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Impacts of the Generalized Uncertainty Principle in Selected Statistical Physics Systems

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Abstract

In this work, we study the effects of the linear–quadratic Generalized Uncertainty Principle (GUP) on canonical ensemble systems. Using the deformed phase-space structure associated with GUP, we derive modified partition functions and investigate the resulting corrections to the Helmholtz free energy, entropy, and internal energy. The formalism is applied to the non-relativistic ideal gas, the ultra-relativistic gas, and the three-dimensional harmonic oscillator. Our results show that the thermodynamic behavior strongly depends on the deformation parameters. In particular, the quadratic contribution generally increases the free energy while reducing entropy and internal energy, whereas the linear term produces opposite effects. The combined linear–quadratic deformation leads to crossover behaviors due to the competition between both contributions. These results highlight the influence of minimal-length effects on statistical systems and provide further insight into possible quantum gravity corrections in high-energy thermodynamics.

ملخص

في هذا العمل، قمنا بدراسة تأثير مبدأ عدم اليقين المعمم الخطي--التربيعي (GUP) على الأنظمة الإحصائية ضمن إطار المجموعة القانونية. بالاعتماد على البنية المشوهة لفضاء الطور الناتجة عن هذا التشوه، اشتققنا دوال التقسيم المعدلة ودرسنا التصحيحات الناتجة على الطاقة الحرة لهلمهولتز، والإنتروبيا، والطاقة الداخلية. تم تطبيق هذا الإطار على عدة أنظمة فيزيائية، من بينها الغاز المثالي غير النسبي، والغاز فائق النسبية، والمذبذب التوافقي ثلاثي الأبعاد. أظهرت النتائج أن السلوك الترموديناميكي يعتمد بشكل كبير على معاملات التشوه. حيث يؤدي الجزء التربيعي عمومًا إلى زيادة الطاقة الحرة مع تقليل الإنتروبيا والطاقة الداخلية، في حين ينتج عن الجزء الخطي تأثير معاكس. أما في حالة التشوه الخطي--التربيعي، فتظهر سلوكيات انتقالية ناتجة عن التنافس بين المساهمتين. توضح هذه النتائج أهمية تأثيرات الطول الأدنى على الأنظمة الإحصائية، وتوفر فهمًا أعمق للتصحيحات المحتملة للجاذبية الكمومية في الأنظمة ذات الطاقات العالية.

Acknowledgment

If you are grateful, I will surely give you more." All praise is due to Allah, who has granted me the success and grace to complete this work, and who has illuminated the path of knowledge for me and guided my steps. I would like to express my deepest gratitude and sincere appreciation to my esteemed professor and supervisor, Dr. Djamel Eddine Zenkhri, who generously provided his valuable guidance and sound advice throughout the research period. His insights had a profound impact on bringing this work to its final form. I also extend my heartfelt thanks to the distinguished members of the defense committee for graciously accepting to review and evaluation of this thesis. To my honorable family, the source of giving and my support in life, my beloved parents: thank you for your unlimited support and your prayers that lit my way. To my colleagues and everyone who assisted me, near or far, at Kasdi Merbah University, Ouargla, and within the Faculty of Physics, may Allah reward you all with goodness.

In conclusion, I pray to Allah Almighty that this work be sincere for His sake and a beneficial contribution to the field of science

Dedication

To those who carry in their hearts a lantern that illuminates my path.

To the symbol of giving and the sanctuary of the soul. To the one whom Allah has adorned with dignity and reverence.

To my dear father, who never wavered for a single day in sacrificing everything precious so that I may reach where I am today. To you, who were my support in times of struggle and my strength in moments of weakness, I dedicate this modest work to you, out of pride in bearing your name and in gratitude for a kindness that words cannot encompass.

To the fountain of tenderness and the paradise of the earth beneath her feet.

To my beloved mother, whose sincere prayers guarded me at dusk and dawn. To the heart that endured, the eyes that stayed awake, and the soul that believed in me when I was overcome by uncertainty. To you, my companion during the long nights of study, I dedicate this success; it is but a drop in the ocean of your inexhaustible giving.

To my backbone, my pride, and those with whom my joys are complete.

To my brothers and sisters, who were the best of companions and the greatest of supporters. To those who shared with me the noise of life and the tranquility of dreams, and who made our home a safe haven for my ambitions.

To the companions of the road and friends during hardships.

To those with whom I shared university benches, hallways, the bitterness of challenges, and the sweetness of achievement. To those who were a mirror of my sincerity and a motivation to persevere whenever my steps faltered.

To my honorable professors.

Who were never stingy with their knowledge, who instilled in me a love for research and learning, and who paved the way of science for me to walk upon with steadfastness.

Finally, to my soul that aspired and persevered.

To the spirit that did not surrender to despair and defied all difficulties to reach this decisive moment. I dedicate this graduation as the fruit of years of sleepless nights, research, and diligence, hoping it will be a cornerstone in building my future and a step toward the elevation of my faith and my nation.

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1 General Introduction

The theory of Statistical Mechanics provides the fundamental framework for understanding the relation between the microscopic dynamics of elementary constituents and the macroscopic behavior of physical systems. Among its various formulations, the Canonical Ensemble occupies a central role in the description of systems in thermal equilibrium with a heat reservoir. Through the partition function, one can derive the principal thermodynamic quantities such as free energy, entropy, internal energy, and heat capacity. Owing to its broad applicability, the canonical formalism has been extensively employed in condensed matter physics, quantum gases, cosmology, and black-hole thermodynamics.

In recent years, increasing attention has been devoted to the study of possible quantum gravitational effects on statistical and thermodynamic systems, especially near the Planck scale where the standard description of spacetime is expected to break down. Several candidate theories of quantum gravity predict the existence of a minimal measurable length, a maximal momentum, or both. Such predictions naturally lead to modifications of the Heisenberg Uncertainty Principle and motivate the introduction of the Generalized Uncertainty Principle (GUP) [1, 2, 3, 4].

The concept of GUP emerges in different approaches to quantum gravity, including string theory, loop quantum gravity, black-hole physics, and doubly special relativity [5, 6]. In these approaches, the standard commutation relations between position and momentum operators are deformed in such a way that a nonzero minimal uncertainty in position appears naturally. Depending on the adopted deformation scheme, the modified algebra may also imply the existence of a maximal observable momentum or modified dispersion relations.

One of the earliest formulations of GUP was introduced by Kempf, Mangano, and Mann [4], where the deformation of the Heisenberg algebra leads to a minimal measurable length. Later, Ali, Das, and Vagenas proposed a linear–quadratic form of the GUP compatible with doubly special relativity and black-hole phenomenology [5]. Their model predicts both a minimal length and a maximal momentum and has attracted considerable interest in different areas of theoretical physics.

The implications of GUP on thermodynamics and statistical mechanics have been investigated in numerous works. In particular, the deformation of phase space induced by GUP modifies the density of states and consequently alters the partition function and the associated thermodynamic quantities. Such corrections become increasingly important at high temperatures and energies approaching the quantum gravity regime.

Several authors have studied canonical systems within the GUP framework. Vakili and Gorji analyzed the effects of minimal length on classical ideal gases and showed that the deformed phase-space measure leads to corrections in entropy and heat capacity [7]. Moussa investigated the thermodynamics of photon gases under GUP corrections and demonstrated the existence of modifications to black-body radiation and thermal spectra [8].

The harmonic oscillator, which represents one of the most fundamental systems in both classical and quantum statistical mechanics, has also been extensively studied within deformed quantum frameworks. Pedram examined the quantum harmonic oscillator in the presence of a minimal length and obtained modified energy spectra and wave functions [9]. Other investigations showed that GUP corrections induce measurable modifications to the partition function and to the thermodynamic behavior of oscillator systems, particularly in the high-temperature regime [10].

In black-hole thermodynamics, Nozari and collaborators demonstrated that GUP corrections modify the entropy–area relation and affect the evaporation process of black holes, leading to the possibility of stable black-hole remnants [11, 12]. Similar corrections have also been reported in studies of Bose–Einstein condensation, degenerate fermion gases, and relativistic quantum systems [13, 14].

These studies indicate that the Generalized Uncertainty Principle not only represents a phenomenological signature of quantum gravity but also provides a powerful framework for studying the deformation of thermodynamic and statistical properties of physical systems.

Motivated by these developments, the present thesis investigates the effects of the linear–quadratic Generalized Uncertainty Principle on canonical ensemble systems. Particular attention is devoted to the modification of the partition function and the resulting corrections to the Helmholtz free energy, entropy, and internal energy for several representative thermodynamic models. The objective is to analyze how linear and quadratic deformation terms influence the equilibrium properties of canonical systems and to identify the regimes where quantum gravitational effects become significant.

This thesis is organized as follows. In the first chapter, we review the basic principles of statistical mechanics and the canonical ensemble formalism. The second chapter is devoted to the theoretical foundations of the Generalized Uncertainty Principle, including different deformation schemes and the corresponding modifications of the phase-space structure. In the third chapter, we apply the linear–quadratic GUP formalism to several canonical systems and investigate the resulting thermodynamic corrections through analytical and numerical approaches.

2

Systems in Canonical Ensembles

2.1 Introduction

The canonical ensemble is considered one of the fundamental concepts in statistical physics, as it is used to describe physical systems that are in thermal contact with an external environment called a heat reservoir. In this description, the number of particles N and the volume V of the system are fixed, while the energy is not strictly fixed, since the system can exchange energy with the heat reservoir until it reaches thermal equilibrium at a constant temperature T [15].

2.2 Ideal Gas

In the canonical ensemble, we model an ideal gas as a collection of N non-interacting particles confined to a fixed volume V and kept in thermal contact with a large reservoir at constant temperature T . Since the particles do not interact, the total energy is purely kinetic, and the Hamiltonian takes the familiar form:

$$H = \sum_{i=1}^{N} \frac{\vec{p}_i^{\,2}}{2m} \tag{2.1}$$

Pi:Momentum./ m:Mass of the particle.

[16] With the Hamiltonian in hand, we can construct the canonical partition function. Following standard conventions, we include the $1/N!$ factor to account for particle indistinguishability and the $1/h^{3N}$ factor to render the phase-space integral dimensionless :

$$Z_N = \frac{1}{h^{3N} N!} \int d^{3N}q \, d^{3N}p \, e^{-\beta H(q,p)} \tag{2.2}$$

Because there are no interaction terms coupling coordinates and momenta, the integral factorizes cleanly into independent spatial and momentum contributions:

$$Z(T,V,N) = \frac{1}{N!h^{3N}} V^N \prod_{\nu=1}^{3N} \int_{-\infty}^{+\infty} dp_\nu \exp\left\{-\beta \frac{p_\nu^2}{2m}\right\} \tag{2.3}$$

The spatial part simply integrates to the available volume for each particle,

$$\int d^3r = V \tag{2.4}$$

while each momentum integral is a standard Gaussian that evaluates to

$$\int e^{-\beta \frac{p^2}{2m}} d^3p = (2\pi m k_B T)^{3/2}, \tag{2.5}$$

Pi:Momentum./ m:Mass of the particle./kB:Boltzmann constant./

and combining these results yields

$$Z(T,V,N) = \frac{V^N}{N!h^{3N}} \left(\frac{2m\pi}{\beta}\right)^{\frac{3N}{2}} = \frac{V^N}{N!} \left(\frac{2\pi mkT}{h^2}\right)^{\frac{3N}{2}} \tag{2.6}$$

It is often convenient to express this result more compactly by introducing the thermal de Broglie wavelength, which sets the quantum length scale of a particle at temperature T. After evaluating the integral over all N particles, we obtain:

$$Z(N,T,V) = \frac{1}{N!} \left(\frac{V}{\lambda^3}\right)^N \tag{2.7}$$

where

$$\lambda = \frac{h}{\sqrt{2\pi m k_B T}}$$

is the thermal de Broglie wavelength.

2.2.1 Free energy:

In statistical mechanics, free energy is defined as the bridge between microscopic properties (such as particle states) and macroscopic properties (thermodynamics), where the number of particles N, volume V, and temperature T are constant.[17]

$$F = -k_B T \ln Z(T,V,N) \tag{2.8}$$

By replacing Eq.(2.6) in Eq.(2.8), the free energy of an ideal gas contracts as follows:[18]

$$F = -NkT\left(1+\ln\left\{\frac{V}{N}\left(\frac{2\pi mkT}{h^2}\right)^{\frac{3}{2}}\right\}\right) \tag{2.9}$$

2.2.2 Internal Energy

In the canonical ensemble, the internal energy corresponds to the ensemble average of the total kinetic energy. Because the particles follow the Boltzmann distribution and do not interact, the internal energy depends solely on temperature and remains completely independent of volume.

$$U = F + TS = \frac{3}{2}Nk_BT \tag{2.10}$$

2.2.3 Entropy

Entropy quantifies the number of accessible microstates compatible with a given macrostate. For an ideal gas, it measures how the particles' positions and momenta can be distributed throughout phase space. Expanding the volume or raising the temperature increases the available phase space, thereby multiplying the number of possible configurations and, consequently, the entropy.

$$S = -\left.\frac{\partial F}{\partial T}\right|_{V,N} = Nk\left[\frac{5}{2}+\ln\left\{\frac{V}{N}\left(\frac{2\pi mkT}{h^2}\right)^{\frac{3}{2}}\right\}\right] \tag{2.11}$$

2.2.4 Chemical Potential

The chemical potential represents the change in the system's free energy when a single particle is added, while holding volume and temperature fixed. It governs how the system exchanges particles with a reservoir and determines the condition for diffusive equilibrium.

$$\mu = \left.\frac{\partial F}{\partial N}\right|_{T,V} = -kT\ln\left\{\frac{V}{N}\left(\frac{2\pi mkT}{h^2}\right)^{\frac{3}{2}}\right\} \tag{2.12}$$

2.2.5 Pressure

Pressure arises from the momentum transfer of gas particles colliding with the container walls. Within the thermodynamic framework, it is naturally obtained by differentiating the free energy with respect to volume, yielding the familiar ideal gas law.

$$P = -\left.\frac{\partial F}{\partial V}\right|_{T,N} = \frac{NkT}{V} \implies PV = NkT \tag{2.13}$$

While the entropy expression above is written in terms of (T,V,N), we can eliminate temperature in favor of internal energy using $U = \frac{3}{2}Nk_BT$. This gives the fundamental microcanonical form:

$$S(U,V,N) = Nk\left[\frac{5}{2} + \ln\left\{\frac{V}{N}\left(\frac{4\pi mU}{3h^2N}\right)^{\frac{3}{2}}\right\}\right] \tag{2.14}$$

The study of the ideal gas is a cornerstone of statistical mechanics and thermodynamics, representing an abstract model that reduces the complexities of molecular interactions to simple mathematical laws with high accuracy at low densities and high energies. Therefore, the success of this model lies not only in its ability to describe classical gaseous systems, but also in its role as a "theoretical laboratory" that allows us to explore the limits of physics when incorporating quantum and gravitational effects, as seen in modifications of the general uncertainty principle (GUP) [19].

2.3 Harmonic Oscillators

Harmonic oscillators have a very important position in classical and statistical physics. For any system that deviates from its stable equilibrium point, there is a restoring force that is proportional to the deviation, resulting in a parabolic potential energy function. This seemingly basic yet incredibly useful approximation helps us describe many processes, and atomic movements in condensed matter due to thermal motion. In statistical physics, the simplest solvable many-body problem is an ensemble of harmonic oscillators.

A classic application of this framework is Einsteins model of a crystalline solid. In this picture, each atom is treated as an independent oscillator vibrating around its fixed lattice site, all sharing a single characteristic frequency ω. While real materials exhibit a broad spectrum of vibrational modes (later captured by Debyes model), Einsteins assumption isolates the core physics of thermal oscillations and provides a clean starting point for deriving thermodynamic quantities from first principles.

Mathematically, the total energy of a system of N independent harmonic oscillators is described by a Hamiltonian that simply sums the kinetic and potential [16] contributions of each mode:

$$H(q_\nu, p_\nu) = \sum_{\nu=1}^{N}\left[\frac{p_\nu^2}{2m} + \frac{1}{2}m\omega^2 q_\nu^2\right]$$

Here, q_ν and p_ν denote the position and momentum of the ν-th oscillator, respectively.

With the Hamiltonian established, we can construct the canonical partition function. As specified, we omit the Gibbs factor $(1/N!)$ since the oscillators are distinguishable by their fixed lattice sites:

$$Z(T,V,N) = \frac{1}{h^3 N} \prod_{\upsilon=1}^{N} \left[\int_{-\infty}^{+\infty} dq_\upsilon \exp\left\{-\beta\frac{1}{2}m\omega^2 q_\upsilon^2\right\} \times \int_{-\infty}^{+\infty} dp_\upsilon \exp\left\{-\beta\frac{p_\upsilon^2}{2m}\right\}\right] \tag{2.15}$$

Because the Hamiltonian is a simple sum of independent single-particle terms, the multidimensional phase-space integral naturally factorizes into a product of identical one-dimensional integrals.

Each of these integrals is a standard Gaussian. By introducing the dimensionless substitutions

$$\sqrt{\beta m\omega^2/2}\, q_\upsilon = x \quad \text{and} \quad \sqrt{\beta/2m}\, p_\upsilon = y, \tag{2.16}$$

the integrals reduce to well-known forms that evaluate analytically. Carrying out the integration yields:

$$Z(T,V,N) = \frac{1}{h^N}\left[\left(\frac{2\pi}{\beta m\omega^2}\right)^{\frac{1}{2}}\left(\frac{2m\pi}{\beta}\right)^{\frac{1}{2}}\right]^N \tag{2.17}$$

Simplifying the constants and expressing the result in terms of the reduced Planck constant, we arrive at the compact form for the partition function:

$$Z(T,V,N) = \left(\frac{kT}{\hbar\omega}\right)^N \tag{2.18}$$

where the reduced Planck constant is defined as

$$\hbar = \frac{h}{2\pi}. \tag{2.19}$$

2.3.1 Free energy

In the canonical ensemble, the Helmholtz free energy captures the balance between internal energy and thermal disorder. For a collection of independent harmonic oscillators, F depends logarithmically on temperature and inversely on the characteristic frequency ω. This functional form reflects how thermal energy progressively populates higher vibrational states as the reservoir heats the system:

$$F = -kT\ln Z = -NkT\ln\left(\frac{kT}{\hbar\omega}\right) \tag{2.20}$$

2.3.2 Internal energy

Physically, the internal energy corresponds to the total energy stored in the microscopic vibrational motion of the oscillators. In the classical limit, each oscillator possesses two quadratic degrees of freedom (kinetic and potential), contributing kT per mode via the equipartition theorem. Thus, the total internal energy scales linearly with both particle number and temperature:

$$U = F + TS = NkT \tag{2.21}$$

Substituting this relation back into the entropy expression eliminates the temperature dependence in favor of the internal energy, yielding the microcanonical form:

$$S(U, V, N) = Nk\left[1 + \ln\left\{\frac{U}{N\hbar\omega}\right\}\right] \tag{2.22}$$

The harmonic oscillator model serves as a universal framework for describing any physical system perturbed slightly from stable equilibrium. Whenever displacements remain small, the restoring force is approximately linear ($F = -kx$), and the potential becomes quadratic. This approximation underpins a wide range of phenomena, from the macroscopic elasticity of springs to atomic-scale vibrations in crystal lattices and the quantized sound waves (phonons) that carry heat through solids. Despite its simplicity, the model captures the essential statistical mechanics of vibrational energy and provides a foundational tool for analyzing mechanical, atomic, and condensed-matter systems.

2.3.3 Entropy

Entropy quantifies how thermal fluctuations distribute the system across its available vibrational states. At low temperatures, the oscillators remain largely confined to their ground states, yielding minimal disorder. As temperature rises, thermal excitation populates higher energy levels, and the entropy grows logarithmically, reflecting the expanding phase space of accessible microstates. In this picture, entropy measures the competition between the oscillator's natural tendency to remain ordered (set by ω) and the randomizing influence of the thermal reservoir:

$$S = -\left(\frac{\partial F}{\partial T}\right)_{V,N} = Nk\left[1 + \ln\left\{\frac{kT}{\hbar\omega}\right\}\right] \tag{2.23}$$

Entropy can be negative, but it should ideally be positive, provided that the quantity

$$\ln\left\{\frac{kT}{\hbar\omega}\right] \tag{2.24}$$

Greater than zero

2.3.4 Chemical potential

The chemical potential represents the free energy cost of introducing an additional independent oscillator mode into the system while holding temperature and volume fixed. Since the oscillators are distinguishable (e.g., pinned to specific lattice sites), μ directly tracks how the partition function scales with N and indicates the system's tendency to exchange vibrational degrees of freedom with a particle reservoir:

$$\mu = \left.\frac{\partial F}{\partial N}\right|_{T,V} = -kT\ln\left(\frac{kT}{\hbar\omega}\right) \tag{2.25}$$

2.3.5 Pressure

Pressure arises from the system's thermodynamic response to volume changes. In the harmonic oscillator model, however, the Hamiltonian contains no explicit volume dependencethe oscillators are either bound to fixed lattice sites or confined by an external potential that does not scale with the container size. Consequently, differentiating the free energy with respect to volume yields zero:

$$P = -\left.\frac{\partial F}{\partial V}\right|_{T,N} = 0 \tag{2.26}$$

This result highlights a key physical distinction from gases: the thermodynamic pressure of a pure oscillator ensemble vanishes because the restoring forces are internal and independent of the system's boundaries,Likewise, due to the absence of external forces, pressure is not a function of the state.

2.4 Ultra-Relativistic Gas

When particles carry kinetic energies far exceeding their rest mass, their dynamics enter the ultra-relativistic regime: the rest energy mc^2 becomes negligible, and the dispersion relation simplifies to $E \approx pc$. This limit is not just a theoretical curiosityit is routinely realized in modern high-energy experiments. In particle accelerators like the Large Hadron Collider, protons and electrons are boosted to Lorentz factors $\gamma \gg 1$, so that their motion is governed almost entirely by momentum rather than mass.

Under these extreme conditions, ensembles of such particles can be treated statistically as an ultra-relativistic gas. A striking example is the quark-gluon plasma produced in heavy-ion collisions: a short-lived, strongly interacting state of deconfined quarks and gluons that, despite its complexity, exhibits collective behavior well-described by relativistic statistical mechanics [20]. Analyzing these systems does more than test the limits of thermodynamics; it offers a window into the state of the universe microseconds after the Big Bang, bridging quantum field theory, special relativity, and statistical physics in a single framework.

From a modeling perspective, the key simplification is the energy-momentum relation. Whereas the non-relativistic ideal gas uses $E = p^2/2m$, the ultra-relativistic limit replaces this with the linear form:

$$E = pc \quad \text{(for massless or highly relativistic particles)},$$

a change that propagates through every thermodynamic quantity we derive.

Accordingly, the Hamiltonian for the system depends exclusively on the magnitude of the particle momenta :

$$H(qV, PV) = \sum_{v=1}^{N} c\,|\vec{p}_v|, \tag{2.27}$$

and the canonical partition function, including the standard Gibbs factor for indistinguishable particles, takes the form [21]:

$$Z(T,V,N) = \frac{1}{N!h^{3N}} \int d^{3N}q\, d^{3N}p \exp\{-\beta H(q,p)\} \tag{2.28}$$

Because the Hamiltonian contains no position-dependent potential terms, the particles are free to explore the entire volume independently. The spatial integration therefore simply contributes a factor of V per particle. As before, the absence of coordinate-momentum coupling allows the multidimensional phase-space integral to factorize into a product of identical single-particle momentum integrals:

$$Z(T,V,N) = \frac{1}{N!h^{3N}} V^N \prod_{v=1}^{N} \int d^3p_v \exp\{-\beta|\vec{p}_v|c\} \tag{2.29}$$

To evaluate the momentum integrals, we switch to spherical polar coordinates. The angular integration yields a geometric factor of 4π, leaving a radial integral over the magnitude of the momentum:

$$Z(T,V,N) = \frac{V^N}{N!h^{3N}} \left(4\pi \int_0^{\infty} p^2 dp\, e^{-\beta c p}\right)^N \tag{2.30}$$

The radial integral is evaluated via the substitution $x = \beta c p$, which transforms it into a standard

Gamma function integral:

$$\int_0^{\infty} p^2 dp\, e^{-\beta cp} = \left(\frac{1}{\beta c}\right)^3 \int_0^{\infty} x^2 dx\, e^{-x} = \left(\frac{1}{\beta c}\right)^3 \Gamma(3) \tag{2.31}$$

Using the identity $\Gamma(n+1) = n!$, we find $\Gamma(3) = 2! = 2$. Substituting this result back and simplifying the constants yields the compact form of the partition function:

$$Z(T,V,N) = \frac{V^N}{N! h^{3N}} \left(8\pi \left(\frac{1}{\beta c}\right)^3\right)^N = \frac{1}{N!} \left(8\pi V \left(\frac{kT}{hc}\right)^3\right)^N \tag{2.32}$$

2.4.1 Free energy

In the ultra-relativistic regime, the Helmholtz free energy captures the thermodynamic potential available for work when particle speeds approach c. Because the partition function scales as T^{3N} rather than $T^{3N/2}$, the free energy acquires a stronger logarithmic dependence on temperature, reflecting the rapid expansion of accessible high-momentum states as the system heats up:

$$F(T,V,N) = -kT \ln Z(T,V,N) = -NkT \left[1 + \ln\left\{\frac{8\pi V}{N}\left(\frac{kT}{hc}\right)^3\right\}\right] \tag{2.33}$$

2.4.2 Internal energy

The internal energy represents the total thermal energy stored in the relativistic motion of the gas. Applying statistical averaging to the linear dispersion relation yields $3kT$ per particletwice the non-relativistic valuebecause the relativistic phase space weights higher momenta more heavily. Thus, the total energy scales linearly with both particle number and temperature:

$$U = F + TS = 3NkT \tag{2.34}$$

Eliminating temperature in favor of internal energy gives the microcanonical entropy expression [22]:

$$S(U,V,N) = Nk \left[4 + \ln\left\{\frac{8\pi V}{N}\left(\frac{U}{3Nhc}\right)^3\right\}\right] \tag{2.35}$$

Real-world systems that exhibit ultra-relativistic behavior include:

1. **High-temperature plasmas:** such as those in stellar cores or supernova remnants, where temperatures reach millions or billions of kelvins, rendering rest mass negligible compared to kinetic energy.

2. **Radiation and nearly massless particles:** including photon gases in thermal equilibrium or high-energy neutrino backgrounds, where $E \approx pc$ holds exactly or to excellent approximation.

3. **Extreme astrophysical environments:** such as the crusts of neutron stars or the early universe microseconds after the Big Bang, where particle energies vastly exceed mc^2.

In all these regimes, the thermodynamics are governed by the linear energy-momentum relation, leading to distinct macroscopic signaturesmost notably the $U = 3NkT$ scaling and the modified entropy densitythat clearly separate ultra-relativistic matter from its classical counterpart.

2.4.3 Entropy

Entropy quantifies how thermal fluctuations distribute the system across its available relativistic momentum states. The T^3 scaling in the partition function leads to a logarithmic term that grows more rapidly with temperature than in the non-relativistic case, capturing how phase-space volume expands when particles become ultra-relativistic and access a broader range of momenta:

$$S = -\left.\frac{\partial F}{\partial T}\right|_{V,N} = Nk\left[4 + \ln\left\{\frac{8\pi V}{N}\left(\frac{kT}{hc}\right)^3\right\}\right] \tag{2.36}$$

2.4.4 Chemical potential

The chemical potential measures the free energy change associated with introducing an additional ultra-relativistic particle into the system at fixed T and V. Its logarithmic dependence on density and temperature reflects how the availability of high-momentum states shifts as the particle number changes, governing the system's tendency to exchange particles with a reservoir:

$$\mu = \left.\frac{\partial F}{\partial N}\right|_{T,V} = -kT\ln\left\{\frac{8\pi V}{N}\left(\frac{kT}{hc}\right)^3\right\} \tag{2.37}$$

2.4.5 Pressure

Pressure emerges from the momentum flux of particles striking the container boundaries. Interestingly, despite the radically different energy-momentum relation, the equation of state retains the familiar ideal gas form. This occurs because the linear dispersion $E = pc$ modifies both the energy density and the momentum transfer in a way that leaves the macroscopic relation unchanged:

$$p = -\left.\frac{\partial F}{\partial V}\right|_{T,N} = \frac{NkT}{V} \quad \text{or} \quad pV = NkT \tag{2.38}$$

2.4.6 Internal energy

The internal energy represents the total thermal energy stored in the relativistic motion of the gas. Applying statistical averaging to the linear dispersion relation yields $3kT$ per particletwice the non-relativistic valuebecause the relativistic phase space weights higher momenta more heavily. Thus, the total energy scales linearly with both particle number and temperature:

$$U = F + TS = 3NkT \tag{2.39}$$

Eliminating temperature in favor of internal energy gives the microcanonical entropy expression [22]:

$$S(U, V, N) = Nk\left[4 + \ln\left\{\frac{8\pi V}{N}\left(\frac{U}{3Nhc}\right)^3\right\}\right] \tag{2.40}$$

Real-world systems that exhibit ultra-relativistic behavior include:

1. **High-temperature plasmas:** such as those in stellar cores or supernova remnants, where temperatures reach millions or billions of kelvins, rendering rest mass negligible compared to kinetic energy.

2. **Radiation and nearly massless particles:** including photon gases in thermal equilibrium or high-energy neutrino backgrounds, where $E \approx pc$ holds exactly or to excellent approximation.

3. **Extreme astrophysical environments:** such as the crusts of neutron stars or the early universe microseconds after the Big Bang, where particle energies vastly exceed mc^2.

In all these regimes, the thermodynamics are governed by the linear energy-momentum relation, leading to distinct macroscopic signaturesmost notably the $U = 3NkT$ scaling and the modified entropy densitythat clearly separate ultra-relativistic matter from its classical counterpart.

2.5 Conclusion

In this chapter we have presented how the canonical ensemble connects microscopic Hamiltonians to macroscopic thermodynamics. By analyzing three representative systemsthe classical ideal gas ($E \propto p^2$), harmonic oscillators ($E \propto q^2 + p^2$), and an ultra-relativistic gas ($E \propto p$)we showed how the energy-momentum relation directly determines the partition function and all derived quantities. Despite their different physical regimes, each model yields to the same statistical approach: factorization of independent degrees of freedom, analytical phase-space integration, and thermodynamic differentiation. Together, they illustrate the power and flexibility of statistical mechanics in describing matter from everyday conditions to the extremes of high-energy physics.

3 The formalism of Generalized Uncertainty Principle

3.1 Introduction and Motivation

The Heisenberg Uncertainty Principle (HUP) is a cornerstone of non-relativistic quantum mechanics, establishing a fundamental limit on the simultaneous precision of conjugate observables. [2] While the HUP successfully describes microscopic phenomena at low energies, it assumes a fixed, classical background spacetime and neglects gravitational backreaction. When attempting to probe distances approaching the Planck scale, this assumption breaks down. [23]

According to general relativity, localizing a particle within a region of size Δx requires a probe energy $E \sim \hbar c/\Delta x$. Concentrating such energy in a sufficiently small volume curves spacetime significantly, and when Δx approaches the Planck length,

$$\ell_{\rm P} = \sqrt{\frac{\hbar G}{c^3}} \approx 1.616 \times 10^{-35}\,\mathrm{m}, \tag{3.1}$$

the system is expected to undergo gravitational collapse into a microscopic black hole. This gedanken argument reveals a fundamental contradiction: standard quantum mechanics permits arbitrarily small Δx by accepting unbounded Δp, whereas gravitational physics imposes a lower bound on spatial resolution.

The Generalized Uncertainty Principle (GUP) resolves this tension by deforming the canonical commutation relations to incorporate Planck-scale gravitational effects. It emerges naturally in several independent approaches to quantum gravity, including string theory (via T-duality and extended objects), black hole thermodynamics (through entropy corrections), and non-commutative geometry. The primary physical consequence of the GUP is the existence of a *minimal measurable length*, reflecting a finite resolution of spacetime at ultra-high energies.

3.2 Heisenberg Uncertainty Principle Revisited

In standard quantum mechanics, the kinematical structure is built upon the canonical commutation relation (CCR)

$$[\hat{x}, \hat{p}] = i\hbar, \tag{3.2}$$

where $\hat{x}$ and $\hat{p}$ are the position and momentum operators acting on a separable Hilbert space $\mathscr{H}$. For any pair of observables $\hat{A}$ and $\hat{B}$, the Robertson–Schrödinger inequality states

$$(\Delta A)^2(\Delta B)^2 \geq \left|\frac{1}{2i}\langle[\hat{A}, \hat{B}]\rangle\right|^2 + \left|\frac{1}{2}\langle\{\hat{A}, \hat{B}\}\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle\right|^2, \tag{3.3}$$

where $\Delta A = \sqrt{\langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2}$ denotes the standard deviation in a given quantum state. Applying this to $\hat{x}$ and $\hat{p}$ and neglecting the covariance term for symmetric minimum-uncertainty states yields the familiar Heisenberg relation

$$\Delta x \Delta p \geq \frac{\hbar}{2}. \tag{3.4}$$

Equality is attained by Gaussian wave packets or harmonic oscillator coherent states. Crucially, Eq. (3.4) expresses a *preparation uncertainty* intrinsic to the quantum state, not a limitation of measurement apparatus. Because the right-hand side is constant, standard quantum mechanics allows $\Delta x \to 0$ provided $\Delta p \to \infty$. This mathematical feature relies entirely on the linearity of the CCR (3.2) and the assumption that spacetime remains smooth and non-dynamical at all scales.

3.3 Mathematical Formulation and Deformed Algebra

3.3.1 Deformed Commutation Relations

To incorporate gravitational backreaction at the Planck scale, the canonical algebra is deformed by replacing the constant $i\hbar$ with a momentum-dependent function:

$$[\hat{x}, \hat{p}] = i\hbar f(\hat{p}), \tag{3.5}$$

where $f(\hat{p})$ satisfies $f(\hat{p}) \to 1$ as $\hat{p} \to 0$, ensuring recovery of standard quantum mechanics in the low-energy limit. Substituting Eq. (3.5) into the Robertson inequality gives the generalized uncertainty relation

$$\Delta x \Delta p \geq \frac{\hbar}{2}\left|\langle f(\hat{p})\rangle\right|. \tag{3.6}$$

The functional form of $f(\hat{p})$ depends on the underlying quantum gravity framework. Parity symmetry and rotational invariance in three dimensions typically restrict $f(\hat{p})$ to even powers of momentum, though odd terms may appear in models with broken Lorentz symmetry or Doubly Special Relativity (DSR).

3.3.2 Quadratic GUP and the Minimal Length

[4] The most widely studied ansatz is the quadratic deformation

$$[\hat{x}, \hat{p}] = i\hbar\left(1 + \beta\hat{p}^2\right), \tag{3.7}$$

where $\beta > 0$ is a small deformation parameter. For a minimum-uncertainty state with $\langle\hat{p}\rangle = 0$, the variance relation $\langle\hat{p}^2\rangle = (\Delta p)^2 + \langle\hat{p}\rangle^2$ simplifies to $\langle\hat{p}^2\rangle = (\Delta p)^2$. Substituting into Eq. (3.6) yields

$$\Delta x \Delta p \geq \frac{\hbar}{2}\left[1 + \beta(\Delta p)^2\right]. \tag{3.8}$$

Solving for Δx and treating the right-hand side as a function of Δp,

$$\Delta x \geq \frac{\hbar}{2}\left(\frac{1}{\Delta p} + \beta\Delta p\right), \tag{3.9}$$

we minimize with respect to Δp:

$$\frac{d}{d(\Delta p)}\left(\frac{1}{\Delta p} + \beta\Delta p\right) = -\frac{1}{(\Delta p)^2} + \beta = 0 \quad \Rightarrow \quad \Delta p_{\min} = \frac{1}{\sqrt{\beta}}. \tag{3.10}$$

Inserting $\Delta p_{\min}$ back gives the fundamental minimal position uncertainty

$$(\Delta x)_{\min} = \hbar\sqrt{\beta}. \tag{3.11}$$

This result establishes a non-vanishing lower bound on spatial resolution, in direct contrast to standard quantum mechanics.

3.3.3 Linear and Linear-Quadratic Deformations

A purely linear deformation of the canonical commutation relation,

$$[\hat{x}, \hat{p}] = i\hbar(1 + \alpha\hat{p}), \tag{3.12}$$

does not, by itself, imply the existence of a minimal length for symmetric states satisfying $\langle \hat{p} \rangle = 0$, since the linear contribution vanishes at the level of expectation values. Linear corrections typically arise in frameworks such as doubly special relativity (DSR) or theories with preferred reference frames, and are physically more consistent when accompanied by higher-order (quadratic) terms.

A widely studied extension is the linear–quadratic generalized uncertainty principle (GUP), defined by

$$[\hat{x}, \hat{p}] = i\hbar\left(1 - \alpha\hat{p} + \beta\hat{p}^2\right), \tag{3.13}$$

where α and β are deformation parameters with dimensions of inverse momentum and inverse momentum squared, respectively.

Using Eq. (3.13), the generalized uncertainty relation can be written as

$$\Delta x \Delta p \geq \frac{\hbar}{2}\left[1 - \alpha\langle \hat{p}\rangle + \beta\left((\Delta p)^2 + \langle \hat{p}\rangle^2\right)\right]. \tag{3.14}$$

In the case of symmetric states ($\langle \hat{p} \rangle = 0$), this reduces to

$$\Delta x \Delta p \geq \frac{\hbar}{2}\left[1 + \beta(\Delta p)^2\right], \tag{3.15}$$

which is formally identical to the purely quadratic GUP and leads to the existence of a nonzero minimal length.

The linear parameter α, however, plays an important role for asymmetric states ($\langle \hat{p} \rangle \neq 0$), modifies the structure of momentum space, and deforms the phase space integration measure. In particular, the associated deformation function,

$$f(p) = 1 - \alpha p + \beta p^2, \tag{3.16}$$

must remain positive to ensure a well-defined algebra and integration measure. This requirement can lead to a bounded momentum domain when the condition $\alpha^2 \geq 4\beta$ is satisfied, in which case a maximal observable momentum emerges,

$$p_{\max} = \frac{\alpha + \sqrt{\alpha^2 - 4\beta}}{2\beta}. \tag{3.17}$$

This feature introduces a natural ultraviolet cutoff and has important consequences for quantum gravity phenomenology and statistical mechanics.

3.3.4 Dimensional Consistency and Phenomenological Parameterization

To ensure dimensional consistency of the deformation terms in Eq. (3.13), the parameters α and β are expressed in terms of the Planck length ℓ_{P} as

$$\alpha = \alpha_0 \frac{\ell_{\mathrm{P}}}{\hbar}, \qquad \beta = \beta_0 \frac{\ell_{\mathrm{P}}^2}{\hbar^2}, \tag{3.18}$$

where α_0 and β_0 are dimensionless constants typically assumed to be of order unity.

Substituting Eq. (3.18) into the expression for the minimal length yields

$$(\Delta x)_{\min} = \hbar\sqrt{\beta} = \ell_{\mathrm{P}}\sqrt{\beta_0}, \tag{3.19}$$

which naturally relates the minimal length scale to the Planck length.

3.4 Jacobian

In classical statistical mechanics, we use canonically conjugate variables (q, p) to compute the partition function.

However, when introducing the Generalized Uncertainty Principle, these variables no longer directly represent the true physical quantities, as the physical variables become deformed compared to the canonical ones.

3.4.1 Jacobian of the Deformed Phase Space

In the presence of the linear-quadratic GUP deformation,

$$[\hat{x}, \hat{p}] = i\hbar f(p), \qquad \text{with} \quad f(p) = 1 - \alpha' p + \alpha p^2, \tag{3.20}$$

the usual phase space structure is modified. As a consequence, the standard measure $d^3x\, d^3p$ must be corrected in order to preserve the consistency of the theory.

From commutator to Poisson structure. In the semiclassical limit, commutators are replaced by Poisson brackets according to

$$\frac{1}{i\hbar}[\hat{x}, \hat{p}] \longrightarrow \{x, p\}. \tag{3.21}$$

Therefore, the deformed algebra implies

$$\{x, p\} = f(p) = 1 - \alpha' p + \alpha p^2. \tag{3.22}$$

Canonical transformation. To construct the phase space measure, we introduce canonical variables (X, P) satisfying

$$\{X, P\} = 1. \tag{3.23}$$

We assume that the physical variables are related to the canonical ones through

$$x = X, \qquad p = p(P), \tag{3.24}$$

so that all the deformation is encoded in the momentum sector.

Using the chain rule for Poisson brackets, we obtain

$$\{x, p\} = \{X, p(P)\} = \frac{dp}{dP}\{X, P\} = \frac{dp}{dP}. \tag{3.25}$$

Comparing with the deformed bracket, we arrive at the fundamental relation

$$\frac{dp}{dP} = f(p) = 1 - \alpha' p + \alpha p^2. \tag{3.26}$$

Jacobian of the transformation. The transformation from canonical variables (X, P) to physical variables (x, p) is characterized by the Jacobian matrix

$$M = \begin{pmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial P} \\ \frac{\partial p}{\partial X} & \frac{\partial p}{\partial P} \end{pmatrix}. \tag{3.27}$$

From $x = X$ and $p = p(P)$, it follows that

$$\frac{\partial x}{\partial X} = 1, \qquad \frac{\partial x}{\partial P} = 0, \qquad \frac{\partial p}{\partial X} = 0, \qquad \frac{\partial p}{\partial P} = \frac{dp}{dP}. \tag{3.28}$$

Thus, the Jacobian determinant is

$$J = \det(M) = \frac{dp}{dP}. \tag{3.29}$$

Using Eq. (3.26), we obtain

$$J = f(p) = 1 - \alpha' p + \alpha p^2. \tag{3.30}$$

Extension to three dimensions. Assuming isotropy, the three-dimensional phase space transformation factorizes, and the total Jacobian becomes

$$J_{(3D)} = \left(\frac{dp}{dP}\right)^3 = \left(1 - \alpha' p + \alpha p^2\right)^3. \tag{3.31}$$

Transformation of the phase space measure. The invariant phase space element is defined in canonical variables as

$$d^3X\, d^3P. \tag{3.32}$$

Using the Jacobian, we relate it to the physical variables:

$$d^3X\, d^3P = \frac{d^3x\, d^3p}{J_{(3D)}}. \tag{3.33}$$

Therefore, the deformed phase space measure takes the form

$$d^3x\, d^3p \longrightarrow \frac{d^3x\, d^3p}{\left(1 - \alpha' p + \alpha p^2\right)^3}. \tag{3.34}$$

Physical interpretation. This result shows that the GUP deformation effectively modifies the density of states by a momentum-dependent factor. In particular, high-momentum contributions are suppressed, reflecting the presence of a fundamental scale and, in certain parameter regimes, a maximal momentum. This modified measure is central to the construction of the partition function and the resulting thermodynamic corrections.

3.5 Conclusion

In this chapter, we have introduced the main forms of the Generalized Uncertainty Principle and discussed their physical implications. We highlighted how the quadratic deformation leads to the existence of a minimal length, while the linear term brings additional features, such as asymmetry in momentum space and the possibility of a maximal momentum.

We then focused on the linear–quadratic case and showed how the deformation modifies the structure of phase space. In particular, we derived the associated Jacobian and obtained the corrected phase space measure, which plays a key role in practical applications.

These results provide the necessary groundwork for the next chapter, where the deformed framework will be applied to canonical ensemble systems in order to study its impact on thermodynamic properties.

4 Canonical ensemble system in deformed spaces with minimal length

4.1 Introduction and Motivation

The existence of a fundamental minimal length, commonly associated with the Planck scale, is a recurring prediction of several quantum gravity approaches. Early arguments based on high-energy scattering and black hole thought experiments [23, 1, 24] suggest that attempts to probe arbitrarily small distances require energies that inevitably lead to gravitational collapse, thereby limiting spatial resolution. This idea is further supported by string theory, where the extended nature of strings prevents the exploration of sub-Planckian distances [2, 25].

A convenient way to incorporate this feature into quantum mechanics is through a deformation of the canonical commutation relations, leading to the so-called Generalized Uncertainty Principle (GUP). Early formulations were proposed by Maggiore [26], while a consistent Hilbert space representation was later developed by Kempf *et al.* [27]. These frameworks provide a systematic way to introduce minimal-length effects and study their consequences in quantum systems.

Beyond formal developments, GUP has been widely applied to investigate modifications in quantum mechanics and statistical physics. In particular, the deformation of the phase space structure leads to corrections in the density of states, which in turn affect thermodynamic quantities. Such effects have been explored in various contexts, including modified oscillators, atomic systems, and quantum gases [14, 28, 29, 30].

In this chapter, we focus on the implications of GUP within the canonical ensemble. We consider a non-relativistic ideal gas and derive the modified partition function using the deformed phase space measure. From this, we obtain corrections to thermodynamic quantities and analyze their behavior across different regimes. Particular attention is given to how minimal-length effects influence the density of states and, consequently, the thermodynamic properties of the system.

4.2 Theoretical Framework

We consider a system of N identical, non-interacting particles described by the Hamiltonian

$$H = \frac{p^2}{2m} + U(x), \tag{4.1}$$

where $U(x)$ represents an external potential. In the canonical ensemble, the partition function is the central quantity from which all thermodynamic properties can be derived. In the semiclassical limit, it is written as

$$Z = \int e^{-\beta H(x,p)}\, d^D x\, d^D p, \tag{4.2}$$

with $\beta = 1/(kT)$.

In the standard formulation, the phase space measure is uniform. However, within the framework of the Generalized Uncertainty Principle (GUP), the structure of phase space is modified. As discussed in the previous section, the deformation introduces a nontrivial Jacobian, which changes the density of states through the replacement

$$d^D x\, d^D p \longrightarrow \frac{d^D x\, d^D p}{\left(1 - \alpha' p + \alpha p^2\right)^D}. \tag{4.3}$$

Accordingly, the partition function takes the form

$$Z = \int \frac{e^{-\beta H(x,p)}}{\left(1 - \alpha' p + \alpha p^2\right)^D}\, d^D x\, d^D p. \tag{4.4}$$

As a result, the partition function of a non-relativistic ideal gas becomes

$$Z(T,V,N) = \frac{1}{N! h^{3N}} \prod_{i=1}^{N} \int\!\!\int \frac{d^3 x_i d^3 p_i \exp\left(-\beta H(p_i, x_i)\right)}{\left(1 - \alpha' p_i + \alpha p_i^2\right)^3}. \tag{4.5}$$

In the present work, we assume that the deformation parameters are sufficiently small,

$$\alpha' p \ll 1, \qquad \alpha p^2 \ll 1, \tag{4.6}$$

which allows the deformation factor to be expanded perturbatively up to second order:

$$(1 - \alpha' p + \alpha p^2)^{-3} \approx 1 + 3\alpha' p + (6\alpha'^2 - 3\alpha) p^2. \tag{4.7}$$

This approximation significantly simplifies the momentum integrals while preserving the leading GUP corrections relevant to the thermodynamic properties of the system.

In the following subsections, this general formalism will be applied to different physical systems by specifying the appropriate form of the Hamiltonian and deriving the corresponding thermodynamic quantities.

4.3 Ideal gas in canonical ensemble under the GUP

In studying an ideal gas within the canonical system and under the influence of the generalized uncertainty principle, we assume a system consisting of N identical and non-interacting particles confined within a volume V. In the non-relativistic limit, the particle dynamics are described by the given Hamiltonian:

$$H(q,p) = \sum_{i=1}^{3N} \frac{p_i^2}{2m}. \tag{4.8}$$

From Eq.(2.2) the partition function in the Canonical set is defined as follows:

$$Z(T,V,N) = \frac{1}{N!h^{3N}} \int d^{3N}q\, d^{3N}p \exp\left(-\beta \sum_{i=1}^{3N} \frac{p_i^2}{2m}\right). \tag{4.9}$$

Where

$$\beta = \frac{1}{kT}. \tag{4.10}$$

We know that a non-relativistic gas has a Hamiltonian

$$H = \frac{P^2}{2m}. \tag{4.11}$$

Where the partition function becomes of the form:

$$Z(T,V) = \frac{V}{h^3} \int d^3p \exp\left(-\beta \sum_{i}^{3} \frac{p_i^2}{2m}\right) \left[1 + 3\alpha' p + (6\alpha'^2 - 3\alpha')p^2\right]. \tag{4.12}$$

The hash function has no unit.

By performing Gaussian integrations in spherical coordinates,we obtain:

$$Z(T,V) = Z_0 \left[1 + 6\sqrt{\frac{2\alpha' mkT}{\pi}} + 9(2\alpha'^2 - \alpha)mkT\right]. \tag{4.13}$$

where:

$$Z_0 = \frac{V}{\lambda^3}. \tag{4.14}$$

is the standard partition function and

$$\lambda = \frac{h}{\sqrt{2\pi mkT}}. \tag{4.15}$$

is the thermal wavelength. In the case of N indistinguishable particles, the partition function is given by:

$$Z(T,V,N) = \frac{[Z(T,V,1)]^N}{N!} = \frac{Z_0^N}{N!}\left[1+6\sqrt{2}\alpha'\sqrt{\frac{mkT}{\pi}}+9(2\alpha'^2-\alpha)mkT[1+6\sqrt{\frac{2\alpha' mkT}{\pi}}+9(2\alpha'^2-\alpha)mkT\right]. \tag{4.16}$$

Note: In an ideal gas, we cannot integrate from positive infinity to infinity because we would then have an infinite volume, making integration difficult.

The generalized uncertainty principle leads to corrections reflected in a decrease in the size of the available phase space, and this effect is only achieved at extremely high temperatures.

The Hamiltonian free energy is derived from the distribution function by the following expression:

$$F = -kT\ln Z(T,V,N). \tag{4.17}$$

By substituting into the distribution function, we find:

$$F_{GUP} = \underbrace{-NkT\left[1+\ln\left(\frac{V}{N}\left(\frac{2\pi mkT}{h^2}\right)^{3/2}\right)\right]}_{F_{std}} - 6NkT\alpha'\sqrt{\frac{2mkT}{\pi}} - 9(2\alpha'^2-\alpha)Nmk^2T^2. \tag{4.18}$$

Figure (4.1) illustrates the variation of the normalized free-energy correction,

$$\frac{\Delta F}{NkT} = \frac{F_{\text{GUP}}-F_{\text{std}}}{NkT}, \tag{4.19}$$

knowing that F_{std} is the Free energy of the ideal gas (before distortion).

as a function of temperature for different values of the deformation parameters α and α'.

For the purely quadratic deformation ($\alpha' > 0$, $\alpha' = 0$), the correction remains positive over the entire temperature range. This indicates that the deformed free energy is always larger than the standard one,

$$F_{\text{GUP}} > F_{\text{std}}. \tag{4.20}$$

Moreover, the difference increases almost linearly with temperature, showing that quadratic GUP effects become progressively more significant at higher thermal energies. Physically, this behavior

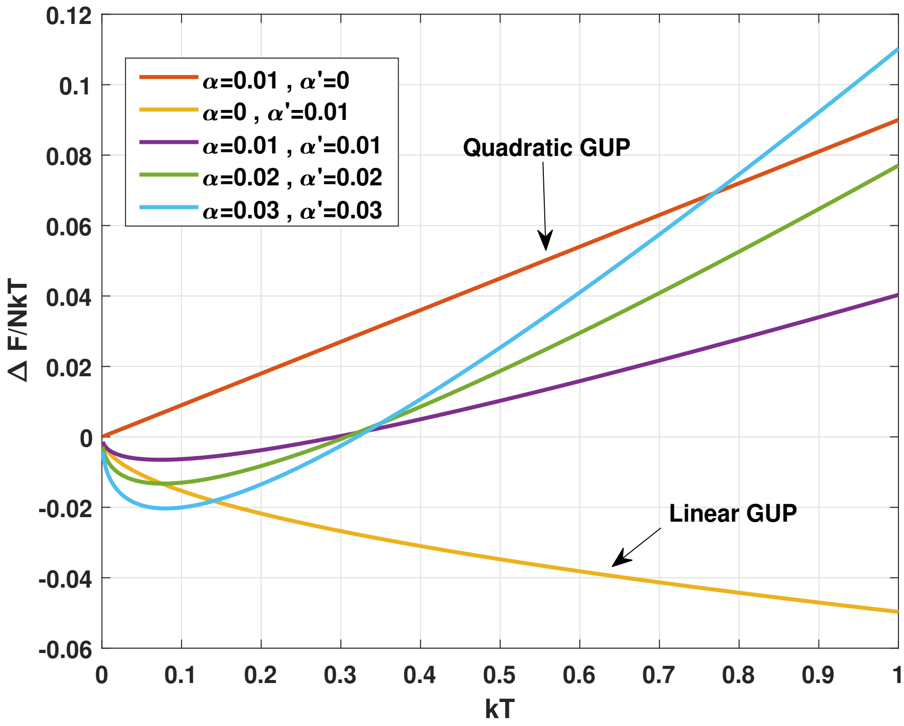


Figure 4.1: Variation of $\Delta F/NkT$ as a function of kT for different values of the deformation parameter α and α' for $m = N = 1$.

reflects the reduction of accessible microstates caused by the existence of a minimal length, which effectively increases the thermodynamic cost of the system.

In contrast, for the purely linear deformation ($\alpha = 0$, $\alpha' \neq 0$), the correction is negative throughout the considered interval,

$$F_{\text{GUP}} < F_{\text{std}}. \tag{4.21}$$

The magnitude of the deviation also grows with temperature, but in the opposite direction. This suggests that the linear contribution modifies the phase space structure differently, leading to an effective decrease in the free energy.

The most interesting behavior appears in the linear–quadratic case ($\alpha \neq 0$, $\alpha' \neq 0$). In this regime, the curves initially take negative values at low temperatures, indicating that the linear contribution dominates in the weak thermal regime. As the temperature increases, the correction gradually approaches zero until reaching a particular point where

$$\Delta F = 0, \tag{4.22}$$

meaning that the deformed and standard free energies become equal. Beyond this transition point, the correction becomes positive, showing that the quadratic contribution eventually over-comes the linear one at higher temperatures.

This crossover behavior highlights the competition between the linear and quadratic sectors of the GUP deformation. At low temperatures, the linear correction governs the thermodynamic response, whereas at higher temperatures the quadratic term, associated with minimal-length effects, becomes dominant. Furthermore, increasing both deformation parameters enhances the magnitude of the correction, making the deviation from the standard thermodynamic behavior more pronounced.

The pressure is obtained from the thermodynamic relation:

$$p = -\left(\frac{\partial F}{\partial V}\right)_{T,N} = \frac{NkT}{V} \quad \text{or} \quad pV = NkT \tag{4.23}$$

We observe that the pressure expression remains unchanged, which indicates that the modifications introduced by GUP focused solely on the energy and thermal behavior of the molecules, but did not alter the way the molecules are distributed within the normal volume. Since the volume relationship in the equations remains the same, the final result remains the same, which is the previously known ideal gas law.

Entropy. The generalized uncertainty principle modifies the structure of phase space by introducing a minimal measurable length. As a consequence, the number of accessible microstates available to the particles is altered compared to the standard case. This modification is reflected in the density of states through the deformed phase space measure, leading to corrections in the thermodynamic quantities of the system.

From a physical point of view, the existence of a minimal length limits the localization of particles at very small scales, effectively reducing the available phase space volume at high momenta. Consequently, the entropy acquires additional GUP-dependent contributions beyond the classical ideal gas expression.

Using the thermodynamic relation

$$S = -\left(\frac{\partial F}{\partial T}\right)_{V,N}, \tag{4.24}$$

we obtain

$$S = Nk\left[\frac{5}{2} + \ln\left(\frac{V}{N}\left(\frac{2\pi mkT}{h^2}\right)^{3/2}\right) + 9\alpha'\sqrt{\frac{2mkT}{\pi}} + 18(2\alpha'^2 - \alpha)mkT\right]. \tag{4.25}$$

The first two terms correspond to the standard entropy of a classical ideal gas, while the remaining terms represent corrections induced by the linear–quadratic GUP deformation. The lin-

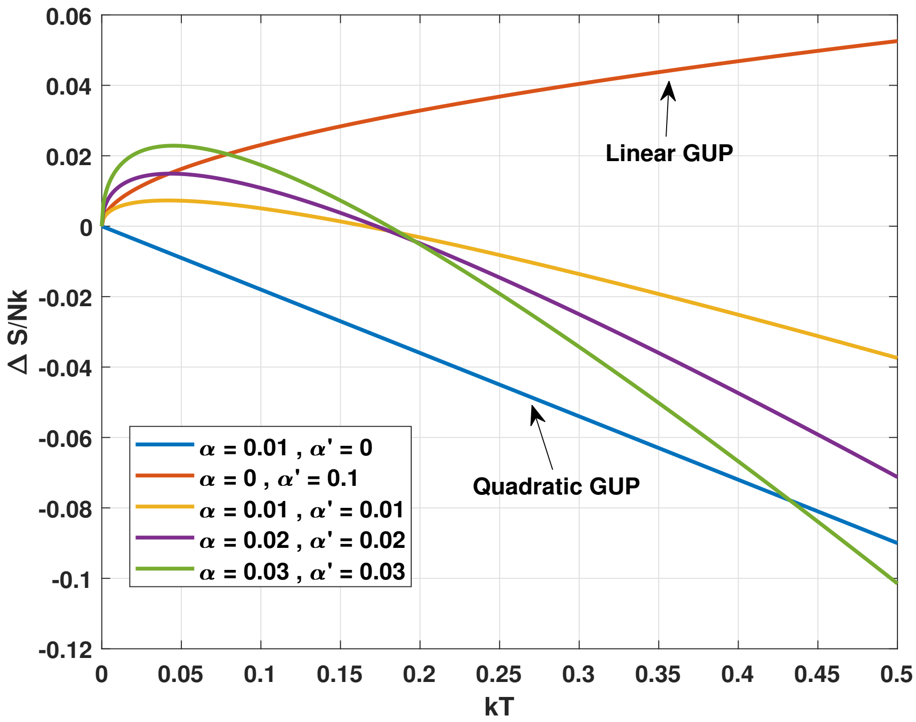


Figure 4.2: Variation of $\Delta S/NkT$ as a function of kT for different values of the deformation parameter α and α' for $m = N = 1$.

ear contribution scales as $\sqrt{T}$, whereas the quadratic correction grows linearly with temperature, indicating that GUP effects become more significant in high-temperature regimes.

The Generalized Uncertainty Principle (GUP) reshapes our understanding of phase space through studies of the coefficients α and α', where the quadratic coefficient alpha reduces the number of available microstates, leading to lower entropy, while the linear coefficient α' increases the density of these states. This mathematical conflict means that quantum gravity not only confines the universe to Planck scales.

The internal energy:U = F + TS evaluates to:

$$U = \frac{3}{2}NkT + 3NkT\alpha'\sqrt{\frac{2mkT}{\pi}} + 9(2\alpha'^2 - \alpha)Nmk^2T^2 \tag{4.26}$$

In this physical context, the increase in internal energy refers to the change or additional amount that occurs in the internal energy of the system as a result of introducing the effects of the generalized uncertainty principle.

Heat capacity can be described as a measure of the elasticity of spacetime in an energy-dense environment. Physically, it is thought that an infinitely large container could accommodate any

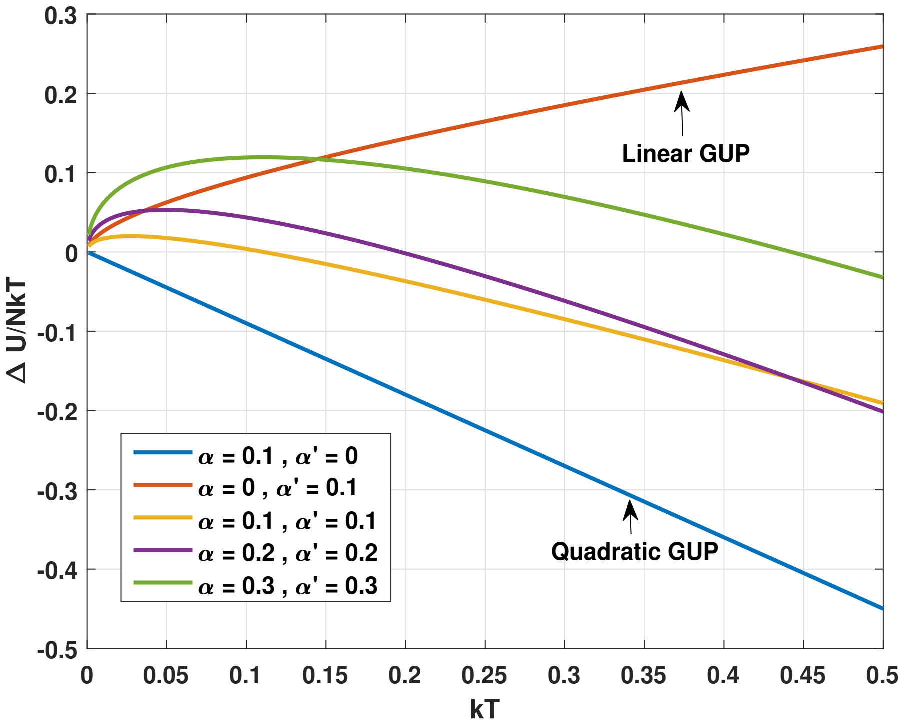


Figure 4.3: Variation of $\Delta U/NkT$ as a function of kT for different values of the deformation parameter α and α'

large flow of momentum. However, the new CV formula tells us that when the limit

$$(\alpha < 2\alpha) \tag{4.27}$$

is exceeded, spacetime begins to exhibit a rigidity that prevents particles from absorbing additional energy, resulting in a decrease in heat capacity.The heat capacity is given by the following equation:

$$C_V = \frac{3}{2}Nk + \frac{9}{2}Nk\alpha'\sqrt{\frac{2mkT}{\pi}} + 18(2\alpha'^2 - \alpha)Nmk^2T \tag{4.28}$$

Figures (4.2) and (4.3) display the temperature dependence of the normalized entropy and internal-energy corrections,

$$\frac{\Delta S}{Nk} = \frac{S_{\text{GUP}} - S_{\text{std}}}{Nk}, \qquad \frac{\Delta U}{NkT} = \frac{U_{\text{GUP}} - U_{\text{std}}}{NkT}, \tag{4.29}$$

for different values of the deformation parameters α and α'. A behavior similar to that observed for the free-energy correction can clearly be identified. In the purely quadratic case ($\alpha \neq 0$, $\alpha' = 0$), both $\Delta S/Nk$ and $\Delta U/NkT$ remain negative and decrease almost linearly with temperature, indicating that the quadratic deformation reduces both the entropy and the internal energy compared to the standard ideal gas. This behavior is consistent with the interpretation of the quadratic GUP as introducing a minimal length that suppresses the number of accessible microstates and restricts the available phase space at high energies. In contrast, the purely linear deformation ($\alpha = 0$, $\alpha' \neq 0$) produces positive corrections over the entire temperature range, showing that the linear contribution enhances both entropy and internal energy.

For the mixed linear–quadratic deformation, the behavior becomes more intricate due to the competition between the two sectors. At low temperatures, the corrections are initially positive, reflecting the dominance of the linear contribution in the weak thermal regime. As the temperature increases, the quadratic term gradually becomes more important, causing the curves to decrease until they cross the zero line at a particular temperature where the deformed and standard quantities become equal. Beyond this transition point, the corrections become negative, signaling the dominance of quadratic minimal-length effects at higher temperatures. Furthermore, increasing the values of α and α' amplifies the magnitude of the deviations, leading to more pronounced departures from the standard thermodynamic behavior. Overall, these results show that the interplay between the linear and quadratic GUP terms strongly influences the thermal properties of the system and becomes increasingly significant as the temperature grows.

4.4 Harmonic oscillators in the canonical ensemble under GUP

We consider a system of N non-interacting quantum harmonic oscillators, each described by the Hamiltonian

$$H = \sum_{i=1}^{N} \left[\frac{p_i^2}{2m} + \frac{1}{2} m\omega^2 q_i^2 \right] \tag{4.30}$$

After introducing corrections for the generalized uncertainty principle, the distribution function for a single particle becomes the following relation;

$$Z(T,V,1) = \frac{1}{h^3} \left[\int d^3q \exp\left(-\frac{\beta m\omega^2}{2} q^2\right) \int \frac{d^3p}{(1-\alpha' p + \alpha p^2)^3} \exp\left(-\frac{\beta p^2}{2m}\right) \right] \tag{4.31}$$

When the scale is expanded to the second degree, it becomes

$$(1-\alpha' p + \alpha p^2)^{-3} \approx 1 + 3\alpha' p + (6\alpha'^2 - 3\alpha) p^2 \tag{4.32}$$

After performing Gauss's integration on spherical coordinates, we find

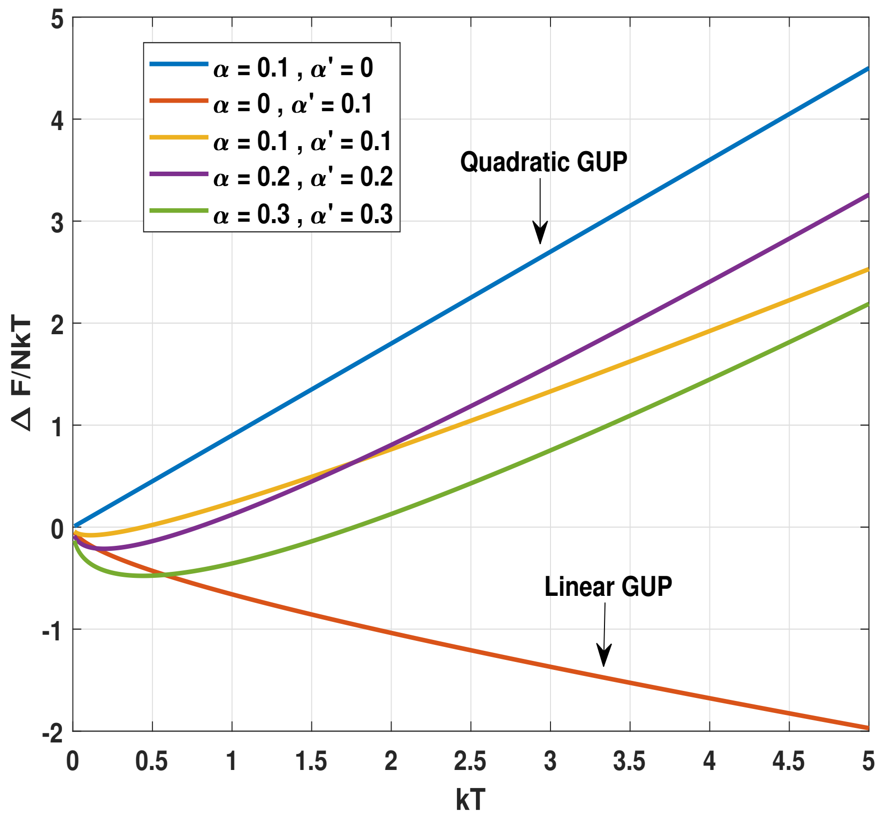


Figure 4.4: Variation of $\Delta F/NkT$ as a function of kT for different values of the deformation parameter α and α' for $m = N = 1$.

$$Z(T,V,1) = \left(\frac{kT}{\hbar\omega}\right)^3 \left[1 + 6\sqrt{2}\alpha'\sqrt{\frac{mkT}{\pi}} + 9(2\alpha'^2 - \alpha)mkT\right] \tag{4.33}$$

The distribution function for a number N harmonic oscillators is given by the following:

$$Z(T,V,N) = [Z(T,V,1)]^N = \left(\frac{kT}{\hbar\omega}\right)^{3N} \left[1 + 6\sqrt{\frac{2mkT}{\pi}}\alpha' + 9(2\alpha'^2 - \alpha)mkT\right]^N \tag{4.34}$$

Therefore, the calculation of free energy, which is expressed by the figure$F = -kT\ln Z$

$$F = -3NkT\ln\left(\frac{kT}{\hbar\omega}\right) - 6\sqrt{2}\alpha' NkT\sqrt{\frac{mkT}{\pi}} - 9(2\alpha'^2 - \alpha)Nmk^2T^2 \tag{4.35}$$

Internal energy is not volume-dependent, therefore thermodynamic pressure vanishes completely

$$p = -\left(\frac{\partial F}{\partial V}\right)_{T,N} = 0 \tag{4.36}$$

The entropy, obtained via

$$S = -\left(\frac{\partial F}{\partial T}\right)_{V,N}$$

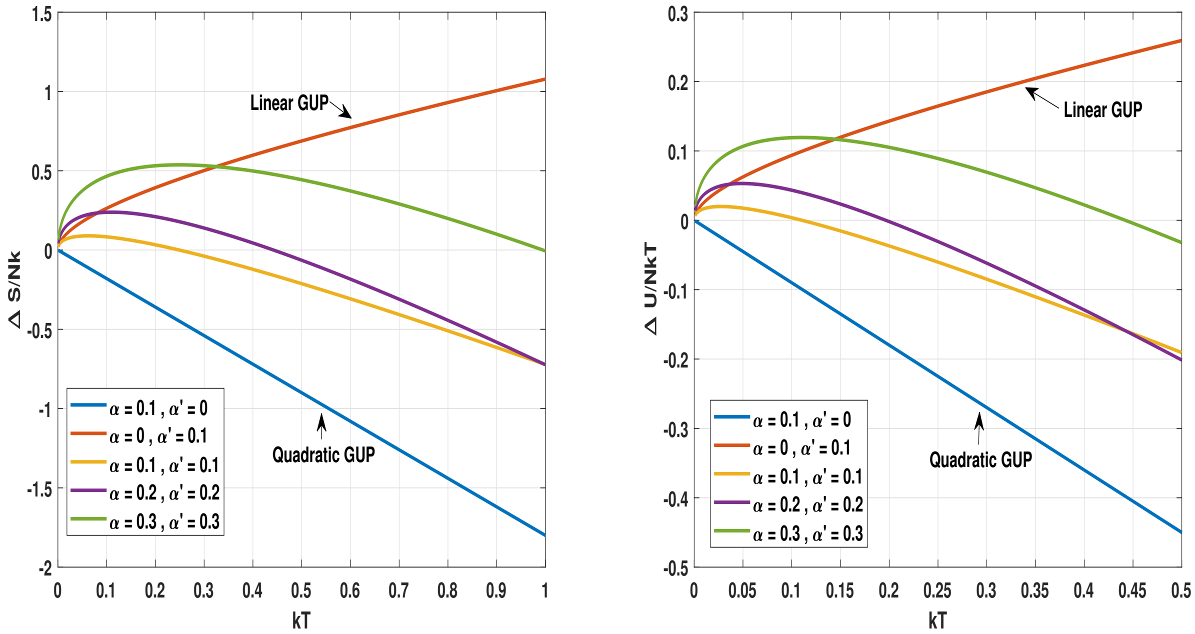


Figure 4.5: Variation of $\Delta S/Nk$ and $\Delta U/Nk$ as a function of kT for different values of the deformation parameter α and α'

:

$$S = -\left(\frac{\partial F}{\partial T}\right)_{V,N} = 3Nk\left[1+\ln\left(\frac{kT}{\hbar\omega}\right)\right] + 9\sqrt{2}\alpha' Nk\sqrt{\frac{mkT}{\pi}} + 18(2\alpha'^2-\alpha)Nmk^2T \tag{4.37}$$

The internal energy

$$U = F + TS$$

evaluates to

$$U = 3NkT + 3\sqrt{2}\alpha' NkT\sqrt{\frac{mkT}{\pi}} + 9(2\alpha'^2-\alpha)Nmk^2T^2 \tag{4.38}$$

$$C_V = 3Nk + \frac{9}{2}\sqrt{2}\alpha' Nk\sqrt{\frac{mkT}{\pi}} + 18(2\alpha'^2-\alpha)Nmk^2T \tag{4.39}$$

The application of the Generalized Uncertainty Principle (GUP) to the three-dimensional harmonic oscillator leads to modified thermodynamic quantities, namely the Helmholtz free energy F, entropy S, and internal energy U. The obtained expressions can be written as the sum of the standard contributions and correction terms induced by the deformation parameters α and α'. In the limit $\alpha = \alpha' = 0$, the ordinary thermodynamic expressions of the harmonic oscillator are recovered, which confirms the consistency of the formalism.

For convenience, the deviations from the standard thermodynamic quantities are defined as

$$\Delta F = F_{\text{GUP}} - F_{\text{std}}, \tag{4.40}$$

$$\Delta S = S_{\text{GUP}} - S_{\text{std}}, \tag{4.41}$$

and

$$\Delta U = U_{\text{GUP}} - U_{\text{std}}. \tag{4.42}$$

The behavior of these corrections as functions of kT is illustrated in Figs. (4.4) (4.5).

The free energy correction $\Delta F/NkT$ exhibits markedly different behaviors depending on the nature of the deformation. In the purely quadratic case ($\alpha' = 0$), the correction remains positive over the whole temperature range. This indicates that the quadratic GUP increases the free energy of the system compared to the standard harmonic oscillator. Moreover, the correction grows almost linearly with temperature, showing that quadratic effects become increasingly important at high thermal energies.

In contrast, for the purely linear deformation ($\alpha = 0$), the correction becomes negative and decreases monotonically with kT. Physically, this means that the linear GUP contribution lowers the free energy relative to the undeformed case. The magnitude of the deviation also increases with temperature, which highlights the sensitivity of the system to linear deformation effects in the high-temperature regime.

The mixed linear–quadratic case ($\alpha \neq 0$, $\alpha' \neq 0$) displays an intermediate and more interesting behavior. At low temperatures, the linear contribution dominates, causing ΔF to take negative values. As the temperature increases, the quadratic contribution becomes progressively stronger and compensates the negative linear term. Consequently, the curves cross the horizontal axis at a specific temperature where

$$F_{\text{GUP}} = F_{\text{std}}, \tag{4.43}$$

that is,

$$\Delta F = 0. \tag{4.44}$$

Beyond this transition point, the quadratic contribution dominates and the correction becomes positive. This crossover behavior clearly reflects the competition between the linear and quadratic deformation terms.

A similar analysis can be made for the entropy correction $\Delta S/Nk$. The quadratic deformation produces negative corrections, meaning that the entropy of the deformed system becomes smaller than the standard entropy. This reduction may be interpreted as a decrease in the number of accessible microstates due to the quadratic GUP effects. On the other hand, the linear deformation leads to positive entropy corrections, indicating an enhancement of the accessible states of the system. For the mixed deformation, the entropy correction first increases, reaches a maximum, and then decreases as temperature grows. This nontrivial behavior again results from the competition between the linear and quadratic contributions.

The internal energy correction $\Delta U/NkT$ follows trends similar to those observed for the entropy. In the quadratic case, the correction is negative, implying that the quadratic GUP lowers the internal energy relative to the standard oscillator. Conversely, the linear deformation yields positive corrections, corresponding to an increase in the internal energy. For the combined linear–quadratic deformation, the correction initially increases due to the dominance of the linear term, then decreases when the quadratic contribution becomes more significant at larger temperatures.

Overall, the figures clearly demonstrate that the thermodynamic properties of the harmonic oscillator are strongly affected by the type of GUP deformation considered. The quadratic deformation tends to increase the free energy while decreasing entropy and internal energy corrections, whereas the linear deformation produces the opposite effect. The mixed deformation reveals a transition between both regimes, emphasizing the interplay between linear and quadratic quantum gravity corrections. These results show that GUP effects can become significant at sufficiently high temperatures and may provide useful insights into quantum gravitational modifications of statistical systems.

4.5 THE ULTRA-RELATIVISTIC GAS UNDER THE GUP

Let's consider a perfect super-relative gas, where the particles move at speeds close to the speed of light, where the relationship between energy and momentum is expressed as follows:

$$E = |\vec{p}|c$$

After applying the job corrections, the financial distribution function is given

$$Z(T,V,1) = \frac{V}{h^3}\int \frac{d^3p}{(1-\alpha' p + \alpha p^2)^3}\exp(-\beta|p|c)$$

After expanding the denominator to second order and transforming to spherical coordinates in momentum space

$$d^3p = 4\pi p^2 dp \tag{4.45}$$

yields

$$Z(T,V,1) = \frac{4\pi V}{h^3}\int p^2 dp \exp(-\beta|p|c)\left[1+3\alpha' p + (6\alpha'^2 - 3\alpha)p^2\right]$$

We know that

$$\int_0^\infty p^n e^{-ap}dp = \frac{\Gamma(n+1)}{a^{n+1}} \tag{4.46}$$

After performing the Gaussian integrations, we find

$$Z(T,V,1) = 8\pi V\left(\frac{kT}{hc}\right)^3\left[1+9\alpha'\frac{kT}{c} + 36(2\alpha'^2-\alpha)\left(\frac{kT}{c}\right)^2\right] \tag{4.47}$$

For N indistinguishable particles, the full partition function is:

$$Z(T,V,N) = \frac{1}{N!}\left\{8\pi V\left(\frac{kT}{hc}\right)^3\left[1+9\alpha'\frac{kT}{c} + 36(2\alpha'^2-\alpha)\left(\frac{kT}{c}\right)^2\right]\right\}^N \tag{4.48}$$

The effect of the Generalized Uncertainty Principle (GUP) is fundamentally demonstrated by modifying the statistical weight of the system. The presence of a minimum quantized length, $\Delta x_{min} \sim \hbar\sqrt{\alpha}$, restricts the density of states at high momentum levels, thus reducing the number of states available to the system in phase space. To derive macroscopic thermodynamic properties, we use the Stirling approximation for large particle numbers, $\ln N! \approx N\ln N - N$, allowing us to write the Helmholtz free energy $F = -kT\ln Z$ as follows:

$$F = -NkT\ln\left[1+\frac{8\pi V}{N}\left(\frac{kT}{hc}\right)^3\right] - 9\alpha'\frac{Nk^2T^2}{c} - 36N(2\alpha'^2-\alpha)\frac{(kT)^3}{c^2} \tag{4.49}$$

The pressure statement is given as follows:

$$P = \frac{NkT}{V} \tag{4.50}$$

It is noteworthy that this derivation reproduces the standard state equation for a super-relativistic ideal gas ($P = \frac{NkT}{V}$). This result is consistent with the physical predictions of GUP models; the

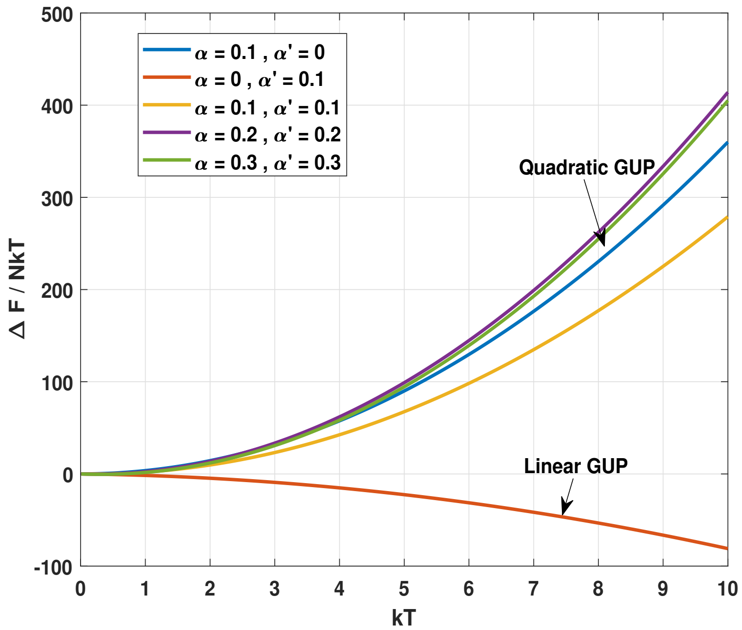
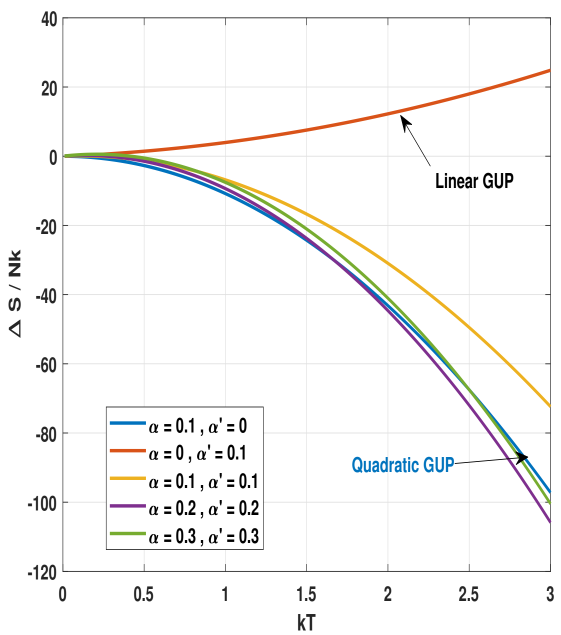


Figure 4.6: Variation of $\Delta S/Nk$ and $\Delta U/Nk$ as a function of kT for different values of the deformation parameter α and α'



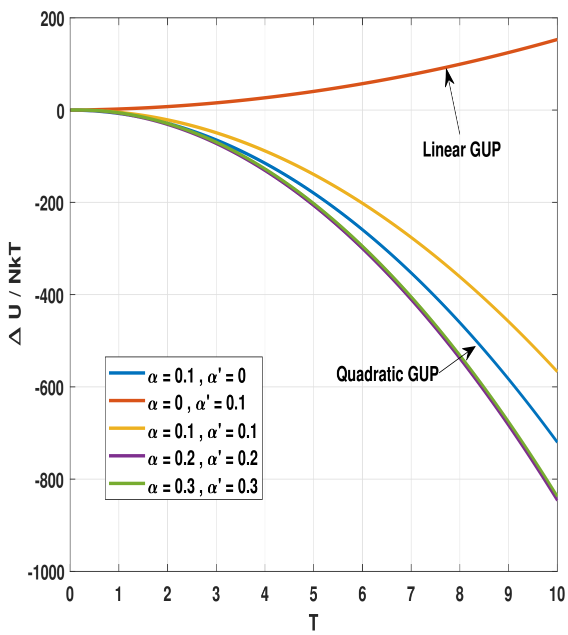


Figure 4.7: Variation of $\Delta S/Nk$ and $\Delta U/Nk$ as a function of kT for different values of the deformation parameter α and α'

correction limits arising from the generalized uncertainty principle depend exclusively on the microscopic variables related to momentum and temperature, and are volume-independent (V-independent). Consequently, the additional contribution of Helmholtz free energy does not generate additional mechanical stress when the derivation is performed with respect to volume, leaving the macroscopic mechanical behavior of the system constant despite quantum distortions in phase space:

$$S = Nk\left[4+\ln\left(\frac{8\pi V}{N}\left(\frac{kT}{hc}\right)^3\right)\right]+18\alpha'\frac{Nk^2T}{c}+108N(2\alpha'^2-\alpha)\frac{k^3T^2}{c^2} \tag{4.51}$$

Internal energy is given in the following form

$$U = 3NkT+9\alpha'\frac{Nk^2T^2}{c}+72N(2\alpha'^2-\alpha)\frac{(kT)^3}{c^2} \tag{4.52}$$

The expression for heat capacity is given as follows:

$$C_V = 3Nk+18\alpha'\frac{Nk^2T}{c}+216N(2\alpha'^2-\alpha)\frac{k^3T^2}{c^2} \tag{4.53}$$

Figures (4.6,4.7) show the temperature dependence of the normalized corrections to the free energy, entropy, and internal energy of the ultra-relativistic gas within the linear–quadratic GUP framework. In all cases, the corrections increase significantly with temperature, indicating that GUP effects become more pronounced in the high-energy regime, where ultra-relativistic systems are expected to be more sensitive to quantum gravity modifications.

For the purely quadratic deformation ($\alpha \neq 0$, $\alpha' = 0$), the correction to the free energy remains positive and grows rapidly with temperature. This means that the deformed free energy is larger than the standard one, with the deviation becoming increasingly important at high thermal energies. On the other hand, both the entropy and internal-energy corrections are negative throughout the considered range. This behavior reflects the restrictive effect of the quadratic GUP term, which is associated with the existence of a minimal length and the suppression of accessible microstates at high momenta. As the value of α increases, the magnitude of these deviations becomes larger, leading to stronger departures from the standard ultra-relativistic behavior.

In contrast, the purely linear deformation ($\alpha = 0$, $\alpha' \neq 0$) produces the opposite effect. In this case, the free-energy correction is negative, while the entropy and internal-energy corrections remain positive and increase with temperature. Physically, this indicates that the linear contribution modifies the phase space structure in a way that effectively enhances the number of accessible states, leading to larger entropy and internal energy compared to the standard system.

For the mixed linear–quadratic deformation ($\alpha \neq 0$, $\alpha' \neq 0$), the thermodynamic behavior results from the competition between the two contributions. At relatively low temperatures, the linear term tends to dominate, producing corrections similar to the purely linear case. However, as the temperature increases, the quadratic contribution becomes progressively more important due to its stronger temperature dependence. Consequently, the curves gradually shift toward the quadratic behavior, with the corrections eventually becoming dominated by minimal-length effects. This transition is clearly visible in the entropy and internal-energy plots, where the initially moderate deviations evolve into large negative corrections at high temperatures.

Overall, the figures demonstrate that the interplay between the linear and quadratic sectors of the GUP deformation has a strong impact on the thermodynamic properties of the ultra-relativistic gas. The corrections are negligible at low temperatures but become increasingly significant in the high-energy regime, emphasizing the relevance of GUP-inspired modifications in systems approaching Planck-scale physics.

4.6 Conclusion

In this chapter, we investigated the effects of the linear–quadratic Generalized Uncertainty Principle on the thermodynamic properties of several canonical ensemble systems. Using the deformed phase space measure, we derived modified partition functions and obtained the corresponding corrections to the free energy, entropy, and internal energy.

The results show that the thermodynamic behavior strongly depends on the type of deformation. The quadratic contribution generally increases the free energy while reducing entropy and internal energy, whereas the linear deformation produces the opposite effect. In the mixed linear–quadratic case, the competition between both contributions leads to crossover behaviors that become more pronounced at high temperatures.

Overall, these results demonstrate that GUP corrections can significantly influence the thermodynamics of statistical systems and provide useful insight into possible quantum gravity effects at high energies.

5 General Conclusion

In this thesis, we studied the implications of the Generalized Uncertainty Principle within the framework of statistical physics and canonical ensemble systems. We began by reviewing the theoretical foundations of GUP and discussing the main types of deformations, namely the linear, quadratic, and linear–quadratic forms. Particular attention was given to the physical interpretation of these deformations, including the emergence of a minimal measurable length and, under certain conditions, a maximal momentum.

We then established the deformed phase space formalism associated with the linear–quadratic GUP by deriving the corresponding Jacobian and modified density of states. This framework allowed us to construct corrected partition functions and systematically investigate the resulting thermodynamic modifications.

The formalism was applied to several physical systems, including the non-relativistic ideal gas, the ultra-relativistic gas, and the three-dimensional harmonic oscillator. Analytical expressions for the Helmholtz free energy, entropy, and internal energy were obtained, together with their leading GUP corrections. The analysis revealed that the thermodynamic behavior strongly depends on the nature of the deformation. In general, the quadratic sector introduces restrictive effects associated with minimal-length physics, while the linear contribution tends to produce opposite corrections. The combined linear–quadratic deformation leads to richer behaviors characterized by transitions between both regimes.

The obtained results confirm that GUP-induced modifications become increasingly important at high temperatures and energies, where quantum gravity effects are expected to emerge. Although these corrections remain extremely small at ordinary scales, they provide valuable insight into the possible interplay between quantum gravity and thermodynamics.

Finally, this work may be extended in several directions. Possible perspectives include the study of interacting systems, relativistic quantum gases in external fields, Bose–Einstein condensation, black-hole thermodynamics, and nonextensive statistical frameworks within GUP-inspired models. Such investigations could further clarify the physical consequences of minimal-length deformations and their role in connecting quantum mechanics, gravity, and statistical physics.

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