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**On the existence and uniqueness of solutions to BSDEs under  
weak monotonicity and general growth conditions**

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## Dedication

After years of diligence and continuous effort, years that carried within them fatigue, patience, and challenges; I stand today on the threshold of graduation to present the fruit of my effort and what I have achieved thanks to God's grace and success.

This accomplishment was not an individual step, but rather the result of the support, love, and assistance of people whose presence had the greatest impact on my journey.

To those who gave me the strength to reach. I dedicate this achievement:

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# Abbreviations and Notations

The different abbreviations and notations used throughout this dissertation are explained below:

$BSDEs$	Backward stochastic differential equations.
$SDE$	Stochastic differential equation.
$(\mathcal{F}_t)_{t \geq 0}$	The filtration.
$(B_t)_{t \geq 0}$	Brownian motion.
$\Omega$	Fundamental space of a random experiment.
$(\Omega, \mathcal{A}, \mathbb{P})$	Probability space.
$\mathcal{B}(\mathbb{R}^d)$	Borel tribe on $\mathbb{R}^d$ .
$\mathbf{1}_B$	Indicator function of the set $B$ .
$\mathbb{E}[X]$	Expectation of $X$ .
$\mathbb{E}[X \mathcal{B}]$	Conditional expectation of $X$ given $\mathcal{B}$ .
$\mathbb{L}^2(\Omega, \mathcal{A}, \mathbb{P})$	Set of random variables, $\mathcal{A}$ -measurable and square integrable.
$\mathbb{R}$	Real numbers set.
$\mathbb{P}$	Probability.
$\mathbb{P}-a.s$	Almost surely in probability.
$a.e.$	Almost everywhere.
$a.s.$	Almost surely.
$r.v$	Random variable.

$(\Omega, \mathcal{A}, \{\mathcal{F}_t\}_{t \in [0, T]}, \mathbb{P})$	Filtered probability space.
<i>i.e</i>	That is.
$\inf$	Infimum.
$\lim$	Limit.
$\sup$	Supremum.
$\max$	Maximum.
$C^2$	Space of twice continuously differentiable functions.
$C^{1,2}([0, T] \times \mathbb{R})$	$\left\{ \begin{array}{l} \text{Space of functions } f(t, x) \text{ continuously differentiable once} \\ \text{with respect to the time variable } t, \text{ and twice} \\ \text{with respect to the space variable } x. \end{array} \right.$
$\langle \cdot, \cdot \rangle$	Scalar product.
$\mathbb{R}^k$	Euclidean real space of $k$ -dimensional.
$\mathbb{R}^{k \times d}$	Set of real matrices $k \times d$ .
$:$	Equal by definition.
$\mathbb{R}^+$	The set of positive real numbers.
$\ f\ _p$	$L^p$ -norm of the function $f$ .

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# Introduction

Backward stochastic differential equations (BSDEs) currently occupy a prominent place in the field of stochastic analysis. The origin of these equations dates back to the pioneering work of Bismut (1973) [2], who introduced them in a linear form within the framework of the stochastic maximum principle. However, it was not until 1990 that the theory truly began to flourish with the seminal work of Pardoux and Peng [11], who proved the existence and uniqueness of solutions for nonlinear BSDEs under a Lipschitz condition, thereby opening the way to a vast field of research.

Interest in these equations has grown considerably due to their concrete applications, particularly in mathematical finance. As demonstrated by El Karoui, Peng, and Quenez [5], BSDEs provide a natural framework for modeling asset pricing and hedging problems, as well as for studying recursive utilities.

A nonlinear BSDE is defined as follows: let  $\xi$  be a square-integrable  $\mathcal{F}_T$ -measurable random variable, and let  $g(t, Y, Z)$  be a progressively measurable function. Then, the associated BSDE is written as:

$$\begin{cases} dY_t = -g(t, Y_t, Z_t)dt + Z_t dB_t, & t \in [0, T] \\ Y_T = \xi, \end{cases}$$

where the pair  $(Y_t, Z_t)_{t \in [0, T]}$  represents the adapted solution,  $g$  is called the generator (or driver), and  $\xi$  is the terminal condition.

In the classical framework established by Pardoux and Peng, the well-posedness of these equations relies on the assumption that the generator  $g$  is uniformly Lipschitz continuous in  $(y, z)$ . Nevertheless, this condition often proves too restrictive for numerous applied models.

Consequently, research has focused on relaxing these hypotheses, moving from the Lipschitz condition to conditions such as linear growth, and later to monotonicity conditions. In this direction, Mao [10] proposed weaker conditions based on a non-Lipschitz approach. More recently, Fan and Jiang [6] introduced the concepts of weak monotonicity and general growth conditions. This generalization encompasses both classical monotonicity and Mao's condition, thereby allowing for the treatment of a broader class of equations.

The main objective of this thesis is to study the solvability of BSDEs when the generator satisfies this weak monotonicity condition. We structure our work around three main axes:

**Chapter 1:** This introductory part lays the necessary theoretical foundations. We recall key concepts of stochastic analysis, ranging from martingales and Brownian motion to stochastic differential equations, without omitting the essential tools of Itô calculus.

**Chapter 2:** We dedicate this chapter to the study of the standard framework. We present the classical existence and uniqueness results obtained by Pardoux and Peng [11] when the coefficients of the equation are Lipschitz, thus providing a benchmark for subsequent extensions.

**Chapter 3:** This final chapter constitutes the core of our contribution. We analyze BSDEs under weak monotonicity and general growth conditions. Building on the recent results of Fan and Jiang [6], we demonstrate that the existence and uniqueness of solutions remain valid under this less restrictive condition.

# Chapter 1

## Preliminaries in stochastic analysis

In this chapter, we present the fundamental probabilistic foundations and analytical tools required for the study of backward stochastic differential equations (BSDEs). These preliminary results constitute the essential framework that will be utilized throughout the subsequent chapters of this dissertation. For a comprehensive treatment and detailed proofs of the theorems presented here, the reader is referred to the standard literature such as [9, 13, 11]

### 1.1 Conditional expectation

We begin by recalling the notion of conditional expectation, which plays a fundamental role in stochastic analysis.

**Definition 1.1 (Random variable)** [9] *Let  $(\Omega, \mathcal{A}, \mathbb{P})$  be a probability space. A random variable  $X$  is a measurable function  $X : \Omega \rightarrow \mathbb{R}^d$  such that for any borel set  $B \in \mathcal{B}(\mathbb{R}^d)$ , the set  $\{w \in \Omega : X(w) \in B\}$  belongs to the  $\sigma$ -algebra  $\mathcal{A}$ . In this work, we primarily consider square-integrable random variables.*

**Definition 1.2** [9] *Let  $(\Omega, \mathcal{A}, \mathbb{P})$  be a probability space. For any measurable set  $B \in \mathcal{A}$ , the indicator function of  $B$ , denoted by  $1_B$ , is a measurable mapping from  $\Omega$  into  $\{0, 1\}$  defined*

as follows:

$$1_B(w) = \begin{cases} 1 & \text{if } w \in B, \\ 0 & \text{if } w \notin B. \end{cases}$$

This function is a vital tool in stochastic analysis, allowing the probability of an event to be expressed as an expectation, such that  $\mathbb{P}(B) = \mathbb{E}[1_B] = \int_{\Omega} 1_B d\mathbb{P}$ .

**Definition 1.3** [9] Let  $(\Omega, \mathcal{A}, \mathbb{P})$  be a probability space and  $B$  a sub- $\sigma$ -field of  $\mathcal{A}$ . For an integrable random variable  $X$ , the conditional expectation of  $X$  given  $B$ , denoted by  $\mathbb{E}[X|\mathcal{B}]$ , is the unique  $\mathcal{B}$ -measurable random variable that satisfies:

$$\mathbb{E}[X1_B] = \mathbb{E}[\mathbb{E}[X|\mathcal{B}]1_B], \quad \forall B \in \mathcal{B}. \quad (1.1)$$

**Theorem 1.1 (Orthogonal projection)** [13] If  $X \in \mathbb{L}^2(\Omega, \mathcal{A}, \mathbb{P})$ , the conditional expectation  $\mathbb{E}[X|\mathcal{B}]$  is the orthogonal projection of  $X$  on to the subspace  $\mathbb{L}^2(\Omega, \mathcal{B}, \mathbb{P})$ . This means that  $\mathbb{E}[X|\mathcal{B}]$  is the best estimate of  $X$  given the information contained in the sub- $\sigma$ -field  $\mathcal{B}$ .

**Properties** [9, 13]

1. **Linearity:**  $\mathbb{E}[aX + bY|\mathcal{B}] = a\mathbb{E}[X|\mathcal{B}] + b\mathbb{E}[Y|\mathcal{B}]$  for any  $a, b \in \mathbb{R}$ .
2. **Monotonicity:** If  $X \leq Y$  a.s., then  $\mathbb{E}[X|\mathcal{B}] \leq \mathbb{E}[Y|\mathcal{B}]$  a.s.
3. If  $\mathcal{H} \subset \mathcal{B}$ , we have  $\mathbb{E}[\mathbb{E}[X|\mathcal{B}]|\mathcal{H}] = \mathbb{E}[X|\mathcal{H}]$ .
4. If  $X$  is  $\mathcal{B}$ -measurable, then  $\mathbb{E}[XY|\mathcal{B}] = X\mathbb{E}[Y|\mathcal{B}]$  a.s.
5. **Independence:** If  $X$  is independent of  $\mathcal{B}$ , then  $\mathbb{E}[X|\mathcal{B}] = \mathbb{E}[X]$  a.s.
6. **Jensen's inequality:** For any concave function  $\kappa$ ,  $\mathbb{E}[\kappa(X)|\mathcal{B}] \leq \kappa(\mathbb{E}[X|\mathcal{B}])$  a.s.
7. **Conditional Cauchy-Schwarz inequality:**  $(\mathbb{E}[XY|\mathcal{B}])^2 \leq \mathbb{E}[X^2|\mathcal{B}]\mathbb{E}[Y^2|\mathcal{B}]$  a.s.

This notion constitutes a basic tool for the study of martingales and BSDEs.

## 1.2 Brownian motion and martingales

We now introduce martingales and Brownian motion, which are central objects in stochastic calculus. This section provides a rigorous treatment of the processes that govern the stochastic component of BSDEs.

**Definition 1.4 (Stochastic process and sample paths)** [9] *A stochastic process is a collection of random variables  $\{X_t\}_{t \in [0, T]}$  defined on  $(\Omega, \mathcal{A}, \mathbb{P})$ . It can be viewed as a mapping  $X : [0, T] \times \Omega \rightarrow \mathbb{R}^d$ . For each fixed  $\omega \in \Omega$ , the function  $t \mapsto X_t(\omega)$  is called the **sample path** or **trajectory** of the process. In the context of BSDEs, we focus on processes with continuous trajectories to ensure the applicability of Itô's calculus.*

**Remark 1.1** [13] *A stochastic process can be viewed as a random evolution indexed by time.*

**Definition 1.5 (Progressively measurable process)** [13] *A stochastic process  $X = \{X_t\}_{t \in [0, T]}$  defined on the filtered probability space  $(\Omega, \mathcal{A}, \{\mathcal{F}_t\}_{t \in [0, T]}, \mathbb{P})$  is said to be progressively measurable if, for every  $t \in [0, T]$ , the mapping  $(s, \omega) \mapsto X_s(\omega)$  from  $[0, t] \times \Omega$  into  $\mathbb{R}^d$  is  $B([0, t]) \otimes \mathcal{F}_t$ -measurable. This property is crucial in the study of BSDEs as it ensures that the generator  $g(t, \omega, Y_t, Z_t)$  is well-defined and integrable.*

**Definition 1.6 (Filtration)** [9] *A filtration on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$  is an increasing collection of sub- $\sigma$ -algebras  $\{\mathcal{F}_t\}_{t \geq 0}$  of  $\mathcal{A}$  (i.e.,  $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{A}$  for all  $0 \leq s \leq t$ ). It represents the information flow available up to time  $t$ . A probability space equipped with a filtration is called a filtered probability space.*

**Definition 1.7 (Filtered space and usual conditions)** [9] *Let  $(\Omega, \mathcal{A}, \mathbb{P})$  be a complete probability space, we equip it with a filtration  $\{\mathcal{F}_t\}_{t \in [0, T]}$ . We assume that the filtration satisfies the **usual conditions**, meaning it is complete (contains all  $\mathbb{P}$ -null sets) and right-continuous ( $\mathcal{F}_t = \bigcap_{s > t} \mathcal{F}_s$ ). This structure is essential to ensure that our processes are non-anticipative and possess regular paths.*

**Definition 1.8 (Adapted process)** [9] *A stochastic process  $X = \{X_t\}_{t \in [0, T]}$  is called adapted to the filtration  $\{\mathcal{F}_t\}_{t \in [0, T]}$  if, for every  $t \in [0, T]$ , the random variable  $X_t$  is  $\mathcal{F}_t$ -measurable.*

*Intuitively, this means the value of the process at time  $t$  is known based on the information available up to that time.*

**Remark 1.2** [9] *The adaptedness condition ensures that the process does not depend on future information.*

**Definition 1.9 (Markov process)** [9] *A stochastic process  $\{X_t\}_{t \geq 0}$  is called a markov process if for any  $s > t$ , the conditional distribution of  $X_s$  given the information up to time  $t$  depends only on the value of  $X$  at time  $t$ . Formally:*

$$\mathbb{P}(X_s \in B | \mathcal{F}_t) = \mathbb{P}(X_s \in B | X_t), \text{ a.s.}$$

*for any borel set  $B$ .*

**Remark 1.3** *This property is fundamental when connecting BSDEs to the theory of partial differential equations (PDEs).*

Among stochastic processes, Brownian motion represents the fundamental driving noise of backward stochastic differential equations.

### 1.2.1 Brownian motion and path properties

**Definition 1.10 (Standard Brownian motion)** [13] *A stochastic process  $B = \{B_t, t \geq 0\}$  defined on  $(\Omega, \mathcal{A}, \mathbb{P})$  is called a standard Brownian motion if:*

1.  $B_0 = 0$  a.s.
2. **Independent increments:** *For any  $0 \leq s < t$ , the increment  $B_t - B_s$  is independent of the  $\sigma$ -field  $\mathcal{F}_s$ .*
3. **Stationary gaussian increments:**  $B_t - B_s \sim \mathcal{N}(0, t - s)$  for  $t > s$ .
4. **Continuity:** *The mapping  $t \mapsto B_t(w)$  is continuous for almost all  $w$ .*

**Remark 1.4** [13] *Brownian motion plays the role of the fundamental source of randomness in stochastic differential equations.*

**Proposition 1.1 (Invariance properties of Brownian motion)** [13] *Let  $B = \{B_t\}_{t \geq 0}$  be a standard Brownian motion. The following transformed processes are also standard Brownian motions:*

1. **Symmetry:**  $\{-B_t\}_{t \geq 0}$ .
2. **Scaling:** For any  $c > 0$ ,  $\left\{\frac{1}{\sqrt{c}}B_{ct}\right\}_{t \geq 0}$ .
3. **Time inversion:** The process  $Y_t$ , defined as  $Y_0 = 0$  and  $Y_t = tB_{\frac{1}{t}}$  for  $t > 0$ .

**Proof.** It is sufficient to check the Gaussian character of these processes and to calculate their expectation and covariance. ■

**The key path properties** [13].

**Non-differentiability:** Almost surely, the Brownian paths are nowhere differentiable. This justifies the need for Itô calculus rather than classical calculus.

**Theorem 1.2 (Quadratic variation)** *Let  $(B)_t$  be a standard Brownian motion. For any fixed  $t > 0$ , define the quadratic variation along a partition  $\Pi_n : 0 = t_0^{(n)} < t_1^{(n)} \dots < t_n^{(n)} = t$  with mesh  $\|\Pi_n\| = \max_i |t_{i+1}^{(n)} - t_i^{(n)}| \rightarrow 0$ , as  $n \rightarrow \infty$ . Then*

$$\lim_{n \rightarrow \infty} \sum_{i=0}^{k_n-1} \left( B_{t_{i+1}^{(n)}} - B_{t_i^{(n)}} \right)^2 = t, \text{ almost surely.}$$

*Equivalently, in probability,*

$$\langle B, B \rangle_t = t,$$

*where  $\langle B, B \rangle_t$  denotes the quadratic variation process.*

**Proof.** We'll prove the theorem for dyadic partitions:  $t_i^{(n)} = \frac{it}{2^n}$  for  $i = 0, 1, \dots, 2^n$ . Then we note that if a sequence of quadratic sums converges a.s. for dyadic partitions, and the Brownian paths are continuous, the limit is the same for any sequence of partitions with

mesh  $\rightarrow 0$ .

Let

$$Q_n(t) = \sum_{i=0}^{2^n-1} \left( B_{\frac{(i+1)t}{2^n}} - B_{\frac{it}{2^n}} \right)^2,$$

Let  $\Delta_i = B_{t_{i+1}} - B_{t_i}$ . Then  $\Delta_i \rightsquigarrow N\left(0, \frac{t}{2^n}\right)$  independently for different  $t$  and

$$\mathbb{E}(Q_n(t)) = \sum_{i=0}^{2^n-1} \mathbb{E}(\Delta_i^2) = 2^n \cdot \frac{t}{2^n} = t,$$

and for the variance (using that for a normal r.v.  $X \rightsquigarrow N(0, \sigma^2)$ ,  $\text{VAR}(X^2) = 2\sigma^4$ ), we get

$$\text{VAR}(Q_n(t)) = \sum_{i=0}^{2^n-1} \text{VAR}(\Delta_i^2) = \sum_{i=0}^{2^n-1} 2 \left( \frac{t}{2^n} \right)^2 = 2^n \cdot 2 \left( \frac{t}{2^n} \right)^2 = \frac{2t^2}{2^n} \rightarrow 0$$

as  $n \rightarrow \infty$ . Thus for any  $\epsilon > 0$ , by Chebyshev's inequality, we have

$$\mathbb{P}(|Q_n(t) - t| > \epsilon) \leq \frac{\text{VAR}(Q_n(t))}{\epsilon^2} = \frac{2t^2}{\epsilon^2 2^n}.$$

Hence,

$$\sum_{n=1}^{\infty} \mathbb{P}(|Q_n(t) - t| > \epsilon) \leq \frac{2t^2}{\epsilon^2} \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty.$$

By the Borel–Cantelli lemma, for each  $\epsilon > 0$

$$\mathbb{P}(|Q_n(t) - t| > \epsilon) = 0,$$

thus,  $Q_n(t) \rightarrow t$  almost surely. ■

**Hölder continuity:** Almost surely, the sample paths of Brownian motion are locally hölder continuous for any exponent  $\alpha \in \left(0, \frac{1}{2}\right)$ , i.e.

$$|B_t - B_s| \leq C |t - s|^\alpha.$$

**Remark 1.5** *This means that while the paths are non-differentiable, they possess a specific degree of regularity that is essential for the construction of stochastic integrals [13].*

## 1.2.2 Martingales and maximal inequalities

In the study of BSDEs, we rely heavily on the stability of martingales to prove the convergence of solutions.

**Definition 1.11** (*Martingales, Submartingales, and Supermartingales*) [13] Let  $\{\mathcal{F}_t\}_{t \geq 0}$  be a filtration. A adapted process  $M$  is a martingale if  $\mathbb{E}[|M_t|] < \infty$  and for each  $s < t$   $\mathbb{E}[M_t | \mathcal{F}_s] = M_s$ .

- If for each  $s < t$ ,  $\mathbb{E}[M_t | \mathcal{F}_s] \geq M_s$ , it is a submartingale.
- If for each  $s < t$ ,  $\mathbb{E}[M_t | \mathcal{F}_s] \leq M_s$ , it is a supermartingale.

**Remark 1.6** A martingale models a fair differential game, a submartingale models a game that is favorable to the player, and a supermartingale models a game that is unfavorable.

**Definition 1.12** (*Stopping time*) [13] A random variable  $\tau : \Omega \rightarrow [0, T] \cup \{+\infty\}$  is called a stopping time with respect to the filtration  $\{\mathcal{F}_t\}$  if for every  $t \geq 0$ , the event  $\{\tau \leq t\}$  belongs to  $\mathcal{F}_t$ . This implies that the decision to 'stop' at time  $t$  depends only on the information accumulated up to that time. Stopping times are vital tools for the localization of martingales and for defining solutions to BSDEs on random intervals.

**Definition 1.13** (*Uniformly integrable martingale*) [13] A martingale  $M = \{M_t\}_{t \in [0, T]}$  is said to be uniformly integrable (UI) if the collection of random variables  $\{M_t\}_{t \in [0, T]}$  is uniformly integrable, i.e.,  $\lim_{k \rightarrow \infty} \sup_{t \in [0, T]} \mathbb{E}[|M_t| 1_{\{|M_t| > k\}}] = 0$ . In the context of BSDEs, every square-integrable martingale on a finite horizon  $[0, T]$  is uniformly integrable, which is a key property for proving the convergence of solution sequences.

**Theorem 1.3** (*Optional stopping theorem*) [13] Let  $M$  be a uniformly integrable martingale and let  $\sigma, \tau$  be two stopping times such that  $0 \leq \sigma \leq \tau \leq T$ . Then, the stopped process maintains the martingale property, and the conditional expectation satisfies:

$$\mathbb{E}[M_\tau | \mathcal{F}_\sigma] = M_\sigma, \quad a.s.$$

*This theorem is a powerful tool for analyzing BSDE solutions over random time horizons.*

**Definition 1.14 (Local martingale)** [13] *An adapted process  $M$  is a local martingale if there exists an increasing sequence of stopping times  $\{\tau_n\}$  such that  $\tau_n \rightarrow \infty$ .a.s as  $n \rightarrow \infty$ , and each stopped process  $M_{t \wedge \tau_n}$  is a uniformly integrable martingale.*

**Theorem 1.4 (Doob-meyer decomposition)** [13] *Let  $\{Y_t\}_{t \in [0, T]}$  be a right-continuous supermartingale. Then there exists a unique decomposition  $Y_t = M_t - A_t$ , where  $M_t$  is a local martingale and  $A_t$  is an adapted, non-decreasing predictable process with  $A_0 = 0$ . In the study of BSDEs, this result provides the theoretical justification for the structure of the equation, separating the martingale part from the generator's integral.*

**Definition 1.15 (Semi-martingale)** [13] *A process  $X = \{X_t\}_{t \geq 0}$  is called a semi-martingale if it can be decomposed as  $X_t = X_0 + M_t + A_t$ , where  $M$  is a local martingale and  $A$  is a process of finite variation, both being adapted and starting at zero. Semi-martingales represent the most general class of processes for which the stochastic integral and Itô's formula can be consistently defined.*

## 1.3 Itô's calculus

To analyze stochastic differential equations, we introduce Itô's stochastic calculus. Itô calculus provides the necessary framework to integrate with respect to processes of unbounded variation, such as Brownian motion. Unlike classical riemann-stieltjes integrals, stochastic integrals require a different approach due to the non-differentiability of Brownian paths.

### 1.3.1 Construction of the Itô integral

The construction of the Itô integral follows a limit procedure from class of simple processes to a more general class of integrands, for more details see for example [9, 13].

1. **Elementary processes:** A process  $H$  is called elementary (or simple) if there exists a partition  $0 = t_0 < t_1 < \dots < t_n = T$  and  $\mathcal{F}_{t_i}$ -measurable random variables  $\xi_i$  such

that:

$$H_t = \sum_{i=0}^{n-1} \xi_i \mathbb{1}_{[t_i, t_{i+1}]}(t).$$

The integral of such a process with respect to  $B$  is defined as:

$$\int_0^T H_s dB_s = \sum_{i=0}^{n-1} \xi_i (B_{t_{i+1}} - B_{t_i}).$$

2. **The Hilbert space  $\mathcal{M}^2(0, T)$ :** To extend this definition, we consider the space of progressively measurable processes  $\phi$  such that:

$$\mathbb{E} \left[ \int_0^T |\phi_s|^2 ds \right] < \infty.$$

The Itô integral for  $\phi \in \mathcal{M}^2(0, T)$  is defined as the  $\mathbb{L}^2$ -limit of integrals of elementary processes.

3. **Key properties (Essential for BSDEs):**

- **Martingale property:** The process  $M_t = \int_0^t \phi_s dB_s$  is a square-integrable martingale. In BSDE theory, this property ensures that  $\mathbb{E} \left[ \int_t^T Z_s dB_s | \mathcal{F}_t \right] = 0$ .
- **Itô isometry:** This is the most crucial property for proving the uniqueness of solutions:

$$\mathbb{E} \left[ \left( \int_0^T \phi_s dB_s \right)^2 \right] = \mathbb{E} \left[ \int_0^T \phi_s^2 ds \right]. \quad (1.2)$$

- **Continuity:** The mapping  $t \mapsto \int_0^t \phi_s dB_s$  is almost surely continuous.

### 1.3.2 Quadratic variation and bracket process

For any continuous local martingale  $M$ , there exists a unique increasing process  $\langle M \rangle$ , called the quadratic variation, such that  $M^2 - \langle M \rangle$  is a local martingale, (see for exemple [\[13\]](#)).

- For Brownian motion:  $\langle B \rangle_t = t$ .
- For an Itô integral:  $\langle \int \phi dB \rangle_t = \int_0^t \phi_s^2 ds$ .

- **Covariation (Crochet):** For two processes  $X$  and  $Y$ ,  $\langle X, Y \rangle$  represents their joint variation. This is used in the integration by parts formula.

**Theorem 1.5 (Brownian martingale representation) [13]** *Let  $\{M_t\}_{t \in [0, T]}$  be a square-integrable martingale with respect to the filtration generated by Brownian motion. Then, there exists a unique progressively measurable process  $Z \in \mathcal{M}^2(0, T)$  such that:*

$$M_t = M_0 + \int_0^t Z_s dB_s, \quad \forall t \in [0, T].$$

*This result is fundamental for BSDE theory as it guarantees the existence of the control process  $Z$  that "compensates" the randomness to reach the terminal value.*

The following result, known as Itô's formula, is one of the main tools of stochastic analysis.

### 1.3.3 Itô's formula (The stochastic chain rule)

Itô's formula is the generalization of the classical chain rule to the stochastic setting. It provides the necessary second-order correction terms to account for the non-zero quadratic variation of Brownian motion. This formula is indispensable for establishing the a priori estimates and deriving the dynamics of BSDE solutions.

**Definition 1.16 (Itô process) [13]** *An Itô process is defined as a stochastic process  $\{X_t\}_{t \in [0, T]}$  that can be decomposed into a sum of an initial value, a lebesgue integral (representing the drift), and a stochastic integral with respect to Brownian motion. Specifically, it takes the form:*

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dB_s \tag{1.3}$$

*where  $b_s$  and  $\sigma_s$  are progressively measurable processes such that  $\int_0^T |b_s| ds < \infty$  and  $\int_0^T \sigma_s^2 ds < \infty$  a.s.*

**Theorem 1.6 (First Itô formula) [13]** *Let  $\{X_t\}_{t \in [0, T]}$  be an Itô process and  $f : \mathbb{R} \rightarrow \mathbb{R}$*

be a  $C^2$  function. The process  $f(X_t)$  is also an Itô process, and its dynamics is given by:

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2} \int_0^t f''(X_s) d\langle X, X \rangle_s.$$

This version is the simplest form, focusing on the second-order correction term  $\frac{1}{2}f''$ .

**Theorem 1.7 (Second Itô formula)** [13] For a function  $f(t, x)$  of class  $C^{1,2}([0, T] \times \mathbb{R})$ , the stochastic differential  $df(t, X_t)$  is expressed as:

$$df(t, X_t) = \frac{\partial f}{\partial t}(t, X_t) dt + \frac{\partial f}{\partial x}(t, X_t) dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, X_t) d\langle X \rangle_t.$$

Setting  $f(t, x) = e^{\beta t} x^2$  in this formula is a standard technique for proving the stability of BSDEs.

**Notation [15]:** The term  $d\langle X_1, X_2 \rangle_s$  denotes the quadratic covariation between two Itô processes. In practice, this term is computed using the following heuristic multiplication rules for stochastic differentials:

$\times$	$dt$	$dB_s$
$dB_s$	0	$dt$
$dt$	0	0

these rules specify that while the square of the Brownian increment  $dB_s^2$  yields  $dt$ , any product involving  $dt$  (such as  $dt^2$  or  $dt \cdot dB_s$ ) is of higher order and thus vanishes in the limit.

**Theorem 1.8 (Multidimensional Itô formula)** [13] Let  $X = (X^1, \dots, X^d)$  be a vector of Itô processes and  $f \in C^{1,2}([0, T] \times \mathbb{R}^d)$ . then:

$$df(t, X_t) = \frac{\partial f}{\partial t} dt + \sum_{i=1}^d \frac{\partial f}{\partial x_i} dX_t^i + \frac{1}{2} \sum_{i,j=1}^d \frac{\partial^2 f}{\partial x_i \partial x_j} d\langle X^i, X^j \rangle_t.$$

**Remark 1.7** This general version is essential for analyzing systems of BSDEs where the solution  $Y_t$  takes values in  $\mathbb{R}^d$ .

**Remark 1.8** *It is important to note that the presence of the term  $d\langle X \rangle_t$  (the quadratic variation) is what distinguishes stochastic calculus from deterministic calculus. In the deterministic case, this term vanishes, and the formula reverts to the classical chain rule.*

### Integration by parts [13]

As a direct consequence of the multidimensional Itô formula, the product of two stochastic processes  $\{X_t\}$  and  $\{Y_t\}$  satisfies the following relation:

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + d\langle X, Y \rangle_t.$$

This rule is frequently applied to compute the squared norm of the difference between two solutions, which is the starting point for proving uniqueness.

## 1.4 Stochastic differential equations

Using the previously introduced stochastic processes, we now define stochastic differential equations (see for more details for example [13]).

### 1.4.1 Formulation and existence via Picard's approximation method

An SDE describes the dynamics of a process  $X_t$  as:

$$dX_t = b(t, X_t) dt + \sigma(t, X_t) dB_t, \quad X_0 = \xi.$$

This differential form is shorthand for the integral equation:

$$X_t = \xi + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s. \quad (1.4)$$

To guarantee the existence of a unique solution for a stochastic differential equation (SDE), certain regularity conditions must be met by its coefficients. Let  $b$  and  $\sigma$  be borel measurable functions (drift and diffusion coefficients, respectively). We postulate the existence of a

constant  $L > 0$  such that the following conditions hold:

1. **Lipschitz continuity:** For all  $t \in [0, T]$  and any two states  $x, \tilde{x} \in \mathbb{R}^n$ , the coefficients satisfy the Lipschitz condition with respect to the spatial variable:

$$|b(t, x) - b(t, \tilde{x})| + \|\sigma(t, x) - \sigma(t, \tilde{x})\| \leq L|x - \tilde{x}|. \quad (1.5)$$

2. **Linear Growth condition:** For all  $t \in [0, T]$  and any state  $x \in \mathbb{R}^n$ , the coefficients exhibit at most linear growth in  $x$ :

$$|b(t, x)| + \|\sigma(t, x)\| \leq L(1 + |x|). \quad (1.6)$$

3. **Initial value constraint:** The initial value  $X(0)$  must possess a finite second moment:

$$\mathbb{E}(|X(0)|^2) < +\infty. \quad (1.7)$$

**Theorem 1.9 (*Existence and Uniqueness*)** [10] *Under the above assumptions, the SDEs has a unique strong solution  $X \in \mathcal{S}^2$ .*

**Proof.** We define  $\mathcal{S}^2$  as the banach space comprising progressively measurable processes  $X_t$ , provided they satisfy the following condition:

$$\mathbb{E} \left( \sup_{0 \leq t \leq T} |X_t|^2 \right) < +\infty,$$

equipped with the norm:

$$\|X\| = \left[ \mathbb{E} \left( \sup_{0 \leq t \leq T} |X_t|^2 \right) < +\infty \right]^{\frac{1}{2}}.$$

We denote by  $\mathcal{S}_c^2$  the subspace of  $\mathcal{S}^2$  consisting of continuous processes.

**Existence part:** We construct the solution using picard's approximation method. We set:

$$\begin{cases} X_t^n = \xi + \int_0^t \sigma(s, X_s^{n-1}) dB_s + \int_0^t b(s, X_s^{n-1}) ds, \\ X_t^1 = \xi + \int_0^t \sigma(s, x) dB_s + \int_0^t b(s, x) ds, \\ X_t^0 = \xi. \end{cases}$$

We verify the well-definedness of the stochastic integrals. By clear induction, we establish that for every  $n$ , the process  $X_t^{(n)}$  is continuous and adapted, which implies the process  $\sigma(s, X_s^{(n-1)})$  is also progressively measurable.

Let us fix a real value  $T > 0$  and proceed with the proof on the interval  $[0, T]$ . First, we verify by induction on  $n$  that there exists a constant  $C_n$  such that for all  $t \in [0, T]$ :

$$\mathbb{E} [(X_t^n)^2] \leq C_n. \quad (1.8)$$

This bound holds true for  $n = 0$ . Subsequently, assuming it holds at order  $n - 1$ , we utilize the bounds from the previous steps to prove it for order  $n$ .

$$|\sigma(s, y)| \leq \kappa' + \kappa |y|, \quad \forall s \in [0, T], y \in \mathbb{R}.$$

To write

$$\begin{aligned} \mathbb{E} [(X_t^n)^2] &\leq 3 \left( |\xi|^2 + \mathbb{E} \left[ \left( \int_0^t \sigma(s, X_s^{n-1}) dB_s \right)^2 \right] + \mathbb{E} \left[ \left( \int_0^t b(s, X_s^{n-1}) ds \right)^2 \right] \right) \\ &\leq 3 \left( |\xi|^2 + \mathbb{E} \left[ \int_0^t (\sigma(s, X_s^{n-1}))^2 ds \right] + t \mathbb{E} \left[ \int_0^t (b(s, X_s^{n-1}))^2 ds \right] \right) \\ &\leq 3 \left( |\xi|^2 + \mathbb{E} \left[ \int_0^t (\sigma(s, X_s^{n-1}))^2 ds \right] + T \mathbb{E} \left[ \int_0^t (b(s, X_s^{n-1}))^2 ds \right] \right) \\ &\leq 3 \left( |\xi|^2 + 4(1+T) \mathbb{E} \left[ \int_0^t (\kappa'^2 + \kappa^2 (X_s^{n-1})^2) ds \right] \right) \\ &\leq C_n \text{ with } C_n = 3 (|\xi|^2 + 4T(1+T) (\kappa'^2 + \kappa^2 C_{n-1})), \end{aligned}$$

to justify the calculation of the second moment of the stochastic integral, we utilized the fact that  $\mathbb{E} \left[ \left( \int_0^t \sigma(s, X_s^{n-1})^2 ds \right) \right] \leq \infty$ .

This result follows directly from the preceding bound for  $\sigma$  and the inductive hypothesis.

The bound (1.5), along with the hypothesis on  $\sigma$ , implies that the local martingale  $\int_0^t \sigma(s, X_s^{n-1}) dB_s$  is, for each  $n$ , a true martingale bounded in  $\mathbb{L}^2$  over the interval  $[0, T]$ . We leverage this remark to majorize by recurrence:

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_t^{n-1} - X_t^n|^2 \right].$$

We have

$$X_t^{n+1} - X_t^n = \int_0^t (\sigma(s, X_s^n) - \sigma(s, X_s^{n-1})) dB_s + \int_0^t (b(s, X_s^n) - b(s, X_s^{n-1})) ds,$$

By applying Doob's inequality, we obtain:

$$\begin{aligned} & \mathbb{E} \left[ \sup_{0 \leq s \leq t} |X_s^{n-1} - X_s^n|^2 \right] \\ & \leq 2 \mathbb{E} \left[ \sup_{0 \leq s \leq t} \left| \int_0^s (\sigma(u, X_u^n) - \sigma(u, X_u^{n-1})) dB_u \right|^2 + \sup_{0 \leq s \leq t} \left| \int_0^s (b(u, X_u^n) - b(u, X_u^{n-1})) du \right|^2 \right] \\ & \leq 2 \left( 4 \mathbb{E} \left[ \left( \int_0^t (\sigma(u, X_u^n) - \sigma(u, X_u^{n-1})) dB_u \right)^2 \right] + \mathbb{E} \left[ \left( \int_0^t (b(u, X_u^n) - b(u, X_u^{n-1})) du \right)^2 \right] \right) \\ & \leq 2 \left( 4 \mathbb{E} \left[ \int_0^t (\sigma(u, X_u^n) - \sigma(u, X_u^{n-1}))^2 du \right] + t \mathbb{E} \left[ \int_0^t (b(u, X_u^n) - b(u, X_u^{n-1}))^2 du \right] \right) \\ & \leq 2 \left( 4 \mathbb{E} \left[ \int_0^t \kappa^2 |X_u^n - X_u^{n-1}|^2 du \right] + t \mathbb{E} \left[ \int_0^t \kappa^2 |X_u^n - X_u^{n-1}|^2 du \right] \right) \\ & \leq 2 \left( 4 \kappa^2 \mathbb{E} \left[ \int_0^t |X_u^n - X_u^{n-1}|^2 du \right] + T \kappa^2 \mathbb{E} \left[ \int_0^t |X_u^n - X_u^{n-1}|^2 du \right] \right) \\ & = 2(4 + T) \kappa^2 \mathbb{E} \left[ \int_0^t |X_u^n - X_u^{n-1}|^2 du \right] \\ & \leq C_T \mathbb{E} \left[ \int_0^t \sup_{0 \leq r \leq u} |X_r^n - X_r^{n-1}|^2 du \right]. \end{aligned}$$

Let's note  $C_T = 2(4+T)\kappa^2$ . If  $g_n(u) = \mathbb{E} \left[ \sup_{0 \leq r \leq u} |X_r^n - X_r^{n-1}|^2 \right]$ , we can therefore see that:

$$g_{n+1} \leq C_T \int_0^t g_n(u) du.$$

Furthermore, inequalities (1.8) together with the preceding results demonstrate that each function  $g_n$  is bounded on  $[0, T]$ . Specifically, there exists a constant  $C_0$  such that  $g_0(t) \leq C_0$  for all  $t \in [0, T]$ . A straightforward induction argument, utilizing inequality (1.8), then proves that for all  $n \geq 0$  and  $t \in [0, T]$ :

$$g_n(t) \leq C'_T (C_T)^n \frac{T^n}{n!}.$$

From this last inequality, we obtain:

$$\begin{aligned} \sum_{n \geq 0} \left\| \sup_{0 \leq t \leq T} |X_t^{n+1} - X_t^n| \right\|_{\mathbb{L}^1} &\leq \sum_{n \geq 0} \left\| \sup_{0 \leq t \leq T} |X_t^{n+1} - X_t^n| \right\|_{\mathbb{L}^2} \\ &\leq \sqrt{C'_T} \sum_{n \geq 0} \frac{(CT)^{\frac{n}{2}}}{\sqrt{n}} < \infty. \end{aligned}$$

Therefore, the series  $\sup_t |X_t^{n+1} - X_t^n|$  converges  $\mathbb{P} - a.s.$ , ensuring that  $X^n$  converges uniformly on  $[0, T]$  to a continuous process  $X$ . Furthermore,  $X \in \mathcal{S}_c^2$  because the convergence takes place within  $\mathcal{S}^2$ . We then verify that  $X$  is a solution to equation (1.4).

By taking the limit in the recurrence equation for  $X^n$ , we find that  $X$  is a (strong) solution of equation (1.4) on  $[0, T]$ .

**Uniqueness part:** Assume that  $X$  and  $\widehat{X}$  are two solutions to equation (1.4). For all  $0 \leq t \leq T$ :

$$X(t) - \widehat{X}(t) = \int_0^t \left( b(s, X_s) - b(s, \widehat{X}_s) \right) ds + \int_0^t \left( \sigma(s, X_s) - \sigma(s, \widehat{X}_s) \right) dB_s.$$

Since  $(a + b)^2 \leq 2a^2 + 2b^2$ , we can estimate:

$$\begin{aligned} & \mathbb{E} \left( \left| X(t) - \widehat{X}(t) \right| \right)^2 \\ & \leq 2\mathbb{E} \left( \left| \int_0^t (b(s, X_s) - b(s, \widehat{X}_s)) ds \right|^2 \right) + 2\mathbb{E} \left( \left| \int_0^t (\sigma(s, X_s) - \sigma(s, \widehat{X}_s)) dB_s \right|^2 \right). \end{aligned}$$

According to the Cauchy-Schwarz inequality, we have:

$$\begin{aligned} \mathbb{E} \left( \left| \int_0^t (b(s, X_s) - b(s, \widehat{X}_s)) ds \right|^2 \right) & \leq \mathbb{E} \left( \left( \int_0^t ds \right) \left( \int_0^t |b(s, X_s) - b(s, \widehat{X}_s)|^2 ds \right) \right) \\ & = \mathbb{E} \left( t \left( \int_0^t |b(s, X_s) - b(s, \widehat{X}_s)|^2 ds \right) \right) \\ & \leq T \mathbb{E} \left( \int_0^t |b(s, X_s) - b(s, \widehat{X}_s)|^2 ds \right) \\ & \leq T \mathbb{E} \left( \int_0^t L^2 |X_s - \widehat{X}_s|^2 ds \right) \\ & \leq L^2 T \int_0^t \mathbb{E} \left( |X_s - \widehat{X}_s|^2 \right) ds. \end{aligned}$$

Similarly, Itô's isometry provides us with:

$$\begin{aligned} \mathbb{E} \left( \left| \int_0^t (\sigma(s, X_s) - \sigma(s, \widehat{X}_s)) dB_s \right|^2 \right) & = \mathbb{E} \left( \int_0^t |\sigma(s, X_s) - \sigma(s, \widehat{X}_s)|^2 ds \right) \\ & \leq L^2 \int_0^t \mathbb{E} \left( |X_s - \widehat{X}_s|^2 \right) ds. \end{aligned}$$

For some constant  $c$ , we have:

$$\begin{aligned} \mathbb{E} \left( \left| X(t) - \widehat{X}(t) \right| \right)^2 & \leq 2L^2 T \int_0^t \mathbb{E} \left( |X_s - \widehat{X}_s|^2 \right) ds + 2L^2 \int_0^t \mathbb{E} \left( |X_s - \widehat{X}_s|^2 \right) ds \\ & = 2L^2 (T + 1) \int_0^t \mathbb{E} \left( |X_s - \widehat{X}_s|^2 \right) ds \\ & = c \int_0^t \mathbb{E} \left( |X_s - \widehat{X}_s|^2 \right) ds, \quad \text{for any } t \in [0, T], \end{aligned}$$

Noting that  $c = 2L^2(T + 1)$ .

Let  $\phi(t) = \mathbb{E} \left( \left| X(t) - \widehat{X}(t) \right|^2 \right)$ , so:

$$\phi(t) \leq c \int_0^t \phi(s) ds \quad \forall 0 \leq t \leq T.$$

Applying Gronwall's lemma, the condition  $c_0 = 0$  implies  $g(t) = 0$ . Therefore,  $X(t) = \widehat{X}(t)$  almost surely for all  $0 \leq t \leq T$ . Since both  $X(t)$  and  $\widehat{X}(t)$  are continuous processes, we deduce that  $X(r) = \widehat{X}(r)$  for all rational numbers  $r$  in  $[0, T]$ , excluding a set of measure zero. Thus,  $X$  and  $\widehat{X}$  have almost surely continuous trajectories,

$$\mathbb{P} \left( \max_{0 \leq t \leq T} |X(t) - \widehat{X}(t)| > 0 \right) = 0.$$

■

# Chapter 2

## Backward stochastic differential equations

The main objective of this chapter is to establish the theoretical foundations for BSDEs. We start by defining the fundamental structure of these equations and clarifying the mathematical notations required for our study. A primary focus is placed on the adaptedness requirement, which distinguishes BSDEs from standard forward equations. Furthermore, we provide a detailed exposition of the classical existence and uniqueness results under the global Lipschitz framework, as originally presented by **Pardoux and Peng** [11]. This standard setting serves as a crucial benchmark for the extensions and generalizations that will be discussed in the subsequent parts of this dissertation.

### 2.1 Vocabulary and notation

#### 2.1.1 Presentation of the problem

Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  be a filtered probability space, and consider an  $\mathcal{F}_T$ -measurable random variable  $\xi$ . Our goal is to find a solution to the following differential equation:

$$\frac{-dY_t}{dt} = f(Y_t), \quad t \in [0, T] \text{ with } Y_T = \xi.$$

Under the requirement that  $Y$  is adapted to the filtration  $\{\mathcal{F}_t\}_{t \geq 0}$ , we ensure that for every time  $t$ ,  $Y_t$  remains independent of information from the future (See [5]).

Consider the simplest scenario where the generator  $g$  is identically zero ( $g \equiv 0$ ). In this case, although  $Y_t = \xi$  appears to be a straightforward candidate, it lacks adaptedness unless  $\xi$  is deterministic. The optimal adapted approximation in the  $\mathbb{L}^2$  framework is given by the martingale  $Y_t = \mathbb{E}(\xi|\mathcal{F}_t)$ . When using the natural filtration of a Brownian motion, the Brownian martingale representation theorem facilitates the construction of an adapted, square-integrable process  $Z$  such that:

$$Y_t = \mathbb{E}[\xi|\mathcal{F}_t] = \mathbb{E}[\xi] + \int_0^t Z_s dB_s.$$

Indeed, this may be expressed differently:

$$Y_t = \mathbb{E}[\xi] + \int_0^t Z_s dB_s, \quad \forall t \in [0, T],$$

from where

$$\begin{aligned} Y_T &= \mathbb{E}[\xi] + \int_0^T Z_s dB_s \\ \Leftrightarrow \xi &= \mathbb{E}[\xi] + \int_0^t Z_s dB_s + \int_t^T Z_s dB_s \\ \Leftrightarrow \xi &= Y_t + \int_t^T Z_s dB_s. \end{aligned}$$

we then have

$$Y_t = \xi - \int_t^T Z_s dB_s, \quad i.e. \quad -dY_t = -Z_t dB_t \quad \text{with} \quad Y_T = \xi.$$

This elementary example highlights the necessity of a second unknown component, the process  $Z$ . The primary function of  $Z$  is to guarantee that the solution process  $Y$  remains adapted to the underlying filtration.

Accordingly, to attain the highest level of generality given the emergence of this second vari-

able, the generator  $g$  is permitted to depend on the process  $Z$ . Consequently, the equation is reformulated as follows:

$$-dY_t = g(t, Y_t, Z_t) dt - Z_t dB_t, \quad \text{with} \quad Y_T = \xi.$$

### Notations

- Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  represent a complete filtered probability space, equipped with a  $d$ -dimensional Brownian motion  $B$  and we denote by  $\{\mathcal{F}_t\}_{t \geq 0}$  the natural filtration generated by the Brownian motion  $B$ , augmented by the  $\mathbb{P}$ -null sets of  $\mathcal{F}$ .
- The Space  $\mathcal{S}^2(\mathbb{R}^k)$ : This vector space consists of all  $\mathbb{R}^k$ -valued, progressively measurable processes  $Y$  that satisfy the following norm condition:

$$\|Y\|_{\mathcal{S}^2}^2 := \mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t|^2 \right] < \infty.$$

Furthermore,  $\mathcal{S}_c^2(\mathbb{R}^k)$  denotes the closed subspace of  $\mathcal{S}^2(\mathbb{R}^k)$  containing processes with continuous trajectories.

- The Space  $\mathcal{M}^2(\mathbb{R}^{k \times d})$ : This space is composed of progressively measurable processes  $Z$  taking values in the set of  $k \times d$  matrices, such that:

$$\|Z\|_{\mathcal{M}^2}^2 := \mathbb{E} \left[ \int_0^T \|Z_t\|^2 dt \right] < \infty.$$

For any matrix  $z \in \mathbb{R}^{k \times d}$ , the norm is defined as  $\|z\|^2 = \text{trace}(zz^*)$ , where  $z^*$  is the transpose of  $z$ .

- The Generator  $g$ . Let  $g: \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \rightarrow \mathbb{R}^k$  be a function such that for every fixed pair  $(Y, Z) \in \mathbb{R}^k \times \mathbb{R}^{k \times d}$ , the stochastic process  $\{g(w, t, Y, Z)\}_{0 \leq t \leq T}$  is progressively measurable.
- Terminal Condition  $\xi$ : We consider a random vector  $\xi \in \mathbb{R}^k$  that is measurable with respect to the  $\sigma$ -field  $\mathcal{F}_T$ .

- It should be noted that the spaces  $\mathcal{S}^2$ ,  $\mathcal{S}_c^2$ , and  $\mathcal{M}^2$  are complete normed spaces, thereby constituting Banach spaces under the norms introduced above.

Let BSDE

$$\begin{cases} -dY_t = g(t, Y_t, Z_t) dt - Z_t dB_t & 0 \leq t \leq T, \\ Y_T = \xi. \end{cases}$$

Alternatively, the equation can be written in its integral representation as follows:

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s \quad 0 \leq t \leq T. \quad (2.1)$$

The function  $g$  is referred to as the generator (or driver) of the BSDE, while  $\xi$  denotes the terminal condition. We shall now formally establish the definition of a solution for equation (2.1).

**Definition 2.1** A pair of stochastic processes  $\{(Y_t, Z_t)\}_{0 \leq t \leq T}$  is called a solution to the BSDE (2.1) if it satisfies the following conditions:

1.  $Y$  and  $Z$  are progressively measurable processes taking values in  $\mathbb{R}^k$  and  $\mathbb{R}^{k \times d}$ , respectively.
2. The integrability condition  $\int_0^T (|g(s, Y_s, Z_s)| + \|Z_s\|^2) ds < \infty$  holds  $\mathbb{P} - a.s.$
3.  $\mathbb{P} - a.s.$ , the processes satisfy the following integral relation:

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s \quad 0 \leq t \leq T.$$

**Proposition 2.1** Assume that there exists a non-negative process  $\{g_t\}_{0 \leq t \leq T}$  belonging to the space  $\mathcal{M}^2(\mathbb{R})$ , along with a strictly positive constant  $\lambda$ , such that:

$$\forall (t, Y, Z) \in [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d}, \quad |g(t, Y, Z)| \leq g_t + \lambda(|Y| + \|Z\|).$$

If  $\{(Y_t, Z_t)\}_{0 \leq t \leq T}$  represents a solution to the BSDE (2.1) such that  $Z$  is an element of  $\mathcal{M}^2$ , then it follows that the process  $Y$  necessarily belongs to the space  $\mathcal{S}_c^2$ .

**Proof.** The conclusion is primarily reached by applying **Gronwall's Lemma** and noting that the initial value  $Y_0$  is deterministic. Specifically, for every  $t \in [0, T]$ , we have:

$$Y_t = Y_0 - \int_0^t g(s, Y_s, Z_s) ds - \int_0^t Z_s dB_s,$$

Then, by applying the assumptions regarding  $g$ ,

$$\begin{aligned} |Y_t| &\leq |Y_0| + \int_0^t |g(s, Y_s, Z_s)| ds + \left| \int_0^t Z_s dB_s \right| \\ &\leq |Y_0| + \int_0^T (g_s + \lambda \|Z_s\|) ds + \sup_{0 \leq t \leq T} \left| \int_0^t Z_s dB_s \right| + \lambda \int_0^t |Y_s| ds. \end{aligned}$$

Let's set

$$\xi = |Y_0| + \int_0^T (g_s + \lambda \|Z_s\|) ds + \sup_{0 \leq t \leq T} \left| \int_0^t Z_s dB_s \right|.$$

Under the assumption that  $Z \in \mathcal{M}^2$ , we invoke **Doob's inequality** to show that the third term is square-integrable. This property is also shared by the process  $\{g_t\}_{0 \leq t \leq T}$ . Furthermore, since  $Y_0$  is a deterministic constant, it is naturally square-integrable, which implies that  $\xi$  is a square-integrable random variable. Given that  $Y$  is a continuous process satisfying,

$$|Y_t| \leq \xi + \lambda \int_0^t |Y_s| ds,$$

Applying **Gronwall's Lemma**, we obtain

$$\mathbb{E} \left( \sup_{0 \leq t \leq T} |Y_t| \right) \leq \mathbb{E}(\xi) e^{\lambda T},$$

since  $\xi$  is square-integrable, it follows that the process  $Y$  belongs to the space  $\mathcal{S}^2$ . ■

**Lemma 2.1** [13] *Consider the processes  $Y \in \mathcal{S}^2(\mathbb{R}^k)$  and  $Z \in \mathcal{M}^2(\mathbb{R}^{k \times d})$ . Then, the stochastic integral defined by  $\left\{ \int_0^t Y_s \cdot Z_s dB_s \right\}_{t \in [0, T]}$  is a uniformly integrable martingale.*

**Proof.** The **Burkholder-Davis-Gundy** (See Theorem **A.7**) inequalities provide:

$$\begin{aligned} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \int_0^t Y_s \cdot Z_s dB_s \right| \right] &\leq C \mathbb{E} \left[ \left\langle |Y_s|^2 \|Z_s\|^2 ds \right\rangle_T^{\frac{1}{2}} \right] \\ &= C \mathbb{E} \left[ \left( \int_0^T |Y_s|^2 \|Z_s\|^2 ds \right)^{\frac{1}{2}} \right] \\ &\leq C \mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t| \left( \int_0^T \|Z_s\|^2 ds \right)^{\frac{1}{2}} \right]. \end{aligned}$$

Consequently, by applying the elementary inequality  $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$ ,

$$\begin{aligned} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \int_0^t Y_s \cdot Z_s dB_s \right| \right] &\leq \frac{C}{2} \left( \mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t|^2 \right] + \mathbb{E} \left[ \int_0^T \|Z_s\|^2 ds \right] \right) \\ &= C' \left( \mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t|^2 \right] + \mathbb{E} \left[ \int_0^T \|Z_s\|^2 ds \right] \right). \end{aligned}$$

Since the latter quantity is finite by assumption, the desired result follows. ■

## 2.2 The lipschitz case

### 2.2.1 The result of Pardoux-Peng

In this section, we present a fundamental existence and uniqueness theorem, which serves as a precursor to the generalized version discussed in the following chapter. This landmark result, established by **E. Pardoux and S. Peng** [11], represents the initial breakthrough in proving the solvability of BSDEs with non-linear generators.

**(L)** There exists a non-negative constant  $\lambda$  such that,  $\mathbb{P} - a.s.$

1. **Lipschitz continuity in  $(Y, Z)$ :** For all  $t \in [0, T]$ , and for any pairs  $(Y, Y')$  and  $(Z, Z')$ , the generator satisfies:

$$|g(t, Y, Z) - g(t, Y', Z')| \leq \lambda (|Y - Y'| + \|Z - Z'\|).$$

2. **Integrability condition:** The terminal condition and the driver at the origin fulfill

the following square-integrability requirement:

$$\mathbb{E} \left[ |\xi|^2 + \int_0^T |g(s, 0, 0)|^2 ds \right] < \infty.$$

We initially consider a simplified scenario where the generator  $g$  is independent of both  $Y$  and  $Z$ . Specifically, given a square-integrable terminal value  $\xi$  and a process  $\{G_t\}_{0 \leq t \leq T}$  belonging to  $\mathcal{M}^2(\mathbb{R}^k)$ , we aim to find the solution to the following BSDE:

$$Y_t = \xi + \int_t^T G_s ds - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T. \quad (2.2)$$

**Lemma 2.2** *Let  $\xi \in \mathbb{L}^2(\mathcal{F}_T)$  and  $\{g_t\}_{0 \leq t \leq T} \in \mathcal{M}^2(\mathbb{R}^k)$ . Then, the BSDE (2.2) admits a unique solution pair  $(Y, Z)$  provided that the condition  $Z \in \mathcal{M}^2$  is satisfied.*

**Proof.** First, let us assume that  $(Y, Z)$  is a solution satisfying the condition  $Z \in \mathcal{M}^2$ . By taking the conditional expectation with respect to the  $\sigma$ -field  $\mathcal{F}_t$ , it follows that:

$$Y_t = \mathbb{E} \left( \xi + \int_t^T G_s ds \middle| \mathcal{F}_t \right).$$

We proceed to define the process  $Y$  according to the aforementioned formula, which leaves the identification of  $Z$  as the next step. It is important to note that, by applying **Fubini's Theorem** and considering the progressive measurability of  $G$ , the integral process  $\int_0^t G_s ds$  is adapted to the filtration  $\{\mathcal{F}_t\}_{t \in [0, T]}$ . Furthermore, since  $G$  is square-integrable, this integral process is an element of  $\mathcal{S}_c^2$ . Thus, for all  $t \in [0, T]$ , we obtain:

$$\begin{aligned} Y_t &= \mathbb{E} \left( \xi + \int_t^T G_s ds \middle| \mathcal{F}_t \right) \\ &= \mathbb{E} \left( \xi + \int_0^T G_s ds \middle| \mathcal{F}_t \right) - \int_0^t G_s ds \\ &=: M_t - \int_0^t G_s ds. \end{aligned}$$

Since  $M$  is a Brownian martingale, the **Brownian martingale representation theorem** allows us to construct a unique process  $Z$  belonging to  $\mathcal{M}^2$  such that:

$$M_t = M_0 + \int_0^t Z_s dB_s,$$

and so

$$\begin{aligned} Y_t &= M_t - \int_0^t G_s ds \\ &= M_0 + \int_0^t Z_s dB_s - \int_0^t G_s ds. \end{aligned}$$

It is straightforward to verify that the pair  $(Y, Z)$  constructed in this manner constitutes a solution to the BSDE under consideration, specifically because the terminal condition  $Y_T = \xi$  is satisfied,

$$\begin{aligned} Y_t - \xi &= M_0 + \int_0^t Z_s dB_s - \int_0^t G_s ds - \left( M_0 + \int_0^T Z_s dB_s - \int_0^T G_s ds \right) \\ &= \int_t^T G_s ds - \int_t^T Z_s dB_s. \end{aligned}$$

Uniqueness is straightforward to establish for any solution pair that satisfies the condition  $Z \in \mathcal{M}^2$ .

We now proceed to establish the existence and uniqueness theorem originally proven by **Pardoux and Peng**. ■

**Theorem 2.1** (*Pardoux and Peng, 1990*) [11]. *Under the validity of hypothesis (L), the BSDE (2.1) admits a unique adapted solution pair  $(Y, Z)$  such that the control process  $Z$  belongs to the space  $\mathcal{M}^2$ .*

**Proof.** We employ a fixed-point argument within the Banach space  $\mathcal{S}^2(\mathbb{R}^k) \times \mathcal{M}^2(\mathbb{R}^{k \times d})$ . To this end, we define a mapping  $\Psi$  from  $\mathcal{S}^2(\mathbb{R}^k) \times \mathcal{M}^2(\mathbb{R}^{k \times d})$  into itself such that a pair  $(Y, Z) \in \mathcal{S}^2(\mathbb{R}^k) \times \mathcal{M}^2(\mathbb{R}^{k \times d})$  solves the BSDE (2.1) if and only if it constitutes a fixed point of the operator  $\Psi$ .

For any pair  $(U, V) \in \mathcal{S}^2(\mathbb{R}^k) \times \mathcal{M}^2(\mathbb{R}^{k \times d})$ , we define  $(Y, Z) = \Psi(U, V)$  as the unique solution to the following BSDE:

$$Y_t = \xi + \int_t^T g(s, U_s, V_s) ds - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T.$$

Observe that this final BSDE possesses a unique solution within the space  $\mathcal{S}^2(\mathbb{R}^k) \times \mathcal{M}^2(\mathbb{R}^{k \times d})$ . Indeed, by setting  $G_s = g(s, U_s, V_s)$ , we can show that this process belongs to  $\mathcal{M}^2$ . This follows from the Lipschitz property of  $g$ , which yields the estimate,

$$\begin{aligned} |G_s|^2 &= |g(s, U_s, V_s)|^2 \\ &= |g(s, U_s, V_s) - g(s, 0, 0) + g(s, 0, 0)|^2 \\ &\leq 2|g(s, U_s, V_s) - g(s, 0, 0)|^2 + 2|g(s, 0, 0)|^2 \\ &\leq 2 \times 2\lambda^2(|U_s|^2 + \|V_s\|^2) + 2|g(s, 0, 0)|^2 \\ &\Leftrightarrow \mathbb{E} \left[ \int_0^t |G_s|^2 ds \right] \leq 4\lambda^2 \mathbb{E} \left[ \int_0^t |U_s|^2 ds \right] + 4\lambda^2 \mathbb{E} \left[ \int_0^t \|V_s\|^2 ds \right] + 2\mathbb{E} \left[ \int_0^t |g(s, 0, 0)|^2 ds \right] \\ &\leq 4\lambda^2 \mathbb{E} \left[ \sup_{0 \leq t \leq T} |U_s|^2 \right] \cdot \int_0^t ds + 4\lambda^2 \mathbb{E} \left[ \int_0^t \|V_s\|^2 ds \right] + 2\mathbb{E} \left[ \int_0^t |g(s, 0, 0)|^2 ds \right] \\ &\leq +\infty, \end{aligned}$$

since these three components are square-integrable, the integrability of  $G_s$  is guaranteed. Consequently, we can apply Lemma [2.2](#) to ensure the existence of a unique solution pair  $(Y, Z)$  with  $Z \in \mathcal{M}^2$ . Furthermore, d'après Proposition [2.1](#), the process  $Y$  resides in  $\mathcal{S}_c^2$ , confirming that  $(Y, Z) \in \mathcal{S}^2(\mathbb{R}^k) \times \mathcal{M}^2(\mathbb{R}^{k \times d})$ . Thus, the mapping  $\Psi : \mathcal{S}^2(\mathbb{R}^k) \times \mathcal{M}^2(\mathbb{R}^{k \times d}) \rightarrow \mathcal{S}^2(\mathbb{R}^k) \times \mathcal{M}^2(\mathbb{R}^{k \times d})$  is well-defined.

Let  $(U, V)$  and  $(U', V')$  be any two elements of the space  $\mathcal{S}^2(\mathbb{R}^k) \times \mathcal{M}^2(\mathbb{R}^{k \times d})$ , and let their respective images under the mapping  $\Psi$  be denoted by  $(Y, Z) = \Psi(U, V)$  and  $(Y', Z') = \Psi(U', V')$ . By defining the differences  $y = Y - Y'$  and  $z = Z - Z'$ , we observe that  $y_T = 0$  and

$$dy_t = -(g(t, U_t, V_t) - g(t, U'_t, V'_t)) dt + z_t dB_t,$$

By applying Itô's formula 2 to the process  $g(t, y) = e^{\alpha t} |y_t|^2$ , we obtain:

$$\begin{aligned} d(e^{\alpha t} |y_t|^2) &= \alpha e^{\alpha t} |y_t|^2 dt + 2e^{\alpha t} |y_t| dy_t + \frac{1}{2} \times 2e^{\alpha t} d\langle Y, Y \rangle_t \\ &= \alpha e^{\alpha t} |y_t|^2 dt + 2e^{\alpha t} |y_t| \{-(g(t, U_t, V_t) - g(t, U'_t, V'_t)) dt\} \\ &\quad + 2e^{\alpha t} |y_t| \cdot z_t dB_t + e^{\alpha t} \|z_t\|^2 dt. \end{aligned}$$

Consequently, by integrating from  $t$  to  $T$ , we obtain

$$\begin{aligned} \int_t^T d(e^{\alpha s} |y_s|^2) &= \alpha \int_t^T e^{\alpha s} |y_s|^2 ds - 2 \int_t^T e^{\alpha s} |y_s| \{(g(s, U_s, V_s) - g(s, U'_s, V'_s)) ds\} \\ &\quad + 2 \int_t^T e^{\alpha s} y_s \cdot z_s dB_s + \int_t^T e^{\alpha s} \|z_s\|^2 ds, \end{aligned}$$

this is equivalent to writing

$$\begin{aligned} e^{\alpha t} |y_T|^2 - e^{\alpha t} |y_t|^2 &= \alpha \int_t^T e^{\alpha s} |y_s|^2 ds - 2 \int_t^T e^{\alpha s} |y_s| \{(g(s, U_s, V_s) - g(s, U'_s, V'_s)) ds\} \\ &\quad + 2 \int_t^T e^{\alpha s} y_s \cdot z_s dB_s + \int_t^T e^{\alpha s} \|z_s\|^2 ds, \end{aligned}$$

and since  $y_T = 0$ , we have

$$\begin{aligned} e^{\alpha t} |y_t|^2 &= -\alpha \int_t^T e^{\alpha s} |y_s|^2 ds + 2 \int_t^T e^{\alpha s} |y_s| \{(g(s, U_s, V_s) - g(s, U'_s, V'_s)) ds\} \\ &\quad - 2 \int_t^T e^{\alpha s} y_s \cdot z_s dB_s - \int_t^T e^{\alpha s} \|z_s\|^2 ds, \end{aligned}$$

then we can rewrite

$$\begin{aligned} e^{\alpha t} |y_t|^2 &+ \int_t^T e^{\alpha s} \|z_s\|^2 ds \\ &= -\alpha \int_t^T e^{\alpha s} |y_s|^2 ds + 2 \int_t^T e^{\alpha s} |y_s| \{(g(s, U_s, V_s) - g(s, U'_s, V'_s)) ds\} - 2 \int_t^T e^{\alpha s} y_s \cdot z_s dB_s, \end{aligned}$$

furthermore, by leveraging the Lipschitz continuity of  $g$  and letting  $u = U - U'$  and  $v = V - V'$  denote the respective differences, we obtain

$$\begin{aligned}
 & e^{\alpha t} |y_t|^2 + \int_t^T e^{\alpha s} \|z_s\|^2 ds \\
 & \leq -\alpha \int_t^T e^{\alpha s} |y_s|^2 ds + 2\lambda \int_t^T e^{\alpha s} |y_s| (|u_s| + \|v_s\|) ds - 2 \int_t^T e^{\alpha s} y_s \cdot z_s dB_s \\
 & = -\alpha \int_t^T e^{\alpha s} |y_s|^2 ds + 2\lambda \int_t^T e^{\alpha s} |y_s| |u_s| ds + 2\lambda \int_t^T e^{\alpha s} |y_s| \|v_s\| ds \\
 & \quad - 2 \int_t^T e^{\alpha s} y_s \cdot z_s dB_s.
 \end{aligned}$$

For every  $\varepsilon > 0$ , we employ the inequality  $2ab \leq \frac{a^2}{\varepsilon} + \varepsilon b^2$ . Consequently, the preceding bound yields

$$\begin{aligned}
 & e^{\alpha t} |y_t|^2 + \int_t^T e^{\alpha s} \|z_s\|^2 ds \\
 & \leq -\alpha \int_t^T e^{\alpha s} |y_s|^2 ds + \frac{\lambda^2}{\varepsilon} \int_t^T e^{\alpha s} |y_s|^2 ds + \varepsilon \int_t^T e^{\alpha s} |u_s|^2 ds \\
 & \quad + \frac{\lambda^2}{\varepsilon} \int_t^T e^{\alpha s} |y_s|^2 ds + \varepsilon \int_t^T e^{\alpha s} \|v_s\|^2 ds - 2 \int_t^T e^{\alpha s} y_s \cdot z_s dB_s \\
 & = -\alpha \int_t^T e^{\alpha s} |y_s|^2 ds + \frac{2\lambda^2}{\varepsilon} \int_t^T e^{\alpha s} |y_s|^2 ds + \varepsilon \int_t^T e^{\alpha s} (|u_s|^2 + \|v_s\|^2) ds \\
 & \quad - 2 \int_t^T e^{\alpha s} y_s \cdot z_s dB_s,
 \end{aligned}$$

by taking  $\alpha = \frac{2\lambda^2}{\varepsilon}$ , we have  $\forall t \in [0, T]$ , the following estimate

$$e^{\alpha t} |y_t|^2 + \int_t^T e^{\alpha s} \|z_s\|^2 ds \leq \varepsilon \int_t^T e^{\alpha s} (|u_s|^2 + \|v_s\|^2) ds - 2 \int_t^T e^{\alpha s} y_s \cdot z_s dB_s. \quad (2.3)$$

According to Lemma 2.1, the local martingale  $\left\{ \int_0^t e^{\alpha s} y_s \cdot z_s dB_s \right\}_{t \in [0, T]}$  is actually a true martingale starting at zero. In particular, by taking the expectation which eliminates the stochastic integral term as established in the previous remark we straightforwardly obtain the following for  $t = 0$ ,

$$\mathbb{E} \left[ e^{\alpha t} |y_t|^2 + \int_t^T e^{\alpha s} \|z_s\|^2 ds \right] \leq \varepsilon \mathbb{E} \left[ \int_t^T e^{\alpha s} (|u_s|^2 + \|v_s\|^2) ds \right].$$

Thus, we get

$$\mathbb{E} \left[ \int_0^T e^{\alpha s} \|z_s\|^2 ds \right] \leq \varepsilon \mathbb{E} \left[ \int_t^T e^{\alpha s} (|u_s|^2 + \|v_s\|^2) ds \right]. \quad (2.4)$$

Returning to inequality (2.3), the **Burkholder-Davis-Gundy (BDG) A.7** inequalities provide where  $C$  denotes a universal constant the following estimate,

$$\begin{aligned} & \mathbb{E} \left[ \sup_{t \in [0, T]} e^{\alpha t} |y_t|^2 \right] \\ & \leq \varepsilon \mathbb{E} \left[ \int_0^T e^{\alpha s} (|u_s|^2 + \|v_s\|^2) ds \right] + 2 \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left( \int_t^T e^{\alpha s} y_s \cdot z_s dB_s \right) \right] \\ & \leq \varepsilon \mathbb{E} \left[ \int_0^T e^{\alpha s} (|u_s|^2 + \|v_s\|^2) ds \right] + 2C_{BDG} \mathbb{E} \left[ \left( \int_0^T e^{2\alpha s} |y_s|^2 \|z_s\|^2 ds \right)^{\frac{1}{2}} \right] \\ & \leq \varepsilon \mathbb{E} \left[ \int_0^T e^{\alpha s} (|u_s|^2 + \|v_s\|^2) ds \right] + 2C_{BDG} \mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\alpha t} |y_t| \left( \int_0^T e^{\alpha s} \|z_s\|^2 ds \right)^{\frac{1}{2}} \right], \end{aligned}$$

then, since  $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$ ,

$$\begin{aligned} & \mathbb{E} \left[ \sup_{t \in [0, T]} e^{\alpha t} |y_t|^2 \right] \\ & \leq \varepsilon \mathbb{E} \left[ \int_0^T e^{\alpha s} (|u_s|^2 + \|v_s\|^2) ds \right] + \frac{1}{2} \mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\alpha t} |y_t|^2 \right] + \frac{4C_{BDG}^2}{2} \mathbb{E} \left[ \int_0^T e^{\alpha s} \|z_s\|^2 ds \right] \\ & = \varepsilon \mathbb{E} \left[ \int_0^T e^{\alpha s} (|u_s|^2 + \|v_s\|^2) ds \right] + \frac{1}{2} \mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\alpha t} |y_t|^2 \right] + 2C_{BDG}^2 \mathbb{E} \left[ \int_0^T e^{\alpha s} \|z_s\|^2 ds \right]. \end{aligned}$$

Taking  $\left\{ \frac{1}{2} \mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\alpha t} |y_t|^2 \right] \right\}$  to the other side, and multiplying the inequality by 2, we find

$$\mathbb{E} \left[ \sup_{t \in [0, T]} e^{\alpha t} |y_t|^2 \right] \leq 2\varepsilon \mathbb{E} \left[ \int_0^T e^{\alpha s} (|u_s|^2 + \|v_s\|^2) ds \right] + 4C_{BDG}^2 \mathbb{E} \left[ \int_0^T e^{\alpha s} \|z_s\|^2 ds \right]. \quad (2.5)$$

By inequalities (2.4) and (2.5), we obtain

$$\begin{aligned}
 & \mathbb{E} \left[ \sup_{t \in [0, T]} e^{\alpha t} |y_t|^2 + \int_0^T e^{\alpha s} \|z_s\|^2 ds \right] \\
 & \leq 3\varepsilon \mathbb{E} \left[ \int_0^T e^{\alpha s} (|u_s|^2 + \|v_s\|^2) ds \right] + 4C_{BDG}^2 \varepsilon \mathbb{E} \left[ \int_0^T e^{\alpha s} (|u_s|^2 + \|v_s\|^2) ds \right] \\
 & = \varepsilon (3 + 4C_{BDG}^2) \mathbb{E} \left[ \int_0^T e^{\alpha s} |u_s|^2 ds + \int_0^T e^{\alpha s} \|v_s\|^2 ds \right] \\
 & \leq \varepsilon (3 + 4C_{BDG}^2) \mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\alpha t} |u_t|^2 \int_0^T ds + \int_0^T e^{\alpha s} \|v_s\|^2 ds \right] \\
 & \leq \varepsilon (3 + 4C_{BDG}^2) (1 \vee T) \mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\alpha t} |u_t|^2 + \int_0^T e^{\alpha s} \|v_s\|^2 ds \right].
 \end{aligned}$$

By selecting  $\varepsilon$  such that  $\varepsilon (3 + 4C_{BDG}^2) (1 \vee T) = \frac{3}{4}$ , the mapping  $\Psi$  becomes a strict contraction on

$$\|(U, V)\|_\alpha = \mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\alpha t} |U_t|^2 + \int_0^T e^{\alpha s} \|V_s\|^2 ds \right]^{\frac{3}{4}},$$

which endows the space with a Banach structure. It is important to note that this specific norm is equivalent to the standard one where  $\alpha = 0$ .

Consequently, the mapping  $\Psi$  possesses a unique fixed point, thereby guaranteeing the existence and uniqueness of a solution to the BSDE (2.1) within the space  $\mathcal{S}^2(\mathbb{R}^k) \times \mathcal{M}^2(\mathbb{R}^{k \times d})$ . Furthermore, we obtain a unique solution satisfying the requirement  $Z \in \mathcal{M}^2$ , as proposition 2.1 ensures that any such solution is necessarily contained in  $\mathcal{S}^2(\mathbb{R}^k) \times \mathcal{M}^2(\mathbb{R}^{k \times d})$ . ■

**Remark 2.1** *The role of the process  $Z$ : The significance of the process  $Z$ , specifically within the stochastic integral term  $\int_t^T Z_s dB_s$ , lies in ensuring the adaptedness of the process  $Y$ . Consequently, in scenarios where such an adjustment is redundant,  $Z$  vanishes.*

## 2.3 Linear BSDEs

In this section, we investigate the specific case of linear BSDEs, for which a closed-form solution can be derived. We focus on the one-dimensional case ( $\kappa = 1$ ), where  $Y$  is a real-valued process and  $Z$  is viewed as a  $1 \times d$  row vector.

**Proposition 2.2** [13] *Let  $\{(a_t, b_t)\}_{t \in [0, T]}$  be a bounded and progressively measurable process taking values in  $\mathbb{R} \times \mathbb{R}^d$ . Consider a real-valued,  $\mathcal{F}_T$ -measurable terminal condition  $\xi \in \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$ , and let  $\{c_t\}_{t \in [0, T]}$  be an element of  $\mathcal{M}^2(\mathbb{R})$ .*

*Then, the following linear BSDE:*

$$Y_t = \xi + \int_t^T \{a_s Y_s + Z_s b_s + c_s\} ds - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T,$$

*admits a unique solution pair  $(Y, Z) \in \mathcal{S}^2 \times \mathcal{M}^2$ , which is explicitly given by the formula:*

$$\forall t \in [0, T], \quad Y_t = \Gamma_t^{-1} \mathbb{E} \left[ \xi \Gamma_T + \int_t^T c_s \Gamma_s ds \mid \mathcal{F}_t \right],$$

*where the process  $\{\Gamma_t\}_{t \in [0, T]}$  is defined as:*

$$\Gamma_t = \exp \left( \int_0^t b_s \cdot dB_s - \frac{1}{2} \int_0^t |b_s|^2 ds + \int_0^t a_s ds \right).$$

**Proof.** We begin by observing that the process  $\Gamma$  satisfies the following:

$$d\Gamma_t = \Gamma_t (a_t dt + b_t dB_t), \quad \Gamma_0 = 1.$$

The boundedness of the process  $b$  ensures that the stochastic exponential  $\Gamma$  is a well-defined martingale. Applying Doob's maximal inequality, we obtain the following estimate:

$$\|\Gamma\|_{\mathcal{S}^2}^2 = \mathbb{E} \left[ \sup_{0 \leq t \leq T} |\Gamma_t|^2 \right] \leq 4 \mathbb{E} [|\Gamma_T|^2],$$

since  $\mathbb{E} [|\Gamma_T|^2] < \infty$  under the given assumptions, it follows that  $\Gamma \in \mathcal{S}^2$ .

Furthermore, the conditions set in this proposition guarantee that the linear BSDE possesses a unique adapted solution  $(Y, Z)$ . This is verified by setting the generator  $g(t, Y, Z) = a_t Y + b_t Z + c_t$ , which satisfies the Lipschitz requirement **(L)**. According to proposition [2.1](#), the process  $Y$  also belongs to  $\mathcal{S}^2$ .

By leveraging the Itô integration by parts formula, we compute the differential of the product

$\Gamma_t Y_t$ :

$$\begin{aligned}
 d\Gamma_t Y_t &= \Gamma_t dY_t + Y_t d\Gamma_t + d\langle \Gamma, Y \rangle_t \\
 &= \Gamma_t [-(a_t Y_t + b_t Z_t + c_t) dt + Z_t dB_t] + Y_t [\Gamma_t a_t dt + \Gamma_t b_t dB_t] + \Gamma_t b_t Z_t dt \\
 &= -\Gamma_t a_t Y_t dt - \Gamma_t b_t Z_t dt - \Gamma_t c_t dt + \Gamma_t Z_t dB_t + Y_t \Gamma_t a_t dt + Y_t \Gamma_t b_t dB_t + \Gamma_t b_t Z_t dt \\
 &= -\Gamma_t c_t dt + \Gamma_t Z_t dB_t + Y_t \Gamma_t b_t dB_t \\
 &= -\Gamma_t c_t dt + \Gamma_t (Z_t + Y_t b_t) dB_t.
 \end{aligned}$$

This differential form implies that the process  $\Gamma_t Y_t + \int_0^t c_s \Gamma_s ds$  is a local martingale. Given that  $c \in \mathcal{M}^2$  and  $\Gamma, Y \in \mathcal{S}^2$ , this process is, in fact, a true martingale. Consequently, the value at time  $t$  can be expressed as the conditional expectation of its terminal value at time  $T$ :

$$\Gamma_t Y_t + \int_0^t \Gamma_s c_s ds = \mathbb{E} \left[ \Gamma_T Y_T + \int_0^T \Gamma_s c_s ds \middle| \mathcal{F}_t \right],$$

this lead directly to the announced closed-form solution. ■

# Chapter 3

## An existence and uniqueness result of solutions for BSDEs under a weak monotonicity and general growth conditions

The main objective of this chapter is to establish an existence and uniqueness result for solutions to multidimensional Backward Stochastic Differential Equations (BSDEs). Unlike classical works that assume a Lipschitz condition or a standard monotonicity condition, we will work here under weaker hypotheses. Specifically, we assume that the generator  $g$  satisfies a **weak monotonicity condition** in  $Y$  as well as a **general growth condition** in  $Y$ .

To conduct this study, we impose the following hypotheses on the generator  $g(w, t, Y, Z)$ :

**(H1) Weak monotonicity condition:** There exists a non-decreasing and concave function  $\kappa(\cdot) : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$  with  $\kappa(0) = 0$ ,  $\kappa(u) > 0$  for  $u > 0$  and  $\int_{0^+} \kappa^{-1}(u) du = +\infty$ , such that  $d\mathbb{P} \times dt - a.e.$ , for all  $Y_1, Y_2 \in \mathbb{R}^k$  and  $Z \in \mathbb{R}^{k \times d}$ .

$$\langle Y_1 - Y_2, g(w, t, Y_1, Z) - g(w, t, Y_2, Z) \rangle \leq \kappa(|Y_1 - Y_2|^2).$$

**(H2) Continuity in  $Y$ :**  $d\mathbb{P} \times dt - a.e.$ , for all  $Z \in \mathbb{R}^{k \times d}$ , the function  $Y \longmapsto g(w, t, Y, Z)$  is

continuous.

**(H3) General growth in  $Y$ :** The generator  $g$  has a general growth with respect to  $Y$ , i.e.,  $d\mathbb{P} \times dt - a.e.$ ,

$$\forall Y \in \mathbb{R}^k, |g(w, t, Y, 0)| \leq |g(w, t, 0, 0)| + \varphi(|Y|),$$

where  $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a continuous and increasing function.

**(H4) Lipschitz condition in  $Z$ :**  $g$  is Lipschitz continuous in  $Z$ , uniformly with respect to  $(w, t, Y)$ , i.e., there exists a constant  $\mu \geq 0$  such that  $d\mathbb{P} \times dt - a.e.$ ,

$$\forall Y \in \mathbb{R}^k, Z_1, Z_2 \in \mathbb{R}^{k \times d}, |g(w, t, Y, Z_1) - g(w, t, Y, Z_2)| \leq \mu |Z_1 - Z_2|.$$

**(H5) Integrability:**  $g(w, t, 0, 0) \in \mathcal{M}^2(0, T; \mathbb{R}^k)$ .

**Theorem 3.1 (Existence and Uniqueness)** *Under the validity of assumptions **(H1)** to **(H5)**, for any terminal variable  $\xi \in \mathbb{L}^2(\mathcal{F}_T; \mathbb{R}^k)$ , the BSDEs (2.1) admits a unique pair of processes  $(Y_t, Z_t)$  belonging to the space  $\mathcal{S}^2(0, T; \mathbb{R}^k) \times \mathcal{M}^2(0, T; \mathbb{R}^{k \times d})$ .*

**Remark 3.1** *The weak monotonicity hypothesis **(H1)** is a strict generalization of several known conditions. In particular:*

1. If  $\kappa(u) = \lambda u$  (linear), we recover the standard monotonicity condition used by Pardoux [12].
2. If  $\kappa(\cdot)$  corresponds to the Mao function (i.e.,  $\int_{0^+} u^{-1} du = +\infty$ ), we recover the Mao condition [10].
3. Hypothesis **(H1)** also covers the Osgood condition and the constantin condition.

Thus, Theorem 3.1 unifies and extends the existence and uniqueness results established in [12, 10, 7, 4].

The proof of this theorem relies on the use of a priori estimates and approximation techniques.

We start by establishing a fundamental a priori estimate.

### 3.1 Estimate for the solutions of BSDEs

To establish this estimate, we introduce an intermediate hypothesis on the generator  $g$ , denoted **(A)**, which will serve as a technical tool (see proposition 3.1 in [6]):

**(A)**  $d\mathbb{P} \times dt - a.e.$ , for all  $(Y, Z) \in \mathbb{R}^k \times \mathbb{R}^{k \times d}$ ,

$$\langle Y, g(w, t, Y, Z) \rangle \leq \psi(|Y|^2) + \lambda |Y| |Z| + |Y| \phi_t,$$

where  $\lambda > 0$  is a constant,  $\phi_t$  is a progressively measurable process satisfying

$$\mathbb{E} \left[ \int_0^T \phi_t^2 dt \right] < +\infty,$$

and  $\psi(\cdot)$  is a non-decreasing concave function such that  $\psi(0) = 0$ .

**Justification of estimation (A):**

To establish this inequality, we begin by decomposing the scalar product  $\langle Y, g(t, Y, Z) \rangle$  by adding and subtracting the intermediate term  $g(t, 0, Z)$ , which gives us:

$$\begin{aligned} \langle Y, g(t, Y, Z) \rangle &= \langle Y, g(t, Y, Z) - g(t, 0, Z) + g(t, 0, Z) \rangle \\ &= \langle Y, g(t, Y, Z) - g(t, 0, Z) \rangle + \langle Y, g(t, 0, Z) \rangle. \end{aligned}$$

Regarding the first term, since  $Y$  can be written as  $Y - 0$ , we are able to apply the weak monotonicity condition **(H1)** with  $Y_1 = Y$  and  $Y_2 = 0$ . This yields:

$$\langle Y - 0, g(t, Y, Z) - g(t, 0, Z) \rangle \leq \kappa(|Y - 0|^2) = \kappa(|Y|^2).$$

Regarding the second term, by adding and subtracting the intermediate term  $g(t, 0, 0)$ , then using the Cauchy-Schwarz inequality and the Lipschitz hypothesis **(H4)** with respect to the

variable  $Z$  (with  $Z_1 = Z$  and  $Z_2 = 0$ ), we obtain the following estimation:

$$\begin{aligned}
 \langle Y, g(t, 0, Z) \rangle &= \langle Y, g(t, 0, Z) - g(t, 0, 0) + g(t, 0, 0) \rangle \\
 &= \langle Y, g(t, 0, Z) - g(t, 0, 0) \rangle + \langle Y, g(t, 0, 0) \rangle \\
 &\leq |Y| |g(t, 0, Z) - g(t, 0, 0)| + |Y| |g(t, 0, 0)| \\
 &\leq \mu |Y| |Z| + |Y| |g(t, 0, 0)|.
 \end{aligned}$$

Finally, by grouping these estimations and identifying  $\psi(\cdot)$  with  $\kappa(\cdot)$ , the constant  $\lambda$  with  $\mu$ , and the process  $\phi_t$  with  $|g(t, 0, 0)|$ , we arrive at the desired inequality:

$$\langle Y, g(t, Y, Z) \rangle \leq \psi(|Y|^2) + \lambda |Y| |Z| + |Y| \phi_t.$$

**Proposition 3.1** *Assume that  $g$  fulfills **(A)** and let  $(Y_t, Z_t)_{t \in [0, T]}$  denote a solution to the BSDE associated with the parameters  $(\xi, T, g)$ . Then, for any  $\theta > 0$ , there exists a positive constant  $c$ , depending solely on  $\lambda$  and  $\theta$  such that, for all  $0 \leq u \leq t \leq T$ :*

$$\begin{aligned}
 &\mathbb{E} \left[ \sup_{r \in [t, T]} |Y_r|^2 | \mathcal{F}_u \right] + \mathbb{E} \left[ \int_t^T |Z_s|^2 | \mathcal{F}_u \right] \\
 &\leq e^{c(T-t)} \left\{ c \mathbb{E} [|\xi|^2 | \mathcal{F}_u] + c \int_t^T \psi(\mathbb{E} [|Y_s|^2 | \mathcal{F}_u]) ds \right\} + e^{c(T-t)} \left\{ \frac{1}{\theta} \mathbb{E} \left[ \int_t^T \phi_s^2 ds | \mathcal{F}_u \right] \right\}. \quad (3.1)
 \end{aligned}$$

**Proof.** To establish this estimate, we first apply Itô's formula to the function  $|Y_t|^2$  over the interval  $t \in [0, T]$ . Considering the terminal condition  $Y_T = \xi$  we obtain:

$$\begin{aligned}
 |Y_T|^2 &= |Y_t|^2 + 2 \int_t^T \langle Y_s, dY_s \rangle + \frac{1}{2} \times 2 \int_t^T d \langle Y, Y \rangle_s \\
 &\Leftrightarrow |\xi|^2 = |Y_t|^2 + 2 \int_t^T \langle Y_s, -g(s, Y_s, Z_s) ds + Z_s dB_s \rangle + \int_t^T |Z_s|^2 ds \\
 &\Leftrightarrow |Y_t|^2 = |\xi|^2 + 2 \int_t^T \langle Y_s, g(s, Y_s, Z_s) \rangle ds - 2 \int_t^T \langle Y_s, Z_s dB_s \rangle - \int_t^T |Z_s|^2 ds \\
 &\Leftrightarrow |Y_t|^2 + \int_t^T |Z_s|^2 ds = |\xi|^2 + 2 \int_t^T \langle Y_s, g(s, Y_s, Z_s) \rangle ds - 2 \int_t^T \langle Y_s, Z_s dB_s \rangle. \quad (3.2)
 \end{aligned}$$

Utilizing assumption **(A)** together with the algebraic inequality  $2ab \leq \theta a^2 + \frac{b^2}{\theta}$  for any  $\theta > 0$ ,

we obtain:

$$\begin{aligned}
 2\langle Y_s, g(s, Y_s, Z_s) \rangle &\leq 2\psi(|Y_s|^2) + 2\lambda |Y_s| |Z_s| + 2 |Y_s| \phi_s \\
 &\leq 2\psi(|Y_s|^2) + 2\lambda^2 |Y_s|^2 + \frac{1}{2} |Z_s|^2 + 290\theta |Y_s|^2 + \frac{\phi_s^2}{290\theta} \\
 &\leq 2\psi(|Y_s|^2) + (2\lambda^2 + 290\theta) |Y_s|^2 + \frac{\phi_s^2}{290\theta} + \frac{1}{2} |Z_s|^2.
 \end{aligned} \tag{3.3}$$

Consequently, combining (3.2) and (3.3) we obtain

$$\begin{aligned}
 &|Y_t|^2 + \int_t^T |Z_s|^2 ds \\
 &\leq |\xi|^2 + \int_t^T \left[ 2\psi(|Y_s|^2) + (2\lambda^2 + 290\theta) |Y_s|^2 + \frac{\phi_s^2}{290\theta} + \frac{1}{2} |Z_s|^2 \right] ds - 2 \int_t^T \langle Y_s, Z_s dB_s \rangle.
 \end{aligned}$$

This allows you to write

$$\begin{aligned}
 &|Y_t|^2 + \frac{1}{2} \int_t^T |Z_s|^2 ds \\
 &\leq |\xi|^2 + \int_t^T \left[ 2\psi(|Y_s|^2) + (2\lambda^2 + 290\theta) |Y_s|^2 + \frac{\phi_s^2}{290\theta} \right] ds - 2 \int_t^T \langle Y_s, Z_s dB_s \rangle,
 \end{aligned}$$

thus, we get, for all  $0 \leq u \leq t \leq T$ ,

$$\frac{1}{2} \mathbb{E} \left[ \int_t^T |Z_s|^2 ds | \mathcal{F}_u \right] \leq X_u^t, \tag{3.4}$$

where

$$\begin{aligned}
 X_u^t &= \mathbb{E} \left[ |\xi|^2 | \mathcal{F}_u \right] + (2\lambda^2 + 290\theta) \mathbb{E} \left[ \int_t^T |Y_s|^2 ds | \mathcal{F}_u \right] \\
 &\quad + \mathbb{E} \left[ \int_t^T \left( 2\psi(|Y_s|^2) + \frac{\phi_s^2}{290\theta} \right) ds | \mathcal{F}_u \right].
 \end{aligned}$$

And since  $|Y_s|^2 \leq \sup_{r \in [s, T]} |Y_r|^2$ , we have:

$$\begin{aligned} X_u^t &= \mathbb{E} [|\xi|^2 | \mathcal{F}_u] + (2\lambda^2 + 290\theta) \int_t^T \mathbb{E} \left[ \sup_{r \in [s, T]} |Y_r|^2 | \mathcal{F}_u \right] ds \\ &\quad + \mathbb{E} \left[ \int_t^T \left( 2\psi(|Y_s|^2) + \frac{\phi_s^2}{290\theta} \right) ds | \mathcal{F}_u \right]. \end{aligned}$$

Moreover, by virtue of the Burkholder–Davis–Gundy inequality, the process

$$\left\{ M_t := \int_0^t \langle Y_s, Z_s dB_s \rangle \right\}_{t \in [0, T]},$$

constitutes a uniformly integrable martingale. Indeed, for every  $0 \leq u \leq t \leq T$ , we have

$$2\mathbb{E} \left[ \sup_{r \in [t, T]} \left| \int_t^T \langle Y_s, Z_s dB_s \rangle \right| | \mathcal{F}_u \right] \leq 12\mathbb{E} \left[ \sup_{r \in [t, T]} |Y_r| \left( \int_t^T |Z_s|^2 ds \right)^{\frac{1}{2}} | \mathcal{F}_u \right].$$

And by applying  $ab \leq a^2\theta + \frac{b^2}{4\theta}$ , we have

$$\begin{aligned} 2\mathbb{E} \left[ \sup_{r \in [t, T]} \left| \int_t^T \langle Y_s, Z_s dB_s \rangle \right| | \mathcal{F}_u \right] &\leq \frac{1}{2}\mathbb{E} \left[ \sup_{r \in [t, T]} |Y_r|^2 | \mathcal{F}_u \right] + \frac{144}{4 \times \frac{1}{2}} \mathbb{E} \left[ \int_t^T |Z_s|^2 ds | \mathcal{F}_u \right] \\ &\leq \frac{1}{2}\mathbb{E} \left[ \sup_{r \in [t, T]} |Y_r|^2 | \mathcal{F}_u \right] + 72\mathbb{E} \left[ \int_t^T |Z_s|^2 ds | \mathcal{F}_u \right] \\ &\leq +\infty. \end{aligned} \tag{3.5}$$

Taking into account (3.3) and (3.5), and using (3.2), we deduce that

$$\begin{aligned} &\mathbb{E} \left[ \sup_{r \in [t, T]} |Y_r|^2 | \mathcal{F}_u \right] + \mathbb{E} \left[ \int_t^T |Z_s|^2 ds | \mathcal{F}_u \right] \\ &\leq \mathbb{E} [|\xi|^2 | \mathcal{F}_u] + \mathbb{E} \left[ \int_t^T 2 \langle Y_s, g(s, Y_s, Z_s) ds \rangle | \mathcal{F}_u \right] + 2\mathbb{E} \left[ \sup_{r \in [t, T]} \int_t^T \langle Y_s, Z_s dB_s \rangle \right] \\ &\leq \mathbb{E} [|\xi|^2 | \mathcal{F}_u] + X_u^t + \frac{1}{2}\mathbb{E} \left[ \int_t^T |Z_s|^2 ds | \mathcal{F}_u \right] + \frac{1}{2}\mathbb{E} \left[ \sup_{r \in [t, T]} |Y_r|^2 | \mathcal{F}_u \right] + 72\mathbb{E} \left[ \int_t^T |Z_s|^2 ds | \mathcal{F}_u \right]. \end{aligned}$$

This allows you to write

$$\mathbb{E} \left[ \sup_{r \in [t, T]} |Y_r|^2 | \mathcal{F}_u \right] + \mathbb{E} \left[ \int_t^T |Z_s|^2 ds | \mathcal{F}_u \right] \leq 2X_u^t + 144 \mathbb{E} \left[ \int_t^T |Z_s|^2 ds | \mathcal{F}_u \right].$$

Combining the previous estimate with (3.4), we obtain

$$\begin{aligned} \mathbb{E} \left[ \sup_{r \in [t, T]} |Y_r|^2 | \mathcal{F}_u \right] + \mathbb{E} \left[ \int_t^T |Z_s|^2 ds | \mathcal{F}_u \right] &\leq 2X_u^t + 144 \times 2X_u^t \\ &\leq 290X_u^t. \end{aligned}$$

Let  $h(t) = \mathbb{E} [\sup_{r \in [t, T]} |Y_r|^2 | \mathcal{F}_u] + \mathbb{E} [\int_t^T |Z_s|^2 ds | \mathcal{F}_u]$  by the definition of  $X_u^t$ , we obtain, for every  $0 \leq u \leq t \leq T$ ,

$$\begin{aligned} h(t) &\leq 290 \left( \mathbb{E} [|\xi|^2 | \mathcal{F}_u] + (2\lambda^2 + 290\theta) \int_t^T \mathbb{E} \left[ \sup_{r \in [t, T]} |Y_r|^2 | \mathcal{F}_u \right] ds \right) \\ &\quad + 290 \left( \mathbb{E} \left[ \int_t^T \left( 2\psi(|Y_s|^2) + \frac{\phi_s^2}{290\theta} \right) ds | \mathcal{F}_u \right] \right) \\ &\leq 290 \mathbb{E} [|\xi|^2 | \mathcal{F}_u] + 290(2\lambda^2 + 290\theta) \int_t^T h(s) ds + \mathbb{E} \left[ \int_t^T \left( 580\psi(|Y_s|^2) + \frac{\phi_s^2}{\theta} \right) ds | \mathcal{F}_u \right], \end{aligned}$$

by combining this with Gronwall's inequality, Fubini's theorem, and Jensen's inequality, and taking into account the concavity condition of  $\psi(\cdot)$ , we obtain inequality (3.1). ■

In the next two subsections, we present the proof of Theorem 3.1. We begin by addressing the uniqueness part.

## 3.2 Proof of uniqueness

**Proof.** Let  $(Y_t^1, Z_t^1)_{t \in [0, T]}$  and  $(Y_t^2, Z_t^2)_{t \in [0, T]}$  be two solutions of the BSDE with parameters  $(\xi, T, g)$ . we have

$$\begin{aligned} Y_t^1 &= \xi + \int_t^T g(s, Y_s^1, Z_s^1) ds - \int_t^T Z_s^1 dB_s, \\ Y_t^2 &= \xi + \int_t^T g(s, Y_s^2, Z_s^2) ds - \int_t^T Z_s^2 dB_s. \end{aligned}$$

Then  $(Y_t^1 - Y_t^2, Z_t^1 - Z_t^2)_{t \in [0, T]}$  is a solution to the following BSDE:

$$Y_t^1 - Y_t^2 = \int_t^T (g(s, Y_s^1, Z_s^1) - g(s, Y_s^2, Z_s^2)) ds - \int_t^T (Z_s^1 - Z_s^2) dB_s,$$

this can be rewritten as,

$$Y_t = \int_t^T \hat{g}(s, Y_s, Z_s) - \int_t^T Z_s dB_s, \quad (3.6)$$

where,  $Y_t = Y_t^1 - Y_t^2$ ,  $Z_t = Z_t^1 - Z_t^2$  and  $\hat{g}(s, Y_s, Z_s) = g(s, Y_s^1, Z_s^1) - g(s, Y_s^2, Z_s^2)$ . By assumptions **(H1)** and **(H4)**. We get

$$\begin{aligned} &\langle Y, \hat{g}(t, Y, Z) \rangle \\ &= \langle Y, \hat{g}(t, Y + Y^2, Z + Z^2) - g(s, Y^2, Z^2) \rangle \\ &= \langle Y, \hat{g}(t, Y + Y^2, Z + Z^2) - g(s, Y^2, Z + Z^2) + g(s, Y^2, Z + Z^2) \rangle - \langle Y, g(s, Y^2, Z^2) \rangle \\ &= \langle Y, \hat{g}(t, Y + Y^2, Z + Z^2) - g(s, Y^2, Z + Z^2) \rangle + \langle Y, g(s, Y^2, Z + Z^2) \rangle - \langle Y, g(s, Y^2, Z^2) \rangle \\ &\leq \kappa(|Y|^2) + \langle Y, g(s, Y^2, Z + Z^2) \rangle - \langle Y, g(s, Y^2, Z^2) \rangle \\ &= \kappa(|Y|^2) + \langle Y, g(s, Y^2, Z + Z^2) - g(s, Y^2, Z^2) \rangle \\ &\leq \kappa(|Y|^2) + |Y| |g(s, Y^2, Z + Z^2) - g(s, Y^2, Z^2)| \\ &\leq \kappa(|Y|^2) + \mu |Y| |Z|. \end{aligned}$$

Consequently, the generator  $\hat{g}(t, Y, Z)$  of BSDE [\(3.6\)](#) fulfills assumption **(A)** by setting  $\psi(u) = \kappa(u)$ ,  $\lambda = \mu$ , and  $\phi_t \equiv 0$ . therefore, applying proposition [3.1](#) with the specific

values  $u = 0$  and  $\theta = 1$ , we deduce the existence of a positive constant  $c$ , which depends solely on  $\mu$  and  $T$ , such that for every  $t \in [0, T]$  :

$$\mathbb{E} \left[ \sup_{r \in [t, T]} |Y_r|^2 | \mathcal{F}_0 \right] + \mathbb{E} \left[ \int_t^T |Z_s|^2 ds | \mathcal{F}_0 \right] \leq e^{cT} \left\{ c \int_t^T \kappa \left( \mathbb{E} [|Y_s|^2 | \mathcal{F}_0] \right) ds \right\},$$

this allows you to write

$$\mathbb{E} \left[ \sup_{r \in [t, T]} |Y_r|^2 + \int_t^T |Z_s|^2 ds \right] \leq c \int_t^T \kappa \left( \mathbb{E} \left[ \sup_{0 \leq r \leq s} |Y_r|^2 \right] \right) ds,$$

using the fact that  $\kappa$  is an increasing and concave function, we obtain

$$\mathbb{E} \left[ \sup_{r \in [t, T]} |Y_r|^2 + \int_t^T |Z_s|^2 ds \right] \leq c \int_t^T \kappa \left( \mathbb{E} \left[ \sup_{0 \leq r \leq s} |Y_r|^2 + \int_s^T |Z_r|^2 dr \right] \right) ds.$$

Given that  $\int_{0+} \kappa^{-1}(u) du = +\infty$ , an application of Bihari's inequality [1] implies that, for any  $t \in [0, T]$ ,

$$\mathbb{E} \left[ \sup_{r \in [t, T]} |Y_r|^2 + \int_t^T |Z_s|^2 ds \right] \leq 0,$$

then we will have the following equality

$$\mathbb{E} \left[ \sup_{r \in [t, T]} |Y_r^1 - Y_r^2|^2 + \int_t^T |Z_s^1 - Z_s^2|^2 ds \right] = 0.$$

The proof of the uniqueness part of Theorem 3.1 is then complete. ■

### 3.3 Proof of existence

The following result is crucial for our proof. We assume that the coefficient  $g$  does not depend on the variable  $z$ .

**Lemma 3.1** *Let for each  $V \in \mathcal{M}^2(0, T; \mathbb{R}^{k \times d})$  and  $\xi \in \mathbb{L}^2(\mathcal{F}_T; \mathbb{R}^k)$  if there existe constant*

$\kappa > 0$  such that

$$d\mathbb{P} - a.s., |\xi| \leq \kappa, d\mathbb{P} \times dt - a.e., |g(t, 0, 0)| \leq \kappa \text{ and } |V_t| \leq \kappa, \quad (3.7)$$

then there exists a unique solution to the following BSDE:

$$\begin{cases} dY_t = -g(t, Y_t, V_t)dt + Z_t dB_t, & t \in [0, T] \\ Y_T = \xi. \end{cases} \quad (3.8)$$

**Proof.** We introduce a standard mollifier  $\psi \in C^\infty(\mathbb{R}^k, \mathbb{R}^+)$  supported on the closed unit ball, and normalized such that  $\int_{\mathbb{R}^k} \psi(Y) dY = 1$ . By means of convolution, for any integer  $n \geq 1$  and given  $(w, t, Y) \in \Omega \times [0, T] \times \mathbb{R}^k$ , we construct the sequence of approximated generators  $g_n$  as follows:

$$g_n(t, Y, V_t) := n^\kappa g(t, Y, V_t) * \psi(nY),$$

by the definition of the convolution product, we obtain:

$$g_n(t, Y, V_t) = n^\kappa \int_{\mathbb{R}^k} g(t, u, V_t) \psi(nY - nu) du, \quad (3.9)$$

By construction, the sequence  $g_n$  preserves the  $(\mathcal{F}_t)$ -progressive measurability for any given  $Y \in \mathbb{R}^k$ . Moreover, taking into account that the mollifier  $\psi$  vanishes outside the unit ball, the convolution integral simplifies to:

$$\begin{aligned} g_n(t, Y, V_t) &= \int_{\mathbb{R}^k} g\left(t, Y - \frac{u}{n}, V_t\right) \psi(u) du \\ &= \int_{\{u: |u| \leq 1\}} g\left(t, Y - \frac{u}{n}, V_t\right) \psi(u) du. \end{aligned} \quad (3.10)$$

By utilizing hypotheses **(H3)** and **(H4)** in conjunction with the bounds established in [\(3.7\)](#), we obtain an upper estimate for the generator  $g$ . For any  $Y \in \mathbb{R}^k$  and  $d\mathbb{P} \times dt - a.e.$ , we begin by adding and subtracting the term  $g(t, Y, 0)$  inside the absolute value, and then applying

the triangle inequality:

$$\begin{aligned}
 |g(t, Y, V_t)| &= |g(t, Y, V_t) - g(t, Y, 0) + g(t, Y, 0)| \\
 &\leq |g(t, Y, V_t) - g(t, Y, 0)| + |g(t, Y, 0)| \\
 &\leq \mu |V_t| + |g(t, Y, 0)| \\
 &\leq \mu |V_t| + |g(t, 0, 0)| + \varphi(|Y|) \\
 &\leq \mu \kappa + \kappa + \varphi(|Y|) \\
 &:= (1 + \mu) \kappa + \varphi(|Y|). \tag{3.11}
 \end{aligned}$$

Consequently, one can readily verify from (3.9) that the function  $g_n(t, Y, V_t)$  satisfies a local Lipschitz condition in  $Y$ , uniformly in  $(t, w)$ , for all  $n \geq 1$ . Moreover, by combining relations (3.10) and (3.11), we deduce that  $d\mathbb{P} \times dt - a.e.$ , for any  $Y \in \mathbb{R}^k$ , the following bound holds:

$$\begin{aligned}
 |g_n(t, Y, V_t)| &= \left| \int_{\{u: |u| \leq 1\}} g\left(t, Y - \frac{u}{n}, V_t\right) \psi(u) du \right| \\
 &\leq \int_{\{u: |u| \leq 1\}} \left| g\left(t, Y - \frac{u}{n}, V_t\right) \right| |\psi(u)| du \\
 &\leq \int_{\{u: |u| \leq 1\}} |(1 + \mu) \kappa + \varphi(|Y|)| |\psi(u)| du \\
 &= (1 + \mu) \kappa + \varphi(|Y|) \int_{\{u: |u| \leq 1\}} |\psi(u)| du \\
 &\leq (1 + \mu) \kappa + \varphi(|Y|) \\
 &\leq (1 + \mu) \kappa + \varphi(|Y| + 1). \tag{3.12}
 \end{aligned}$$

Let us introduce a smooth cut-off function  $\theta_r$  for a sufficiently large positive integer  $r$ , which will be specified later. This function satisfies  $0 \leq \theta_r \leq 1$ , taking the value 1 on  $\{|Y| \leq r\}$  and vanishing for  $|Y| \geq r + 1$ . Consequently, for any  $n \geq 1$ , the product  $\theta_r(Y)g_n(t, Y, V_t)$  becomes globally Lipschitz with respect to  $Y$ , uniformly in  $(t, w)$ . To verify this, consider two arbitrary points  $Y, Y' \in \mathbb{R}^k$ . The Lipschitz condition holds trivially if both  $|Y| > r + 1$  and  $|Y'| > r + 1$ ; thus, it suffices to examine the case where  $|Y'| > r + 1$ . Recognizing that  $\theta_r$  is globally Lipschitz and  $g_n(t, Y, V_t)$  is locally Lipschitz in  $Y$  (uniformly in  $t$  and  $w$ ),

we can deduce from (3.12) the existence of strictly positive constants  $C_1$  and  $C_2$  such that,  $d\mathbb{P} \times dt - a.e.$ ,

$$\begin{aligned}
& |\theta_r(Y)g_n(t, Y, V_t) - \theta_r(Y')g_n(t, Y', V_t)| \\
&= |\theta_r(Y) \{g_n(t, Y, V_t) - g_n(t, Y', V_t)\} - \{(\theta_r(Y') - \theta_r(Y))g_n(t, Y', V_t)\}| \\
&\leq |\theta_r(Y)| |(g_n(t, Y, V_t) - g_n(t, Y', V_t))| \\
&+ |\theta_r(Y') - \theta_r(Y)| |g_n(t, Y', V_t)| \\
&\leq C_1 |Y - Y'| + |\theta_r(Y') - \theta_r(Y)| [(1 + \mu)\kappa + \varphi(|Y'| + 1)] \\
&\leq C_1 |Y - Y'| + C_2 [(1 + \mu)\kappa + \varphi(|Y'| + 1)] |Y - Y'| \\
&\leq C_1 |Y - Y'| + C_2 [(1 + \mu)\kappa + \varphi(|Y'| + 2)] |Y - Y'|.
\end{aligned}$$

Consequently, it follows from (II) that for every  $n \geq 1$ , the following BSDE admits a unique solution  $(Y_t^n, Z_t^n)_{t \in [0, T]}$  :

$$Y_t^n = \xi + \int_t^T \theta_r(Y_s^n)g_n(s, Y_s^n, V_s)ds - \int_t^T Z_s^n dB_s, \quad t \in [0, T]. \quad (3.13)$$

By virtue of (H1) and (3.10), we deduce that for all  $n \geq 1$  and any  $Y_1, Y_2 \in \mathbb{R}^k$ , it holds  $d\mathbb{P} \times dt - a.e.$ ,

$$\begin{aligned}
& \langle Y_1 - Y_2, g_n(t, Y_1, V_t) - g_n(t, Y_2, V_t) \rangle \\
&= \langle Y_1 - Y_2, \int_{\mathbb{R}^k} g(t, Y_1 - \frac{u}{n}, V_t) \psi(u) du - \int_{\mathbb{R}^k} g(t, Y_2 - \frac{u}{n}, V_t) \psi(u) du \rangle \\
&= \int_{\mathbb{R}^k} \langle Y_1 - Y_2, g(t, Y_1 - \frac{u}{n}, V_t) - g(t, Y_2 - \frac{u}{n}, V_t) \rangle \psi(u) du \\
&\leq \kappa (|Y_1 - Y_2|^2) \int_{\mathbb{R}^k} \psi(u) du \\
&= \kappa (|Y_1 - Y_2|^2).
\end{aligned} \quad (3.14)$$

By putting together (3.14) and (3.12), we deduce that for every  $n \geq 1$  and any  $Y \in \mathbb{R}^k$ , it holds  $d\mathbb{P} \times dt - a.e.$ ,

$$\begin{aligned} \langle Y^n, \theta_r(Y^n)g_n(t, Y^n, V_t) \rangle &= \theta_r(Y^n)\langle Y^n, g_n(t, Y^n, V_t) - g_n(t, 0, V_t) \rangle + \theta_r(Y^n)\langle Y^n, g_n(t, 0, V_t) \rangle \\ &\leq \theta_r(Y^n)\kappa(|Y^n|^2) + \theta_r(Y^n)|Y^n||g_n(t, 0, V_t)| \\ &\leq \kappa(|Y^n|^2) + |Y^n|\{(1 + \mu)\kappa + \varphi(1)\}. \end{aligned}$$

Consequently, the generator  $\theta_r(Y^n)g_n(t, Y^n, V_t)$  associated with BSDE (3.13) fulfills condition **(A)** when setting  $\psi(u) = \kappa(u)$ ,  $\lambda = 0$ , and  $\phi_t \equiv (1 + \mu)\kappa + \varphi(1)$ . Therefore, by applying proposition 3.1 with  $\theta = 1$  and taking (3.7) into account, we can find a strictly positive constant  $c$ —which depends exclusively on  $T$ —such that for all integers  $n \geq 1$  and any  $0 \leq u \leq t \leq T$ ,

$$\begin{aligned} &\mathbb{E} \left[ \sup_{0 \leq s \leq r} |Y_s^n|^2 | \mathcal{F}_u \right] + \mathbb{E} \left[ \int_t^T |Z_s^n|^2 ds | \mathcal{F}_u \right] \\ &\leq e^{cT} \left\{ c\mathbb{E}(|\xi|^2 | \mathcal{F}_u) + c \int_t^T \kappa(\mathbb{E}[|Y_s^n|^2 | \mathcal{F}_u]) ds \right\} + e^{cT} \left\{ \mathbb{E} \left[ \int_t^T ((1 + \mu)\kappa + \varphi(1))^2 ds | \mathcal{F}_u \right] \right\} \\ &\leq e^{cT} \left\{ c\kappa^2 + c \int_t^T \kappa(\mathbb{E}[|Y_s^n|^2 | \mathcal{F}_u]) ds \right\} + e^{cT} \{((1 + \mu)\kappa + \varphi(1))^2(T - t)\} \\ &\leq e^{cT} \left\{ c\kappa^2 + c \int_t^T \kappa(\mathbb{E}[|Y_s^n|^2 | \mathcal{F}_u]) ds \right\} + e^{cT} \{((1 + \mu)\kappa + \varphi(1))^2 T\}. \end{aligned}$$

Moreover, the fact that  $\kappa(\cdot)$  is concave, nondecreasing, and vanishes at the origin ( $\kappa(0) = 0$ ) implies that its growth is at most linear. In other words, we can find a strictly positive constant  $A$  satisfying  $\kappa(x) \leq A(x + 1)$  for all  $x \geq 0$ . An application of Gronwall's inequality then leads to

$$\begin{aligned} \mathbb{E} \left[ \sup_{0 \leq s \leq r} |Y_s^n|^2 + \int_t^T |Z_s^n|^2 ds | \mathcal{F}_u \right] &\leq e^{cT} \left\{ c\kappa^2 + cA \int_t^T (\mathbb{E}[|Y_s^n|^2 | \mathcal{F}_u] + 1) ds + ((1 + \mu)\kappa + \varphi(1))^2 T \right\} \\ &\leq e^{cT} \left\{ c\kappa^2 + cA \int_t^T (\mathbb{E}[|Y_s^n|^2 + \int_t^T |Z_r^n|^2 dr | \mathcal{F}_u]) ds \right\} \\ &\quad + e^{cT} \left\{ cA \int_t^T ds + ((1 + \mu)\kappa + \varphi(1))^2 T \right\}. \end{aligned}$$

On note  $f(t) = \mathbb{E} \left[ \sup_{0 \leq s \leq r} |Y_s^n|^2 + \int_t^T |Z_s^n|^2 ds | \mathcal{F}_t \right]$ . Then, we have

$$\begin{aligned} f(t) &\leq e^{cT} \left\{ c\kappa^2 + cA(T-t) + ((1+\mu)\kappa + \varphi(1))^2 T + cA \int_t^T f(s) ds \right\} \\ &\leq \{c\kappa^2 + cAT + ((1+\mu)\kappa + \varphi(1))^2 T\} e^{cAT} =: r^2. \end{aligned}$$

By setting  $u = t$  in the preceding inequality, we obtain that for all  $n \geq 1$ ,

$$\forall t \in [0, T], \quad |Y_t^n| \leq r \text{ and } \mathbb{E} \left[ \int_0^T |Z_s^n|^2 ds \right] \leq r^2. \quad (3.15)$$

Combining (3.13) and (3.15), we deduce that the pair  $(Y_t^n, Z_t^n)_{t \in [0, T]}$  satisfies the following BSDE:

$$Y_t^n = \xi + \int_t^T g_n(s, Y_s^n, V_s) ds - \int_t^T Z_s^n dB_s, \quad t \in [0, T]. \quad (3.16)$$

We now proceed to demonstrate that the sequence of processes  $\left\{ (Y_t^n, Z_t^n)_{t \in [0, T]} \right\}_{n=1}^\infty$  constitutes a Cauchy sequence within the product space  $\mathcal{S}(0, T; \mathbb{R}^k) \times \mathcal{M}^2(0, T; \mathbb{R}^{k \times d})$ . To this end, for any integers  $n, m \geq 1$ , we introduce the notations  $\widehat{Y}^{n,m} = Y^n - Y^m$  and  $\widehat{Z}^{n,m} = Z^n - Z^m$  to represent the respective differences. Thus, we have

$$\widehat{Y}_t^{n,m} = \int_t^T \widehat{g}^{n,m}(s, \widehat{Y}_s^{n,m}, V_s) ds - \int_t^T \widehat{Z}_s^{n,m} dB_s, \quad t \in [0, T]. \quad (3.17)$$

In this framework, the driver increment  $\widehat{g}^{n,m}(s, Y, V_s) = g_n(s, Y + Y_s^m, V_s) - g_m(s, Y_s^m, V_s)$  for any  $Y \in \mathbb{R}^k$ . By leveraging the result established in (3.14) and applying the Cauchy-Schwarz inequality  $\langle X, Y \rangle \leq |X| |Y|$ , it follows that for every  $Y \in \mathbb{R}^k$ , the following holds true  $d\mathbb{P} \times dt - a.e.$ ,

$$\begin{aligned} \langle Y, \widehat{g}^{n,m}(t, Y, V_t) \rangle &= \langle Y, g_n(t, Y + Y_t^m, V_t) - g_m(t, Y_t^m, V_t) \rangle \\ &= \langle Y, g_n(t, Y + Y_t^m, V_t) - g_m(t, Y_t^m, V_t) - g_n(t, Y_t^m, V_t) + g_n(t, Y_t^m, V_t) \rangle \\ &= \langle Y, g_n(t, Y + Y_t^m, V_t) - g_n(t, Y_t^m, V_t) \rangle + \langle Y, g_n(t, Y_t^m, V_t) - g_m(t, Y_t^m, V_t) \rangle \\ &\leq \kappa (|Y|^2) + |Y| |g_n(t, Y_t^m, V_t) - g_m(t, Y_t^m, V_t)|. \end{aligned}$$

Consequently, the generator  $\widehat{g}^{n,m}(t, Y, V_t)$  associated with BSDE (3.17) complies with the requirements of assumption **(A)** by identifying  $\psi(u) = \kappa(u)$ ,  $\lambda = 0$ , and

$$\phi_t = |g_n(t, Y_t^m, V_t) - g_m(t, Y_t^m, V_t)|.$$

By invoking proposition 3.1 with the parameters  $u = 0$  and  $\theta = 1$  we can establish the existence of a positive constant  $c$ , which depends solely on  $T$  such that for every  $t \in [0, T]$ ,

$$\begin{aligned} & \mathbb{E} \left[ \sup_{r \in [t, T]} \left| \widehat{Y}_r^{n,m} \right|^2 \right] + \mathbb{E} \left[ \int_t^T \left| \widehat{Z}_s^{n,m} \right|^2 ds \right] \\ & \leq c \int_t^T \kappa \left( \mathbb{E} \left[ \sup_{r \in [s, T]} \left| \widehat{Y}_r^{n,m} \right|^2 \right] \right) ds + c \mathbb{E} \left[ \int_0^T |g_n(s, Y_s^m, V_s) - g_m(s, Y_s^m, V_s)|^2 ds \right]. \end{aligned} \quad (3.18)$$

Conversely, (3.10) implies that for all integers  $n, m \geq 1$ , every  $s \in [0, T]$ , and  $Y \in \mathbb{R}^k$ , it holds  $d\mathbb{P} \times dt - a.e.$ , that:

$$\begin{aligned} |g_n(s, Y_s^m, V_s) - g_m(s, Y_s^m, V_s)| &= \left| \int_{\{u: |u| \leq 1\}} \left( g\left(s, Y_s^m - \frac{u}{n}, V_s\right) - g\left(s, Y_s^m - \frac{u}{m}, V_s\right) \right) \psi(u) du \right| \\ &\leq \int_{\{u: |u| \leq 1\}} \left| g\left(s, Y_s^m - \frac{u}{n}, V_s\right) - g\left(s, Y_s^m - \frac{u}{m}, V_s\right) \right| \psi(u) du. \end{aligned}$$

In light of assumption **(H2)**, the estimate (3.15), and the uniform continuity theorem, it is straightforward to see that  $\lim_{n,m \rightarrow \infty} |g(s, Y_s^m - \frac{u}{n}, V_s) - g(s, Y_s^m - \frac{u}{m}, V_s)| = 0$  holds  $d\mathbb{P} \times dt - a.e.$ , for any  $u \in \mathbb{R}^k$ . Furthermore, applying (3.11) alongside (3.15) ensures that for all  $s \in [0, T]$ ,  $n, m \geq 1$ , and  $u \in \mathbb{R}^k$  satisfying  $|u| \leq 1$ ,

$$\begin{aligned} \left| g\left(s, Y_s^m - \frac{u}{n}, V_s\right) - g\left(s, Y_s^m - \frac{u}{m}, V_s\right) \right| &\leq \left| g\left(s, Y_s^m - \frac{u}{n}, V_s\right) \right| + \left| g\left(s, Y_s^m - \frac{u}{m}, V_s\right) \right| \\ &\leq \kappa + \varphi\left(\left|Y_s^m - \frac{u}{n}\right|\right) + \mu\kappa + \kappa + \varphi\left(\left|Y_s^m - \frac{u}{m}\right|\right) + \mu\kappa \\ &= 2(\kappa + \mu\kappa) + \varphi\left(\left|r + \frac{u}{n}\right|\right) + \varphi\left(\left|r + \frac{u}{m}\right|\right) \\ &\leq 2(\kappa + \mu\kappa + \varphi(r + 1)). \end{aligned}$$

Then, we obtain

$$\begin{aligned}
 |g_n(s, Y_s^m, V_s) - g_m(s, Y_s^m, V_s)| &\leq 2 \int_{\{u: |u| \leq 1\}} (\kappa + \mu\kappa + \varphi(r+1)) \psi(u) du \\
 &= 2(\kappa + \mu\kappa + \varphi(r+1)) \int_{\{u: |u| \leq 1\}} \psi(u) du \\
 &:= 2(\kappa + \mu\kappa + \varphi(r+1)).
 \end{aligned}$$

Consequently, two successive applications of Lebesgue's dominated convergence theorem lead to the following:

$$\lim_{n,m \rightarrow \infty} \mathbb{E} \left[ \int_0^T |g_n(s, Y_s^m, V_s) - g_m(s, Y_s^m, V_s)|^2 ds \right] = 0. \quad (3.19)$$

Next, for any  $t \in [0, T]$ , let us define

$$h^{n,m}(t) := \mathbb{E} \left[ \sup_{s \in [t, T]} |\widehat{Y}_s^{n,m}|^2 \right] + \mathbb{E} \left[ \int_t^T |\widehat{Z}_s^{n,m}|^2 ds \right] \text{ and } h(t) = \lim_{n,m \rightarrow \infty} \sup h^{n,m}(t).$$

Recalling the estimate (3.18) and using the non-decreasing property of  $\kappa(\cdot)$ , we can inject the positive integral of  $Z$  inside  $\kappa$ . Denoting this sum by  $h^{n,m}(\cdot)$ , we obtain:

$$\begin{aligned}
 &\mathbb{E} \left[ \sup_{r \in [t, T]} |\widehat{Y}_r^{n,m}|^2 \right] + \mathbb{E} \left[ \int_t^T |\widehat{Z}_s^{n,m}|^2 ds \right] \\
 &\leq c \int_t^T \kappa \left( \mathbb{E} \left[ \sup_{r \in [s, T]} |\widehat{Y}_r^{n,m}|^2 \right] \right) ds + c \mathbb{E} \left[ \int_0^T |g_n(s, Y_s^m, V_s) - g_m(s, Y_s^m, V_s)|^2 ds \right] \\
 &\leq c \int_t^T \kappa \left( \mathbb{E} \left[ \sup_{r \in [s, T]} |\widehat{Y}_r^{n,m}|^2 \right] \right) ds + c \int_t^T \kappa \left( \mathbb{E} \left[ \int_s^T |\widehat{Z}_u^{n,m}|^2 du \right] \right) ds \\
 &+ c \mathbb{E} \left[ \int_0^T |g_n(s, Y_s^m, V_s) - g_m(s, Y_s^m, V_s)|^2 ds \right].
 \end{aligned}$$

Now, applying the  $\lim_{n,m \rightarrow \infty} \sup$  to (3.18) and taking into account relation (3.19), the second term vanishes, which leaves us with:

$$\begin{aligned} & \lim_{n,m \rightarrow \infty} \sup \mathbb{E} \left( \sup_{r \in [t, T]} |\widehat{Y}_r^{n,m}|^2 + \int_t^T |\widehat{Z}_s^{n,m}|^2 ds \right) \\ & \leq c \lim_{n,m \rightarrow \infty} \sup \int_t^T \kappa \left( \mathbb{E} \left[ \sup_{r \in [s, T]} |\widehat{Y}_r^{n,m}|^2 + \int_s^T |\widehat{Z}_u^{n,m}|^2 du \right] \right) ds, \end{aligned}$$

this last one allows us to write the following

$$\lim_{n,m \rightarrow \infty} \sup h^{n,m}(t) \leq c \lim_{n,m \rightarrow \infty} \sup \int_t^T \kappa(h^{n,m}(s)) ds.$$

Furthermore, the function  $h(t)$  is properly defined by (3.15). By means of Fatou's lemma, combined with the fact that  $\kappa(\cdot)$  is continuous and non-decreasing, we obtain the following integral bound for each  $t \in [0, T]$ :

$$h(t) \leq c \int_t^T \kappa \left( \lim_{n,m \rightarrow \infty} \sup h^{n,m}(s) \right) ds \leq c \int_t^T \kappa(h(s)) ds.$$

Consequently, in light of the condition  $\int_0^+ \kappa^{-1}(u) du = +\infty$ , Bihari's inequality dictates that  $h(t) \leq 0$ . Coupled with the obvious positivity  $h(t) \geq 0$ , we conclude that  $h(t) = 0$  for all  $t$ . Thus, at  $t = 0$ , we have:

$$\lim_{n,m \rightarrow \infty} \left\{ \mathbb{E} \left[ \sup_{r \in [0, T]} |\widehat{Y}_r^{n,m}|^2 \right] + \mathbb{E} \left[ \int_0^T |\widehat{Z}_s^{n,m}|^2 ds \right] \right\} = 0.$$

This implies that the sequence of processes  $\left\{ (Y_t^n, Z_t^n)_{t \in [0, T]} \right\}_{n \geq 1}$  forms a Cauchy sequence within the product space  $\mathcal{S}^2(0, T; \mathbb{R}^k) \times \mathcal{M}^2(0, T; \mathbb{R}^{k \times d})$ .

To conclude the proof of this lemma, we denote  $(Y_t, Z_t)_{t \in [0, T]}$  as the limit of the convergent sequence  $\left\{ (Y_t^n, Z_t^n)_{t \in [0, T]} \right\}_{n=1}^\infty$  within the product space  $\mathcal{S}^2(0, T; \mathbb{R}^k) \times \mathcal{M}^2(0, T; \mathbb{R}^{k \times d})$ . By taking the limit as  $n \rightarrow \infty$  in the sense of uniform convergence in probability (ucp) for equation (3.16), and relying on Lebesgue's dominated convergence theorem alongside assumption (H2) and estimates (3.10) – (3.12) and (3.15), we verify that the pair  $(Y_t, Z_t)_{t \in [0, T]}$  is indeed

the unique solution to BSDE (3.8). This completes the proof of the lemma 3.1. ■

**Lemma 3.2** *Let  $V \in \mathcal{M}^2(0, T; \mathbb{R}^{k \times d})$  and  $\xi \in \mathbb{L}^2(\mathcal{F}_T; \mathbb{R}^k)$ . We assume that there exists a constant  $\kappa > 0$  such that*

$$d\mathbb{P} - a.s., |\xi| \leq \kappa \text{ and } d\mathbb{P} \times dt - a.e., |g(t, 0, 0)| \leq \kappa, \quad (3.20)$$

then the BSDE (3.8) has a unique solution.

**Proof.** Let us introduce the truncation function  $q_n(Z) = \frac{Zn}{(|Z| \vee n)}$ , for any integer  $n \geq 1$  and  $Z \in \mathbb{R}^{k \times d}$ . It is easy to observe that this function satisfies the bound  $|q_n(Z)| \leq |Z| \wedge n$ . Relying on the result established in the previous lemma 3.1, we can guarantee that for every  $n \geq 1$ , there exists a solution pair  $(Y_t^n, Z_t^n)_{t \in [0, T]}$  satisfying the subsequent BSDE:

$$Y_t^n = \xi + \int_t^T g(s, Y_s^n, q_n(V_s)) ds - \int_t^T Z_s^n dB_s, \quad t \in [0, T]. \quad (3.21)$$

We now proceed to demonstrate that the sequence of processes  $\left\{ (Y_t^n, Z_t^n)_{t \in [0, T]} \right\}_{n=1}^\infty$  constitutes a Cauchy sequence within the product space  $\mathcal{S}^2(0, T; \mathbb{R}^k) \times \mathcal{M}^2(0, T; \mathbb{R}^{k \times d})$ . To this end, for any integers  $n, m \geq 1$ , we introduce the notations  $\widehat{Y}^{n,m} = Y^n - Y^m$  and  $\widehat{Z}^{n,m} = Z^n - Z^m$  to represent the respective differences. Thus, we have

$$\widehat{Y}_t^{n,m} = \int_t^T \widehat{g}^{n,m}(s, \widehat{Y}_s^{n,m}, V_s) ds - \int_t^T \widehat{Z}_s^{n,m} dB_s, \quad t \in [0, T]. \quad (3.22)$$

In this framework, the driver increment

$$\widehat{g}^{n,m}(s, Y, V_s) = g(s, Y + Y_s^n, q_n(V_s)) - g(s, Y_s^m, q_m(V_s)) \text{ for any } Y \in \mathbb{R}^k.$$

By leveraging hypotheses **(H1)** and **(H4)** and applying the cauchy-schwarz inequality  $\langle X, Y \rangle \leq |X| |Y|$ , it follows that for every  $Y \in \mathbb{R}^k$ , the following holds true  $d\mathbb{P} \times dt - a.e.$ ,

$$\begin{aligned}
 & \langle Y, \widehat{g}^{n,m}(t, Y, V_t) \rangle \\
 &= \langle Y, g(t, Y + Y_t^m, q_n(V_t)) - g(t, Y_t^m, q_m(V_t)) \rangle \\
 &= \langle Y, g(t, Y + Y_t^m, q_n(V_t)) - g(t, Y_t^m, q_n(V_t)) - g(t, Y_t^m, q_n(V_t)) + g(t, Y_t^m, q_n(V_t)) \rangle \\
 &= \langle Y, g(t, Y + Y_t^m, q_n(V_t)) - g(t, Y_t^m, q_n(V_t)) \rangle + \langle Y, g(t, Y_t^m, q_n(V_t)) - g(t, Y_t^m, q_m(V_t)) \rangle \\
 &\leq \kappa (|Y|^2) + |Y| |g(t, Y_t^m, q_n(V_t)) - g(t, Y_t^m, q_m(V_t))| \\
 &\leq \kappa (|Y|^2) + \mu |Y| |q_n(V_t) - q_m(V_t)|.
 \end{aligned}$$

Consequently, the generator  $\widehat{g}^{n,m}(t, Y, V_t)$  associated with BSDE [\(3.22\)](#) complies with the requirements of assumption **(A)** by identifying  $\psi(u) = \kappa(u)$ ,  $\lambda = 0$  and  $\phi_t = \mu |q_n(V_t) - q_m(V_t)|$ . By invoking proposition [3.1](#) with the parameters  $u = 0$  and  $\theta = 1$ , we can establish the existence of a positive constant  $c > 0$ , which depends solely on  $T$  such that for every  $t \in [0, T]$ ,

$$\begin{aligned}
 \mathbb{E} \left[ \sup_{r \in [t, T]} |\widehat{Y}_r^{n,m}|^2 + \int_t^T |\widehat{Z}_s^{n,m}|^2 ds \right] &\leq c \int_t^T \kappa \left( \mathbb{E} \left[ \sup_{r \in [s, T]} |\widehat{Y}_r^{n,m}|^2 \right] \right) ds \\
 &+ c\mu^2 \mathbb{E} \left[ \int_0^T |q_n(V_s) - q_m(V_s)|^2 ds \right]. \tag{3.23}
 \end{aligned}$$

Taking into account the linear growth bound  $\kappa(x) \leq A(x+1)$  valid for all  $x \geq 0$ , we have the following estimate for every  $t \in [0, T]$  and integers  $n, m \geq 1$ ,

$$\begin{aligned}
& \mathbb{E} \left[ \sup_{r \in [t, T]} |\widehat{Y}_r^{n, m}|^2 + \int_t^T |\widehat{Z}_s^{n, m}|^2 ds \right] \\
& \leq c \int_t^T \kappa \left( \mathbb{E} \left[ \sup_{r \in [s, T]} |\widehat{Y}_r^{n, m}|^2 \right] \right) ds + c\mu^2 \mathbb{E} \left[ \int_0^T |q_n(V_s) - q_m(V_s)|^2 ds \right] \\
& \leq c \int_t^T \left( A \mathbb{E} \left[ \sup_{r \in [s, T]} |\widehat{Y}_r^{n, m}|^2 \right] + A \right) ds + c\mu^2 \mathbb{E} \left[ \int_0^T |q_n(V_s) - q_m(V_s)|^2 ds \right] \\
& \leq cA \int_t^T \mathbb{E} \left[ \sup_{r \in [s, T]} |\widehat{Y}_r^{n, m}|^2 \right] ds + cA(T-t) + c\mu^2 \mathbb{E} \left[ \int_0^T (|q_n(V_s)|^2 + |q_m(V_s)|^2) ds \right] \\
& \leq cA \int_t^T \mathbb{E} \left[ \sup_{r \in [s, T]} |\widehat{Y}_r^{n, m}|^2 \right] ds + cAT + 2c\mu^2 \mathbb{E} \left[ \int_0^T |V_s|^2 ds \right] \\
& \leq cA \int_t^T \mathbb{E} \left[ \sup_{r \in [s, T]} |\widehat{Y}_r^{n, m}|^2 + \int_s^T |\widehat{Z}_u^{n, m}|^2 du \right] ds + cAT + 2c\mu^2 \mathbb{E} \left[ \int_0^T |V_s|^2 ds \right].
\end{aligned}$$

An application of Gronwall's inequality leads to the following estimate for every  $t \in [0, T]$  and integers  $n, m \geq 1$ ,

$$\begin{aligned}
\mathbb{E} \left[ \sup_{r \in [t, T]} |\widehat{Y}_r^{n, m}|^2 + \int_t^T |\widehat{Z}_s^{n, m}|^2 ds \right] & \leq \left( cAT + 2c\mu^2 \mathbb{E} \left[ \int_0^T |V_s|^2 ds \right] \right) \cdot e^{cA \int_t^T ds} \\
& \leq \left( cAT + 2c\mu^2 \mathbb{E} \left[ \int_0^T |V_s|^2 ds \right] \right) \cdot e^{cAT}.
\end{aligned}$$

The inequality [\(3.23\)](#) allows us to write the following

$$\begin{aligned}
& \mathbb{E} \left[ \sup_{r \in [t, T]} |\widehat{Y}_r^{n, m}|^2 + \int_t^T |\widehat{Z}_s^{n, m}|^2 ds \right] \\
& \leq c \int_t^T \kappa \left( \mathbb{E} \left[ \sup_{r \in [s, T]} |\widehat{Y}_r^{n, m}|^2 \right] \right) ds + c\mu^2 \lim_{n, m \rightarrow \infty} \sup \mathbb{E} \left[ \int_0^T |q_n(V_s) - q_m(V_s)|^2 ds \right] \\
& \leq c \int_t^T \kappa \left( \mathbb{E} \left[ \sup_{r \in [s, T]} |\widehat{Y}_r^{n, m}|^2 + \int_s^T |\widehat{Z}_u^{n, m}|^2 du \right] \right) ds + c\mu^2 \mathbb{E} \left[ \int_0^T |q_n(V_s) - q_m(V_s)|^2 ds \right]
\end{aligned}$$

Now, applying the  $\lim_{n,m \rightarrow \infty} \sup$  to inequality (3.23), and noting that the truncation difference  $|q_n(V_s) - q_m(V_s)|$  vanishes almost everywhere, then, we obtain

$$\begin{aligned} & \lim_{n,m \rightarrow \infty} \sup \mathbb{E} \left[ \sup_{r \in [t, T]} |\widehat{Y}_r^{n,m}|^2 + \int_t^T |\widehat{Z}_s^{n,m}|^2 ds \right] \\ & \leq c \lim_{n,m \rightarrow \infty} \sup \int_t^T \kappa \left( \mathbb{E} \left[ \sup_{r \in [s, T]} |\widehat{Y}_r^{n,m}|^2 + \int_s^T |\widehat{Z}_u^{n,m}|^2 du \right] \right) ds \\ & \Leftrightarrow \lim_{n,m \rightarrow \infty} \sup h^{n,m}(t) \leq c \lim_{n,m \rightarrow \infty} \sup \int_t^T \kappa(h^{n,m}(s)) ds. \end{aligned}$$

By means of Fatou's lemma, combined with the fact that  $\kappa(\cdot)$  is continuous and non-decreasing, we can pass the limit inside the integral. This yields the following bound for each  $t \in [0, T]$  :

$$h(t) \leq c \int_t^T \kappa \left( \lim_{n,m \rightarrow \infty} \sup h^{n,m}(s) \right) ds \leq c \int_t^T \kappa(h(s)) ds.$$

Consequently, in light of the condition  $\int_{0+} \kappa^{-1}(u) du = +\infty$ , Bihari's inequality dictates that  $h(t) \leq 0$ . Coupled with the obvious positivity  $h(t) \geq 0$ , we conclude that  $h(t) = 0$  for all  $t$ . Thus, evaluating at  $t = 0$ , we have:

$$\lim_{n,m \rightarrow \infty} \left\{ \mathbb{E} \left[ \sup_{r \in [0, T]} |\widehat{Y}_r^{n,m}|^2 \right] + \mathbb{E} \left[ \int_0^T |\widehat{Z}_s^{n,m}|^2 ds \right] \right\} = 0.$$

This rigorously demonstrates that the sequence of processes  $\left\{ (Y_t^n, Z_t^n)_{t \in [0, T]} \right\}_{n \geq 1}$  forms a Cauchy sequence within the product space  $\mathcal{S}^2(0, T; \mathbb{R}^k) \times \mathcal{M}^2(0, T; \mathbb{R}^{k \times d})$ . To conclude the proof of this lemma, we denote  $(Y_t, Z_t)_{t \in [0, T]}$  as the limit of the convergent sequence  $\left\{ (Y_t^n, Z_t^n)_{t \in [0, T]} \right\}_{n=1}^{\infty}$  within the product space  $\mathcal{S}^2(0, T; \mathbb{R}^k) \times \mathcal{M}^2(0, T; \mathbb{R}^{k \times d})$ . By taking the limit as  $n \rightarrow \infty$  in the sense of uniform convergence in probability (ucp) for equation (3.21), and relying on Lebesgue's dominated convergence theorem alongside assumption (H2)–(H4), we verify that the pair  $(Y_t, Z_t)_{t \in [0, T]}$  is indeed the unique solution to BSDE (3.8). This completes the proof of the lemma. ■

**Lemma 3.3** *Let  $\xi \in \mathbb{L}^2(\mathcal{F}_T; \mathbb{R}^k)$ , if condition (3.20) holds, then there exists a unique solu-*

tion to the BSDE (3.8).

**Proof.** Building upon the conclusion of the preceding lemma 3.2, we construct the Picard iterative sequence. For any integer  $n \geq 1$ , this sequence is defined recursively by the following system:

$$\begin{cases} Y_t^n = \xi + \int_t^T g(s, Y_s^n, Z_s^{n-1}) ds - \int_t^T Z_s^n dB_s, & t \in [0, T], \\ (Y_t^0, Z_t^0) = (0, 0). \end{cases} \quad (3.24)$$

By (H1) and (H4), we have  $d\mathbb{P} \times dt - a.e.$ , for each  $Y \in \mathbb{R}^k$

$$\begin{aligned} \langle Y, g(t, Y, Z_t^{n-1}) \rangle &= \langle Y, g(t, Y, Z_t^{n-1}) - g(t, 0, Z_t^{n-1}) + g(t, 0, Z_t^{n-1}) \rangle \\ &= \langle Y, g(t, Y, Z_t^{n-1}) - g(t, 0, Z_t^{n-1}) \rangle + \langle Y, g(t, 0, Z_t^{n-1}) \rangle \\ &\leq \kappa (|Y|^2) + |Y| |g(t, 0, Z_t^{n-1})| \\ &\leq \kappa (|Y|^2) + |Y| |g(t, 0, Z_t^{n-1}) - g(t, 0, 0) + g(t, 0, 0)| \\ &\leq \kappa (|Y|^2) + |Y| (|g(t, 0, Z_t^{n-1}) - g(t, 0, 0)| + |g(t, 0, 0)|) \\ &\leq \kappa (|Y|^2) + |Y| (\mu |Z_t^{n-1}| + |g(t, 0, 0)|). \end{aligned}$$

Consequently, the generator associated with BSDE (3.24) fulfills condition (A) by setting  $\psi(u) = \kappa(u)$ ,  $\lambda = 0$  and  $\phi_t = \mu |Z_t^{n-1}| + |g(t, 0, 0)|$ . Therefore, applying proposition 3.1 with  $u = 0$  and  $\theta = 16\mu^2$ , alongside the algebraic inequality  $(x + y)^2 \leq 2x^2 + 2y^2$ , ensures the

existence of a positive constant  $c > 0$ , depending solely on  $\mu$ , such that for every  $t \in [0, T]$  :

$$\begin{aligned}
& \mathbb{E} \left[ \sup_{r \in [t, T]} |Y_r^n|^2 + \int_t^T |Z_s^n|^2 ds \right] \\
& \leq e^{c(T-t)} \left\{ c \mathbb{E} (|\xi|^2) + c \int_t^T \kappa \left( \mathbb{E} \left[ \sup_{r \in [s, T]} |Y_r^n|^2 \right] \right) ds \right\} \\
& + e^{c(T-t)} \left\{ \frac{1}{16\mu^2} \mathbb{E} \left[ \int_t^T |\mu |Z_s^{n-1}| + |g(s, 0, 0)|^2 ds \right] \right\} \\
& \leq e^{c(T-t)} \left\{ c \mathbb{E} (|\xi|^2) + c \int_t^T \kappa \left( \mathbb{E} \left[ \sup_{r \in [s, T]} |Y_r^n|^2 \right] \right) ds \right\} \\
& + e^{c(T-t)} \left\{ \frac{1}{16\mu^2} \left[ 2\mu^2 \mathbb{E} \left( \int_t^T |Z_s^{n-1}|^2 ds \right) \right] \right\} + e^{c(T-t)} \left\{ \frac{1}{16\mu^2} \left[ 2 \mathbb{E} \left( \int_t^T |g(s, 0, 0)|^2 ds \right) \right] \right\} \\
& \leq e^{c(T-t)} M(t) + ce^{c(T-t)} \int_t^T \kappa \left( \mathbb{E} \left[ \sup_{r \in [s, T]} |Y_r^n|^2 \right] \right) ds + e^{c(T-t)} \frac{1}{8} \mathbb{E} \left[ \int_t^T |Z_s^{n-1}|^2 ds \right],
\end{aligned}$$

where, in view of condition [\(3.20\)](#),

$$\begin{aligned}
M(t) &= c \mathbb{E} (|\xi|^2) + \frac{1}{8\mu^2} \mathbb{E} \left[ \int_t^T |g(s, 0, 0)|^2 ds \right] \\
&\leq c\kappa^2 + \frac{1}{8\mu^2} \kappa^2 \mathbb{E} \left[ \int_t^T ds \right] \leq c\kappa^2 + \frac{\kappa^2 T}{8\mu^2} =: M.
\end{aligned}$$

We introduce the time threshold  $T_1 = \max \left\{ T - \frac{\ln 2}{c}, T - \frac{\ln 2}{2cA}, 0 \right\}$ . Consequently, whenever  $t \in [T_1, T]$ , the specific choice of  $T_1$  ensures the validity of the exponential bounds  $e^{c(T-t)} \leq 2$  and  $e^{2cA(T-t)} \leq 2$ . Furthermore, we obtain

$$\mathbb{E} \left[ \sup_{r \in [t, T]} |Y_r^n|^2 + \int_t^T |Z_s^n|^2 ds \right] \leq 2M + 2c \int_t^T \kappa \left( \mathbb{E} \left[ \sup_{r \in [s, T]} |Y_r^n|^2 \right] \right) ds + \frac{1}{4} \mathbb{E} \left[ \int_t^T |Z_s^{n-1}|^2 ds \right].$$

Additionally, keeping in mind the linear growth condition  $\kappa(x) = A(x+1)$  valid for all  $x \geq 0$ , an application of Gronwall's lemma on the interval  $[T_1, T]$  leads to the following

estimate

$$\begin{aligned}
& \mathbb{E} \left[ \sup_{r \in [t, T]} |Y_r^n|^2 + \int_t^T |Z_s^n|^2 ds \right] \\
& \leq 2M + 2c \int_t^T \kappa \left( \mathbb{E} \left[ \sup_{r \in [s, T]} |Y_r^n|^2 \right] \right) ds + \frac{1}{4} \mathbb{E} \left[ \int_t^T |Z_s^{n-1}|^2 ds \right] \\
& \leq 2M + 2c \int_t^T \left( A \mathbb{E} \left[ \sup_{r \in [s, T]} |Y_r^n|^2 \right] + A \right) ds + \frac{1}{4} \mathbb{E} \left[ \int_t^T |Z_s^{n-1}|^2 ds \right] \\
& \leq 2M + 2cA \int_t^T \mathbb{E} \left[ \sup_{r \in [s, T]} |Y_r^n|^2 \right] ds + 2cA \int_t^T ds + \frac{1}{4} \mathbb{E} \left[ \int_t^T |Z_s^{n-1}|^2 ds \right] \\
& \leq 2M + 2cA \int_t^T \mathbb{E} \left[ \sup_{r \in [s, T]} |Y_r^n|^2 + \int_s^T |Z_u^n|^2 du \right] ds + 2cA(T-t) + \frac{1}{4} \mathbb{E} \left[ \int_t^T |Z_s^{n-1}|^2 ds \right] \\
& \leq \left( 2M + 2cAT + \frac{1}{4} \mathbb{E} \left[ \int_t^T |Z_s^{n-1}|^2 ds \right] \right) e^{2cA \int_t^T ds} \\
& \leq \left( 2M + 2cAT + \frac{1}{4} \mathbb{E} \left[ \int_t^T |Z_s^{n-1}|^2 ds \right] \right) e^{2cA(T-t)} \\
& \leq \left( 4M + 4cAT + \frac{1}{2} \mathbb{E} \left[ \int_t^T |Z_s^{n-1}|^2 ds \right] \right),
\end{aligned}$$

proceeding by mathematical induction, we can conclude from the above estimate that for all integers  $n \geq 1$  and every  $t \in [T_1, T]$ , we obtain

$$\mathbb{E} \left[ \sup_{r \in [t, T]} |Y_r^n|^2 \right] + \mathbb{E} \left[ \int_t^T |Z_s^n|^2 ds \right] \leq 4M + 4cAT + \frac{1}{2} \mathbb{E} \left[ \int_t^T |Z_s^{n-1}|^2 ds \right].$$

By dropping the non-negative supremum term on the left-hand side, we deduce a recursive inequality for  $Z^n$ , which we then iterate successively down to the initial term  $n = 0$  to obtain:

$$\begin{aligned}
 & \int_t^T |Z_s^n|^2 ds \\
 & \leq 4M + 4cAT + \frac{1}{2} \mathbb{E} \left[ \int_t^T |Z_s^{n-1}|^2 ds \right] \\
 & \leq 4M + 4cAT + \frac{1}{2} \left\{ 4M + 4cAT + \frac{1}{2} \mathbb{E} \left[ \int_t^T |Z_s^{n-2}|^2 ds \right] \right\} \\
 & \leq (4M + 4cAT) + \frac{1}{2} (4M + 4cAT) + \frac{1}{4} \left\{ 4M + 4cAT + \frac{1}{2} \mathbb{E} \left[ \int_t^T |Z_s^{n-3}|^2 ds \right] \right\} \\
 & \leq (4M + 4cAT) + \frac{1}{2} (4M + 4cAT) + \frac{1}{4} (4M + 4cAT) + \frac{1}{8} \left\{ 4M + 4cAT + \frac{1}{2} \mathbb{E} \left[ \int_t^T |Z_s^{n-4}|^2 ds \right] \right\} \\
 & \vdots \\
 & \leq (4M + 4cAT) + \frac{1}{2} (4M + 4cAT) + \frac{1}{4} (4M + 4cAT) + \frac{1}{8} (4M + 4cAT) + \cdots + \frac{1}{2^n} \mathbb{E} \left[ \int_t^T |Z_s^0|^2 ds \right],
 \end{aligned}$$

recognizing the partial sum of a geometric progression, this bound can be explicitly computed as:

$$\begin{aligned}
 (4M + 4cAT) \times \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} &= (4M + 4cAT) \times \frac{1 - \left(\frac{1}{2}\right)^n}{\frac{1}{2}} \\
 &= 2(4M + 4cAT) \left(1 - \left(\frac{1}{2}\right)^n\right).
 \end{aligned}$$

Substituting this upper bound back into the initial estimate, and using the fact that  $1 - \left(\frac{1}{2}\right)^n < 1$  for all  $n \geq 1$ , we finally arrive at the uniform bound:

$$\begin{aligned}
 \mathbb{E} \left[ \sup_{r \in [t, T]} |Y_r^n|^2 \right] + \mathbb{E} \left[ \int_t^T |Z_s^n|^2 ds \right] &\leq 4M + 4cAT + \frac{1}{2} (2(4M + 4cAT)) \\
 &:= 8M + 8cAT.
 \end{aligned} \tag{3.25}$$

Subsequently, for any pair of indices  $n, m \geq 1$ , we introduce the notations  $\widehat{Y}^{n,m} = Y^n - Y^m$  and  $\widehat{Z}^{n,m} = Z^n - Z^m$  to represent the respective differences. Consequently, we obtain

$$\widehat{Y}_r^{n,m} = \int_t^T \widehat{g}^{n,m}(s, \widehat{Y}_s^{n,m}) ds - \int_t^T \widehat{Z}_s^{n,m} dB_s, \quad t \in [0, T], \quad (3.26)$$

in this framework, the driver increment  $\widehat{g}^{n,m}(s, Y) = g(s, Y + Y_s^m, Z_s^{n-1}) - g(s, Y_s^m, Z_s^{m-1})$  for any  $Y \in \mathbb{R}^k$ . By leveraging hypotheses **(H1)** and **(H4)** and applying the Cauchy-Schwarz inequality  $\langle X, Y \rangle \leq |X| |Y|$ , it follows that for every  $Y \in \mathbb{R}^k$ , the following holds true  $d\mathbb{P} \times dt - a.e.$ ,

$$\begin{aligned} \langle Y, \widehat{g}^{n,m}(t, Y) \rangle &= \langle Y, g(t, Y + Y_t^m, Z_t^{n-1}) - g(t, Y_t^m, Z_t^{m-1}) \rangle \\ &= \langle Y, g(t, Y + Y_t^m, Z_t^{n-1}) - g(t, Y_t^m, Z_t^{m-1}) - g(t, Y_t^m, Z_t^{n-1}) + g(t, Y_t^m, Z_t^{n-1}) \rangle \\ &\leq \langle Y, g(t, Y + Y_t^m, Z_t^{n-1}) - g(t, Y_t^m, Z_t^{n-1}) \rangle + \langle Y, g(t, Y_t^m, Z_t^{n-1}) - g(t, Y_t^m, Z_t^{m-1}) \rangle \\ &\leq \kappa (|Y|^2) + |Y| |g(t, Y_t^m, Z_t^{n-1}) - g(t, Y_t^m, Z_t^{m-1})| \\ &\leq \kappa (|Y|^2) + \mu |Y| \left| \widehat{Z}_t^{n-1, m-1} \right|. \end{aligned}$$

Consequently, the generator  $\widehat{g}^{n,m}(t, Y)$  associated with BSDE (3.26) complies with the requirements of assumption **(A)** by identifying  $\psi(u) = \kappa(u)$ ,  $\lambda = 0$  and  $\phi_t = \mu \left| \widehat{Z}_t^{n-1, m-1} \right|$ . By invoking proposition 3.1 with the parameters  $u = 0$  and  $\theta = 8\mu^2$ , we can establish the existence of a positive constant  $\bar{c} > 0$ , which depends solely on  $\mu$  such that for every  $t \in [0, T]$ ,

$$\begin{aligned} &\mathbb{E} \left[ \sup_{r \in [t, T]} \left| \widehat{Y}_r^{n,m} \right|^2 \right] + \mathbb{E} \left[ \int_t^T \left| \widehat{Z}_s^{n,m} \right|^2 ds \right] \\ &\leq e^{\bar{c}(T-t)} \left\{ \bar{c} \int_t^T \kappa \left( \mathbb{E} \left[ \sup_{r \in [s, T]} \left| \widehat{Y}_r^{n,m} \right|^2 \right] \right) ds \right\} + e^{\bar{c}(T-t)} \left\{ \frac{1}{8\mu^2} \mu^2 \mathbb{E} \left[ \int_t^T \left| \widehat{Z}_s^{n-1, m-1} \right|^2 ds \right] \right\} \\ &\leq \bar{c} e^{\bar{c}(T-t)} \int_t^T \kappa \left( \mathbb{E} \left[ \sup_{r \in [s, T]} \left| \widehat{Y}_r^{n,m} \right|^2 \right] \right) ds + \frac{e^{\bar{c}(T-t)}}{8} \mathbb{E} \left[ \int_t^T \left| \widehat{Z}_s^{n-1, m-1} \right|^2 ds \right]. \end{aligned}$$

Without loss of generality, we may assume that  $\bar{c} = c$ . Reintroducing the time threshold  $T_1 = \max \left\{ T - \frac{\ln 2}{c}, T - \frac{\ln 2}{2cA}, 0 \right\}$ . Consequently, whenever  $t \in [T_1, T]$ , the specific choice of  $T_1$  ensures the validity of the exponential bounds  $e^{c(T-t)} \leq 2$  and  $e^{2cA(T-t)} \leq 2$ . Furthermore,

we obtain

$$\begin{aligned} & \mathbb{E} \left[ \sup_{r \in [t, T]} \left| \widehat{Y}_r^{n, m} \right|^2 \right] + \mathbb{E} \left[ \int_t^T \left| \widehat{Z}_s^{n, m} \right|^2 ds \right] \\ & \leq 2c \int_t^T \kappa \left( \mathbb{E} \left[ \sup_{r \in [s, T]} \left| \widehat{Y}_r^{n, m} \right|^2 \right] \right) ds + \frac{1}{4} \mathbb{E} \left[ \int_t^T \left| \widehat{Z}_s^{n-1, m-1} \right|^2 ds \right]. \end{aligned}$$

Our next objective is to demonstrate that the sequence  $\{(Y_t^n, Z_t^n)_{t \in [T_1, T]}\}_{n \geq 1}$  forms a Cauchy sequence within the space  $\mathcal{S}^2(T_1, T; \mathbb{R}^k) \times \mathcal{M}^2(T_1, T; \mathbb{R}^{k \times d})$ . To this end, for any integers  $n, m \geq 1$ , we define:

$$h^{n, m}(t) = \mathbb{E} \left[ \sup_{r \in [t, T]} \left| \widehat{Y}_r^{n, m} \right|^2 \right] + \mathbb{E} \left[ \int_t^T \left| \widehat{Z}_s^{n, m} \right|^2 ds \right].$$

In view of the uniform estimate established in (3.25) and the preceding derivation, we have for every  $t \in [T_1, T]$ :

$$\begin{aligned} & \mathbb{E} \left[ \sup_{r \in [t, T]} \left| \widehat{Y}_r^{n, m} \right|^2 \right] + \mathbb{E} \left[ \int_t^T \left| \widehat{Z}_s^{n, m} \right|^2 ds \right] \\ & \leq 2c \int_t^T \kappa \left( \mathbb{E} \left[ \sup_{r \in [s, T]} \left| \widehat{Y}_r^{n, m} \right|^2 \right] \right) ds + \frac{1}{4} \mathbb{E} \left[ \int_t^T \left| \widehat{Z}_s^{n-1, m-1} \right|^2 ds \right] \\ & \leq 2c \int_t^T \kappa \left( \mathbb{E} \left[ \sup_{r \in [s, T]} \left| \widehat{Y}_r^{n, m} \right|^2 \right] \right) ds + 2c \int_t^T \kappa \left( \mathbb{E} \left[ \int_s^T \left| \widehat{Z}_r^{n, m} \right|^2 dr \right] \right) ds + \frac{1}{4} \mathbb{E} \left[ \int_t^T \left| \widehat{Z}_s^{n-1, m-1} \right|^2 ds \right]. \end{aligned}$$

Let  $h(t) = \limsup_{n, m \rightarrow \infty} h^{n, m}(t)$ . Since the sequence is uniformly bounded by (3.25),  $h(t)$  is well-defined and finite. By taking the  $\limsup_{n, m \rightarrow \infty}$  on both sides of the previous inequality it follows that:

$$\begin{aligned} & \limsup_{n, m \rightarrow \infty} \mathbb{E} \left[ \sup_{r \in [t, T]} \left| \widehat{Y}_r^{n, m} \right|^2 \right] + \mathbb{E} \left[ \int_t^T \left| \widehat{Z}_s^{n, m} \right|^2 ds \right] \\ & \leq 2c \limsup_{n, m \rightarrow \infty} \int_t^T \kappa \mathbb{E} \left[ \sup_{r \in [s, T]} \left| \widehat{Y}_r^{n, m} \right|^2 + \int_s^T \left| \widehat{Z}_u^{n, m} \right|^2 du \right] ds \\ & \quad + \frac{1}{4} \limsup_{n, m \rightarrow \infty} \mathbb{E} \left[ \int_t^T \left| \widehat{Z}_s^{n-1, m-1} \right|^2 ds \right], \end{aligned}$$

the latter is equivalent to the following

$$\lim_{n,m \rightarrow \infty} \sup h^{n,m}(t) \leq 2c \lim_{n,m \rightarrow \infty} \sup \int_t^T \kappa(h^{n,m}(s)) ds + \frac{1}{4} \lim_{n,m \rightarrow \infty} \sup h^{n-1,m-1}(t),$$

thus, we obtain

$$h(t) \leq 2c \lim_{n,m \rightarrow \infty} \sup \int_t^T \kappa(h^{n,m}(s)) ds + \frac{1}{4}h(t).$$

By virtue of Fatou's lemma and the continuity of the function  $\kappa(\cdot)$ , we can pass the limit inside the integral:

$$h(t) \leq 2c \int_t^T \kappa \left( \lim_{n,m \rightarrow \infty} \sup h^{n,m}(s) \right) ds + \frac{1}{4}h(t) \leq 2c \int_t^T \kappa(h(s)) ds + \frac{1}{4}h(t).$$

Rearranging the terms, we obtain:

$$\frac{3}{4}h(t) \leq 2c \int_t^T \kappa(h(s)) ds, \quad \forall t \in [T_1, T],$$

then, we get

$$h(t) \leq \frac{8c}{3} \int_t^T \kappa(h(s)) ds, \quad \forall t \in [T_1, T].$$

Given that  $\kappa(\cdot)$  is a non-decreasing concave function with  $\kappa(0) = 0$ , and since it satisfies the Osgood condition  $\int_{0^+} \kappa^{-1}(u) du = +\infty$  from hypothesis **(H1)**, an application of the uniqueness case of Bihari's inequality dictates that  $h(t) \leq 0$ . Coupled with the obvious positivity  $h(t) \geq 0$ , we conclude that  $h(t) = 0$  for all  $t \in [T_1, T]$ . Thus, we have

$$\lim_{n,m \rightarrow \infty} \left\{ \mathbb{E} \left[ \sup_{r \in [t, T]} |\widehat{Y}_r^{n,m}|^2 \right] + \mathbb{E} \left[ \int_t^T |\widehat{Z}_s^{n,m}|^2 ds \right] \right\} = 0.$$

This rigorously demonstrates that  $\{(Y_t^n, Z_t^n)_{t \in [T_1, T]}\}_{n \geq 1}$  is indeed a Cauchy sequence. By the completeness of the process space  $\mathcal{S}^2(T_1, T; \mathbb{R}^k) \times \mathcal{M}^2(T_1, T; \mathbb{R}^{k \times d})$ , the sequence converges to a limit  $(Y_t, Z_t)$ , which constitutes the unique solution to the BSDE over the interval  $[T_1, T]$ . By taking the limit in the ucp sense for the BSDE [\(3.24\)](#) and employing Lebesgue's dominated convergence theorem alongside assumptions **(H2)**–**(H4)**, it follows that the limit process

$(Y_t, Z_t)_{t \in [T_1, T]}$  satisfies the BSDE with parameters  $(\xi, T, g)$  over the interval  $[T_1, T]$ . The step size  $T - T_1$  is a positive constant determined solely by  $\mu$  and  $A$ . Consequently, this construction can be extended to the entire interval  $[0, T]$  by repeating the same argument over a finite number of successive sub-intervals  $[T_{i+1}, T_i]$ . ■

Now, let us address the proof of the most important results of this section, namely Theorem [3.1](#).

**Proof.** Let us approximate the generator  $g(t, Y, Z)$  and the terminal condition  $\xi$  to satisfy the bounds assumed in the preceding lemma [3.3](#). For any integer  $n \geq 1$ , we introduce the truncation function  $q_n(x) = \frac{xn}{|x|\sqrt{n}}$  for each  $x \in \mathbb{R}^k$ , and we define:

$$\xi_n := q_n(\xi) \text{ and } g_n(t, Y, Z) := g(t, Y, Z) - g(t, 0, 0) + q_n(g(t, 0, 0)). \quad (3.27)$$

Clearly, the sequence of generators  $g_n$  satisfies the assumptions of the previous lemma, and by virtue of **(H5)** and Lebesgue's dominated convergence theorem, we obtain as  $n \rightarrow \infty$ :

$$\mathbb{E} [|\xi_n - \xi_m|^2] \rightarrow 0 \text{ and } \mathbb{E} \left[ \int_0^T |q_n(g(s, 0, 0)) - g(s, 0, 0)|^2 ds \right] \rightarrow 0. \quad (3.28)$$

Relying on the result established in the previous lemma, we can guarantee that for every  $n \geq 1$ , there exists a solution pair  $(Y_t^n, Z_t^n)_{t \in [0, T]}$  satisfying the subsequent BSDE:

$$Y_t^n = \xi_n + \int_t^T g_n(s, Y_s^n, Z_s^n) ds - \int_t^T Z_s^n dB_s, \quad t \in [0, T]. \quad (3.29)$$

Subsequently, for any pair of indices  $n, m \geq 1$ , we introduce the notations  $\widehat{Y}^{n,m} = Y^n - Y^m$  and  $\widehat{Z}^{n,m} = Z^n - Z^m$  to represent the respective differences. Consequently, we obtain

$$\widehat{Y}_t^{n,m} = \xi_n - \xi_m + \int_t^T \widehat{g}^{n,m}(s, \widehat{Y}_s^{n,m}, \widehat{Z}_s^{n,m}) ds - \int_t^T \widehat{Z}_s^{n,m} dB_s, \quad t \in [0, T]. \quad (3.30)$$

In this framework, the driver increment for each pair  $(Y, Z) \in \mathbb{R}^k \times \mathbb{R}^{k \times d}$

$$\widehat{g}^{n,m}(s, Y_s, Z_s) := g_n(s, Y + Y_s^m, Z + Z_s^m) - g_m(s, Y_s^m, Z_s^m)$$

By considering the formulation provided in (3.27), leveraging hypotheses **(H1)** and **(H4)**, and applying the cauchy-schwarz inequality  $\langle X, Y \rangle \leq |X| |Y|$ , it follows that for every  $(Y, Z)$ , the following holds true  $d\mathbb{P} \times dt - a.e.$ ,

$$\begin{aligned}
 & \langle Y, \widehat{g}^{n,m}(t, Y, Z) \rangle \\
 &= \langle Y, g_n(t, Y + Y_t^m, Z + Z_t^m) - g_m(t, Y_t^m, Z_t^m) \rangle \\
 &= \langle Y, g_n(t, Y + Y_t^m, Z + Z_t^m) - g_n(t, Y_t^m, Z + Z_t^m) \rangle + \langle Y, g_n(t, Y_t^m, Z + Z_t^m) - g_m(t, Y_t^m, Z_t^m) \rangle \\
 &\leq \kappa(|Y|^2) + \langle Y, g_n(t, Y_t^m, Z + Z_t^m) - g_m(t, Y_t^m, Z_t^m) - g_n(t, Y_t^m, Z_t^m) + g_n(t, Y_t^m, Z_t^m) \rangle \\
 &= \kappa(|Y|^2) + \langle Y, g_n(t, Y_t^m, Z + Z_t^m) - g_n(t, Y_t^m, Z_t^m) \rangle + \langle Y, g_n(t, Y_t^m, Z_t^m) - g_m(t, Y_t^m, Z_t^m) \rangle \\
 &\leq \kappa(|Y|^2) + |Y| |g_n(t, Y_t^m, Z + Z_t^m) - g_n(t, Y_t^m, Z_t^m)| + |Y| |g_n(t, Y_t^m, Z_t^m) - g_m(t, Y_t^m, Z_t^m)| \\
 &\leq \kappa(|Y|^2) + \mu |Y| |Z| + |Y| |q_n(g(t, 0, 0)) - q_m(g(t, 0, 0))|.
 \end{aligned}$$

Consequently, the generator  $\widehat{g}^{n,m}(t, Y, Z)$  associated with BSDE (3.30) complies with the requirements of assumption **(A)** by identifying  $\psi(u) = \kappa(u)$ ,  $\lambda = \mu$ , and

$$\phi_t = |q_n(g(t, 0, 0)) - q_m(g(t, 0, 0))|.$$

By invoking proposition 3.1 with the parameters  $u = 0$  and  $\theta = 1$ , we can establish the existence of a positive constant  $c > 0$ , which depends solely on  $T$  and  $\mu$ , such that for every  $t \in [0, T]$  :

$$\begin{aligned}
 \mathbb{E} \left[ \sup_{r \in [t, T]} |\widehat{Y}_r^{n,m}|^2 \right] + \mathbb{E} \left[ \int_t^T |\widehat{Z}_s^{n,m}|^2 ds \right] &\leq c \mathbb{E} [|\xi_n - \xi_m|^2] + c \int_t^T \kappa \left( \mathbb{E} \left[ \sup_{r \in [s, T]} |\widehat{Y}_r^{n,m}|^2 \right] \right) ds \\
 &\quad + c \mathbb{E} \left[ \int_0^T |q_n(g(s, 0, 0)) - q_m(g(s, 0, 0))|^2 ds \right].
 \end{aligned} \tag{3.31}$$

Taking into account the linear growth bound  $\kappa(x) \leq A(x + 1)$  valid for all  $x \geq 0$ , an application of Gronwall's inequality leads to the following estimate for every  $t \in [0, T]$  and

integers  $n, m \geq 1$ ,

$$\begin{aligned}
& \mathbb{E} \left[ \sup_{r \in [t, T]} |\widehat{Y}_r^{n, m}|^2 + \int_t^T |\widehat{Z}_s^{n, m}|^2 ds \right] \\
& \leq c \mathbb{E} [|\xi_n - \xi_m|^2] + c \int_t^T \kappa \left( \mathbb{E} \left[ \sup_{r \in [s, T]} |\widehat{Y}_r^{n, m}|^2 \right] \right) ds + c \mathbb{E} \left[ \int_0^T |q_n(g(s, 0, 0)) - q_m(g(s, 0, 0))|^2 ds \right] \\
& \leq 2c \mathbb{E} [|\xi|^2] + c \int_t^T \left( A \mathbb{E} \left[ \sup_{r \in [s, T]} |\widehat{Y}_r^{n, m}|^2 \right] + A \right) ds + c \mathbb{E} \left[ \int_0^T (|q_n(g(s, 0, 0))|^2 - |q_m(g(s, 0, 0))|^2) ds \right] \\
& \leq 2c \mathbb{E} [|\xi|^2] + cA \int_t^T \mathbb{E} \left[ \sup_{r \in [s, T]} |\widehat{Y}_r^{n, m}|^2 \right] ds + cA(T-t) + 2c \mathbb{E} \left[ \int_0^T |g(s, 0, 0)|^2 ds \right] \\
& \leq 2c \mathbb{E} [|\xi|^2] + cA \int_t^T \mathbb{E} \left[ \sup_{r \in [s, T]} |\widehat{Y}_r^{n, m}|^2 + \int_s^T |\widehat{Z}_u^{n, m}|^2 du \right] ds + cAT + 2c \mathbb{E} \left[ \int_0^T |g(s, 0, 0)|^2 ds \right] \\
& \leq \left( 2c \mathbb{E} [|\xi|^2] + cAT + 2c \mathbb{E} \left[ \int_0^T |g(s, 0, 0)|^2 ds \right] \right) \cdot e^{cAT}.
\end{aligned}$$

Now, applying the  $\lim_{n, m \rightarrow \infty} \sup$  to estimate (3.31), and noting that the truncation difference  $|\xi_n - \xi_m|^2$  and  $|q_n(g(s, 0, 0)) - q_m(g(s, 0, 0))|^2$  vanish almost everywhere as  $n, m \rightarrow \infty$  due to the convergence properties established in (3.28), we are left with

$$\begin{aligned}
& \lim_{n, m \rightarrow \infty} \sup \mathbb{E} \left[ \sup_{r \in [t, T]} |\widehat{Y}_r^{n, m}|^2 + \int_t^T |\widehat{Z}_s^{n, m}|^2 ds \right] \\
& \leq c \lim_{n, m \rightarrow \infty} \sup \int_t^T \kappa \mathbb{E} \left[ \sup_{r \in [s, T]} |\widehat{Y}_r^{n, m}|^2 + \int_s^T |\widehat{Z}_u^{n, m}|^2 du \right] ds \\
& \lim_{n, m \rightarrow \infty} \sup h^{n, m}(t) \leq c \lim_{n, m \rightarrow \infty} \sup \int_t^T \kappa(h^{n, m}(s)) ds.
\end{aligned}$$

By means of Fatou's lemma, combined with the fact that  $\kappa(\cdot)$  is continuous and non-decreasing, we can pass the limit inside the integral. This yields the following bound for each  $t \in [0, T]$  :

$$\begin{aligned}
h(t) & \leq c \int_t^T \kappa \left( \lim_{n, m \rightarrow \infty} \sup h^{n, m}(s) \right) ds \\
& \leq c \int_t^T \kappa(h(s)) ds.
\end{aligned}$$

Consequently, in light of the condition  $\int_{0+} \kappa^{-1}(u) = +\infty$  from hypothesis **(H1)**, Bihari's inequality dictates that  $h(t) \leq 0$ . Coupled with the obvious positivity  $h(t) \geq 0$ , we conclude that  $h(t) = 0$  for all  $t$ . Thus, evaluating at  $t = 0$ , we have:

$$\lim_{n,m \rightarrow \infty} \left\{ \mathbb{E} \left[ \sup_{r \in [t, T]} |\widehat{Y}_r^{n,m}|^2 \right] + \mathbb{E} \left[ \int_t^T |\widehat{Z}_s^{n,m}|^2 ds \right] \right\} = 0.$$

This rigorously demonstrates that the sequence of processes  $\left\{ (Y_t^n, Z_t^n)_{t \in [0, T]} \right\}_{n \geq 1}$  forms a Cauchy sequence within the product Banach space  $\mathcal{S}^2(0, T; \mathbb{R}^k) \times \mathcal{M}^2(0, T; \mathbb{R}^{k \times d})$ . To conclude the proof of this lemma, we denote  $(Y_t, Z_t)_{t \in [0, T]}$  as the limit of the convergent sequence  $\left\{ (Y_t^n, Z_t^n)_{t \in [0, T]} \right\}_{n=1}^{\infty}$  within the product space  $\mathcal{S}^2(0, T; \mathbb{R}^k) \times \mathcal{M}^2(0, T; \mathbb{R}^{k \times d})$ . by taking the limit as  $n \rightarrow \infty$  in the sense of uniform convergence in probability (ucp) for equation [\(3.29\)](#), and relying on Lebesgue's dominated convergence theorem alongside assumptions **(H2)**–**(H4)** and estimates [\(3.28\)](#), we verify that the pair  $(Y_t, Z_t)_{t \in [0, T]}$  is indeed the unique solution to the BSDE associated with parameters  $(\xi, T, g)$ . This completes the proof of the theorem

[3.1](#) ■

# Conclusion

In this work, we studied the existence and uniqueness of solutions for BSDEs within a framework of relaxed assumptions. Unlike classical approaches that rely solely on the Lipschitz condition, our objective was to expand the validity of these results to a broader class of equations by considering the weak monotonicity condition.

We first laid the necessary theoretical foundations, covering essential concepts from stochastic analysis, including conditional expectation, martingales, and Itô's calculus. We then reviewed the classical existence and uniqueness results for BSDEs under the standard Lipschitz hypothesis, which served as a reference point for our extensions.

The core contribution of this thesis lies in analyzing BSDEs under the hypothesis of weak monotonicity and general growth. We demonstrated that the existence and uniqueness of solutions remain valid under this less constraining condition, which notably generalizes both Mao's monotonicity condition and the Osgood condition.

To achieve these findings, we leveraged a combination of powerful mathematical tools, including a priori estimates and generalized Gronwall's inequalities. These techniques were crucial in overcoming the analytical challenges posed by relaxing the Lipschitz constraint on the generator function.

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# Appendix A

## Some mathematical tools

**Definition A.1 (Convolution Product)** [3] Let  $f$  be a locally integrable function on  $\mathbb{R}^d$  and  $\phi$  be a continuous function with compact support. The convolution of  $f$  and  $\phi$ , denoted by  $f * \phi$ , is the function defined for all  $x \in \mathbb{R}^d$  by:

$$(f * \phi)(x) = \int_{\mathbb{R}^d} f(y)\phi(x - y) dy.$$

**Definition A.2** [3] A sequence of functions  $(\psi_n)_{n \in \mathbb{N}^*}$  is called a sequence of standard mollifiers if it is generated from a "prototype" function  $\psi \in C_c^\infty(\mathbb{R}^d)$  satisfying:

1.  $\psi(x) \geq 0$  for all  $x \in \mathbb{R}^d$ .
2. The support of  $\psi$  is contained in the unit ball  $\{x \in \mathbb{R}^d : |x| \leq 1\}$ .
3.  $\int_{\mathbb{R}^d} \psi(x) dx = 1$ .

The sequence is then defined as:

$$\psi_n(x) = n^d \psi(nx).$$

**Approximation by Convolution** As applied in the current work, for a given generator  $g(t, Y, V)$ , the sequence of approximated generators  $g_n$  is constructed using the convolution

product:

$$g_n(t, Y, V) = \int_{\mathbb{R}^k} g(t, u, V) \psi_n(Y - u) du.$$

By a change of variables  $u = Y - \frac{z}{n}$ , this integral is equivalent to:

$$g_n(t, Y, V) = \int_{|z| \leq 1} g\left(t, Y - \frac{z}{n}, V\right) \psi_n(z) dz.$$

This approximation ensures that  $g_n$  is locally Lipschitz with respect to  $Y$ , even if the original function  $g$  is only continuous.

**Theorem A.1 (Lebesgue's Dominated Convergence Theorem)** [9] *Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. Suppose that  $(f_n)_{n \in \mathbb{N}}$  is a sequence of measurable functions that converges pointwise to  $f$  for almost every  $w \in \Omega$ . If there exists an integrable function  $g$  such that:*

$$|f_n(w)| \leq g(w), \text{ for almost all } w \in \Omega,$$

*then the limit function  $f$  is also integrable, and we have:*

$$\lim_{n \rightarrow \infty} \int_{\Omega} |f_n - f| d\mathbb{P} = 0.$$

**Lemma A.1 (Fatou's Lemma)** [9] *Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space and  $\{X_n\}_{n \in \mathbb{N}}$  be a sequence of non-negative random variables (or measurable functions). Then:*

$$\mathbb{E} \left[ \liminf_{n \rightarrow \infty} X_n \right] \leq \liminf_{n \rightarrow \infty} \mathbb{E} [X_n].$$

*In terms of integration, it is expressed as:*

$$\int_{\Omega} \liminf_{n \rightarrow \infty} X_n(w) d\mathbb{P}(w) \leq \liminf_{n \rightarrow \infty} \int_{\Omega} X_n(w) d\mathbb{P}(w).$$

**Remark A.1 (Superior and inferior limits):** *For any sequence of real numbers  $\{a_n\}$ , we define*

1. **Limit inferior** ( $\liminf$ ): The smallest limit point of the sequence. It represents the "eventual lower bound" and is defined as:

$$\liminf_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left( \inf_{k \geq n} a_k \right).$$

2. **Limit superior** ( $\limsup$ ): The largest limit point of the sequence. It represents the "eventual upper bound" and is defined as:  $\lim_{n \rightarrow \infty} \sup a_n = \lim_{n \rightarrow \infty} \left( \sup_{k \geq n} a_k \right)$ .

### The Relationship between them:

1. **Always:**  $\lim_{n \rightarrow \infty} \inf a_n \leq \lim_{n \rightarrow \infty} \sup a_n$ .

2. **Convergence:** A sequence  $\{a_n\}$  converges to a limit  $L$  if and only if  $\lim_{n \rightarrow \infty} \inf a_n = \lim_{n \rightarrow \infty} \sup a_n = L$ .

**Lemma A.2 (Gronwall's Inequality)** [8] Let  $g: [0, T] \rightarrow \mathbb{R}^+$  be a non-negative, measurable, and Bounded function. Suppose there exist strictly positive constants  $a$  and  $b$  satisfying the following integral inequality for all  $t \in [0, T]$ :

$$g(t) \leq a + b \int_t^T g(s) ds.$$

Then, we have  $g(t) \leq ae^{b(T-t)}$  for every  $t \in [0, T]$ .

**Proof.** We begin by defining an auxiliary function

$$v(t) = \int_t^T g(s) ds.$$

From the hypothesis, we obtain:

$$g(t) \leq a + bv(t),$$

by the fundamental theorem of calculus, the derivative of  $v(t)$  with respect to  $t$  is given by  $v'(t) = -g(t)$ . Substituting this into the inequality from the hypothesis yields:

$$-v'(t) \leq a + bv(t) \Leftrightarrow v'(t) + bv(t) \geq -a.$$

To solve this inequality, we multiply both sides by the integrating factor  $e^{bt}$ :

$$e^{bt}v'(t) + be^{bt}v(t) \geq -ae^{bt},$$

and the latter is as follows

$$\frac{d}{dt}(v(t)e^{bt}) \geq -ae^{bt},$$

now, we integrate both sides from  $t$  to  $T$ :

$$\begin{aligned} \int_t^T \frac{d}{ds}(v(s)e^{bs}) ds &\geq \int_t^T -ae^{bs} ds, \\ \Leftrightarrow v(T)e^{bT} - v(t)e^{bt} &\geq -a \left[ \frac{e^{bs}}{b} \right]_t^T, \\ \Leftrightarrow v(T)e^{bT} - v(t)e^{bt} &\geq \frac{-a}{b} (e^{bT} - e^{bt}), \end{aligned}$$

from the definition of  $v(t)$ , we know that  $v(T) = \int_T^T g(s)ds = 0$ . So we have:

$$-v(t)e^{bt} \geq \frac{-a}{b} (e^{bT} - e^{bt}).$$

By multiplying the inequality by  $-1$ , and dividing by  $e^{bt}$ , we find:

$$v(t) \leq \frac{a}{b} \left( \frac{e^{bT}}{e^{bt}} - 1 \right) = \frac{a}{b} (e^{b(T-t)} - 1),$$

by substituting  $v(t)$  into the hypothesis  $g(t) \leq a + bv(t)$ , we find:

$$g(t) \leq a + b \left[ \frac{a}{b} (e^{b(T-t)} - 1) \right] = a + ae^{b(T-t)} - a = ae^{b(T-t)}.$$

■

**Lemma A.3 (Bihari's inequality)** [\[1\]](#) *Let  $u(t)$  be a non-negative continuous function defined on the interval  $[0, T]$ , and let  $a \geq 0$  be a constant. Let  $f(t)$  be a non-negative integrable function on  $[0, T]$ . Suppose that  $\kappa : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a continuous and non-decreasing function such that  $\kappa(0) = 0$  and  $\kappa(r) > 0$  for  $r > 0$ . If  $u(t)$  satisfies the following integral*

inequality:

$$u(t) \leq a + \int_t^T f(s)\kappa(u(s))ds, \quad t \in [0, T]$$

then, the following results hold:

1. If  $a > 0$ , then  $u(t) \leq G^{-1}(G(a) + \int_t^T f(s)ds)$ , where  $G(x) = \int_{x_0}^x \frac{1}{\kappa(r)}dr$ .
2. If  $a = 0$  and the function  $\kappa$  satisfies the condition:

$$\int_{0^+} \frac{1}{\kappa(r)}dr = +\infty$$

then  $u(t) = 0$  for all  $t \in [0, T]$ .

**Theorem A.2 (Hölder's inequality)** [3] For  $X \in \mathbb{L}^p$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , we have

$$\mathbb{E}[|XY|] \leq \|X\|_p \|Y\|_q.$$

**Theorem A.3 (Stochastic Fubini theorem)** [13] Under suitable integrability conditions, the order of expectation and time integration can be interchanged:

$$\mathbb{E} \left[ \int_0^T f(s) ds \right] = \int_0^T \mathbb{E}[f(s)] ds.$$

**Theorem A.4 (Banach fixed point theorem)** [3] Let  $(E, \|\cdot\|_E)$  be a non-empty Banach space. A mapping  $T : E \rightarrow E$  is said to be a contraction if there exists a constant  $\kappa \in [0, 1)$  such that for all  $x, y \in E$ :

$$\|T(x) - T(y)\|_E \leq \kappa \|x - y\|_E.$$

Under this condition, the mapping  $T$  possesses a unique fixed point  $x^* \in E$ , such that  $T(x^*) = x^*$ . This fixed point is the unique limit of the sequence defined by  $x_{n+1} = T(x_n)$ , for any arbitrary starting point  $x_0 \in E$ .

**Remark A.2 (Application of fixed point theorem to BSDEs)** In the proof of existence

and uniqueness of solutions to BSDEs, this theorem is applied to the picard iteration sequence  $(Y^{(n)}, Z^{(n)})$ . By showing that the map associated with the BSDE generator is a contraction on the Banach space  $\mathcal{S}^2 \times \mathcal{M}^2$ , we guarantee the convergence of the iterations to the unique solution  $(Y, Z)$ .

**Lemma A.4 (Young's inequality)** [8] Let  $a$  and  $b$  be non-negative real numbers. For any positive parameter  $\varepsilon > 0$ , the following relationship holds:

$$2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2.$$

**Lemma A.5 (Elementary power inequality)** [8] The subsequent inequality provides a useful bound for the  $p$ -th power of a sum of two real numbers  $a$  and  $b$ . For any  $p \geq 1$ , the following holds:

$$|a + b|^p \leq 2^{p-1} (|a|^p + |b|^p).$$

**Theorem A.5 (Cauchy-Schwarz inequality)** [8] Consider two vectors  $x$  and  $y$  within the  $d$ -dimensional euclidean space  $\mathbb{R}^d$ . The absolute value of their inner product is consistently Bounded by the product of their respective norms:

$$|\langle x, y \rangle| \leq |x| \cdot |y|.$$

**Theorem A.6 (Doob's  $\mathbb{L}^2$  maximal inequality)** [13] Let  $M$  be a continuous martingale. A fundamental result in martingale theory states that the expected value of the squared running supremum of  $M$  over the interval  $[0, T]$  is dominated by the second moment of its terminal value:

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |M_t|^2 \right] \leq 4 \mathbb{E} [|M_T|^2].$$

**Theorem A.7 (Burkholder-Davis-Gundy inequalities)** [13] For any continuous martingale  $M$  and any power  $p > 0$ , there exist universal positive constants  $c_p$  and  $C_p$  such that

*the following two-sided estimate holds:*

$$c_p \mathbb{E} \left[ \langle M \rangle_T^{\frac{p}{2}} \right] \leq \mathbb{E} \left[ \sup_{0 \leq t \leq T} |M_t|^p \right] \leq C_p \mathbb{E} \left[ \langle M \rangle_T^{\frac{p}{2}} \right].$$

## Abstract

In this work, we study the existence and uniqueness of solutions for backward stochastic differential equations under relaxed assumptions on the generator. We specifically focus on the case where the generator satisfies a weak monotonicity and a general growth conditions with respect to the variable  $Y$ , while remaining Lipschitz continuous in  $Z$ . By employing a priori estimates, approximation techniques and Bihari's inequality, we demonstrate that these equations admit a unique solution under these weakened hypotheses. These results provide a significant generalization of classical frameworks based on standard Lipschitz or monotonicity conditions, allowing for the treatment of a broader class of equations.

**Keywords:** *Backward stochastic differential equations; Weak monotonicity; Existence and uniqueness; General growth condition; Bihari's inequality.*

## المخلص

في هذه المذكرة، قمنا بدراسة وجود ووحدانية الحلول للمعادلات التفاضلية العشوائية التراجعية في إطار شروط مخففة على المولد. ركزنا بشكل أساسي على الحالة التي يحقق فيها المولد شرط "الرتابة الضعيفة" ونموياً عاماً بالنسبة للمتغير  $Y$ ، مع كونه يحقق شرط ليبشيتز بالنسبة للمتغير  $Z$ . من خلال الاعتماد على التقديرات القبلية وتقنيات التقريب، وباستخدام متباينة بيهاري، أثبتنا أن هذه المعادلات تمتلك حلاً وحيداً تحت هذه الافتراضات. وتعد هذه النتائج تعميماً هاماً للنتائج الكلاسيكية التي تعتمد على شروط ليبشيتز أو الرتابة القياسية، مما يسمح بمعالجة فئة أوسع من المعادلات.

**الكلمات المفتاحية:** *المعادلات التفاضلية العشوائية التراجعية، الرتابة الضعيفة، الوجود والوحدانية، شرط النمو العام، متباينة بيهاري.*

## Résumé

Dans ce mémoire, nous étudions l'existence et l'unicité des solutions pour les équations différentielles stochastiques rétrogrades sous des hypothèses affaiblies sur le générateur. Nous nous concentrons principalement sur le cas où le générateur satisfait une condition de monotonie faible et une croissance générale par rapport à la variable  $Y$ , tout en étant Lipschitzien par rapport à  $Z$ . En nous appuyant sur des estimations a priori, des techniques d'approximation et l'inégalité de Bihari, nous démontrons que ces équations admettent une solution unique sous ces hypothèses relaxées. Ces résultats constituent une généralisation importante des cadres classiques basés sur les conditions de Lipschitz ou de monotonie standard, permettant ainsi de traiter une classe plus large d'équations.

**Mots-clés :** *Equations différentielles stochastiques rétrogrades; Monotonie faible; Existence et unicité; Croissance générale; Inégalité de Bihari.*