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Department of Physics

MASTER THESIS

Specialty : **Theoretical Physics**

Study and Simulation of a Bidirectional Cyclic Quantum Teleportation Protocol via a Noisy Channel

Presented by :

GHOUINI Cherifa

Jury Committee :

Pr. Lamine Khodja	President	Univ. Ouargla
Dr. Khaled Khalfaoui	Supervisor	Univ. Jijel
Pr. Tahar Boudjedaa	Co-supervisor	Univ. Jijel
Pr. Benzaier Hedjira	Examiner	Univ. Ouargla

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Abstract

This master's thesis investigates the implementation and performance of a Bidirectional Cyclic Quantum Teleportation (BCQT) protocol within an 11-qubit quantum system. Unlike standard unidirectional protocols, this cyclic approach enables multiple parties (Alice, Bob, and Charlie) to exchange unknown quantum states simultaneously using a pre-shared entangled resource. A primary focus of this work is evaluating the protocol's robustness under realistic conditions, specifically through a noisy channel modeled as a Werner state. The theoretical framework employs density operator formalism to track the evolution of the global system state during Bell-state measurements and controller-dependent operations. Furthermore, we provide a symbolic simulation using Wolfram Mathematica to verify the operational validity of the protocol. Our results analyze how the entanglement parameter (λ) and environmental noise affect the quality of quantum information transfer. This study contributes to the field of quantum communication networks by demonstrating the scalability of cyclic protocols and the limitations imposed by decoherence in multi-party quantum information processing.

Résumé

Ce mémoire de master étudie la mise en œuvre et les performances d'un protocole de Téléportation Quantique Cyclique Bidirectionnelle (BCQT) au sein d'un système à 11 qubits. Contrairement aux protocoles unidirectionnels standards, cette approche cyclique permet à plusieurs parties (Alice, Bob et Charlie) d'échanger simultanément des états quantiques inconnus en utilisant une ressource intriquée préalablement partagée. L'objectif principal de ce travail est d'évaluer la robustesse du protocole dans des conditions réalistes, notamment à travers un canal bruité modélisé par un état de Werner.

Le cadre théorique utilise le formalisme de l'opérateur densité pour suivre l'évolution de l'état global du système lors des mesures de l'état de Bell et des opérations dépendantes du contrôleur. De plus, nous proposons une simulation symbolique utilisant Wolfram Mathematica afin de vérifier la validité opérationnelle du protocole. Nos résultats analysent comment le paramètre d'intrication (λ) et le bruit environnemental affectent la qualité du transfert d'information quantique. Cette étude contribue au domaine des réseaux de communication quantique en démontrant la faisabilité des protocoles cycliques et les limitations imposées par la décohérence dans le traitement de l'information quantique multi-parties.

الملخص

تدرس هذه المذكرة تنفيذ وأداء بروتوكول الانتقال الآني الكمي الحلقي ثنائي الاتجاه ضمن نظام مكون من ١١ كيوبت. على عكس البروتوكولات أحادية الاتجاه القياسية، يسمح هذا النهج الحلقي لعدة أطراف (أليس، وبوب، وتشارلي) بتبادل حالات كمية مجهولة بشكل متزامن باستخدام مورد متشابك مشترك مسبقاً. الهدف الرئيسي من هذا العمل هو تقييم متانة البروتوكول في ظل ظروف واقعية، لا سيما من خلال قناة مشوشة تمت نمذجتها بواسطة حالة ويرنر .

يستخدم الإطار النظري شكليات مشغل الكثافة لمتابعة تطور الحالة الإجمالية للنظام أثناء قياسات حالة بيل والعمليات التي تعتمد على المراقب. بالإضافة إلى ذلك، نقترح محاكاة رمزية باستخدام برنامج ويلفرام ماتيماتيكا من أجل التحقق من الصلاحية التشغيلية للبروتوكول. تُحلل نتائجنا كيف يؤثر معامل التشابك (λ) والضوضاء البيئية على جودة نقل المعلومات الكمية. تساهم هذه الدراسة في مجال شبكات الاتصالات الكمية من خلال إثبات جدوى البروتوكولات الحلقية والقيود التي يفرضها تماسك الطور في معالجة المعلومات الكمية متعددة الأطراف.

Dedication and Acknowledgements

Dedication

To Allah Almighty,
My Creator and Guide, for granting me the strength and perseverance to complete this work. All praise is to Him for His infinite blessings.

To my beloved parents,
For their unconditional love, endless sacrifices, and prayers that have been my source of strength throughout this journey.

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My constant support and companion, for always being there for me.

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My esteemed teacher and role model, for her invaluable support, guidance, and for being a beacon of knowledge in my academic path.

To myself,
For the patience, resilience, and determination in the face of complex equations and the challenges of theoretical physics. I dedicate this

achievement to my own persistence.

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Chapitre 1

General Introduction

Quantum Information is a cutting-edge field that combines the principles of quantum mechanics with information theory to process and transmit data in ways that classical computers cannot. Unlike classical bits (0 or 1), quantum information is processed via Qubits, which can exist in multiple states simultaneously due to Superposition.

In the realm of Communications, quantum information plays a transformative role :

Absolute Security : It enables the creation of unhackable communication channels through Quantum Key Distribution (QKD).

Quantum Teleportation : This allows the transfer of quantum states between distant parties using Entanglement, forming the backbone of the future Quantum Internet.

Bidirectional Cyclic Quantum Teleportation Protocol

The Bidirectional Cyclic Quantum Teleportation protocol is an advanced scheme where multiple parties (Alice, Bob, and Charlie) exchange quantum information simultaneously in a circular or cyclic manner.

Mechanism : It utilizes a multi-qubit entangled state (such as a GHZ state, cluster state, or a composite channel) as a shared resource.

Cyclic Exchange : Instead of a simple one-way transfer, the protocol allows Alice to send a state to Bob, Bob to Charlie, and Charlie back to Alice in a single synchronized process.

Efficiency : This approach is more resource-efficient than performing multiple independent one-way teleportation steps, as it reuses the entangled resource for the entire network loop.

Realistic Scenarios and Noise Models

In a real-world quantum network, communication is often hindered by environmental interactions. To simulate this, our study incorporates :

Operational Scenarios : We analyze the protocol under a three-party (Alice-Bob-Charlie) configuration, ensuring that the cyclic exchange is maintained even under complex channel conditions.

Clockwise cyclic controlled teleportation : $|\Psi\rangle_A$ is sent to Charlie, $|\Psi\rangle_B$ is sent

to Alice and $|\Psi\rangle_C$ is sent to Bob.

Counter-clockwise cyclic controlled teleportation : $|\Psi\rangle_A$ is sent to Bob,

$|\Psi\rangle_B$ is sent to Charlie and $|\Psi\rangle_C$ is sent to Alice.

Werner Noise Model : To account for quantum decoherence, the protocol is implemented using the Werner State as the communication channel. This model introduces a mixing parameter (λ) that represents the purity of the state, allowing us to evaluate the protocol's robustness against noise.

$$\bar{\rho}_{channel} = \lambda \rho_{channel} + \frac{(1-\lambda)}{2^{11}} I^{\otimes 11} \tag{1.1}$$

Implementation in Wolfram MathematicaWolfram Mathematica is the preferred tool for simulating these protocols because of its powerful symbolic computation and matrix manipulation capabilities. The implementation involves :

Defining the Quantum Resource :

Using KroneckerProduct to construct the multi-qubit entangled state (the quantum channel) in matrix form.State Preparation : Defining the unknown input states as symbolic vectors :

$$|\Psi\rangle = \alpha |0\rangle + \beta |1\rangle \tag{1.2}$$

Applying Quantum Gates : Implementing local operations using matrix multiplication (Dot) with standard gates like CNOT and Hadamard.

Measurement Simulation : Using projection operators to simulate the measurement in the Bell Basis, which collapses the system into specific outcomes.

Unitary Transformations : Applying corrective Pauli gates (X, Y, Z) based on the measurement results to recover the original state at the destination.

Simplification : Utilizing functions like FullSimplify to verify that the output state mathematically matches the input state, confirming the protocol's success.

Chapitre 2

Quantum Calculation

2.1 Introduction

The transition from classical to quantum information processing is fundamentally a transition in mathematical representation. While classical computing relies on Boolean algebra and discrete logic, quantum computing is built upon the rigorous framework of Linear Algebra in complex Hilbert spaces.Understanding the behavior of quantum systems, particularly the intricate nature of entanglement in 3-qubit systems, requires a robust set of mathematical tools designed to describe both pure and mixed states.

In this chapter, we establish the essential formalism required to analyze multipartite quantum systems. We begin with the Density Matrix ρ representation, which provides a comprehensive description of quantum states, including the Reduced Density Matrix obtained through the partial trace. This tool is crucial for quantifying correlations in bipartitions such as A/BC. Furthermore, we introduce the Quantum Circuit model as a framework for implementing quantum gates and operations that generate entanglement.

Central to our study is the geometric approach to entanglement, where the parallelism of vectors and the Wedge product ($\wedge$) serve as indicators for separability. These tools allow us to compute the squared concurrence and the 3-tangle (τ), providing a quantitative measure of how entanglement is distributed according to the Coffman-Kundu-Wootters (CKW) inequality. This grounding is indispensable for characterizing the "Monogamy of Entanglement" in multi-party architectures.

2.2 Basic Concebts

Qubit

A qubit, or quantum bit, is the basic unit of quantum information. Unlike a classical bit, which can only be 0 or 1, a qubit can exist in a superposition of both states at the same time.This means that a qubit can be partially 0 and partially 1 until it is measured.

The state of a qubit is described by two numbers called probability amplitudes. One amplitude corresponds to the probability of finding [1] the qubit in state 0, and the other corresponds to the probability of finding it in state 1. The probabilities are obtained by taking the square of the amplitudes, and the total probability is always equal to 1.

For example, if the amplitudes are $3b1$ and $3b2$, the probabilities of measuring the qubit in state 0 or 1 are $|3b1|$ and $|3b2|$, respectively, and they satisfy the condition :

$$|\alpha|^2 + |\beta|^2 = 1 \tag{2.1}$$

A qubit can also be represented as a simple two-component column vector, where the first component represents state 0 and the second component represents state 1. The basis states are :

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.2}$$

These basic states form the foundation for all operations on qubits in quantum computing.

Quantum State

A quantum state provides a complete description of a quantum system. It contains all the information that can be known about the system and determines the probabilities of all possible measurement outcomes. In general, a quantum state can be represented as a vector in a complex vector space called a Hilbert space.

For a single qubit, the quantum state is a combination of the basis states 0 and 1. This combination is called a superposition and is expressed using probability amplitudes. The squared magnitudes of these amplitudes give the probabilities of measuring the qubit in each of the basis states.

More generally, for a quantum system with multiple states, the quantum state can be written as a sum over all possible basis states, with each term multiplied by its corresponding probability amplitude. This formalism allows quantum mechanics to describe phenomena such as interference and entanglement, which have no classical counterparts.

In summary, a quantum state is the fundamental representation of a quantum system, encoding both the possible outcomes and their likelihoods, and providing the framework for [2] predicting the behavior of the system under measurement.

Evolution of quantum state

The evolution of a quantum state describes how the state of a quantum system changes over time.In quantum mechanics, this evolution is deterministic and is governed by the Schrödinger equation.For a system with a time-independent Hamiltonian, the evolution can be expressed as a unitary transformation of the initial state.

If $|\Psi(0)\rangle$ represents the state of the system at time $t = 0$, then at a later time t the state $|\Psi(t)\rangle$ is given by :

$$|\Psi(t)\rangle = U(t)\,|\Psi(0)\rangle \tag{2.3}$$

where $U(t)$ is a unitary operator that depends on the system's Hamiltonian. The unitarity of $U(t)$ ensures that the total probability is conserved during the evolution.

For a single qubit, this means that the probability amplitudes of being in state 0 or 1 may change over time, but the sum of their squared magnitudes always remains equal to one. The ability of quantum states to evolve coherently under unitary transformations is the basis for quantum computation, allowing qubits to perform complex operations while maintaining superposition and entanglement.

In summary, the evolution of a quantum state is the fundamental mechanism by which quantum information changes over time, fully determined by the system's dynamics and described mathematically through unitary transformations.

Quantum circuit

A quantum circuit is a model for performing computation with qubits, where quantum gates are applied in sequence to manipulate the qubits' states.It is the quantum analogue of a classical logic circuit and forms the basis for implementing quantum algorithms.

The NOT gate :

The NOT gate flips a qubit : 0 becomes 1 and 1 becomes 0.It is represented by the matrix :

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.4}$$

Pauli Matrices :

The Pauli matrices are a set of three fundamental 22 matrices used to describe single-qubit operations in quantum computing. They are :

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.5}$$

σ_x : flips the qubit (NOT gate).

σ_y : flips the qubit and adds a phase.

σ_z : changes the phase of state 1.

These matrices are unitary and form the basis for many quantum operations.

The Hadamard gate :

The Hadamard gate creates a superposition of a single qubit. It transforms $|0\rangle$ into $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|1\rangle$ into $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.Its matrix representation is :

$$H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}. \tag{2.6}$$

It is widely used to initialize qubits in superposition for quantum algorithms.

The SWAP gate :

The SWAP gate exchanges the states of two qubits.If the first qubit is in state $|a\rangle$ and the second in $|b\rangle$, after the gate the first qubit becomes $|b\rangle$ and the second becomes $|a\rangle$. Its matrix representation is :

$$SWAP = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.7}$$

The SWAP gate is often used to reorder qubits in a quantum circuit.

The C_Not gate :

The C_Not gate is a two-qubit gate. It flips the second qubit (target) if the first qubit (control) is $|1\rangle$, and does nothing if the control is $|1\rangle$.Its matrix representation is :

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{2.8}$$

It is widely used to create entanglement between qubits in quantum circuits.

The Postulates of Quantum Mechanics

Postulate 1 :The Quantum State Postulate

Every isolated physical system is associated with a complex vector space called a Hilbert space. The state of the system is completely described by a unit vector in this space.

For example, a single qubit is represented by a vector :

$$|\Psi\rangle = \alpha |0\rangle + \beta |1\rangle . \tag{2.9}$$

where α β are complex probability amplitudes satisfying $|\alpha|^2 + |\beta|^2 = 1$.

This postulate establishes the fundamental framework for describing all quantum systems.

Postulate 2 :Observables and Operators

Every measurable physical quantity (observable) in a quantum system is represented by a Hermitian operator acting on the system's Hilbert space. The possible outcomes of measuring the observable are given by the operator's eigenvalues, and the system collapses into the corresponding eigenstate after the measurement.

For example, if A is an observable with eigenstates $|a_i\rangle$ and eigenvalues a_i, then measuring A on a state $|\Psi\rangle$ gives the outcome a_i with probability $|\langle a_i| |\Psi\rangle|^2$.

This postulate links the mathematical formalism of quantum mechanics to physically measurable quantities.

Postulate 3 :The Measurement Postulate

When a measurement is performed on a quantum system, the outcome corresponds to one of the eigenvalues of the observable being measured.After the measurement, the system collapses into the corresponding eigenstate.

The probability of obtaining a specific measurement result is given by the squared magnitude of the projection of the quantum state onto the eigenstate :

$$P(a_i) = |\langle a_i| |\Psi\rangle|^2 . \tag{2.10}$$

This postulate connects the mathematical description of quantum states to actual experimental observations.

Postulate 4 :Time Evolution

The time evolution of a closed quantum system is deterministic and is governed by the Schrödinger equation. If $|\Psi(t_0)\rangle$ is the state at time t_0, then at a later time t, the state $|\Psi(t)\rangle$ is given by :

$$|\Psi(t)\rangle = U(t, t_0)\,|\Psi(t_0)\rangle\,. \tag{2.11}$$

where $U(t, t_0)$ is a unitary operator that depends on the system's Hamiltonian. The unitarity ensures that the total probability is conserved during evolution.

This postulate describes how quantum states change over time and forms the basis for predicting the dynamics of quantum systems.

Projective Measurement

Projective measurement is a fundamental concept in quantum mechanics used to determine the state of a quantum system.When a measurement is performed, the quantum state collapses onto one of the eigenstates of the observable being measured.

If the system is in the state $|\Psi\rangle$, the probability of obtaining the outcome associated with eigenstate $|a_i\rangle$ is given by :

$$P(a_i) = |\langle a_i|\,|\Psi\rangle|^2\,. \tag{2.12}$$

After the measurement, the system collapses into the corresponding eigenstate $|a_i\rangle$.

Projective measurements therefore determine both the measurement outcome and the new state of the quantum system after the measurement.

Expectation Value of Measurement

The expectation value of a measurement represents the average value of an observable obtained from many repeated measurements on identically prepared quantum systems. If A is an observable represented by the operator A, and the system is in the state $|\Psi\rangle$, the expectation value is defined as :

$$\langle A\rangle = \langle\Psi|\,A\,|\Psi\rangle\,. \tag{2.13}$$

This quantity corresponds to the statistical mean of the measurement outcomes and provides important information about the physical properties of the system.

To justify this expression, consider that the observable A has eigenstates $|a_i\rangle$ with corresponding eigenvalues a_i. According to the measurement postulate, the probability of obtaining the outcome a_i when measuring A in the state $|\Psi\rangle$ is :

$$P(a_i) = |\langle a_i| |\Psi\rangle|^2 . \tag{2.14}$$

Therefore, the average value of many measurements can be written as :

$$\langle A\rangle = \sum_i a_i P(a_i) = \sum_i a_i |\langle a_i| |\Psi\rangle|^2 . \tag{2.15}$$

Using the expansion of the state $|\Psi\rangle$ in the eigenbasis of A, this expression can be rewritten in the compact operator form.

$$\langle A\rangle = \langle\Psi| A |\Psi\rangle . \tag{2.16}$$

This result establishes the standard formula for the expectation value of an observable in quantum mechanics.

The quantum deviation (or standard deviation) of an observable M in the quantum state $|\Psi\rangle$ is defined by :

$$\Delta(M) = \sqrt{\langle M^2\rangle_\Psi - \langle M\rangle_\Psi^2}. \tag{2.17}$$

where $\langle M\rangle_\Psi$ is the expectation value of the observable M in the state Ψ, and $\langle M^2\rangle_\Psi$ is the expectation value of the operator M^2 in the same state.

The quantity $\Delta(M)$ measures the spread of the measurement results of the observable around its mean value.

The Density Operator

Quantum mechanics can also be formulated using the density operator formalism (or density matrix), which is particularly useful when the state [3] of the system is not completely known. In many physical situations, a system may be found in several possible pure states $|\Psi_i\rangle$ with associated classical probabilities P_i.The ensemble of states can therefore be written as $\{P_i, |\Psi_i\rangle\}$.

In this case, the quantum state of the system is described by the density operator ρ, defined as :

$$\rho = \sum_i P_i \left|\Psi_i\right\rangle \left\langle\Psi_i\right| . \tag{2.18}$$

which acts in the Hilbert space of the system.

$$\rho : \mathcal{H} \to \mathcal{H} \tag{2.19}$$

This operator contains all the statistical information about the quantum system.

If the system is in a pure state $|\Psi\rangle$, the density operator reduces to :

$$\rho = \left|\Psi\right\rangle \left\langle\Psi\right| \tag{2.20}$$

To verify the property of a pure state, we calculate the square of the density operator.

$$\begin{aligned} \rho^2 &= \rho\rho = \left(\left|\Psi\right\rangle \left\langle\Psi\right|\right)\left(\left|\Psi\right\rangle \left\langle\Psi\right|\right) . \\ &= \left|\Psi\right\rangle \left(\left\langle\Psi\right| \left|\Psi\right\rangle\right) \left\langle\Psi\right| . \end{aligned} \tag{2.21}$$

Since the state is normalized,$\left\langle\Psi\right| \left|\Psi\right\rangle = 1$, therefore.

$$\rho^2 = \left|\Psi\right\rangle \left\langle\Psi\right| = \rho.$$

Hence, if

$$\rho^2 = \rho. \tag{2.22}$$

the system is in a pure state, while if

$$\rho^2 \neq \rho. \tag{2.23}$$

the system is in a mixed state.

Let U be a quantum operation acting on the system.Each state vector evolves according to

$$|\Psi_i\rangle \rightarrow \left|\tilde{\Psi}_i\right\rangle = U\,|\Psi_i\rangle \tag{2.24}$$

The conjugate vector transforms as

$$\left\langle\tilde{\Psi}_i\right| = \langle\Psi_i|\,U^\dagger \tag{2.25}$$

Substituting these transformations into the definition of the density operator gives :

$$\begin{aligned}\rho\,(U) &= \sum_i P_i \left|\tilde{\Psi}_i\right\rangle\left\langle\tilde{\Psi}_i\right| \\ &= \sum_i P_i\,(U\,|\Psi_i\rangle)\left(\langle\Psi_i|\,U^\dagger\right) \\ &= U\left(\sum_i P_i\,|\Psi_i\rangle\langle\Psi_i|\right)U^\dagger.\end{aligned} \tag{2.26}$$

Since the term inside the parentheses is simply ρ, the evolution of the density operator becomes.

$$\rho\,(U) = U\rho U^\dagger. \tag{2.27}$$

Thus, the density matrix evolves by multiplying the unitary operator from the left and its Hermitian conjugate from the right.

If the system is initially in the state $|\Psi\rangle$, the probability of obtaining the measurement outcome m associated with operator M_m is given by

$$P\,(m \mid i) = \langle\Psi|\,M_m^\dagger M_m\,|\Psi\rangle\,. \tag{2.28}$$

When the initial state is not completely known, the total probability is obtained by averaging over all possible states.

$$P\,(m) = \sum_i P_i P\,(m \mid i)\,. \tag{2.29}$$

Substituting the expression for $P\,(m \mid i)$ gives :

$$P\,(m) = \sum_i P_i\,\langle\Psi|\,M_m^\dagger M_m\,|\Psi\rangle\,.$$

Using the definition of the density operator

$$\rho = \sum_i P_i \left|\Psi_i\right\rangle \left\langle\Psi_i\right| .$$

the probability can be written compactly as

$$P\left(m\right) = Tr\left(M_m^\dagger M_m \rho\right) .$$

This expression shows that the density matrix alone contains all the information needed to compute measurement probabilities.

After obtaining the measurement result m, the system collapses into a new state. In the density operator formalism, the post-measurement state is

$$\rho\left(m\right) = \frac{M_m \rho M_m^\dagger}{Tr\left(M_m^\dagger M_m \rho\right)}.$$

The denominator ensures normalization and corresponds to the probability of obtaining the outcome m

Trace Property.

Finally, the density operator must satisfy the normalization condition.

$$Tr\left(\rho\right) = 1. \tag{2.30}$$

the trace becomes

$$Tr\left(\rho\right) = \sum_i P_i Tr\left(\left|\Psi_i\right\rangle \left\langle\Psi_i\right|\right) . \tag{2.31}$$

Since $Tr\left(\left|\Psi_i\right\rangle \left\langle\Psi_i\right|\right) = 1$, we obtain

$$Tr\left(\rho\right) = \sum_i P_i = 1. \tag{2.32}$$

which reflects the conservation of total probability.

Reduced Density Matrix

In a quantum system composed of two parts, A and B, the total system can be described by a density matrix ρ_{AB}, which contains all the information about the combined system.

Often, we are only interested in one part, for example A, and we do not have access to the full system.In this case, we define the reduced density matrix of A, denoted ρ_A, by summing over all possible states of B. Mathematically, this is the partial trace over subsystem B :

$$\rho_A = Tr_B\left(\rho_{AB}\right) = \sum_{j}^{N_B} \langle j|_B \, \rho_{AB} \, |j\rangle_B \, . \tag{2.33}$$

where $\{|j\rangle_B\}$ is an orthonormal basis of subsystem B. Similarly, the reduced density matrix of subsystem B is

$$\rho_B = Tr_A\left(\rho_{AB}\right) = \sum_{j}^{N_A} \langle j|_A \, \rho_{AB} \, |j\rangle_A \, . \tag{2.34}$$

This construction ensures that ρ_A and ρ_B are valid density matrices, because the partial trace preserves the properties of Hermiticity and normalization.

To see this explicitly, we first verify Hermiticity.Since $\rho_{AB}^{\dagger} = \rho_{AB}$, we have

$$\begin{aligned} \rho_A^{\dagger} &= \left(\sum_{j}^{N_B} \langle j|_B \, \rho_{AB} \, |j\rangle_B\right)^{\dagger} = \sum_{j}^{N_B} \left(\langle j|_B \, \rho_{AB} \, |j\rangle_B\right)^{\dagger} \\ &= \sum_{j}^{N_B} \langle j|_B \, \rho_{AB} \, |j\rangle_B = \rho_A. \end{aligned} \tag{2.35}$$

Next, for normalization, the trace of ρ_A is

$$\begin{aligned} Tr_A\left(\rho_A\right) &= \sum_{i=1}^{N_A} \langle i|_A \, \rho_{AB} \, |i\rangle_A = \sum_{i=1}^{N_A} \sum_{j=1}^{N_B} \langle i|_A \, \langle j|_B \, \rho_{AB} \, |i\rangle_A \, |j\rangle_B \\ &= Tr_{AB}\left(\rho_{AB}\right) = 1. \end{aligned} \tag{2.36}$$

showing that the total probability is preserved.

The reduced density matrix also reproduces expectation values of local observables. If an operator acts only on A, written as $O_A \otimes I_B$, then

$$\begin{aligned}\langle O_A \rangle &= Tr_{AB}\left(\rho_{AB}\left(O_A \otimes I_B\right)\right) \\ &= \sum_{i,j} \langle i|_A \langle j|_B \, \rho_{AB}\left(O_A \otimes I_B\right) |i\rangle_A \, |j\rangle_B \\ &= \sum_{i} \langle i|_A \, \rho_A O_A \, |i\rangle_A \\ &= Tr_A\left(\rho_A O_A\right).\end{aligned} \tag{2.37}$$

This proves that all measurements on subsystem A can be computed from ρ_A alone, without knowing the full system.

Finally, even if the total system is in a pure state, the reduced density matrix may describe a mixed state if the subsystems are entangled.For example, consider the Bell state,$|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$. The total density matrix is

$$\rho_{AB} = |\Psi\rangle\langle\Psi| = \frac{1}{2}\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right). \tag{2.38}$$

Taking the partial trace over B, we obtain the reduced density matrix for A :

$$\rho_A = Tr_B\left(\rho_{AB}\right) = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right). \tag{2.39}$$

which is a mixed state. This shows that entanglement naturally produces mixed local states even if the total system is pure. The reduced density matrix thus provides a complete, self-contained description of a subsystem, allowing us to compute probabilities, expectation values, and correlations in an intuitive and mathematically consistent way.

Quantum entanglement

Quantum entanglement is a uniquely quantum phenomenon in which two or more subsystems become correlated in such a way that the state of each subsystem cannot be described independently of the others. This correlation persists even when the subsystems are spatially separated, demonstrating the nonlocal nature of quantum mechanics. Entanglement is central to many quantum information protocols, including quantum teleportation, superdense coding, and quantum key distribution.

Consider a bipartite system composed of two subsystems,Alice(A) and Bob(B), described by the Hilbert space $\mathcal{H} \otimes \mathcal{H}$. Let the total system be in a pure state $|\Psi_{AB}\rangle \in \mathcal{H} \otimes \mathcal{H}$. The state $|\Psi_{AB}\rangle$ is said to be entangled or non-separable if it cannot be written as a simple product of subsystem states :

$$|\Psi_{AB}\rangle \neq |\Psi_A\rangle \otimes |\Psi_B\rangle \,. \tag{2.40}$$

For any $|\Psi_A\rangle \in \mathcal{H}_A$ and $|\Psi_B\rangle \in \mathcal{H}_B$.

Definitions and Properties of Entanglement :

A pure bipartite state $|\Psi_{AB}\rangle$ is entangled if its Schmidt rank is greater than 1.

According to the Schmidt decomposition, any pure bipartite state can be written as :

$$|\Psi_{AB}\rangle = \sum_k \sqrt{\lambda_k}\, |a_k\rangle \otimes |b_k\rangle \,. \tag{2.41}$$

where $|a_k\rangle$ and $|b_k\rangle$ are orthonormal bases for Alice and Bob,and $\lambda_k \geq 0$ are the Schmidt coefficients satisfying $\sum_k \lambda_k = 1$. The state is entangled if more than one λ_k is nonzero, it is separable if only one $\lambda_k = 1$.

Property : If $|\Psi_{AB}\rangle$ is separable, there exist $|\Psi_A\rangle$ and $|\Psi_B\rangle$ such that $|\Psi_{AB}\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle$. Otherwise $|\Psi_{AB}\rangle$ exhibits entanglement, meaning measurements on Alice's subsystem are correlated with Bob's outcomes.

If the dimensions of the subsystems satisfy $dim\,(\mathcal{H}_A) \leq dim\,(\mathcal{H}_B)$, a state $|\Psi_{AB}\rangle$ is maximally entangled when the reduced states are completely mixed, and the Schmidt.

coefficients are equal :

$$\lambda_k = \frac{1}{dim\,(\mathcal{H}_A)} \forall k. \tag{2.42}$$

A well-known example is the Bell state for two qubits :

$$\left|\Phi^+\right\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B\right). \tag{2.43}$$

For mixed states, a bipartite density matrix ρ_{AB} is non-entangled (separable) if it can be expressed as a convex combination of product states :

$$\rho_{AB} = \sum_j P_j \rho_A^{(j)} \otimes \rho_B^{(j)}, P_j \geq 0, \sum_j P_j = 1. \tag{2.44}$$

where $\rho_A^{(j)}$ and $\rho_B^{(j)}$ are density operators of subsystems A and B, respectively. If no such decomposition exists, ρ_{AB} is entangled.

Property : Entanglement implies correlations beyond classical statistics, and it is responsible for quantum phenomena that have no classical analogue, such as violation of Bell inequalities.

Von Neumann Entropy

The Von Neumann entropy measures the uncertainty or mixedness of a quantum state. For a system described by a density matrix ρ, it is defined as :

$$S(\rho) = -Tr\left[\rho . \log(\rho)\right]. \tag{2.45}$$

If ρ has eigenvalues λ_i, this becomes :

$$S(\rho) = -\sum_i \lambda_i . \log \lambda_i . \tag{2.46}$$

$S(\rho) \geq 0$,with $S(\rho) = 0$ for pure states.

Maximum entropy $S(\rho) = \log d$ occurs for a maximally mixed state in a d-dimensional space.

For a bipartite pure state $|\Psi_{AB}\rangle$, the entropies of subsystems are equal :$S(\rho_A) = S(\rho_B)$.

Von Neumann entropy therefore provides a direct measure of uncertainty and entanglement in quantum systems.

The exterior product

The exterior product (or wedge product) is a mathematical operation used to [3, 4]

extend vectors to higher-dimensional geometric objects. It is denoted by the symbol $\wedge$.

Let $\vec{a}$ and $\vec{b}$ be two vectors in a complex vector space of dimension m. They can be written in the basis $\{e_1, e_2, ..., e_m\}$as

$$\vec{a} = \sum_{i=1}^{m} a_i e_i, \vec{b} = \sum_{j=1}^{m} b_j e_j . \tag{2.47}$$

The exterior product of $\vec{a}$ and $\vec{b}$ is obtained by applying the wedge product between the two vectors :

$$\vec{a} \wedge \vec{b} = \left(\sum_{i=1}^{m} a_i e_i \right) \wedge \left(\sum_{j=1}^{m} b_j e_j \right). \tag{2.48}$$

Using the bilinearity of the wedge product, we expand :

$$\vec{a} \wedge \vec{b} = \sum_{i=1}^{m} \sum_{j=1}^{m} a_i b_j \left(e_i \wedge e_j \right). \tag{2.49}$$

The wedge product satisfies the antisymmetry property :

$$e_i \wedge e_j = -e_j \wedge e_i. \tag{2.50}$$

and

$$e_i \wedge e_i = 0. \tag{2.51}$$

Therefore, the terms with $i = j$ vanish, and the terms (i, j) and (j, i) combine as

$$a_i b_j \left(e_i \wedge e_j \right) + a_j b_i \left(e_j \wedge e_i \right) = \left(a_i b_j - a_j b_i \right) \left(e_i \wedge e_j \right). \tag{2.52}$$

To avoid counting the same pair twice, we keep only the terms with $i < j$.Thus we obtain

$$\vec{a} \wedge \vec{b} = \sum_{i=1}^{m-1} \sum_{j=i=1}^{m} \left(a_i b_j - a_j b_i \right) \left(e_i \wedge e_j \right). \tag{2.53}$$

The result is called a bivector, which represents an oriented surface element and is neither a scalar nor an ordinary vector.

The coefficients multiplying the basis bivectors $e_i \wedge e_j$ give the components of $\vec{a} \wedge \vec{b}$. These components are the antisymmetric combinations

$$\begin{pmatrix} a_1 b_2 - a_2 b_1, a_1 b_3 - a_3 b_1, ..., a_1 b_m - a_m b_1, \\ a_2 b_3 - a_3 b_2, ..., a_{m-1} b_m - a_m b_{m-1} \end{pmatrix}. \tag{2.54}$$

These terms correspond to all possible antisymmetric pairs $a_i b_j - a_j b_i$ formed from the components of the vectors $\vec{a}$ and $\vec{b}$.

Lagrange's Identity

Let $\vec{a}$ and $\vec{b}$ be two vectors in a complex vector space of dimension m, written as

$$\vec{a} = (a_1, a_2, ..., a_m), \vec{b} = (b_1, b_2, ..., b_m). \tag{2.55}$$

Lagrange's identity relates the norm of the exterior produc $\vec{a} \wedge \vec{b}$ to the. The sum over $i < j$ can be written as

$$\left\| \vec{a} \wedge \vec{b} \right\|^2 = \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} |a_i b_j - a_j b_i|^2. \tag{2.56}$$

Expanding the modulus,

$$|a_i b_j - a_j b_i|^2 = (a_i b_j - a_j b_i) - \left(\bar{a}_i \bar{b}_j - \bar{a}_j \bar{b}_i\right). \tag{2.57}$$

which gives

$$|a_i|^2 |b_j|^2 - 2Re\left(a_i b_j \bar{a}_j \bar{b}_i\right) + |a_j|^2 |b_i|^2.$$

Summing over $_i$ and $_j$,

$$\left\| \vec{a} \wedge \vec{b} \right\|^2 = \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \left(|a_i|^2 |b_j|^2 - 2Re\left(a_i b_j \bar{a}_j \bar{b}_i\right) + |a_j|^2 |b_i|^2\right). \tag{2.58}$$

Rearranging the sums leads to

$$\left\| \vec{a} \wedge \vec{b} \right\|^2 = \left(\sum_{i=1}^{m} |a_i|^2\right) \left(\sum_{j=1}^{m} |b_j|^2\right) - \left|\sum_{i=1}^{m} a_i \bar{b}_i\right|. \tag{2.59}$$

Recognizing the norms and the scalar product,

$$\|\vec{a}\|^2 = \sum_{i=1}^{m} |a_i|^2, \left\|\vec{b}\right\|^2 = \sum_{i=1}^{m} |b_i|^2, \vec{a}.\vec{b} = \sum_{i=1}^{m} a_i \bar{b}_i. \tag{2.60}$$

we obtain Lagrange's identity

$$\left\| \vec{a} \wedge \vec{b} \right\|^2 = \|\vec{a}\|^2 \left\|\vec{b}\right\|^2 - \left|\vec{a}.\vec{b}\right|^2. \tag{2.61}$$

Two-Qubit Concurrence Using the Lagrange Identity and the Exterior Product

For a pure two–qubit state

$$|\Psi\rangle = p\,|0_A 0_B\rangle + q\,|0_A 1_B\rangle + r\,|1_A 0_B\rangle + s\,|1_A 1_B\rangle\,. \tag{2.62}$$

the concurrence is defined as

$$C\,(\Psi) = 2\,\|a \wedge b\|\,. \tag{2.63}$$

where the vectors

$$a = (p, q)\,, b = (r, s)\,. \tag{2.64}$$

represent the coefficients associated with the basis states of Alice.

The exterior product of two vectors in $\mathbb{C}^2$ is defined by

$$a \wedge b = (a_1 b_2 - a_2 b_1)\,(e_1 \wedge e_2)\,. \tag{2.65}$$

Substituting

$$a_1 = p, a_2 = q, b_1 = r, b_2 = s. \tag{2.66}$$

we obtain

$$a \wedge b = (ps - qr)\,(e_1 \wedge e_2)\,. \tag{2.67}$$

Since the basis bivector satisfies

$$\|e_1 \wedge e_2\| = 1. \tag{2.68}$$

the magnitude of the exterior product becomes

$$\|a \wedge b\| = |ps - qr|\,.$$

Substituting this result into the definition of concurrence gives

$$C(\Psi) = 2\left|ps - qr\right|. \tag{2.69}$$

The quantity $ps - qr$ is the determinant of the coefficient matrix

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix}. \tag{2.70}$$

If $ps - qr = 0$, the vectors a are b linearly dependent and the state is separable.

If $ps - qr \neq 0$, the vectors are independent and the state is entangled.

Thus, the concurrence measures the degree of entanglement of the two–qubit state.

Three-Qubit State

Three-Qubit State is

$$\left|\Psi\right\rangle = \sum_{i,j,k=0}^{1} a_{i,j,k}\left|ijk\right\rangle. \tag{2.71}$$

with the normalization condition

$$\sum_{i,j,k=0}^{1} \left|a_{i,j,k}\right|^2 = 1. \tag{2.72}$$

We rewrite the state grouping Alice's basis :

$$\left|\Psi\right\rangle = \left|0\right\rangle_A \left|\Phi_0\right\rangle_{BC} + \left|1\right\rangle_A \left|\Phi_1\right\rangle_{BC}. \tag{2.73}$$

where

$$\begin{aligned} \left|\Phi_0\right\rangle &= p\left|000\right\rangle + q\left|001\right\rangle + r\left|010\right\rangle + s\left|011\right\rangle, \\ \left|\Phi_1\right\rangle &= t\left|100\right\rangle + u\left|101\right\rangle + v\left|q10\right\rangle + w\left|111\right\rangle. \end{aligned} \tag{2.74}$$

These two vectors describe the state of subsystem BC. It shows that a system of 3 qubits lives in an 8-dimensional Hilbert space, and the coefficients p,q,r,s,t,u,v,w, are the probability amplitudes of the computational basis states the condition of separability for the bipartition A/BC :

$$\frac{p}{t} = \frac{q}{u} = \frac{r}{v} = \frac{s}{w}. \tag{2.75}$$

The vectors (p, q, r, s) and (t, u, v, w) must be parallel. If this condition holds, the state can be written as a product state.If not, the system is entangled.

Four-Qubit States

Consider a quantum system composed of four qubits denoted by A,B,C, and$D$. Let $|\Psi\rangle$ be a normalized pure state of the system. In the computational basis used in quantum computing, the most general form of this state can be written as :

$$|\Psi\rangle = \sum_{i,j,k,l=0}^{1} c_{ijkl} |i_A j_B k_C l_D\rangle . \tag{2.76}$$

where $c_{ijkl} \in \mathbb{C}$ are complex probability amplitudes associated with each basis state and satisfy the normalization condition.

$$\sum_{i,j,k,l=0}^{1} |c_{ijkl}|^2 = 1. \tag{2.77}$$

To study the entanglement properties of the system, particularly the separability of qubit A from the remaining subsystem BCD, the state can be rewritten by grouping the terms A with respect to the basis of qubit

$$E_A = |0_A\rangle \left|\Phi_0^A\right\rangle + |1_A\rangle \left|\Phi_1^A\right\rangle . \tag{2.78}$$

Here, the vectors $\left|\Phi_0^A\right\rangle$ and $\left|\Phi_1^A\right\rangle$ belong to the Hilbert space. To analyze the separability of the pair of qubits(AB) from the subsystem (CD), we consider the four conditional vectors $\langle 0_A 0_B| |\Psi\rangle$, $\langle 0_A 1_B| |\Psi\rangle$, $\langle 1_A 0_B| |\Psi\rangle$, $\langle 1_A 1_B| |\Psi\rangle$.

Which belong to the Hilbert space $\mathcal{H}_{CD}$. The subsystem (AB) is separable from (CD) if these vectors are mutually parallel. Based on this geometric approach, the entanglement measure associated with the pair AB is defined by

$$E_{AB}^2 = 4 \sum_{i<j} \| \langle i| |\Psi\rangle \wedge \langle i| |\Psi\rangle \| . \tag{2.79}$$

where the sum runs over the four basis states of the subsystem AB. This expression provides a geometric quantification of entanglement between the pair of qubits AB and. The remaining subsystem CD in the four-qubit system.

Geometric Interpretation of Entanglement based on Vector Parallelism

Consider a quantum system composed of three qubits, labeled A,B,C,. The general pure state of the system in the computational basis can be written as

$$|\Psi\rangle = a\,|000\rangle + b\,|001\rangle + c\,|010\rangle + d\,|011\rangle + p\,|100\rangle + q\,|101\rangle + r\,|110\rangle + s\,|111\rangle\,. \quad (2.80)$$

where :a,b,c,d,p,q,r,$s \in \mathbb{C}$ are complex probability amplitudes. We consider a bipartition A/BC this allows us to analyze whether qubit A is entangled with the subsystem BC, after measuring the BC system in the computational basis $|00\rangle$,$|01\rangle$,$|10\rangle$,$|11\rangle$, the unnormalized post-measurement vectors for qubit A are

$$\begin{aligned} \Phi_A^0 &= a\,|0\rangle + p\,|1\rangle\,, \Phi_A^1 = b\,|0\rangle + q\,|1\rangle\,. \\ \Phi_A^2 &= c\,|0\rangle + r\,|1\rangle\,, \Phi_A^3 = d\,|0\rangle + s\,|1\rangle\,. \end{aligned}$$

These vectors represent the states of qubit A conditioned on the measurement outcome of BC. If Φ_A^0,Φ_A^1,Φ_A^2,Φ_A^3 are all parallel, the state factorizes :$|\Psi\rangle = |\Phi_A\rangle \otimes |\Phi_{BC}\rangle$ and qubit A is not entangled with BC. If the vectors are not parallel, there exists quantum entanglement between A and BC.

The concurrence squared for this bipartition is

$$C^2_{A/BC} = 4\left(\left|\Phi_A^0 \wedge \Phi_A^1\right|^2 + \left|\Phi_A^0 \wedge \Phi_A^2\right|^2 + \left|\Phi_A^0 \wedge \Phi_A^3\right|^2 + \left|\Phi_A^1 \wedge \Phi_A^2\right|^2 + \left|\Phi_A^1 \wedge \Phi_A^3\right|^2 + \left|\Phi_A^2 \wedge \Phi_A^3\right|^2\right). \quad (2.81)$$

If vectors are parallel : $\Phi_i \wedge \Phi_j = 0 \Longrightarrow C_{A/BC} = 0$,no entanglement.

This inequality describes how entanglement is distributed in multipartit esystem :

$$C^2_{A/BC} \geq C^2_{A/B} + C^2_{A/C}. \quad (2.82)$$

Entanglement is monogamo. If A is strongly entangled with B its entanglement with C is limited.

The 3-tangle quantifies the genuine three-way entanglement :

$$\tau = C^2_{A/BC} - C^2_{A/B} - C^2_{A/C}. \quad (2.83)$$

If $\tau = 0$, the entanglement is purely bipartite and if $\tau > 0$, there exists true tripartite entanglement.

Since :

$$C^2_{A/B} = 4\left|\left(\chi_0^A \wedge \chi_2^A\right) + \left(\chi_1^A \wedge \chi_3^A\right)\right|^2 . \tag{2.84}$$

$$C^2_{A/C} = 4\left|\left(\chi_0^A \wedge \chi_1^A\right) + \left(\chi_2^A \wedge \chi_3^A\right)\right| .$$

$$\tau = 4[\left|\chi_0^A \wedge \chi_3^A\right|^2 + \left|\chi_1^A \wedge \chi_2^A\right|^2 - 2\left(\chi_0^A \wedge \chi_1^A\right) . \left(\chi_2^A \wedge \chi_3^A\right) - 2\left(\chi_0^A \wedge \chi_2^A\right) . \left(\chi_1^A \wedge \chi_3^A\right)]. \tag{2.85}$$

This gives a geometric measure of three-way entanglement using the post-measurement vectors of qubit A.

Chapitre 3

Quantum Teleportation

3.1 Introduction

Quantum teleportation, first proposed by Bennett, Brassard, et al., has become a cornerstone of quantum communication and networking. This protocol enables the transmission of an unknown quantum state between distant nodes without the physical transfer of the particle itself. The process relies on the unique properties of quantum entanglement and the exchange of classical information.

Over the years, teleportation has evolved into various specialized schemes to meet the demands of secure quantum networks :

Standard (Simple) Teleportation : The original model where a single qubit is transferred from a sender (Alice) to a receiver (Bob) through a shared Bell state.

Controlled Quantum Teleportation (CQT) : A tripartite scheme where the teleportation is governed by a controller (Charlie). In this setup, Bob cannot reconstruct the teleported state unless Charlie performs a measurement and shares the result, providing a layer of authority and security.

Bidirectional Quantum Teleportation (BQT) : A more sophisticated protocol that allows Alice and Bob to exchange quantum states simultaneously. This bidirectional flow maximizes the efficiency of the entangled resource and is vital for full-duplex quantum communication.

The implementation of these protocols is deeply linked to entanglement distribution. By utilizing multi-party entangled states, such as GHZ states, we can precisely determine and distribute correlations across the system. This structural approach is essential for understanding how quantum particles are correlated and, more importantly, for preventing the unlimited or unauthorized sharing of entanglement between multiple parties, thereby maintaining the integrity of the quantum network.

3.2 Simple Quantum Teleportation Protocol

The Simple Quantum Teleportation Protocol (Brassard et al.) is a method that allows a sender (Alice) to transmit an unknown qubit to a receiver (Bob) using a shared entangled pair and classical communication, where Bob applies a unitary correction to recover the original state.

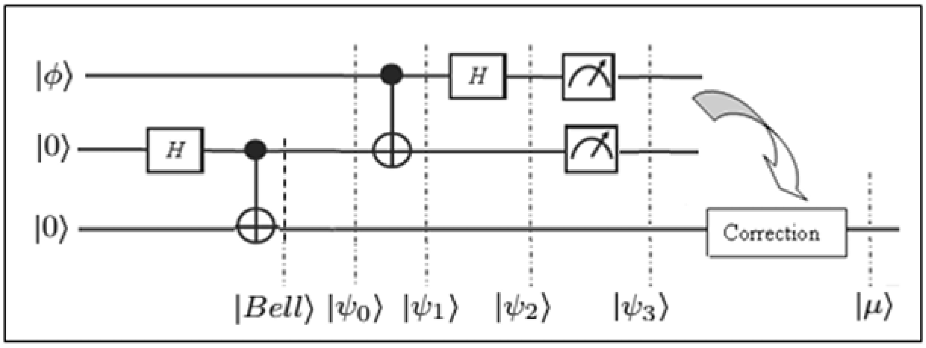

FIGURE 3.1: The Original Brassard Protocol

Matrix Formalism :

The protocol begins with a three-qubit system in a Hilbert space of dimension $2^3 = 8$. Alice possesses the unknown state $|Q\rangle_A = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ and the first half of a shared entangled Bell state$|Bell\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$.

The total initial state $|\Phi\rangle_0$ is the tensor product of the unknown qubit and the Bell pair :

$$|\Psi\rangle_0 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{\alpha}{\sqrt{2}} \\ \frac{\beta}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{\beta}{\sqrt{2}} \end{pmatrix}. \quad (3.1)$$

Now, we apply the CNOT operation to Alice's two qubits, where the first qubit (the one to be teleported) acts as the control, and the second qubit (entangled with the channel) acts as the target.

$$U_1 = CNOT_{AA} \otimes I_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.2}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

We calculate the state $|\Psi\rangle_1$ following the application of the CNOT gate.

$$|\Psi\rangle_1 = U_1 |\Psi\rangle_0 \tag{3.3}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\alpha}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{\alpha}{\sqrt{2}} \\ \frac{\beta}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{\beta}{\sqrt{2}} \end{pmatrix}.$$

$$= \begin{pmatrix} \frac{\alpha}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{\alpha}{\sqrt{2}} \\ 0 \\ \frac{\beta}{\sqrt{2}} \\ \frac{\beta}{\sqrt{2}} \\ 0 \end{pmatrix}.$$

We apply a Hadamard gate to Alice's qubit that is to be teleported.

$$U_2 = H \otimes I_2 \otimes I_2. \tag{3.4}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$U_2 = \begin{pmatrix} 1\frac{1}{\sqrt{2}} & 0 & 0 \; 0 & \frac{1}{\sqrt{2}} & 0 & 0 \; 0 \\ 0 & 1\frac{1}{\sqrt{2}} & 0 \; 0 & 0 & \frac{1}{\sqrt{2}} & 0 \; 0 \\ 0 \; 0 & \frac{1}{\sqrt{2}} & 0 & 0 \; 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 \; 0 & 0 & \frac{1}{\sqrt{2}} & 0 \; 0 & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 \; 0 & \frac{-1}{\sqrt{2}} & 0 & 0 \; 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \; 0 & 0 & \frac{-1}{\sqrt{2}} & 0 \; 0 \\ 0 \; 0 & \frac{1}{\sqrt{2}} & 0 & 0 \; 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 \; 0 & 0 & \frac{1}{\sqrt{2}} & 0 \; 0 & 0 & \frac{-1}{\sqrt{2}} \end{pmatrix}.$$

After calculating $|\Psi\rangle_1$, we now calculate $|\Psi\rangle_2$ following the application of the Hadamard gate to Alice's qubit that is to be teleported.

$$|\Psi\rangle_2 = U_2 \, |\Psi\rangle_1 \, . \tag{3.5}$$

$$|\Psi\rangle_2 = \begin{pmatrix} 1\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 1\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{\alpha}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{\alpha}{\sqrt{2}} \\ 0 \\ \frac{\beta}{\sqrt{2}} \\ \frac{\beta}{\sqrt{2}} \\ 0 \end{pmatrix}.$$

$$= \frac{1}{2} \begin{pmatrix} \alpha \\ \beta \\ \beta \\ \alpha \\ \alpha \\ -\beta \\ -\beta \\ \alpha \end{pmatrix}.$$

If the measurement result is 00, the state before normalization is calculated using the projection operator :

$$P_{00} = \langle 00| \otimes I_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.6}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Before Normalization :

$$|\Psi\rangle_2 \to^{P_{00}} |\Psi_3\rangle . \tag{3.7}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \beta \\ \alpha \\ \alpha \\ -\beta \\ -\beta \\ \alpha \end{pmatrix}.$$

$$= \frac{1}{2}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

After Normalization :

$|\Psi_3\rangle = \frac{P_{00}|\Psi_2\rangle}{\|P_{00}|\Psi_2\rangle\|}$, $\|P_{00}\,|\Psi_2\rangle\| = \sqrt{\left(\frac{\alpha}{\sqrt{2}}\right)^2 + \left(\frac{\beta}{\sqrt{2}}\right)^2} = \sqrt{(\alpha^2+\beta^2)\,\frac{1}{4}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$

$$|\Psi_3\rangle = \frac{P_{00}\,|\Psi_2\rangle}{\|P_{00}\,|\Psi_2\rangle\|} = \frac{\frac{1}{2}}{\frac{1}{2}}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{3.8}$$

Therefore in this case, there is no correction.

If the measurement result is 01, the state before normalization is calculated using the projection operator

$$P_{01} = \langle 01| \otimes I_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.9}$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Before Normalization :

$$|\Psi_3\rangle = P_{01} |\Psi\rangle_2 . \tag{3.10}$$

$$= \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \beta \\ \alpha \\ \alpha \\ -\beta \\ -\beta \\ \alpha \end{pmatrix} .$$

$$= \frac{1}{2} \begin{pmatrix} \beta \\ \alpha \end{pmatrix} .$$

After Normalization :

$|\Psi_3\rangle = \frac{P_{01}|\Psi_2\rangle}{\|P_{01}|\Psi_2\rangle\|}, \|P_{01} |\Psi_2\rangle\| = \sqrt{\left(\frac{\alpha}{\sqrt{2}}\right)^2 + \left(\frac{\beta}{\sqrt{2}}\right)^2} = \sqrt{(\alpha^2 + \beta^2) \frac{1}{4}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$

$$|\Psi_3\rangle = \frac{P_{01} |\Psi_2\rangle}{\|P_{01} |\Psi_2\rangle\|} = \frac{\frac{1}{2}}{\frac{1}{2}} \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix} . \tag{3.11}$$

$$|\Psi_3\rangle = \begin{pmatrix} \beta \\ \alpha \end{pmatrix} .$$

In this case, Bob applies the X gate.

$$X \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} . \tag{3.12}$$

If the measurement result is 10, the state before normalization is calculated using the projection operator.

$$P_{10} = \langle 10| \otimes I_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{3.13}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} .$$

Before Normalization :

$$|\Psi_3\rangle = P_{10}\,|\Psi\rangle_2 . \tag{3.14}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \beta \\ \alpha \\ \alpha \\ -\beta \\ -\beta \\ \alpha \end{pmatrix}.$$

$$= \frac{1}{2}\begin{pmatrix} \alpha \\ -\beta \end{pmatrix}.$$

After Normalization :

$|\Psi_3\rangle = \frac{P_{10}|\Psi_2\rangle}{\|P_{10}|\Psi_2\rangle\|}, \|P_{10}\,|\Psi_2\rangle\| = \sqrt{\left(\frac{\alpha}{\sqrt{2}}\right)^2 + \left(\frac{\beta}{\sqrt{2}}\right)^2} = \sqrt{(\alpha^2+\beta^2)\,\frac{1}{4}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$

$$|\Psi_3\rangle = \frac{P_{10}\,|\Psi_2\rangle}{\|P_{10}\,|\Psi_2\rangle\|} = \frac{\frac{1}{2}}{\frac{1}{2}}\begin{pmatrix} \alpha \\ -\beta \end{pmatrix} = \begin{pmatrix} \alpha \\ -\beta \end{pmatrix}. \tag{3.15}$$

$$|\Psi_3\rangle = \begin{pmatrix} \alpha \\ -\beta \end{pmatrix}. \tag{3.16}$$

In this case, Bob applies the Z gate.

$$Z\begin{pmatrix} \alpha \\ -\beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \alpha \\ -\beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{3.17}$$

If the measurement result is 11, the state before normalization is calculated using the projection operator.

$$P_{11} = \langle 11| \otimes I_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.18}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Before Normalization :

$$|\Psi_3\rangle = P_{11}\,|\Psi\rangle_2\,. \tag{3.19}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \beta \\ \alpha \\ \alpha \\ -\beta \\ -\beta \\ \alpha \end{pmatrix}.$$

$$= \frac{1}{2}\begin{pmatrix} -\beta \\ \alpha \end{pmatrix}.$$

After Normalization :

$|\Psi_3\rangle = \frac{P_{11}|\Psi_2\rangle}{\|P_{11}|\Psi_2\rangle\|}, \|P_{11}\,|\Psi_2\rangle\| = \sqrt{\left(\frac{\alpha}{\sqrt{2}}\right)^2 + \left(\frac{\beta}{\sqrt{2}}\right)^2} = \sqrt{(\alpha^2+\beta^2)\,\frac{1}{4}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$

$$|\Psi_3\rangle = \frac{P_{11}\,|\Psi_2\rangle}{\|P_{11}\,|\Psi_2\rangle\|} = \frac{\frac{1}{2}}{\frac{1}{2}}\begin{pmatrix} -\beta \\ \alpha \end{pmatrix} = \begin{pmatrix} -\beta \\ \alpha \end{pmatrix}. \tag{3.20}$$

$$|\Psi_3\rangle = \begin{pmatrix} -\beta \\ \alpha \end{pmatrix}.$$

In this case, Bob applies the Z X gate.

$$Z.X\begin{pmatrix} \alpha \\ -\beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ -\beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{3.21}$$

Bra-Ket Formalism :

Alice has a qubit in an unknown state :

$$|Q\rangle_A = \alpha\,|0\rangle + \beta\,|1\rangle\,. \tag{3.22}$$

Alice and Bob also share a maximally entangled Bell pair :

$$|Bell\rangle = \frac{1}{\sqrt{2}}\,(|00\rangle + |11\rangle)\,. \tag{3.23}$$

The total initial state of the three qubits is

$$|\Phi\rangle_0 = |Q\rangle_A \otimes |Bell\rangle . \quad (3.24)$$

$$\begin{aligned} |\Phi\rangle_0 &= (\alpha|0\rangle + \beta|1\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) . \quad (3.25)\\ &= \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle) . \end{aligned}$$

Applying a CNOT gate between Alice's two qubits.

$$|\Phi\rangle_0 \rightarrow^{CNOT_{12}} |\Phi\rangle_1 = \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle) . \quad (3.26)$$

Applying a Hadamard gate to Alice's target qubit results in the state :

$$|\Phi\rangle_1 \rightarrow^{H_1} |\Phi\rangle_2 = \frac{1}{2}(\alpha|000\rangle + \alpha|100\rangle + \alpha|011\rangle + \alpha|111\rangle + \beta|010\rangle - \beta|110\rangle + \beta|001\rangle - \beta|101\rangle) . \quad (3.27)$$

Alice measures her two qubits in the standard basis

$$|00\rangle , |01\rangle , |10\rangle , |11\rangle . \quad (3.28)$$

If Alice obtains the measurement outcome 00.

$$|\Phi\rangle_2 \rightarrow^{P_{00}} |\Phi\rangle_3 = (\langle 00| \otimes I_2) |\Phi\rangle_2 = \frac{1}{2}(\alpha|0\rangle + \beta|1\rangle) . \quad (3.29)$$

The state after normalization is given by :

$$|\Phi\rangle_4 = \frac{|\Phi\rangle_3}{\||\Phi\rangle_3\|} = \frac{\frac{1}{2}(\alpha|0\rangle + \beta|1\rangle)}{\left\|\sqrt{\left(\frac{1}{2}\right)^2(\alpha)^2 + (\beta)^2}\right\|} = \frac{\frac{1}{2}(\alpha|0\rangle + \beta|1\rangle)}{\frac{1}{2}} . \quad (3.30)$$

$$|\Phi\rangle_4 = \alpha|0\rangle + \beta|1\rangle . \quad (3.31)$$

Bob's state matches Alice's initial state, so no unitary correction is needed.

If Alice obtains the measurement outcome 01.

$$|\Phi\rangle_2 \rightarrow^{P_{01}} |\Phi\rangle_3 = (\langle 01| \otimes I_2) |\Phi\rangle_2 = \frac{1}{2}(\beta|0\rangle + \alpha|1\rangle) . \quad (3.32)$$

The state after normalization is given by :

$$|\Phi\rangle_4 = \beta |0\rangle + \alpha |1\rangle . \tag{3.33}$$

Bob performs a Pauli-X operation to restore the original state.

$$X |\Phi\rangle_4 = \alpha |0\rangle + \beta |1\rangle . \tag{3.34}$$

If Alice obtains the measurement outcome 10.

$$|\Phi\rangle_2 \to^{P_{10}} |\Phi\rangle_3 = (\langle 10| \otimes I_2) |\Phi\rangle_2 = \frac{1}{2} (\alpha |0\rangle - \beta |1\rangle) . \tag{3.35}$$

The state after normalization is given by :

$$|\Phi\rangle_4 = \alpha |0\rangle - \beta |1\rangle . \tag{3.36}$$

Bob applies the Z gate to correct the state.

$$Z |\Phi\rangle_4 = \alpha |0\rangle + \beta |1\rangle . \tag{3.37}$$

If Alice obtains the measurement outcome 11.

$$|\Phi\rangle_2 \to^{P_{11}} |\Phi\rangle_3 = (\langle 11| \otimes I_2) |\Phi\rangle_2 = \frac{1}{2} (\alpha |1\rangle - \beta |0\rangle) . \tag{3.38}$$

The state after normalization is given by :

$$|\Phi\rangle_4 = \alpha |1\rangle - \beta |0\rangle . \tag{3.39}$$

Bob applies both Z and X gates to correct the state.

$$ZX |\Phi\rangle_4 = \alpha |0\rangle + \beta |1\rangle . \tag{3.40}$$

FIGURE 3.2: The quantum states prior to Alice's measurement

$\|Q\rangle$	$\alpha \|0\rangle + \beta \|1\rangle$
$\|Bell\rangle$	$\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$
$\|\psi_0\rangle$	$\frac{1}{\sqrt{2}}(\alpha \|000\rangle + \alpha \|011\rangle + \beta \|100\rangle + \beta \|111\rangle)$
$\|\psi_1\rangle$	$\frac{1}{\sqrt{2}}(\alpha \|000\rangle + \alpha \|011\rangle + \beta \|101\rangle + \beta \|110\rangle)$
$\|\psi_2\rangle$	$\frac{1}{2}(\alpha \|000\rangle + \beta \|001\rangle + \alpha \|011\rangle + \beta \|010\rangle + \alpha \|100\rangle - \beta \|101\rangle - \beta \|110\rangle + \alpha \|111\rangle)$

FIGURE 3.3: Bob's unitary corrections following Alice's measurement

Résultats de mesure d'Alice	$\|\psi_3\rangle$	Qubit de Bob ($\|\mu\rangle$)	Correction appropriée
$\|00\rangle$	$\alpha\|000\rangle + \beta\|001\rangle$	$\alpha\|0\rangle + \beta\|1\rangle$	I
$\|01\rangle$	$\alpha\|011\rangle + \beta\|010\rangle$	$\beta\|0\rangle + \alpha\|1\rangle$	X
$\|10\rangle$	$\alpha\|100\rangle - \beta\|101\rangle$	$\alpha\|0\rangle - \beta\|1\rangle$	Z
$\|11\rangle$	$-\beta\|110\rangle + \alpha\|111\rangle$	$\beta\|0\rangle - \alpha\|1\rangle$	Z X

3.3 Controlled Quantum Teleportation Protocol

The Controlled Quantum Teleportation protocol, proposed by Brassard et al., is an extension of the standard quantum teleportation scheme in which the successful transfer of an unknown quantum state from a sender (Alice) to a receiver (Bob) requires the participation of a third party, called the controller (Charlie).

Matrix Formalism :

Alice wants to teleport an arbitrary qubit

$$|Q\rangle_A = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{3.41}$$

using a shared GHZ state with Bob and Charlie :

$$|GHZ\rangle_{ACB} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{3.42}$$

The total initial state is :

$$|\Phi\rangle_0 = |Q\rangle_A \otimes |GHZ\rangle_{ACB} . \tag{3.43}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} .$$

$$|\Phi\rangle_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \alpha \\ \beta \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \beta \end{pmatrix} .$$

Starting from the global initial state $|\Phi\rangle_0$ and after Alice's operations, we can derive the first collapsed state shared between Charlie and Bob for each measurement outcome.

$$|\Phi\rangle_1 = U_1 |\Phi\rangle_0 . \tag{3.44}$$

Alice applies CNOT (control : qubit A, target : qubit 1)

$$U_1 = C\_NOT_{AA} \otimes I_4. \tag{3.45}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$U_1 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

The transformed state is

$$|\Phi\rangle_1 = U_1 |\Phi\rangle_0 . \tag{3.46}$$

$$|\Phi\rangle_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \alpha \\ 0 \\ 0 \\ 0 \\ \beta \\ \beta \\ 0 \\ 0 \\ 0 \end{pmatrix} . \tag{3.47}$$

To prepare the system for the Bell-state measurement, Alice applies a Hadamard gate (H) to her first qubit (A). This step transforms $|\Phi\rangle_1$ into the second evolved state $|\Phi\rangle_2$.

$$|\Phi\rangle_2 = U_2 |\Phi\rangle_1 . \tag{3.48}$$

$$U_2 = H \otimes I_8. \tag{3.49}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} .$$

After applying the CNOT gate, the expression for $|\Phi\rangle_2$ becomes :

$$|\Phi\rangle_2 = U_2 |\Phi\rangle_1 . \tag{3.50}$$

$$|\Phi\rangle_2 = \frac{1}{2}\begin{pmatrix} \alpha \\ 0 \\ 0 \\ \beta \\ \beta \\ 0 \\ 0 \\ \alpha \\ \alpha \\ 0 \\ 0 \\ -\beta \\ -\beta \\ 0 \\ 0 \\ \alpha \end{pmatrix}.$$

In this protocol, Alice performs a joint Bell-state measurement (BSM) on her two qubits, resulting in one of the four possible Bell states $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, while Charlie acts as the controller by measuring his qubit in the diagonal basis $\{|+\rangle, |-\rangle\}$ to authorize the teleportation process."

Alice's Projection and Normalization :

Alice Measures $|00\rangle$

$$P_{00} = \langle 00| \otimes I_4 \tag{3.51}$$

$$|\Phi\rangle_2 \rightarrow^{P_{00}} |\Phi^*\rangle_3 = \frac{P_{00}|\Phi\rangle_2}{\|P_{00}|\Phi\rangle_2\|} \tag{3.52}$$

$$\Longrightarrow P_{00}|\Phi\rangle_2 = \frac{1}{2\sqrt{2}}\begin{pmatrix} \alpha \\ 0 \\ 0 \\ \beta \end{pmatrix},$$

$$\|P_{00}|\Phi\rangle_2\| = \sqrt{\left(\frac{1}{2\sqrt{2}}\right)^2 \left\{|\alpha|^2 + |\beta|^2\right\}} = \frac{1}{2\sqrt{2}}$$

If Charlie measures $|-\rangle$:

The normalized state shared between Charlie and Bob is

$$|\Phi^*\rangle_3 = \begin{pmatrix} \alpha \\ 0 \\ 0 \\ \beta \end{pmatrix} \tag{3.53}$$

Charlie performs a measurement in the diagonal basis $\{|+\rangle, |-\rangle\}$. This requires projecting the state onto the $|\Phi\rangle_3$.

If Charlie measures $|+\rangle$:

$$|\Phi^*\rangle_3 \to^{P_+} |\Phi\rangle_3 = \frac{P_+ |\Phi^*\rangle_3}{\|P_+ |\Phi^*\rangle_3\|} \tag{3.54}$$

$$\Longrightarrow P_+ |\Phi^*\rangle_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ 0 \\ 0 \\ \beta \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

$$\|P_+ |\Phi^*\rangle_3\| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 \left\{|\alpha|^2 + |\beta|^2\right\}} = \frac{1}{\sqrt{2}}$$

After final normalization, Bob receives :

$$|\Phi\rangle_3 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{3.55}$$

Correction : I.

If Charlie measures $|-\rangle$:

$$|\Phi^*\rangle_3 \to^{P_-} |\Phi\rangle_3 = \frac{P_- |\Phi^*\rangle_3}{\|P_- |\Phi^*\rangle_3\|} \tag{3.56}$$

$$|\Phi\rangle_3 = \begin{pmatrix} \alpha \\ -\beta \end{pmatrix} \tag{3.57}$$

After final normalization, Bob receives :

$$|\Phi\rangle_3 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{3.58}$$

Correction : Z.

Alice Measures $|01\rangle$

$$|\Phi\rangle_2 \rightarrow^{P_{01}} |\Phi^*\rangle_3 = \frac{P_{01}\,|\Phi\rangle_2}{\|P_{01}\,|\Phi\rangle_2\|} \tag{3.59}$$

The normalized state shared between Charlie and Bob is

$$|\Phi^*\rangle_3 = \begin{pmatrix} \beta \\ 0 \\ 0 \\ \alpha \end{pmatrix} \tag{3.60}$$

If Charlie measures $|+\rangle$:

$$|\Phi^*\rangle_3 \rightarrow^{P_+} |\Phi\rangle_3 = \frac{P_+\,|\Phi^*\rangle_3}{\|P_+\,|\Phi^*\rangle_3\|} \tag{3.61}$$

$$|\Phi\rangle_3 = \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \tag{3.62}$$

After final normalization, Bob receives :

$$|\Phi\rangle_3 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{3.63}$$

Correction : X.

If Charlie measures $|-\rangle$:

$$|\Phi^*\rangle_3 \rightarrow^{P_-} |\Phi\rangle_3 = \frac{P_-\,|\Phi^*\rangle_3}{\|P_-\,|\Phi^*\rangle_3\|} \tag{3.64}$$

$$|\Phi\rangle_3 = \begin{pmatrix} \beta \\ -\alpha \end{pmatrix} \tag{3.65}$$

After final normalization, Bob receives :

$$|\Phi\rangle_3 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{3.66}$$

Correction : XZ.

Alice Measures $|10\rangle$

$$|\Phi\rangle_2 \to^{P_{10}} |\Phi^*\rangle_3 = \frac{P_{10}\,|\Phi\rangle_2}{\|P_{10}\,|\Phi\rangle_2\|} \tag{3.67}$$

The normalized state shared between Charlie and Bob is

$$|\Phi^*\rangle_3 = \begin{pmatrix} \alpha \\ 0 \\ 0 \\ -\beta \end{pmatrix} \tag{3.68}$$

If Charlie measures $|+\rangle$:

$$|\Phi^*\rangle_3 \to^{P_+} |\Phi\rangle_3 = \frac{P_+\,|\Phi^*\rangle_3}{\|P_+\,|\Phi^*\rangle_3\|} \tag{3.69}$$

$$|\Phi\rangle_3 = \begin{pmatrix} \alpha \\ -\beta \end{pmatrix} \tag{3.70}$$

After final normalization, Bob receives :

$$|\Phi\rangle_3 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{3.71}$$

Correction : Z.

If Charlie measures $|-\rangle$:

$$|\Phi^*\rangle_3 \to^{P_-} |\Phi\rangle_3 = \frac{P_-\,|\Phi^*\rangle_3}{\|P_-\,|\Phi^*\rangle_3\|} \tag{3.72}$$

$$|\Phi\rangle_3 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{3.73}$$

After final normalization, Bob receives :

$$|\Phi\rangle_3 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{3.74}$$

Correction : I.

Alice Measures $|11\rangle$

$$|\Phi\rangle_2 \rightarrow^{P_{11}} |\Phi^*\rangle_3 = \frac{P_{11}|\Phi\rangle_2}{\|P_{11}|\Phi\rangle_2\|} \tag{3.75}$$

The normalized state shared between Charlie and Bob is

$$|\Phi^*\rangle_3 = \begin{pmatrix} -\beta \\ 0 \\ 0 \\ \alpha \end{pmatrix} \tag{3.76}$$

If Charlie measures $|+\rangle$:

$$|\Phi^*\rangle_3 \rightarrow^{P_+} |\Phi\rangle_3 = \frac{P_+|\Phi^*\rangle_3}{\|P_+|\Phi^*\rangle_3\|} \tag{3.77}$$

$$|\Phi\rangle_3 = \begin{pmatrix} -\beta \\ \alpha \end{pmatrix} \tag{3.78}$$

After final normalization, Bob receives :

$$|\Phi\rangle_3 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{3.79}$$

Correction : XZ.

If Charlie measures $|-\rangle$:

$$|\Phi^*\rangle_3 \rightarrow^{P_-} |\Phi\rangle_3 = \frac{P_-|\Phi^*\rangle_3}{\|P_-|\Phi^*\rangle_3\|} \tag{3.80}$$

$$|\Phi\rangle_3 = \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \tag{3.81}$$

After final normalization, Bob receives :

$$|\Phi\rangle_3 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{3.82}$$

Correction : X.

Bra-Ket Formalism :

In this section, we describe the quantum teleportation process using the Bra–Ket formalism. The protocol relies on a multipartite entangled state shared between the communicating parties.

First, we define the Greenberger–Horne–Zeilinger (GHZ) state, which represents a maximally entangled three-qubit state. It is written as

$$|GHZ\rangle_{ACB} = \frac{1}{\sqrt{2}} \left(|000\rangle + |111\rangle\right), \tag{3.83}$$

Alice possesses an arbitrary qubit that she wishes to transmit to Bob. This qubit can be expressed as

$$|Q\rangle_{A} = \alpha\,|0\rangle + \beta\,|1\rangle\,, \tag{3.84}$$

The total quantum state of the system is obtained by taking the tensor product of Alice's qubit with the shared GHZ state. The initial global state is therefore

$$|\Phi\rangle_{0} = |Q\rangle_{A} \otimes |GHZ\rangle\,, \tag{3.85}$$

Substituting the expressions of the states gives

$$|\Phi\rangle_{0} = \left(\alpha\,|0\rangle + \beta\,|1\rangle\right) \otimes \left(\frac{1}{\sqrt{2}} \left(|000\rangle + |111\rangle\right)\right), \tag{3.86}$$

By expanding the tensor product, we obtain

$$|\Phi\rangle_{0} = \frac{1}{\sqrt{2}} \left(\alpha\,|0000\rangle + \alpha\,|0111\rangle + \beta\,|1000\rangle + \beta\,|1111\rangle\right). \tag{3.87}$$

This state represents the initial configuration of the system before any measurement is performed.

The teleportation protocol requires measurements in the Bell basis, which consists of four maximally entangled two-qubit states defined as

$$\left|\Psi^{\mp}\right\rangle = \frac{1}{\sqrt{2}} \left(|00\rangle \mp |11\rangle\right), \tag{3.88}$$

$$\left|\Phi^{\mp}\right\rangle = \frac{1}{\sqrt{2}} \left(|01\rangle \mp |10\rangle\right). \tag{3.89}$$

In addition, we define the following single-qubit basis states

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right), \tag{3.90}$$

$$|-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right). \tag{3.91}$$

Alice performs a measurement in the Bell basis on her qubits. Suppose that the measurement outcome corresponds to the Bell state $|\Psi^+\rangle$. The associated projection operator is given by

$$P_{\Psi^+} = \left|\Psi^+\right\rangle\left\langle\Psi^+\right| \otimes I_4. \tag{3.92}$$

Applying this projector to the initial state yields

$$|\Phi\rangle_0 \to^{P_{\Psi^+}} |\Phi\rangle_1 = \frac{1}{2}\left(\alpha\,|00\rangle + \beta\,|11\rangle\right). \tag{3.93}$$

After normalization, the post-measurement state becomes

$$\begin{aligned} |\Phi\rangle_1 &= \frac{P_{\Psi^+}|\Phi\rangle_0}{\|P_{\Psi^+}|\Phi\rangle_0\|} = \frac{\frac{1}{2}\left(\alpha\,|00\rangle + \beta\,|11\rangle\right)}{\sqrt{\left(\frac{1}{2}\right)^2\left(|\alpha|^2 + |\beta|^2\right)}} \\ &= \frac{\frac{1}{2}\left(\alpha\,|00\rangle + \beta\,|11\rangle\right)}{\frac{1}{2}} \\ &= \alpha\,|00\rangle + \beta\,|11\rangle\,. \end{aligned} \tag{3.94}$$

Charlie then performs a measurement on his qubit in the basis $\{|+\rangle, |-\rangle\}$.

Case 1 : Charlie obtains $|+\rangle$.

The corresponding projection operator is

$$P_+ = |+\rangle\langle+| \otimes I_2. \tag{3.95}$$

Applying this projector to the state $|\Phi\rangle_1$ gives

$$P_+\,|\Phi\rangle_1 \to^{P_+} |\Phi\rangle_2 = \frac{1}{\sqrt{2}}\left(\alpha\,|0\rangle + \beta\,|1\rangle\right). \tag{3.96}$$

After normalization, the resulting state is

$$
\begin{aligned}
|\Phi\rangle_2 &= \frac{P_+ |\Phi\rangle_1}{\|P_+ |\Phi\rangle_1\|} \\
&= \frac{\frac{1}{\sqrt{2}} (\alpha |0\rangle + \beta |1\rangle)}{\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 \left(|\alpha|^2 + |\beta|^2\right)}} = \frac{\frac{1}{\sqrt{2}} (\alpha |0\rangle + \beta |1\rangle)}{\frac{1}{\sqrt{2}}} \\
&= \alpha |0\rangle + \beta |1\rangle .
\end{aligned} \tag{3.97}
$$

$$
|\Phi\rangle_2 = \alpha |0\rangle + \beta |1\rangle . \tag{3.98}
$$

In this case, Bob already possesses the original quantum state and no corrective operation is required.

Case 2 : Charlie obtains $|-\rangle$.

The corresponding projection operator is

$$
P_- = |-\rangle \langle -| \otimes I_2. \tag{3.99}
$$

Applying this operator results in

$$
|\Phi\rangle_1 \to^{P_-} |\Phi\rangle_2 = \frac{1}{\sqrt{2}} (\alpha |0\rangle - \beta |1\rangle) \tag{3.100}
$$

After normalization, the resulting state is

$$
\begin{aligned}
|\Phi\rangle_2 &= \frac{P_- |\Phi\rangle_1}{\|P_- |\Phi\rangle_1\|} \\
&= \frac{\frac{1}{\sqrt{2}} (\alpha |0\rangle - \beta |1\rangle)}{\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 \left(|\alpha|^2 + |\beta|^2\right)}} = \frac{\frac{1}{\sqrt{2}} (\alpha |0\rangle - \beta |1\rangle)}{\frac{1}{\sqrt{2}}} \\
&= \alpha |0\rangle - \beta |1\rangle
\end{aligned} \tag{3.101}
$$

$$
|\Phi\rangle_2 = \alpha |0\rangle - \beta |1\rangle \tag{3.102}
$$

To recover the original state, Bob applies the Pauli-Z gate

$$
|\Phi\rangle_2 = Z (\alpha |0\rangle - \beta |1\rangle) = \alpha |0\rangle + \beta |1\rangle \tag{3.103}
$$

After this correction, Bob successfully reconstructs the initial qubit $|Q\rangle_A$.

Alice performs a measurement in the Bell basis on her qubits. Suppose that the measurement outcome corresponds to the Bell state $|\Psi^-\rangle$. The associated projection operator is given by

$$P_{\Psi^-} = \left|\Psi^-\right\rangle \left\langle\Psi^-\right| \otimes I_{4.} \tag{3.104}$$

To find the state of the remaining qubits after Alice's measurement, we apply the projection operator to the initial global state as follows

$$|\Phi\rangle_0 \rightarrow^{P_{\Psi^-}} |\Phi\rangle_1 = \frac{1}{2}\left(\alpha\,|00\rangle - \beta\,|11\rangle\right). \tag{3.105}$$

Since the resulting vector is not normalized, we divide by its norm to obtain the post-measurement state $|\Phi\rangle_1$.

$$\begin{aligned} |\Phi\rangle_1 &= \frac{P_{\Psi^-}\,|\Phi\rangle_0}{\|P_{\Psi^-}\,|\Phi\rangle_0\|} \\ &= \frac{\frac{1}{2}\left(\alpha\,|00\rangle - \beta\,|11\rangle\right)}{\sqrt{\left(\frac{1}{2}\right)^2\left(|\alpha|^2 + |\beta|^2\right)}} = \frac{\frac{1}{2}\left(\alpha\,|00\rangle - \beta\,|11\rangle\right)}{\frac{1}{2}} \\ &= \alpha\,|00\rangle - \beta\,|11\rangle\,. \end{aligned} \tag{3.106}$$

In the next stage of the protocol, Charlie performs a measurement on his qubit. Consider the case where Charlie obtains the outcome $|+\rangle$. The corresponding projection is

$$P_+ = |+\rangle\,\langle+| \otimes I_2. \tag{3.107}$$

By applying the projector P_+, we obtain

$$|\Phi\rangle_1 \rightarrow^{P_+} |\Phi\rangle_2 = \frac{1}{\sqrt{2}}\left(\alpha\,|0\rangle - \beta\,|1\rangle\right). \tag{3.108}$$

After applying Charlie's projection and normalizing the state again, the system collapses to

$$\begin{aligned}
|\Phi\rangle_2 &= \frac{P_+ |\Phi\rangle_1}{\|P_+ |\Phi\rangle_1\|} \\
&= \frac{\frac{1}{\sqrt{2}} (\alpha |0\rangle + \beta |1\rangle)}{\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 \left(|\alpha|^2 + |\beta|^2\right)}} = \frac{\frac{1}{\sqrt{2}} (\alpha |0\rangle - \beta |1\rangle)}{\frac{1}{\sqrt{2}}} \\
&= \alpha |0\rangle - \beta |1\rangle .
\end{aligned} \tag{3.109}$$

$$|\Phi\rangle_2 = \alpha |0\rangle - \beta |1\rangle$$

Finally, to recover the original state sent by Alice, Bob must apply a local unitary operation based on the previous measurement outcomes. In this specific case, Bob applies the Pauli-Z gate :

$$|\Phi\rangle_2 = Z (\alpha |0\rangle - \beta |1\rangle) = \alpha |0\rangle + \beta |1\rangle . \tag{3.110}$$

Alternatively, we consider the second possibility where Charlie's measurement yields the state$|-\rangle$:

$$p_- = |-\rangle \langle -| \otimes I_2. \tag{3.111}$$

The projection operator associated with this outcome is applied to the intermediate state $|\Phi\rangle_1$ as follows :

$$|\Phi\rangle_1 \to^{P_-} |\Phi\rangle_2 = \frac{1}{\sqrt{2}} (\alpha |0\rangle + \beta |1\rangle) \tag{3.112}$$

Since the resulting vector is not normalized, we divide by its norm to obtain the post-measurement state $|\Phi\rangle_2$.

$$\begin{aligned}
|\Phi\rangle_2 &= \frac{P_+ |\Phi\rangle_1}{\|P_+ |\Phi\rangle_1\|} \\
&= \frac{\frac{1}{\sqrt{2}} (\alpha |0\rangle + \beta |1\rangle)}{\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 \left(|\alpha|^2 + |\beta|^2\right)}} = \frac{\frac{1}{\sqrt{2}} (\alpha |0\rangle + \beta |1\rangle)}{\frac{1}{\sqrt{2}}} \\
&= \alpha |0\rangle + \beta |1\rangle .
\end{aligned} \tag{3.113}$$

$$|\Phi\rangle_2 = \alpha |0\rangle + \beta |1\rangle .$$

In this specific case, the state naturally collapses back to the original qubit state without the need for further unitary corrections from Bob.

$$|\Phi\rangle_2 = \alpha\,|0\rangle + \beta\,|1\rangle \tag{3.114}$$

Now, we examine another scenario where Alice's measurement outcome corresponds to the Bell state $|\Phi^+\rangle$ **:**

$$P_{\Phi^+} = \left|\Phi^+\right\rangle \left\langle\Phi^+\right| \otimes I_4 \tag{3.115}$$

$$|\Phi\rangle_0 \to^{P_{\Phi^+}} |\Phi\rangle_1 = \frac{1}{2}\left(\beta\,|00\rangle + \alpha\,|11\rangle\right) \tag{3.116}$$

Since the resulting vector is not normalized, we divide by its norm to obtain the post-measurement state $|\Phi\rangle_1$.

$$\begin{aligned} |\Phi\rangle_1 &= \frac{P_{\Phi^+}\,|\Phi\rangle_0}{\|P_{\Phi^+}\,|\Phi\rangle_0\|} \\ &= \frac{\frac{1}{2}\left(\beta\,|00\rangle + \alpha\,|11\rangle\right)}{\sqrt{\left(\frac{1}{2}\right)^2\left(|\alpha|^2 + |\beta|^2\right)}} = \frac{\frac{1}{2}\left(\beta\,|00\rangle + \alpha\,|11\rangle\right)}{\frac{1}{2}} \\ &= \beta\,|00\rangle + \alpha\,|11\rangle \end{aligned} \tag{3.117}$$

Following Alice's measurement, Charlie proceeds to measure his qubit. If he obtains the state $|+\rangle$, the global state is further projected as follows :

$$P_+ = |+\rangle\,\langle+| \otimes I_2 \tag{3.118}$$

Applying this operator results in

$$|\Phi\rangle_1 \to^{P_+} |\Phi\rangle_2 = \frac{1}{\sqrt{2}}\left(\beta\,|0\rangle + \alpha\,|1\rangle\right). \tag{3.119}$$

The normalized state becomes

$$|\Phi\rangle_2 = \frac{P_+ |\Phi\rangle_1}{\|P_+ |\Phi\rangle_1\|} \tag{3.120}$$

$$= \frac{\frac{1}{\sqrt{2}} (\beta |0\rangle + \alpha |1\rangle)}{\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 (|\alpha|^2 + |\beta|^2)}} = \frac{\frac{1}{\sqrt{2}} (\beta |0\rangle + \alpha |1\rangle)}{\frac{1}{\sqrt{2}}} \tag{3.121}$$

$$= \beta |0\rangle + \alpha |1\rangle \tag{3.122}$$

$$|\Phi\rangle_2 = \beta |0\rangle + \alpha |1\rangle$$

"In this case, the resulting state at Bob's side is a bit-flipped version of the original qubit. Therefore, Bob must apply the Pauli-X gate to recover the initial state $|Q\rangle_A$.

$$|\Phi\rangle_2 = X (\beta |0\rangle + \alpha |1\rangle) = \alpha |0\rangle + \beta |1\rangle$$

Alternatively, if Charlie measures the qubit in the state $|-\rangle$, the projection leads to the following state :

$$P_- = |-\rangle \langle -| \otimes I_2 \tag{3.123}$$

Applying this projector to the state $|\Phi\rangle_1$ leads to the following projected state

$$|\Phi\rangle_1 \to^{P_-} |\Phi\rangle_2 = \frac{1}{\sqrt{2}} (\beta |0\rangle - \alpha |1\rangle) . \tag{3.124}$$

In order to obtain the normalized state, we divide by the norm of the projected vector. Thus, the normalized state becomes

$$|\Phi\rangle_2 = \frac{P_- |\Phi\rangle_1}{\|P_- |\Phi\rangle_1\|} \tag{3.125}$$

$$= \frac{\frac{1}{\sqrt{2}} (\beta |0\rangle - \alpha |1\rangle)}{\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 (|\alpha|^2 + |\beta|^2)}} = \frac{\frac{1}{\sqrt{2}} (\beta |0\rangle - \alpha |1\rangle)}{\frac{1}{\sqrt{2}}}$$

$$= \beta |0\rangle - \alpha |1\rangle .$$

After normalization, we obtain

$$|\Phi\rangle_2 = \beta |0\rangle - \alpha |1\rangle .$$

To rectify both the bit-flip and the phase-flip introduced by this specific measurement outcome, Bob must apply a combination of Pauli gates, specifically ZX.

$$|\Phi\rangle_2 = ZX\left(\beta\,|0\rangle - \alpha\,|1\rangle\right) = \alpha\,|0\rangle + \beta\,|1\rangle \tag{3.126}$$

Suppose that the result of Alice's measurement is the state $|\Phi^-\rangle$. The corresponding projection operator is given by :

$$P_{\Phi^-} = \left|\Phi^-\right\rangle\left\langle\Phi^-\right| \otimes I_4 \tag{3.127}$$

Applying this projector to the initial state $|\Phi\rangle_0$ gives

$$|\Phi\rangle_0 \to^{P_{\Phi^-}} |\Phi\rangle_1 = \frac{1}{2}\left(\beta\,|00\rangle - \alpha\,|11\rangle\right). \tag{3.128}$$

To obtain the normalized state, we divide by the norm of the projected vector. Thus, the post-measurement state becomes

$$\begin{aligned} |\Phi\rangle_1 &= \frac{P_{\Phi^-}\,|\Phi\rangle_0}{\|P_{\Phi^-}\,|\Phi\rangle_0\|} \\ &= \frac{\frac{1}{2}\left(\beta\,|00\rangle - \alpha\,|11\rangle\right)}{\sqrt{\left(\frac{1}{2}\right)^2\left(|\alpha|^2 + |\beta|^2\right)}} = \frac{\frac{1}{2}\left(\beta\,|00\rangle - \alpha\,|11\rangle\right)}{\frac{1}{2}} \\ &= \beta\,|00\rangle - \alpha\,|11\rangle\,. \end{aligned} \tag{3.129}$$

Next,if Charlie measures the state $|+\rangle$, the associated projection operator is.

$$P_+ = |+\rangle\langle+| \otimes I_2 \tag{3.130}$$

Applying this projector to the state $|\Phi\rangle_1$ yields

$$|\Phi\rangle_1 \to^{P_+} |\Phi\rangle_2 = \frac{1}{\sqrt{2}}\left(\beta\,|0\rangle - \alpha\,|1\rangle\right). \tag{3.131}$$

The normalized state is therefore

$$\begin{aligned}
|\Phi\rangle_2 &= \frac{P_+ |\Phi\rangle_1}{\|P_+ |\Phi\rangle_1\|} \qquad (3.132)\\
&= \frac{\frac{1}{\sqrt{2}} (\beta |0\rangle - \alpha |1\rangle)}{\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 (|\alpha|^2 + |\beta|^2)}} = \frac{\frac{1}{\sqrt{2}} (\beta |0\rangle - \alpha |1\rangle)}{\frac{1}{\sqrt{2}}}\\
&= \beta |0\rangle - \alpha |1\rangle .
\end{aligned}$$

To recover the original quantum state, Bob must apply ZX gate. Therefore

$$|\Phi\rangle_2 = ZX (\beta |0\rangle - \alpha |1\rangle) = \alpha |0\rangle + \beta |1\rangle \qquad (3.133)$$

If Charlie measures the qubit in the state $|-\rangle$, the projection operator is given by.

$$P_+ = |-\rangle \langle -| \otimes I_2. \qquad (3.134)$$

Applying this projector to the state $|\Phi\rangle_1$ leads to

$$|\Phi\rangle_1 \to^{P_-} |\Phi\rangle_2 = \frac{1}{\sqrt{2}} (\beta |0\rangle + \alpha |1\rangle) . \qquad (3.135)$$

After normalization, the resulting state becomes

$$\begin{aligned}
|\Phi\rangle_2 &= \frac{P_+ |\Phi\rangle_1}{\|P_+ |\Phi\rangle_1\|} \qquad (3.136)\\
&= \frac{\frac{1}{\sqrt{2}} (\beta |0\rangle + \alpha |1\rangle)}{\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 (|\alpha|^2 + |\beta|^2)}} = \frac{\frac{1}{\sqrt{2}} (\beta |0\rangle + \alpha |1\rangle)}{\frac{1}{\sqrt{2}}}\\
&= \beta |0\rangle + \alpha |1\rangle
\end{aligned}$$

$$|\Phi\rangle_2 = \beta |0\rangle + \alpha |1\rangle$$

To recover the original state, Bob applies the Pauli-X gate.

$$|\Phi\rangle_2 = X(\beta |0\rangle + \alpha |1\rangle) = \alpha |0\rangle + \beta |1\rangle \qquad (3.137)$$

FIGURE 3.4: Controlled Quantum Teleportation : Results obtained after Alice's measurement.

Résultat de la mesure d'Alice	Qubits de Charlie et Bob
$\|\psi^+\rangle$	$\frac{1}{\sqrt{2}}(\alpha\|00\rangle + \beta\|11\rangle)$
$\|\psi^-\rangle$	$\frac{1}{\sqrt{2}}(\alpha\|00\rangle - \beta\|11\rangle)$
$\|\phi^+\rangle$	$\frac{1}{\sqrt{2}}(\alpha\|11\rangle + \beta\|00\rangle)$
$\|\phi^-\rangle$	$\frac{1}{\sqrt{2}}(\alpha\|11\rangle - \beta\|00\rangle)$

FIGURE 3.5: Controlled Quantum Teleportation : Outcomes after Alice's measurement

Rest Mes d'Alice	Qubits de Charlie et Bob	Rest Mes de Charlie	Qubit de Bob	Corr Bob
$\|\psi^+\rangle$	$(\alpha\|00\rangle + \beta\|11\rangle)$	$\|+\rangle$	$\alpha\|0\rangle + \beta\|1\rangle$	I
$\|\psi^+\rangle$	$(\alpha\|00\rangle + \beta\|11\rangle)$	$\|-\rangle$	$\alpha\|0\rangle - \beta\|1\rangle$	Z
$\|\psi^-\rangle$	$(\alpha\|00\rangle - \beta\|11\rangle)$	$\|+\rangle$	$\alpha\|0\rangle - \beta\|1\rangle$	Z
$\|\psi^-\rangle$	$(\alpha\|00\rangle - \beta\|11\rangle)$	$\|-\rangle$	$\alpha\|0\rangle + \beta\|1\rangle$	I
$\|\phi^+\rangle$	$(\alpha\|11\rangle + \beta\|00\rangle)$	$\|+\rangle$	$\beta\|0\rangle + \alpha\|1\rangle$	X
$\|\phi^+\rangle$	$(\alpha\|11\rangle + \beta\|00\rangle)$	$\|-\rangle$	$\beta\|0\rangle - \alpha\|1\rangle$	Z X
$\|\phi^-\rangle$	$(\alpha\|11\rangle - \beta\|00\rangle)$	$\|+\rangle$	$\beta\|0\rangle - \alpha\|1\rangle$	Z X
$\|\phi^-\rangle$	$(\alpha\|11\rangle - \beta\|00\rangle)$	$\|-\rangle$	$\beta\|0\rangle + \alpha\|1\rangle$	X

3.4 Controlled bidirectional quantum teleportation

Controlled bidirectional quantum teleportation is a protocol that enables two transmitting parties (T_1 and T_2) to simultaneously exchange unknown quantum states with two receiving parties. This process is strictly governed by a third party, the controller, who supervises the quantum channel and grants permission for the successful completion of the teleportation.

In this protocol, we consider a configuration involving two transmitters, T_1 and T_2, each possessing an arbitrary single-qubit state to be teleported :

Two transmitters (T_1 and T_2), each holding an arbitrary qubit :

$$|Q\rangle_1 = \alpha_1|0\rangle + \beta_1|1\rangle \tag{3.138}$$

$$|Q\rangle_2 = \alpha_2|0\rangle + \beta_2|1\rangle \tag{3.139}$$

Two receivers (R_1 and R_2), who are the intended destinations for the teleported states.

The controller (C) acts as a quantum switch, routing the teleportation process according to the following two scenarios :

Scenario 1 :$|Q\rangle_1$ is sent to receiver$_1$,and $|Q\rangle_2$ is sent to receiver$_2$.

Scenario 2 :$|Q\rangle_1$ is sent to receiver$_2$,and $|Q\rangle_2$ is sent to receiver$_1$.

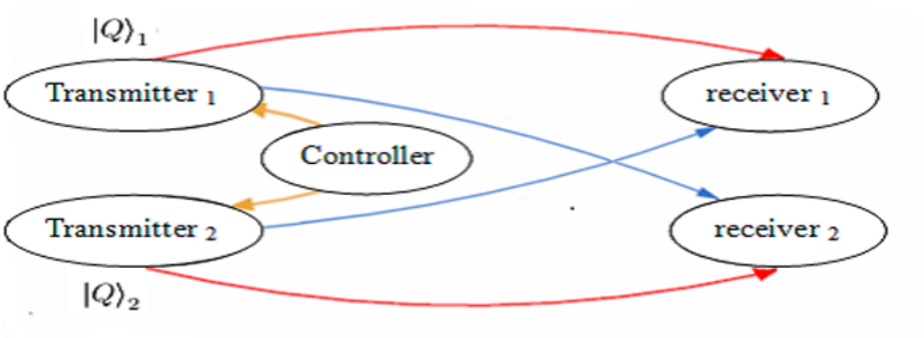

FIGURE 3.6: Bidirectional quantum teleportation protocol

The five participants share the following entangled quantum channel :

$$|Channel\rangle_{T_1T_2C_{tr_1}C_{tr_2}R_1R_2} = \frac{1}{2\sqrt{2}}\begin{bmatrix} |000000\rangle + |000111\rangle + |011001\rangle + |011110\rangle \\ +|101010\rangle + |101101\rangle + |110011\rangle + |110100\rangle \end{bmatrix} \tag{3.140}$$

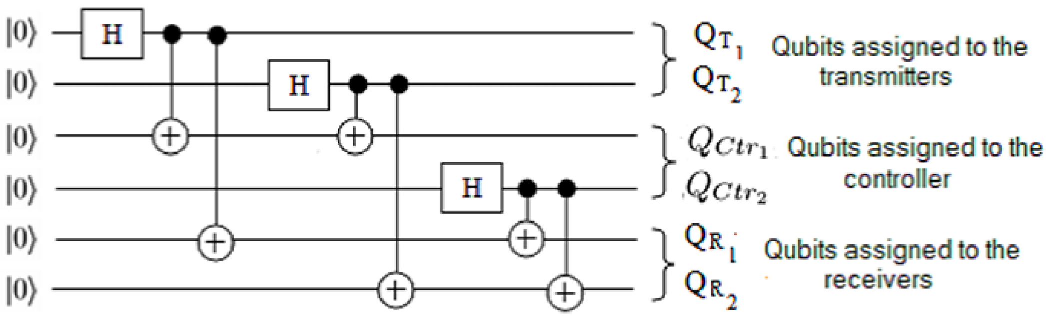

FIGURE 3.7: The quantum teleportation channel

The initial state is given as follows :

$$|\Phi_0\rangle = |Q_1\rangle \otimes |Q_2\rangle \otimes |\textbf{Channel}\rangle_{T_1T_2C_{tr_1}C_{tr_2}R_1R_2} \tag{3.141}$$

First, we begin by defining the independent quantum states of qubits Q_1 and Q_2 that the parties wish to transmit. They are represented as a linear superposition as follows :

$$|Q_1\rangle \otimes |Q_2\rangle = (\alpha_1 |0\rangle + \beta_1 |1\rangle) \otimes (\alpha_2 |0\rangle + \beta_2 |1\rangle) \\ = \alpha_1\alpha_2 |00\rangle + \alpha_1\beta_2 |01\rangle + \beta_1\alpha_2 |10\rangle + \beta_1\beta_2 |11\rangle \tag{3.142}$$

Now, we couple the qubits to be teleported with the communication channel to form the complete initial state of the system before any measurements are performed. The general equation is :

$$|\Phi_0\rangle = \frac{1}{2\sqrt{2}} (\alpha_1\alpha_2 |00\rangle + \alpha_1\beta_2 |01\rangle + \beta_1\alpha_2 |10\rangle + \beta_1\beta_2 |11\rangle) \\ \otimes \begin{pmatrix} |000000\rangle + |000111\rangle + |011001\rangle + |011110\rangle \\ + |101010\rangle + |101101\rangle + |110011\rangle + |110100\rangle \end{pmatrix} \tag{3.143}$$

$$|\Phi_0\rangle = \alpha_1\alpha_2 \begin{pmatrix} |00000000\rangle + |00000111\rangle + |00011001\rangle + |00011110\rangle \\ + |00101010\rangle + |00101101\rangle + |00110011\rangle + |00110100\rangle \end{pmatrix} \\ + \alpha_1\beta_2 \begin{pmatrix} |01000000\rangle + |01000111\rangle + |01011001\rangle + |01011110\rangle \\ + |01101010\rangle + |01101101\rangle + |01110011\rangle + |01110100\rangle \end{pmatrix} \\ + \beta_1\alpha_2 \begin{pmatrix} |10000000\rangle + |10000111\rangle + |10011001\rangle + |10011110\rangle \\ + |10101010\rangle + |10101101\rangle + |10110011\rangle + |10110100\rangle \end{pmatrix} \\ + \beta_1\beta_2 \begin{pmatrix} |11000000\rangle + |11000111\rangle + |11011001\rangle + |11011110\rangle \\ + |11101010\rangle + |11101101\rangle + |11110011\rangle + |11110100\rangle \end{pmatrix} \tag{3.144}$$

The two transmitters perform Bell measurements on their qubits. This basis is defined by the following four well-known kets :

$$|Bell_{ij}\rangle = \frac{1}{\sqrt{2}} \left[|0j\rangle + (-1)^i |1\bar{j}\rangle \right], i, j = 01$$

In a more detailed way

$$|Bell_{00}\rangle = \frac{1}{\sqrt{2}} [|00\rangle + |11\rangle] \tag{3.145}$$

$$|Bell_{01}\rangle = \frac{1}{\sqrt{2}} [|01\rangle + |10\rangle] \tag{3.146}$$

$$|Bell_{10}\rangle = \frac{1}{\sqrt{2}} [|00\rangle - |11\rangle] \tag{3.147}$$

$$|Bell_{11}\rangle = \frac{1}{\sqrt{2}} [|01\rangle - |10\rangle] \tag{3.148}$$

After performing this measurement, the initial state $|\Phi_0\rangle$ becomes the state $|\Phi_1\rangle$. The projector onto the Bell states is written as follows

$$M_{ij} = |Bell_{ij}\rangle \langle Bell_{ij}| \, , i, j = 0, 1 \tag{3.149}$$

Operations Performed by the Controller

Based on the scenarios mentioned above, the controller performs the following operations :

Technical Breakdown of the Scenarios :

Scenario 1 : The controller applies a Hadamard gate to its first qubit, denoted as $H(C_1)$

$$|\Phi_1\rangle \rightarrow^{H(C_1)} |\Phi_2\rangle = \left(I^{\otimes 4} \otimes H \otimes I^{\otimes 3}\right) |\Phi_1\rangle \tag{3.150}$$

Scenario 2 : The controller applies a CNOT gate between its two qubits, $CNOT(C_1, C_2)$, followed by a Hadamard gate on its first qubit$H(C_1)$.

$$|\Phi_1\rangle \rightarrow^{CNOT} |\Phi_2\rangle = \left(I^{\otimes 4} \otimes U_{CNOT}(C_1, C_2) \otimes I^{\otimes 2}\right) |\Phi_1\rangle \tag{3.151}$$

$$\rightarrow^{H(C_1)} |\Phi_3\rangle = \left(I^{\otimes 4} \otimes H \otimes I^{\otimes 3}\right) |\Phi_2\rangle \tag{3.152}$$

Senders (T_1 and T_2), Suppose we have two senders who have measured their systems and obtained specific results (which are the Bell states $|Bell_{00}\rangle$ and $|Bell_{01}\rangle$,Before applying the measurements, a SWAP operation must be performed between qubit Q_2 and qubit T_1.

$$|\Phi_1\rangle = \frac{M_{00,01}\mathbf{S} |\Phi_0\rangle}{\| M_{00,01} S |\Phi_0\rangle \|} \tag{3.153}$$

$$\mathbf{S} = I_2 \otimes S \otimes I^{\otimes 5} \tag{3.154}$$

The SWAP gate is given as the following matrix :

$$SWAP = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.155}$$

$$M_{00,01} = \langle Bell_{00}| \otimes \langle Bell_{01}| \otimes I^{\otimes 4} \tag{3.156}$$

$$S|\Phi_0\rangle = I_2 \otimes S \otimes I^{\otimes 5}|\Phi_0\rangle \tag{3.157}$$

$$\begin{aligned} S|\Phi_0\rangle = \frac{1}{2\sqrt{2}}\alpha_1\alpha_2 &\begin{pmatrix} |00000000\rangle + |00000111\rangle + |00011001\rangle + |00011110\rangle \\ + |01001010\rangle + |01001101\rangle + |01010011\rangle + |01010100\rangle \end{pmatrix} \\ + \alpha_1\beta_2 &\begin{pmatrix} |00100000\rangle + |00100111\rangle + |00111001\rangle + |00111110\rangle \\ + |01101010\rangle + |01101101\rangle + |01110011\rangle + |01110100\rangle \end{pmatrix} \\ + \beta_1\alpha_2 &\begin{pmatrix} |10000000\rangle + |10000111\rangle + |10011001\rangle + |10011110\rangle \\ + |11001010\rangle + |11001101\rangle + |11010011\rangle + |11010100\rangle \end{pmatrix} \\ + \beta_1\beta_2 &\begin{pmatrix} |10100000\rangle + |10100111\rangle + |10111001\rangle + |10111110\rangle \\ + |11101010\rangle + |11101101\rangle + |11110011\rangle + |11110100\rangle \end{pmatrix} \end{aligned} \tag{3.158}$$

To determine the resulting state and its associated probability, we apply the measurement operator $M\{00,01\}$ to the state $S|\Phi_0\rangle$. The calculation of the squared norm, which represents the success probability, is carried out as follows :

$$M_{00,01} = \langle Bell_{00}| \otimes \langle Bell_{01}| \otimes I^{\otimes 4} = \frac{1}{\sqrt{2}}[\langle 00| + \langle 11|] \otimes \frac{1}{\sqrt{2}}[\langle 01| + \langle 10|] \otimes I^{\otimes 4} \tag{3.159}$$

$$M_{00,01} = \frac{1}{2}(\langle 0001| + \langle 0010| + \langle 1101| + \langle 1110|) \otimes I^{\otimes 4} \tag{3.160}$$

$$\begin{aligned} M_{00,01}S|\Phi_0\rangle = \frac{1}{4\sqrt{2}}&[\alpha_1\alpha_2(|1001\rangle + |1110\rangle) + \alpha_1\beta_2(|0000\rangle + |0111\rangle) \\ &+ \beta_1\alpha_2(|0011\rangle + |0100\rangle) + \beta_1\beta_2(|1010\rangle + |1101\rangle)] \end{aligned} \tag{3.161}$$

The squared norm of the projected state is calculated by taking the inner product of the state with itself, which determines the probability of obtaining this specific measurement outcome :

$$\|M_{00,01}S\,|\Phi_0\rangle\|^2 = \frac{1}{4\sqrt{2}}\left\{\begin{array}{c}\alpha_1^*\alpha_2^*\left(\langle 1001| + \langle 1110|\right) + \alpha_1^*\beta_2^*\left(\langle 0000| + \langle 0111|\right)\\ +\beta_1^*\alpha_2^*\left(\langle 0011| + \langle 0100|\right) + \beta_1^*\beta_2^*\left(\langle 1010| + \langle 1101|\right)\end{array}\right\} \tag{3.162}$$

$$\frac{1}{4\sqrt{2}}\left\{\begin{array}{c}\alpha_1\alpha_2\left(|1001\rangle + |1110\rangle\right) + \alpha_1\beta_2\left(|0000\rangle + |0111\rangle\right)\\ +\beta_1\alpha_2\left(|0011\rangle + |0100\rangle\right) + \beta_1\beta_2\left(|1010\rangle + |1101\rangle\right)\end{array}\right\}$$
$$= \frac{1}{32}\left(2\|\alpha_1\|^2\|\alpha_2\|^2 + 2\|\alpha_1\|^2\|\beta_2\|^2 + 2\|\beta_1\|^2\|\alpha_1\|^2 + 2\|\beta_1\|^2\|\beta_2\|^2\right)$$
$$= \frac{1}{32}\left(2\|\alpha_1\|^2 + 2\|\beta_1\|^2\right)$$
$$\Longrightarrow \|M_{00,01}S\,|\Phi_0\rangle\|^2 = \frac{1}{4}$$

By substituting the calculated normalization factor back into the state equation, the normalized state $|\Phi_1\rangle$ is obtained as follows :

$$|\Phi_1\rangle = \frac{1}{\sqrt{2}}\left(\begin{array}{c}\alpha_1\alpha_2\left\{|1001\rangle + |1110\rangle\right\} + \alpha_1\beta_2\left\{|0000\rangle + |0111\rangle\right\}\\ +\beta_1\alpha_2\left\{|0011\rangle + |0100\rangle\right\} + \beta_1\beta_2\left\{|1010\rangle + |1101\rangle\right\}\end{array}\right) \tag{3.163}$$

Scenario 1

Apply the Hadamard gate to the first control qubit (Ctr_1)to transform the first state into the second state :

$$|\Phi_1\rangle \rightarrow^{H_{Ctr_1}} |\Phi_2\rangle = \frac{1}{2}\left(\begin{array}{c}\alpha_1\alpha_2\left\{|0001\rangle - |1001\rangle + |0110\rangle - |1110\rangle\right\}\\ +\alpha_1\beta_2\left\{|0000\rangle + |1000\rangle + |0111\rangle + |1111\rangle\right\}\\ +\beta_1\alpha_2\left\{|0011\rangle + |1011\rangle + |0100\rangle + |1100\rangle\right\}\\ +\beta_1\beta_2\left\{|0010\rangle - |1010\rangle + |0101\rangle - |1101\rangle\right\}\end{array}\right) \tag{3.164}$$

After the controller applies the Hadamard gate to the first control qubit, a measurement is performed in the computational basis. If the measurement outcome yields $|01\rangle$, the system collapses into the state $|\Phi_3\rangle$, which is defined by the following projection :

$$|\Phi_2\rangle \rightarrow^{P_{01}} |\Phi_3\rangle = \frac{\langle 01| \otimes I^{\otimes 2}\,|\Phi_2\rangle}{\|\langle 01| \otimes I^{\otimes 2}\,|\Phi_2\rangle\|^2} \tag{3.165}$$

$$|\Phi_3\rangle = \frac{\frac{1}{2}\left(\alpha_1\alpha_2\,|10\rangle + \alpha_1\beta_2\,|11\rangle + \beta_1\alpha_2\,|00\rangle + \beta_1\beta_2\,|01\rangle\right)}{\frac{1}{2}\sqrt{\|\alpha_1\|^2\left(\|\alpha_2\|^2 + \|\beta_2\|^2\right) + \|\beta_1\|^2\left(\|\alpha_2\|^2 + \|\beta_2\|^2\right)}} \tag{3.166}$$

$$|\Phi_3\rangle = \frac{\frac{1}{2}\left(\alpha_1\alpha_2\,|10\rangle + \alpha_1\beta_2\,|11\rangle + \beta_1\alpha_2\,|00\rangle + \beta_1\beta_2\,|01\rangle\right)}{\frac{1}{2}}$$

$$|\Phi_3\rangle = \alpha_1\alpha_2\,|10\rangle + \alpha_1\beta_2\,|11\rangle + \beta_1\alpha_2\,|00\rangle + \beta_1\beta_2\,|01\rangle$$

Finally, the receivers' state $|R_1R_2\rangle$ is extracted from the normalized state $|\Phi_3\rangle$. This state can be decomposed into a tensor product of the two reconstructed qubits, demonstrating the successful teleportation of the original information :

$$|R_1R_2\rangle = \alpha_1\alpha_2\,|10\rangle + \alpha_1\beta_2\,|11\rangle + \beta_1\alpha_2\,|00\rangle + \beta_1\beta_2\,|01\rangle \tag{3.167}$$

$$|R_1R_2\rangle = (\beta_1\,|0\rangle + \alpha_1\,|1\rangle) \otimes (\alpha_2\,|0\rangle + \beta_2\,|1\rangle) \tag{3.168}$$

To ensure that the teleported information matches the original qubits Q_1 and Q_2, a final local unitary correction is applied. In this case, applying the Pauli-X gate to R_1 and the Identity operator to R_2 yields the desired result :

$$(X \otimes I)\,|R_1R_2\rangle = |Q_1\rangle \otimes |Q_2\rangle \tag{3.169}$$

Scenario 2

In the second scenario, the controller performs a different set of operations to enable the teleportation. This involves increasing the entanglement between the control qubits before the final measurement.

Apply the CNOT gate to the control qubits (Ctr_1, Ctr_2), then the Hadamard gate to the first control qubit $(Ctr_1)$to transform the first state into the second state

$$|\Phi_1\rangle \rightarrow^{CNOT_{Ctr_1,Ctr_2}} |\Phi_2^*\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha_1\alpha_2\,\{|1101\rangle + |1010\rangle\} \\ +\alpha_1\beta_2\,\{|0000\rangle + |0111\rangle\} \\ +\beta_1\alpha_2\,\{|0011\rangle + |0100\rangle\} \\ +\beta_1\beta_2\,\{|1110\rangle + |1001\rangle\} \end{pmatrix} \tag{3.170}$$

Following the application of the CNOT gate between (Ctr_1, Ctr_2), a Hadamard gate is applied to the first control qubit. This transforms the intermediate state $|\Phi_2^*\rangle$ into $|\Phi_2\rangle$ as follows :

$$|\Phi_2^*\rangle \rightarrow^{H_{Ctr_1}} |\Phi_2\rangle = \frac{1}{2} \begin{pmatrix} \alpha_1\alpha_2\,\{|0101\rangle - |1101\rangle + |0010\rangle - |1010\rangle\} \\ +\alpha_1\beta_2\,\{|0000\rangle + |1000\rangle + |0111\rangle + |1111\rangle\} \\ +\beta_1\alpha_2\,\{|0011\rangle + |1011\rangle + |0100\rangle + |1100\rangle\} \\ +\beta_1\beta_2\,\{|0110\rangle - |1110\rangle + |0001\rangle\} - |1001\rangle \end{pmatrix} \tag{3.171}$$

After the controller applies the Hadamard gate to the first control qubit, a measurement is performed in the computational basis. If the measurement outcome yields $|11\rangle$, the system collapses into the state $|\Phi_3\rangle$, which is defined by the following projection :

$$|\Phi_2\rangle \to^{P_{11}} |\Phi_3\rangle = \frac{\langle 11| \otimes I_2 |\Phi_2\rangle}{\| \langle 11| \otimes I_2 |\Phi_2\rangle \|^2} \tag{3.172}$$

$$|\Phi_3\rangle = \frac{\frac{1}{2}\left(\beta_1\alpha_2 |00\rangle - \alpha_1\alpha_2 |01\rangle - \beta_1\beta_2 |10\rangle + \alpha_1\beta_2 |11\rangle\right)}{\frac{1}{2}\sqrt{\|\alpha_1\|^2\left(\|\alpha_2\|^2 + \|\beta_2\|^2\right) + \|\beta_1\|^2\left(\|\alpha_2\|^2 + \|\beta_2\|^2\right)}}$$

$$|\Phi_3\rangle = \frac{\frac{1}{2}\left(\beta_1\alpha_2 |00\rangle - \alpha_1\alpha_2 |01\rangle - \beta_1\beta_2 |10\rangle + \alpha_1\beta_2 |11\rangle\right)}{\frac{1}{2}}$$

$$|\Phi_3\rangle = \beta_1\alpha_2 |00\rangle - \alpha_1\alpha_2 |01\rangle - \beta_1\beta_2 |10\rangle + \alpha_1\beta_2 |11\rangle$$

The final state of the receivers, denoted as $|R_1R_2\rangle$, is extracted from the collapsed system state. By rearranging the terms, we can express it as a tensor product of the two reconstructed qubits :

$$|R_1R_2\rangle = \beta_1\alpha_2 |00\rangle - \alpha_1\alpha_2 |01\rangle - \beta_1\beta_2 |10\rangle + \alpha_1\beta_2 |11\rangle \tag{3.173}$$

$$|R_1R_2\rangle = (\alpha_2 |0\rangle - \beta_2 |1\rangle) \otimes (\beta_1 |0\rangle - \alpha_1 |1\rangle)$$

To recover the original quantum information, the receivers must apply specific unitary transformations based on the measurement outcomes. In this scenario, the Z and XZ operations are applied to qubits R_1 and R_2 respectively to correct the phase and bit-flip errors :

$$(Z \otimes XZ) |R_1R_2\rangle = |Q_2\rangle \otimes |Q_1\rangle \tag{3.174}$$

Finally, by applying the unitary correction $(Z \otimes XZ)$, the original states are perfectly reconstructed at the destination, completing the bidirectional teleportation process.

Chapitre 4

Controlled Cyclic Quantum Teleportation Protocol Pure State

4.1 Introduction

Quantum teleportation stands as a fundamental pillar in the field of Quantum Information Processing (QIP), enabling the transfer of unknown quantum states between distant nodes using pre-shared entanglement and classical communication. As quantum networks evolve toward multi-user architectures, the demand for more sophisticated protocols capable of managing complex information flow has significantly increased.In this section, we investigate the Double-direction Cyclic Controlled Teleportation protocol. Unlike standard teleportation schemes that involve a single sender and receiver, this advanced [6] protocol facilitates a Cyclic Permutation of quantum states $|\Psi\rangle_A$, $|\Psi\rangle_B$, and $|\Psi\rangle_C$ among three legitimate parties : Alice, Bob, and Charlie.

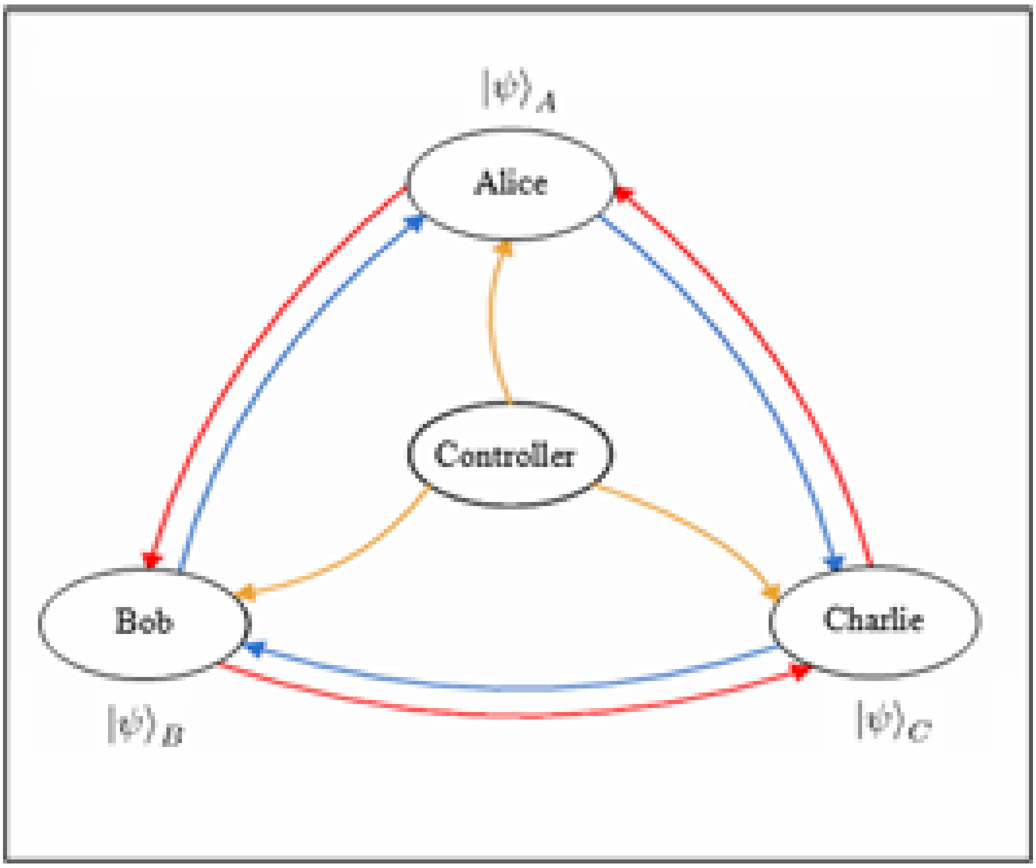

FIGURE 4.1: Double-direction cyclic controlled teleportation protocol

To establish the framework of the protocol, we define the participating parties and their respective roles. The system involves three primary actors : Alice, Bob, and Charlie. A fundamental characteristic of this protocol is the dual nature of these actors :

Transmitters : At the initial stage, each actor holds a qubit containing the information to be sent.

Receivers : By the conclusion of the protocol, each actor receives a new teleported state from another participant.

Initially, each participant is assumed to possess an arbitrary single-qubit state, which serves as the input for the cyclic exchange. These states, denoted as , $|\Psi\rangle_A$, $|\Psi\rangle_B$, and $|\Psi\rangle_C$, are defined in the computational basis as :

$$|\Psi\rangle_A = \alpha_a |0\rangle + \beta_a |1\rangle . \tag{4.1}$$

$$|\Psi\rangle_B = \alpha_b |0\rangle + \beta_b |1\rangle . \tag{4.2}$$

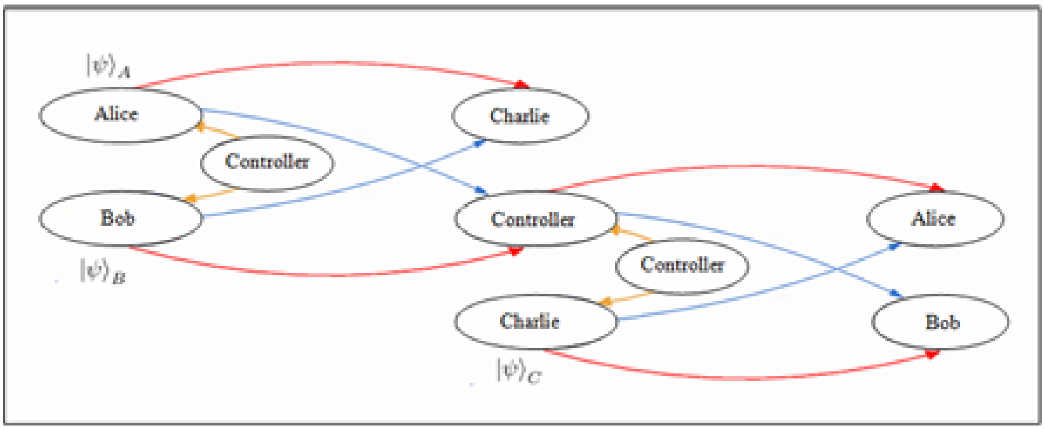

FIGURE 4.2: Double-direction cyclic controlled teleportation protocol : Composition process

$$|\Psi\rangle_C = \alpha_c |0\rangle + \beta_c |1\rangle . \tag{4.3}$$

The Controller serves as the protocol's supervisor, acting as a quantum switch to dictate the direction of the permutation. The Controller selects between two operational modes :

Clockwise Cyclic Controlled Teleportation : In this mode, the quantum states are shifted forward in the cycle. Specifically :Alice's state $|\Psi\rangle_A$ is teleported to Charlie, Bob's state $|\Psi\rangle_B$ is teleported to Alice, Charlie's state $|\Psi\rangle_C$ is teleported to Bob.

Counter-clockwise Cyclic Controlled Teleportation : In this mode, the Controller reverses the flow of information. Specifically :Alice's state $|\Psi\rangle_A$ is teleported to Bob, Bob's state $|\Psi\rangle_B$ is teleported to Charlie, Charlie's state $|\Psi\rangle_C$ is teleported to Alice.

4.2 Protocol Design and Implementation

The implementation of the proposed protocol is executed through two successive and interconnected stages, as illustrated in Fig 4.2. This phased approach ensures the structured routing of quantum information across the network.

Step 1 : First Switch-Controlled Bidirectional Phase

In the initial stage, a bidirectional teleportation protocol is established under the supervision of the Controller. The roles are assigned as follows :

Transmitters : Alice and Bob initiate the process by sending their respective qubits.

Receivers : Charlie and the Controller act as the primary recipients.

Technical Note : During this step, the Controller utilizes an auxiliary qubit (Q_{Ctr_3}). This qubit serves as a temporary quantum buffer or relay, which is prepared to be transmitted to its designated destination during the subsequent stage of the protocol.

Configuration of the Sub-Quantum Channel :

In the first phase of the protocol, the Controller acts as both a participant and a supervisor, managing a 6-qubit entangled sub-channel. This resource is distributed among the actors such that Alice and Bob hold the transmission qubits Q_{A_1} and Q_{B_1}, Charlie holds the reception qubit Q_{C_2}, and the Controller manages qubits Q_{Ctr_1} and Q_{Ctr_2} for supervision, along with the auxiliary qubit Q_{Ctr_3} which serves as a temporary receiver.

In the second stage of the protocol, a subsequent switch-controlled bidirectional teleportation is executed. During this phase, the roles shift as follows :

Transmitters : Charlie and the Controller (utilizing the auxiliary qubit Q_{Ctr_3} prepared in Stage 1) act as the senders.

Receivers : Alice and Bob serve as the recipients.

Supervisor : The Controller continues to oversee the process using a second 6-qubit sub-channel consisting of $Q_{C_2}, Q_{Ctr_3}, Q_{Ctr_4}, Q_{Ctr_5}, Q_{A_2}, Q_{B_2}$.

The Global 11-Qubit Channel Composition :

The complete quantum resource is constructed by merging the two bidirectional protocols into a single 11-qubit global channel. This integration involves two critical adaptations to ensure the continuity of entanglement :

The Relay Mechanism : The auxiliary qubit Q_{Ctr_3} acts as a quantum bridge, it is a receiving qubit in the first stage and becomes a transmitting qubit in the second. This dual role ensures it is entangled with every qubit in the global system.

Circuit Optimization : To preserve the pre-existing entanglement, the Hadamard gate typically applied at the start of the second entanglement circuit is omitted in this global composition.

Final Qubit DistributionThe resulting 11-qubit architecture is distributed among the four participants as summarized below :

Alice, Bob, and Charlie : Each holds two qubits (Q_{i_1}, Q_{i_2}), where the first is dedicated to transmission and the second to reception.

The Controller : Manages five qubits in total : (Q_{Ctr_1}, Q_{Ctr_2}) to guide the first phase, the pivotal auxiliary qubit Q_{Ctr_3}, and (Q_{Ctr_4}, Q_{Ctr_5}) to supervise the second phase.Protocol Execution and Measurement Procedure

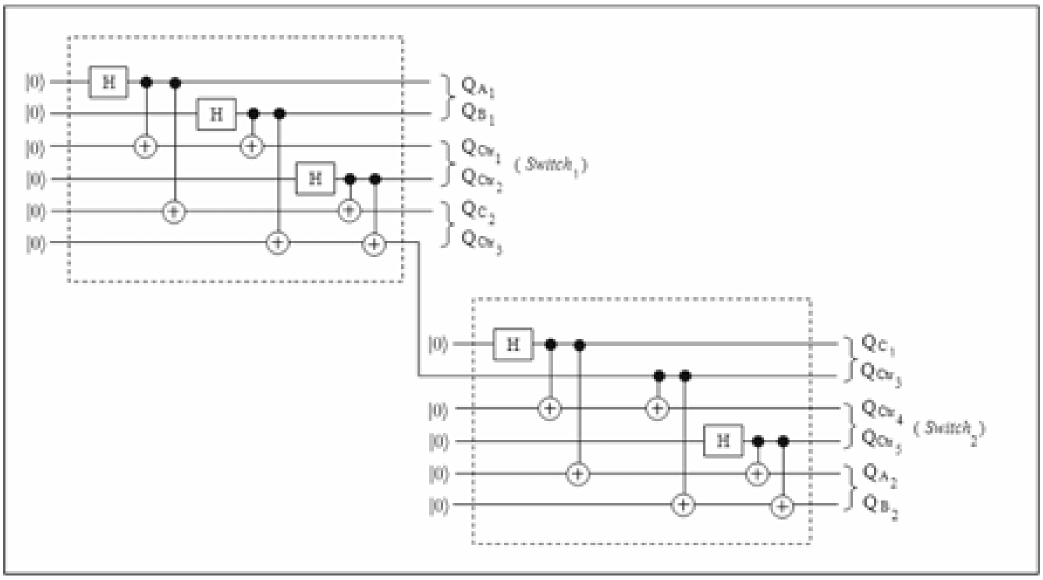

FIGURE 4.3: Double-direction cyclic controlled teleportation protocol : Quantum channel

Once the 11-qubit quantum channel is prepared and distributed, the teleportation process proceeds through a sequence of synchronized operations and measurements. The execution is divided into three main phases :

Phase 1 : Local Bell-State Measurements (BSM)

Alice measures the pair : $(|\Psi\rangle_A, Q_{A_1})$, Bob measures the pair : $(|\Psi\rangle_B, Q_{B_1})$, Charlie measures the pair : $(|\Psi\rangle_C, Q_{C_1})$

Phase 2 : Controller's Directional SwitchingThe Controller determines the permutation's direction by applying specific quantum gates and performing measurements in the computational basis $\{|0\rangle, |1\rangle\}$.

Case A : Clockwise Permutation

To route the states in a clockwise direction, the Controller executes the following sequence :

Apply a Hadamard gate $H(Q_{Ctr_1})$, followed by measurements of (Q_{Ctr_1}, Q_{Ctr_2}).

Apply a C_NOT gate on (Q_{Ctr_4}, Q_{Ctr_5}) followed by $H(Q_{Ctr4})$, then measure (Q_{Ctr_4}, Q_{Ctr_5}).

Finally, apply $H(Q_{Ctr_3})$ and measure the auxiliary qubit Q_{Ctr_3}.

Case B : Counter-clockwise Permutation

To reverse the flow, the Controller modifies the operations :

Apply a C_NOT gate on (Q_{Ctr_1}, Q_{Ctr_2}) followed by $H(Q_{Ctr_1})$, then measure (Q_{Ctr_1}, Q_{Ctr_2}).

Apply $H\left(Q_{Ctr4}\right)$ and measure $\left(Q_{Ctr_4}, Q_{Ctr_5}\right)$.

Finally, apply $H\left(Q_{Ctr_3}\right)$ and measure Q_{Ctr_3}.

In the final phase, Alice, Bob, and Charlie apply the required unitary corrections to their reception qubits $(Q_{A_2}, Q_{B_2}, Q_{C_2})$ based on the classical measurement results shared by all four participants. This step ensures the successful reconstruction of the teleported states at their designated destinations.

4.3 Application example

The quantum channel employed in this protocol, as illustrated in Fig. 2, is defined as follows :

$$|Channel\rangle_{A_1B_1C_{tr_1}C_{tr_2}C_2C_{tr_3}C_1C_{tr_4}C_{tr_5}A_2B_2} \tag{4.4}$$
$$= \frac{1}{4\sqrt{2}} \begin{pmatrix} |00000000000\rangle + |00000000111\rangle + |00000011010\rangle \\ + |00000011101\rangle + |00011101001\rangle + |00011101110\rangle \\ + |00011110011\rangle + |00011110100\rangle + |01100101001\rangle \\ + |01100101110\rangle + |01100110011\rangle + |01100110100\rangle \\ + |01111000000\rangle + |01111000111\rangle + |01111011010\rangle \\ + |01111011101\rangle + |10101000000\rangle + |10101000111\rangle \\ + |10101011010\rangle + |10101011101\rangle + |10110101001\rangle \\ + |10110101110\rangle + |10110110011\rangle + |10110110100\rangle \\ + |11001101001\rangle + |11001101110\rangle + |11001110011\rangle \\ + |11001110100\rangle + |11010000000\rangle + |11010000111\rangle \\ + |11010011010\rangle + |11010011101\rangle \end{pmatrix}$$

The initial global state is :

$$|\Psi\rangle_A \otimes |\Psi\rangle_B \otimes |\Psi\rangle_C \otimes |Channel\rangle \tag{4.5}$$

Due to the complexity of manual calculations and the high dimensionality of the Hilbert space, we provide the following Mathematica implementation to compute the Global State :

```
(* Construction de l'etat global *)

(* Kronecker product of the first two qubits *)
QubitsAlice01Alice02 = KroneckerProduct[QubitAlice01,
   QubitAlice02];
```

```
(* Adding the third qubit *)
QubitsAlice01Alice02Alice03 = KroneckerProduct[
   QubitsAlice01Alice02, QubitAlice03];

(* Final Global State including the Quantum Channel *)
EtatGlobal = KroneckerProduct[QubitsAlice01Alice02Alice03, Canal
   ];
```

The local measurements performed by Alice, Bob, and Charlie project the global quantum state onto the Bell basis. Given the specific outcomes $|\Psi^+\rangle, |\Psi^-\rangle$ and $|\Phi^+\rangle$, the corresponding projection operators are defined as follows :

Alice's Projector :

The projector P_{Ψ^+} is given by :

$$\begin{aligned} P_{\Psi^+} &= \left|\Psi^+\right\rangle\left\langle\Psi^+\right| \\ &= \frac{1}{2}\left(|01\rangle\langle 01| + |01\rangle\langle 10| + |10\rangle\langle 01| + |10\rangle\langle 10|\right) \end{aligned} \tag{4.6}$$

Bob's Projector :

The projector P_{Ψ^-} is given by :

$$\begin{aligned} P_{\Psi^-} &= \left|\Psi^-\right\rangle\left\langle\Psi^-\right| \\ &= \frac{1}{2}\left(|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10|\right) \end{aligned} \tag{4.7}$$

Charlie's Projector :

The projector P_{Φ^+} is given by :

$$\begin{aligned} P_{\Phi^+} &= \left|\Phi^+\right\rangle\left\langle\Phi^+\right| \\ &= \frac{1}{2}\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right) \end{aligned} \tag{4.8}$$

Suppose that Aliceâe™s, Bobâe™s and Charlieâe™s measurement outcomes

are $|\Psi^+\rangle, |\Psi^-\rangle$ and $|\Phi^+\rangle$, respectively. The obtained quantum state is :

$$\frac{1}{2}\left(\begin{array}{c}\alpha_a\alpha_b\beta_c\left|00000000\right\rangle+\alpha_a\alpha_b\beta_c\left|00000111\right\rangle+\alpha_a\alpha_b\alpha_c\left|00001010\right\rangle\\+\alpha_a\alpha_b\alpha_c\left|00001101\right\rangle-\alpha_c\beta_a\beta_b\left|001100\right\rangle-\alpha_c\beta_a\beta_b\left|00110100\right\rangle\\-\beta_a\beta_b\beta_c\left|00111001\right\rangle-\beta_a\beta_b\beta_c\left|00111110\right\rangle\beta_a\beta_b\beta_c\left|01000000\right\rangle\\-\beta_a\beta_b\beta_c\left|01000111\right\rangle+\alpha_c\beta_a\beta_b\left|01001010\right\rangle-\alpha_c\beta_a\beta_b\left|01001101\right\rangle\\+\alpha_a\alpha_b\alpha_c\left|01110011\right\rangle+\alpha_a\alpha_b\alpha_c\left|01110100\right\rangle+\alpha_a\alpha_b\beta_c\left|01111001\right\rangle\\+\alpha_a\alpha_b\beta_c\left|01111110\right\rangle-\alpha_a\alpha_c\beta_b\left|10010011\right\rangle-\alpha_a\alpha_c\beta_b\left|10010100\right\rangle\\-\alpha_a\beta_b\beta_c\left|10011001\right\rangle-\alpha_a\beta_b\beta_c\left|10011110\right\rangle+\alpha_b\beta_a\beta_c\left|10100000\right\rangle\\+\alpha_b\beta_a\beta_c\left|10100111\right\rangle+\alpha_b\alpha_c\beta_a\left|10100111\right\rangle+\alpha_b\alpha_c\beta_a\left|10101101\right\rangle\\+\alpha_b\alpha_c\beta_a\left|11010011\right\rangle+\alpha_b\alpha_c\beta_a\left|11010100\right\rangle+\alpha_b\beta_a\beta_c\left|11011001\right\rangle\\+\alpha_b\beta_a\beta_c\left|11011110\right\rangle-\alpha_a\beta_b\beta_c\left|11100000\right\rangle-\alpha_a\beta_b\beta_c\left|11100111\right\rangle\\-\alpha_a\alpha_c\beta_b\left|11101010\right\rangle-\alpha_a\alpha_c\beta_b\left|11101101\right\rangle\end{array}\right) \tag{4.9}$$

Due to the high dimensionality of the Hilbert space and the significant complexity of the global density matrices, manual calculations are no longer practical. Therefore, we provide the following Mathematica implementation to verify the state evolution and the projection outcomes :

```
(* --- 1. Alice's Projection --- *)
ProjecteurAlice = Bell00;
ProjecteurGlobal = KroneckerProduct[ProjecteurAlice,
    IdentityMatrix[4096]];
EtatGlobal = ProjecteurGlobal . EtatGlobal;

(* --- 2. Bob's Projection --- *)
ProjecteurBob = Bell00;
ProjecteurGlobal = KroneckerProduct[ProjecteurBob, IdentityMatrix
    [1024]];
EtatGlobal = ProjecteurGlobal . EtatGlobal;

(* --- 3. Charlie's Projection --- *)
ProjecteurCharlie = Bell00;
ProjecteurGlobal = KroneckerProduct[ProjecteurCharlie,
    IdentityMatrix[256]];
EtatGlobal = ProjecteurGlobal . EtatGlobal;
```

In the case of a clockwise permutation, the Controller initiates the process by executing a series of quantum gates and measurements to steer the state transfer. The sequence of operations is as follows :

Apply a Hadamard gate $H\left(Q_{Ctr_1}\right)$, followed by a measurement of qubits $\left(Q_{Ctr_1}, Q_{Ctr_2}\right)$ in the computational basis. Due to the complexity of manual calculations and the high dimensionality of the Hilbert space, we provide the following Mathematica implementation to verify the state evolution and the projection outcomes :

```
(* Choice Gate for Controller 01 *)
PorteChoix01Controleur = KroneckerProduct[H, IdentityMatrix
    [128]];
EtatGlobal = PorteChoix01Controleur . EtatGlobal;

(* //////////////////////////////////////////// *)

(* Projection 01 of the Controller *)
Projecteur01Controleur = ConjugateTranspose[Projecteur00];

ProjecteurGlobal = KroneckerProduct[Projecteur01Controleur,
    IdentityMatrix[64]];
EtatGlobal = ProjecteurGlobal . EtatGlobal;
```

Apply a C_NOT gate (Q_{Ctr_4}, Q_{Ctr_5}) and a Hadamard gate $H\,(Q_{Ctr_4})$, then measure (Q_{Ctr_4}, Q_{Ctr_5}) in the computational basis. Due to the complexity of manual calculations and the high dimensionality of the Hilbert space, we provide the following Mathematica implementation to compute the evolution and the second projection :

```
(* Portes du Choix 02 *)

Porte01Choix02Controleur = KroneckerProduct[IdentityMatrix[4],
    CNot];
Porte01Choix02Controleur = KroneckerProduct[
    Porte01Choix02Controleur, IdentityMatrix[4]];
EtatGlobal = Porte01Choix02Controleur . EtatGlobal;

Porte02Choix02Controleur = KroneckerProduct[IdentityMatrix[4], H
    ];
Porte02Choix02Controleur = KroneckerProduct[
    Porte02Choix02Controleur, IdentityMatrix[8]];
EtatGlobal = Porte02Choix02Controleur . EtatGlobal;

(* //////////////////////////////////////////// *)

(* Projection 02 du Controleur *)

Projecteur02Controleur = ConjugateTranspose[Projecteur00];

ProjecteurGlobal = KroneckerProduct[IdentityMatrix[4],
    Projecteur02Controleur];
```

```
ProjecteurGlobal = KroneckerProduct[ProjecteurGlobal,
   IdentityMatrix[4]];
EtatGlobal = ProjecteurGlobal . EtatGlobal;
```

Finally, apply a Hadamard gate $H\left(Q_{Ctr_3}\right)$ and measure the auxiliary qubit Q_{Ctr_3} in the computational basis. To ensure the precision of the multi-party entanglement distribution, we implement the following Mathematica code to perform the remaining controller operations and the third projection :

```
(* Autres operations du Controleur *)

PorteHControleur = KroneckerProduct[IdentityMatrix[2], H];
PorteHControleur = KroneckerProduct[PorteHControleur,
   IdentityMatrix[4]];
EtatGlobal = PorteHControleur . EtatGlobal;

(* //////////////////////////////////////////// *)

(* Projection 03 du Controleur *)

Projecteur03Controleur = ConjugateTranspose[QubitZero];

ProjecteurGlobal = KroneckerProduct[IdentityMatrix[2],
   Projecteur03Controleur];
ProjecteurGlobal = KroneckerProduct[ProjecteurGlobal,
   IdentityMatrix[4]];
QubitsCharlieAliceBob = ProjecteurGlobal . EtatGlobal;
```

Assuming the Controller's measurement outcomes are $|10\rangle$, $|00\rangle$ and $|0\rangle$, respectively, the receiving qubits $(Q_{C_2}, Q_{A_2}, Q_{B_2})$ held by Charlie, Alice, and Bob collapse into the following quantum state :

$$\begin{aligned} &\alpha_a\alpha_b\beta_c\,|000\rangle + \alpha_a\alpha_b\alpha_c\,|001\rangle + \alpha_a\beta_b\beta_c\,|010\rangle + \alpha_a\beta_c\beta_b\,|011\rangle \\ &- \alpha_b\beta_a\beta_c\,|100\rangle - \alpha_b\alpha_c\beta_a\,|101\rangle - \beta_a\beta_b\beta_c\,|110\rangle - \alpha_c\beta_a\beta_b\,|111\rangle \end{aligned} \tag{4.10}$$

This expression is mathematically equivalent to :

$$(\alpha_a\,|0\rangle - \beta_a\,|1\rangle) \otimes (\alpha_b\,|0\rangle + \beta_b\,|1\rangle) \otimes (\beta_c\,|0\rangle + \alpha_c\,|1\rangle) \tag{4.11}$$

To successfully reconstruct the original states, the three receivers must apply local unitary operations based on the aforementioned results. In this specific case, the required corrections are :

Charlie : Applies the Z gate.

Alice : Applies Identity (No operation required).

Bob : Applies the X gate.

Consequently, the final combined state of the reception qubits $(Q_{C_2}, Q_{A_2}, Q_{B_2})$ becomes :

$$|\Psi\rangle_A \otimes |\Psi\rangle_B \otimes |\Psi\rangle_C \tag{4.12}$$

This outcome confirms the successful implementation of the clockwise cyclic controlled teleportation protocol, as each state has been accurately transferred to its next designated neighbor in the cycle. To demonstrate the success of our protocol and the effective recovery of the transmitted states for each participant, we provide the following Mathematica implementation to extract the individual qubits :

```
(* Il reste trois Qubits dans l'ordre : Charlie, Alice et Bob *)

(* Recuperation du Qubit de Charlie *)
ProjecteurAlice = ConjugateTranspose[QubitZero];
ProjecteurGlobal = KroneckerProduct[IdentityMatrix[2],
   ProjecteurAlice];
ProjecteurBob = ConjugateTranspose[QubitZero];
ProjecteurGlobal = KroneckerProduct[ProjecteurGlobal,
   ProjecteurBob];
QubitCharlie = ProjecteurGlobal . QubitsCharlieAliceBob;

(* Recuperation du Qubit d'Alice *)
ProjecteurCharlie = ConjugateTranspose[QubitZero];
ProjecteurGlobal = KroneckerProduct[ProjecteurCharlie,
   IdentityMatrix[2]];
ProjecteurBob = ConjugateTranspose[QubitZero];
ProjecteurGlobal = KroneckerProduct[ProjecteurGlobal,
   ProjecteurBob];
QubitAlice = ProjecteurGlobal . QubitsCharlieAliceBob;

(* Recuperation du Qubit de Bob *)
ProjecteurCharlie = ConjugateTranspose[QubitZero];
ProjecteurAlice = ConjugateTranspose[QubitZero];
```

```
ProjecteurGlobal = KroneckerProduct[ProjecteurCharlie,
    ProjecteurAlice];
ProjecteurGlobal = KroneckerProduct[ProjecteurGlobal,
    IdentityMatrix[2]];
QubitBob = ProjecteurGlobal . QubitsCharlieAliceBob;
```

Alternatively, if the Controller decides to execute a counter-clockwise permutation, the states are routed in the reverse direction : $|\Psi\rangle_B$ →Charlie, $|\Psi\rangle_C$ →Alice, and $|\Psi\rangle_A$ →Bob. The Controller initiates this process by performing the following quantum operations :

Apply a C_NOT gate (Q_{Ctr_1}, Q_{Ctr_2}) and a Hadamard gate $H\,(Q_{Ctr_1})$, followed by a measurement of (Q_{Ctr_1}, Q_{Ctr_2}) in the computational basis. To verify the state evolution and ensure the precise distribution of entanglement within the multi-party system, we provide the following Mathematica implementation to compute the first choice gates and the corresponding projection :

```
(* Portes du Choix 01 *)

Porte01Choix01Controleur = KroneckerProduct[CNot, IdentityMatrix
    [64]];
EtatGlobal = Porte01Choix01Controleur . EtatGlobal;

Porte02Choix01Controleur = KroneckerProduct[H, IdentityMatrix
    [128]];
EtatGlobal = Porte02Choix01Controleur . EtatGlobal;

(* ////////////////////////////////////////////// *)

(* Projection 01 du Controleur *)

Projecteur01Controleur = ConjugateTranspose[Projecteur00];

ProjecteurGlobal = KroneckerProduct[Projecteur01Controleur,
    IdentityMatrix[64]];
EtatGlobal = ProjecteurGlobal . EtatGlobal;

NormeEtatGlobal = Norm[EtatGlobal];
```

Apply a Hadamard gate $H\,(Q_{Ctr_4})$, then measure (Q_{Ctr_4}, Q_{Ctr_5}) in the computational basis. To further verify the protocol's evolution and the redistribution of entanglement, we provide the following Mathematica implementation to calculate the second set of choice gates and the subsequent projection :

```
(* Portes du Choix 02 *)
```

```
PorteChoix02Controleur = KroneckerProduct[IdentityMatrix[4], H];
PorteChoix02Controleur = KroneckerProduct[PorteChoix02Controleur,
    IdentityMatrix[8]];
EtatGlobal = PorteChoix02Controleur . EtatGlobal;

(* ////////////////////////////////////////////// *)

(* Projection 02 du Controleur *)

Projecteur02Controleur = ConjugateTranspose[Projecteur00];

ProjecteurGlobal = KroneckerProduct[IdentityMatrix[4],
   Projecteur02Controleur];
ProjecteurGlobal = KroneckerProduct[ProjecteurGlobal,
   IdentityMatrix[4]];
EtatGlobal = ProjecteurGlobal . EtatGlobal;
```

Finally, apply a Hadamard gate $H\left(Q_{Ctr_3}\right)$ and measure the auxiliary qubit Q_{Ctr_3} in the computational basis. To complete the controller's sequence and perform the third projection, we implement the following Mathematica code to ensure the precise distribution of entanglement within the multi-party system :

```
(* Autres operations du Controleur *)

PorteHControleur = KroneckerProduct[IdentityMatrix[2], H];
PorteHControleur = KroneckerProduct[PorteHControleur,
   IdentityMatrix[4]];
EtatGlobal = PorteHControleur . EtatGlobal;

(* ////////////////////////////////////////////// *)

(* Projection 03 du Controleur *)

Projecteur03Controleur = ConjugateTranspose[QubitZero];

ProjecteurGlobal = KroneckerProduct[IdentityMatrix[2],
   Projecteur03Controleur];
ProjecteurGlobal = KroneckerProduct[ProjecteurGlobal,
   IdentityMatrix[4]];
QubitsCharlieAliceBob = ProjecteurGlobal . EtatGlobal;
```

Assuming the measurement outcomes are $|10\rangle$, $|11\rangle$ and $|1\rangle$, respectively, the reception qubits ($Q_{C_2}, Q_{A_2}, Q_{B_2}$) held by Charlie, Alice, and Bob collapse into the quantum state :

$$\begin{aligned} &\alpha_b\alpha_c\beta_a |000\rangle - \alpha_a\alpha_b\alpha_c |001\rangle - \alpha_b\beta_a\beta_c |010\rangle + \alpha_a\alpha_b\beta_c |011\rangle \\ &+ \alpha_c\beta_a\beta_b |100\rangle - \alpha_a\alpha_c\beta_b |101\rangle - \beta_a\beta_b\beta_c |110\rangle + \alpha_a\beta_b\beta_c |111\rangle \end{aligned} \tag{4.13}$$

This state is mathematically equivalent to the factored form :

$$(\alpha_b |0\rangle + \beta_b |1\rangle) \otimes (\alpha_c |0\rangle - \beta_c |1\rangle) \otimes (\beta_a |0\rangle - \alpha_a |1\rangle) \tag{4.14}$$

To achieve the final state reconstruction, the three receivers apply their respective local unitary corrections based on the Controller's results :

Charlie : Applies Identity (No correction required).

Alice : Applies the Z gate.

Bob : Applies the composite XZ gate.

Following these operations, the final state of the reception qubits ($Q_{C_2}, Q_{A_2}, Q_{B_2}$) is successfully transformed into :

$$|\Psi\rangle_B \otimes |\Psi\rangle_C \otimes |\Psi\rangle_A \tag{4.15}$$

This result confirms the successful execution of the counter-clockwise cyclic controlled teleportation, effectively reversing the flow of quantum information under the supervisor's control.

To demonstrate the final success of the protocol in the reverse direction and the effective recovery of the transmitted states for each participant, we provide the following Mathematica implementation :

```
(* Recuperation des trois Qubits : Charlie, Alice et Bob *)

(* Recuperation du Qubit de Charlie *)
ProjecteurAlice = ConjugateTranspose[QubitZero];
ProjecteurGlobal = KroneckerProduct[IdentityMatrix[2],
   ProjecteurAlice];
ProjecteurBob = ConjugateTranspose[QubitZero];
ProjecteurGlobal = KroneckerProduct[ProjecteurGlobal,
   ProjecteurBob];
QubitCharlie = ProjecteurGlobal . QubitsCharlieAliceBob;
```

```
(* Recuperation du Qubit d'Alice *)
ProjecteurCharlie = ConjugateTranspose[QubitZero];
ProjecteurGlobal = KroneckerProduct[ProjecteurCharlie,
   IdentityMatrix[2]];
ProjecteurBob = ConjugateTranspose[QubitZero];
ProjecteurGlobal = KroneckerProduct[ProjecteurGlobal,
   ProjecteurBob];
QubitAlice = ProjecteurGlobal . QubitsCharlieAliceBob;

(* Recuperation du Qubit de Bob *)
ProjecteurCharlie = ConjugateTranspose[QubitZero];
ProjecteurAlice = ConjugateTranspose[QubitZero];
ProjecteurGlobal = KroneckerProduct[ProjecteurCharlie,
   ProjecteurAlice];
ProjecteurGlobal = KroneckerProduct[ProjecteurGlobal,
   IdentityMatrix[2]];
QubitBob = ProjecteurGlobal . QubitsCharlieAliceBob;
```

Chapitre 5

Effects of Noise

5.1 Introduction

After establishing the theoretical framework for the proposed quantum protocols in the previous chapters under ideal conditions, this chapter is dedicated to studying these protocols using the Density Matrix formalism. The importance of this mathematical tool lies in its comprehensive ability to describe quantum systems in Mixed States, allowing us to track the effects of interaction with the surrounding environment through the Werner-like noise model.

The primary focus of this chapter is to analyze the impact of noise on the 11-qubit system by considering the shared entanglement as a Werner State. Unlike pure states, the Werner state provides a more realistic representation by mixing the maximally entangled state with white noise (the maximally mixed state). We will formulate the density matrix for each protocol and monitor the structural changes that occur as the purity of the [7] [6] state varies. This analysis aims to derive the final analytical expressions for the output density matrices, providing a precise physical description of the system's state before the measurement phase, while accounting for the factors leading to the dissipation of quantum information in a non-ideal environment.

5.2 Analysis of the Simple Parasad Protocol under Werner Noise

In this section, we provide a detailed analytical derivation of the Simple Parasad Protocol using the Density Matrix formalism. This transition from state vectors to density matrices (ρ) is essential for evaluating the protocol's performance in realistic environments where quantum systems interact with external noise.To simulate these environmental effects, we incorporate the Werner Noise model into the shared entangled channel. By introducing the purity parameter p, we can quantify the impact of isotropic white noise on the teleportation process. The following steps track the evolution of the global density matrix through unitary transformations and measurements, ultimately determining the fidelity of the recovered state at Bob's end.

Initial Density Matrix Representation :

We begin by defining the initial state of the message qubit $|\Psi\rangle_A$ and the shared Bell state $|Bell\rangle_{12}$. The corresponding density matrices are constructed using the outer product :

$$\rho_0 = \rho_{\Psi_A} \otimes \rho_{Bell_{12}} \tag{5.1}$$

The density matrix representing Alice's initial message state is defined as :

$$\begin{aligned} \rho_{\Psi_A} &= |\Psi\rangle_A \langle\Psi|_A \\ &= (\alpha|0\rangle + \beta|1\rangle)_A (\alpha^* \langle 0| + \beta^* \langle 1|)_A \\ &= |\alpha|^2 |0\rangle\langle 0| + \alpha\beta^* |0\rangle\langle 1| + \beta\alpha^* |1\rangle\langle 0| + |\beta|^2 |1\rangle\langle 1| . \end{aligned} \tag{5.2}$$

We define the density matrix representing the shared entanglement resource (the channel) as :

$$\begin{aligned} \rho_{Bell_{12}} &= |Bell\rangle_{12} \langle Bell|_{12} \\ &= \left(\frac{1}{\sqrt{2}}\{|00\rangle + |11\rangle\}\right)\left(\frac{1}{\sqrt{2}}\{\langle 00| + \langle 11|\}\right) \\ &= \frac{1}{2}\{|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\} . \end{aligned} \tag{5.3}$$

By substituting the expressions of ρ_{Ψ_A} and $\rho_{Bell_{12}}$ into equation (62764464563962762f644629&¿ 627644631642645), we obtain :

$$\begin{aligned} \rho_0 &= \left\{|\alpha|^2 |0\rangle\langle 0| + \alpha\beta^* |0\rangle\langle 1| + \beta\alpha^* |1\rangle\langle 0| + |\beta|^2 |1\rangle\langle 1|\right\} \\ &\otimes \frac{1}{2}\{|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\} \\ &= \frac{1}{2}\begin{pmatrix} |\alpha|^2 \{|000\rangle\langle 000| + |000\rangle\langle 011| + |011\rangle\langle 000| + |011\rangle\langle 011|\} + \\ \alpha\beta^* \{|000\rangle\langle 100| + |000\rangle\langle 111| + |011\rangle\langle 100| + |011\rangle\langle 111|\} + \\ \beta\alpha^* \{|100\rangle\langle 000| + |100\rangle\langle 011| + |111\rangle\langle 000| + |111\rangle\langle 011|\} + \\ |\beta|^2 \{|100\rangle\langle 100| + |100\rangle\langle 111| + |111\rangle\langle 100| + |111\rangle\langle 111|\} \end{pmatrix} . \end{aligned} \tag{5.4}$$

Once the initial density matrix is established, we proceed to the first stage of the quantum circuit. A Controlled-NOT (C_NOT) gate is applied, using the message qubit (the first qubit) as the control and the second qubit as the target.

The evolution of the system's state is described by the following unitary transformation :

$$\rho_1 = C\_NOT \rho_{in} C\_NOT^{\dagger}. \tag{5.5}$$

$$\rho_1 = \frac{1}{2} \begin{pmatrix} |\alpha|^2 \{|000\rangle\langle 000| + |000\rangle\langle 011| + |011\rangle\langle 000| + |011\rangle\langle 011|\} + \\ \alpha\beta^* \{|000\rangle\langle 110| + |000\rangle\langle 101| + |011\rangle\langle 110| + |011\rangle\langle 101|\} + \\ \beta\alpha^* \{|110\rangle\langle 000| + |110\rangle\langle 011| + |101\rangle\langle 000| + |101\rangle\langle 011|\} + \\ |\beta|^2 \{|110\rangle\langle 110| + |110\rangle\langle 101| + |101\rangle\langle 110| + |101\rangle\langle 101|\} \end{pmatrix}. \tag{5.6}$$

After establishing the correlations with the C_NOT gate, the next stage involves applying a Hadamard (H) gate to the first qubit (the message qubit). This transformation is essential for shifting the basis of the system into a superposition state, which allows the quantum information to be distributed effectively for the measurement phase.

Mathematically, the evolution of the density matrix is give by :

$$\rho_2 = (H \otimes I \otimes I)\, \rho_1 \left((H \otimes I \otimes I)\right)^{\dagger}. \tag{5.7}$$

$$\rho_2 = \frac{1}{4}\left(\begin{array}{c} |\alpha|^2 \left\{\begin{array}{c} |000\rangle\langle 000| + |000\rangle\langle 100| + |100\rangle\langle 000| + |100\rangle\langle 100| \\ + |000\rangle\langle 011| + |000\rangle\langle 111| + |100\rangle\langle 011| + |100\rangle\langle 111| \\ + |011\rangle\langle 000| + |011\rangle\langle 100| + |111\rangle\langle 000| + |111\rangle\langle 100| \\ + |011\rangle\langle 011| + |011\rangle\langle 111| + |111\rangle\langle 011| + |111\rangle\langle 111| \end{array}\right\} + \\ \alpha\beta^* \left\{\begin{array}{c} |000\rangle\langle 010| - |000\rangle\langle 110| + |100\rangle\langle 010| - |100\rangle\langle 110| \\ + |000\rangle\langle 001| - |000\rangle\langle 101| + |100\rangle\langle 001| - |100\rangle\langle 101| \\ + |011\rangle\langle 010| - |011\rangle\langle 110| + |111\rangle\langle 010| - |111\rangle\langle 110| \\ + |011\rangle\langle 001| - |011\rangle\langle 101| + |111\rangle\langle 001| - |111\rangle\langle 101| \end{array}\right\} + \\ \beta\alpha^* \left\{\begin{array}{c} |010\rangle\langle 000| + |010\rangle\langle 100| - |110\rangle\langle 000| - |110\rangle\langle 100| \\ + |010\rangle\langle 011| + |010\rangle\langle 111| - |110\rangle\langle 011| - |110\rangle\langle 111| \\ + |001\rangle\langle 000| + |001\rangle\langle 100| - |101\rangle\langle 000| - |101\rangle\langle 100| \\ + |001\rangle\langle 001| + |001\rangle\langle 111| - |101\rangle\langle 011| - |101\rangle\langle 111| \end{array}\right\} + \\ |\beta|^2 \left\{\begin{array}{c} |010\rangle\langle 010| - |010\rangle\langle 110| - |110\rangle\langle 010| + |110\rangle\langle 110| \\ + |010\rangle\langle 001| - |010\rangle\langle 101| - |110\rangle\langle 001| + |110\rangle\langle 101| \\ + |001\rangle\langle 010| - |001\rangle\langle 110| - |101\rangle\langle 010| + |101\rangle\langle 110| \\ + |001\rangle\langle 001| - |001\rangle\langle 101| - |101\rangle\langle 001| + |101\rangle\langle 101| \end{array}\right\} \end{array}\right). \tag{5.8}$$

State Projection and Density Matrix Normalization

In this stage, we analyze the scenario where Alice's measurement in the computational basis yields the result $|00\rangle$. This measurement causes the global state ρ_2 to collapse into a specific subspace.

he resulting density matrix at Bob's end is determined by applying the projection operator

$$P_{00} = \langle 00| \otimes I_2 \tag{5.9}$$

By substituting the global density matrix ρ_2 into the projection formula and performing the partial trace over the measured qubits, the normalized state at Bob's end is obtained as follows :

$$\begin{aligned} \rho_{3_{(00)}} &= \frac{P_{00}\rho_2 P_{00}^\dagger}{Tr\left(P_{00}\rho_2 P_{00}^\dagger\right)}. \qquad (5.10) \\ &\Longrightarrow P_{00}\rho_2 P_{00}^\dagger = \frac{1}{4}\left(|\alpha|^2 |0\rangle\langle 0| + \alpha\beta^* |0\rangle\langle 1| + \beta\alpha^* |1\rangle\langle 0| + |\beta|^2 |1\rangle\langle 1|\right), \\ Tr\left(P_{00}\rho_2 P_{00}^\dagger\right) &= \frac{1}{4}\left(|\alpha|^2 + |\beta|^2\right) = \frac{1}{4}, \left(|\alpha|^2 + |\beta|^2\right) = 1. \end{aligned}$$

$$\rho_{3_{(00)}} = |\alpha|^2 |0\rangle\langle 0| + \alpha\beta^* |0\rangle\langle 1| + \beta\alpha^* |1\rangle\langle 0| + |\beta|^2 |1\rangle\langle 1|. \tag{5.11}$$

Similarly, we consider the second possible outcome where Alice's measurement yields the result $|01\rangle$.

$$P_{01} = \langle 01| \otimes I_2 \tag{5.12}$$

By applying the corresponding projection operator to the global state ρ_2, we derive the new reduced density matrix at Bob's end

$$\begin{aligned}
\rho_{3_{(01)}} &= \frac{P_{01}\rho_2 P_{01}^\dagger}{Tr\left(P_{01}\rho_2 P_{01}^\dagger\right)} \qquad (5.13)\\
&\Longrightarrow P_{01}\rho_2 P_{01}^\dagger = \frac{1}{4}\left(|\alpha|^2 |1\rangle\langle 1| + \alpha\beta^* |1\rangle\langle 0| + \beta\alpha^* |0\rangle\langle 1| + |\beta|^2 |0\rangle\langle 0|\right),\\
Tr\left(P_{01}\rho_2 P_{01}^\dagger\right) &= \frac{1}{4}
\end{aligned}$$

$$\rho_{3_{(01)}} = |\alpha|^2 |1\rangle\langle 1| + \alpha\beta^* |1\rangle\langle 0| + \beta\alpha^* |0\rangle\langle 1| + |\beta|^2 |0\rangle\langle 0| \tag{5.14}$$

In the third scenario, we analyze the case where the measurement outcome is $|10\rangle$.

$$P_{10} = \langle 10| \otimes I_2 \tag{5.15}$$

The state projection is performed using the operator , which leads to a phase shift in the resulting density matrix at Bob's end :

$$\begin{aligned}
\rho_{3_{(10)}} &= \frac{P_{10}\rho_2 P_{10}^\dagger}{Tr\left(P_{10}\rho_2 P_{10}^\dagger\right)} \qquad (5.16)\\
&\Longrightarrow P_{10}\rho_2 P_{10}^\dagger = \frac{1}{4}\left(|\alpha|^2 |0\rangle\langle 0| - \alpha\beta^* |0\rangle\langle 1| - \beta\alpha^* |1\rangle\langle 0| + |\beta|^2 |1\rangle\langle 1|\right),\\
Tr\left(P_{10}\rho_2 P_{10}^\dagger\right) &= \frac{1}{4}
\end{aligned}$$

$$\rho_{3_{(10)}} = |\alpha|^2 |0\rangle\langle 0| - \alpha\beta^* |0\rangle\langle 1| - \beta\alpha^* |1\rangle\langle 0| + |\beta|^2 |1\rangle\langle 1| \tag{5.17}$$

Finally, we examine the fourth possible outcome where the measurement result is $|11\rangle$.

$$P_{11} = \langle 11| \otimes I_2 \tag{5.18}$$

By applying the projection operator we obtain the following reduced density matrix at Bob's end :

$$\rho_{3_{(11)}} = \frac{P_{11}\rho_2 P_{11}^\dagger}{Tr\left(P_{11}\rho_2 P_{11}^\dagger\right)} \tag{5.19}$$

$$\Longrightarrow \quad P_{11}\rho_2 P_{11}^\dagger = \frac{1}{4}\left(|\alpha|^2 |1\rangle\langle 1| - \alpha\beta^* |1\rangle\langle 0| - \beta\alpha^* |0\rangle\langle 1| + |\beta|^2 |0\rangle\langle 0|\right),$$

$$Tr\left(P_{11}\rho_2 P_{11}^\dagger\right) = \frac{1}{4}.$$

$$\rho_{3_{(11)}} = |\alpha|^2 |1\rangle\langle 1| - \alpha\beta^* |1\rangle\langle 0| - \beta\alpha^* |0\rangle\langle 1| + |\beta|^2 |0\rangle\langle 0| \tag{5.20}$$

To reconstruct the original state, Bob performs the necessary unitary operations based on the classical information received from Alice.

For the outcome $|00\rangle$:

Since the state is already in the correct form, Bob applies the identity operator I to maintain the teleported qubit :

$$\tilde{\rho}_{3_{(00)}} = I\rho_{3_{(00)}} I^\dagger \tag{5.21}$$

$$\tilde{\rho}_{3_{(00)}} = |\alpha|^2 |0\rangle\langle 0| + \alpha\beta^* |0\rangle\langle 1| + \beta\alpha^* |1\rangle\langle 0| + |\beta|^2 |1\rangle\langle 1|.$$

For the outcome $|01\rangle$:

To reverse the bit-flip induced by the measurement, Bob applies the Pauli-X gate as a correction operator :

$$\tilde{\rho}_{3_{(01)}} = X\rho_{3_{(01)}} X^\dagger \tag{5.22}$$

$$\tilde{\rho}_{3_{(01)}} = |\alpha|^2 (X|1\rangle\langle 1|X) + \alpha\beta^* (X|1\rangle\langle 0|X) + \beta\alpha^* (X|0\rangle\langle 1|X) + |\beta|^2 (X|0\rangle\langle 0|X)$$

$$\tilde{\rho}_{3_{(01)}} = |\alpha|^2 |0\rangle\langle 0| + \alpha\beta^* |0\rangle\langle 1| + \beta\alpha^* |1\rangle\langle 0| + |\beta|^2 |1\rangle\langle 1|.$$

For the outcome $|10\rangle$:

To correct the phase-flip and restore the relative phase of the state, Bob applies the Pauli-Z gate :

$$\begin{aligned}
\tilde{\rho}_{3_{(10)}} &= Z\rho_{3_{(10)}}Z^{\dagger} \quad (5.23)\\
\tilde{\rho}_{3_{(10)}} &= |\alpha|^2 (Z|0\rangle\langle 0|Z) - \alpha\beta^* (Z|0\rangle\langle 1|Z) - \beta\alpha^* (Z|1\rangle\langle 0|Z) + |\beta|^2 (Z|1\rangle\langle 1|Z)\\
\tilde{\rho}_{3_{(10)}} &= |\alpha|^2 |0\rangle\langle 0| - \alpha\beta^* (|0\rangle(-\langle 1|)) - \beta\alpha^* ((-|1\rangle)\langle 0|) + |\beta|^2 ((-|1\rangle)(-\langle 1|))\\
\tilde{\rho}_{3_{(10)}} &= |\alpha|^2 |0\rangle\langle 0| + \alpha\beta^* |0\rangle\langle 1| + \beta\alpha^* |1\rangle\langle 0| + |\beta|^2 |1\rangle\langle 1|.
\end{aligned}$$

For the outcome $|11\rangle$:

In this case, where both bit and phase inversions occur, Bob applies the Pauli-Y gate (or the combined XZ operation) to recover the message

$$\begin{aligned}
\tilde{\rho}_{3_{(11)}} &= (ZX)\,\rho_{3_{(11)}}\,(ZX)^{\dagger} \quad (5.24)\\
\tilde{\rho}_{3_{(11)}} &= (ZX)\,\rho_{3_{(11)}}X^{\dagger}Z^{\dagger}
\end{aligned}$$

Applying the Pauli-X gate performs a bit-flip operation, effectively swapping the basis states and aligning the amplitudes α and β with their original positions.

$$\begin{aligned}
\grave{\rho}_{3_{(11)}} &= X\rho_{3_{(11)}}X^{\dagger} \quad (5.25)\\
&= |\alpha|^2 (X|1\rangle\langle 1|X) - \alpha\beta^* (X|1\rangle\langle 0|X) - \beta\alpha^* (X|0\rangle\langle 1|X) + |\beta|^2 (X|0\rangle\langle 0|X)\\
\grave{\rho}_{3_{(11)}} &= |\alpha|^2 |0\rangle\langle 0| - \alpha\beta^* |0\rangle\langle 1| - \beta\alpha^* |1\rangle\langle 0| + |\beta|^2 |1\rangle\langle 1|
\end{aligned}$$

Finally, the Pauli-Z gate is applied to correct the relative phase. This transformation eliminates the negative signs and restores the coherence between the superposition states.

$$\begin{aligned}
\tilde{\rho}_{3_{(11)}} &= Z\grave{\rho}_{3_{(11)}}Z^{\dagger} \quad (5.26)\\
&= |\alpha|^2 (Z|0\rangle\langle 0|Z) - \alpha\beta^* (Z|0\rangle\langle 1|Z) - \beta\alpha^* (Z|1\rangle\langle 0|Z) + |\beta|^2 (Z|1\rangle\langle 1|Z)
\end{aligned}$$

As shown above, the resulting density matrix is identical to the initial state , confirming the successful recovery of the quantum information through the unitary correction process.

$$\tilde{\rho}_{3_{(11)}} = |\alpha|^2 |0\rangle\langle 0| + \alpha\beta^* |0\rangle\langle 1| + \beta\alpha^* |1\rangle\langle 0| + |\beta|^2 |1\rangle\langle 1| \quad (5.27)$$

Defining the Noisy Channel :

We begin by defining the state of the quantum channel under the influence of Werner noise. The ideal Bell state is replaced by a mixed state, which is expressed as a linear combination of the pure entangled state and the maximally mixed state I_4 :

$$\rho_{noisy} = \lambda \rho_{Bell} + \frac{(1-\lambda)}{2^2} I_2^{\otimes 2}. \tag{5.28}$$

The first component represents the ideal Bell state. Having already derived its evolution, we simply scale the result by λ.

The second part represents the maximally mixed noise. As shown in the following calculations, this component evolves through the same gates but results in a completely mixed state at Bob's end.

$$\rho_0^* = \rho_{\Psi_A} \otimes I_4 \tag{5.29}$$

Recalling the density matrix ρ_{Ψ_A} from the previous section, the total state is expressed as :

$$\rho_{\Psi_A} = |\alpha|^2 |0\rangle \langle 0| + \alpha\beta^* |0\rangle \langle 1| + \beta\alpha^* |1\rangle \langle 0| + |\beta|^2 |1\rangle \langle 1| . \tag{5.30}$$

The noise contribution of the Werner channel is represented by the identity matrix I_4. This component describes a maximally mixed state, where the information is uniformly distributed and lacks any quantum correlation :

$$\begin{aligned} I_4 &= I_2 \otimes I_2 \\ &= (|0\rangle \langle 0| + |1\rangle \langle 1|) \otimes (|0\rangle \langle 0| + |1\rangle \langle 1|) \\ &= |00\rangle \langle 00| + |01\rangle \langle 01| + |10\rangle \langle 10| + |11\rangle \langle 11| . \end{aligned} \tag{5.31}$$

The initial state of the total system is then formed by the tensor product of the message qubit and the noisy entangled channel. This allowed us to analyze the evolution of the signal and the noise components independently due to the linearity of the quantum operations :

$$\begin{aligned}
\rho_{\Psi_A} \otimes I_4 &= \left(|\alpha|^2 |0\rangle \langle 0| + \alpha\beta^* |0\rangle \langle 1| + \beta\alpha^* |1\rangle \langle 0| + |\beta|^2 |1\rangle \langle 1|\right) \qquad (5.32) \\
&\otimes (|00\rangle \langle 00| + |01\rangle \langle 01| + |10\rangle \langle 10| + |11\rangle \langle 11|) . \\
\rho_0^* &= |\alpha|^2 (|000\rangle \langle 000| + |001\rangle \langle 001| + |010\rangle \langle 010| + |011\rangle \langle 011|) \qquad (5.33) \\
&+\alpha\beta^* (|000\rangle \langle 100| + |001\rangle \langle 101| + |010\rangle \langle 110| + |011\rangle \langle 111|) \\
&+\beta\alpha^* (|100\rangle \langle 000| + |101\rangle \langle 001| + |110\rangle \langle 010| + |111\rangle \langle 011|) \\
&+ |\beta|^2 (|100\rangle \langle 100| + |101\rangle \langle 101| + |110\rangle \langle 110| + |111\rangle \langle 111|) .
\end{aligned}$$

Step 1 : Application of the C_NOT Gate

We apply the C_NOT operation to the initial noise component. Since the identity matrix represents a state of maximum entropy, it remains invariant under the action of the control gate :

$$\rho_1^* = (C\_NOT)\, \rho_0^* \,(C\_NOT)^\dagger \qquad (5.34)$$

$$\begin{aligned}
\rho_1^* &= |\alpha|^2 (|000\rangle \langle 000| + |001\rangle \langle 001| + |010\rangle \langle 010| + |011\rangle \langle 011|) \\
&+\alpha\beta^* (|000\rangle \langle 110| + |001\rangle \langle 111| + |010\rangle \langle 100| + |011\rangle \langle 101|) \qquad (5.35) \\
&+\beta\alpha^* (|110\rangle \langle 000| + |111\rangle \langle 001| + |100\rangle \langle 010| + |101\rangle \langle 011|) \\
&+ |\beta|^2 (|110\rangle \langle 110| + |111\rangle \langle 111| + |100\rangle \langle 100| + |101\rangle \langle 101|) .
\end{aligned}$$

Step 2 : Application of the Hadamard Gate

Similarly, the Hadamard gate is applied to the first qubit. The identity component I_4 continues to be invariant under this local unitary transformation :

$$\rho_2^* = H \rho_1^* H^\dagger \qquad (5.36)$$

$$\rho_2^* = \frac{1}{2}\left(\begin{array}{c} |\alpha|^2 \left\{\begin{array}{c} |000\rangle\langle 000| + |000\rangle\langle 100| + |100\rangle\langle 000| + |100\rangle\langle 100| \\ + |001\rangle\langle 001| + |001\rangle\langle 101| + |101\rangle\langle 101| + |001\rangle\langle 101| \\ + |110\rangle\langle 110| + |010\rangle\langle 110| + |110\rangle\langle 010| + |110\rangle\langle 110| \\ + |011\rangle\langle 011| + |011\rangle\langle 111| + |111\rangle\langle 011| + |111\rangle\langle 111| \end{array}\right\} + \\ \beta\alpha^* \left\{\begin{array}{c} |000\rangle\langle 010| - |000\rangle\langle 110| + |100\rangle\langle 010| - |100\rangle\langle 110| \\ + |001\rangle\langle 011| - |001\rangle\langle 111| + |101\rangle\langle 011| - |101\rangle\langle 111| \\ + |010\rangle\langle 000| - |010\rangle\langle 100| + |110\rangle\langle 000| - |110\rangle\langle 100| \\ + |011\rangle\langle 001| - |011\rangle\langle 101| + |111\rangle\langle 001| - |111\rangle\langle 101| \end{array}\right\} + \\ \beta\alpha^* \left\{\begin{array}{c} |010\rangle\langle 010| + |010\rangle\langle 100| - |110\rangle\langle 000| - |110\rangle\langle 100| \\ + |011\rangle\langle 001| + |011\rangle\langle 101| - |111\rangle\langle 001| - |111\rangle\langle 101| \\ + |000\rangle\langle 010| + |000\rangle\langle 110| - |100\rangle\langle 010| - |100\rangle\langle 110| \\ + |001\rangle\langle 011| + |001\rangle\langle 111 - |101\rangle\langle 011|| - |101\rangle\langle 111| \end{array}\right\} + \\ |\beta|^2 \left\{\begin{array}{c} |010\rangle\langle 010| - |010\rangle\langle 110| - |110\rangle\langle 010| + |110\rangle\langle 110| \\ + |011\rangle\langle 011| - |011\rangle\langle 111| - |111\rangle\langle 011| + |111\rangle\langle 111| \\ + |000\rangle\langle 000| - |000\rangle\langle 100| - |100\rangle\langle 000| + |100\rangle\langle 100| \\ + |001\rangle\langle 001| - |001\rangle\langle 101| - |101\rangle\langle 001| + |101\rangle\langle 101| \end{array}\right\} \end{array}\right). \quad (5.37)$$

After the unitary evolution of the background noise part (I_4), Alice performs measurements in the computational basis. We analyze the resulting state at Bob's end for each specific outcome.

For the measurement outcome $|00\rangle$:

If Alice obtains the result $|00\rangle$, the state is projected using the operator

$$\begin{aligned} P_{00} &= \langle 00| \otimes I_2 \\ \Longrightarrow & P_{00}^\dagger = |00\rangle \otimes I_2 \end{aligned} \quad (5.38)$$

The reduced density matrix at Bob's side is found by normalizing the projected state with its trace. As shown in the calculation below, the result is a maximally mixed state :

$$\begin{aligned} \rho^*_{3_{(00)}} &= \frac{P_{00}\rho_2^* P_{00}^\dagger}{Tr\left(P_{00}\rho_2^* P_{00}^\dagger\right)} \\ \Longrightarrow & P_{00}\rho_2^* P_{00}^\dagger = \frac{1}{2}\left(|\alpha|^2\{|0\rangle\langle 0| + |1\rangle\langle 1|\} + |\beta|^2\{|0\rangle\langle 0| + |1\rangle\langle 1|\}\right) \\ &= \frac{1}{2}\left(|\alpha|^2 + |\beta|^2\right)\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) \end{aligned} \quad (5.39)$$

The normalization factor, determined by the trace of the projected matrix, remains 1/2.

$$Tr\left(P_{00}\rho_2^* P_{00}^\dagger\right) = \frac{1}{2} \tag{5.40}$$

This leads to the following reduced density matrix at Bob's side

$$\begin{aligned} \rho^*_{3_{(00)}} &= \frac{\frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)}{\frac{1}{2}} \\ &= |0\rangle\langle 0| + |1\rangle\langle 1|. \end{aligned} \tag{5.41}$$

For the measurement outcome $|01\rangle$:

In the case where Alice's measurement yields the result $|01\rangle$, we apply the projection operator . This operation isolates the corresponding subspace of the noise component :

$$\begin{aligned} P_{01} &= \langle 01| \otimes I_2 \\ &\Longrightarrow P_{01}^\dagger = |01\rangle \otimes I_2 \end{aligned} \tag{5.42}$$

Following the same normalization procedure, the trace of the projected noise component is calculated as follows :

$$\begin{aligned} \rho^*_{3_{(01)}} &= \frac{P_{01}\rho_2^* P_{01}^\dagger}{Tr\left(P_{01}\rho_2^* P_{01}^\dagger\right)} \\ &\Longrightarrow P_{01}\rho_2^* P_{01}^\dagger = \frac{1}{2}\left(|\alpha|^2\left\{|0\rangle\langle 0| + |1\rangle\langle 1|\right\} + |\beta|^2\left\{|0\rangle\langle 0| + |1\rangle\langle 1|\right\}\right) \\ &= \frac{1}{2}\left(|\alpha|^2 + |\beta|^2\right)\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) \\ Tr\left(P_{01}\rho_2^* P_{01}^\dagger\right) &= \frac{1}{2} \end{aligned} \tag{5.43}$$

$$\rho^*_{3_{(01)}} = |0\rangle\langle 0| + |1\rangle\langle 1| \tag{5.44}$$

For the measurement outcome $|10\rangle$:

In the case where Alice's measurement yields the result $|10\rangle$, the noise component is projected using the operator

$$\begin{aligned} P_{10} &= \langle 10| \otimes I_2 \\ &\Longrightarrow P_{10}^{\dagger} = |10\rangle \otimes I_2 \end{aligned} \tag{5.45}$$

We follow the same procedure to determine the resulting state at Bob's side :

$$\begin{aligned} \rho^*_{3_{(10)}} &= \frac{P_{10}\rho_2^* P_{10}^{\dagger}}{Tr\left(P_{10}\rho_2^* P_{10}^{\dagger}\right)} \\ &\Longrightarrow P_{10}\rho_2^* P_{10}^{\dagger} = \frac{1}{2}\left(|\alpha|^2 \{|0\rangle\langle 0| + |1\rangle\langle 1|\} + |\beta|^2 \{|0\rangle\langle 0| + |1\rangle\langle 1|\}\right) \\ &= \frac{1}{2}\left(|\alpha|^2 + |\beta|^2\right)\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) \\ Tr\left(P_{10}\rho_2^* P_{10}^{\dagger}\right) &= \frac{1}{2} \end{aligned} \tag{5.46}$$

$$\rho^*_{3_{(10)}} = |0\rangle\langle 0| + |1\rangle\langle 1| \tag{5.47}$$

For the measurement outcome $|11\rangle$:

Finally, we consider the case where Alice's measurement yields the result $|11\rangle$. The corresponding projection operator is defined as :

$$\begin{aligned} P_{11} &= \langle 11| \otimes I_2 \\ &\Longrightarrow P_{11}^{\dagger} = |11\rangle \otimes I_2 \end{aligned} \tag{5.48}$$

The evolution of the noise component through this branch is calculated as :

$$\begin{aligned} \rho^*_{3_{(11)}} &= \frac{P_{11}\rho_2^* P_{11}^{\dagger}}{Tr\left(P_{11}\rho_2^* P_{11}^{\dagger}\right)} \\ &\Longrightarrow P_{11}\rho_2^* P_{11}^{\dagger} = \frac{1}{2}\left(|\alpha|^2 \{|0\rangle\langle 0| + |1\rangle\langle 1|\} + |\beta|^2 \{|0\rangle\langle 0| + |1\rangle\langle 1|\}\right) \\ &= \frac{1}{2}\left(|\alpha|^2 + |\beta|^2\right)\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) \\ Tr\left(P_{11}\rho_2^* P_{11}^{\dagger}\right) &= \frac{1}{2} \end{aligned} \tag{5.49}$$

$$\rho^*_{3_{(11)}} = |0\rangle\langle 0| + |1\rangle\langle 1| \tag{5.50}$$

In summary, all four measurement outcomes for the identity component I_4. lead to the same output state

$$\rho_3^* = \frac{1}{2} I_2 \tag{5.51}$$

This consistency allows us to simplify the final reconstruction of the noisy teleported qubit.

Final Reconstructed Density Matrix

Having analyzed the evolution of both the entangled signal and the background noise components independently, we now reconstruct the final state at Bob's side. Using the linearity of the quantum channel, the total density matrix is obtained by summing the weighted contributions of the ideal teleported state and the identity component :

$$\rho_{Bob}^{Final} = \lambda \rho_{\Psi_A} + \frac{(1-\lambda)}{2} I_2. \tag{5.52}$$

This general expression represents the final state at Bob's end as a convex combination of the ideal teleported qubit and the isotropic noise component. The weights are determined by the purity parameter λ of the Werner state.

$$\begin{aligned} \rho_{Bob}^{Final} &= \lambda \left(|\alpha|^2 |0\rangle \langle 0| + \alpha\beta^* |0\rangle \langle 1| + \beta\alpha^* |1\rangle \langle 0| + |\beta|^2 |1\rangle \langle 1| \right) \\ &+ \frac{(1-\lambda)}{2} \left(|0\rangle \langle 0| + |1\rangle \langle 1| \right). \end{aligned} \tag{5.53}$$

By substituting the explicit density matrix of the input state ρ_{Alice} and the identity matrix I_2, we obtain the following expanded form. This step allows us to see how the noise floor is distributed across the computational basis.

$$\rho_{Bob}^{Final} = \left(\lambda + \frac{(1-\lambda)}{2} \right) |0\rangle \langle 0| + \lambda\alpha\beta^* |0\rangle \langle 1| + \lambda\beta\alpha^* |1\rangle \langle 0| + \left(\lambda + \frac{(1-\lambda)}{2} \right) |1\rangle \langle 1|. \tag{5.54}$$

The final reconstructed density matrix shows that the diagonal elements are shifted by the noise contribution, while the off-diagonal coherences are attenuated by the factor λ. This matrix serves as the basis for calculating the efficiency of the protocol.

5.3 Analytical Evaluation of the Controlled Protocol under Werner Noise

To further investigate the limitations of quantum communication in realistic environments, we extend our study to the Controlled Quantum Teleportation Protocol. In this sophisticated configuration, the successful transmission of a quantum state is not merely a transaction between the sender (Alice) and the receiver (Bob), but is strictly governed by a third party, the controller (Charlie). The fundamental motivation behind analyzing this protocol lies in its essential role in building secure, hierarchical quantum networks where information access must be authorized.For this purpose, we utilize a three-qubit GHZ (Greenberger-Horne-Zeilinger) state as the primary entangled resource. However, in any practical implementation, such high-dimensional entanglement is highly susceptible to environmental decoherence. To account for these effects, we model the shared quantum channel as a Werner state, which allows us to trace the transition from a purely entangled tripartite system to a state affected by isotropic white noise.To maintain mathematical rigor while managing the resulting (1616) density matrices, we adopt a linearity-based decomposition approach. This methodology allows us to separate the protocol's evolution into two distinct branches : the signal branch (pure GHZ) and the noise branch (identity matrix). This systematic separation ensures a clear derivation of the final reconstructed state and provides a precise measure of the protocol's robustness against decoherence.

The noisy initial channel is defined as :

$$\bar{\rho}_{noise} = \lambda \rho_{GHZ} + \frac{(1-\lambda)}{8} I_8. \tag{5.55}$$

The teleportation process is initiated by defining the composite state of the entire system. This involves the tensor product of the unknown input state ρ_{Ψ_A} and the entangled Werner resource ρ_{noise}. The resulting global density matrix ρ_{Total} serves as the starting point for the subsequent unitary transformations.

$$\bar{\rho}_0 = \rho_{\Psi_A} \otimes \bar{\rho}_{noise} \tag{5.56}$$

To calculate the global density matrix by performing the Kronecker product of the initial state and the quantum channel, we use the following Mathematica implementation :

```
//////////////////////////////////////////////////////////////////

RhoEtatGlobal = KroneckerProduct[RhoPsi, RhoCanal] ;

//////////////////////////////////////////////////////////////////
```

Measurement and Projection Stage

Once the unitary transformations are complete, the system is prepared for the measurement phase. To extract the teleported information, two simultaneous operations are performed :

Alice's Measurement : Alice performs a Bell-state measurement (BSM) on her two qubits (the message qubit and her part of the shared channel). This joint measurement projects the remaining qubits of the system into a specific state corresponding to one of the four Bell outcomes. To obtain the reduced density matrix for Charlie and Bob by applying Alice's projection to the global state, we use the following Mathematica implementation :

```
ProjecteurAlice = Bell01† ;
 ProjecteurAlice = Bell00† ;
 ProjecteurAlice = Bell10† ;
 ProjecteurAlice = Bell11† ;

ProjecteurGlobal = KroneckerProduct[ProjecteurAlice,
   IdentityMatrix[4]] ;

RhoCharlieBob = ProjecteurGlobal . RhoEtatGlobal ;
RhoCharlieBob = RhoCharlieBob . ProjecteurGlobal† ;
```

Charlie's Measurement : Acting as the controller, Charlie performs a measurement on his qubit in the transversal basis $\{|-\rangle, |+\rangle\}$ to authorize the state reconstruction at Bob's end.

To extract the final reduced density matrix for Bob by applying Charlie's projection to the previously obtained state, we use the following Mathematica implementation :

```
ProjecteurCharlie = HPlus† ;
 ProjecteurCharlie = HMoins† ;

ProjecteurGlobal = KroneckerProduct[ProjecteurCharlie,
   IdentityMatrix[2]] ;
```

After Alice and Charlie perform their measurements and Bob applies the necessary unitary correction gates (Pauli gates) based on the received classical information, the final state of Bob's qubit is obtained. By summing the contributions from the signal branch and the noise branch, the reduced density matrix ρ_{Bob}^{Final} is expressed in its analytical form as :

$$\rho_{Bob}^{Final} = \lambda \rho_{\Psi_A} + \frac{(1-\lambda)}{2} I_2 \tag{5.57}$$

This expression represents a linear combination of the ideal teleported state and a maximally mixed state, where the purity of the final qubit is directly governed by the decoherence parameter λ.

To extract the final reduced density matrix for Bob by applying the global projector to the Charlie-Bob state, we use the following Mathematica implementation :

```
RhoBob = ProjecteurGlobal.RhoCharlieBob ;
RhoBob = RhoBob.ProjecteurGlobal† ;
```

5.4 Analytical Evaluation of the Bidirectional Controlled Protocol under Werner Noise

In this section, we extend our study to the Switch-controlled Bidirectional Quantum Teleportation (BCQT) protocol under realistic conditions. Unlike ideal scenarios, quantum networks face the challenge of decoherence, which degrades the shared entangled resource.

To evaluate the robustness of this bidirectional exchange, we model the quantum channel using the Werner Noise model. By transitioning the shared resource from a maximally entangled state to a partially mixed state, we aim to precisely derive the final reconstructed states and assess how the "Switch" functionality and teleportation fidelity are affected by environmental noise.

In this protocol, we consider two senders, Transmitter$_1$ (T_1) and Transmitter$_2$ (T_2), who possess two arbitrary unknown qubits, Q_1 and Q_2, respectively. Under the influence of noise, these qubits are no longer described by pure state vectors, but are instead defined by their Density Matrices :

Transmitter$_1$ (T_1) disposes of the qubit Q_1, represented by the density matrix :

$$\begin{aligned} \rho_{Q_1} &= |Q_1\rangle \langle Q_1| \\ &= |\alpha_1|^2 |0\rangle \langle 0| + \alpha_1\beta_1^* |0\rangle \langle 1| + \beta_1\alpha_1^* |1\rangle \langle 0| + |\beta_1|^2 |1\rangle \langle 1| \end{aligned} \tag{5.58}$$

Transmitter$_2$ (T_2) disposes of the qubit Q_2, represented by the density matrix :

$$\begin{aligned} \rho_{Q_2} &= |Q_2\rangle \langle Q_2| \\ &= |\alpha_2|^2 |0\rangle \langle 0| + \alpha_2\beta_2^* |0\rangle \langle 1| + \beta_2\alpha_2^* |1\rangle \langle 0| + |\beta_2|^2 |1\rangle \langle 1| \end{aligned} \tag{5.59}$$

Two receivers Receiver$_1$ (R_1) and Receiver$_2$ (R_2). In the presence of environmental noise, the receivers are expected to retrieve mixed quantum states, where the final fidelity depends on the level of decoherence in the channel.

A controller (C_{tr}) who acts as a switch. The controller's role is to guide the noisy quantum information through two possible scenarios. Under the influence of Werner noise, the switching logic is defined by the following scenarios :

Scenario 1 : The noisy state ρ_{Q_1} is routed to Receiver$_1$, and ρ_{Q_2} is routed to Receiver$_2$.

Scenario 2 : The noisy state ρ_{Q_1} is routed to Receiver$_2$, and ρ_{Q_2} is routed to Receiver$_1$.

Using our platform as a protocol research tool, several teleportation schemes were evaluated under noisy conditions. The most suitable scheme for maintaining robustness in the design of the cyclic protocol is presented in the following subsections, where we analyze the system using the Density Matrix formalism to account for the loss of purity.

To represent the noise in the 6-qubit quantum channel shared between the participants, we employ the Werner state model. In this framework, the ideal entangled state transitions into a mixed state due to environmental interactions. The noisy density matrix of the channel is defined as :

$$\bar{\rho}_{noise} = \lambda \rho_{channel} + \frac{(1-\lambda)}{64} I_{64} \tag{5.60}$$

The components of the noisy channel are defined as :

The ideal channel density matrix :

$$\rho_{channel} = |Channel\rangle \langle channel| \tag{5.61}$$

The maximally mixed state (Identity matrix) :

$$I_{64} = \sum_{i=1}^{64} |i\rangle \langle i| \tag{5.62}$$

The Initial Global

$$\bar{\rho}_0 = \rho_{Q_1} \otimes \rho_{Q_2} \otimes \bar{\rho}_{noise} \tag{5.63}$$

To construct the global density matrix of the system by combining the initial states of the participants with the quantum channel through successive Kronecker products, we use the following Mathematica implementation :

```
Construction de l'etat global
```

```
Print["Construction␣de␣l'etat␣global␣"]
RhoAlice01Alice02 = KroneckerProduct[RhoAlice01 , RhoAlice02 ] ;
RhoEtatGlobal = KroneckerProduct[RhoAlice01Alice02 , RhoCanal ] ;

////////////////////////////////////////////////////////////////
```

Bell-State Measurements (BSM)

T_1 and T_2 perform joint measurements using the projection basis in its operator form :

$$\left\{ \left|\Psi^{\pm}\right\rangle \left\langle \Psi^{\pm}\right|, \left|\Phi^{\pm}\right\rangle \left\langle \Phi^{\pm}\right| \right\} \tag{5.64}$$

The system collapses into a mixed state $\bar{\rho}_1$. Due to the Werner noise, the information of the unknown qubits becomes entangled with the depolarized components of the channel.

$$\bar{\rho}_1 = \frac{MS\bar{\rho}_0 S^{\dagger} M^{\dagger}}{Tr\left(MS\bar{\rho}_0 S^{\dagger} M^{\dagger}\right)} \tag{5.65}$$

To perform the sequential projections for Alice's qubits and update the global density matrix accordingly, we use the following Mathematica implementation :

```
Projection d'Alice 01
ProjecteurAlice = Bell00† ;
ProjecteurGlobal = KroneckerProduct[ProjecteurAlice ,
   IdentityMatrix[64]] ;
RhoEtatGlobal = ProjecteurGlobal.RhoEtatGlobal ;
RhoEtatGlobal = RhoEtatGlobal.ProjecteurGlobal† ;

Projection d'Alice 02
ProjecteurBob = Bell00† ;
ProjecteurGlobal = KroneckerProduct[ProjecteurBob ,
   IdentityMatrix[16]] ;
RhoEtatGlobal = ProjecteurGlobal.RhoEtatGlobal ;
RhoEtatGlobal = RhoEtatGlobal.ProjecteurGlobal† ;
```

Controller's Switching Operations

The Controller chooses the scenario by applying unitary operations on qubits $\left(Q_{C_{tr_1}}\right)$ and $\left(Q_{C_{tr_2}}\right)$. The density matrix evolves as follows :

Scenario 1 (Direct) : The controller applies $H\left(Q_{C_{tr_1}}\right)$.

$$H\left(Q_{C_{tr_1}}\right)\bar{\rho}_1 H^{\dagger}\left(Q_{C_{tr_1}}\right) \tag{5.66}$$

To apply the selection gates for the first controller and update the global density matrix accordingly, we use the following Mathematica implementation :

```
 Portes du Choix 01
PorteChoix01Controleur = KroneckerProduct[H, IdentityMatrix[8]];
RhoEtatGlobal = PorteChoix01Controleur.RhoEtatGlobal ;
RhoEtatGlobal = RhoEtatGlobal.PorteChoix01Controleur† ;
```

The controller then performs a measurement in the computational basis :

$$\{|00\rangle\langle 00|, |01\rangle\langle 01|, |10\rangle\langle 10|, |11\rangle\langle 11|\} \tag{5.67}$$

To perform the controller's measurement and reduce the system's dimensionality before the final state recovery, we use the following Mathematica implementation :

```
 Projection du Controleur
Print["Projection␣du␣Controleur␣"]
ProjecteurControleur = Projecteur00† ;
ProjecteurGlobal = KroneckerProduct[ProjecteurControleur ,
   IdentityMatrix[4]] ;
```

Final Reconstructed State at the Receivers

After Receiver$_1$ (R_1) and Receiver$_2$ (R_2) apply the necessary unitary corrections, the final state of the system is :

$$\bar{\rho}_{R_1R_2} = \lambda\rho_{Q_1}\otimes\rho_{Q_2} + \frac{(1-\lambda)}{4}I_4 \tag{5.68}$$

To demonstrate the final success of the protocol and the effective recovery of the transmitted states for Bob's qubits, we provide the following Mathematica implementation :

```
RhoBob01Bob02 = ProjecteurGlobal.RhoEtatGlobal ;
RhoBob01Bob02 = RhoBob01Bob02.ProjecteurGlobal† ;
```

Scenario 2 (Crossed) : The controller applies $C\_NOT\left(Q_{C_{tr_1}}, Q_{C_{tr_2}}\right)$ and $H\left(Q_{C_{tr_1}}\right)$.

$$\bar{\rho}_2 = C\_NOT\left(Q_{C_{tr_1}}, Q_{C_{tr_2}}\right)\bar{\rho}_1 C\_NOT^{\dagger}\left(Q_{C_{tr_1}}, Q_{C_{tr_2}}\right) \tag{5.69}$$

To implement the second controller's choice gates and update the system's global density matrix, we use the following Mathematica implementation :

```
 Portes du Choix 02
Porte01Choix02Controleur = KroneckerProduct[CNot, IdentityMatrix
   [4]] ;
RhoEtatGlobal = Porte01Choix02Controleur.RhoEtatGlobal ;
RhoEtatGlobal = RhoEtatGlobal.Porte01Choix02Controleur† ;
```

$$\bar{\rho}_3 = H\bar{\rho}_2 H^\dagger \tag{5.70}$$

To apply the second choice gate for the controller and update the global density matrix of the system, we use the following Mathematica implementation :

```
 Porte 02 Choix 02 Controleur
Porte02Choix02Controleur = KroneckerProduct[H, IdentityMatrix[8]]
    ;
RhoEtatGlobal = Porte02Choix02Controleur.RhoEtatGlobal ;
RhoEtatGlobal = RhoEtatGlobal.Porte02Choix02Controleur† ;
```

The controller then performs a measurement in the computational basis.

To perform the controller's measurement and reduce the system's dimensionality before the final state recovery, we use the following Mathematica implementation :

```
 Projection du Controleur
Projecteur02Controleur = Projecteur00† ;
ProjecteurGlobal = KroneckerProduct[ IdentityMatrix[4] ,
   Projecteur02Controleur] ;
ProjecteurGlobal = KroneckerProduct[ ProjecteurGlobal ,
   IdentityMatrix[4] ] ;
```

Final Reconstructed State at the Receivers

After Receiver$_1$ (R_1) and Receiver$_2$ (R_2) apply the necessary unitary corrections, the final state of the system is :

$$\bar{\rho}_{R_1R_2} = \lambda\rho_{Q_2} \otimes \rho_{Q_1} + \frac{(1-\lambda)}{4} I_4 \tag{5.71}$$

The following result demonstrates the successful protocol implementation and the calculation of the final density matrix in Mathematica :

```
RhoBob01Bob02 = ProjecteurGlobal . RhoEtatGlobal ;
RhoBob01Bob02 = RhoBob01Bob02 . ProjecteurGlobal† ;
```

5.5 Analysis of Double-Direction Cyclic Controlled Teleportation under

In this section, we extend our analytical framework to the Double-Direction Cyclic Controlled Teleportation protocol. This advanced configuration enables a circular exchange of quantum information among multiple participants, regulated by a central controller.To model the effects of environmental decoherence on this system, we utilize an 11-qubit quantum channel subjected to Werner Noise. The transition from a 6-qubit to an 11-qubit resource significantly increases the complexity of the density matrix, which now operates in a Hilbert space of dimension $2^{11} = 2048$.Using the Density Matrix formalism, we aim to derive the analytical expressions for the final states as they circulate through this large-scale noisy resource. We will specifically evaluate how the purity parameter λ affects the fidelity of the cyclic transfer and whether the Switch mechanism remains robust when scaling up the number of entangled qubits in the channel.

In the double-direction cyclic controlled teleportation protocol, we define three main actors : Alice, Bob, and Charlie. Their roles evolve throughout the process :

At the beginning : They act as transmitters, each holding an unknown quantum state.

At the end : They act as receivers, retrieving the teleported states according to the controller's decision.

Initially, we assume that the three transmitters (Alice, Bob, and Charlie) dispose of three arbitrary unknown qubits Q_A, |Q$_B$ and Q_C, respectively. In a noisy environment, these states are described by their Density Matrices to account for the loss of purity during the teleportation process :

For Alice :

$$\rho_{Q_A} = |Q_A\rangle \langle Q_A| \tag{5.72}$$

For Bob :

$$\rho_{Q_B} = |Q_B\rangle \langle Q_B| \tag{5.73}$$

For Charlie :

$$\rho_{Q_C} = |Q_C\rangle \langle Q_C| \tag{5.74}$$

The Controller as a Quantum Switch

The Controller (Ctr) plays a vital role in determining the direction of the cyclic permutation. In our analysis under Werner noise, the controller's operations are modeled as unitary transformations applied to the global density matrix ρ. The controller can choose between two main noisy scenarios :

Scenario 1 : Clockwise Cyclic Controlled TeleportationThe controller applies specific quantum gates to route the information such that :ρ_{Q_A} →Charlie, ρ_{Q_B} →Alice, and ρ_{Q_C} →Bob.In this case, the final density matrix at each receiver's node will be a mixture of the target state and the background noise defined by λ.

Scenario 2 : Counter-Clockwise Cyclic Controlled TeleportationThe controller acts as a switch to reverse the permutation flow :ρ_{Q_A} →Bob, ρ_{Q_B} →Charlie, and ρ_{Q_C} →Alice.

In this cyclic protocol, the shared resource is an 11-qubit quantum channel distributed among the three transmitters (Alice, Bob, and Charlie) and the controller. To model a realistic communication environment, we assume the channel is affected by Werner Noise, which transforms the initial maximally entangled state into a mixed state.

The noisy density matrix of this 11-qubit channel is defined as :

$$\bar{\rho}_{channel} = \lambda \rho_{channel} + \frac{(1-\lambda)}{2048} I_{2048} \tag{5.75}$$

The Initial Global

$$\rho_0 = \rho_{Q_A} \otimes \bar{\rho}_{channel} \tag{5.76}$$

Here, we calculate the global state of the system by performing the Kronecker product between Alice's state and the quantum channel state :

```
RhoEtatGlobal = KroneckerProduct[RhoAlice, RhoCanal];
```

Alice perform joint Bell-state measurements on their respective qubits :

$$\left\{ \left|\Psi^{\pm}\right\rangle \left\langle \Psi^{\pm}\right|, \left|\Phi^{\pm}\right\rangle \left\langle \Phi^{\pm}\right| \right\} \tag{5.77}$$

The measurement process is implemented in Wolfram Mathematica as follows

```
ProjecteurAlice = Bell00†;
ProjecteurGlobal = KroneckerProduct[ProjecteurAlice,
   IdentityMatrix[1024]];
RhoEtatGlobal = ProjecteurGlobal . RhoEtatGlobal;
RhoEtatGlobal = RhoEtatGlobal . ProjecteurGlobal†;
```

The Initial Global

$$\rho_0 = \rho_{Q_B} \otimes \bar{\rho}_{channel} \tag{5.78}$$

The Wolfram Mathematica implementation is as follows

```
RhoEtatGlobal = KroneckerProduct[RhoBob, RhoEtatGlobal];
```

Bob perform joint Bell-state measurements on their respective qubits :

$$\left\{ \left|\Psi^{\pm}\right\rangle \left\langle \Psi^{\pm}\right|, \left|\Phi^{\pm}\right\rangle \left\langle \Phi^{\pm}\right| \right\} \tag{5.79}$$

The Wolfram Mathematica implementation is as follows :

```
ProjecteurBob = Bell00†;
ProjecteurGlobal = KroneckerProduct[ProjecteurBob,
   IdentityMatrix[512]];
RhoEtatGlobal = ProjecteurGlobal . RhoEtatGlobal;
RhoEtatGlobal = RhoEtatGlobal . ProjecteurGlobal†;
```

The Initial Global

$$\rho_0 = \rho_{Q_C} \otimes \bar{\rho}_{channel} \tag{5.80}$$

The Wolfram Mathematica implementation is as follows

```
RhoEtatGlobal = KroneckerProduct[RhoCharlie, RhoEtatGlobal];
```

Charlie perform joint Bell-state measurements on their respective qubits :

$$\left\{ \left|\Psi^{\pm}\right\rangle \left\langle \Psi^{\pm}\right|, \left|\Phi^{\pm}\right\rangle \left\langle \Phi^{\pm}\right| \right\} \tag{5.81}$$

The Wolfram Mathematica implementation is as follows :

```
ProjecteurCharlie = Bell00†;
ProjecteurGlobal = KroneckerProduct[ProjecteurCharlie,
   IdentityMatrix[256]];
RhoEtatGlobal = ProjecteurGlobal . RhoEtatGlobal;
RhoEtatGlobal = RhoEtatGlobal . ProjecteurGlobal†;
```

$$\bar{\rho}_1 = \frac{MS\bar{\rho}_0 S^\dagger M^\dagger}{Tr\left(MS\bar{\rho}_0 S^\dagger M^\dagger\right)} \tag{5.82}$$

Controller's Operations in the Clockwise Scenario

The controller initiates the routing by applying the Hadamard gate to the first controller qubit :

$$\bar{\rho}_2 = H\left(Q_{C_{tr_1}}\right)\bar{\rho}_1 H^\dagger\left(Q_{C_{tr_1}}\right) \tag{5.83}$$

The following implementation shows the application of the Hadamard gate on the controller and the subsequent update of the global density matrix in Mathematica :

PorteChoix01Controleur = KroneckerProduct[H, IdentityMatrix[128]];
RhoEtatGlobal = PorteChoix01Controleur.RhoEtatGlobal;
RhoEtatGlobal = RhoEtatGlobal.PorteChoix01Controleur† ;

Measurement : A measurement is performed in the computational basis on qubits $Q_{C_{tr_1}}$ $Q_{C_{tr_2}}$.

The following result demonstrates the implementation of the controller's projection and the reduction of the global density matrix in Mathematica :

Projecteur01Controleur = Projecteur00† ;
ProjecteurGlobal = KroneckerProduct[Projecteur01Controleur , IdentityMatrix[64]];
RhoEtatGlobal = ProjecteurGlobal.RhoEtatGlobal;
RhoEtatGlobal = RhoEtatGlobal.ProjecteurGlobal† ;

The controller applies a C_NOT gate followed by a Hadamard gate :

$$\bar{\rho}_3 = C\_NOT_{\left(Q_{C_{tr_4}},Q_{C_{tr_5}}\right)}\bar{\rho}_2 C\_NOT^\dagger_{\left(Q_{C_{tr_4}},Q_{C_{tr_5}}\right)} \tag{5.84}$$

The following result demonstrates the application of the CNOT gate within the controlled system and the update of the global density matrix in Mathematica :

Porte01Choix02Controleur = KroneckerProduct[IdentityMatrix[4] , CNot];
Porte01Choix02Controleur = KroneckerProduct[Porte01Choix02Controleur , IdentityMatrix[4]];
RhoEtatGlobal = Porte01Choix02Controleur.RhoEtatGlobal;
RhoEtatGlobal = RhoEtatGlobal.Porte01Choix02Controleur† *;*

$$\bar{\rho}_4 = H\left(Q_{C_{tr_4}}\right)\bar{\rho}_3 H^\dagger\left(Q_{C_{tr_4}}\right) \tag{5.85}$$

The following result demonstrates the application of the Hadamard gate to the second controller choice and the corresponding update of the global density matrix in Mathematica :

Porte02Choix02Controleur = KroneckerProduct[IdentityMatrix[4] , H];
Porte02Choix02Controleur = KroneckerProduct[Porte02Choix02Controleur , IdentityMatrix[8]];
RhoEtatGlobal = Porte02Choix02Controleur.RhoEtatGlobal;
RhoEtatGlobal = RhoEtatGlobal.Porte02Choix02Controleur† *;*

Measurement : A second measurement is carried out in the computational basis on qubits $\left(Q_{C_{tr_4}}, Q_{C_{tr_5}}\right)$.

The following result demonstrates the construction of the global projector for the second controller and the subsequent update of the density matrix in Mathematica :

Projecteur02Controleur = Projecteur00† *;*
ProjecteurGlobal = KroneckerProduct[IdentityMatrix[4] , Projecteur02Controleur];
ProjecteurGlobal = KroneckerProduct[ProjecteurGlobal , IdentityMatrix[4]];
RhoEtatGlobal = ProjecteurGlobal.RhoEtatGlobal;
RhoEtatGlobal = RhoEtatGlobal.ProjecteurGlobal† *;*

Finally, the controller applies a Hadamard gate :

$$\bar{\rho}_5 = H\left(Q_{C_{tr_3}}\right)\bar{\rho}_4 H^\dagger\left(Q_{C_{tr_3}}\right) \tag{5.86}$$

The following result demonstrates the application of the Hadamard gate to the controller and the subsequent update of the global density matrix in Mathematica :

PorteHControleur = KroneckerProduct[IdentityMatrix[2], H];
PorteHControleur = KroneckerProduct[PorteHControleur , IdentityMatrix[4]];
RhoEtatGlobal = PorteHControleur.RhoEtatGlobal;
RhoEtatGlobal = RhoEtatGlobal.PorteHControleur† *;*

A final measurement on qubit $Q_{C_{tr_3}}$ is performed in the computational basis.

The following result demonstrates the construction of the global projector for the third controller and the specific arrangement of the system's identity matrices in Mathematica :

Projecteur03Controleur = QubitZero† *;*
ProjecteurGlobal = KroneckerProduct[IdentityMatrix[2] , Projecteur03Controleur];
ProjecteurGlobal = KroneckerProduct[ProjecteurGlobal , IdentityMatrix[4]];
RhoCharlieAliceBob = ProjecteurGlobal.RhoEtatGlobal;
RhoCharlieAliceBob = RhoCharlieAliceBob.ProjecteurGlobal† *;*

After receiving the classical measurement outcomes from both the transmitters and the controller, the receivers (Alice, Bob, and Charlie) apply the necessary unitary correction operations to their respective qubits. In the presence of Werner noise, the final reconstructed state of the system $\rho_{R_1R_2R_3}$is no longer a pure product state, but a mixed density matrix :

$$\rho_{R_1R_2R_3} = \lambda\left(\rho_{Q_B} \otimes \rho_{Q_C} \otimes \rho_{Q_A}\right) + \frac{(1-\lambda)}{8} I_8 \tag{5.87}$$

This final expression confirms the successful execution of the clockwise cyclic controlled teleportation

The following result demonstrates the calculation of the final density matrix for the Charlie-Alice-Bob system using the adjoint of the global projector in Mathematica :

RhoCharlieAliceBob = ProjecteurGlobal.RhoEtatGlobal;
RhoCharlieAliceBob = RhoCharlieAliceBob.ProjecteurGlobal† *;*

Controller's Operations in the Counter-Clockwise Scenario

To initiate the counter-clockwise flow, the controller applies a C_NOT gate followed by a Hadamard gate to the first set of controller qubits :

$$\bar{\rho}_2 = C\_NOT_{\left(Q_{C_{tr_1}},Q_{C_{tr_2}}\right)} \bar{\rho}_1 C\_NOT_{\left(Q_{C_{tr_1}},Q_{C_{tr_2}}\right)} \tag{5.88}$$

The following result demonstrates the state evolution after applying the controlled unitary operations :

Porte01Choix01Controleur = KroneckerProduct[CNot , IdentityMatrix[64]];
RhoEtatGlobal = Porte01Choix01Controleur.RhoEtatGlobal;
RhoEtatGlobal = RhoEtatGlobal.Porte01Choix01Controleur† *;*

$$\bar{\rho}_3 = H\left(Q_{C_{tr_1}}\right) \bar{\rho}_2 H^\dagger \left(Q_{C_{tr_1}}\right) \tag{5.89}$$

The following steps describe the application of the second Hadamard gate operation on the global state :

Porte02Choix01Controleur = KroneckerProduct[H, IdentityMatrix[128]];
RhoEtatGlobal = Porte02Choix01Controleur.RhoEtatGlobal;
RhoEtatGlobal = RhoEtatGlobal.Porte02Choix01Controleur† *;*

Measurement : The qubits $\left(Q_{C_{tr_1}}, Q_{C_{tr_2}}\right)$ are measured in the computational basis.

The following steps illustrate the construction of the global projector and the subsequent update of the density matrix :

Projecteur01Controleur = Projecteur00† *;*
ProjecteurGlobal = KroneckerProduct[Projecteur01Controleur , IdentityMatrix[64]];
RhoEtatGlobal = ProjecteurGlobal.RhoEtatGlobal;
RhoEtatGlobal = RhoEtatGlobal.ProjecteurGlobal† *;*

The controller applies a Hadamard gate :

$$\bar{\rho}_4 = H\left(Q_{C_{tr_4}}\right) \bar{\rho}_3 H^\dagger \left(Q_{C_{tr4}}\right) \tag{5.90}$$

The next step involves applying the second choice of the controlled gate using the Hadamard matrix, as shown in the following evolution :

PorteChoix02Controleur = KroneckerProduct[IdentityMatrix[4] , H];
PorteChoix02Controleur = KroneckerProduct[PorteChoix02Controleur ,
IdentityMatrix[8]];
RhoEtatGlobal = PorteChoix02Controleur.RhoEtatGlobal;
RhoEtatGlobal = RhoEtatGlobal.PorteChoix02Controleur† *;*

Qubits $Q_{C_{tr_4}}$ and $Q_{C_{tr_5}}$ are measured in the computational basis.

The following steps detail the construction of the second global projector and the subsequent evolution of the system's density matrix :

Projecteur02Controleur = Projecteur00† *;*
ProjecteurGlobal = KroneckerProduct[IdentityMatrix[4] , Projecteur02Controleur];
ProjecteurGlobal = KroneckerProduct[ProjecteurGlobal , IdentityMatrix[4]];
RhoEtatGlobal = ProjecteurGlobal.RhoEtatGlobal;
RhoEtatGlobal = RhoEtatGlobal.ProjecteurGlobal† *;*

As with the first scenario, the final phase adjustment is performed :

$$\bar{\rho}_5 = H\left(Q_{C_{tr_3}}\right)\bar{\rho}_4 H^\dagger\left(Q_{C_{tr3}}\right) \tag{5.91}$$

The following lines describe the application of the Hadamard gate as a controlled operator on the global density matrix :

PorteHControleur = KroneckerProduct[IdentityMatrix[2], H];
PorteHControleur = KroneckerProduct[PorteHControleur , IdentityMatrix[4]];
RhoEtatGlobal = PorteHControleur.RhoEtatGlobal;
RhoEtatGlobal = RhoEtatGlobal.PorteHControleur† *;*

A final measurement on qubit $Q_{C_{tr_3}}$ is performed in the computational basis. The following lines describe the initialization of the third controller projector and its expansion into the global system space :

Projecteur03Controleur = QubitZero† *;*
ProjecteurGlobal = KroneckerProduct[IdentityMatrix[2] , Projecteur03Controleur];
ProjecteurGlobal = KroneckerProduct[ProjecteurGlobal , IdentityMatrix[4]];

After the receivers (Alice, Bob, and Charlie) apply their specific unitary corrections based on the controller's results, the system collapses into the final noisy state :

$$\rho_{R_1R_2R_3} = \lambda\left(\rho_{Q_C} \otimes \rho_{Q_A} \otimes \rho_{Q_B}\right) + \frac{(1-\lambda)}{8} I_8 \tag{5.92}$$

The equation describes the output density matrix of the protocol under the influence of Werner noise. It consists of two main parts :

The Signal Part (λ) : This term represents the desired quantum state (the tensor product of the states at Q_A, Q_B, and Q_C. It is scaled by the purity parameter λ, which indicates the probability that the system remained in its ideal state.

The Noise Part $\left(\frac{(1-\lambda)}{8} I_8\right)$: This term represents the Werner noise contribution. Since we are dealing with a 3-qubit system, the noise is distributed over a Hilbert space of dimension $2^3 = 8$. The term I_8 (the identity matrix) represents the maximally mixed state, which signifies total randomness or white noise introduced by the environment. The following result demonstrates the calculation of the final density matrix for the Charlie-Alice-Bob system using the adjoint of the global projector in Mathematica :

RhoCharlieAliceBob = ProjecteurGlobal.RhoEtatGlobal ;
RhoCharlieAliceBob = RhoCharlieAliceBob.ProjecteurGlobal† *;*

5.6 Program MATHEMATICA

Chapitre 6

General conclusion

In conclusion, this research has explored the complex dynamics of the Double Cyclic quantum teleportation protocol in realistic conditions. By transitioning from ideal mathematical models to noisy environments, we have gained a deeper understanding of how quantum information behaves when subjected to environmental decoherence.

The rigorous analysis conducted in this thesis, using the Werner Noise model, has demonstrated that while noise remains a significant challenge, the Double Cyclic architecture offers a promising framework for multi-party quantum networking. Through our numerical simulations in Mathematica, we successfully identified the critical thresholds of fidelity, providing clear insights into the protocol's robustness and efficiency.

The findings of this work emphasize that the success of a Quantum Internet depends not only on the design of the protocols but also on our ability to optimize them against noise. As a perspective for future research, we suggest extending this analysis to more complex noise models and larger network scales. Ultimately, this thesis serves as a contribution toward bridging the gap between theoretical quantum mechanics and the future of secure, global quantum communication.

Annexe A

Appendix A

Simulation and Study of a Mixed Cyclic Quantum Teleportation Protocol via a Noisy Channel

In this section, we present the Mathematica implementation of the 3-party cyclic protocol in the presence of noise. The simulation confirms the success of the protocol for both orientations.

1. First Orientation (Direction 01)

This part describes the implementation when the quantum information flows in the first direction (Alice →Charlie → Bob → Alice) :

```
(* ============================================== *)
(* ============================================== *)
X = 0, 1, 1, 0; Y = 0, -I, I, 0;
Z = 1, 0, 0, -1;
H = 1/√2, 1/√2, 1/√2, -1/√2;
CNot = 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0;
Swap = 1, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 1;

(* ============================================== *)
(* 2. Initialisation du Canal (10 Qubits) *)
(* ============================================== *)
Canal = KroneckerProduct[QubitZero, QubitZero];
(*        *)
Do[Canal = KroneckerProduct[Canal, QubitZero], 8];

Porte01 = KroneckerProduct[H, IdentityMatrix[1024]];
```

```
Canal = Porte01 . Canal; Remove[Porte01];
Porte02Swap = KroneckerProduct[IdentityMatrix[2], Swap,
  IdentityMatrix[256]];
Porte02CNot = KroneckerProduct[CNot, IdentityMatrix[512]];
Canal = Porte02Swap . Porte02CNot . Porte02Swap . Canal;

(* ============================================== *)
(* 3. Matrices de Densité et Bruit (Werner) *)
(* ============================================== *)
RhoCanal = Canal . Canal† ;
RhoCanal = λ RhoCanal +
  ((1 - λ)/2048) * IdentityMatrix[2048];

RhoAlice = QubitAlice . QubitAlice† ;
RhoBob = QubitBob . QubitBob† ;
RhoCharlie = QubitCharlie . QubitCharlie† ;

(* ============================================== *)
(* 4. Projections et Mesures de Bell *)
(* ============================================== *)
RhoEtatGlobal = KroneckerProduct[RhoAlice, RhoCanal];
ProjecteurGlobal = KroneckerProduct[Bell00†,
  IdentityMatrix[1024]];
RhoEtatGlobal = ProjecteurGlobal . RhoEtatGlobal .
  ProjecteurGlobal† ;

(* ============================================== *)
(* 5. Simplification et Résultat Final *)
(* ============================================== *)
TraceFinal = Tr[RhoCharlieAliceBob];
RhoFinal = FullSimplify[RhoCharlieAliceBob / TraceFinal,
  TransformationFunctions → Automatic,
  # /. α1 Conjugate[α1] + β1 Conjugate[β1] → 1 &];

Print["Matrice de Densité Finale (Simulation) :"];
RhoFinal // MatrixForm
```

2. Second Orientation (Direction 02)

This part describes the implementation for the reverse cyclic direction (Alice →Bob → Charlie → Alice) :

```
(* ============================================== *)
```

```
(* 1. Definitions des Operateurs de Base *)
(* ================================================ *)
X = 0, 1, 1, 0; Y = 0, -I, I, 0;
Z = 1, 0, 0, -1; R = 1, 0, 0, Exp[I Pi / 4];
H = 1/√2, 1/√2, 1/√2, -1/√2;
CNot = 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0;
Swap = 1, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 1;

(* ================================================ *)
(* 2. Initialisation du Canal (10 Qubits) *)
(* ================================================ *)
Canal = KroneckerProduct[QubitZero, QubitZero];
Do[Canal = KroneckerProduct[Canal, QubitZero], 8];
(* Sequence des Portes (01 a 09) *)
Porte01 = KroneckerProduct[H, IdentityMatrix[1024]];
Canal = Porte01 . Canal;
Porte02Swap = KroneckerProduct[IdentityMatrix[2], Swap,
  IdentityMatrix[256]];
Canal = Porte02Swap . (KroneckerProduct[CNot, IdentityMatrix[512]])
  . Porte02Swap . Canal;

(* ================================================ *)
(* 3. Matrice de Densite et Bruit de Werner *)
(* ================================================ *)
RhoCanal = Canal . Canal† ;
RhoCanal = λ RhoCanal + ((1 - λ)/2048) * IdentityMatrix[2048];
RhoAlice = QubitAlice . QubitAlice† ;
RhoBob = QubitBob . QubitBob† ;
RhoCharlie = QubitCharlie . QubitCharlie† ;

(* ================================================ *)
(* 4. Projections et Mesures (Orientation 02) *)
(* ================================================ *)
RhoEtatGlobal = KroneckerProduct[RhoAlice, RhoCanal];
ProjecteurGlobalA = KroneckerProduct[Bell00†,
  IdentityMatrix[1024]];
RhoEtatGlobal = ProjecteurGlobalA . RhoEtatGlobal .
  ProjecteurGlobalA† ;
(* Measurement of Bob and Charlie *)
RhoEtatGlobal = KroneckerProduct[RhoBob, RhoEtatGlobal];
ProjecteurGlobalB = KroneckerProduct[Bell00†,
```

```
  IdentityMatrix[512]] ;
RhoEtatGlobal = ProjecteurGlobalB . RhoEtatGlobal .
  ProjecteurGlobalB† ;

(* ================================================= *)
(* 5. Actions du Controleur et Etat Final *)
(* ================================================= *)
RhoEtatGlobal = PorteHControleur . RhoEtatGlobal .
  PorteHControleur† ;
RhoCharlieAliceBob = ProjecteurGlobalCtrl . RhoEtatGlobal .
  ProjecteurGlobalCtrl† ;

(* ================================================= *)
(* 6. Simplification et Normalisation *)
(* ================================================= *)
RhoCharlieAliceBob = FullSimplify[RhoCharlieAliceBob,
  TransformationFunctions → Automatic,
  # /. α1 Conjugate[α1] + β1 Conjugate[β1] → 1 &,
  # /. α2 Conjugate[α2] + β2 Conjugate[β2] → 1 &,
  # /. α3 Conjugate[α3] + β3 Conjugate[β3] → 1 &] ;

TraceFinal = Tr[RhoCharlieAliceBob] ;
Print["Trace Finale : ", FullSimplify[TraceFinal]] ;
```

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