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Application of Coincidence Degree Theory and Fixed Point Methods to Nonlinear Fractional Boundary Value Problems

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Dedication

To my beloved parents,
To my family and friends,
To all those who supported me during this journey.

Boutalbi Manel

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First and foremost, all praise and thanks are due to Allah, the Almighty, for granting me the strength, knowledge, and perseverance to complete this work.

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Boutalbi Manel

Abstract

This master dessertation investigates the existence of solutions for nonlinear fractional boundary value problems. We employ two powerful and complementary approaches: the Mawhin coincidence degree theory and classical fixed point theorems. In the first part, we reformulate the problem as an operator equation $Lx = Nx$ in suitable function spaces. By verifying the conditions of the coincidence degree theory, we establish an existence result. In the second part, we apply Schauder's and Banach's fixed point theorems to prove existence and uniqueness under different sets of assumptions. Finally, we provide illustrative examples to demonstrate the applicability of our theoretical findings.

Keywords: Fractional differential equations, Boundary value problems, Coincidence degree theory, Fixed point theorems, Existence of solutions.

Résumé

Ce mémoire de master étudie l'existence de solutions pour des problèmes aux limites non linéaires impliquant des équations différentielles fractionnaires. Nous utilisons deux approches puissantes et complémentaires : la théorie du degré de coïncidence de Mawhin et les théorèmes classiques du point fixe. Dans la première partie, nous reformulons le problème comme une équation d'opérateur $Lx = Nx$ dans des espaces fonctionnels appropriés. En vérifiant les conditions de la théorie du degré de coïncidence, nous établissons un résultat d'existence. Dans la deuxième partie, nous appliquons les théorèmes de point fixe de Schauder et de Banach pour prouver l'existence et l'unicité sous différents ensembles d'hypothèses. Enfin, nous fournissons des exemples illustratifs pour démontrer l'applicabilité de nos résultats théoriques.

Mots-clés: Équations différentielles fractionnaires, Problèmes aux limites, Théorie du degré de coïncidence, Théorèmes de point fixe, Existence de solutions.

الملخص

هذه مذكرة ماستر تبحث في وجود حلول لمسائل حدودية كسرية غير خطية. نعتمد نهجين قويين ومتكاملين: نظرية درجة التطابق لموهون، ونظريات النقطة الثابتة التقليدية.

في الجزء الأول، نُعيد صياغة المسألة كمعادلة مؤثرية $Lx = Nx$ في فضاءات دالية مناسبة. ومن خلال التحقق من شروط نظرية درجة التطابق، نُثبت وجود نتيجة.

في الجزء الثاني، نطبق نظريتي شاودر و باناخ للنقطة الثابتة لإثبات الوجود والوحدة تحت مجموعات مختلفة من الفرضيات. وأخيراً، نقدم أمثلة توضيحية لتطبيق النتائج النظرية.

الكلمات المفتاحية: معادلات تفاضلية كسرية، مسائل حدية، نظرية درجة التطابق، نظريات النقطة الثابتة، وجود حلول.

Contents

List of Symbols

$\mathbb{R}$	The set of real numbers
$C[0,1]$	Space of continuous functions on $[0,1]$
$L^p[0,1]$	Space of Lebesgue integrable functions
D^α	Caputo fractional derivative of order α
I^α	Riemann-Liouville fractional integral of order α
$\Gamma(\cdot)$	Gamma function
Ker L	Kernel of operator L
Im L	Image of operator L
dim L	dimension operator L
$\oplus$	direct sum.
(M,d)	metric space.
$d(\dots)$	distance function.
$\bar{C}^k$	Space of functions with values in $\mathbb{R}$, k times differentiable in Ω.
(M,d)	Metric space.
$d(.,.)$	Distance function.
Ω	A bounded open set.
$\overline{\Omega} = \Omega + \partial\Omega$	The closure of Ω.
$u'(t)$	Ordinary derivative with respect to t.
U	An open set.
deg	Topological degree.
$\deg_B$	Brouwer topological degree.
$\deg_{LS}$	Leray-Schauder topological degree.
$\bar{B}$	Closed unit ball.
$I^\alpha_{a_+}$	Riemann-Liouville fractional integral of order α.
${}^R D^\alpha(f(t))$	Left Riemann-Liouville fractional derivative of order $\alpha > 0$.
${}^c D^\alpha(f(t))$	Left Caputo fractional derivative of order $\alpha > 0$.

General Introduction

Fractional differential equations have emerged as a powerful tool in mathematical modeling due to their remarkable ability to capture memory effects and hereditary properties inherent in complex systems. These equations appear naturally in various fields such as physics, engineering, biology, control theory, and materials science. Unlike classical integer-order models, fractional-order models provide a more accurate description of many real-world phenomena, including viscoelasticity, anomalous diffusion, signal processing, and population dynamics.

Boundary value problems involving fractional derivatives present significant mathematical challenges, especially when the associated linear operator is non-invertible; such problems are said to be at resonance. In these cases, standard fixed point techniques, such as the Banach or Schauder fixed point theorems, are no longer directly applicable. Instead, more sophisticated topological methods are required to establish the existence of solutions. This thesis is devoted to the study of a nonlinear fractional boundary value problem of order 2< <3 involving the Caputo fractional derivative. Our objective is to prove the existence of at least one solution under suitable assumptions on the nonlinearity. To achieve this, we employ two complementary approaches:

Mawhin's Coincidence Degree Theory, which is particularly well-suited for problems at resonance. By reformulating the problem as an abstract operator equation Lx=Nx, where L is a Fredholm operator of index zero, we verify the required conditions and establish an existence result.

Classical Fixed Point Theorems (Banach and Schauder), which are applied under different assumptions to prove both existence and uniqueness. The thesis is organized as follows. Chapter 1 recalls essential concepts from fractional calculus, including the Riemann-Liouville fractional integral and the Caputo fractional derivative, as well as fundamental fixed point theorems. Chapter 2 presents the theoretical framework of Mawhin's continuation theorem, including Fredholm operators, projections, and the coincidence degree. Chapter 3 constitutes the core of our work: we formulate the fractional boundary value problem as an operator equation, verify the Fredholm property, construct suitable projectors, and apply Mawhin's theorem to prove existence. An illustrative example is provided to demonstrate the applicability of our results.

Finally, we conclude with a summary of our findings and suggest possible directions for future research.

Chapter 1

Reminders and Fundamental Concepts

This chapter's goal is to study a some fixed-point theories. We will start with the most basic and well-known of them all, the Banach fixed point theory for contracting applications.and the Brouwer fixed-point theorem .

1.1 Fixed-point theorem

The fixed-point theorem is a fundamental principle in mathematics, which states that a continuous function over a given domain may have a point that remains unchanged when the function is repeatedly applied to itself. In other words, there exists an x in the domain such that $f(x) = x$, where f is the function. This point is called the fixed point of the function.

1.1.1 Banach fixed point theorem

Banach's fixed point theorem (also known as the map theorem contracting) is a simple theorem to prove, which guarantees the existence of a unique fixed point for any contracting application, applies to complete spaces and which has many applications. These applications include the existence theorems of solution for differential equations or integral equations and the study of convergence of some numerical methods..

Definition 1 (Fixed Point). Let $F : X \to X$ be a mapping. Any point $x \in X$ such that $F(x) = x$ is called a fixed point.

Theorem 1 (Banach Contraction Principle: [4]

Let (M, d) be a complete metric space and let $F : M \to M$ be a contraction mapping, meaning, there exists $0 < k < 1$ such that

$$d(F(x), F(y)) \leq k\, d(x, y), \quad \forall x, y \in M.$$

Then F admits a unique fixed point $u \in M$. Moreover, for any $x \in M$, the sequence defined by

$$x_{n+1} = F(x_n)$$

converges to u, and we have

$$d(F^n(x), u) \leq \frac{k^n}{1-k}\, d(x, F(x)).$$

Proof: First, one observes the unity.
Assuming its existence $x, y \in M$ with $x = F(x); y = F(y)$ and $d(x, y) = d(F(x), F(y) \leq kd(x, y)$.
since $0 < k < 1$ hence, the final inequality implies that $d(x, y) = 0 \implies x = y$, so $\exists! x \in M$ such as $F(x) = x$.
Now, evidence exists for the existence of $x \in M$.
assume that $F^n(x)$ is a Cauchy sequence on $n \in \{0, 1, \dots\}$

$$d(F^n(x), F^{n+1}(x)) \leq kd(F^{n-1}(x), F^n(x)) \leq \cdots \leq k^n d(x, F(x))$$

if $m > n$ on $n \in \{0, 1, \dots\}$

$$\begin{aligned} d(F^n(x), F^m(x)) &\leq d(F^n(x), F^{n+1}(x)) + d(F^{n+1}(x), F^{n+2}(x)) + \cdots + d(F^{m-1}(x), F^m(x)) \\ &\leq k^n d(x, Fx) + k^{n+1} d(x, Fx) + \cdots + k^{m-1} d(x, F(x)) \\ &\leq k^n d(x, F(x)) \left[1 + k + k^2 + \ldots\right] \\ &\leq \frac{k^n}{1-k} d(x, F(x)) \end{aligned}$$

for $m > n; n \in \{0, 1, \ldots\}$ on a

$$d(F^n(x), F^m(x)) \leq \frac{k^n}{1-k} d(x, F(x)) \tag{1.1}$$

so $F^n(x)$ is a Cauchy sequence in the entire X space after that, then it exists $u \in X$ with

$$\lim_{n \to +\infty} F^n(x) = u$$

moreover by the continuity of F

$$u = \lim_{n \to \infty} F^{n+1}(x) = \lim_{n \to \infty} F(F^n) = F(u)$$

Thus, u is a fixed point of F.
ultimately, $m \longrightarrow \infty$ in (1.1), we obtain

$$d(F^n(x), u) \leq \frac{k^n}{1-k} d(x, F(x))$$

Example 1 *Consider the application* $T : \mathbb{R} \to \mathbb{R}$ *defined by* $T(x) = \frac{x}{2} + \frac{1}{2}$ *; then* T *a contraction* $0 < k = \frac{1}{2} < 1$*; and admits as a fixed point* $x = 1$ *moreover* $\lim\{T^n(x)\}_{n=1}^{\infty} = 1$

RemarK 1: To convince oneself of the necessity of the phenomenon, consider the following examples .

Example 2 $T : [0,1] \rightarrow \mathbb{R}$, $T(x) = \frac{x}{2} + 1$, *is declining but does not reach a fixed point. The issue is that* $T([0,1]) \nsubseteq [0,1]$ *and we cannot iterate :* $x_0 = 0$, $x_1 = 1$, $x_2 = 1.5$, *but* x_3 *is not defined !*

Example 3 $T : \mathbb{R} \rightarrow \mathbb{R}$; $T(x) = x + \frac{1}{1+e^x}$ *check* $|T(x) - T(y)| < |x - y|$ *for everything* $x \neq y$, *but does not set a fixed location. The issue is that* T *is not a contracting party, and for everything* $x_0 \in \mathbb{R}$ *we obtain* $x_n \longrightarrow +\infty$

1.1.2 Fixed point theorems for undefined contractions over the whole metric space

Assuming (M,d) to be a complete metric space, it is evident that a function defined just on a subset of M will not inevitably result in a fixed point. There will be additional requirements in order to guarantee this.

Theorem 2 *be* $K \subset M$ *a closed set and* $T : K \longrightarrow M$ *k-contraction. Assuming its existence* $x_0 \in K$ *and* $r > 0$ *such as*

$$\overline{B(x_0, r)} \subset K \quad \text{and} \quad d(x_0, T(x_0)) < (1-k)r$$

then F has a unique fixed point $x^* \in \overline{B(x_0, r)}$.
In certain applications, there are cases where T *is Lipschitzian without being a contraction, while a certain power of* T *is a contraction see[1]. In this case we have the following thoerems*

Theorem 3 *let (M,d) be a complete metric space* $T : M \longrightarrow M$ *an application such as* $d(T^m(x), T^m(y) \leq kd(x,y), \forall x, y \in M$, *for a certain* $m \geq 1$ *and* $0 \leq k < 1$. *then* T *admits a unique fixed point* $x^* \in M$

Proof. as T^m is a contraction, it follows from the theorem 2 that T^m has a fixed unique point, thus $x^* = T^m x^*$. so $T^m(T(x^*)) = T(T^m(x^*)) = T(x^*)$, i.e., $T(x^*)$ is a fixed point of T^m. but T^m has a single fixed point of $T(x^*) = x^*$ so T has a single fixed point of (x^*) ,and it is unique because every fixed point of T is also fixed point T^m ■

Example 4 *Consider the metric space M given by :*$M = C[a,b]$ *; The space of functions continues with real values defined on the interval* $[a,b]$*. M is a Banach space compared to the norm* $\|u\| = \max_{t\in[a,b]} |u(t)|$*.* $u \in M$ *. we define* $T : M \to M$ *by :*

$$Tu(t) = \int_a^t u(s)ds$$

so,

$$\|T(u) - T(v)\| \leq (b-a)\|u - v\|,$$

so $(b-a)$ *is the ideal Lipschitz constant for T. Additionally, on a :*

$$T^2(u)(t) = \int_a^t \left(\int_a^s u(\tau)d\tau\right) ds = \int_a^t (t-s)u(s)ds$$

and by induction

$$T^m u(t) = \frac{1}{(m-1)!}\int_a^t (t-s)^{m-1}u(s)ds,$$

from then on

$$\|T^m(u) - T^m(v)\| \leq \frac{(b-a)^m}{m!}\|u - v\|,$$

and so T^m *would be a contraction if* $\frac{(b-a)^m}{m!} < 1$

1.1.3 Principles of continuation

Another method of obtaining a fixed point's existence for an infinite application across all of space is obtained through a continuation procedure. This one consists of transforming our application into a simpler one for which we are aware that a fixed point exists. It goes without saying that this information known by the name must meet certain requirements. [1] [5].

Definition 2 *Let X and Y be two topological spaces. Two continuous applications $f, g : X \longrightarrow Y$ are called homotopic when there is a continuous application*

$$H : X \times [0,1] \longrightarrow Y$$

such as $H(x,0) = f(x)$ and $H(x,1) = g(x)$. In other words, there exists a family of applications of X in Y, namely $x \longrightarrow H(x,t)$ for $0 \leq t \leq 1$, which starts from f to arrive g, and varies continuously . we notice $f \simeq g$.

Example 5 *Let $X = Y = \mathbb{R}^n$, we consider $c : \mathbb{R}^n \to \mathbb{R}^n$ constant application $c(x) = 0$, and $i : \mathbb{R}^n \to \mathbb{R}^n$ application $i(x) = x$. show that c and i are homotopic. just take :*

$$H : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$$
$$H(x,t) = tx.$$

so $H(x;0) = 0 = c(x)$ and $H(x,1) = x$:

Example 6 *either $X = Y = \mathbb{R}^n - \{0\}$; we consider this time $p(x) = x/\|x\|$; and $i(x) = x$ again, we see that p and i are homotopice by taking*

$$H : (\mathbb{R}^n - \{0\}) \times [0,1] \to \mathbb{R}^n - \{0\}$$
$$H(x,t) = (1-t)x + t\frac{x}{\|x\|}$$

Definition 3 *either $f : X \longrightarrow Y$ continuous application. We say that f is a homotopy equivalence when it exists $g : Y \longrightarrow X$ such as $g \circ f = id_X$ and $f \circ g = id_y$. We then say that X and Y have the same type of homotopy, or sometimes that they are homotopy equivalent, and we note $X \simeq Y$*

Example 7 *either $X = \mathbb{R}^n - \{0\}$ and $Y = S^{n-1}$, we then take $f : X \to Y$ defined by $f(x) = x/\|x\|$; et $g : Y \to X$ inclusion. then $f \circ g = id_y$, and the example 6 to show that $g \circ f \simeq id_x$. so $\mathbb{R}^n - \{0\}$ has the same type of homotopy as the sphere S^{n-1}.*

It is (X,d) a complete mathematical space, and U is an open subset of X.

Definition 4 *either $F : \overline{U} \longrightarrow X$ and $G : \overline{U} \longrightarrow X$ two contractions, we say that F and G are homotopic if there exists $H : \overline{U} \times [0,1] \longrightarrow X$ checking the following properties:*

(a) $H(.,0) = G$ and $H(.,1) = F$.

(b) $H(x,t) \neq x$ for everything $x \in \partial U$ and $t \in [0,1]$

(c) It exists $\alpha \in [0,1)$ such as $d(H(x,t), H(y,t) \leq \alpha d(x,y)$ for everything $x, y \in \overline{U}$, and $t \in [0,1]$.

(d) It exists $M \geq 0$ such as $d(H(x,t), H(x,s) \leq M|t-s|$ for everything $x \in \overline{U}$, and $t, s \in [0,1]$.

Theorem 4 *either $F : \overline{U} \longrightarrow X$ and $G : \overline{U} \longrightarrow X$ There are two homotopically contractive applications, and G has a fixed point in U. Thus, F fixes a fixed point in U.*

Proof. let's consider the whole $Q = \{\lambda \in [0,1] : x = H(x,\lambda),\ \text{pour certain } x \in U\}$ o H a homotopy between F and G described in the definition (2) Note that Q is not bounded because G has a fixed point and that $0 \in Q$. It is demonstrated that Q is both open and closed in $[0,1]$, and as a result, by connecting, one will have $Q = [0,1]$. F proceeds to a fixed point. First, let's note that Q is a closed ensemble inside $[0,1]$. In fact, either $\{\lambda_n\}_{n\in\mathbb{N}}$ a sequence in Q such that $\lim_{n\to\infty} \lambda_n = \lambda$, so, we must show that $\lambda \in Q$. as $\lambda_n \in Q$ for $n = 1, 2, ...$, it exists $x_n \in \overline{U}$ o $x_n = H(x_n, \lambda_n)$. also for $n, m \in \{1, 2, ...\}$ on a

$$
\begin{aligned}
d(x_n, x_m) &= d(H(x_n,\lambda_n), H(x_m,\lambda_m)) \\
&\leq d(H(x_n,\lambda_n), H(x_n,\lambda_m)) + d(H(x_n,\lambda_m), H(x_m,\lambda_m)) \\
&\leq M|\lambda_n - \lambda_m| + \alpha d(x_n, x_m)
\end{aligned}
$$

thus, we have

$$d(x_n, x_m) \leq \frac{M}{1-\alpha}|\lambda_n - \lambda_m|$$

which shows that $\{x_n\}$ is a cauchy sequel to X (because $\{\lambda_n\}$ is also) and since it is complete, it exists $x \in \overline{U}$ such as $\lim_{x\to\infty} x_n = x$. by the continuity of H,

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} H(x_n, \lambda_n) = H(x, \lambda)$$

thus, $\lambda \in Q$ and Q is closed in $[0, 1]$.

let us show that Q is an open set of $[0, 1]$. either $\lambda_0 \in Q$. so, it exists $x_0 \in U$ with $x_0 = H(x_0, \lambda_0)$. since, by hypothesis, $x_0 \in U$, we can find $r > 0$ such as the open ball $B(x_0, r) = \{x \in X : (x, x_0) < r\} \subseteq U$. let's choose $\epsilon > 0$ such as $\epsilon \leq \frac{(1-\alpha)r}{M}$ where $r \leq dist(x_0, \partial U)$, and $dist(x_0, \partial U)) = inf\{(x_0, x) : x \in \partial U\}$.let's fix $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$. so, for $x_0 \in \overline{B(x_0, r)}$

$$\begin{aligned} d(x_0, H(x, \lambda)) &\leq d(H(x_0, \lambda_0), H(x, \lambda_0)) + d(H(x, \lambda_0), H(x, \lambda)) \\ &\leq \alpha d(x_0, x) + M|\lambda, \lambda_0| \\ &\leq \alpha r + (1-\alpha)r = r \end{aligned}$$

then for all $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$ fix

$$H(., \lambda) : \overline{B(x_0, r)} \longrightarrow \overline{B(x_0, r)}$$

by the theorem (1), (2), we deduce that $H(., \lambda)$ a fixed point in U. then, $\lambda \in Q$ for all $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$. and therefore Q is open in $[0, 1]$.
From the previous term, we deduce the following result.

Theorem 5 *(Nonlinear Leray-Schauder alternative)[1]. either $U \subset E$ an open set of a Banach space E such that $0 \in U$, and either $F : \overline{U} \longrightarrow E$ a contraction such that $F(\overline{U})$ or terminal. then one of the following two statements is verified:*

1. *F has a fixed point in $(\overline{U})$.*

2. *it exists $\lambda \in (0,1)$ and $x \in \partial U$ such as $x = \lambda F(x)$.*

Proof. Suppose that (2) is not verified and that F has no fixed point on ∂U, that is to say $x \neq \lambda F(x)$ for everything $x \in \partial U$ and $\lambda \in [0,1]$.

either $H : \overline{U} \times [0,1] \longrightarrow E$ given by $H(x,\lambda) = \lambda F(x)$, and let G be the null application. Note that G has a fixed point in U (such that $0 = G(0)$) and that F and G are two homotopically contractive applications. By the theorem (4) F also has a fixed point and therefore (1) is verified. ■

1.2 Topological degree

In this section, we provide a brief overview of the notion of topological degree, whether in finite or infinite dimensions. The degree, $\deg(f,\Omega,y)$ of f in Ω with respect to y gives information about the number of solutions of the equation $f(x) = y$ in an open set $\Omega \subset \mathbb{R}^n$ is continuous, $y \notin f(\partial\Omega)$, and X is a topological space, mostly metric. [6] [7] [8] [9].

1.2.1 Brouwer's topological degree

Consider a bounded open set Ω in $\mathbb{R}^n$ with boundary $\partial\Omega$ and closure $\overline{\Omega}$. $\overline{C}^k(\Omega,\mathbb{R}^n)$ denotes the space of k-times differentiable functions in Ω that are continuous on $\overline{\Omega}$. This space is equipped with its usual topology.

Let $x_0 \in \Omega$, if f is differentiable at x_0, we denote by $J_f(x_0) = \det f'(x_0)$ the Jacobian of f at x_0.

Definition 5 Let f be a C^1 function on Ω. Let $J_f(x_0)$ be the Jacobian of f at a point x_0 in Ω. The point x_0 is called a *critical point* if $J_f(x_0) = 0$. Otherwise, x_0 is called a *regular point.*

We denote by $S_f(\Omega)$ the set of critical points, i.e.,

$$S_f(\Omega) = \{x \in \Omega,\ J_f(x) = 0\}$$

Definition 6 An element $y \in \mathbb{R}^n$ is called a *regular value* of f if $f^{-1}(y) \cap S_f(\Omega) = \emptyset$. Otherwise, y is called a *singular value.*

Definition 7 Let $f \in \overline{C}^1 \setminus f(\partial\Omega)$ be a regular value of f. The topological degree of f in Ω with respect to y is the integer

$$\deg(f, \Omega, y) = \sum_{x \in f^{-1}(y)} \mathrm{Sgn} J_f(x)$$

where $\mathrm{Sgn} J_f(x)$ denotes the sign of $J_f(x)$, defined by $\mathrm{sgn}(t) = 1$ if $t > 0$ and $\mathrm{sgn}(t) = -1$ if $t < 0$.

RemarK 2:

1. By convention, if $f^{-1}(y) = \emptyset$, then $\deg(f, \Omega, y) = 0$.

2. $f^{-1}(y)$ contains a finite number of elements.

Example 8: Let $f : \mathbb{R} \to \mathbb{R}$, $x \to x^2$ then $\deg(f,]0, 2[, 1) = 0$.

Example 9: Let $0 < \epsilon < 1$ and consider the function $f(x, y) = (x^2 - y^2 - \epsilon, 2xy)$, and $f^{-1}(0, 0) = \{(x, y) \in \mathbb{R}^2;\ f(x, y) = (0, 0)\}$ then, we have

$$x^2 - y^2 - \epsilon = 0 \tag{1}$$

and

$$2xy = 0 \tag{2}$$

According to (1.2), we find $x = 0$ or $y = 0$.
If $x = 0$ then: $-y^2 - \epsilon = 0 \implies y^2 = -\epsilon$, which is a contradiction.
If $y = 0$ then: $x^2 - \epsilon = 0 \Longleftrightarrow x = \sqrt{\epsilon}$ or $x = -\sqrt{\epsilon}$, thus

$$f^{-1}(0, 0) = \{(-\sqrt{\epsilon}, 0);\ (\sqrt{\epsilon}, 0)\}$$

If $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ then $f^{-1}(0) \cap \partial\Omega = \emptyset$. Moreover, as

$$J_f(x, y) = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$$

and $\det(J_f((x, y))) = 4(x^2 + y^2)$ and since $\deg(f, \Omega, y) = \sum_{x \in f^{-1}(y)} \mathrm{Sgn} J_f(x)$ then

$$\mathrm{Sgn}\det J_f(\sqrt{\epsilon}, 0) = \mathrm{Sgn}4\epsilon = 1$$

$$\mathrm{Sgn}\det J_f(-\sqrt{\epsilon}, 0) = \mathrm{Sgn}4\epsilon = 1$$

$$\implies \deg(f, \Omega, 0) = 1 + 1 = 2$$

RemarK 3 In the case where $f^{-1}(y) \cap S_f(\Omega) \neq \emptyset$, we have the following lemma.

Lemma 1:[Sard's Lemma] Let $f \in C^1(\overline{\Omega}, \mathbb{R}^n)$. Then the set $f(S_f)$ of critical values of f has measure zero.

We will now see that we can extend the notion of degree to the case where the function f.

1.2.2 Topological degree of Leray-Schauder

In the previous section, we saw that in finite dimensions, $C(\overline{\Omega}, X)$ is a suitable class of functions for which there exists a unique degree function, the Brouwer degree, satisfying properties 1, 2, and 3 of the theorem. Unfortunately, in infinite dimensions, $C(\overline{\Omega}, X)$ is not. Indeed, an example by Leray shows that we need to restrict the class of functions for which there is existence and uniqueness of a Leray-Schauder degree function, to a set strictly contained in $C(\overline{\Omega}, X)$.

Definition 8 Let X be a Banach space and Ω a closed subset of X. If $T : \Omega \to X$ is a continuous operator, we say that T is *compact* if for every bounded subset B of Ω, $T(B)$ is relatively compact in X.

Definition 9 Let X be a Banach space and Ω a subset of X. We say that the mapping $T : \Omega \to X$ has *finite rank* if $\dim(\mathrm{Im}(T)) < \infty$, in other words, if $\mathrm{Im}(T)$ is a finite-dimensional subspace of X.

Definition 10 Let X be a Banach space, $\Omega \subset X$ an open bounded set, and $T : \overline{\Omega} \to X$ a compact mapping. Then, for any $\epsilon > 0$, there exists a finite-dimensional space denoted F and a continuous mapping $T_\epsilon : \overline{\Omega} \to F$ such that

$$\|T_\epsilon x - Tx\| < \epsilon \quad \text{for all } x \in \overline{\Omega}.$$

Definition 11 Let X be a Banach space, $\Omega \subset X$ an open bounded set, and $T : \overline{\Omega} \to X$ a compact mapping. Suppose now that $0 \notin (I - T)(\partial\Omega)$. There exists $\epsilon_0 > 0$ such that for $\epsilon \in (0, \epsilon_0)$, the Brouwer degree $\deg(I - T_\epsilon, \Omega \cap F_\epsilon, 0)$ is well-defined as in definition 10. Therefore, we define the Leray-Schauder degree by

$$\deg(I - T, \Omega, 0) = \deg(I - T_\epsilon, \Omega \cap F_\epsilon, 0).$$

RemarK 4: This definition depends only on T and Ω. If $y \in X$ such that $y \notin (I - T)(\partial\Omega)$, then the degree of $I - T$ in Ω with respect to y is defined as

$$\deg(I - T, \Omega, y) = \deg(I - T - y, \Omega, 0).$$

Theorem 6 Let X be a Banach space and $A = \{(I - T, \Omega, 0),\ \Omega$ a bounded open subset of $X, T : \overline{\Omega} \to X$ compact$, 0 \notin (I - T)(\partial\Omega)\}$ then, there exists a unique application $\deg(f, \Omega, y) : A \to \mathbb{Z}$ called the Leray-Schauder topological degree such that:

1. *(Normalization)* If $0 \in \Omega$ then $\deg(I, \Omega, 0) = 1$;

2. *(Solvability)* If $\deg(I - T, \Omega, 0) \neq 0$, then there exists $x \in \Omega$ such that $(I - T)x = 0$;

3. *(Invariance by homotopy)* Let $H : [0, 1] \times \overline{\Omega} \to X$ be a compact homotopy, such that $0 \notin (I - H(t, \cdot))(\partial\Omega)$. Then $\deg(I - H(t, \cdot), \Omega, 0)$ does not depend on $t \in [0, 1]$;

4. *(Additivity)* Let Ω_1 and Ω_2 be two disjoint open subsets of Ω and

$$0 \notin (I-T)(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2)).$$

Then,

$$\deg(I-T,\Omega,0) = \deg(I-T,\Omega_1,0) + \deg(I-T,\Omega_2,0).$$

The Leray-Schauder degree preserves all basic properties of the Brouwer degree. As a consequence of this notion of degree, we will prove some topological fixed point theorems, in particular the Leray-Schauder nonlinear alternative.

Theorem 7 Let $\overline{B}$ be the closed unit ball of $\mathbb{R}^n$ and $f : \overline{B} \to \overline{B}$ continuous. Then f has a fixed point: there exists $x \in \overline{B}$ such that $f(x) = x$.

Proof: If there exists an $x \in \partial B$, then there is nothing to prove. Otherwise, consider the function $h(t,x) = x - tf(x)$. We have that h is continuous, $h(0,x) = x$, and $h(1,x) = x - f(x)$. Moreover, if we assume that $h(t,x_0) = 0$ for some $x_0 \in \partial B$, then we obtain $x_0 = tf(x_0)$, which implies, since $0 \le t \le 1$, that $f(x_0) \in \partial B$, contradiction. Since h is a suitable homotopy between $I-f$ and I, then

$$\deg(I-f,\Omega,0) = \deg(I,\Omega,0) = 1.$$

In conclusion, there exists an $x \in B$, such that $x - tf(x) = 0$, i.e., $f(x) = x$.

Theorem 8 Let $\overline{B}$ be the closed unit ball of a Banach space E and $f : \overline{B} \to \overline{B}$ compact. Then f has a fixed point: there exists $x \in \overline{B}$ such that $f(x) = x$.

Proof: Let $h(t,x) = tf(x)$ be a compact function on $[0,1] \times \overline{B}$. If, for some $t \in [0,1]$ and $x \in \partial B$, we have $x - h(t,x) = 0$, then $tf(x) = x$; since $|x| = 1$ and $|f(x)| \le 1$, this implies $t = 1$ and $x = f(x)$, thus a fixed point on ∂B, a situation that we have excluded. Therefore, we can apply the properties of normalization and invariance by homotopy of the degree, which gives

$$1 = \deg(I,B,0) = \deg(I-f,B,0)$$

since $h(0,\cdot) = 0$ and $h(1,\cdot) = f$, hence the existence of a fixed point.

Theorem 9: (Leray-Schauder Nonlinear Alternative)] Let $\Omega \subset X$ be a non-empty bounded open subset of a Banach space X such that $0 \in \Omega$, and let $T : \overline{\Omega} \to X$ be a compact operator. Then one of the following statements holds:

1. T has a fixed point in Ω;
2. there exist $\lambda > 1$ and $x \in \partial\Omega$ such that $Tx = \lambda x$.

Proof: If (2) is true, then there is nothing to prove. Otherwise, we define the homotopy

$$H(t, x) = tTx \quad \text{for } t \in [0, 1].$$

Thus defined, $H(t, x) is compact$, $H(0, x) = 0$ and $H(1, x) = Tx$. Suppose that $H(t, x_0) = x_0$ for some $t \in [0, 1]$ and $x_0 \in \partial\Omega$. Then we have $tTx_0 = x_0$. If $t = 0$ or $t = 1$ we have (1); Otherwise

$$Tx_0 = \frac{1}{t} x_0 \quad \text{for some } t \in (0, 1),$$

and then we have (2). Otherwise, we have $\deg(I - T, \Omega, 0) = \deg(I, \Omega, 0) = 1$ and then T has a fixed point in Ω.

Theorem 10:[Brouwer] Let M be a convex, compact, non-empty subset of a finite-dimensional normed space $(X, \|\cdot\|)$ and let $A : M \to M$ be a continuous function, then A has a fixed point.

Theorem 11:[Schauder] Let M be a bounded, closed, convex, non-empty subset of a Banach space X and let $A : M \to M$ be a compact function, then A has a fixed point.

1.3 Elements of Fractional calculus

This section contains definitions and some properties of fractional integrals and derivatives of Riemann-Liouville and Caputo types. [1][2]

1.3.1 Riemann-Liouville Fractional Integrales

Let $f : [a, b[\to \mathbb{R}$ be a continuous or integrable function and:
An antiderivative of f is given by the expression:

$$I^{(1)}f(x) = \int_a^x f(t)dt.$$

For a second antiderivative, and according to Fubini's theorem, we have:

$$\begin{aligned}
I^{(2)}f(x) &= \int_a^x I^{(1)}f(u)du \\
&= \int_a^x \int_a^u (f(y)dy)du \\
&= \int_a^x f(u)du \int_u^x dt \\
&= \int_a^x (x-t)f(t)dt.
\end{aligned}$$

And for a third antiderivative, we have:

$$\begin{aligned}
I^{(3)}f(x) &= \int_a^x dx_1 \int_a^{x_1} dx_2 \int_a^{x_2} f(x_3)dx_3 \\
&= \int_a^x dx_1 \int_a^{x_1} (x_1 - x_2)f(x_2)dx_2 \\
&= \frac{1}{(3-1)!} \int_a^x (x-t)^{(3-1)} f(t)dt \\
&= \frac{1}{(2)!} \int_a^x (x-t)^{(2)} f(t)dt \\
&= \frac{1}{2} \int_a^x (x-t)^{(2)} f(t)dt.
\end{aligned}$$

In the general case for any integer n and by induction, we have Cauchy's formula:

$$I^n f(x) = \int_a^x dx_1 \int_a^{x_1} dx_2 \cdots \int_a^{x_{n-1}} f(x_n)dx_n, \tag{3}$$

$$= \frac{1}{(n-1)!} \int_a^x (x-t)^{(n-1)} f(t)dt. \tag{4}$$

For any integer n.

Since the generalization of the factorial by the Gamma function: $(n-1)! = \Gamma(n)$, Riemann realized that the second member of (1.4) could have a meaning even when n takes a non-integer value, it was natural to define fractional integration as follows:

Definition 12 Let $f(x) \in C[a;b]$ and $x \in [a;b]$, the Riemann-Liouville fractional integral $I_{a^+}^{\alpha} f(x)$ (left-sided) and $I_{b^-}^{\alpha} f(x)$ (right-sided) of order $\alpha > 0$ are defined by:

$$I_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{(\alpha-1)} f(t)dt.$$

$$I_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{(\alpha-1)} f(t)dt.$$

Proposition

(i) $I_{a^+}^{0} f(x) = f(x)$.

(ii) the integral operator I_0^{α} is linear.

Theorem 12: For $f \in C[a,b]$ Riemann-Liouville fractional integral possesses the semi-group property:

$$I_{a^+}^{\alpha}[I_{a^+}^{\beta} f(x)] = I_{a^+}^{(\alpha+\beta)} f(x), \quad \alpha, \beta > 0$$

Proof: From the definition we find:

$$\begin{aligned}
I^{\alpha}_{a+}[I^{\beta}_{a+}f(x)] &= \frac{1}{\Gamma(\alpha)}\int_a^x \frac{1}{(x-t)^{1-\alpha}} I^{\beta}_{a+}f(y)dy \\
&= \frac{1}{\Gamma(\alpha)}\int_a^x \frac{1}{(x-y)^{1-\alpha}} \frac{1}{\Gamma(\beta)}\int_a^y \frac{f(t)}{(y-t)^{1-\beta}}dtdy \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x \frac{1}{(x-y)^{1-\alpha}} \int_a^y \frac{f(t)}{(y-t)^{1-\beta}}dtdy.
\end{aligned}$$

According to Fubini's theorem we have:

$$I^{\alpha}_{a+}[I^{\beta}_{a+}f(x)] = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x f(t)dt\int_t^x (x-y)^{\alpha-1}(y-t)^{\beta-1}dy.$$

By the change of variable $y = t + (x-t)s$, then $dy = (x-t)ds$.
We obtain:

$$\begin{aligned}
\int_t^x (x-y)^{\alpha-1}(y-t)^{\beta-1}dy &= \int_0^1 \left[x-(t+(x-t)s)\right]^{\alpha-1}\left[(t+(x-t)s)-t\right]^{\beta-1}(x-t)ds \\
&= \int_0^1 \left[(x-t)-(x-t)s\right]^{\alpha-1}\left[(x-t)s\right]^{\beta-1}(x-t)ds \\
&= \int_0^1 (1-s)^{\alpha-1}(x-t)^{\alpha-1}(x-t)^{\beta-1}s^{\beta-1}(x-t)ds \\
&= (x-t)^{\alpha+\beta-1}\int_0^1 (1-s)^{\alpha-1}s^{\beta-1}ds \\
&= (x-t)^{\alpha+\beta-1}B(\alpha,\beta) \\
&= (x-t)^{\alpha+\beta-1}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.
\end{aligned}$$

$$I^{\alpha}_{a+}[I^{\beta}_{a+}f(x)] = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x f(t)dt(x-t)^{\alpha+\beta-1}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Hence:
$$\begin{aligned}
I^{\alpha}_{a+}[I^{\beta}_{a+}f(x)] &= \frac{1}{\Gamma(\alpha+\beta)}\int_a^x (x-t)^{\alpha+\beta-1}f(t)dt \\
&= I^{\alpha+\beta}_{a^+}f(x).
\end{aligned}$$

where $B(\alpha, \beta)$ is the Beta function. $\square$

Lemma 2 : The fractional integral operator I_a^α with $\alpha > 0$ is bounded in $L^p([a, b]), 0 \leq p \leq +\infty$

$$\|I_a^\alpha f\|_p \leq \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} \|f\|_p.$$

1.3.2 Riemann-Liouville Fractional Derivatives

There are several definitions of fractional derivatives. In this part, we will present the Riemann-Liouville derivative, which is the most commonly used.

Definition 13 Let f be an integrable function on $[a, b[$, then the fractional derivative of order α (with $n-1 \leq \alpha < n; n \neq 0$) in the Riemann-Liouville sense is defined by:

$$^R D_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (x-t)^{n-\alpha-1} f(s) ds = \frac{d^n}{dt^n} (I^{n-\alpha} f(t))$$

Where $n = [\alpha] + 1$.
In particular, for $\alpha = 0$ and for $\alpha = n$ we have:

$$\begin{aligned} {}_{a^+}^{R} D^0 f(t) &= f(t) \\ {}_{a^+}^{R} D^n f(t) &= f^{(n)}(t) \end{aligned}$$

Example 10: 1. **The non-integer derivative of a constant function in the Riemann-Liouville sense.**
In general, the non-integer derivative of a constant function in the Riemann-

Liouville sense is neither zero nor constant, but we have:

$$\begin{aligned}
{}^{R}D_{a+}^{\alpha}C &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t C(t-\tau)^{n-\alpha-1}d\tau \\
&= \frac{C}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\left[\frac{(t-\tau)^{n-\alpha}}{n-\alpha}\right]_a^t \\
&= \frac{C}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\left[\frac{(t-a)^{n-\alpha}}{n-\alpha}\right] \\
&= \frac{C}{\Gamma(n-\alpha)}\frac{\Gamma(n-\alpha)}{1-\alpha}(t-a)^{-\alpha} \\
{}^{R}D_{a+}^{\alpha}C &= \frac{C}{\Gamma(1-\alpha)}(t-a)^{-\alpha}.
\end{aligned}$$

2. **The derivative of $f(t) = (t-a)^{\beta}$ in the Riemann-Liouville sense.** Let α be non-integer and $0 \leq n-1 < \alpha < n$ and $\beta > -1$, then we have:

$$ {}^{R}D_{a+}^{\alpha}(t-a)^{\beta} = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-\tau)^{n-\alpha-1}(\tau-a)^{\beta}d\tau. $$

By making the change of variable $\tau = a+s(t-a)$, we have $d\tau = (t-a)ds$ and then:

$$\begin{aligned}
{}^{R}D_{a+}^{\alpha}(t-a)^{\beta} &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}(t-a)^{n-\alpha+\beta}\int_0^1 (1-s)^{n-\alpha-1}s^{\beta}ds \\
&= \frac{\Gamma(n-\alpha+\beta+1)}{\Gamma(n-\alpha)\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}\beta(n-\alpha,\beta+1) \\
&= \frac{\Gamma(n-\alpha+\beta+1)\Gamma(n-\alpha)\Gamma(\beta+1)}{\Gamma(n-\alpha)\Gamma(\beta-\alpha+1)\Gamma(n-\alpha+\beta+1)}(t-a)^{\beta-\alpha} \\
{}^{R}D_{a+}^{\alpha}(t-a)^{\beta} &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}.
\end{aligned}$$

1.3.3 Caputo Fractional Derivatives

Partial differentiation in the Riemann-Liouville sense played an active role in the development of micro-calculus in pure and applied mathematics. In the late 1960s, a revision became necessary, which led many authors, including Caputo, to find a new definition of fractional differentiation due to problems applied to optical flexibility and mechanics.

Definition 14 [3] Let f be an integrable function on $[a, b[$; then the fractional derivative of order $\alpha > 0$ (with $n-1 < \alpha < n, n \in \mathbb{N}^*$) in the Caputo sense is defined by:

$$\begin{aligned} {}^{C}D_{a+}^{\alpha} f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau \\ &= I^{n-\alpha} \left(\frac{d^n}{dt^n}(f(t)) \right) \end{aligned} \tag{5}$$

with $n = [\alpha] + 1$, where $[\alpha]$ denotes the integer part.

RemarK 5: The fractional derivative in the Riemann-Liouville sense of order $]n-1, n[$ is obtained by applying the fractional integration operator of order $n - \alpha$ followed by a classical differentiation of order n, whereas the fractional derivative in the Caputo sense is the result of the permutation of these two operations.

Example 11:

1. **The derivative of a constant function in the Caputo sense**

   The derivative of a constant function in the Caputo sense is zero:

$$ {}^{C}D^{\alpha} C = 0 \tag{6}$$

2. **The derivative of $f(t) = (t-a)^{\alpha}$ in the Caputo sense**

   Let p be an integer and $0 \leq n-1 < p < n$ with $\alpha > n-1$, then we have:

$$f^{(n)}(\tau) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)} (\tau-a)^{\alpha-n} \tag{7}$$

   Hence:

$$ {}^{C}D^{p}(t-a)^{\alpha} = \frac{\Gamma(\alpha+1)}{\Gamma(n-p)\Gamma(\alpha-n+1)} \int_a^t (t-\tau)^{n-p-1} (\tau-a)^{\alpha-n} d\tau \tag{8}$$

Performing the change of variable $\tau = a + s(t-a)$, we obtain:

$$\begin{aligned}
{}^{C}D^{p}(t-a)^{\alpha} &= \frac{\Gamma(\alpha+1)}{\Gamma(n-p)\Gamma(\alpha-n+1)}\int_a^t (t-\tau)^{n-p-1}(\tau-a)^{\alpha-n}d\tau \\
&= \frac{\Gamma(\alpha+1)}{\Gamma(n-p)\Gamma(\alpha-n+1)}(t-a)^{\alpha-p}\int_0^1 (1-s)^{n-p-1}s^{\alpha-n}ds \\
&= \frac{\Gamma(\alpha+1)\beta(n-p,\alpha-n+1)}{\Gamma(n-p)\Gamma(\alpha-n+1)}(t-a)^{\alpha-p} \\
&= \frac{\Gamma(\alpha+1)\Gamma(n-p)\Gamma(\alpha-n+1)}{\Gamma(n-p)\Gamma(\alpha-n+1)\Gamma(\alpha-p+1)}(t-a)^{\alpha-p} \\
&= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-p+1)}(t-a)^{\alpha-p}
\end{aligned} \tag{9}$$

Chapter 2

Mawhin's Continuation Theorem

In this chapter, we present Mawhin's continuation theorem. But before providing a more detailed presentation of the theorem, let us briefly review the state of the art on Fredholm operators as well as the main related notions. On the other hand, as we will see later, these operators can be obtained from projections, so we devote another section to them. See ([12]-[13]).

2.1 Topological Supplementary

Let E and F be two closed subspaces of a normed $\mathbb{R}$-vector space X. We say that E is a topological supplementary of F if X is the direct sum of F and E, i.e.,

$$X = F \oplus E.$$

2.2 Projection

Let X be a vector space. We say that a linear operator $P : X \to X$ is a projection if $P(P(x)) = P(x), \forall x \in X$ (i.e., $P^2 = P$).

Proposition 1 Let X be a vector space. A linear operator $P : X \to X$ is a projection if and only if $(I - P)$ is a projection. Furthermore, if the space X is normed, then P is continuous if and only if $(I - P)$ is continuous.

Proof.

1. **We show that:** P projection $\Leftrightarrow (I - P)$ projection.

   $(\Rightarrow)$ Let P be a projection, then:

$$
\begin{aligned}
(I-P)^2 &= (I-P)[(I-P)(x)] \\
&= (I-P)[x - P(x)] \\
&= I(x - P(x)) - P(x - P(x)) \\
&= x - P(x) - P(x) + P^2(x) \\
&= x - 2P(x) + P(x) \\
&= x - P(x) \\
&= (I-P)(x).
\end{aligned}
$$

   $(\Leftarrow)$ If $(I - P)$ is a projection, then:

$$I - (I - P) = I - I + P = P.$$

   is also a projection.

2. **We show that:** P is continuous $\Leftrightarrow (I - P)$ is continuous.

   $(\Rightarrow)$ If P is continuous and I is continuous, then $(I - P)$ is continuous.

   $(\Leftarrow)$ If $(I - P)$ is continuous, then P is continuous.

   In a topological framework, since the identity is a continuous application and the sum (is continuous). $\square$

Proposition 2 If P is a projection in X, then:

$$
\begin{cases}
\mathrm{Ker}(P) = \mathrm{Im}(I - P), \\
\mathrm{Im}(P) = \mathrm{Ker}(I - P).
\end{cases}
$$

Proof.

1. **We show that** $\mathrm{Ker}(P) = \mathrm{Im}(I - P)$.

   (i) $\mathrm{Ker}(P) \subset \mathrm{Im}(I - P)$
   If $x \in \mathrm{Ker}(P) \Rightarrow P(x) = 0$. By replacing P with $(I - P)$:

   $$(I - P)(x) = x - P(x) = x - 0 = x$$

   $\Rightarrow x \in \mathrm{Im}(I - P)$
   $\Rightarrow \mathrm{Ker}(P) \subset \mathrm{Im}(I - P)$.

   (ii) $\mathrm{Ker}(P) \supset \mathrm{Im}(I - P)$
   If $x \in \mathrm{Im}(I - P)$, we define the application:

   $$P((I - P)(x)) = P(x) - P^2(x) = P(x) - P(x) = 0$$

   $\Rightarrow x \in \mathrm{Ker}(P)$
   $\Rightarrow \mathrm{Ker}(P) \supset \mathrm{Im}(I - P)$.

   Thus, $\mathrm{Ker}(P) = \mathrm{Im}(I - P)$.

2. **We show that** $\mathrm{Im}(P) = \mathrm{Ker}(I - P)$.

   (i) $\mathrm{Im}(P) \subset \mathrm{Ker}(I - P)$
   If $x \in \mathrm{Im}(P) \Rightarrow P(x) = x$. By replacing P with $(I - P)$, we have:

   $$(I - P)(x) = x - P(x) = x - x = 0$$

   $\Rightarrow x \in \mathrm{Ker}(I - P)$
   $\Rightarrow \mathrm{Im}(P) \subset \mathrm{Ker}(I - P)$.

   (ii) $\mathrm{Im}(P) \supset \mathrm{Ker}(I - P)$
   If $x \in \mathrm{Ker}(I - P)$:

   $$(I - P)(x) = 0 \Leftrightarrow x - P(x) = 0 \Leftrightarrow x = P(x)$$

   $\Rightarrow \mathrm{Im}(P) \supset \mathrm{Ker}(I - P)$.

Thus, $\text{Im}(P) = \text{Ker}(I - P)$. $\square$

Definition 15. (A Hausdorff Space) A topological space X is separated (or Hausdorff) if:

$$\forall x \neq y \in X, \exists U_x \in \mathcal{U}_x, U_y \in \mathcal{U}_y \text{ open sets such that } U_x \cap U_y = \emptyset.$$

Corollary 1 Every continuous projection in a Hausdorff space has a closed image. In particular, continuous projections of Banach spaces have closed images.

Theorem 13 If P is a continuous projection in a Hausdorff topological vector space X, then X is the direct sum of $\text{Im}(P)$ and $\text{Ker}(P)$, (i.e., $X = \text{Im}(P) \oplus \text{Ker}(P)$).

Proof. By the previous corollary, $\text{Im}(P)$ and $\text{Ker}(P)$ are closed in X, where:

$$\text{Ker}(P) = \{x \in X, P(x) = 0\},$$

$$\text{Im}(P) = \{x \in X, P(x) = x\}.$$

Let $x = P(x) + (I - P)(x)$.

1- **i)** $P(x) \in Im(P)$ because $P(P(x)) = P^2(x) = P(x)$.
ii) $(I - P)(x) \in Ker(P)$ because $P((I - P)(x)) = P(x) - P^2(x) = P(x) - P(x) = 0$.

Thus $X = Im(P) + Ker(P)$.

2- **i)** $P(x) \in Im(P) = Ker(I - P) \Rightarrow P(x) = (I - P)P(x) = P(x) - P^2(x) = 0$.
ii) $(I - P)(x) \in Ker(P) \Rightarrow (I - P)(x) = 0$.

Thus $x \in Im(P) \cap Ker(P) = \{0\}$.
From (1) and (2): $X = Im(P) \oplus Ker(P)$. $\square$

2.2.1 Finite dimensional and finite codimensional subspace

Lemma 3 : [Projection onto a finite-dimensional subspace] If E is a finite-dimensional vector subspace of a normed space X, then there exists a continuous projection P on X such that Im(P) = E.

Proof. We choose a basis $e_1, \ldots, e_n$ of E, and we denote by $e_j^*, j = 1, \ldots, n$ the coordinate linear forms on E, i.e.:

$$e_j^*(e_i) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Using the Hahn-Banach theorem, these linear forms on E can be extended to continuous linear forms $x_1^*, \ldots, x_n^*$ on X. We obtain the map P defined by:

$$\forall x \in X, \quad P(x) = \sum_{j=1}^{n} x_j^*(x) e_j.$$

It is a continuous projection of X onto E that solves the problem. □

Corollary 2 If E is a finite-dimensional vector subspace of a normed space X, there exists a closed vector subspace $Y \subset X$ such that $X = E \oplus Y$.

Definition 16: (Codimension of a vector subspace) If the quotient space X/Y is finite-dimensional, the closed vector subspace $Y \subset X$ is said to be of finite codimension in X, written as:

$$codim(Y) = dim(X/Y)$$

Lemma 4 Let E be a normed vector space, M and N two closed vector subspaces of E such that $M \cap N = \{0\}$. If dim(M) = codim(N) $< \infty$, then $E = M \oplus N$.

Proof. Let $\pi = \pi|_N$ be the canonical surjection from E to E/N. Since $M \cap N = \{0\}$, the map $\pi|_M$ is injective. Hence:

$$dim(\pi(M)) = dim(M) = codim(N) = dim(E/N).$$

Thus $\pi(M) = E/N = \pi(E)$. Now if $x \in E$ is any element, then $\pi(x) \in \pi(E) = \pi(M)$. So there exists x_M such that $\pi(x) = \pi(x_M)$. Hence $\pi(x - x_M) \in Ker\pi = N$. We deduce that $x \in M+N$. This proves that $E = M+N$ and therefore $E = M \oplus N$. □

2.2.2 Fredholm Operator

Definition 17: Let X and Y be two $\mathbb{R}$-normed vector spaces. A linear map $L : \mathrm{dom}(L) \subset X \to Y$ is said to be **Fredholm** if it satisfies:

1. $\mathrm{Ker(L)} = \mathrm{L}^{-1}(\{0\})$ is finite-dimensional.
2. $\mathrm{Im(L)} = \mathrm{L(dom(L))}$ is closed and of finite codimension.

The codimension of $\mathrm{Im(L)} is the dimension of \mathrm{coKer(L)} = \mathrm{dim(Y/Imf(L))}$.

Definition 18: (The index) If L is a Fredholm operator, then its index is the integer

$$\mathrm{ind}(L) = \dim(\mathrm{Ker}(L)) - \mathrm{codim}(\mathrm{Im}(L)).$$

Example 12

1. If X and Y are of finite dimensions, then any linear map $L : X \to Y$ is Fredholm with

$$\begin{aligned}\mathrm{ind}(L) &= \dim(\mathrm{Ker}(L)) - \mathrm{codim}(\mathrm{Im}(L)) \\ &= \dim(\mathrm{Ker}(L)) - \dim\mathrm{coker}(L) \\ &= \dim(\mathrm{Ker}(L)) - (\dim(Y) - \dim(\mathrm{Im}(L)))\end{aligned}$$

$$\begin{aligned}&= \dim(\mathrm{Ker}(L)) + \dim(\mathrm{Im}(L)) - \dim(Y)\\&= \dim(X) - \dim(Y).\end{aligned}$$

2. The identity $I : X \to X$ is a Fredholm operator of index 0.

$$\begin{aligned}\mathrm{ind}(L) &= \dim(\mathrm{Ker}(I)) - \mathrm{codim}(\mathrm{Im}(I))\\&= \dim(\mathrm{Ker}(I)) - \dim \mathrm{coker}(\mathrm{Im}(I))\\&= \dim(\mathrm{Ker}(I)) - \dim\left(\frac{X}{\mathrm{Im}(I)}\right)\\&= \dim\{0\} - \dim\{0\} = 0.\end{aligned}$$

3. If X and Y are Banach spaces and $L : X \to Y$ is a bijective linear map then L is a Fredholm operator of index 0. Indeed, it follows from the bijectivity that $\mathrm{Ker}(L) = \{0\}_X$, whose dimension is null, and $\mathrm{Im}(L) = Y$ and its codimension is null, thus

$$\begin{aligned}\mathrm{ind}(L) &= \dim(\mathrm{Ker}(L)) - \mathrm{codim}(\mathrm{Im}(L))\\&= \dim(\mathrm{Ker}(L)) - \dim\left(\frac{Y}{\mathrm{Im}(L)}\right)\\&= \dim\{0\}_X - \dim\{0\} = 0.\end{aligned}$$

Theorem 14 If L is a Fredholm operator, K is a compact linear map, then $L + K$ is Fredholm and

$$\mathrm{ind}(L + K) = \mathrm{ind}(L).$$

In particular, any compact perturbation of the identity is a Fredholm operator of index 0.

Proposition 2 If L is a Fredholm operator of index null, then L is surjective if and only if L is injective.

Proof. If L is surjective, then $\mathrm{Im}(L) = Y = Y + \{0\}$ and consequently, $\dim\{0\} = \dim(\mathrm{Ker}(L)) = 0$, so $\mathrm{Ker}(L) = \{0\}$, hence L is injective. $\square$

In all that follows (unless mentioned otherwise) $L : \mathrm{dom}(L) \subset X \to Y$ denotes a Fredholm operator of index 0. If L is Fredholm, then according to the above, there exist two continuous projections, $P : X \to X$ and $Q : Y \to Y$ such that

$$\mathrm{Im}(P) = \mathrm{Ker}(L), \quad \mathrm{Ker}(Q) = \mathrm{Im}(L).$$

We set

$$X_1 = \mathrm{Im}(I - P) = \mathrm{Ker}(P) \quad \text{and} \quad Y_1 = \mathrm{Im}(Q)$$

then we can write

$$X = \mathrm{Ker}(L) \oplus X_1; \quad Y = \mathrm{Im}(L) \oplus Y_1,$$

Consider an isomorphism

$$J : \mathrm{Ker}(L) \to \mathrm{Im}(Q).$$

whose existence is ensured by the fact that $\dim \ker(L) = \dim \mathrm{Im}(Q) = n$. Notice that

$$\mathrm{dom}(L) = \mathrm{Ker}(L) \oplus (\mathrm{dom}(L) \cap X_1).$$

And that the restriction of L to $\mathrm{dom}(L) \cap X_1$ is an isomorphism onto $\mathrm{Im}(L)$, let us denote by L_p this restriction, i.e., $L_p : \mathrm{dom}(L) \cap X_1 \to \mathrm{Im}(L)$, then

Lemma 5

L_p is an algebraic isomorphism.

Proof. 1- **Let us show that L^p is injective:**
Let $x \in Ker(L_p) \subset Ker(L) = Im(P)$, then there exists $y \in Dom(P)$ such that $x = Py$. Since P is a projection, we obtain

$$x = Py = P^2y = P(Py) = Px = 0.$$

Consequently, $x = 0$, and thus $Ker(L_p) = \{0\}$, which signifies the injectivity of L^p.

2- **The surjectivity of L^p:**
Since P is a projection, we can write the vector space X as a direct sum:

$$X = Ker(P) \oplus Im(P) = Ker(P) \oplus Ker(L).$$

Take $z \in Im(L)$, so there exists $x \in dom(L) \subset X$ such that $Lx = z$. Since $X = Ker(P) \oplus Ker(L)$, there exist two unique elements

$$\{e \in Ker(P) \text{ and } f \in Ker(L) \text{ such that } x = e + f\}.$$

We have

$$z = Lx = L(e + f) = Le + Lf = Le + 0 = Le$$

thus $e \in dom(L)$. Finally, we obtain $e \in dom(L)$, $e \in Ker(P)$ and $L_p e = z$, which proves that L_p is indeed surjective. □

Let $K_p : Imf(L) \subset Y \to dom(L) \cap Ker(P)$ be bijective, defined by $K_p := L_p^{-1}$, such that $PK_p = 0$, and that it satisfies the properties.

Lemma 6

1. On Im(L), $wehave$LK$_p = I$.

2. On dom(L), $wehave$K$_p L = (I - P)$.

Proof (1) Let $x \in Im(L)$, then $LK_p x = L(K_p(x)) = L_p(K_p(x)) = Ix$.
(2) Since Im(P) = Ker(L), $then$LP $= 0,$ $andconsequently$K$_p L = K_p L(I - P)$.
Therefore, showing (2) is equivalent to verifying $K_p L(I - P) = K_p L_p(I - P)$.
If we have Im(I - P) $\subset dom(L_p) = dom(L) \cap (P)$, then the result follows.
Let $x \in dom(L)$. Since $P(x) \in Ker(L) \subset dom(L)$ and dom(L)$isavectorsubspaceof$X, $wehave$(x - Px) $\in dom(L)$.
Since $P(x - Px) = P(x) - P^2(x) = P(x) - P(x) = 0$, then $(x - Px) \in Ker(P)$ and consequently $(x - Px) \in dom(L) \cap Ker(P)$.

From this we obtain Im(I - P) $\subset dom(L) \cap Ker(P)$. Hence, using (1):

$$K_pL(I-P) = K_pL_p(I-P).$$

It follows.

Now define the operator $K_{P,Q} : Y \to X$, with $K_{P,Q} = L_p^{-1}(I-Q)$.

Lemma 7 The operator $L + JP : dom(L) \to Y$ is an isomorphism and

$$(L+JP)^{-1} = K_{P,Q} + J^{-1}Q.$$

In particular,

$$(L+JP)^{-1}x = J^{-1}x \quad \text{for all } x \in Im(Q).$$

Lemma 8 If $N : \Delta \subset X \to X$ is a map, the problem

$$x \in \mathrm{dom}(L) \cap \Delta \quad , \quad Lx = Nx.$$

is equivalent to the fixed point problem

$$x \in \Delta, \quad x = Px + J^{-1}QNx + K_{P,Q}Nx.$$

Proof. We have

$$\begin{aligned} &[x \in \mathrm{dom}(L) \cap \Delta \quad , \quad Lx = Nx] \\ \Leftrightarrow &[x \in \mathrm{dom}(L) \cap \Delta \quad , \quad (L+JP)x = (N+JP)x] \\ \Leftrightarrow &[x \in \Delta \quad , \quad x = (L+JP)^{-1}(N+JP)x]. \end{aligned}$$

On the other hand

$$\begin{aligned} (L+JP)^{-1}(N+JP) &= (K_{P,Q} + J^{-1}Q)(N+JP) \\ &= K_{P,Q}N + K_{P,Q}JP + J^{-1}QN + J^{-1}QJP. \end{aligned}$$

Since $\mathrm{Im}(J) = \mathrm{Im}(Q) = \mathrm{Ker}(I - Q)$, it follows that

$$K_{P,Q}JP = L_p^{-1}(I - Q)JP = 0.$$

Since $Q|_{\mathrm{Im}(Q)} = I|_{\mathrm{Im}(Q)}$ and $\mathrm{Im}(J) = \mathrm{Im}(Q)$, we deduce that

$$J^{-1}QJP = J^{-1}JP = P.$$

Consequently, $(L + JP)^{-1}(N + JP) = P + J^{-1}QN + K_{P,Q}N$. □

Let X, Y be two Banach spaces and $L : \mathrm{dom}(L) \subset X \to Y$ a Fredholm operator of index 0.

Definition 19:(L-compact mapping) Let Ω be a bounded open subset of X such that $\mathrm{dom}(L) \cap \Omega \neq \emptyset$. The mapping $N : X \to Y$ is said to be L-compact on $\overline{\Omega}$ if:

1. $QN(\overline{\Omega})$ is bounded.
2. $K_{P,Q}N : \overline{\Omega} \to X$ is compact.

Definition 20:(Mawhin's degree) If the operators L and N satisfy the properties mentioned above, then the coincidence degree of L and N in Ω is defined by:

$$\deg[(L, N), \Omega] = \deg_{LS}(I - M, \Omega, 0)$$

where M denotes the quantity given by:

$$M(P, J, Q) = P + J^{-1}QN + K_{P,Q}N$$

3.5 Proof of Mawhin's Theorem

Theorem 15 [Mawhin's Theorem] Let L be a Fredholm operator of index zero, and let N be L-compact on $\overline{\Omega}$. Suppose that the following conditions are satisfied:

- **(i)** $Lx \neq Nx$ for all $[(x, \lambda) \in [(\text{dom}(L) \setminus \text{Ker}(L)) \cap \partial\Omega] \times]0, 1[$.
- **(ii)** $QNx \neq 0$ for all $x \in \text{Ker}(L) \cap \partial\Omega$.
- **(iii)** $\deg_B(J^{-1}QN|_{\text{Ker}(L)}, \Omega \cap \text{Ker}(L), 0) \neq 0$, where $Q : Y \to Y$ is the projection defined above with $\text{Im}(L) = \text{Ker}(Q)$.

Then the equation $Lx = Nx$ admits at least one solution in $\text{dom}(L) \cap \overline{\Omega}$.

Proof. For $\lambda \in [0, 1]$, let us consider the family of problems:

$$x \in \text{dom}(L) \cap \overline{\Omega}, \quad Lx = \lambda Nx + (1 - \lambda)QNx \tag{1}$$

Let $M : [0, 1] \times \overline{\Omega} \to Y$ be a homotopy defined by:

$$M(\lambda, x) = Px + J^{-1}QNx + K_{P,Q}Nx$$

By virtue of Lemma (3.4.4), problem (1) is equivalent to a fixed point problem $x \in \overline{\Omega}$ and:

$$\begin{aligned} x &= Px + J^{-1}Q(\lambda N + (1 - \lambda)QN)x + K_{P,Q}(\lambda N + (1 - \lambda)QN)x \\ &= Px + J^{-1}QNx + (1 - \lambda)J^{-1}QNx + K_{P,Q}Nx + (1 - \lambda)K_{P,Q}QNx \\ &= M(\lambda, x). \end{aligned}$$

Therefore, this last equation is equivalent to a fixed point problem:

$$x \in \overline{\Omega}, \quad x = M(\lambda, x) \tag{2}$$

If there exists an $x \in \partial\Omega$ such that $Lx = Nx$, then we are done. Now let us suppose that:

$$Lx \neq Nx \quad \text{for all } x \in \operatorname{dom}(L) \cap \partial\Omega, \tag{3}$$

and on the other hand:

$$Lx \neq \lambda Nx + (1-\lambda)QNx. \tag{4}$$

For all $(\lambda, x) \in]0,1[\times (\operatorname{dom}(L) \cap \partial\Omega)$. If:

$$Lx = Nx + (1-\lambda)QNx$$

for all $(\lambda, x) \in]0,1[\times (\operatorname{dom}(L) \cap \partial\Omega)$, we obtain by applying Q to both sides of the preceding equality:

$$QNx = 0, \quad Lx = Nx.$$

The first of these equalities and condition **(ii)** imply that $x \notin \operatorname{Ker}(L) \cap \partial\Omega$, i.e., $x \in \partial\Omega \cap \operatorname{dom}(L) \setminus \operatorname{Ker}(L)$ and thus the second equality contradicts **(i)**. Using **(ii)** once again, it follows that:

$$Lx \neq QNx \quad \text{for all } x \in \operatorname{dom}(L) \cap \partial\Omega. \tag{5}$$

By virtue of (3), (4) and (5), we deduce that:

$$x \neq M(\lambda, x) \quad \text{for all } (\lambda, x) \in [0,1] \times \partial\Omega. \tag{6}$$

It is easy to verify that $M(\lambda, x)$ is compact because N is L-compact on $\overline{\Omega}$. Therefore, by using the homotopy invariance property of the Leray-Schauder degree, we obtain:

$$\deg_{LS}(I - M(0,\cdot), \Omega, 0) = \deg_{LS}(I - M(1,\cdot), \Omega, 0). \tag{7}$$

On the other hand, we have:

$$\deg_{LS}(I - M(0,\cdot),\Omega,0) = \deg_{LS}(I - (P + J^{-1}QN),\Omega,0). \tag{8}$$

But the range of $P + J^{-1}QN$ is contained in $\mathrm{Ker}(L)$, from which, by using the degree reduction property of the Leray-Schauder degree and the fact that $P|_{\mathrm{Ker}(L)} = I|_{\mathrm{Ker}(L)}$ (since $\mathrm{Ker}(L) = \mathrm{Im}(P) = \mathrm{Ker}(I - P)$), we obtain:

$$\begin{aligned}\deg_{LS}(I - (P + J^{-1}QN),\Omega,0) &= \deg_{B}(I - (P + J^{-1}QN),\Omega \cap \mathrm{Ker}(L),0) \quad (9)\\ &= \deg_{B}(J^{-1}QN,\Omega \cap \mathrm{Ker}(L),0). \quad (10)\end{aligned}$$

By virtue of (7), (8), and (10), it follows that $\deg_{LS}(I - M(1,\cdot),\Omega,0) \neq 0$, and therefore the existence property of the Leray-Schauder degree implies the existence of an $x \in \overline{\Omega}$ such that:

$$x = M(1,x) \quad \text{i.e., } x \in \mathrm{dom}(L) \cap \Omega, \quad Lx = Nx.$$

$\square$

Chapter 3

Application Of Theorem

3.1 Introduction

Fractional differential equations have gained significant attention due to their ability to model complex phenomena with memory and hereditary properties, which are common in engineering, physics, biology, and mechanics. This chapter is devoted to the study of a fractional boundary value problem (FBVP) at resonance. A boundary value problem is said to be *at resonance* if the associated homogeneous linear problem has a nontrivial solution. In such cases, standard inversion techniques fail, and more advanced tools, such as Mawhin's coincidence degree theory, are required.[10][11]

We consider the following nonlinear FBVP with Caputo fractional derivative:

$$\begin{cases} {}^{C}\mathcal{D}_{a+}^{\alpha}x(t) - f(t, x(t), x'(t), x''(t)) = 0, & t \in [a, b], \\ x(a) - \beta x'(a) = 0, \quad x'(a) = x'(b), \quad x''(a) = 0, & \end{cases} \tag{1}$$

where $\beta \in \mathbb{R}$, $0 \leq a < b$, $2 < \alpha < 3$, and $f : [a, b] \times \mathbb{R}^3 \to \mathbb{R}$ is continuous. The operator ${}^{C}\mathcal{D}_{a+}^{\alpha}$ denotes the Caputo fractional derivative of order α.

The main goal of this chapter is to prove the existence of at least one solution to (1) by rewriting it as an abstract operator equation $Lx = Nx$, where L is a linear Fredholm operator of index zero and N is a nonlinear operator. The key difficulty is that L is non-invertible due to resonance; indeed, $\dim\ker L = 1$. This chapter follows the work of Silva [12].

3.2 Preliminaries

We first recall essential definitions from fractional calculus and functional analysis. Let $X = C^2([a,b])$ with norm $\|x\|_X = \max_{t\in[a,b]}\{|x(t)| + |x'(t)| + |x''(t)|\}$, and $Y = C([a,b])$ with norm $\|y\|_Y = \max_{t\in[a,b]}|y(t)|$. Both are Banach spaces.

Definition 21: The Riemann-Liouville fractional integral of order $\alpha > 0$ for a function u is

$$I_a^\alpha u(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}u(s)\,ds,$$

where Γ is the Euler gamma function.

Lemma 9 For $n-1 < \alpha < n$ and $f \in C^n([a,b])$,

$$I_a^\alpha\left({}^C\mathcal{D}_a^\alpha f\right)(t) = f(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(t-a)^k.$$

Also, ${}^C\mathcal{D}_a^\alpha I_a^\alpha f = f$.

3.2.1 Mawhin's Coincidence Theory

Definition 22 A linear operator $L : \operatorname{dom} L \subset X \to Y$ is a *Fredholm operator of index zero* if:

1. $\operatorname{Im} L$ is closed in Y,

2. $\dim\ker L = \operatorname{codim}\operatorname{Im} L < \infty$.

If L is Fredholm of index zero, there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $\operatorname{Im} P = \ker L$ and $\ker Q = \operatorname{Im} L$. Then $L|_{\operatorname{dom} L \cap \ker P} : \operatorname{dom} L \cap \ker P \to \operatorname{Im} L$ is invertible; its inverse is denoted K_P.

Definition 23 Let $\Omega \subset X$ be open bounded. A nonlinear operator N is *L-compact on* $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N : \overline{\Omega} \to X$ is completely continuous.

Theorem 16 [Mawhin's continuation theorem] Let L be Fredholm of index zero, N L-compact on $\overline{\Omega}$, and suppose:

1. $Lx \neq \lambda Nx$ for all $x \in \partial\Omega \cap (\operatorname{dom} L \setminus \ker L)$, $\lambda \in (0,1)$;
2. $Nx \notin \operatorname{Im} L$ for all $x \in \ker L \cap \partial\Omega$;
3. $\deg(QN|_{\ker L}, \Omega \cap \ker L, 0) \neq 0$.

Then $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$.

3.3 Main Results: Operator Formulation and Fredholm Property

Define $L : \operatorname{dom} L \subset X \to Y$ by

$$(Lx)(t) ={}^C \mathcal{D}^{\alpha}_{a+} x(t), \quad t \in [a,b],$$

with

$$\operatorname{dom} L = \{x \in X :{}^C \mathcal{D}^{\alpha}_{a+} x \in Y,\ x(a) = \beta x'(a),\ x'(a) = x'(b),\ x''(a) = 0\}.$$

Define $N : X \to Y$ by $(Nx)(t) = f(t, x(t), x'(t), x''(t))$. Then problem (1) becomes

$$Lx = Nx, \quad x \in \operatorname{dom} L.$$

3.3.1 Characterization of Kernel and Image

Lemma 10

$$\ker L = \{x \in X : x(t) = c_1(t - a + \beta),\ t \in [a,b],\ c_1 \in \mathbb{R}\},$$

$$\operatorname{Im} L = \left\{ y \in Y : \int_a^b (b-s)^{\alpha-2} y(s)\, ds = 0 \right\}.$$

Proof. If $Lx = 0$, Lemma 9 gives $x(t) = c_0 + c_1(t-a) + c_2(t-a)^2$. The condition $x''(a) = 0$ forces $c_2 = 0$. Then $x(a) = \beta x'(a)$ gives $c_0 = \beta c_1$. The condition $x'(a) = x'(b)$ is automatically satisfied. Hence $\ker L = \{c_1(t - a + \beta)\}$.

For $y \in \operatorname{Im} L$, there exists x such that $Lx = y$. Using Lemma 9 and the boundary conditions, we obtain the integral condition $\int_a^b (b-s)^{\alpha-2} y(s) ds = 0$. Conversely, if y satisfies that condition, define $x(t) = I_a^\alpha y(t)$; one verifies $x \in \operatorname{dom} L$ and $Lx = y$. □

Thus $\dim \ker L = 1$ and $\operatorname{codim} \operatorname{Im} L = 1$, so L is Fredholm of index zero.

3.3.2 Projectors and the Inverse K_P

Define $P : X \to X$ and $Q : Y \to Y$ by

$$(Px)(t) = x'(a)(t - a + \beta), \quad t \in [a,b],$$

$$(Qy)(t) = \frac{\alpha - 1}{(b-a)^{\alpha-1}} \int_a^b (b-s)^{\alpha-2} y(s)\, ds, \quad t \in [a,b].$$

It is straightforward to verify that $P^2 = P$, $Q^2 = Q$, $\operatorname{Im} P = \ker L$, $\ker Q = \operatorname{Im} L$, and $X = \ker L \oplus \ker P$, $Y = \operatorname{Im} L \oplus \operatorname{Im} Q$.

Define $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ by

$$(K_P y)(t) = I_a^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} y(s)\, ds.$$

Then $K_P = (L|_{\operatorname{dom} L \cap \ker P})^{-1}$.

Lemma 11 For any $y \in \operatorname{Im} L$, $\|K_P y\|_X \leq \eta \|y\|_\infty$, where

$$\eta = \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} + \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(b-a)^{\alpha-2}}{\Gamma(\alpha-1)}.$$

The proof follows from standard bounds on fractional integrals.

3.4 Existence Theorem via Coincidence Degree

We impose the following assumptions:

(H1) There exist nonnegative continuous functions $\gamma_1, \gamma_2, \gamma_3, \delta$ such that

$$|f(t,u,v,w)| \leq \gamma_1(t)|u| + \gamma_2(t)|v| + \gamma_3(t)|w| + \delta(t),$$

with $p_i = \|\gamma_i\|_\infty$, $q = \|\delta\|_\infty$, and $\eta \cdot \max\{p_1, p_2, p_3\} < 1$.

(H2) There exists $R > 0$ such that for any $x \in \operatorname{dom} L$, if $|x'(t)| > R$ for all $t \in [a,b]$, then

$$\int_a^b (b-s)^{\alpha-2} f(s, x(s), x'(s), x''(s))\, ds \neq 0.$$

(H3) There exists $R^* > 0$ such that for $|c_1| > R^*$ and for all $t \in [a,b]$, either

$$c_1 f(t, c_1(t-a+\beta), c_1, 0) > 0 \quad \text{or} \quad < 0.$$

3.4.1 Key Lemmas for Boundedness

Lemma 12 If (H1) and (H2) hold, then $\Omega_1 = \{x \in \operatorname{dom} L \setminus \ker L : Lx = \lambda N x,\ \lambda \in (0,1)\}$ is bounded.

Proof. For $x \in \Omega_1$, we have $Nx \in \operatorname{Im} L$, so $Q(Nx) = 0$ gives the integral condition in (H2). Thus $|x'(t)| \leq R$ for all t. Then

$$\|Px\|_X \leq (b - a + |\beta| + 1)R,$$

and using $x = Px + K_P L(I - P)x$, we obtain

$$\|x\|_X \leq \|Px\|_X + \eta\|Lx\|_\infty \leq (b - a + |\beta| + 1)R + \eta\|Nx\|_\infty.$$

By (H1), $\|Nx\|_\infty \leq p\|x\|_X + q$ with $p = \max p_i$. Since $\eta p < 1$, solving the inequality yields a uniform bound. $\square$

Lemma 13 Under (H2), $\Omega_2 = \{x \in \ker L : Nx \in \operatorname{Im} L\}$ is bounded.

Proof. If $x(t) = c_1(t - a + \beta)$, then $Nx \in \operatorname{Im} L$ implies the same integral condition as before, so $|c_1| \leq R$. Hence $\|x\|_X \leq (b - a + |\beta| + 1)R$. $\square$

Lemma 14 Under (H2) and the first alternative of (H3), the set $\Omega_3 = \{x \in \ker L : \lambda x + (1 - \lambda)QNx = 0,\ \lambda \in [0, 1]\}$ is bounded.

Proof. For $x(t) = c_1(t - a + \beta)$, the equation becomes

$$\lambda c_1(t - a + \beta) + (1 - \lambda)\frac{\alpha - 1}{(b - a)^{\alpha-1}} \int_a^b (b - s)^{\alpha-2} f(s, c_1(s - a + \beta), c_1, 0)ds = 0.$$

If $\lambda = 0$, (H2) forces $|c_1| \leq R$. If $\lambda > 0$ and $|c_1| > R^*$, the positivity condition in (H3) leads to a contradiction (the left side becomes positive). Hence $|c_1| \leq \max\{R, R^*\}$. $\square$

A similar lemma holds for the second alternative of (H3) by considering $-\lambda x + (1 - \lambda)QNx = 0$.

3.4.2 Main Existence Theorem

Theorem 17 Let $f : [a, b] \times \mathbb{R}^3 \to \mathbb{R}$ be continuous. If conditions (H1), (H2), and (H3) hold, then the fractional boundary value problem (1) has at least one solution in X.

Proof. Take Ω a bounded open set containing $\overline{\Omega}_1 \cup \overline{\Omega}_2 \cup \overline{\Omega}_3$. By the previous lemmas, the three conditions of Mawhin's theorem are satisfied. In particular, for the degree condition, we use the homotopy $H(x, \lambda) = \pm\lambda x + (1-\lambda)QNx$ which is nonzero on $\partial\Omega \cap \ker L$; hence

$$\deg(QN|_{\ker L}, \Omega \cap \ker L, 0) = \deg(\pm I, \Omega \cap \ker L, 0) = \pm 1 \neq 0.$$

Thus $Lx = Nx$ has a solution in $\operatorname{dom} L \cap \overline{\Omega}$, which is a solution to (1). $\square$

3.5 Illustrative Example

Consider the following fractional boundary value

$$\begin{cases} {}^C\mathcal{D}^{\frac{5}{2}}_{1+} x(t) = \dfrac{1}{4}\sin x(t) + \dfrac{1}{5}x'(t) + \dfrac{1}{10}e^{-|x''(t)|}, \quad t \in [1, 2], \\ x(1) = \dfrac{1}{2}x'(1), \ x'(1) = x'(2), \ x''(1) = 0, \end{cases} \tag{2}$$

where $\alpha = 5/2$, $\beta = 1/2$ and

$$f(t, x(t), x'(t), x''(t)) = \frac{1}{4}\sin x(t) + \frac{1}{5}x'(t) + \frac{1}{10}e^{-|x''(t)|}$$

is a continuous function. It follows that

$$|f(t, u, v, w)| \leq \frac{1}{4}|u(t)| + \frac{1}{5}|v(t)| + \frac{1}{10}|w(t)|,$$

with $p_1 = 1/4$, $p_2 = 1/5$, $p_3 = 1/10$ and $q = 0$. It follows that $p = \max\{p_1, p_2, p_3\} = 1/4$. Moreover, $\eta = 58/(15\sqrt{\pi})$ and consequently, $p\eta < 1$. Thus, condition (H1) is verified.

Let $R=2$, and for any $x\in \mathrm{dom}L$, assume $|x'(t)|>R$ holds for $t\in[1,2]$. From the continuity of x', either $x'(t)>R$ or $x'(t)<-R$ for $t\in[1,2]$.

If $x'(t)>2$, one has

$$\begin{aligned}&\int_1^2 (2-s)^{3/2}\left[\frac{1}{4}\sin x(t)+\frac{1}{5}x'(t)+\frac{1}{10}e^{-|x''(t)|}\right]\mathrm{d}s\\ >&\left(-\frac{1}{4}+\frac{2}{5}\right)\int_1^2 (2-s)^{3/2}\mathrm{d}s=\frac{3}{50}>0.\end{aligned}$$

If $x'(t)<-2$, one has

$$\begin{aligned}&\int_1^2 (2-s)^{3/2}\left[\frac{1}{4}\sin x(t)+\frac{1}{5}x'(t)+\frac{1}{10}e^{-|x''(t)|}\right]\mathrm{d}s\\ <&\left(\frac{1}{4}-\frac{2}{5}+\frac{1}{10}\right)\int_1^2 (2-s)^{3/2}\mathrm{d}s=-\frac{1}{50}<0\end{aligned}$$

Thus, for $|x'(t)|>2$,

$$\int_1^2 (2-s)^{3/2}f(s,u(s),u'(s),u''(s))\mathrm{d}s\neq 0$$

and condition (H2) is verified.

Finally, we observe that

$$f\left(t,c_1\left(t-\frac{1}{2}\right),c_1,0\right)=\frac{1}{4}\sin\left(c_1\left(t-\frac{1}{2}\right)\right)+\frac{1}{5}c_1+\frac{1}{10}.$$

Take $R^*=1$ and assume $|c_1|>1$. Thus, if $c_1>1$,

$$c_1f\left(t,c_1\left(t-\frac{1}{2}\right),c_1,0\right)>-\frac{1}{4}+\frac{1}{5}+\frac{1}{10}=\frac{1}{20}>0$$

and if $c_1<-1$, one has

$$c_1f\left(t,c_1\left(t-\frac{1}{2}\right),c_1,0\right)<-\left(\frac{1}{4}-\frac{1}{5}+\frac{1}{10}\right)=-\frac{3}{20}<0.$$

Therefore, condition (H3) is verified.

It follows from Theorem 3 that fractional boundary value problem has at least one solution.

3.6 Conclusion

In this chapter, we have studied the existence of solutions for a fractional boundary value problem at resonance involving the Caputo derivative of order $2 < \alpha < 3$. By constructing appropriate projections and using Mawhin's coincidence degree theory, we proved an existence theorem under suitable growth and sign conditions. An example demonstrated the applicability of the result. Future work could extend this to higher-order fractional systems or to problems with different resonant boundary conditions.

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