}) \tag{3.7}$$

where μ is the magnetic permeability.

3.3 The Notion of a Waveguide

A structure is called a **waveguide** when the propagation model is invariant under translation. This fundamental property leads to two essential characteristics:

3.3.1 Geometric Characterization

Definition 3.3.1 (Waveguide Geometry) *The propagation domain is a cylinder:*

$$\Omega = S \times \mathbb{R}$$

where $S \subset \mathbb{R}^2 = \{\mathbf{x} = (x, y)\}$ *is a connected open set called the* ***cross-section****, and* $z \in \mathbb{R}$*is the* ***longitudinal variable****.*

3.3.2 Classification of Waveguides

Corollary 3.3.2 (Closed Waveguide) *If the cross-section* S *is bounded, the waveguide is said to be* ***closed****.*

Corollary 3.3.3 (Open Waveguide) *If* S *is unbounded, the waveguide is* ***open****, allowing energy leakage through radiation.*

3.3.3 Boundary Conditions

The waveguide model must be completed by appropriate boundary conditions on $\partial\Omega = (\partial S) \times \mathbb{R}$. For our two principal examples:

Acoustic waveguides:

$$\partial_\nu p = 0 \quad \text{on } \partial\Omega \quad \text{(rigid boundary)}$$

Electromagnetic waveguides:

$$\nu \times (\mathbf{E} \times \nu) = 0 \quad \text{on } \partial\Omega \quad \text{(PEC boundary)}$$

where ν is the outward unit normal vector to ∂S.

3.4 Problem Formulation For waveguides

The study of waveguides naturally leads to two complementary problem formulations, each suited to different physical situations and mathematical analyses.

3.4.1 The Evolution Problem (P_t)

The evolution problem describes the complete time-dependent behavior of the wave field given initial conditions.

Problem 3.4.1 (Evolution Problem) *Find* $\mathbf{U}(\mathbf{x}, t)$ *satisfying:*

$$\begin{cases} \alpha(\mathbf{x})\partial_t^2\mathbf{U} + \mathcal{L}(\mathbf{x}, \nabla, \partial_z)\mathbf{U} = \mathbf{F}(\mathbf{x}, t), & (\mathbf{x}, t) \in \Omega \times \mathbb{R}^+ \\ \mathbf{U}(\mathbf{x}, 0) = \mathbf{U}_0(\mathbf{x}), \quad \partial_t\mathbf{U}(\mathbf{x}, 0) = \mathbf{U}_1(\mathbf{x}), & \mathbf{x} \in \Omega \\ \textit{Boundary conditions on } \partial\Omega \end{cases} \tag{3.8}$$

3.4.2 The Time-Harmonic Problem (P_ω)

When the source is monochromatic, we seek time-harmonic solutions.

Problem 3.4.2 (Time-Harmonic Problem) *Given frequency* $\omega > 0$ *and assuming:*

$$\mathbf{F}(\mathbf{x}, t) = \mathbb{F}(\mathbf{x})e^{-i\omega t}$$

look for solutions of the form:

$$\mathbf{U}(\mathbf{x}, t) = \mathbf{U}_\omega(\mathbf{x})e^{-i\omega t}$$

Substituting into (3.1) yields the time-harmonic equation:

$$\begin{cases} -\alpha(\mathbf{x})\omega^2\mathbf{U}_\omega + \mathcal{L}(\mathbf{x}, \nabla, \partial_z)\mathbf{U}_\omega = \mathbb{F}(\mathbf{x}), \quad \mathbf{x} \in \Omega \\ \textit{Boundary conditions on } \partial\Omega \\ \textit{Radiation conditions as } z \to \pm\infty \end{cases} \tag{3.9}$$

3.4.3 Radiation Conditions in Wave Propagation

The Wave Propagation Problem

Consider a typical wave equation in an unbounded domain:

$$\partial_{tt}u(x, t) - c^2\Delta u(x, t) = f(x, t), \qquad x \in \mathbb{R}^d, \ t \in \mathbb{R}. \tag{3.10}$$

Applying the Fourier transform with respect to time:

$$\hat{u}(x, \omega) = \int_{-\infty}^{+\infty} u(x, t)e^{i\omega t}dt, \tag{3.11}$$

leads to a frequency-domain formulation.

Helmholtz Equation in the Frequency Domain

After Fourier transform, we obtain

$$-\omega^2\hat{u}(x,\omega) - c^2\Delta\hat{u}(x,\omega) = \hat{f}(x,\omega). \tag{3.12}$$

For each frequency ω, this can be written as a Helmholtz equation:

$$-\Delta\hat{u} - k^2\hat{u} = \hat{f}, \qquad k = \frac{\omega}{c}. \tag{3.13}$$

This equation is posed on the unbounded domain $\mathbb{R}^d$.

Why Additional Conditions Are Needed

The Helmholtz equation on $\mathbb{R}^d$ admits many solutions:

* **Outgoing waves**
* **Incoming waves**
* **Standing waves**
* **Non-physical growing solutions**

Radiation conditions are imposed to select the physically relevant solution corresponding to wave propagation away from the sources.

Physical Interpretation

Radiation conditions express the fundamental principle:
Waves generated by sources propagate outward and no energy enters the domain from infinity. This condition:

- Excludes incoming waves from infinity
- Ensures outward energy propagation
- Encodes causality of the time-dependent problem

The Sommerfeld Radiation Condition

The Sommerfeld radiation condition is defined as the constraint that ensures that the propagating waves are purely **outgoing waves**.

In three dimensions ($d = 3$), it takes the following form:

$$\lim_{r\to\infty} r^{\frac{d-1}{2}} \left(\frac{\partial\hat{u}}{\partial r} - ik\hat{u} \right) = 0, \qquad r = |x|. \tag{3.14}$$

This enforces the asymptotic behavior:

$$\hat{u}(x,\omega) \sim \frac{e^{ikr}}{r^{(d-1)/2}}, \qquad r \to \infty, \tag{3.15}$$

corresponding to outgoing spherical waves.

Time-Domain Interpretation

The phrase "radiation conditions at $\pm\infty$" is commonly understood in two equivalent ways:

Time-domain interpretation:

- $t \to +\infty$: waves propagate forward in time
- $t \to -\infty$: no waves exist before the source acts

This reflects the principle of causality.

Frequency-Domain Causality

In the frequency domain, causality is enforced by the prescription:

$$\omega^2 \longrightarrow (\omega + i0)^2, \tag{3.16}$$

known as the limiting absorption principle.

This choice:

- Selects outgoing solutions
- Excludes incoming or non-physical modes
- Guarantees uniqueness of the solution

One-Dimensional Example

Consider the one-dimensional Helmholtz equation:

$$-u''(x) - k^2 u(x) = f(x), \qquad x \in \mathbb{R}. \tag{3.17}$$

The homogeneous solution is:

$$u(x) = Ae^{ikx} + Be^{-ikx}. \tag{3.18}$$

Radiation conditions impose:

- as $x \to +\infty$: only e^{ikx} (right-going wave)
- as $x \to -\infty$: only e^{-ikx} (left-going wave)

Equivalence Between Causality and Radiation

Radiation conditions in space are equivalent to causality in time:

$$\text{causality in time} \Longleftrightarrow \text{outgoing radiation condition in space} \tag{3.19}$$

They ensure:

- Uniqueness of the frequency-domain problem
- Physical relevance of the solution
- Well-defined inverse Fourier transform

Radiation conditions are essential for both physical and mathematical consistency of wave propagation problems in unbounded domains.

3.4.4 Limiting Amplitude Principle

Time-dependent wave equation

We consider the wave equation in $\mathbb{R}^d$:

$$\begin{cases} \partial_{tt}\mathbf{U}(x,t) - \Delta\mathbf{U}(x,t) = \mathbf{f}(x)e^{-i\omega t}, & \omega > 0, \\ \mathbf{U}(\cdot,0) = \partial_t\mathbf{U}(\cdot,0) = 0. \end{cases} \tag{3.20}$$

The source oscillates forever at frequency ω

What happens as $t \to +\infty$?

The solution consists of two parts:

$$\mathbf{U}(x,t) = \mathbf{U}_{\text{transient}}(x,t) + \Re\big(\mathbf{U}_\omega(x)e^{-i\omega t}\big)\,.$$

- Transient waves are generated initially,
- A steady oscillation at frequency ω is continuously forced.

The question is: which part survives for large time?

Statement of the Limiting Amplitude Principle

The Limiting Amplitude Principle states that

$$\lim_{t\to+\infty} e^{i\omega t}\,\mathbf{U}(x,t) = \mathbf{U}_\omega(x), \tag{3.21}$$

locally in space (or in suitable norms).

This means that, after removing the oscillation $e^{-i\omega t}$, the solution converges to a *time-independent amplitude* [8][6].

Interpretation of the limit

The limit implies that for large t,

$$\mathbf{U}(x,t) \;\approx\; \Re\big(\mathbf{U}_\omega(x)e^{-i\omega t}\big)\,.$$

- The transient part vanishes locally,
- The solution oscillates purely at frequency ω,
- The spatial profile $\mathbf{U}_\omega(x)$ is fixed.

Why do transients disappear? (Physical intuition)

- The wave equation is dispersive,
- Energy carried by transient waves propagates to infinity,
- No energy enters the domain from infinity.

As a result, near any fixed observation point x,

$$\mathbf{U}_{\text{transient}}(x,t) \longrightarrow 0 \quad \text{as } t \to +\infty.$$

The limiting amplitude solves a Helmholtz problem

The limiting amplitude $\mathbf{U}_\omega(x)$ satisfies

$$-\Delta \mathbf{U}_\omega - \omega^2 \mathbf{U}_\omega = \mathbf{f}(x) \quad \text{in} \quad \mathbb{R}^d. \tag{3.22}$$

This is the Helmholtz equation associated with the frequency ω.

Role of radiation conditions

The Helmholtz equation alone is not uniquely solvable in $\mathbb{R}^d$. Radiation conditions are imposed to select the physical solution:

- waves propagate outward,
- no incoming energy from infinity.

The limiting amplitude is the *outgoing solution.*

Connection with the Limiting Absorption Principle

The outgoing Helmholtz solution can be obtained by solving

$$-\Delta \mathbf{U} - (\omega^2 + i\varepsilon)\mathbf{U} = \mathbf{f}, \qquad \varepsilon > 0,$$

and letting $\varepsilon \to 0^+$

- absorption damps waves at infinity,
- the limit selects the outgoing solution.

Amplitude vs absorption

- **Limiting absorption principle**: selects the correct frequency-domain solution.
- **Limiting amplitude principle**: explains why this solution is observed as $t \to +\infty$.

Both principles describe the same physical selection mechanism.

What the principle does not imply

- It does not imply decay of $\mathbf{U}(x,t)$ itself,
- It does not imply convergence in global energy norms,
- Convergence is typically local in space.

Waves do not disappear, they propagate away.

3.4.5 Relation Between the Two Formulations

The two problems are intimately connected through Fourier analysis and the limiting amplitude principle.

Theorem 3.4.3 (Fourier Representation) *The solution of (P_t) can be obtained as a superposition of time-harmonic solutions:*

$$\mathbf{U}(\mathbf{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{U}_\omega(\mathbf{x})e^{-i\omega t}\,d\omega \tag{3.23}$$

Theorem 3.4.4 (Limiting Amplitude Principle) *For any initial data $(\mathbf{U}_0, \mathbf{U}_1)$ with compact support:*

$$\lim_{t\to+\infty} e^{i\omega t}\mathbf{U}(\mathbf{x},t) = \mathbf{U}_\omega(\mathbf{x}) \tag{3.24}$$

where $\mathbf{U}_\omega$ solves the time-harmonic problem with $\mathbb{F} = 0$.

Proof. The proof follows from the Limiting Absorption Principle [11] together with the spectral decomposition of the operator A. It consists in expressing the solution via Duhamel's formula and passing to the limit as $t \to +\infty$. We refer to [4], Section 2.5, for full details. ■

3.5 Guided Modes

Definition 3.5.1 (Guided Mode) *A guided mode is a particular non-zero solution of the homogeneous equation obtained by separation of variables:*

$$\mathbf{U}(\mathbf{x},t) = \mathcal{U}(x,y)e^{i(\beta z-\omega t)}, \quad (\omega,\beta)\in\mathbb{R}, \quad \mathcal{U}\neq 0 \tag{3.25}$$

Where:
$\mathcal{U}(x,y)$ is the ***transverse mode profile***

Substituting (3.25)into the homogeneous equation ($\mathbf{F} = 0$) yields:

$$-\alpha(\mathbf{x})\omega^2\mathcal{U} + \mathcal{L}(\mathbf{x},\nabla,i\beta)\mathcal{U} = 0 \tag{3.26}$$

Definition 3.5.2 (Reduced Operator) *Define:*

$$\mathcal{L}_\beta(\mathbf{x},\nabla) := \mathcal{L}(\mathbf{x},\nabla,i\beta) \tag{3.27}$$

Equation (3.26) becomes:

$$\mathcal{L}_\beta\mathcal{U} = \alpha\omega^2\mathcal{U} \tag{3.28}$$

3.5.1 The Dispersion Relation

Definition 3.5.3 (Dispersion Relation) *The existence of non-trivial solutions $\mathcal{U} \neq 0$ to (3.28) requires:*

$$\ker\left[\mathcal{L}_\beta - \alpha\omega^2\right] \neq \{0\} \tag{3.29}$$

This condition defines an equivalence relation between ω and β:

$$D(\omega, \beta) = 0 \tag{3.30}$$

*called the **dispersion relation** of the waveguide.*

3.5.2 Operator Formulation in Hilbert Space

Let $\mathcal{H}$ be an appropriate Hilbert space with inner product $(\cdot,\cdot)$e,g $\mathcal{H} = L^2(S)^m$ and after incorporating the boundary conditions in the domain of the operator, the modal equation(3.26) can be written as a quadratic eigenvalue problem:

$$A(\beta)\mathcal{U} = \omega^2\mathcal{U} \tag{3.31}$$

Proposition 3.5.4 (Structure of $A(\beta)$) *[12] The operator $A(\beta)$ has the quadratic form:*

$$A(\beta) = A_2 + i\beta A_1 + \beta^2 A_0 \tag{3.32}$$

where:

- *A_2, A_0 are self-adjoint and positive*
- *A_1 is skew-adjoint: $A_1^* = -A_1$*
- *A_0 is bounded*

Proof. This follows directly from the expansion of $\mathcal{L}_\beta$ in powers of β and the properties of $\mathcal{L}$. ■

Proposition 3.5.5 (Self-adjoint for Real β) *[12] For $\beta \in \mathbb{R}$, $A(\beta)$ is self-adjoint on its domain $D(A(\beta))$.*

Proof. Compute the adjoint:

$$\begin{aligned}
A(\beta)^* &= (A_2 + i\beta A_1 + \beta^2 A_0)^* \\
&= A_2^* + (i\beta A_1)^* + (\beta^2 A_0)^* \\
&= A_2 - i\beta A_1^* + \beta^2 A_0 \\
&= A_2 - i\beta(-A_1) + \beta^2 A_0 \\
&= A_2 + i\beta A_1 + \beta^2 A_0 = A(\beta)
\end{aligned}$$

■

3.6 Two Complementary Viewpoints

The eigenvalue problem $A(\beta)\mathcal{U} = \omega^2\mathcal{U}$ can be approached from two complementary perspectives.

3.6.1 Given β, Find ω

This approach is natural for the evolution problem, as Fourier transform in z fixes β as a parameter.

Problem 3.6.1 (β-eigenvalue problem) *For each $\beta \in \mathbb{R}$, find eigenvalues ω^2 and eigenvectors $\mathcal{U}$ satisfying:*

$$A(\beta)\mathcal{U} = \omega^2\mathcal{U} \tag{3.33}$$

3.6.2 Given ω, Find β

This approach is natural for the time-harmonic problem and is essential for defining radiation conditions.

Problem 3.6.2 (ω-eigenvalue problem) *For each $\omega \in \mathbb{R}$, find eigenvalues β^2 and eigenvectors $\mathcal{U}$ satisfying a quadratic eigenvalue problem.*

When $\beta \notin \mathbb{R}$, $A(\beta)$ is no longer self-adjoint, making this problem more challenging.

3.7 The Special Case $A_1 = 0$, $A_0 > 0$

A significant simplification occurs when the skew-adjoint term vanishes and the zero-order term is positive.

3.7.1 Simplified Operator Structure

When $A_1 = 0$, the operator reduces to:

$$A(\beta) = A_2 + \beta^2 A_0 \tag{3.34}$$

The eigenvalue problem becomes:

$$(A_2 + \beta^2 A_0)\mathcal{U} = \omega^2\mathcal{U} \tag{3.35}$$

Proposition 3.7.1 (Dual Problem) *When $A_1 = 0$ and $A_0 > 0$, define:*

$$\tilde{A}(\omega) := A_0^{-1}(A_2 - \omega^2) \tag{3.36}$$

Then the eigenvalue problem is equivalent to:

$$\tilde{A}(\omega)\mathcal{U} = -\beta^2\mathcal{U} \tag{3.37}$$

Proof. Starting from (3.35):

$$\begin{aligned} A_2\mathcal{U} + \beta^2 A_0\mathcal{U} &= \omega^2\mathcal{U} \\ A_2\mathcal{U} - \omega^2\mathcal{U} &= -\beta^2 A_0\mathcal{U} \end{aligned}$$

Multiplying by A_0^{-1} (which exists and is bounded since $A_0 > 0$):

$$A_0^{-1}(A_2 - \omega^2)\mathcal{U} = -\beta^2\mathcal{U}$$

which is exactly (3.37). ■

3.7.2 Functional Framework for the Dual Problem

Definition 3.7.2 (Weighted Inner Product) *Define:*

$$(u, v)_{A_0} := (A_0 u, v) \tag{3.38}$$

Since $A_0 > 0$ and bounded, this inner product is equivalent to the original one.

Proposition 3.7.3 (Self-adjoint of $\tilde{A}(\omega)$) *The operator $\tilde{A}(\omega)$ is self-adjoint with respect to the inner product $(\cdot, \cdot)_{A_0}$.*

Proof. For $u, v \in D(\tilde{A}(\omega)) = D(A_2)$:

$$\begin{aligned} (\tilde{A}(\omega)u, v)_{A_0} &= (A_0 A_0^{-1}(A_2 - \omega^2)u, v) \\ &= ((A_2 - \omega^2)u, v) \\ &= (u, (A_2 - \omega^2)v) \\ &= (u, A_0 A_0^{-1}(A_2 - \omega^2)v) \\ &= (u, \tilde{A}(\omega)v)_{A_0} \end{aligned}$$

■

Proposition 3.7.4 (Boundedness from Below) *$\tilde{A}(\omega)$ is bounded from below.*

Proof. Since $A_2 \geq 0$:

$$A_2 - \omega^2 \geq -\omega^2 I$$

Multiplying by $A_0^{-1} > 0$:

$$\tilde{A}(\omega) \geq -\omega^2 A_0^{-1}$$

The right-hand side is bounded because A_0^{-1} is bounded. ■

3.7.3 Compact Resolvent Property

Theorem 3.7.5 (Compact Resolvent for Closed Waveguides) *In closed waveguides, the embedding $V \subset \mathcal{H}$ is compact, where $V = D(A_2^{1/2})$. Consequently, A_2 has a compact resolvent, and both $A(\beta)$ and $\tilde{A}(\omega)$ have compact resolvents.*

Proof. The compact embedding property follows from Rellich's theorem when S is bounded. The remainder follows from the theory of compact perturbations. ∎

3.8 The Acoustic Waveguide: A Complete Example

3.8.1 Spectral Analysis

Since the embedding $V \subset \mathcal{H}$ is compact (S bounded), the resolvent of A_2 is compact.

Theorem 3.8.1 (Spectrum of Acoustic Waveguide) *The spectrum of $A(\beta)$ consists of a discrete set of eigenvalues:*

$$\sigma(A(\beta)) = 0 \leq \lambda_1(\beta) \leq \lambda_2(\beta) \leq \cdots \to +\infty \tag{3.39}$$

The associated eigen functions $\{\mathcal{U}_{n,\beta}, n \geq 1\}$ constitute an orthonormal basis.

3.8.2 Constant Sound Speed Case

When the sound speed c is constant, further simplifications occur.

Proposition 3.8.2 (Constant Velocity Simplification) *If $c = \sqrt{\lambda/\rho}$ is constant, then $A_0 = c^2 I$ and:*

$$A_2 p_{n,\beta} = \lambda_n p_{n,\beta}, \quad \lambda_n = \omega_n(\beta)^2 - \beta^2 c^2 \tag{3.40}$$

where $\{\lambda_n\}_{n=1}^{\infty}$ are the eigenvalues of A_2 (independent of β).

Corollary 3.8.3 (Dispersion Relation) *For constant sound speed:*

$$\omega_n(\beta) = c\sqrt{\lambda_n + \beta^2} \tag{3.41}$$

$$\beta_n(\omega) = \begin{cases} \frac{1}{c}\sqrt{\omega^2 - \lambda_n}, & \lambda_n \leq \omega^2 \quad \text{(propagative)} \\ \frac{i}{c}\sqrt{\lambda_n - \omega^2}, & \lambda_n > \omega^2 \quad \text{(evanescent)} \end{cases} \tag{3.42}$$

3.8.3 Non-Constant Sound Speed

For variable sound speed, we have explicit bounds:

Proposition 3.8.4 (Dispersion Bounds) *If $c_-^2 I \leq A_0 \leq c_+^2 I$, then:*

$$c_- \sqrt{\lambda_n + \beta^2} \leq \omega_n(\beta) \leq c_+ \sqrt{\lambda_n + \beta^2} \tag{3.43}$$

3.9 Solution of the Evolution Problem

We now demonstrate how guided modes provide an explicit solution to the evolution problem.

3.9.1 Fourier Transform in the Longitudinal Direction

Consider the homogeneous evolution problem with zero initial velocity:

$$\begin{cases} \alpha(\mathbf{x}_T)\partial_t^2\mathbf{U} + \mathcal{L}(\mathbf{x}_T, \nabla_T, \partial_z)\mathbf{U} = 0, & (\mathbf{x}, t) \in \Omega \times \mathcal{H}^+ \\ \mathbf{U}(\mathbf{x}, 0) = \mathbf{U}_0(\mathbf{x}), \quad \partial_t\mathbf{U}(\mathbf{x}, 0) = 0, & \mathbf{x} \in \Omega \end{cases} \tag{3.44}$$

Apply Fourier transform in the longitudinal variable z:

$$\hat{\mathbf{U}}(\mathbf{x}_T, \beta, t) := \int_{-\infty}^{\infty} \mathbf{U}(\mathbf{x}_T, z, t)e^{-i\beta z}\, dz \tag{3.45}$$

Proposition 3.9.1 (Transformed Equation) *The Fourier transform satisfies:*

$$\partial_t^2\hat{\mathbf{U}}_\beta + A(\beta)\hat{\mathbf{U}}_\beta = 0 \tag{3.46}$$

with initial conditions $\hat{\mathbf{U}}_\beta(0) = \hat{\mathbf{U}}_{0,\beta}$, $\partial_t\hat{\mathbf{U}}_\beta(0) = 0$.

3.9.2 Mode Decomposition

Since $A(\beta)$ is self-adjoint with compact resolvent (for closed waveguides), its eigenfunctions $\{\mathcal{U}_{n,\beta}\}_{n=1}^{\infty}$ form an orthonormal basis of $\mathcal{H}$.

Expand the transformed field in this basis:

$$\hat{\mathbf{U}}_\beta(\cdot, t) = \sum_{n=1}^{\infty} u_{n,\beta}(t)\mathcal{U}_{n,\beta} \tag{3.47}$$

Proposition 3.9.2 (Mode Amplitude Equations) *The coefficients* $u_{n,\beta}(t)$ *satisfy:*

$$u''_{n,\beta}(t) + \omega_n(\beta)^2 u_{n,\beta}(t) = 0 \tag{3.48}$$

where $\omega_n(\beta) = \sqrt{\lambda_n(\beta)}$ *and* $\lambda_n(\beta)$ *are the eigenvalues of* $A(\beta)$.

Proof. Substitute (3.46) into (3.45), use linearity and the eigenfunction relation $A(\beta)\mathcal{U}_{n,\beta} = \omega_n(\beta)^2\mathcal{U}_{n,\beta}$, and take inner product with $\mathcal{U}_{n,\beta}$. ■

Proposition 3.9.3 (Explicit Solution) *With zero initial velocity:*

$$u_{n,\beta}(t) = u_{0,n,\beta}\cos(\omega_n(\beta)t) \tag{3.49}$$

where $u_{0,n,\beta} = (\hat{\mathbf{U}}_{0,\beta}, \mathcal{U}_{n,\beta})$.

3.9.3 Inverse Fourier Transform

Theorem 3.9.4 (Solution of the Evolution Problem) *The solution of (3.46) is:*

$$\mathbf{U}(\mathbf{x}, z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} (\hat{\mathbf{U}}_{0,\beta}, \mathcal{U}_{n,\beta}) \cos(\omega_n(\beta)t) \mathcal{U}_{n,\beta}(\mathbf{x}) e^{i\beta z} \, d\beta \tag{3.50}$$

Using $\cos(\omega t) = \frac{1}{2}(e^{i\omega t} + e^{-i\omega t})$, we obtain the physically illuminating form:

$$\mathbf{U}(\mathbf{x}, z, t) = \frac{1}{4\pi} \sum_{\pm} \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} (\hat{\mathbf{U}}_{0,\beta}, \mathcal{U}_{n,\beta}) \mathcal{U}_{n,\beta}(\mathbf{x}_T) e^{i(\beta z \pm \omega_n(\beta)t)} \, d\beta \tag{3.51}$$

Chapter 4

Propagation of Acoustic Waves in an infinite Waveguide

4.1 Introduction

In this work, we aim to study wave propagation in a one dimensional periodic medium containing a local perturbation, where the propagation domain Ω is unbounded, while the geometric and material properties remain periodic outside a bounded region denoted by Ω^i

4.1.1 Statement of the Problem

Let the basic mathematical model:

$$\rho(x)\frac{\partial^2 U}{\partial^2 t} - \Delta U = F(x,t), \quad \forall x \in \mathbb{R} \tag{4.1}$$

where :

- ρ is periodic function
- $F(x,t) = \Re(f(x)e^{-i\omega t})$

it is natural to look for :

$$U(x,t) \;\approx\; \Re\left(u(x)e^{-i\omega t}\right). \tag{4.2}$$

We consider the one-dimensional scalar wave equation on $\mathbb{R}$ in the harmonic regime with variable coefficients

$$-\frac{\partial^2 u}{\partial x^2} - \rho(x)\,\omega^2 u = f. \tag{$\mathcal{P}$}$$

Remark 4.1.1 *The physical solution is the outgoing wave. However, problem ($\mathcal{P}$) is not well-posed in $L^2(\mathbb{R})$*

- *In $L^2(\mathbb{R})$: The homogeneous equation ($f = 0$) admits no nontrivial solution. for the inhomogeneous equation, either no solution exists, or if a solution exists, it is not unique in general.*
- *In $L^2_{loc}(\mathbb{R})$: The problem admits infinitely many solutions .*

4.1.2 Example 1D

Consider the case of a homogeneous reference medium
where :

- $\rho(x)$ is constant
- $\operatorname{supp} f \subset (-a, a)$.

For $x > a$, the solution to the homogeneous equation takes the form:

$$u(x) \sim \alpha e^{ikx}, \quad \text{where } k = \sqrt{\rho\omega^2} \tag{4.3}$$

- $u \notin L^2(\mathbb{R})$: Therefore, $L^2(\mathbb{R})$ is not the appropriate space for finding the physical solution.
- If $u \in L^2_{\text{loc}}(\mathbb{R})$, the general solution can be written as:

$$u(x) = \alpha e^{ikx} + \beta e^{-ikx} \tag{4.4}$$

This leads to an infinite number of solutions meaning that uniqueness is lost in $L^2_{\text{loc}}(\mathbb{R})$.

For a homogeneous medium, we can ensure uniqueness by applying the *Sommerfeld Radiation Condition* at infinity ($x \to +\infty$):"

$$u(x) \sim \alpha e^{ikx} \tag{4.5}$$

which is explicitly characterized by the differential relation:

$$\frac{du}{dx} - iku = 0, \quad \text{for } x > a \tag{4.6}$$

Consequently, establishing the problem within the constrained framework:

$$\Big\{ u \in L^2_{\text{loc}}(\mathbb{R}) + \tfrac{du}{dn} - iku = 0 \implies \text{Uniqueness} \tag{4.7}$$

Remark 4.1.2 *For periodic case, we do not use the direct application of the classical Somerfield radiation condition, which necessitates replacing it with the limited absorption principle (LAP)[7].*

The approach consists in seeking the solution u of $(\mathcal{P})$ as the limit, if it exists, of (u_ε) when ε tends to 0 (in a norm weaker than the L^2 norm, which remains to be determined):

$$u = \lim_{\varepsilon \searrow 0} u_\varepsilon,$$

where $u_\varepsilon \in H^2(\mathbb{R})$ is the unique solution of

$$-\frac{\partial^2 u_\varepsilon}{\partial x^2} - \rho(x)(\omega^2 + i\varepsilon\omega)\, u_\varepsilon = f. \tag{$\mathcal{P}_\varepsilon$}$$

We recall the assumptions of the study:

1. $$0 < \rho_- \leq \rho(x) \leq \rho_+,$$

2. the medium is a compact perturbation of a periodic medium $\rho_p(x)$:

   $$\exists L > 0, \qquad \rho_p(x \pm L) = \rho_p(x), \qquad \forall x \in \mathbb{R},$$

   and

   $$\operatorname{supp}(\rho - \rho_p) \subset \Omega^i = [a^-, a^+],$$

3. $\operatorname{supp} f \subset (-a, a), \qquad f \in L^2(\mathbb{R})$

We also determine, for every $\varepsilon > 0$, the two NtD coefficients $\lambda_\varepsilon^\pm$ such that the restriction of u_ε to Ω^i, denoted by u_ε^i, is the unique solution of the problem

$$\begin{cases} -\dfrac{\partial^2 u_\varepsilon^i}{\partial x^2} - \rho(x)(\omega^2 + i\varepsilon\omega)\, u_\varepsilon^i = f & \text{in } \Omega^i, \\ u_\varepsilon^i - \lambda_\varepsilon^- \dfrac{\partial u_\varepsilon^i}{\partial x} = 0 & \text{at } x = a^-, \\ u_\varepsilon^i + \lambda_\varepsilon^+ \dfrac{\partial u_\varepsilon^i}{\partial x} = 0 & \text{at } x = a^+. \end{cases} \tag{$\mathcal{P}_\varepsilon^i$}$$

The NtD coefficients are defined as follows. Let $u_\varepsilon^\pm \in H^1$ be the unique solution of

$$\begin{cases} -\dfrac{\partial^2 u_\varepsilon^\pm}{\partial x^2} - \rho_p(x)(\omega^2 + i\varepsilon\omega)\, u_\varepsilon^\pm = 0 & \text{in } \Omega^\pm = (a^\pm, \pm\infty), \\ \mp\dfrac{\partial u_\varepsilon^\pm}{\partial x}(a^\pm) = 1. \end{cases} \tag{$\mathcal{P}_\varepsilon^\pm$}$$

Then

$$u_\varepsilon^\pm(a^\pm) = \lambda_\varepsilon^\pm.$$

The idea is then to show that the sequences (λ_ε^+) and (λ_ε^-) indeed admit limits, denoted respectively by λ^+ and λ^-,

$$\lambda^\pm = \lim_{\varepsilon \searrow 0} \lambda_\varepsilon^\pm,$$

and to study the well-posedness of the problem $(\mathcal{P}^i)$ with the resulting NtD conditions:

$$\begin{cases} -\dfrac{\partial^2 u^i}{\partial x^2} - \rho(x)\,\omega^2 u^i = f & \text{in } \Omega^i, \\ u^i - \lambda^- \dfrac{\partial u^i}{\partial x} = 0 & \text{at } x = a^-, \\ u^i + \lambda^+ \dfrac{\partial u^i}{\partial x} = 0 & \text{at } x = a^+. \end{cases} \tag{$\mathcal{P}^i$}$$

The solution of this problem, if it exists, is the restriction to Ω^i of the physical "solution" of $(\mathcal{P})$ that we seek. The limit, if it exists, depends on the choice $\varepsilon > 0$ or $\varepsilon < 0$

4.1.3 The homogeneous medium case

Let us first clarify some ideas with the homogeneous medium case, which is a very particular case of a periodic medium and for which explicit calculations can be carried out.

Consider the case where ρ_p is constant:

$$\rho_p(x) = \alpha^2, \qquad (\alpha > 0).$$

In this case, we seek the physical solution of problem $(\mathcal{P})$. This naturally leads to the proper definition of the coefficients $\lambda^\pm$.

The unique solution $u_\varepsilon^+ \in H^2([a^+, +\infty[)$ of the problem

$$\begin{cases} -\dfrac{\partial^2 u_\varepsilon^+}{\partial x^2} - \alpha^2(\omega^2 + i\varepsilon\omega)\, u_\varepsilon^+ = 0, & \text{for } x > a^+, \\ -\dfrac{\partial u_\varepsilon^+}{\partial x}(a^+) = 1, \end{cases}$$

is of the form

$$u_\varepsilon(x) = \frac{1}{\gamma_\varepsilon} e^{-\gamma_\varepsilon (x - a^+)},$$

where γ_ε is the root with positive real part of the equation

$$\gamma_\varepsilon^2 + \alpha^2(\omega^2 + i\varepsilon\omega) = 0.$$

That is,

$$\gamma_\varepsilon = -i\alpha\sqrt{\omega^2 + i\varepsilon\omega}, \qquad \text{with } \operatorname{Im}\bigl(\sqrt{\omega^2 + i\varepsilon\omega}\bigr) > 0.$$

Hence,

$$\lambda_\varepsilon^+ = \frac{1}{\gamma_\varepsilon}.$$

The family $(\lambda_\varepsilon^+)_\varepsilon$ has a limit as ε tends to 0, which we denote by λ^+:

$$\lambda^+ = \frac{1}{\gamma_0} = -\frac{1}{i\alpha\omega}.$$

Similarly, the family (u_ε^+) has a limit in L^2_{loc} as ε tends to 0, defined by

$$u^+(x) = -\frac{1}{i\alpha\omega} e^{i\alpha\omega(x-a^+)},$$

which corresponds to a wave propagating in the direction of increasing x, under the convention

$$u(x,t) = u(x)e^{-i\omega t}.$$

The general solutions of the Helmholtz equation in Ω^+

$$-\frac{\partial^2 u}{\partial x^2} - \alpha^2\omega^2 u = f, \qquad (x > a^+),$$

are all linear combinations of the functions

$$u(x) = e^{i\alpha\omega x} \qquad \text{and} \qquad u(x) = e^{-i\alpha\omega x}.$$

The absorption principle therefore allows us to select the physical solution, namely the one propagating in the direction of increasing x. If, instead, we had considered the limit by replacing $\varepsilon > 0$ with $\varepsilon < 0$, we would have selected the wave propagating in the direction of decreasing x, that is, the one "coming from infinity".

In problem $(\mathcal{P}^i)$, we therefore impose the following NtD condition:

$$u^i + \lambda^+ \frac{\partial u^i}{\partial x} = 0, \qquad \text{at } x = a^+,$$

and similarly:

$$u^i - \lambda^- \frac{\partial u^i}{\partial x} = 0, \qquad \text{at } x = a^-,$$

with

$$\lambda^- = \lambda^+ = -\frac{1}{i\alpha\omega}.$$

One can easily show (except in the limiting case $\omega = 0$) that problem $(\mathcal{P}^i)$ is well posed, that its unique solution u^i is the limit in H^2 of the solution u_ε^i of the interior problem with absorption, and finally that the physical solution u of problem $(\mathcal{P})$, defined as the limit of the solutions $(u_\varepsilon)_\varepsilon$ of the problems $(\mathcal{P}_\varepsilon)$, is characterized by:

$$\begin{cases} u(x) = u^i(x), & x \in \Omega^i, \\ u(x) = +\dfrac{1}{i\alpha\omega}\dfrac{\partial u^i}{\partial x}(a^+)e^{i\alpha\omega(x-a^+)}, & x \geq a^+, \\ u(x) = -\dfrac{1}{i\alpha\omega}\dfrac{\partial u^i}{\partial x}(a^-)e^{-i\alpha\omega(x-a^-)}, & x \leq a^-. \end{cases}$$

Remark 4.1.3 (Zero-frequency case) *For the limiting case $\omega = 0$, the previous calculations show that the limiting absorption principle is not applicable. To see this, one must replace the absorption term $\varepsilon\omega$ by an absorption independent of ω, denoted by η. In this case, the solutions u_η^+ of the half-line problems at zero frequency with absorption satisfy*

$$u_\eta = \mathcal{O}\left(\frac{1}{\sqrt{\eta}}\right).$$

Hence, they do not admit a limit. One can similarly show that the NtD coefficients $\lambda^\pm$ do not admit a limit in this case. We shall see that, in the general setting, this type of situation may occur and not only for the zero frequency case.

4.1.4 Back to the General Case

Unlike homogeneous media, where phase and group velocities coincide, periodic structures introduce fundamental ambiguities in defining wave directionality. At each local variation of the refractive index, partial reflections occur, which can reverse the sign of the phase velocity while the overall wave energy continues propagating outward. This discrepancy sparked historical debates about whether the outgoing radiation condition should be based on phase velocity or group velocity (see [7]).

In contrast, the present work avoids these complexities entirely. By confining all non-homogeneities to a bounded domain Ω^i and assuming a purely homogeneous reference medium outside $[a^-, a^+]$, we can safely apply the classical Sommerfeld radiation condition, where both velocity notions coincide.

To illustrate the difficulty of defining an outgoing wave *a priori* in a periodic setting, we perform a numerical simulation of the time-domain wave equation:

$$\begin{cases} \rho(x)\dfrac{\partial^2 U}{\partial t^2} - \dfrac{\partial^2 U}{\partial x^2} = 0, \qquad x \in \mathbb{R}, \quad t \geq 0, \\ U(x, t = 0) = U_0(x), \\ \dfrac{\partial U}{\partial t}(x, t = 0) = 0. \end{cases}$$

where ρ satisfies

$$\begin{cases} \rho(x) = \rho_-, \qquad \forall x < 0, \\ \rho(x + L) = \rho(x), \qquad \forall x > 0, \end{cases}$$

and

$$\begin{cases} \rho(x) = \rho_+, \qquad \forall x \in \left[0, \dfrac{L}{2}\right], \\ \rho(x) = \rho_-, \qquad \forall x \in \left[\dfrac{L}{2}, L\right]. \end{cases}$$

Numerical simulations of the time-dependent wave equation are performed using a standard θ-scheme (with $\theta = 1/2$). To approximate the solution in an unbounded domain, the problem is initially restricted to a large but bounded spatial domain $[-M, M]$ enforced with homogeneous Dirichlet conditions at the endpoints. This approximation remains physically valid for early times ($t < M\sqrt{\rho_-}$), before the wave fronts reach the boundaries.

Within the localized non-homogeneous region, the wave fields exhibit a highly complex structure due to successive back scattering events induced by the spatial variations of the refractive index $\rho(x)$. Such local scattering phenomena complicate the direct implementation of classical radiation conditions or the immediate application of the limiting absorption principle.

To resolve this, we adopt a variational framework that characterizes the outgoing physical solution, transitioning systematically from the absorbed Helmholtz problem ($\epsilon > 0$) to the exact undamped limit ($\epsilon \to 0^+$).

4.2 The Damped Helmholtz Problem

Consider the equation

$$-u_\varepsilon'' - \rho(x)\big(\omega^2 + i\varepsilon\omega\big)u_\varepsilon = f \qquad \text{in } \mathbb{R}, \tag{4.8}$$

together with the condition

$$u_\varepsilon(\pm\infty) = 0.$$

We assume that

$$0 < \rho_0 \le \rho(x) \le \rho_1 < +\infty,$$

with $\varepsilon > 0$ and $\omega \neq 0$.

4.2.1 Weak formulation

We introduce the space $V := H^1(\mathbb{R})$,Multiplying the equation by a test function $v \in H^1(\mathbb{R})$, integrating over $\mathbb{R}$, and using integration by parts, we obtain:

$$\int_{\mathbb{R}} u_\varepsilon' \,\overline{v'}\, dx - \int_{\mathbb{R}} \rho(x)\big(\omega^2 + i\varepsilon\omega\big)u_\varepsilon \overline{v}\, dx = \int_{\mathbb{R}} f\,\overline{v}\, dx.$$

Hence, the weak formulation reads:

$$\begin{cases} \text{Find} \quad u_\varepsilon \in H^1(\mathbb{R}) & \text{such that} \\ a_\varepsilon(u_\varepsilon, v) = \ell(v), & \forall v \in H^1(\mathbb{R}), \end{cases} \tag{4.9}$$

where

$$a_\varepsilon(u, v) = \int_{\mathbb{R}} u'\overline{v'}\, dx - \int_{\mathbb{R}} \rho(x)\big(\omega^2 + i\varepsilon\omega\big)u\overline{v}\, dx, \tag{4.10}$$

$$\ell(v) = \int_{\mathbb{R}} f\,\overline{v}\, dx. \tag{4.11}$$

4.2.2 Existence and uniqueness

Lemma 4.2.1 *The problem (4.9) admits unique solution $u_\varepsilon inV$*

Proof. We apply the Lax-Milgram theorem .

- V is a Hilbert space
- a_ε is a continuous on $H^1(\mathbb{R})$,
  For all $u, v \in H^1(\mathbb{R})$,

$$|a_\varepsilon(u,v)| \leq \|u'\|_{L^2(\mathbb{R})}\|v'\|_{L^2(\mathbb{R})} + \rho_1|\omega^2 + i\varepsilon\omega|\|u\|_{L^2(\mathbb{R})}\|v\|_{L^2(\mathbb{R})}.$$

  Therefore,

$$|a_\varepsilon(u,v)| \leq C\|u\|_{H^1(\mathbb{R})}\|v\|_{H^1(\mathbb{R})},$$

- a_ε is a sesquilinear form coercive ,indeed $\forall u \in H^1(\mathbb{R})$,

$$a_\varepsilon(u,u) = \|u'\|^2_{L^2(\mathbb{R})} - \omega^2 \int_{\mathbb{R}} \rho|u|^2\,dx - i\varepsilon\omega \int_{\mathbb{R}} \rho|u|^2\,dx.$$

  Taking the imaginary part gives

$$\big|\mathrm{Im}\,a_\varepsilon(u,u)\big| = \varepsilon|\omega| \int_{\mathbb{R}} \rho|u|^2\,dx.$$

  Using the bound $\rho(x) \geq \rho_0$, we obtain

$$\big|\mathrm{Im}\,a_\varepsilon(u,u)\big| \geq \varepsilon|\omega|\rho_0\|u\|^2_{L^2(\mathbb{R})}.$$

  Moreover,

$$\mathrm{Re}\,a_\varepsilon(u,u) = \|u'\|^2_{L^2(\mathbb{R})} - \omega^2 \int_{\mathbb{R}} \rho|u|^2\,dx.$$

  Hence,

$$|a_\varepsilon(u,u)| \geq c\Big(\|u'\|^2_{L^2(\mathbb{R})} + \|u\|^2_{L^2(\mathbb{R})}\Big),$$

  for some constant $c > 0$ depending on ρ_0, ρ_1, ω, and ε.
  We explain in detail why the quantity

$$|a_\varepsilon(u,u)|$$

  controls the full $H^1(\mathbb{R})$ norm.
  Recall that

$$a_\varepsilon(u,u) = \|u'\|^2_{L^2} - \omega^2 \int_{\mathbb{R}} \rho|u|^2\,dx - i\varepsilon\omega \int_{\mathbb{R}} \rho|u|^2\,dx.$$

Set

$$X := \|u'\|^2_{L^2(\mathbb{R})}, \qquad Y := \int_{\mathbb{R}} \rho |u|^2 \, dx.$$

Then

$$a_\varepsilon(u,u) = X - \omega^2 Y - i\varepsilon\omega Y.$$

Therefore,

$$|a_\varepsilon(u,u)|^2 = (X - \omega^2 Y)^2 + \varepsilon^2\omega^2 Y^2.$$

Hence,

$$|a_\varepsilon(u,u)| = \sqrt{(X - \omega^2 Y)^2 + \varepsilon^2\omega^2 Y^2}. \tag{4.12}$$

Since both terms under the square root are nonnegative, we obtain

$$|a_\varepsilon(u,u)| \geq \varepsilon|\omega| \, Y.$$

That is,

$$|a_\varepsilon(u,u)| \geq \varepsilon|\omega| \int_{\mathbb{R}} \rho|u|^2 \, dx.$$

Using the lower bound $\rho(x) \geq \rho_0 > 0$, we deduce

$$|a_\varepsilon(u,u)| \geq \varepsilon|\omega|\rho_0 \|u\|^2_{L^2(\mathbb{R})}.$$

Thus, the L^2 norm is controlled.

Next, we estimate the derivative term. From

$$X = (X - \omega^2 Y) + \omega^2 Y,$$

we get

$$X \leq |X - \omega^2 Y| + \omega^2 Y.$$

Since

$$|X - \omega^2 Y| \leq |a_\varepsilon(u,u)|,$$
$$Y \leq \frac{1}{\varepsilon|\omega|} |a_\varepsilon(u,u)|,$$

it follows that

$$X \leq |a_\varepsilon(u,u)| + \frac{\omega^2}{\varepsilon|\omega|}|a_\varepsilon(u,u)| = \left(1 + \frac{|\omega|}{\varepsilon}\right) |a_\varepsilon(u,u)|.$$

Therefore,

$$\|u'\|^2_{L^2(\mathbb{R})} \leq C \, |a_\varepsilon(u,u)|.$$

Combining the two estimates yields

$$\|u'\|^2_{L^2(\mathbb{R})} + \|u\|^2_{L^2(\mathbb{R})} \leq C \, |a_\varepsilon(u,u)|.$$

Equivalently,

$$|a_\varepsilon(u,u)| \geq c\left(\|u'\|^2_{L^2(\mathbb{R})} + \|u\|^2_{L^2(\mathbb{R})}\right),$$

for some constant $c > 0$ depending on ρ_0, ρ_1, ω, and ε.

Therefore,

$$|a_\varepsilon(u,u)| \geq c\|u\|^2_{H^1(\mathbb{R})}.$$

- The linear form $\ell(v)$ is continuous on $H^1(\mathbb{R})$.

Therefore, the variational problem (4.9) admits unique solution $u_\varepsilon \in H^1(\mathbb{R})$ ■

Remark 4.2.2 *When $\varepsilon \to 0$, Lax-Milgram is not applicable since the coercivity constant tends to zero.*

4.3 Analytic solution via Fourier transform

Let the source term be the characteristic function of the interval $[-1,1]$:

$$f(x) = \begin{cases} 1, & |x| \leq 1, \\ 0, & |x| > 1. \end{cases}$$

Using the Fourier transform convention

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-i\xi x}\,dx,$$

we obtain

$$\widehat{f}(\xi) = \int_{-1}^{1} e^{-i\xi x}\,dx.$$

For $\xi \neq 0$,

$$\widehat{f}(\xi) = \left[\frac{e^{-i\xi x}}{-i\xi}\right]_{-1}^{1} = \frac{e^{-i\xi} - e^{i\xi}}{-i\xi} = \frac{2\sin\xi}{\xi}.$$

At $\xi = 0$, $\widehat{f}(0) = 2$.

Thus,

$$\widehat{f}(\xi) = \frac{2\sin\xi}{\xi}.$$

Applying the Fourier transform to the damped Helmholtz equation

$$-u_\varepsilon'' - (\omega^2 + i\varepsilon\omega)u_\varepsilon = f,$$

gives

$$\big(\xi^2 - (\omega^2 + i\varepsilon\omega)\big)\widehat{u}_\varepsilon(\xi) = \widehat{f}(\xi).$$

Hence,

$$\widehat{u}_\varepsilon(\xi) = \frac{2\sin\xi}{\xi\big(\xi^2 - k_\varepsilon^2\big)}, \qquad k_\varepsilon = \sqrt{\omega^2 + i\varepsilon\omega}, \quad \operatorname{Im}(k_\varepsilon) > 0.$$

Using contour integration in the complex plane, one obtains the explicit formula

$$u_\varepsilon(x) = -\frac{i}{2k_\varepsilon}\int_{-1}^{1} e^{ik_\varepsilon|x-y|}\,dy.$$

As $\varepsilon \to 0^+$, we have $k_\varepsilon \to \omega + i0$, and therefore

$$u_\varepsilon(x) \longrightarrow u(x) = -\frac{i}{2\omega}\int_{-1}^{1} e^{i\omega|x-y|}\,dy.$$

This limit satisfies the **outgoing Sommerfeld radiation conditions**:

$$u'(x) - i\omega u(x) \to 0 \quad \text{as } x \to +\infty, \qquad u'(x) + i\omega u(x) \to 0 \quad \text{as } x \to -\infty.$$

For the particular source term $f(x) = \mathbf{1}_{[-1,1]}(x)$, the explicit solution is

$$u(x) = \begin{cases} \dfrac{1}{\omega^2}\big(1 - e^{i\omega}\cos(\omega x)\big), & |x| \le 1, \\ -\dfrac{i\sin(\omega)}{\omega^2}\, e^{i\omega|x|}, & |x| > 1. \end{cases}$$

The outgoing Green's function for the Helmholtz operator $-\frac{d^2}{dx^2} - \omega^2$ is

$$G(x) = -\frac{i}{2\omega}e^{i\omega|x|},$$

and the solution can be written as the convolution $u = G * f$.

Remark 4.3.1 *Unlike homogeneous media where the Fourier transform gives explicit solutions, periodic media break translation invariance and require the Floquet-Bloch transform.*
This tool decomposes the solution into Bloch waves $u(x) = e^{i\kappa x}\phi(x)$ with ϕ periodic, and diagonalizes the operator over a single cell [14]. In the constant-coefficient limit, it reduces to the standard Fourier transform.

4.4 Construction of NtD conditions in the presence of Absorption

4.4.1 A first characterization of the solution

In this section, we seek to solve the following problem:

$$-\frac{\partial^2 u_\varepsilon}{\partial x^2} - \rho(x)(\omega^2 + i\varepsilon\omega)\,u_\varepsilon = f, \qquad x \in \mathbb{R}. \tag{$\mathcal{P}_\varepsilon$}$$

Lemma 4.4.1 *The problem $(\mathcal{P}_\varepsilon)$ admits a unique solution $u_\varepsilon \in H^2(\mathbb{R})$, for every $\varepsilon > 0$.*

Proof. According to the Lax–Milgram lemma,and Elliptic Regularity Theory problem $(\mathcal{P}_\varepsilon)$ admits a unique solution $u_\varepsilon \in H^2(\mathbb{R})$, for every $\varepsilon > 0$. ■
To solve $(\mathcal{P}_\varepsilon)$ numerically, the method we present relies on the following (trivial) proposition.

Proposition 4.4.2 *The problem* $(\mathcal{P}_\varepsilon)$ *is equivalent to the problem*

$$\begin{cases} -\dfrac{\partial^2 u^i_\varepsilon}{\partial x^2} - \rho(x)(\omega^2 + i\varepsilon\omega)\, u^i_\varepsilon = f, & \text{in } \Omega^i =]a^-, a^+[, \\ u^i_\varepsilon - \lambda^-_\varepsilon \dfrac{\partial u^i_\varepsilon}{\partial x} = 0, & \text{at } x = a^-, \\ u^i_\varepsilon + \lambda^+_\varepsilon \dfrac{\partial u^i_\varepsilon}{\partial x} = 0, & \text{at } x = a^+. \end{cases} \tag{$\mathcal{P}^i_\varepsilon$}$$

provided that λ^+_ε and λ^-_ε are defined by

$$\lambda^\pm_\varepsilon = u^\pm_\varepsilon(a^\pm),$$

where $u^\pm_\varepsilon$ is the unique solution in $H^2(\mathbb{R}^\pm)$ of the boundary value problem

$$\begin{cases} -\dfrac{\partial^2 u^\pm_\varepsilon}{\partial x^2} - \rho_p(x)(\omega^2 + i\varepsilon\omega)\, u^\pm_\varepsilon = 0 & \text{in } \Omega^\pm = (a^\pm, \pm\infty), \\ \mp \dfrac{\partial u^\pm_\varepsilon}{\partial x}(a^\pm) = 1. \end{cases} \tag{$\mathcal{P}^\pm_\varepsilon$}$$

More precisely, if u_ε is a solution of $(\mathcal{P}_\varepsilon)$, then the restriction of u_ε to Ω^i is a solution of $(\mathcal{P}^i_\varepsilon)$. Conversely, if u^i_ε is a solution of $(\mathcal{P}^i_\varepsilon)$, then we construct u_ε by

$$\begin{cases} u_\varepsilon(x) = u^i_\varepsilon(x), & x \in \Omega^i, \\ u_\varepsilon(x) = -\dfrac{\partial u^i_\varepsilon}{\partial x}(a^+)\, u^+_\varepsilon(x), & x \geq a^+, \\ u_\varepsilon(x) = +\dfrac{\partial u^i_\varepsilon}{\partial x}(a^-)\, u^-_\varepsilon(x), & x \leq a^-. \end{cases}$$

We have therefore reduced the resolution of problem (P_ϵ) to the resolution of a problem posed on a bounded domain, provided that the Neumann-to-Dirichlet (NtD) coefficients are known. The computation of these coefficients requires solving problems posed on half-lines where ρ is constant. In this case, explicit formulas can be derived.

Remark 4.4.3 *In the general periodic case, the periodicity of* ρ_p *is exploited to compute* $\lambda^\pm_\epsilon$ *using only one period. However, in our study where* ρ *is constant, these coefficients are obtained explicitly.*

4.4.2 Simplified calculation of DtN coefficients

the DtN coefficient is defined at $x = a$ as:

$$u^{0\prime}_\varepsilon(a) = \frac{u^{+\prime}_\varepsilon(a)}{u^+_\varepsilon(a)} u^0_\varepsilon(a) \tag{4.13}$$

where $\lambda^+ = \frac{u_\varepsilon^{+\prime}(a)}{u_\varepsilon^+(a)}$ is the DtN coefficient.

$$u_\varepsilon^{0\prime}(a) = \frac{u_\varepsilon^{+\prime}(a)}{u_\varepsilon^+(a)} u_\varepsilon^0(a) \tag{4.14}$$

where $\lambda^+ = \frac{u_\varepsilon^{+\prime}(a)}{u_\varepsilon^+(a)}$ is the DtN coefficient.

By linearity, we can write $u_\varepsilon^+ = u_\varepsilon^+(a)v_\varepsilon^+$, where v_ε^+ is the solution of the following half-line problem (P_ε^+):

$$\begin{cases} -v_\varepsilon^{+\prime\prime} - \rho_p(\omega^2 + i\varepsilon\omega)v_\varepsilon^+ = 0 \\ v_\varepsilon^+(0) = 1 \end{cases}$$

The same formulation holds for the negative half-line $(-\infty, -a)$ by replacing $+$ with $-$. The incoming and outgoing fields are matched at the boundaries:

$$v_\varepsilon^-(-a) = 1, \quad v_\varepsilon^+(a) = 1 \tag{4.15}$$

Thus, the DtN coefficients are given by:

$$\lambda^- = v_\varepsilon^{-\prime}(-a), \quad \lambda^+ = v_\varepsilon^{+\prime}(a) \tag{4.16}$$

Reduced Problem on a Bounded Interval

The global problem can be reduced to a problem (P_ε^0) on the bounded interval $(-a, a)$ with transparent boundary conditions:

$$(P_\varepsilon^0) \quad \begin{cases} -u_\varepsilon^{0\prime\prime} - \rho(\omega^2 + i\varepsilon\omega)u_\varepsilon^0 = f & \text{in } (-a, a) \\ -u_\varepsilon^{0\prime}(-a) + \lambda^- u_\varepsilon^0(-a) = 0 \\ u_\varepsilon^{0\prime}(a) + \lambda^+ u_\varepsilon^0(a) = 0 \end{cases}$$

Theorem 4.4.4 *If u_ε is the solution of the global problem (P_ε), then its restriction $u_\varepsilon|_{(-a,a)}$ is the solution of (P_ε^0). Conversely, if u_ε^0 is the solution of (P_ε^0), then the global solution can be reconstructed as:*

$$u_\varepsilon = \begin{cases} u_\varepsilon^0 & \textit{in } (-a, a) \\ u_\varepsilon^0(\pm a)v_\varepsilon^\pm & \textit{outside } (-a, a) \end{cases} \tag{4.17}$$

Solving the Half-Line Problem (ρ_p Constant)

To compute the coefficients $\lambda^\pm$, we solve the half-line problem (P_ε^+). For simplicity, suppose first that $a = 0$:

$$\begin{cases} -v_\varepsilon^{+\prime\prime} - \rho_p(\omega^2 + i\varepsilon\omega)v_\varepsilon^+ = 0 \\ v_\varepsilon^+(0) = 1 \end{cases} \tag{4.18}$$

Assuming $\rho_p = \rho_0$ is constant, the solution is a plane wave of the form:

$$v_\varepsilon^+(x) = e^{ik_\varepsilon x} \tag{4.19}$$

where the wavenumber k_ε satisfies:

$$k_\varepsilon^2 = \rho_0(\omega^2 + i\varepsilon\omega) \tag{4.20}$$

When $a \neq 0$, the solution shifting due to the boundary condition $v_\varepsilon^+(a) = 1$ yields:

$$v_\varepsilon^+(x) = e^{ik_\varepsilon(x-a)} \tag{4.21}$$

Conclusion

In this work, we have studied a general wave propagation model. The approach used here can be useful for the case of the wave equation with periodic coefficients posed on unbounded domains. The main idea is to consider an auxiliary regular perturbed problem, which can solved by solving three coupled problems: two problems posed on unbounded domains, but can be solved analytically, by using Fourier transform for the constant coefficients case or by using Floquet-Bloch transform for the periodic case, the trace values of their solutions are used to solve the third problem which is posed on a bounded domain. This last problem can be solved by using several known numerical methods prescribed in chapter 2.

As perspectives of the present work, we can treat by the same technique some non-linear generalizations of the wave equation and also to consider some specific geometries like plates or shells.

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ملخص

في هذا العمل نتناول المعادلات الديناميكية وظواهر انتشار الامواج في وسائط هيكلية. بالاستعانة ب مبدا الامتصاص المحدود (the Limiting Absorption Principle (LAP)) تمت معالجة الادلة الموجية الغير محدودة ذات اضطرابات محلية. وبعد ذلك تم تطبيق تحويلات فوريه (Fourier transforms) في حالة المعادلات ذات المعاملات الثابتة ، وتحويلات فلوكي_بلوك (Floquet-Bloch transforms) في حالة المعادلات ذات المعاملات الدورية . تهدف هذه الخطوة إلى حل المسائل الخارجية تحليلياً واستخلاص مؤثرات ديريكليه-إلى-نيومان (NtD) لربطها عدديًا بالمجال الداخلي المحدود.

الكلمات المفتاحية: انتشار الأمواج، وسائط هيكلية، مبدأ الامتصاص المحدود، أدلة موجية غير محدودة، تحويل فوريه، تحويل فلوكي-بلوخ، مؤثرات ديريكليه-إلى-نيومان(NtD)

Résumé

Ce travail est dédié à l'étude des équations dynamiques et des phénomènes de propagation d'ondes dans les milieux structurés. En se basant sur le principe d'absorption limitée (LAP), nous traitons le cas des guides d'ondes infinis présentant des perturbations locales. Ensuite, la transformation de Fourier est configurée pour les équations à coefficients constants, tandis que la transformation de Floquet-Bloch est appliquée aux équations à coefficients périodiques. Cette étape vise à résoudre analytiquement les problèmes extérieurs et à déduire les opérateurs de Dirichlet-to-Neumann (DtN) afin de les coupler numériquement avec le domaine intérieur borné.

Mots-clés:Propagation d'ondes, milieux structurés, principe d'absorption limitée (LAP), guides d'ondes infinis, transformation de Fourier, transformation de Floquet-Bloch, opérateurs Dirichlet-to-Neumann (DtN)

Abstract

This work investigates the dynamic equations and wave propagation phenomena in structured media. By applying the Limiting Absorption Principle (LAP), we analyze unbounded waveguides featuring localized perturbations. Subsequently, Fourier transforms are employed for equations with constant coefficients, while Floquet-Bloch transforms are applied to those with periodic coefficients. This step aims to analytically solve the exterior problems and derive the Dirichlet-to-Neumann (DtN) operators, enabling their numerical coupling with the bounded interior domain.

Keywords: Wave propagation, structured media, Limiting Absorption Principle (LAP), unbounded waveguides, Fourier transform, Floquet-Bloch transform, Dirichlet-to-Neumann (DtN) operators