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Dedication

In the name of Allah, the Most Gracious, the Most Merciful. Praise be to Allah, by whose grace good deeds are accomplished.

First and foremost, by the grace of Allah, and thanks to the prayers and support of those who loved and stood by me, I dedicate the fruit of this effort to my beloved parents, may Allah protect them and grant them long lives, who have always been my support and constant source of prayers.

To my dear family, my elder sister, my beloved brothers, my uncles from both sides, thank you for your support, encouragement, and for always standing by my side.

To my beloved grandmother, may Allah bless her with a long and healthy life, whose prayers accompanied me in every step of my journey.

To the pure soul of my grandfather, may Allah shower you with His mercy and grant you the highest place in Paradise. You were a source of prayers, support, and unforgettable memories.

To my friends, and to everyone who accompanied and supported me throughout my academic journey.

To my respected teachers, from primary school to university, may Allah reward you abundantly for the knowledge and guidance you have provided.

And to everyone who stood by me and supported me, whether I mentioned your name or not, I dedicate this humble work to you and pray that Allah makes it beneficial and blessed knowledge.

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Finally, I would like to extend my thanks to everyone who contributed, directly or indirectly, to the completion of this work, whether mentioned by name or not.

May Allah reward everyone who supported me and make this work beneficial and meaningful.

Abstract

In this work, we studied the Gamma function and its fundamental properties. We began by defining the Gamma function and its analytic continuation, then we discussed the main results related to it, especially its connection with the Beta function.

We also presented some applications concerning the computation of integrals and certain finite products, in order to highlight the importance of these special functions in mathematical analysis.

Keywords: Gamma function, Beta function, Integral.

Résumé

Dans ce mémoire, nous avons étudié la fonction Gamma ainsi que ses propriétés fondamentales. Nous avons commencé par définir la fonction Gamma et son prolongement analytique, puis nous avons abordé les principaux résultats qui lui sont liés, notamment sa relation avec la fonction Bêta.

Nous avons également présenté quelques applications concernant le calcul d'intégrales et de certains produits finis, afin de mettre en évidence l'importance de ces fonctions spéciales en analyse mathématique.

Mots-clés : Fonction Gamma, Fonction Bêta, intégrale.

ملخص

في هذه المذكرة قمنَا بدرَاسة دَالة ڤَامَا وخصَائِصحَا الاَسَاسية. بدأَنَا بتريف الدَالة ڤَامَ وتمديدهَا التحليلي، ثم تطرقنَا إِلَى أَهم النتَائِج المتلّقة بهَا، خَاصة ارتبَاطهَا بَالدَالة بيتَا.

كمَا قمنَا بتقديم بعض التطبيقَات المتعلقة بحسَاب التكَاملَات وبعض الجدَاءَات المتهية، وذلك لَابرَاز اهمية هذه الدوَال الخَاصة في التحليل الريَاضي.

الكلمَات المفتَاحية: دَالة ڤَامَا، دَالة بيتَا، اَلتكَاملَات.

Contents

Introduction

The Gamma function is one of the most fundamental special functions in mathematical analysis. Its origin lies in the attempt to extend the factorial function beyond the set of positive integers. While the factorial $n!$ is defined only for natural numbers, the development of calculus during the seventeenth and eighteenth centuries created the need for an analytic extension of this concept to real and later complex numbers.

The first major contribution to this problem is attributed to Leonhard Euler. In 1738, Euler introduced analytic expressions that generalized the factorial function and established an integral representation that is now regarded as the classical definition of the Gamma function:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt, \quad x > 0.$$

This definition satisfies the fundamental functional equation:

$$\Gamma(x+1) = x\Gamma(x),$$

which ensures that $\Gamma(n+1) = n!$ for every n natural number . Euler's work provided the first rigorous analytic extension of the factorial function to the positive real numbers.

In 1811, Adrien-Marie Legendre introduced the notation $\Gamma(x)$, which remains in use today. Later, Carl Friedrich Gauss extended the function to the complex plane through a limit representation, transforming it into a meromorphic function with simple poles at the non-positive integers.

Further development was achieved by Karl Weierstrass, who derived an infinite product representation of the Gamma function. This representation played a crucial role in the development of complex analysis and contributed to the Weierstrass Factorization Theorem.

A decisive characterization was given in 1922 by Harald Bohr and Johannes Mollerup. The Bohr–Mollerup Theorem states that the Gamma function is the unique logarithmically convex function on $(0, \infty)$ satisfying:

$$f(x+1) = xf(x), \quad \text{and} \quad f(1) = 1.$$

This result confirms the uniqueness of the Gamma function as the natural extension of the factorial function. This thesis is organized into three chapters.

In the first chapter, we present the fundamental properties of the Gamma function, including its definition, analytic continuation, recurrence relation, reflection formula, and infinite product representations.

In the second chapter, we study the Beta function and its main properties, as well as the fundamental relation between the Beta and Gamma functions and some important applications.

In the third chapter, we discuss several applications of the Gamma function in the evaluation of improper integrals and finite products.

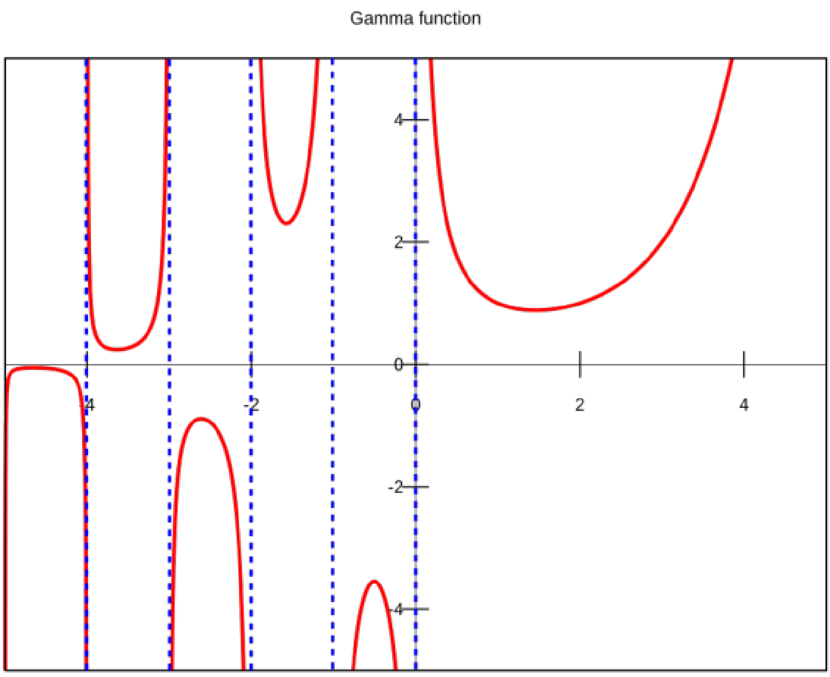
Chapter 1

Fundamental Properties of the Gamma Function

In this chapter, we introduce the Gamma function and study its fundamental properties. We begin with its integral definition and establish its convergence and analyticity. Then, we present several important properties such as the recurrence relation, the reflection formula, and the infinite product representation of Weierstrass. These results play a central role in the theory of special functions and complex analysis.

Definition 1. *Let* $z \in \mathbb{C}$ *with* $\mathrm{Re}(z) > 0$. *The Gamma function is defined by*

$$\Gamma(z) := \int_0^\infty e^{-t}\, t^{z-1}\, dt,$$



Lemma 1. *The integral*

$$\int_0^\infty e^{-t}\, t^{z-1}\, dt,$$

is convergent for all $z \in \mathbb{C}$ *such that* $\mathrm{Re}(z) > 0$.

Proof. Let $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$. We study the convergence of the integral

$$\int_0^\infty e^{-t}\, t^{z-1}\, dt$$

by splitting it into two parts as follows:

$$\int_0^\infty e^{-t} t^{z-1} dt = \int_0^1 e^{-t} t^{z-1} dt + \int_1^\infty e^{-t} t^{z-1} dt,$$

1. Convergence on $(0, 1)$:

For $0 < t \leq 1$, we have $e^{-t} \leq 1$, hence

$$|e^{-t} t^{z-1}| \leq t^{\mathrm{Re}(z)-1}.$$

Thus,

$$\int_0^1 |e^{-t} t^{z-1}| dt \leq \int_0^1 t^{\mathrm{Re}(z)-1} dt.$$

The latter integral converges if and only if $\mathrm{Re}(z) > 0$. Hence,

$$\int_0^1 e^{-t} t^{z-1} dt$$

is convergent.

2. Convergence on $(1, \infty)$:

For $t \geq 1$, the exponential term e^{-t} decays rapidly. Moreover, there exists a constant $C > 0$ such that

$$t^{\mathrm{Re}(z)-1} \leq C e^{t/2} \quad \text{for sufficiently large } t.$$

Thus,

$$|e^{-t} t^{z-1}| \leq C e^{-t/2}.$$

Since

$$\int_1^\infty e^{-t/2} dt < \infty,$$

it follows by comparison that

$$\int_1^\infty e^{-t} t^{z-1} dt$$

is convergent.

Conclusion:

Both integrals converge, hence

$$\int_0^\infty e^{-t} t^{z-1} dt$$

is convergent for all z such that $\mathrm{Re}(z) > 0$. □

Theorem 1. *The Gamma function* $\Gamma(z)$ *defines an analytic function on the domain*

$$\{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}.$$

In order to prove this theorem we need the following lemma.

Lemma 2. *Let* $0 < \delta < R$ *and define the strip*

$$U_{\delta,R} := \{z \in \mathbb{C} : \delta < \mathrm{Re}(z) < R\}.$$

Then the function

$$f(t, z) := e^{-t} t^{z-1}$$

satisfies the following properties:

(i) $F(t, z)$ *is continuous on* $\mathbb{R}_{>0} \times U_{\delta,R}$*;*

(ii) For each fixed $t > 0$*, the function* $z \mapsto f(t, z)$ *is analytic on* $U_{\delta,R}$*;*

(iii) There exists an integrable function $M(t)$ *such that*

$$|F(t, z)| \leq M(t) \quad \textit{for all } z \in U_{\delta,R}.$$

Proof. (i) The function e^{-t} is continuous for $t > 0$, and $t^{z-1} = e^{(z-1)\log t}$ is continuous in both variables t and z. Hence $f(t, z)$ is continuous on $\mathbb{R}_{>0} \times U_{\delta,R}$.

(ii) For fixed $t > 0$, the function

$$z \mapsto t^{z-1} = e^{(z-1)\log t},$$

is an exponential of a linear function in z, hence it is analytic. Since e^{-t} is constant with respect to z, it follows that $f(t, z)$ is analytic in z.

(iii) Let $z \in U_{\delta,R}$. Then:

$$|t^{z-1}| = t^{\mathrm{Re}(z)-1}.$$

Hence

$$|f(t, z)| = e^{-t} t^{\mathrm{Re}(z)-1}.$$

We estimate this expression in two regions:

For $0 < t \leq 1$:

$$|f(t,z)| \leq t^{\delta-1}.$$

For $t > 1$:

$$|f(t,z)| \leq e^{-t}t^{R-1}.$$

Since t^{R-1} grows at most polynomially while e^{-t} decays exponentially, there exists a constant $C > 0$ such that

$$e^{-t}t^{R-1} \leq Ce^{-t/2}.$$

Thus we define

$$M(t) := \begin{cases} t^{\delta-1}, & 0 < t \leq 1, \\ Ce^{-t/2}, & t > 1. \end{cases}$$

Finally,

$$\int_0^\infty M(t)\,dt = \int_0^1 t^{\delta-1}dt + C\int_1^\infty e^{-t/2}dt = \frac{1}{\delta} + 2C < \infty.$$

Hence $M(t)$ is integrable. □

Proof of Theorem 1. From the lemma, all the hypotheses of the theorem on differentiation under the integral sign (or analytic parameter integrals) are satisfied.

Therefore, the function

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt,$$

is analytic on $U_{\delta,R}$.

Since δ and R are arbitrary with $0 < \delta < R$, it follows that $\Gamma(z)$ is analytic on the entire domain

$$\{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}.$$

. □

1.0.1 Infinite Product Representation of the Gamma Function (Weierstrass Form)

One of the fundamental representations of the Gamma function is given by the Weierstrass infinite product. This representation is essential in the study of the analytic properties of the Gamma function and plays a key role in deriving several important identities.

The Gamma function can be expressed as:

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}, \quad z \in \mathbb{C}, \tag{1.1}$$

where γ denotes the Euler–Mascheroni constant defined by:

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n \right). \tag{1.2}$$

This representation is particularly useful for establishing important properties of the Gamma function, such as the reflection formula and the location of its zeros.

Theorem 2 (Meromorphic Continuation and Fundamental Properties)**.** *There exists a unique meromorphic function* Γ *on* $\mathbb{C}$ *satisfying the following properties:*

(i) $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$ *for* $\mathrm{Re}(z) > 0$.

(ii) Γ *is analytic on* $\mathbb{C} \setminus \{0, -1, -2, \dots\}$.

(iii) Γ *has a simple pole at each* $z = -n$ *(*$n = 0, 1, 2, \dots$*) with residue*

$$\mathrm{Res}(\Gamma, -n) = \frac{(-1)^n}{n!}.$$

(iv) $\Gamma(z+1) = z\Gamma(z)$ *for all* $z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$.

(v) $\Gamma(n) = (n-1)!$ *for all* $n \in \mathbb{Z}_{>0}$.

Lemma 3 (Recurrence Relation)**.** *For* $\mathrm{Re}(z) > 0$*, we have*

$$\Gamma(z+1) = z\Gamma(z).$$

Proof. Let $\mathrm{Re}(z) > 0$. By definition,

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

We apply integration by parts. Let

$$u = t^{z-1}, \quad dv = e^{-t} dt.$$

Then

$$du = (z-1)t^{z-2} dt, \quad v = -e^{-t}.$$

Thus,

$$\Gamma(z) = \left[-e^{-t} t^{z-1} \right]_0^\infty + \int_0^\infty e^{-t} (z-1) t^{z-2} dt.$$

The boundary term vanishes:

- as $t \to \infty$, $e^{-t} t^{z-1} \to 0$.

- as $t \to 0$, $t^{z-1} \to 0$ since $\text{Re}(z) > 0$.

Hence,

$$\Gamma(z) = (z-1) \int_0^\infty e^{-t} t^{z-2} dt = (z-1)\Gamma(z-1).$$

Replacing z by $z+1$, we obtain

$$\Gamma(z+1) = z\Gamma(z).$$

□

Lemma 4 (Extension Formula)**.** *For* $\text{Re}(z) > 0$ *and* $n \in \mathbb{Z}_{>0}$.

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)\cdots(z+n-1)}.$$

Proof. We prove the result by induction using the recurrence relation.

For $n = 1$, we have

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}.$$

Assume the formula holds for some $n \geq 1$, i.e.,

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)\cdots(z+n-1)}.$$

Using the recurrence relation,

$$\Gamma(z+n) = \frac{\Gamma(z+n+1)}{z+n}.$$

Substituting into the induction hypothesis, we obtain

$$\Gamma(z) = \frac{\Gamma(z+n+1)}{z(z+1)\cdots(z+n)}.$$

Thus the formula holds for $n+1$, completing the induction. □

Proof of Theorem 2. We construct the extension of Γ to $\mathbb{C} \setminus \{0, -1, -2, \dots\}$ using the formula

$$\Gamma(z) := \frac{\Gamma(z+n)}{z(z+1)\cdots(z+n-1)},$$

where n is chosen such that $\text{Re}(z+n) > 0$.

Well-definedness: One verifies that the definition does not depend on the choice of n by repeatedly applying the extension formula.

Analyticity: Since $\Gamma(z+n)$ is analytic for $\text{Re}(z+n) > 0$ and the denominator is nonzero away from $z = 0, -1, -2, \dots$, it follows that Γ is analytic on $\mathbb{C} \setminus \{0, -1, -2, \dots\}$.

Poles and residues: From the denominator $z(z+1)\cdots(z+n-1)$, it follows that Γ has simple poles at $z=-n$. A direct limit computation gives

$$\mathrm{Res}(\Gamma, -n) = \frac{(-1)^n}{n!}.$$

Functional equation: The identity $\Gamma(z+1) = z\Gamma(z)$ holds on $\mathrm{Re}(z) > 0$ and extends to the whole domain by analyticity.

Values at integers: Since

$$\Gamma(1) = \int_0^\infty e^{-t} dt = 1,$$

we obtain recursively

$$\Gamma(n) = (n-1)! \quad \text{for all } n \in \mathbb{Z}_{>0}.$$

This completes the proof. $\square$

Theorem 3 (Reflection Formula)**.** *For all $z \in \mathbb{C} \setminus \mathbb{Z}$, we have*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$

Lemma 5. *For all $z \in (\mathbb{C} \setminus \mathbb{Z}) \cup \{0\}$, the identity*

$$\Gamma(1+z)\Gamma(1-z) = \frac{\pi z}{\sin(\pi z)}$$

holds.

Proof. We first note that both sides define analytic functions on the domain

$$A := (\mathbb{C} \setminus \mathbb{Z}) \cup \{0\}.$$

Indeed, by the meromorphic properties of the Gamma function, the function

$$z \mapsto \Gamma(1+z)\Gamma(1-z).$$

is analytic on A.

On the other hand, the function

$$z \mapsto \frac{\pi z}{\sin(\pi z),}$$

is analytic on A, since $\sin(\pi z)$ has simple zeros at integers and the factor z cancels the zero at $z=0$. Moreover,

$$\lim_{z\to 0} \frac{\pi z}{\sin(\pi z)} = 1,$$

so the function extends analytically at $z = 0$.

By the identity theorem for analytic functions, it suffices to verify the identity on a subset of A having a limit point in A.

Let

$$S := \left\{ \frac{1}{2n} : n \in \mathbb{Z}_{>0} \right\}.$$

Then S has the limit point $0 \in A$. Hence, it is enough to prove that for all $n \in \mathbb{Z}_{>0}$,

$$\Gamma\left(1 + \frac{1}{2n}\right) \Gamma\left(1 - \frac{1}{2n}\right) = \frac{\pi/(2n)}{\sin(\pi/(2n))}.$$

Using the definition of the Gamma function, we write

$$\Gamma\left(1 + \frac{1}{2n}\right) = \int_0^\infty e^{-s} s^{\frac{1}{2n}} ds, \quad \Gamma\left(1 - \frac{1}{2n}\right) = \int_0^\infty e^{-t} t^{-\frac{1}{2n}} dt.$$

Thus,

$$\Gamma\left(1 + \frac{1}{2n}\right) \Gamma\left(1 - \frac{1}{2n}\right) = \int_0^\infty \int_0^\infty e^{-s-t} \left(\frac{s}{t}\right)^{\frac{1}{2n}} ds\, dt.$$

We perform the change of variables:

$$u = s + t, \quad v = \frac{s}{t}.$$

Then

$$s = \frac{uv}{v+1}, \quad t = \frac{u}{v+1},$$

and the Jacobian determinant is

$$\left| \frac{\partial(s,t)}{\partial(u,v)} \right| = \frac{u}{(v+1)^2}.$$

Hence,

$$\Gamma\left(1 + \frac{1}{2n}\right) \Gamma\left(1 - \frac{1}{2n}\right) = \int_0^\infty \int_0^\infty e^{-u} v^{\frac{1}{2n}} \frac{u}{(v+1)^2} du\, dv.$$

This separates into a product:

$$= \left(\int_0^\infty e^{-u} u\, du \right) \left(\int_0^\infty \frac{v^{\frac{1}{2n}}}{(v+1)^2} dv \right).$$

The first integral equals

$$\int_0^\infty e^{-u} u\, du = 1.$$

The second integral can be computed explicitly and yields

$$\int_0^\infty \frac{v^{\frac{1}{2n}}}{(v+1)^2} dv = \frac{\pi/(2n)}{\sin(\pi/(2n))}.$$

Thus,

$$\Gamma\left(1+\frac{1}{2n}\right)\Gamma\left(1-\frac{1}{2n}\right) = \frac{\pi/(2n)}{\sin(\pi/(2n))}.$$

This proves the identity on S, and hence on A. $\square$

Proof of Theorem 3. Using the functional equation of the Gamma function,

$$\Gamma(1+z) = z\Gamma(z),$$

we rewrite the identity in the lemma as

$$z\Gamma(z)\Gamma(1-z) = \frac{\pi z}{\sin(\pi z)}.$$

Cancelling $z \neq 0$, we obtain

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$

This holds for all $z \in \mathbb{C} \setminus \mathbb{Z}$. $\square$

Corollary 1. *We have*

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Proof. We substitute $z = \frac{1}{2}$ into the reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$

Thus,

$$\Gamma\left(\frac{1}{2}\right)\Gamma\left(1-\frac{1}{2}\right) = \frac{\pi}{\sin\left(\frac{\pi}{2}\right)}.$$

Since

$$\sin\left(\frac{\pi}{2}\right) = 1,$$

we obtain

$$\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \pi.$$

Hence,

$$\Gamma\left(\frac{1}{2}\right)^2 = \pi.$$

Since $\Gamma(x) > 0$ for all $x > 0$, it follows that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

$\square$

Corollary 2. *The Gamma function satisfies the following properties:*

(i) $\Gamma(z) \neq 0$ *for all* $z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$;

(ii) the function $\dfrac{1}{\Gamma(z)}$ *is entire (analytic on* $\mathbb{C}$*), and it has simple zeros at* $z = 0, -1, -2, \dots$.

Lemma 6. *For all* $z \in \mathbb{C} \setminus \mathbb{Z}$*, we have*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \neq 0.$$

Proof. From the reflection formula,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$

For $z \notin \mathbb{Z}$, we have $\sin(\pi z) \neq 0$, hence the right-hand side is well-defined and nonzero. Therefore,

$$\Gamma(z)\Gamma(1-z) \neq 0.$$

It follows that neither $\Gamma(z)$ nor $\Gamma(1-z)$ can be zero for $z \notin \mathbb{Z}$. □

Proof of Corollary 2. **(i) Non-vanishing of Γ:**

For $z \in \mathbb{C} \setminus \mathbb{Z}$, the lemma shows that $\Gamma(z) \neq 0$.

For positive integers $n \in \mathbb{Z}_{>0}$, we have

$$\Gamma(n) = (n-1)! \neq 0.$$

Thus,

$$\Gamma(z) \neq 0 \quad \text{for all } z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}.$$

(ii) Analyticity of $1/\Gamma$:

Since $\Gamma(z)$ is analytic and nonzero on $\mathbb{C} \setminus \{0, -1, -2, \dots\}$, it follows that

$$\frac{1}{\Gamma(z)},$$

is analytic on this domain.

At the points $z = 0, -1, -2, \dots$, the Gamma function has simple poles. Therefore, its reciprocal $\frac{1}{\Gamma(z)}$ has simple zeros at these points and extends analytically there.

Hence, $\frac{1}{\Gamma(z)}$ is an entire function with simple zeros at

$$z = 0, -1, -2, \dots$$

□

Theorem 4 (Alternative Representations of the Gamma Function)**.** *For all $z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$, we have*

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)\cdots(z+n)} = e^{-\gamma z} z^{-1} \prod_{n=1}^{\infty} \frac{e^{z/n}}{1+\frac{z}{n}},$$

where γ denotes the Euler–Mascheroni constant.

Lemma 7. *The infinite product*

$$h(z) := \prod_{n=1}^{\infty} \left(\left(1 + \frac{z}{n}\right) e^{-z/n} \right)$$

converges and defines an analytic function on $\mathbb{C}$, and $h(z) \neq 0$ for all $z \notin \{0, -1, -2, \dots\}$.

Proof. We show that the infinite product converges uniformly on compact subsets.

Let $|z| < R$. Using the Taylor expansion of the exponential function,

$$e^{-z/n} = 1 - \frac{z}{n} + \frac{z^2}{2!n^2} - \frac{z^3}{3!n^3} + \cdots,$$

we obtain

$$\left(1 + \frac{z}{n}\right) e^{-z/n} = 1 - \frac{z^2}{2n^2} + \left(\frac{1}{2!} - \frac{1}{3!}\right) \frac{z^3}{n^3} + \cdots.$$

Hence,

$$\left| \left(1 + \frac{z}{n}\right) e^{-z/n} - 1 \right| \leq \sum_{k=2}^{\infty} \left(\frac{1}{(k-1)!} - \frac{1}{k!} \right) \left(\frac{|z|}{n} \right)^k.$$

This yields the estimate

$$\left| \left(1 + \frac{z}{n}\right) e^{-z/n} - 1 \right| \leq \frac{R^2 e^R}{n^2}.$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, it follows that the infinite product converges absolutely.

Therefore, $h(z)$ defines an analytic function on $\mathbb{C}$, and it is nonzero except at $z = 0, -1, -2, \dots$ where one of the factors vanishes. □

Proof of Theorem 4. We first show the equivalence of the two expressions.

Using the definition of the Euler–Mascheroni constant

$$\gamma = \lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{1}{n} - \log N \right),$$

we write

$$e^{-\gamma z} = \lim_{N\to\infty} e^{z(\log N - \sum_{n=1}^{N} \frac{1}{n})}.$$

Thus,

$$e^{-\gamma z} z^{-1} \prod_{n=1}^{N} \frac{e^{z/n}}{1+\frac{z}{n}} = \frac{e^{z \log N}}{z} \prod_{n=1}^{N} \frac{1}{1+\frac{z}{n}} = \frac{N^z}{z} \prod_{n=1}^{N} \frac{n}{n+z}.$$

Hence,

$$= \frac{N^z N!}{z(z+1)\cdots(z+N)}.$$

Taking the limit as $N \to \infty$, we obtain

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)\cdots(z+n)}.$$

Finally, one proves that this limit coincides with the integral definition of $\Gamma(z)$ for $\mathrm{Re}(z) > 0$, and by analytic continuation, the identity holds on $\mathbb{C} \setminus \{0, -1, -2, \dots\}$. □

Corollary 3 (Infinite Product Representation of $\sin(\pi z)$)**.** *For all $z \in \mathbb{C}$, we have*

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

Lemma 8. *For all $z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$, the Gamma function admits the infinite product representation*

$$\Gamma(z) = e^{-\gamma z} z^{-1} \prod_{n=1}^{\infty} \frac{e^{z/n}}{1+\frac{z}{n}}.$$

Proof. This follows from the previous theorem on the alternative representations of the Gamma function. □

Proof of Corollary 3. Using the identity

$$\sin(\pi z) = \frac{\pi}{\Gamma(z)\Gamma(1-z)},$$

and substituting the infinite product representations of $\Gamma(z)$ and $\Gamma(1-z)$, we obtain

$$\Gamma(z)\Gamma(1-z) = \left(e^{-\gamma z} z^{-1} \prod_{n=1}^{\infty} \frac{e^{z/n}}{1+\frac{z}{n}}\right) \left(e^{\gamma z} (-z)^{-1} \prod_{n=1}^{\infty} \frac{e^{-z/n}}{1-\frac{z}{n}}\right).$$

Simplifying, we get

$$\Gamma(z)\Gamma(1-z) = \frac{-1}{z^2} \prod_{n=1}^{\infty} \frac{1}{\left(1-\frac{z}{n}\right)\left(1+\frac{z}{n}\right)}.$$

Thus,

$$\Gamma(z)\Gamma(1-z) = \frac{1}{z} \prod_{n=1}^{\infty} \frac{1}{1-\frac{z^2}{n^2}}.$$

Taking reciprocals and multiplying by π, we obtain

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

This completes the proof. $\square$

Corollary 4 (Duplication Formula)**.** *For all* $z \in \mathbb{C} \setminus \{0, -\frac{1}{2}, -1, -\frac{3}{2}, -2, \dots\}$*, we have*

$$\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\big(z + \tfrac{1}{2}\big).$$

Lemma 9. *Define the function*

$$F(z) := \frac{2^{2z}\Gamma(z)\Gamma\big(z + \frac{1}{2}\big)}{\Gamma(2z)}.$$

Then $F(z)$ *is constant on its domain of definition.*

Proof. Let z belong to the domain

$$A := \mathbb{C} \setminus \{0, -\tfrac{1}{2}, -1, -\tfrac{3}{2}, -2, \dots\}.$$

Using the infinite product / limit representations of the Gamma function,

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)\cdots(z+n)},$$

we write similar expressions for $\Gamma(z + \frac{1}{2})$ and $\Gamma(2z)$.

After rewriting all three terms with compatible denominators (by passing to even/odd indexing), we obtain

$$F(z) = \lim_{n\to\infty} \left\{ \frac{2^{2n+2}(n!)^2\sqrt{n}}{(2n+1)!} \right\}.$$

The key point is that the dependence on z cancels out in the limit. Indeed,

$$\lim_{n\to\infty} \frac{2^{2z}n^{2z}}{(2n+1)^{2z}} = 1.$$

Hence $F(z)$ does not depend on z, and is therefore constant on A. $\square$

Proof of Corollary 4. Since $F(z)$ is constant, we evaluate it at a convenient value. Let $z = \frac{1}{2}$. Then

$$F\big(\tfrac{1}{2}\big) = \frac{2\,\Gamma\big(\frac{1}{2}\big)\,\Gamma(1)}{\Gamma(1)}.$$

Using $\Gamma(1) = 1$ and $\Gamma\big(\frac{1}{2}\big) = \sqrt{\pi}$, we obtain

$$F\big(\tfrac{1}{2}\big) = 2\sqrt{\pi}.$$

Thus,

$$F(z) = 2\sqrt{\pi}.$$

Rewriting the definition of $F(z)$, we get

$$\frac{2^{2z}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)}{\Gamma(2z)} = 2\sqrt{\pi}.$$

Dividing both sides by 2, we obtain

$$\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\left(z+\tfrac{1}{2}\right).$$

This completes the proof. □

Chapter 2

The Beta Function

The Beta function is one of the classical special functions that arises naturally in various areas of mathematics, particularly in analysis, probability theory, and mathematical physics. It plays a fundamental role in evaluating definite integrals and appears in many applications involving continuous distributions and integral transforms. Moreover, it is closely related to the Gamma function, forming an important link between different branches of mathematical analysis.

2.1 Beta Function

The Beta function, also known as Euler's integral of the first kind, is defined by the following definite integral:

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx, \quad \text{where } m>0,\ n>0. \tag{2.1}$$

It is important to note that the Beta function depends on the parameters m and n, while the variable x is only a dummy variable of integration. Thus, $\beta(m,n)$ represents a function of two variables and can be interpreted geometrically as an area under a curve on the interval $[0,1]$.

Using the definition of the Beta function,

$$\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx,$$

we compute some simple examples.

Example 1: Compute $\beta(1,1)$.

We have

$$\beta(1,1) = \int_0^1 x^{1-1}(1-x)^{1-1}\,dx.$$

Since $x^0 = 1$ and $(1-x)^0 = 1$, the integral becomes

$$\beta(1,1) = \int_0^1 1\,dx.$$

Therefore,

$$\beta(1,1) = [x]_0^1 = 1.$$

Thus,

$$\boxed{\beta(1,1) = 1}.$$

2.1.1 Fundamental Relation Between Beta and Gamma Functions

Theorem. Let $m > 0$ and $n > 0$. Then the Beta and Gamma functions satisfy:

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

Proof.
We start from the definition of the Gamma function:

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt, \quad s > 0.$$

Consider the product:

$$\Gamma(m)\Gamma(n) = \left(\int_0^\infty e^{-x} x^{m-1} dx\right)\left(\int_0^\infty e^{-y} y^{n-1} dy\right).$$

This can be written as a double integral:

$$\Gamma(m)\Gamma(n) = \int_0^\infty \int_0^\infty e^{-(x+y)} x^{m-1} y^{n-1}\,dx\,dy.$$

We perform the change of variables:

$$x = ru, \quad y = r(1-u), \quad \text{with } r > 0,\ 0 < u < 1.$$

The Jacobian of this transformation is:

$$\left|\frac{\partial(x,y)}{\partial(r,u)}\right| = r.$$

Thus:

$$\Gamma(m)\Gamma(n) = \int_0^1 \int_0^\infty e^{-r} (ru)^{m-1} (r(1-u))^{n-1} r\,dr\,du.$$

Simplifying:

$$= \int_0^1 u^{m-1}(1-u)^{n-1}du \cdot \int_0^\infty e^{-r}r^{m+n-1}dr.$$

Recognizing the integrals:

$$= \beta(m,n)\,\Gamma(m+n).$$

Hence:

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

Example

Let us compute:

$$\beta(2,3)$$

Using the fundamental relation:

$$\beta(2,3) = \frac{\Gamma(2)\Gamma(3)}{\Gamma(2+3)}$$

First, we calculate each Gamma function.

We know that for any positive integer n:

$$\Gamma(n) = (n-1)!$$

Therefore:

$$\Gamma(2) = 1! = 1$$

$$\Gamma(3) = 2! = 2$$

$$\Gamma(5) = 4! = 24$$

Substituting these values into the formula:

$$\beta(2,3) = \frac{1 \times 2}{24}$$

$$\beta(2,3) = \frac{2}{24}$$

Finally:

$$\boxed{\beta(2,3) = \frac{1}{12}}$$

■

2.1.2 Recurrence Relation of the Beta Function

Theorem. Let $m > 0$ and $n > 0$. The Beta function satisfies the recurrence relation:

$$\beta(m, n) = \beta(m, n + 1) + \beta(m + 1, n).$$

Proof.

We start from the definition of the Beta function:

$$\beta(m, n) = \int_0^1 x^{m-1}(1 - x)^{n-1}dx.$$

Using the identity:

$$1 = x + (1 - x),$$

we write:

$$\beta(m, n) = \int_0^1 x^{m-1}(1 - x)^{n-1}(x + (1 - x))dx.$$

Expanding the integrand, we obtain:

$$= \int_0^1 x^m(1 - x)^{n-1}dx + \int_0^1 x^{m-1}(1 - x)^n dx.$$

Recognizing the Beta function in each term:

$$= \beta(m + 1, n) + \beta(m, n + 1).$$

Thus:

$$\beta(m, n) = \beta(m, n + 1) + \beta(m + 1, n).$$

Example

Let us take:

$$m = 2, \qquad n = 1$$

Then the recurrence relation becomes:

$$\beta(2, 1) = \beta(2, 2) + \beta(3, 1)$$

We calculate each term using:

$$\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m + n)}$$

First term:

$$\beta(2,1) = \frac{\Gamma(2)\Gamma(1)}{\Gamma(3)}$$

Since:

$$\Gamma(2) = 1!, \qquad \Gamma(1) = 0!, \qquad \Gamma(3) = 2!$$

we obtain:

$$\beta(2,1) = \frac{1 \times 1}{2} = \frac{1}{2}$$

Second term:

$$\beta(2,2) = \frac{\Gamma(2)\Gamma(2)}{\Gamma(4)}$$

Thus:

$$\beta(2,2) = \frac{1 \times 1}{6} = \frac{1}{6}$$

Third term:

$$\beta(3,1) = \frac{\Gamma(3)\Gamma(1)}{\Gamma(4)}$$

Therefore:

$$\beta(3,1) = \frac{2 \times 1}{6} = \frac{1}{3}$$

Now we verify the recurrence relation:

$$\beta(2,2) + \beta(3,1) = \frac{1}{6} + \frac{1}{3}$$

$$= \frac{1}{6} + \frac{2}{6}$$

$$= \frac{3}{6} = \frac{1}{2}$$

Hence:

$$\beta(2,1) = \beta(2,2) + \beta(3,1)$$

which verifies the recurrence relation.

■

2.1.3 Relations Between Shifted Beta Functions

Theorem. Let $m > 0$ and $n > 0$. The Beta function satisfies:

$$\frac{\beta(m+1,n)}{m} = \frac{\beta(m,n+1)}{n} = \frac{\beta(m,n)}{m+n}.$$

Proof.
We use the fundamental relation between the Beta and Gamma functions:

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

First, compute:

$$\frac{\beta(m+1,n)}{m}.$$

We have:

$$\beta(m+1,n) = \frac{\Gamma(m+1)\Gamma(n)}{\Gamma(m+n+1)}.$$

Using the property $\Gamma(m+1) = m\Gamma(m)$, we obtain:

$$\beta(m+1,n) = \frac{m\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)}.$$

Thus:

$$\frac{\beta(m+1,n)}{m} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)}.$$

Now, using $\Gamma(m+n+1) = (m+n)\Gamma(m+n)$, we get:

$$= \frac{\Gamma(m)\Gamma(n)}{(m+n)\Gamma(m+n)}.$$

Hence:

$$\frac{\beta(m+1,n)}{m} = \frac{\beta(m,n)}{m+n}.$$

Similarly, compute:

$$\frac{\beta(m,n+1)}{n}.$$

Using $\Gamma(n+1) = n\Gamma(n)$, we obtain:

$$\frac{\beta(m,n+1)}{n} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)} = \frac{\beta(m,n)}{m+n}.$$

Therefore:

$$\frac{\beta(m+1,n)}{m} = \frac{\beta(m,n+1)}{n} = \frac{\beta(m,n)}{m+n}.$$

Example

Let us take:

$$m = 2, \qquad n = 1$$

Then the relation becomes:

$$\frac{\beta(3,1)}{2} = \frac{\beta(2,2)}{1} = \frac{\beta(2,1)}{3}$$

We calculate each Beta function using:

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

First, compute:

$$\beta(3,1) = \frac{\Gamma(3)\Gamma(1)}{\Gamma(4)}$$

Since:

$$\Gamma(3) = 2!, \qquad \Gamma(1) = 0!, \qquad \Gamma(4) = 3!$$

we obtain:

$$\beta(3,1) = \frac{2 \times 1}{6} = \frac{1}{3}$$

Thus:

$$\frac{\beta(3,1)}{2} = \frac{1}{3 \times 2} = \frac{1}{6}$$

Next, compute:

$$\beta(2,2) = \frac{\Gamma(2)\Gamma(2)}{\Gamma(4)}$$

Hence:

$$\beta(2,2) = \frac{1 \times 1}{6} = \frac{1}{6}$$

Therefore:

$$\frac{\beta(2,2)}{1} = \frac{1}{6}$$

Finally, compute:

$$\beta(2,1) = \frac{\Gamma(2)\Gamma(1)}{\Gamma(3)}$$

Thus:

$$\beta(2,1) = \frac{1 \times 1}{2} = \frac{1}{2}$$

Then:

$$\frac{\beta(2,1)}{3} = \frac{1}{2 \times 3} = \frac{1}{6}$$

Therefore:

$$\frac{\beta(3,1)}{2} = \frac{\beta(2,2)}{1} = \frac{\beta(2,1)}{3} = \frac{1}{6}$$

■

2.1.4 Applications of the Beta Function

Theorem. Let $p > -1$ and $q > -1$. Then:

$$\int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)}.$$

Proof.
We start from the integral representation of the Beta function:

$$\beta(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1} \theta \cos^{2n-1} \theta \, d\theta, \quad m > 0,\ n > 0.$$

On the other hand, we have the relation:

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

Equating the two expressions, we obtain:

$$2\int_0^{\frac{\pi}{2}} \sin^{2m-1} \theta \cos^{2n-1} \theta \, d\theta = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

Now, we introduce the change of variables:

$$p = 2m - 1, \quad q = 2n - 1.$$

Thus:

$$m = \frac{p+1}{2}, \quad n = \frac{q+1}{2}.$$

Substituting into the integral, we get:

$$\int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta \, d\theta = \frac{1}{2} \cdot \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

Replacing m and n by their expressions:

$$= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}.$$

Hence:

$$\int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)}.$$

Example: Evaluate the integral

$$I = \int_0^{\frac{\pi}{2}} \sin^3 \theta \cos^2 \theta \, d\theta$$

Using the Beta function property:

$$\int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

we take:

$$p = 3, \qquad q = 2$$

Then:

$$I = \frac{\Gamma\left(\frac{3+1}{2}\right)\Gamma\left(\frac{2+1}{2}\right)}{2\Gamma\left(\frac{3+2+2}{2}\right)}$$

Simplifying:

$$I = \frac{\Gamma(2)\Gamma\left(\frac{3}{2}\right)}{2\Gamma\left(\frac{7}{2}\right)}$$

Using the known Gamma function values:

$$\Gamma(2) = 1!, \qquad \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}, \qquad \Gamma\left(\frac{7}{2}\right) = \frac{15\sqrt{\pi}}{8}$$

Substituting:

$$I = \frac{1 \times \frac{\sqrt{\pi}}{2}}{2 \times \frac{15\sqrt{\pi}}{8}}$$

After simplification:

$$I = \frac{2}{15}$$

Therefore,

$$\boxed{\int_0^{\frac{\pi}{2}} \sin^3\theta \cos^2\theta \, d\theta = \frac{2}{15}}$$

■

2.1.5 Euler Reflection Formula

Theorem. Let $z \in \mathbb{C} \setminus \mathbb{Z}$. Then the Gamma function satisfies:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$

Proof.
We start from the relation between the Beta and Gamma functions:

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Setting $x = z$ and $y = 1 - z$, we obtain:

$$\beta(z, 1-z) = \frac{\Gamma(z)\Gamma(1-z)}{\Gamma(1)}.$$

Since $\Gamma(1) = 1$, it follows that:

$$\beta(z, 1-z) = \Gamma(z)\Gamma(1-z).$$

On the other hand, the Beta function admits the integral representation:

$$\beta(z, 1-z) = \int_0^\infty \frac{t^{z-1}}{1+t} \, dt, \quad 0 < \Re(z) < 1.$$

It is a known result that:

$$\int_0^\infty \frac{t^{z-1}}{1+t} \, dt = \frac{\pi}{\sin(\pi z)}, \quad 0 < \Re(z) < 1.$$

Thus:

$$\beta(z, 1-z) = \frac{\pi}{\sin(\pi z)}.$$

Combining the two expressions of $\beta(z, 1-z)$, we obtain:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$

Finally, by analytic continuation, the identity holds for all $z \in \mathbb{C} \setminus \mathbb{Z}$.

Example: Use Euler's Reflection Formula to compute

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right).$$

Euler's Reflection Formula states that:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$

Taking:

$$z = \frac{1}{4}$$

we obtain:

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right) = \frac{\pi}{\sin\left(\frac{\pi}{4}\right)}.$$

Since:

$$1 - \frac{1}{4} = \frac{3}{4}$$

then:

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\pi}{\sin\left(\frac{\pi}{4}\right)}.$$

Using the trigonometric value:

$$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

Substituting:

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\pi}{\frac{\sqrt{2}}{2}}.$$

After simplification:

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \pi\sqrt{2}.$$

■

2.1.6 Symmetry Property of the Beta Function

Theorem 5 (Symmetry Property)**.** *Let* $m > 0$ *and* $n > 0$*. Then:*

$$\beta(m, n) = \beta(n, m).$$

Proof.

From the definition of the Beta function,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx.$$

Using the substitution:

$$x = 1 - t,$$

we obtain:

$$dx = -dt.$$

When $x = 0$, we have $t = 1$, and when $x = 1$, we have $t = 0$.

Thus:

$$\beta(m,n) = \int_1^0 (1-t)^{m-1}t^{n-1}(-dt).$$

Hence:

$$\beta(m,n) = \int_0^1 t^{n-1}(1-t)^{m-1}dt.$$

Recognizing the Beta function:

$$\beta(m,n) = \beta(n,m).$$

■

Corollary 5. *For all $m > 0$,*

$$\beta(m,1) = \beta(1,m) = \frac{1}{m}.$$

Example: Verify the symmetry property of the Beta function for

$$m = 2, \qquad n = 3$$

The symmetry property states that:

$$\beta(m,n) = \beta(n,m)$$

First, compute:

$$\beta(2,3) = \int_0^1 x^{2-1}(1-x)^{3-1}\,dx$$

Thus:

$$\beta(2,3) = \int_0^1 x(1-x)^2\,dx$$

Expanding:

$$(1-x)^2 = 1 - 2x + x^2$$

Then:

$$\beta(2,3) = \int_0^1 (x - 2x^2 + x^3)\,dx$$

Integrating term by term:

$$\beta(2,3) = \left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right]_0^1$$

Substituting the limits:

$$\beta(2,3) = \frac{1}{2} - \frac{2}{3} + \frac{1}{4}$$

Taking the common denominator 12:

$$\beta(2,3) = \frac{6-8+3}{12} = \frac{1}{12}$$

Now compute:

$$\beta(3,2) = \int_0^1 x^{3-1}(1-x)^{2-1}\,dx$$

Thus:

$$\beta(3,2) = \int_0^1 x^2(1-x)\,dx$$

Expanding:

$$\beta(3,2) = \int_0^1 (x^2 - x^3)\,dx$$

Integrating:

$$\beta(3,2) = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$

Substituting the limits:

$$\beta(3,2) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

Therefore,

$$\boxed{\beta(2,3) = \beta(3,2) = \frac{1}{12}}$$

Chapter 3

Applications of Gamma Function

In this chapter, several important applications of Gamma function are presented. Different types of improper integrals are evaluated step by step by transforming them into Gamma form. The purpose of this chapter is to demonstrate the effectiveness of this special function in simplifying mathematical computations and solving integrals encountered in mathematical physics and some finite product.

3.1 Evaluation of Integrals Using Gamma Function

In this section, different improper integrals are evaluated by using the properties of the Gamma function. The main idea consists in transforming the given integral into the standard Gamma form.

Example 1. *Evaluate the following integral:*

$$\int_0^\infty x^4 e^{-x}\,dx.$$

Solution

Comparing the given integral with the standard Gamma form

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\,dx,$$

we obtain $n = 5$. Hence,

$$\int_0^\infty x^4 e^{-x}\,dx = \Gamma(5).$$

Using the property:

$$\Gamma(n+1) = n!$$

we get:

$$\Gamma(5) = 4! = 24.$$

Therefore, the value of the integral is:

$$\int_0^\infty x^4 e^{-x}\, dx = 24.$$

Example 2. *Evaluate the integral:*

$$\int_0^\infty x^8 e^{-4x}\, dx.$$

Solution

We use the Gamma function property:

$$\Gamma(n) = z^n \int_0^\infty x^{n-1} e^{-zx}\, dx.$$

Rearranging the formula gives:

$$\int_0^\infty x^{n-1} e^{-zx}\, dx = \frac{\Gamma(n)}{z^n}.$$

Comparing with the given integral we get $n = 9$ and $z = 4$. Thus,

$$\int_0^\infty x^8 e^{-4x}\, dx = \frac{\Gamma(9)}{4^9}.$$

Since $\Gamma(9) = 8! = 40320$ and $4^9 = 262144$, then

$$\int_0^\infty x^8 e^{-4x}\, dx = \frac{40320}{262144}.$$

Finally

$$\int_0^\infty x^8 e^{-4x}\, dx \approx 0.153.$$

Example 3. *Evaluate the integral:*

$$\int_0^\infty t^{\frac{7}{2}} e^{-t}\, dt.$$

Solution

By comparing with the definition of the Gamma function:

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t}\, dt$$

we get $n = \frac{9}{2}$. Hence,

$$\int_0^\infty t^{7/2} e^{-t}\, dt = \Gamma\left(\frac{9}{2}\right).$$

Using the recurrence property:

$$\Gamma(n+1) = n\Gamma(n)$$

we obtain:

$$\Gamma\left(\frac{9}{2}\right) = \frac{7}{2}\Gamma\left(\frac{7}{2}\right),\ \Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right),\ \Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right),\ \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right).$$

Since

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi},$$

then

$$\Gamma\left(\frac{9}{2}\right) = \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2}\sqrt{\pi} = \frac{105}{16}\sqrt{\pi}.$$

Hence,

$$\int_0^\infty t^{7/2} e^{-t}\, dt = \frac{105}{16}\sqrt{\pi}.$$

Example 4. *Evaluate the integral:*

$$\int_0^\infty x^{m-1} e^{-ax} \sin(bx)\, dx$$

in terms of the Gamma function.

Solution

Let

$$I = \int_0^\infty x^{m-1} e^{-ax} \sin(bx)\, dx.$$

Using Euler's formula:

$$\sin(bx) = \frac{e^{ibx} - e^{-ibx}}{2i}.$$

Substituting into the integral:

$$I = \frac{1}{2i} \int_0^\infty x^{m-1} e^{-ax} \left(e^{ibx} - e^{-ibx}\right) dx.$$

Hence:

$$I = \frac{1}{2i}\left[\int_0^\infty x^{m-1} e^{-(a-ib)x} dx - \int_0^\infty x^{m-1} e^{-(a+ib)x} dx\right].$$

Using the Gamma integral formula:

$$\int_0^\infty x^{m-1}e^{-cx}dx = \frac{\Gamma(m)}{c^m},$$

we obtain:

$$I = \frac{\Gamma(m)}{2i}\left[(a-ib)^{-m} - (a+ib)^{-m}\right].$$

Let $a = r\cos\theta$ and $b = r\sin\theta$ with $r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}\left(\frac{b}{a}\right)$. Then

$$a \pm ib = re^{\pm i\theta}.$$

Thus,

$$(a \pm ib)^{-m} = r^{-m}e^{\mp im\theta}.$$

Substituting these, we get

$$I = \frac{\Gamma(m)}{2i}r^{-m}\left(e^{im\theta} - e^{-im\theta}\right).$$

Now, using the identity

$$e^{ix} - e^{-ix} = 2i\sin x$$

to obtain

$$I = \Gamma(m)r^{-m}\sin(m\theta).$$

Therefore,

$$\int_0^\infty x^{m-1}e^{-ax}\sin(bx)\,dx = \frac{\Gamma(m)\sin(m\theta)}{(a^2+b^2)^{m/2}}.$$

3.2 Evaluation of Some Products

In this section we use the properties of Gamma function to evaluate some products.

Example 5. *Show that:*

$$2\cdot 4\cdot 6\cdots(2n) = 2^n\Gamma(n+1)$$

.

Solution

Consider the left-hand side:

$$LHS = 2\cdot 4\cdot 6\cdots(2n).$$

Taking 2 as a common factor from each term, we obtain:

$$LHS = 2^n(1\cdot 2\cdot 3\cdots n).$$

Since,

$$1 \cdot 2 \cdot 3 \cdots n = n!,$$

then:

$$LHS = 2^n n!.$$

Using the Gamma function property:

$$\Gamma(n+1) = n!,$$

we get:

$$2 \cdot 4 \cdot 6 \cdots (2n) = 2^n \Gamma(n+1).$$

Example 6. *Show that:*

$$1 \cdot 3 \cdot 5 \cdots (2n-1) = \frac{\Gamma(2n)}{2^{n-1}\Gamma(n)}.$$

Solution

Consider:

$$1 \cdot 3 \cdot 5 \cdots (2n-1)$$

Multiplying and dividing by:

$$2 \cdot 4 \cdot 6 \cdots (2n)$$

gives:

$$1 \cdot 3 \cdot 5 \cdots (2n-1) = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots (2n)}{2 \cdot 4 \cdot 6 \cdots (2n)}.$$

Thus:

$$1 \cdot 3 \cdot 5 \cdots (2n-1) = \frac{(2n)!}{2^n n!}.$$

Using the Gamma function identities:

$$(2n)! = \Gamma(2n+1)$$

and

$$n! = \Gamma(n+1)$$

we get

$$1 \cdot 3 \cdot 5 \cdots (2n-1) = \frac{\Gamma(2n+1)}{2^n \Gamma(n+1)}.$$

Finally using

$$\Gamma(n+1) = n\Gamma(n)$$

to obtain:

$$1 \cdot 3 \cdot 5 \cdots (2n-1) = \frac{\Gamma(2n)}{2^{n-1}\Gamma(n)}.$$

3.3 Fractional Derivatives and the Gamma Function

Fractional calculus extends the notion of derivatives to non-integer orders. One classical definition is the Riemann–Liouville fractional derivative.

For $\alpha > 0$, the fractional integral is defined by

$$(I^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt.$$

The fractional derivative of order α (with $n-1 < \alpha < n$) is defined by

$$(D^\alpha f)(x) = \frac{d^n}{dx^n} I^{n-\alpha} f(x).$$

Example 7. *Let $f(x) = x^\beta$ with $\beta > -1$. Then the fractional derivative satisfies*

$$D^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}\, x^{\beta-\alpha}.$$

Example 8 (Special case)**.** *Take $\beta = 1$ and $\alpha = \frac{1}{2}$. Then*

$$D^{1/2} x = \frac{\Gamma(2)}{\Gamma\left(\frac{3}{2}\right)} x^{1/2}.$$

Since

$$\Gamma(2) = 1!, \quad \Gamma\left(\tfrac{3}{2}\right) = \frac{\sqrt{\pi}}{2},$$

we obtain

$$D^{1/2} x = \frac{1}{\sqrt{\pi}/2}\, x^{1/2} = \frac{2}{\sqrt{\pi}} x^{1/2}.$$

Conclusion

In this thesis, we have studied the Gamma function as one of the most important special functions in mathematical analysis. We began by presenting its definition and fundamental properties, including its analytic continuation, recurrence relation, reflection formula, and infinite product representation. These properties demonstrate the central role of the Gamma function as a natural extension of the factorial function to real and complex numbers.

We then introduced the Beta function and established its fundamental relationship with the Gamma function. Several important properties and applications of the Beta function were also presented, highlighting the strong connection between these two special functions.

Finally, we explored various applications of the Gamma function in the evaluation of improper integrals, finite products, and fractional derivatives. These applications illustrate the effectiveness of the Gamma function as a powerful mathematical tool that appears in different branches of analysis and applied mathematics.

The study carried out in this work shows that the Gamma function is not only of theoretical interest but also of great practical importance in solving mathematical problems. This topic remains an active area of research, with further applications in probability theory, mathematical physics, differential equations, and many other fields.

We hope that this work provides a clear introduction to the Gamma function and encourages further study of special functions and their applications.

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