PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH
University Kasdi Merbah Ouargla

Faculty of Mathematics and Material Sciences
Department of Mathematics
Thesis Submitted for the Degree: **Master**
Specialty: **Mathematics**
Option: **Functional Analysis**
Presented by: **Meriem Abdelouahab**

Entitled:

Double and Triple Integrals and its applications

Thesis publicly defended on 04/06 before the jury

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Academic Year: 2025 – 2026

Dedication

In the Name of Allah, the Most Gracious, the Most Merciful

“And say: My Lord, increase me in knowledge.”

All praise is due to Allah, by whose grace good deeds are accomplished. To Him belongs all gratitude for granting me the strength and guidance to complete this work.

I dedicate this humble effort to my beloved parents: to my dear father, who has always been my support and encouragement throughout my academic journey, and to my precious mother, whose prayers, patience, and endless love were the reason behind every success. May Allah protect them and bless them with health and happiness.

I also dedicate this work to the memory of my grandparents, may Allah have mercy upon them, and to my sisters, family members, and aunts who continuously supported and encouraged me throughout my studies.

My sincere gratitude also goes to my respected teachers, who inspired in me the love of knowledge and learning. May Allah reward them abundantly for their efforts and guidance.

To my friends and everyone who stood by my side with sincere support and prayers, I express my deepest appreciation and affection.

Finally, I pray that Allah makes this work beneficial and blessed, and grants me success in what pleases Him.

Acknowledgments

In the Name of Allah, the Most Gracious, the Most Merciful

“And whatever blessing you have is from Allah.”

All praise is due to Allah for granting me the strength, patience, and guidance to complete this humble work.

I would like to express my sincere gratitude to my respected supervisor, Mr. Brahim Mittou, for his valuable guidance, support, and encouragement throughout this work. May Allah reward him abundantly.

I would also like to express my sincere thanks to the honorable members of the examination committee:

- Mr. Mohammed Kouidri, President of the jury.
- Mr. Lotfi Kasmi, Examiner.
- Mr. Brahim Mittou, Supervisor.

for their valuable remarks, constructive comments, and the time devoted to evaluating this dissertation.

I also extend my thanks to the honorable members of the examination committee for accepting to evaluate this dissertation and for their valuable remarks.

My sincere thanks also go to all my respected teachers, especially Professor Lachhab Ilyas, Professor Belaid Malik, Professor Khoudja, and Professor Zenkhri Louiza, for their support and encouragement.

I would also like to thank all the professors of the Department of Mathematics at Kasdi Merbah University of Ouargla for their efforts and dedication throughout our academic journey.

Special thanks go to my beloved family, friends, and everyone who supported me with sincere prayers, encouragement, and kind words.

May Allah make this work beneficial and grant us more knowledge, guidance, and success.

"And say: My Lord, increase me in knowledge."

Abstract: In this work, we study the theory of integration from one-dimensional integrals to multiple integrals. We begin with the fundamental concepts of partitions, Darboux sums, Riemann sums, and integrability conditions. We then introduce functions of several variables, including limits, continuity, partial derivatives, and differentiability. The main part of this work is devoted to double and triple integrals, their properties, iterated integrals, and coordinate transformations such as polar, cylindrical, and spherical coordinates. Several applications are presented, including the computation of areas, volumes, masses, centers of gravity.

Keywords: Double integrals, Triple integrals, Multivariable functions, Polar coordinates.

Résumé: Dans ce mémoire, nous étudions la théorie de l'intégration depuis les intégrales à une variable jusquaux intégrales multiples. Nous commençons par les notions fondamentales de partitions, sommes de Darboux, sommes de Riemann et conditions dintégrabilité. Ensuite, nous introduisons les fonctions de plusieurs variables, notamment les limites, la continuité, les dérivées partielles et la différentiabilité. La partie principale de ce travail est consacrée aux intégrales doubles et triples, à leurs propriétés, aux intégrales itérées ainsi quaux changements de coordonnées tels que les coordonnées polaires, cylindriques et sphériques. Plusieurs applications sont présentées, notamment le calcul des aires, volumes, masses, centres de gravité.

Mots-clés : Intégrales doubles, Intégrales triples, Fonctions de plusieurs variables, Coordonnées polaires.

ملخص

في هذه المذكرة قمنَا بدرَاسة نظرية التكَامل، بدَاية من التكَاملَات وحيدة المتغير وصولًا إِلَى التكَاملَات المضَاعفة. بدأَنَا بتعريف المفَاهيم الأَسَاسية مثل التقسيمَات، مجَاميع دَاربو، مجَاميع ريمَان، وشروط قَابلية التكَامل. ثم تطرقنَا إِلَى الدوَال متعدَدة المتغيرَات، بمَا في ذلك النهَايَات، الَاستمرَارية، المشتقَات الجزئِية، وَالقَابلية لَلَاشتقَاق. أَمَا الجزء الرئِيسي فقَد خصَص لدرَاسة التكَاملَات الثنَائِية وَالثلَاثية، وخصَائِصهَا، وتحويل الإِحدَاثيَات مثل الإِحدَاثيَات القطبية، الَاسطوَانية، وَالكروية. كمَا قدمنَا عديد التطبيقَات، منهَا حسَاب المسَاحَات، الحجوم، الكتل، ومركز الثقل.

الكلمَات المفتَاحية: التكَاملَات الثنَائِية، التكَاملَات الثلَاثية، الدوَال متعدَدة المتغيرَات الإِحدَاثيَات القطبية.

Contents

Introduction

Mathematical analysis constitutes one of the fundamental pillars of modern mathematics. Among its most important branches, integration theory occupies a central position due to its deep theoretical structure and its numerous applications in geometry, physics, engineering, mechanics, and probability theory. The concept of integration first appeared as a method for computing areas, volumes, and accumulated quantities. Over time, it evolved into a rigorous mathematical theory through the works of Newton, Leibniz, Cauchy, Riemann, and Darboux.

The classical Riemann integral provides a precise framework for defining the area under a curve and studying the accumulation of infinitesimal quantities. The development of Darboux sums and refinement processes introduced a rigorous approach to integrability based on upper and lower approximations. These notions form the foundation of modern integration theory and constitute the starting point for the study of more advanced concepts in mathematical analysis.

The extension of integration to functions of several variables represents a natural and essential generalization of one-dimensional analysis. Functions depending on two or three variables arise naturally in many scientific domains, especially in geometry and mathematical physics. This transition from one-dimensional spaces to multidimensional spaces introduces new analytical and geometric difficulties related to neighborhoods, continuity, limits, differentiability, and coordinate transformations.

Double and triple integrals play a crucial role in the study of multidimensional phenomena. Double integrals allow the computation of areas, masses, and surface-related quantities over planar regions, while triple integrals are fundamental in the evaluation of volumes, densities, centers of gravity, and moments of inertia in three-dimensional spaces. Moreover, coordinate transformations such as polar, cylindrical, and spherical coordinates considerably simplify many computations involving circular, cylindrical, and spherical symmetries.

Introduction

The objective of this thesis is to present a rigorous and progressive study of double and triple integrals together with their principal applications. The work begins with the basic concepts of integration theory, including partitions, bounded functions, Darboux sums, Riemann sums, and integrability criteria. We then introduce functions of several variables and study their limits, continuity, partial derivatives, and differentiability in Euclidean spaces.

The second part of this thesis is devoted to the theory of double and triple integrals. We present their definitions, fundamental properties, iterated integrals, and the role of Fubini's theorem in reducing multidimensional integrals to repeated one-dimensional integrals. Special attention is given to coordinate transformations and Jacobian determinants in polar, cylindrical, and spherical systems.

Finally, several applications are discussed in order to illustrate the importance of multiple integration in mathematics and applied sciences. These applications include the computation of areas, volumes, masses, centers of gravity, moments of inertia, and surface areas.

This thesis is mainly based on classical references in mathematical analysis and multivariable calculus, including the works of Apostol [1], Stewart [6], Thomas [7], Courant [2], and Monier [5]. These references provide both the theoretical foundations and practical methods necessary for understanding multiple integration and its applications.

Chapter 1

Introduction to Integration Theory

Integration theory represents one of the fundamental tools of mathematical analysis. It provides rigorous methods for studying accumulation processes, areas, volumes, and many applications arising in mathematics, physics, and engineering.

1.1 Partitions and Refinement Process

Definition 1.1.1. *A partition of the interval* $[a, b]$ *is a finite ordered set*

$$P = \{x_0, x_1, \dots, x_n\},$$

such that

$$a = x_0 < x_1 < \cdots < x_n = b.$$

A partition divides the interval into smaller subintervals:

$$[x_{i-1}, x_i], \quad i = 1, \dots, n.$$

Definition 1.1.2. *A refinement of a partition* P *is another partition* P' *obtained by adding new points to* P*.*

Definition 1.1.3. *The mesh of a partition* P *is defined by*

$$\|P\| = \max_{1 \le i \le n} (x_i - x_{i-1}).$$

Remark 1.1.1. The condition $\|P\| \to 0$ guarantees that all subintervals become arbitrarily small.

1.2 Bounded Functions and Local Behavior

Definition 1.2.1. *A function $f : [a, b] \to \mathbb{R}$ is bounded if there exists a constant $M > 0$ such that*

$$|f(x)| \le M, \quad \forall x \in [a, b].$$

Example 1.2.1. The function

$$f(x) = x^2,$$

is bounded on $[0, 1]$ since

$$0 \le x^2 \le 1.$$

Definition 1.2.2 (Oscillation)**.** *The oscillation of a function f on an interval I is defined by*

$$\omega(f, I) = \sup_{x \in I} f(x) - \inf_{x \in I} f(x).$$

Lemma 1.2.1. *If f is continuous on $[a, b]$, then for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$|I| < \delta,$$

implies

$$\omega(f, I) < \varepsilon.$$

1.3 Upper and Lower Darboux Sums

Let f be bounded on $[a, b]$ and let

$$P = \{x_0, x_1, \ldots, x_n\},$$

be a partition.

For each subinterval define

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x),$$

and

$$m_i = \inf_{x \in [x_{i-1}, x_i]} f(x).$$

The upper Darboux sum is

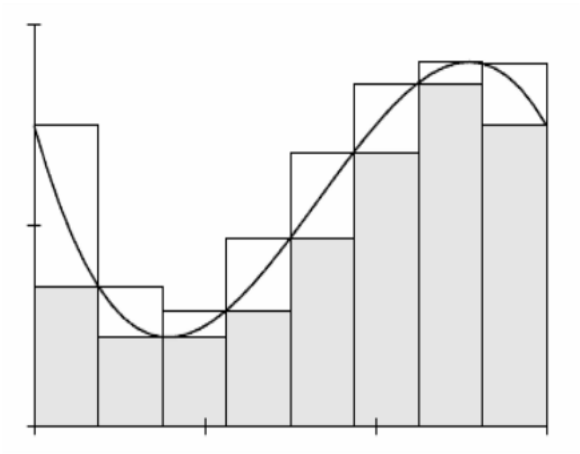
$$U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i,$$

while the lower Darboux sum is

$$L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i,$$

where

$$\Delta x_i = x_i - x_{i-1}.$$



Upper and Lower Darboux Sums

Proposition 1.3.1. *For every partition P,*

$$L(f,P) \leq U(f,P).$$

Proposition 1.3.2. *If P' is a refinement of P, then*

$$L(f,P) \leq L(f,P') \leq U(f,P') \leq U(f,P).$$

Example 1.3.1. Consider

$$f(x) = x^2,$$

on $[0,1]$ with partition

$$P = \left\{0, \frac{1}{2}, 1\right\}.$$

For the interval $\left[0, \frac{1}{2}\right]$,

$$m_1 = 0, \qquad M_1 = \frac{1}{4}.$$

For the interval $\left[\frac{1}{2}, 1\right]$,

$$m_2 = \frac{1}{4}, \qquad M_2 = 1.$$

Hence,

$$L(f,P) = 0 \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8},$$

and

$$U(f,P) = \frac{1}{4} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{5}{8}.$$

As the partition becomes finer, both sums converge toward

$$\int_0^1 x^2 dx = \frac{1}{3}.$$

1.4 Darboux Integral and Convergence Gap

Definition 1.4.1. *The upper integral of f is*

$$\overline{\int_a^b} f = \inf_P U(f, P).$$

Definition 1.4.2. *The lower integral of f is*

$$\underline{\int_a^b} f = \sup_P L(f, P).$$

Theorem 1.4.1. *A bounded function is Darboux integrable if and only if*

$$\overline{\int_a^b} f = \underline{\int_a^b} f.$$

Theorem 1.4.2. *A bounded function f is integrable on $[a, b]$ if and only if*

$$\forall \varepsilon > 0, \quad \exists P \quad \textit{such that} \quad U(f, P) - L(f, P) < \varepsilon.$$

Definition 1.4.3 (Step Function)**.** *A step function is a function that is constant on each subinterval of a partition.*

Lemma 1.4.1. *For every bounded function f, there exist step functions s_n and t_n such that*

$$s_n \leq f \leq t_n,$$

and

$$t_n - s_n \to 0.$$

1.5 Riemann Sums

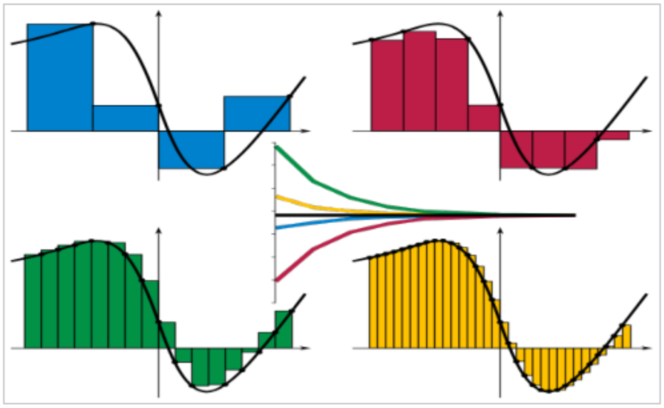
Definition 1.5.1. *Let*

$$\xi_i \in [x_{i-1}, x_i].$$

The Riemann sum associated with f and the partition P is

$$S(f, P, \xi) = \sum_{i=1}^{n} f(\xi_i)\Delta x_i.$$

Riemann Sums



Theorem 1.5.1. *If f is integrable, then all Riemann sums converge to the same limit:*

$$\lim_{\|P\|\to 0} S(f, P, \xi) = \int_a^b f(x)\,dx.$$

The following theorem gives the properties of the integral

Theorem 1.5.2. *Let f and g be integrable functions and let $\lambda \in \mathbb{R}$.*

1. *Linearity:*

$$\int_a^b (\lambda f) = \lambda \int_a^b f.$$

2. *Additivity:*

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

3. *Monotonicity: If*

$$f(x) \le g(x),$$

then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

4. *Interval additivity: For every* $c \in [a,b]$,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

5. *Triangle inequality:*

$$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx.$$

Theorem 1.5.3 (Mean Value Theorem for Integrals)**.** *If* f *is continuous on* $[a,b]$*, then there exists a point* $c \in [a,b]$ *such that*

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

1.6 Applications of Integration

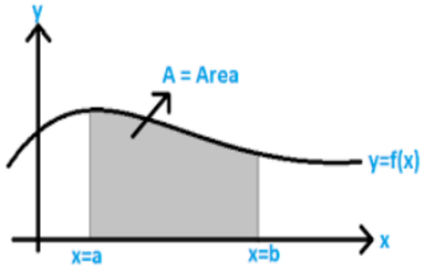
1.6.1 Area Under a Curve

If $f(x) \geq 0$ on $[a,b]$, then

$$A = \int_a^b f(x)\,dx,$$

represents the area between the curve and the x-axis.

Area under curve



Example 1.6.1. Compute the area under

$$y = \sin x,$$

on $[0, \pi]$.

We have

$$A = \int_0^\pi \sin x \, dx = [-\cos x]_0^\pi = 2.$$

1.6.2 Area Between Two Curves

If

$$f(x) \geq g(x),$$

then the area between the curves is

$$A = \int_a^b (f(x) - g(x)) dx.$$

Example 1.6.2. Find the area between

$$y = x$$

and

$$y = x^2.$$

The intersection points satisfy

$$x = x^2,$$

thus

$$x = 0 \quad \text{or} \quad x = 1.$$

Hence,

$$A = \int_0^1 (x - x^2)dx = \frac{1}{6}.$$

1.6.3 Volumes of Revolution

If a region rotates around the x-axis, the generated volume is

$$V = \pi \int_a^b [f(x)]^2 dx.$$

Example 1.6.3. The volume generated by rotating the semicircle

$$x^2 + y^2 = a^2$$

around the x-axis is

$$V = \frac{4}{3}\pi a^3,$$

which is precisely the volume of a sphere of radius a.

1.7 Transition to Higher Dimensions

Integration naturally extends to functions of several variables.

For a function $f(x, y)$ defined on a region $D \subset \mathbb{R}^2$,

$$\iint_D f(x, y)\, dx\, dy$$

represents a generalized accumulation process over two-dimensional regions.

This extension leads to multiple integration, vector calculus, and fundamental results such as Fubini's theorem and the divergence theorem.

Chapter 2

Functions of Several Variables

The study of functions of several variables constitutes one of the fundamental branches of mathematical analysis. Unlike ordinary real functions depending on a single variable, multivariable functions describe phenomena involving several independent parameters simultaneously.

The transition from one-dimensional analysis to multidimensional analysis introduces new geometric and analytical difficulties related to neighborhoods, limits, continuity, partial derivatives, and differentiability.

2.1 Euclidean Space $\mathbb{R}^n$

Definition 2.1.1. *The Euclidean space* $\mathbb{R}^n$ *is the set of all ordered* n*-tuples*

$$x = (x_1, x_2, \ldots, x_n),$$

where each coordinate belongs to $\mathbb{R}$*.*

In particular,

$$\mathbb{R}^2 = \{(x, y) \ ; \ x, y \in \mathbb{R}\},$$

and

$$\mathbb{R}^3 = \{(x, y, z) \ ; \ x, y, z \in \mathbb{R}\}.$$

Definition 2.1.2 (Distance)**.** *For two points*

$$X = (x_1, x_2, \dots, x_n), \qquad Y = (y_1, y_2, \dots, y_n),$$

the Euclidean distance between X *and* Y *is defined by*

$$d(X, Y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

Definition 2.1.3 (Norm)**.** *The Euclidean norm of a vector* X *is defined by*

$$\|X\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

In $\mathbb{R}^2$,

$$\|(x, y)\| = \sqrt{x^2 + y^2}.$$

Definition 2.1.4 (Open Ball)**.** *Let* $X_0 \in \mathbb{R}^n$ *and let* $r > 0$*.*

The open ball centered at X_0 *with radius* r *is*

$$B(X_0, r) = \{X \in \mathbb{R}^n \ ; \ \|X - X_0\| < r\}.$$

Example 2.1.1. In $\mathbb{R}^2$,

$$B((0,0), 1) = \{(x, y) \in \mathbb{R}^2 \ ; \ x^2 + y^2 < 1\},$$

which represents the open unit disk.

Definition 2.1.5 (Neighborhood)**.** *A neighborhood of a point* X_0 *is any subset of* $\mathbb{R}^n$ *containing an open ball centered at* X_0*.*

Proposition 2.1.1. *Every point of an open ball is itself the center of another open ball contained inside the first one.*

2.2 Limits of Functions of Several Variables

Let

$$f : D \subset \mathbb{R}^2 \to \mathbb{R}.$$

Definition 2.2.1. *We say that*

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L,$$

if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta,$$

implies

$$|f(x,y) - L| < \varepsilon.$$

Remark 2.2.1. The limit must be independent of the path used to approach the point.

Example 2.2.1. Consider

$$f(x,y) = x^2 + y^2.$$

We prove that

$$\lim_{(x,y)\to(1,1)} f(x,y) = 2.$$

Indeed,

$$|f(x,y) - 2| = |x^2 + y^2 - 2|.$$

We write

$$x^2 - 1 = (x-1)(x+1),$$

and

$$y^2 - 1 = (y-1)(y+1).$$

Hence,

$$|f(x,y) - 2| \leq |x-1||x+1| + |y-1||y+1|.$$

If (x,y) is sufficiently close to $(1,1)$, then $|x+1|$ and $|y+1|$ remain bounded.

Therefore,

$$\lim_{(x,y)\to(1,1)} (x^2 + y^2) = 2.$$

Definition 2.2.2. *The polar coordinate transformation is defined by*

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

where

$$r \geq 0, \qquad \theta \in [0, 2\pi].$$

Remark 2.2.2. We have

$$(x,y) \to (0,0) \quad \Longleftrightarrow \quad r \to 0.$$

Now, we give an example of a nonexistent limit.

Example 2.2.2. Consider the function

$$f(x,y) = \frac{xy}{x^2 + y^2}, \qquad (x,y) \neq (0,0).$$

Using polar coordinates:

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Then,

$$f(r\cos\theta, r\sin\theta) = \frac{r^2\cos\theta\sin\theta}{r^2} = \cos\theta\sin\theta.$$

Since the result depends on θ, the limit is not unique.

Therefore,

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2},$$

does not exist.

2.3 Continuity

Definition 2.3.1. *A function*

$$f : D \subset \mathbb{R}^2 \to \mathbb{R},$$

is continuous at (x_0, y_0) *if*

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0).$$

Remark 2.3.1. Polynomial and trigonometric functions are continuous on their domains.

Example 2.3.1. Consider

$$f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

Using polar coordinates:

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Then,

$$f(r\cos\theta, r\sin\theta) = \frac{r^4\cos^2\theta\sin^2\theta}{r^2} = r^2\cos^2\theta\sin^2\theta.$$

Since

$$|f(r\cos\theta, r\sin\theta)| \le r^2,$$

we obtain

$$\lim_{r\to 0} f(r\cos\theta, r\sin\theta) = 0.$$

Thus,

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0),$$

which proves continuity at the origin.

2.4 Partial Derivatives

Definition 2.4.1. *Let*

$$f : \mathbb{R}^2 \to \mathbb{R}.$$

The partial derivative of f with respect to x at (x_0, y_0) is

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$

provided the limit exists.

Definition 2.4.2. *The partial derivative of f with respect to y at (x_0, y_0) is*

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h},$$

provided the limit exists.

Example 2.4.1. Consider

$$f(x,y) = x^2 + xy - y^2.$$

We compute

$$\frac{\partial f}{\partial x} = 2x + y,$$

and

$$\frac{\partial f}{\partial y} = x - 2y.$$

2.5 Differentiability

Definition 2.5.1. *A function*

$$f : \mathbb{R}^2 \to \mathbb{R}$$

is differentiable at (x_0, y_0) *if there exist constants* a *and* b *such that*

$$f(x_0 + h, y_0 + k) = f(x_0, y_0) + ah + bk + \varepsilon(h, k),$$

where

$$\lim_{(h,k)\to(0,0)} \frac{\varepsilon(h, k)}{\sqrt{h^2 + k^2}} = 0.$$

Remark 2.5.1. Differentiability implies continuity.

Remark 2.5.2. The existence of partial derivatives alone is not sufficient for differentiability.

The following theorem gives sufficient condition for differentiability.

Theorem 2.5.1. *If the first-order partial derivatives of* f *exist and are continuous in a neighborhood of* (x_0, y_0)*, then* f *is differentiable at* (x_0, y_0)*.*

Example 2.5.1. Consider

$$f(x, y) = x^2 + y^2.$$

We have

$$\frac{\partial f}{\partial x} = 2x, \qquad \frac{\partial f}{\partial y} = 2y.$$

Both partial derivatives are continuous on $\mathbb{R}^2$.

Therefore, the function is differentiable everywhere in $\mathbb{R}^2$.

2.6 Geometric Interpretation

Partial derivatives describe the variation of the function in coordinate directions.

Differentiability means that the graph of the function admits a tangent plane at the considered point.

Let

$$X = (x, y)^T, \qquad X_0 = (x_0, y_0)^T.$$

The gradient vector of f at (x_0, y_0) is

$$\nabla f(x_0, y_0) = \left(\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0) \right)^T.$$

The tangent plane equation is

$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0).$$

Equivalently, using vector notation,

$$z = f(X_0) + \nabla f(X_0) \cdot (X - X_0).$$

This expression shows that the tangent plane is determined by the gradient vector at the point X_0. The gradient indicates the direction of the steepest increase of the function and is orthogonal to the level curves of f.

Chapter 3

Triple Integrals and Applications

Multiple integration extends the classical notion of integration to higher-dimensional spaces. Double and triple integrals play a central role in geometry, physics, and applied sciences through the computation of areas, volumes, masses, and other physical quantities.

3.1 Double Integrals

Definition 3.1.1. *Let $D \subset \mathbb{R}^2$ be a bounded region and let $f : D \to \mathbb{R}$ be a bounded function. We divide the region D into small subregions*

$$\Delta A_1, \Delta A_2, \ldots, \Delta A_n.$$

Inside each subregion, choose a point (x_i^*, y_i^*). The associated Riemann sum is $\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta A_i$. If the limit exists when the norm of the subdivision tends to zero, then the function f is said to be integrable over D.

Definition 3.1.2. *The double integral of f over the region D is defined by*

$$\iint_D f(x,y)\,dx\,dy = \lim_{\max \Delta A_i \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta A_i.$$

Remark 3.1.1. $f(x,y) = 1$, then $\iint_D dx\,dy$ represents the area of the region D.

3.1.1 Properties of Double Integrals

Double integrals satisfy several important properties.

Theorem 3.1.1 (Linearity)**.** *Let f and g be integrable functions on D, and let $\lambda \in \mathbb{R}$. Then*

$$\iint_D (\lambda f)\, dx\, dy = \lambda \iint_D f\, dx\, dy,$$

and

$$\iint_D (f+g)\, dx\, dy = \iint_D f\, dx\, dy + \iint_D g\, dx\, dy.$$

Theorem 3.1.2 (Positivity)**.** *If*

$$f(x,y) \geq 0,$$

for all $(x,y) \in D$, then

$$\iint_D f(x,y)\, dx\, dy \geq 0.$$

Theorem 3.1.3 (Monotonicity)**.** *If*

$$f(x,y) \leq g(x,y),$$

for all $(x,y) \in D$, then

$$\iint_D f(x,y)\, dx\, dy \leq \iint_D g(x,y)\, dx\, dy.$$

3.1.2 Iterated Double Integrals

In practice, double integrals are computed by transforming them into repeated one-dimensional integrals. This method is based on Fubini's theorem.

Theorem 3.1.4 (Fubini's Theorem)**.** *Let $f(x,y)$ be continuous on the rectangle*

$$R = [a,b] \times [c,d].$$

Then,

$$\iint_R f(x,y)\, dx\, dy = \int_a^b \left(\int_c^d f(x,y)\, dy \right) dx.$$

Similarly,

$$\iint_R f(x,y)\,dx\,dy = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy.$$

Remark 3.1.2. The order of integration may be changed whenever the function is continuous on the considered region.

3.1.3 General Domains

Suppose the region D is described by $a \le x \le b$, and $\varphi_1(x) \le y \le \varphi_2(x)$. Then,

$$\iint_D f(x,y)\,dx\,dy = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\,dx.$$

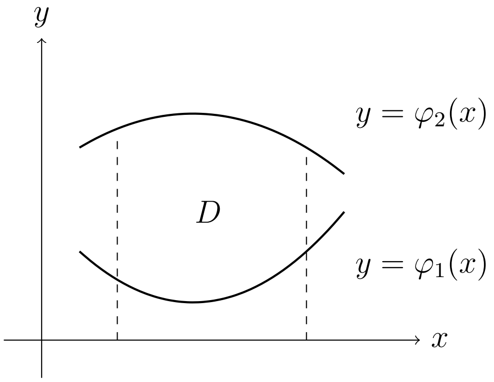


Figure 3.1: A vertically simple region

Example 3.1.1. Compute

$$\iint_D (x+y)\,dx\,dy, \text{ where} D = [0,1] \times [0,2].$$

Solution: Using Fubini's theorem,

$$I = \int_0^1 \int_0^2 (x+y)\,dy\,dx.$$

First, integrate with respect to y:

$$\int_0^2 (x+y)\,dy = \left[xy + \frac{y^2}{2} \right]_0^2 = 2x + 2.$$

Therefore,

$$I = \int_0^1 (2x+2)\,dx.$$

We obtain

$$I = \left[x^2 + 2x\right]_0^1 = 3.$$

Example 3.1.2. Evaluate

$$\iint_D xy\,dx\,dy,$$

where

$$D = \left\{(x, y) \in \mathbb{R}^2 : 0 \le x \le 1,\ 0 \le y \le x\right\}.$$

Solution: The region is triangular. Therefore,

$$I = \int_0^1 \int_0^x xy\,dy\,dx.$$

Integrating with respect to y,

$$\int_0^x xy\,dy = x\left[\frac{y^2}{2}\right]_0^x = \frac{x^3}{2}.$$

Hence,

$$I = \frac{1}{2}\int_0^1 x^3\,dx.$$

Therefore,

$$I = \frac{1}{2}\left[\frac{x^4}{4}\right]_0^1 = \frac{1}{8}.$$

3.1.4 Jacobian Determinant

Under the transformation

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

the area element changes according to

$$dx\,dy = r\,dr\,d\theta.$$

The factor r is called the Jacobian determinant.

Theorem 3.1.5. *Let D be a region described in polar coordinates. Then,*

$$\iint_D f(x, y)\,dx\,dy = \iint_{D'} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

Proof. The transformation $x = r\cos\theta$ and $y = r\sin\theta$ has Jacobian matrix

$$J = \begin{pmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{pmatrix}.$$

Hence,

$$J = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}.$$

Its determinant is

$$|J| = r\cos^2\theta + r\sin^2\theta.$$

Using $\cos^2\theta + \sin^2\theta = 1$, we obtain $|J| = r$. Therefore, $dx\,dy = r\,dr\,d\theta$. □

Example 3.1.3. Compute

$$\iint_D (x^2 + y^2)\,dx\,dy,$$

where D is the unit disk

Solution Using polar coordinates $x^2 + y^2 = r^2$, The disk becomes

$$0 \le r \le 1, \qquad 0 \le \theta \le 2\pi.$$

Thus,

$$I = \int_0^{2\pi} \int_0^1 r^2 \cdot r\,dr\,d\theta.$$

Hence,

$$I = \int_0^{2\pi} \int_0^1 r^3\,dr\,d\theta.$$

Integrating with respect to r,

$$\int_0^1 r^3\,dr = \frac{1}{4}.$$

Therefore,

$$I = \int_0^{2\pi} \frac{1}{4}\,d\theta = \frac{\pi}{2}.$$

Remark 3.1.3. Polar coordinates are especially useful when the domain contains circles, disks, cylinders, or expressions involving $x^2 + y^2$.

3.2 Triple Integrals

Triple integrals extend the concept of double integrals to functions of three variables over regions in space. Let $V \subset \mathbb{R}^3$ be a bounded solid region and let $f : V \to \mathbb{R}$ be a bounded function. We divide the region V into small subregions

$$\Delta V_1, \Delta V_2, \ldots, \Delta V_n.$$

Inside each subregion choose a point

$$(x_i^*, y_i^*, z_i^*).$$

The associated Riemann sum is

$$\sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta V_i.$$

If the limit exists when the maximum diameter of the subdivision tends to zero, then the function f is integrable over V.

Definition 3.2.1. *The triple integral of f over the region V is defined by*

$$\iiint_V f(x,y,z)\,dV = \lim_{\max \Delta V_i \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta V_i.$$

Remark 3.2.1. If $f(x,y,z) = 1$, then $\iiint_V dV$ represents the volume of the solid region V.

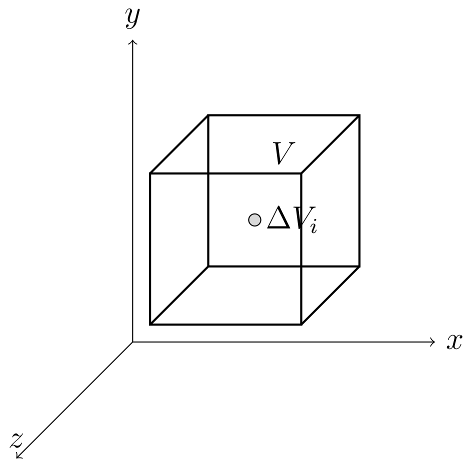


Figure 3.2: Subdivision of a solid region

3.2.1 Properties of Triple Integrals

Triple integrals satisfy properties analogous to those of double integrals.

Theorem 3.2.1 (Linearity)**.** *Let f and g be integrable on V, and let $\lambda \in \mathbb{R}$. Then*

$$\iiint_V (\lambda f)\, dV = \lambda \iiint_V f\, dV,$$

and

$$\iiint_V (f+g)\, dV = \iiint_V f\, dV + \iiint_V g\, dV.$$

Theorem 3.2.2 (Positivity)**.** *If*

$$f(x,y,z) \geq 0$$

for all $(x,y,z) \in V$, then

$$\iiint_V f(x,y,z)\, dV \geq 0.$$

Theorem 3.2.3 (Monotonicity)**.** *If*

$$f(x,y,z) \leq g(x,y,z)$$

for all $(x,y,z) \in V$, then

$$\iiint_V f(x,y,z)\, dV \leq \iiint_V g(x,y,z)\, dV.$$

3.2.2 Iterated Triple Integrals

In practice, triple integrals are evaluated as repeated integrals.

Suppose the solid region V is described by

$$(x,y) \in D \text{ and } \varphi_1(x,y) \leq z \leq \varphi_2(x,y),$$

then,

$$\iiint_V f(x,y,z)\, dV = \iint_D \left(\int_{\varphi_1(x,y)}^{\varphi_2(x,y)} f(x,y,z)\, dz \right) dx\, dy.$$

Example 3.2.1. Evaluate

$$\iiint_V (x+y+z)\,dV,$$

where

$$V = [0,1]\times[0,1]\times[0,2].$$

Solution We write

$$I = \int_0^1 \int_0^1 \int_0^2 (x+y+z)\,dz\,dy\,dx.$$

Integrating with respect to z,

$$\int_0^2 (x+y+z)\,dz = 2x+2y+2.$$

Thus,

$$I = \int_0^1 \int_0^1 (2x+2y+2)\,dy\,dx.$$

Integrating with respect to y,

$$\int_0^1 (2x+2y+2)\,dy = 2x+3.$$

Hence,

$$I = \int_0^1 (2x+3)\,dx.$$

Therefore,

$$I = \left[x^2+3x\right]_0^1 = 4.$$

Example 3.2.2. Find the volume bounded by the paraboloids

$$z = x^2+y^2 \text{ and } z = 8-x^2-y^2.$$

Solution The surfaces intersect when

$$x^2+y^2 = 8-x^2-y^2.$$

Thus,

$$x^2+y^2 = 4.$$

The projection on the xy-plane is the disk

$$x^2+y^2 \le 4.$$

Hence,

$$V = \iint_D \left[(8 - x^2 - y^2) - (x^2 + y^2)\right] dx\,dy.$$

Therefore,

$$V = \iint_D (8 - 2x^2 - 2y^2)\,dx\,dy.$$

Using polar coordinates,

$$V = \int_0^{2\pi} \int_0^2 (8 - 2r^2)\,r\,dr\,d\theta.$$

Thus,

$$V = 2\pi \int_0^2 (8r - 2r^3)\,dr.$$

We obtain

$$V = 2\pi \left[4r^2 - \frac{r^4}{2}\right]_0^2 = 16\pi.$$

3.2.3 Jacobian Determinant - cylindrical transformation

Under the transformation

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z,$$

the volume element becomes

$$dV = r\,dr\,d\theta\,dz.$$

Theorem 3.2.4. *Let V be a solid region described in cylindrical coordinates. Then,*

$$\iiint_V f(x,y,z)\,dV = \iiint_{V'} f(r\cos\theta, r\sin\theta, z)\,r\,dr\,d\theta\,dz.$$

Proof. The Jacobian matrix is

$$J = \begin{pmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} & \dfrac{\partial x}{\partial z} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial y}{\partial z} \\ \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial \theta} & \dfrac{\partial z}{\partial z} \end{pmatrix}.$$

Hence,

$$J = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Its determinant is

$$|J| = r(\cos^2\theta + \sin^2\theta) = r.$$

Therefore,

$$dV = r\,dr\,d\theta\,dz.$$

□

Example 3.2.3. Compute the volume of the cylinder

$$x^2 + y^2 \le 4, \qquad 0 \le z \le 3.$$

Solution Using cylindrical coordinates,

$$0 \le r \le 2, \qquad 0 \le \theta \le 2\pi, \qquad 0 \le z \le 3.$$

Hence,

$$V = \int_0^{2\pi}\int_0^2\int_0^3 r\,dz\,dr\,d\theta.$$

Integrating with respect to z,

$$V = \int_0^{2\pi}\int_0^2 3r\,dr\,d\theta.$$

Then,

$$V = 3\int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^2 d\theta.$$

Thus,

$$V = 3\int_0^{2\pi} 2\,d\theta = 12\pi.$$

Example 3.2.4. Evaluate

$$\iiint_V (x^2 + y^2)\,dV,$$

where V is the cylinder

$$x^2 + y^2 \le 1, \qquad 0 \le z \le 2.$$

Solution Using cylindrical coordinates,

$$x^2 + y^2 = r^2.$$

Thus,

$$I = \int_0^{2\pi}\int_0^1\int_0^2 r^2 \cdot r\,dz\,dr\,d\theta.$$

Hence,

$$I = \int_0^{2\pi} \int_0^1 2r^3 \, dr \, d\theta.$$

Integrating with respect to r,

$$\int_0^1 2r^3 \, dr = \frac{1}{2}.$$

Therefore,

$$I = \int_0^{2\pi} \frac{1}{2} \, d\theta = \pi.$$

3.2.4 Jacobian Determinant - spherical transformation

Under the spherical transformation,

$$x = \rho \sin\varphi \cos\theta, \qquad y = \rho \sin\varphi \sin\theta, \qquad z = \rho \cos\varphi,$$

the volume element becomes

$$dV = \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta.$$

Theorem 3.2.5. *Let V be a solid region described in spherical coordinates. Then,*

$$\iiint_V f(x, y, z) \, dV = \iiint_{V'} f(\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi) \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta.$$

Example 3.2.5. Compute the volume of the sphere

$$x^2 + y^2 + z^2 \leq R^2.$$

Solution In spherical coordinates,

$$0 \leq \rho \leq R, \qquad 0 \leq \varphi \leq \pi, \qquad 0 \leq \theta \leq 2\pi.$$

Hence,

$$V = \int_0^{2\pi} \int_0^{\pi} \int_0^{R} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta.$$

Integrating with respect to ρ,

$$\int_0^R \rho^2 \, d\rho = \frac{R^3}{3}.$$

Therefore,

$$V = \frac{R^3}{3} \int_0^{2\pi} \int_0^{\pi} \sin\varphi \, d\varphi \, d\theta.$$

Since

$$\int_0^{\pi} \sin\varphi\, d\varphi = 2,$$

we obtain

$$V = \frac{2R^3}{3}\int_0^{2\pi} d\theta = \frac{4}{3}\pi R^3.$$

Example 3.2.6. Find the volume cut from the sphere

$$\rho \le a \text{ by the cone } \varphi \le \alpha.$$

Solution The limits are

$$0 \le \rho \le a, \qquad 0 \le \varphi \le \alpha, \qquad 0 \le \theta \le 2\pi.$$

Thus,

$$V = \int_0^{2\pi}\int_0^{\alpha}\int_0^{a} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta.$$

Integrating with respect to ρ,

$$V = \frac{a^3}{3}\int_0^{2\pi}\int_0^{\alpha} \sin\varphi\, d\varphi\, d\theta.$$

Since

$$\int_0^{\alpha} \sin\varphi\, d\varphi = 1 - \cos\alpha,$$

we obtain

$$V = \frac{a^3}{3}\int_0^{2\pi}(1-\cos\alpha)\, d\theta = \frac{2\pi a^3}{3}(1-\cos\alpha).$$

Remark 3.2.2. Spherical coordinates are very effective when the domain involves spheres, cones, or expressions containing $x^2 + y^2 + z^2$.

3.2.5 Applications of Triple Integrals

Triple integrals possess numerous applications in geometry, physics, engineering, and mechanics. They are widely used in the computation of masses, centers of gravity, and moments of inertia.

Mass of a Solid

Definition 3.2.2. *Let a solid occupy a region V with density function $\gamma(x, y, z)$. The mass of the solid is*

$$M = \iiint_V \gamma(x, y, z)\, dV.$$

Example 3.2.7. Find the mass of the hemisphere

$$x^2 + y^2 + z^2 \le R^2, \qquad z \ge 0,$$

if the density is proportional to the distance from the plane $z = 0$,

$$\gamma(x, y, z) = kz.$$

Solution Using spherical coordinates,

$$z = \rho \cos\varphi.$$

The hemisphere is described by

$$0 \le \rho \le R, \qquad 0 \le \varphi \le \frac{\pi}{2}, \qquad 0 \le \theta \le 2\pi.$$

Hence,

$$M = \int_0^{2\pi} \int_0^{\pi/2} \int_0^R k\rho \cos\varphi \cdot \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta.$$

Thus,

$$M = k \int_0^{2\pi} d\theta \int_0^{\pi/2} \cos\varphi \sin\varphi \, d\varphi \int_0^R \rho^3 \, d\rho.$$

Now,

$$\int_0^R \rho^3 \, d\rho = \frac{R^4}{4},$$

and

$$\int_0^{\pi/2} \cos\varphi \sin\varphi \, d\varphi = \frac{1}{2}.$$

Therefore,

$$M = k(2\pi) \left(\frac{1}{2}\right) \left(\frac{R^4}{4}\right) = \frac{\pi k R^4}{4}.$$

Moment of Inertia

Definition 3.2.3. *The moments of inertia of a solid body with density* $\gamma(x, y, z)$ *are defined by*

$$I_x = \iiint_V (y^2 + z^2)\gamma(x, y, z)\, dV,$$

$$I_y = \iiint_V (x^2 + z^2)\gamma(x, y, z)\, dV,$$

$$I_z = \iiint_V (x^2 + y^2)\gamma(x, y, z)\, dV.$$

Example 3.2.8. Find the moment of inertia of the cylinder

$$x^2 + y^2 \leq R^2, \qquad -h \leq z \leq h,$$

with constant density λ, about the z-axis.

Solution Using cylindrical coordinates,

$$x^2 + y^2 = r^2.$$

Hence,

$$I_z = \iiint_V r^2 \lambda\, dV.$$

Thus,

$$I_z = \lambda \int_0^{2\pi} \int_0^R \int_{-h}^h r^2 \cdot r\, dz\, dr\, d\theta.$$

Therefore,

$$I_z = \lambda \int_0^{2\pi} \int_0^R 2hr^3\, dr\, d\theta.$$

Integrating,

$$I_z = 2h\lambda \int_0^{2\pi} \left[\frac{r^4}{4}\right]_0^R d\theta.$$

Hence,

$$I_z = \frac{h\lambda R^4}{2} \int_0^{2\pi} d\theta = \pi h \lambda R^4.$$

Center of Gravity

Definition 3.2.4. *Let a solid body occupy the region* V *with density* $\gamma(x, y, z)$*. The coordinates of the center of gravity of* V *are*

$$x_c = \frac{\iiint_V x\gamma(x, y, z)\, dV}{\iiint_V \gamma(x, y, z)\, dV},$$

$$y_c = \frac{\iiint_V y\gamma(x, y, z)\, dV}{\iiint_V \gamma(x, y, z)\, dV},$$

$$z_c = \frac{\iiint_V z\gamma(x, y, z)\, dV}{\iiint_V \gamma(x, y, z)\, dV}.$$

Example 3.2.9. Determine the center of gravity of the homogeneous hemisphere

$$x^2 + y^2 + z^2 \leq R^2, \qquad z \geq 0.$$

Solution By symmetry,

$$x_c = 0, \qquad y_c = 0.$$

We only compute z_c. Since the density is constant,

$$z_c = \frac{\iiint_V z\, dV}{\iiint_V dV}.$$

Using spherical coordinates $z = \rho\cos\varphi$. Hence,

$$z_c = \frac{\displaystyle\int_0^{2\pi}\int_0^{\pi/2}\int_0^R \rho\cos\varphi \cdot \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta}{\displaystyle\int_0^{2\pi}\int_0^{\pi/2}\int_0^R \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta}.$$

After integration, we obtain

$$z_c = \frac{3R}{8}.$$

Surface Area

Let a surface be represented by $z = f(x, y)$, over a region D.

Then the surface area is

$$S = \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dx\, dy.$$

Example 3.2.10. Find the surface area of the paraboloid

$$z = x^2 + y^2,$$

inside the cylinder

$$x^2 + y^2 \leq 1.$$

Solution We have

$$f(x, y) = x^2 + y^2.$$

Hence,

$$\frac{\partial f}{\partial x} = 2x, \qquad \frac{\partial f}{\partial y} = 2y.$$

Thus,

$$S = \iint_D \sqrt{1 + 4x^2 + 4y^2}\, dx\, dy.$$

Using polar coordinates,

$$x^2 + y^2 = r^2, \qquad dx\, dy = r\, dr\, d\theta.$$

Therefore,

$$S = \int_0^{2\pi} \int_0^1 \sqrt{1 + 4r^2}\, r\, dr\, d\theta.$$

Integrating,

$$S = \frac{\pi}{6}\left(5\sqrt{5} - 1\right).$$

Conclusion

In this work, we have presented a progressive study of integration theory, beginning with the fundamental notions of Riemann and Darboux integrals and extending toward multiple integration in higher-dimensional spaces. The concepts of partitions, refinement processes, upper and lower sums, and integrability criteria provided the rigorous foundation necessary for understanding the theory of integration.

We also studied functions of several variables and introduced the essential analytical tools related to limits, continuity, partial derivatives, and differentiability in Euclidean spaces. These notions constitute the theoretical basis for the study of double and triple integrals. The main part of this work was devoted to multiple integrals and their applications. We presented the definitions and properties of double and triple integrals together with Fubini's theorem, which allows the transformation of multidimensional integrals into iterated one-dimensional integrals. Furthermore, we studied coordinate transformations and Jacobian determinants in polar, cylindrical, and spherical coordinate systems, which simplify many computations involving symmetric regions.

Several applications were discussed throughout this thesis, including the computation of areas, volumes, masses, centers of gravity, moments of inertia, and surface areas. These applications demonstrate the importance of multiple integration not only in pure mathematics but also in physics, engineering, and applied sciences.

The study of double and triple integrals represents an essential step toward more advanced topics in mathematical analysis, vector calculus, differential geometry, and mathematical physics. Indeed, multiple integration plays a fundamental role in the formulation of many physical laws and mathematical models describing natural phenomena.

Finally, this work highlights the richness and significance of integration theory as one of the central tools of modern mathematical analysis.

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