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Faculty of Mathematics and Material Sciences

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Specialty: Functional Analysis

Study of a non-linear Volterra integro-differential equation with a non-linear unknown source term

Prepared by: Hamada Zineb Kouloud

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Chairperson : Agoun Rachid

Supervisor : Karek Mohamed

Discussant : Abbassi Houssein

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Contents

Dedication:

First and foremost, all praise and thanks are due to God for His guidance and support, and for granting me the strength and patience to reach this moment.
I dedicate this achievement to myself, for every effort that went unseen and for every time I chose to keep going despite the challenges.
I extend my sincere gratitude to my beloved family, my constant source of support, for their love, encouragement, and trust, which have been my greatest motivation.
Special thanks to my dear aunt for standing by my side, supporting and encouraging me.
I would also like to thank my friends, who shared this journey with me through their support, laughter, and presence during the most difficult times.
To all of you, I express my deepest thanks and appreciation, as this achievement would not have been possible without you.

Abstract :

In this dissertation, we study the non-linear Volterra integro-differential equation. This equation plays a very important role in modeling phenomena in various sciences.

We prove the existence and uniqueness of the solution of the proposed equation by applying Schauder's fixed point theorem. We also use Krasnoselskii's fixed point theorem to handle cases where Schauder's compactness assumptions are not directly satisfied. The numerical solution of the equation is obtained using the Nyström approximation method.

Keywords: *Volterra integral equation; integro-differential equation; Schauder fixed point theorem; Krasnoselskii's fixed point theorem; numerical approximation; Nyström method.*

Résumé :

Dans ce mémoire, nous étudions l'équation intégro-différentielle non linéaire de Volterra. Cette équation joue un rôle très important dans la modélisation des phénomènes dans diverses sciences.

Nous prouvons l'existence et l'unicité de la solution de l'équation proposée en appliquant le théorème du point fixe de Schauder. Nous utilisons également le théorème du point fixe de Krasnoselskii pour traiter les cas où les hypothèses de compacité de Schauder ne sont pas directement satisfaites. La solution numérique de l'équation est obtenue en utilisant la méthode d'approximation de Nyström.

Mots clés :*Équation intégrale de Volterra, Équation intégro-différentielle, Théorème du point fixe de Schauder, Théorème du point fixe de Krasnoselskii, Approximation numérique, Méthode de Nyström.*

ملخص

في هذه المذكرة، ندرس معادلة فولتيرا التفاضلية الصغرى غير الخطية. تلعب هذه المعادلة دورًا مهمًا جدًا في نمذجة الظواهر في مختلف العلوم.

نثبت وجود ووحدانية الحل للمعادلة المقترحة من خلال تطبيق نظرية النقطة الثابتة لشاودر. نستخدم أيضًا نظرية النقطة الثابتة لكراسنوسيلسكي للتعامل مع الحالات التي لا تكون فيها شروط التراص لشاودر محققة بشكل مباشر. يتم الحصول على الحل العددي للمعادلة باستخدام طريقة تقريب نيستروم.

الكلمات المفتاحية: معادلة فولتيرا التكاملية، معادلة تفاضلية صغرى، نظرية النقطة الثابتة لشاودر، نظرية النقطة الثابتة لكراسنوسيلسكي، تقريب عددي.

Introduction

Since the late nineteenth century, a new type of equation known as Fredholm or Volterra integral equations has emerged. This type of equation plays a very important role compared to differential equations due to its wide applications [4, 5, 6], in all fields of science: physics, chemistry, biology, engineering, electronics, signal processing, quantum mechanics, and others. Moreover, most of these phenomena can be modeled as integral equations for the purpose of studying them in an easy and inexpensive way.

It is also well known that linear and nonlinear Volterra and Fredholm integral equations play a fundamental role in many other scientific and applied fields, such as population dynamics, epidemic modeling, and semiconductor physics, due to their ability to describe systems that depend on the entire history of their evolution ("memory")[5, 9, 14, 15] . As a result, researchers have dedicated significant resources to understanding the different types of these equations [2, 6, 7, 11] and improving techniques for studying them [4, 10, 12, 13].

In this work, we consider a new class of Volterra integro-differential equations of the second kind, where the kernel depends on both the unknown function and its first derivative. We study two cases:

- **First case (Chapter 2):** The source term is linear, denoted by $f(t)$, depending only on t:

$$u(t) = \int_a^t k(t, s, u(s), u'(s))\, ds + f(t), \quad \forall t \in [a, b]$$

- **Second case (Chapter 3):** The source term is nonlinear, denoted by $f(t, u(t))$, depending also on the solution u:

$$u(t) = \int_0^t k(t, s, u(s), u'(s))\, ds + f(t, u(t)), \quad \forall t \in [0, 1]$$

where:

- $u(t)$ is the unknown function to be determined,
- f is the source term (linear or nonlinear),
- k is a known kernel, generally nonlinear, depending on t, s, $u(s)$ and $u'(s)$.

Therefore, this dissertation is divided into three chapters as follows:

- **Chapter 1:** We present the basic concepts, tools, and theorems that will be used in the following chapters (such as Schauder's theorem, Krasnoselskii's theorem, and the Nyström technique).
- **Chapter 2:** We study the equation in the linear case where the source term $f(t)$ is independent of $u(t)$. We prove existence and uniqueness using Schauder's fixed point theorem, then we treat this equation numerically using the Nyström technique to find an approximate solution.
- **Chapter 3:** We study the equation in the nonlinear case where $f(t, u(t))$ depends explicitly on $u(t)$. We prove existence and uniqueness using Krasnoselskii's fixed point theorem, then we also apply the Nyström technique to obtain the numerical approximate solution.

At the end of each of the two chapters, we present numerical examples illustrating the obtained results.

Chapter 1

Fundamental notions and preliminaries

In this chapter, we recall some basic notions.

1.1 Banach spaces

1.1.1 Normed space

Definition 1.1.1. *Let V be a vector space. A norm is a function defined on V with values in $\mathbb{R}^+$, denoted by $\|\cdot\|$, such that the following three properties are satisfied:*

(1) $\forall v \in V, \|v\|_V = 0 \Leftrightarrow v = 0.$

(2) $\forall \lambda \in \mathbb{R}, \forall v \in V, \|\lambda v\|_V = |\lambda| \|v\|_V.$

(3) $\forall v, w \in V, \|v + w\|_V \leq \|v\|_V + \|w\|_V.$ *(Triangle inequality)*

We then say that V is a normed vector space (n.v.s).

1.1.2 Cauchy sequence

Definition 1.1.2. *Let $(U_n)_{n\in\mathbb{N}}$ be a sequence in V, We say that $(U_n)_{n\in\mathbb{N}}$ is a Cauchy sequence for the norm $\|\cdot\|_V$ if:*

$$\forall \varepsilon > 0, \exists N \geq 0, \forall n > N, \forall p \geq 0, \|U_{n+p} - U_n\|_V \leq \varepsilon.$$

Proposition 1.1.3. *Every Cauchy sequence is bounded.*

Proposition 1.1.4. *Every convergent sequence is Cauchy.*

Remark: Not every Cauchy sequence is necessarily convergent in an arbitrary space. Hence, the interest in introducing a new definition: complete spaces.

1.1.3 Banach space

Definition 1.1.5 (Complete space or Banach space)**.** *A normed vector space V is said to be complete for the norm $\| \cdot \|_V$ if every Cauchy sequence is convergent (for this norm). Such a space is also called a Banach space.*

Example 1.1.6. *Consider $C^1([a,b],\mathbb{R})$, the set of real functions of class C^1 on $[a,b]$. We equip this space with the norm*

$$\|f\|_{C^1} = \|f\|_\infty + \|f'\|_\infty, \quad f \in C^1([a,b],\mathbb{R}).$$

Then, $(C^1([a,b],\mathbb{R}), \| \cdot \|_1)$ is a Banach space.

1.2 Fixed point theorems

For nonlinear equations, fixed point theorems are generally used to show the existence of their solutions. We mention here the Banach and Schauder fixed point theorems.

Definition 1.2.1. *Let X be a Banach space and $T : X \longrightarrow X$. We say that x is a fixed point of T if and only if:*

$$x = T(x).$$

1.2.1 Krasnoselskii's fixed point theorem

Let X be a Banach space, and let M be a **closed**, **convex**, and **non-empty** subset of X. Consider two operators $A : M \to X$ and $B : M \to X$ that satisfy the following conditions:[5]

- **(Contraction)** A is a contraction mapping. There exists a constant $k \in [0,1)$ such that for all $x, y \in M$:

  $$\|A(x) - A(y)\| \leq k\|x - y\|.$$

- **(Compactness)** B is **compact** (i.e., it is continuous and maps bounded subsets into relatively compact subsets).

- **(Invariance)** For all $x, y \in M$, the following holds:

$$A(x) + B(y) \in M.$$

If these conditions are satisfied, then the operator $A + B$ has at least one **fixed point** $x^* \in M$. In other words, there exists $x^* \in M$ such that:

$$A(x^*) + B(x^*) = x^*.$$

Proof. For more details on the proof of this theorem, see reference[5]. ☐

1.2.2 Schauder fixed point theorem

Theorem 1.2.2. *Let S be a closed convex subset of a Banach space X and $T : S \to S$. If T is continuous and $T(S)$ is relatively compact, then T admits a fixed point [27].*

1.2.3 Banach fixed point theorem

Theorem 1.2.3. *Let X be a Banach space. If T is a contraction mapping from X into X, then T possesses a unique fixed point [27].*

1.3 Useful analytical properties

1.3.1 Lipschitz function

Let I be an interval of $\mathbb{R}$ and f a function from I to $\mathbb{R}$. We say that f is Lipschitz with constant $k > 0$ if for all x, y in I,

$$|f(x) - f(y)| \leq k|x - y|.$$

More generally, one can define a Lipschitz map from a metric space to another. The map is called a contraction if $k < 1$.

1.3.2 Mean value theorem

Let $f : [a, b] \to \mathbb{R}$ be a function continuous on $[a, b]$ and differentiable on $]a, b[$. Then there exists c belonging to $]a, b[$ such that

$$f(b) - f(a) = f'(c)(b - a).$$

1.3.3 Derivative of an integral function

Let $G(x)$ be a function defined by the following integral formula:

$$G(x) = \int_{a(x)}^{b(x)} g(x,t)\,dt,$$

then its derivative is given by:

$$\frac{dG}{dx}(x) = (\int_{a(x)}^{(b(x)} g(t)\,dt)' = b'(x)g(x,b(x)) - a'(x)g(x,a(x)) + \int_{a(x)}^{b(x)} \frac{dg}{dx}(x,t)\,dt.$$

Example 1.3.1. *Consider the following Volterra equation:*

$$u(t) = \int_a^t k(t,s,u(s))\,ds + f(t),$$

then the derivative of the previous equation is given by:

$$u'(t) = k(t,t,u(t)) + \int_a^t \frac{\partial k}{\partial t}(t,s,u(s))\,ds + f'(t).$$

Lemma 1.3.2. *[8] Let $\varphi(t)$ be a positive continuous function on $[a,b]$ such that:*

$$\exists L > 0; \quad \phi(t) \le L \int_a^t \phi(s)\,ds,$$

then

$$\forall t \in [a,b], \phi(t) = 0.$$

Proof. Since $\phi(t)$ is a positive continuous function on $[a,b]$, then $\exists \eta > 0$ such that:

$$\phi(t) \le \eta \quad \forall t \in [a,b],$$

then,

$$\phi(t) \le L\eta \int_a^t ds = L\eta(t-a).$$

By induction, we find:

$$\phi(t) \le L^n \eta \frac{(t-a)^n}{n!} \xrightarrow[n \to +\infty]{} 0.$$

□

1.4 Numerical integration methods

The main goal of numerical integration methods is to estimate the value of the definite integral over a particular domain for a given function. There are several techniques; we mention the Trapezoidal method and Simpson's method [30, 11].

1.4.1 Trapezoidal method

The Trapezoidal method consists of dividing the integration interval $[a, b]$ into n equal segments, thus obtaining $(n+1)$ equally spaced points. We set:

$$x_i = a + ih, \quad i = 0, 1 \ldots n, \quad \text{with} \quad h = \frac{b-a}{n}.$$

We approximate the area of each "slice" by a trapezoid constructed from the values of the function at the boundaries of each sub-interval.

$$\int_{x_{n-1}}^{x_n} f(x)dx = h\frac{f(x_{n-1}) + f(x_n)}{2}.$$

The function f is thus replaced by a straight line (polynomial of degree 1) on each sub-interval. By adding the sum of the areas of all the trapezoids, we obtain:

$$\begin{aligned}
\int_a^b f(x)\,dx &= \int_{x_0}^{x_n} f(x)\,dx, \\
&= \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)\,dx, \\
&= \frac{h}{2}(f(x_0) + f(x_1)) + \frac{h}{2}(f(x_1) \\
&+ f(x_2)) + \cdots + \frac{h}{2}(f(x_{n-1}) + f(x_n)).
\end{aligned}$$

we can simplify this expression as follows:

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(a) + f(b) + 2\sum_{i=1}^{n-1} f(x_i)\right].$$

1.4.2 Simpson's method

Simpson's method consists of dividing the integration interval $[a, b]$ into n equal segments with n an even number ($n = 2m$), thus obtaining $(2m + 1)$ equally spaced points:

$$x_n = a + ih, \quad i = 0, 1...n, \quad \text{with} \quad h = \frac{b-a}{n},$$

Approximate the function f on each "slice" by a parabola constructed from three consecutive points.

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx + \int_{x2}^{x_4} f(x)\,dx + \cdots + \int_{x_{2m-2}}^{x_{2m}} f(x)\,dx.$$

Between x_0 and x_2 passing through x_1, there are three interpolation points, so we can replace the function $f(x)$ by a polynomial of degree 2:

$$\int_{x_0}^{x_2} f(x)\,dx = \int_{x_0}^{x_2} P_2(x)\,dx.$$

According to Lagrange interpolation, this polynomial can be written as:

$$P_2(x) = f(x_0)L_0(x) + f(x_1)L_1(x) + f(x_2)L_2(x),$$

with

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)},$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)},$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}.$$

If we set $x - x_1 = th$, so

$$x - x_2 = (x - x_1) - (x_2 - x_1) = th - h = h(t - 1)$$

$$x - x_0 = (x - x_1) + (x_1 - x_0) = th + h = h(t + 1)$$

which yields

$$L_0(x) = \frac{1}{2}(t^2 - t), L_1(x) = 1 - t^2, L_2(x) = \frac{1}{2}(t^2 + t)$$

. by replacing $p_2(x)$ with its expression, we obtain:

$$\begin{aligned}\int_{x_0}^{x_2} f(x)\,dx &= \frac{f(x_0)}{2}h\int_{-1}^{1}(t^2-t)\,dt + f(x_1)h\int_{-1}^{1}(1-t^2)\,dt\\ &+ \frac{f(x_2)}{2}h\int_{-1}^{1}(t^2-t)\,dt,\\ &= \frac{f(x_0)}{3}h + \frac{4f(x_1)}{3}h + \frac{f(x_2)}{3}h,\\ &= \frac{h}{3}\Big(f(x_0)+4f(x_1)+f(x_2)\Big).\end{aligned}$$

we deduce that:

$$\int_a^b f(x)\,dx = \frac{h}{3}\left(f(a)+f(b)+4\sum_{i=1}^{m} f(x_{2i-1}) + 2\sum_{i=1}^{m-1} f(x_{2i})\right).$$

Chapter 2

Analytical and numerical study of the first model (with linear source term f(t))

2.1 Analytical Study of the First Model

In this chapter, we follow the same approach described in [8], in order to demonstrate the existence and uniqueness of the solution of the proposed equation.

2.1.1 Problem formulation and hypotheses

We consider the following nonlinear Volterra equation of integro-differential type:

$$u(t) = \int_a^t k(t, s, u(s), u'(s))\, ds + f(t), \quad \forall t \in [a, b] \tag{2.1}$$

We assume that k is defined by:

$$k : [a, b]^2 \times \mathbb{R}^2 \to \mathbb{R}, \quad (t, s, x, y) \to k(t, s, x, y)$$

satisfies the following hypotheses:

$$(H_1) \begin{cases} 1)\dfrac{\partial k}{\partial t} \in C([a, b]^2 \times \mathbb{R}^2), \\ 2)\exists M \in \mathbb{R}_+^*, \quad \forall t, s \in [a, b], \forall x, y \in \mathbb{R}, \\ \quad \max\left(|k(t, s, x, y)|, \left|\frac{\partial k}{\partial t}(t, s, x, y)\right|\right) \leq M. \end{cases}$$

Furthermore, for $f \in C^1([a, b], \mathbb{R})$, we differentiate equation (2.1) to obtain another equation that contains more information about the solution u, given by:

$$u'(t) = k(t,t,u(t),u'(t)) + \int_a^t \frac{\partial k}{\partial t}(t,s,u(s),u'(s))\,ds + f'(t), \quad \forall t \in [a,b]. \tag{2.2}$$

In the next section, we will see that the hypothesis (H_1) allows us to ensure the existence of the solution of equation (2.1).

2.1.2 Existence via Schauder's fixed point theorem

We consider the notation $U = (u, u') \in \mathfrak{B} = C([a,b],\mathbb{R}) \times C([a,b],\mathbb{R})$. Therefore, we can group equations (2.1) and (2.2) into the problem:

$$U = \phi_f(U),$$

where ϕ_f is the functional defined from $\mathfrak{B}$ into itself by:

$$\phi_f(U) = (\phi_1(U), \phi_2(U)), \quad \forall f \in C^1([a,b],\mathbb{R}),$$

with

$$\phi_1(U)(t) = \int_a^t k(t,s,u(s),u'(s))\,ds + f(t), \quad \forall t \in [a,b],$$

$$\phi_2(U)(t) = k(t,t,u(t),u'(t)) + \int_a^t \frac{\partial k}{\partial t}(t,s,u(s),u'(s))\,ds + f'(t), \quad \forall t \in [a,b].$$

It is clear that if ϕ_f has a fixed point, then equation (2.1) admits a solution. This means we must prove that ϕ_f satisfies the conditions of the Schauder fixed point theorem.

Proposition 2.1.1. *For all $f \in C^1([a,b],\mathbb{R})$, ϕ_f is continuous from $\mathfrak{B}$ into itself.*

Proof. For more details, see reference [8]. □

Theorem 2.1.2. *Let F be the set defined by:*
$F = \{\xi \in C^1([a,b],\mathbb{R}), \xi(a) = f(a), \forall t \in [a,b], |\xi(t)-f(t)| \leq M(a-b), |\xi'(t)-f'(t)| \leq M(b-a+1)\}$.
Thus, $\phi_f(F)$ is relatively compact.

Proof. We will show that $\phi_f(F)$ is relatively compact.

First, it is clear that the set F is closed and convex. For any $\xi \in F$ and any $t \in [a, b]$, we have:

$$\begin{aligned}
|\phi_1(\xi)(t) - f(t)| &= \left| \int_a^t k(t, s, \xi(s), \xi'(s))\, ds + f(t) - f(t) \right|, \\
&\leq \int_a^t k(t, s, \xi(s), \xi'(s))\, ds, \\
&\leq M \int_a^t 1\, ds, \\
&\leq M \int_a^b 1\, ds, \\
&\leq M(b - a).
\end{aligned}$$

Similarly, we find:

$$\begin{aligned}
|\phi_2(\xi)(t) - f'(t)| &= \left| k(t, t, \xi(t), \xi'(t)) + \int_a^t \frac{\partial k}{\partial t}(t, s, \xi(s), \xi'(s))\, ds + f(t)' - f'(t) \right|, \\
&\leq |k(t, t, \xi(t), \xi'(t))| + \left| \int_a^t \frac{\partial k}{\partial t}(t, s, \xi(s), \xi'(s))\, ds \right|, \\
&\leq M + M \int_a^b 1\, ds, \\
&\leq M + M(b - a), \\
&\leq M(b - a + 1).
\end{aligned}$$

We deduce $\phi_f(F) \subset F$, which confirms that $\phi_f(F)$ is uniformly bounded. It remains to prove that $\phi_f(F)$ is equicontinuous. For all $t_1, t_2 \in [a, b]$ with $t_1 > t_2$, we have:

$$\begin{aligned}
|\phi_1(U)(t_1) - \phi_1(U)(t_2)| \leq& \left| \int_a^{t_1} k(t_1, s, u(s), u'(s))\, ds - \int_a^{t_2} k(t_2, s, u(s), u'(s))\, ds \right| \\
&+ |f(t_1) - f(t_2)|, \\
\leq& \left| \int_a^{t_2} k(t_1, s, u(s), u'(s))\, ds + \int_{t_2}^{t_1} k(t_1, s, u(s), u'(s))\, ds \right. \\
&\left. - \int_a^{t_2} k(t_2, s, u(s), u'(s))\, ds + f(t_1) - f(t_2) \right|, \\
\leq& \left| \int_a^{t_2} k(t_1, s, u(s), u'(s))\, ds - k(t_2, s, u(s), u'(s))\, ds \right| \\
&+ \left| \int_{t_2}^{t_1} k(t_1, s, u(s), u'(s))\, ds \right| + |f(t_1) - f(t_2)|,
\end{aligned}$$

the mean value theorem gives us:

$$\begin{aligned}
|\phi_1(U)(t_1) - \phi_1(U)(t_2)| &\leq M\,|t_1 - t_2| \int_a^{t_2} 1\,ds + M \int_{t_2}^{t_1} 1\,ds \\
&+ \max_{s\in[a,b]} |f'(s)|\,|t_1 - t_2|\,, \\
&\leq M\,|t_1 - t_2|\,(t_2 - a) + M\,|t_1 - t_2| \\
&+ \max_{s\in[a,b]} |f'(s)|\,|t_1 - t_2|\,, \\
&\leq (M(b - a + 1) + \max_{s\in[a,b]} |f'(s)|)\,|t_1 - t_2|\,, \\
&\leq C_1\,|t_1 - t_2|\,.
\end{aligned}$$

Similarly one can write:

$$\begin{aligned}
|\phi_2(U)(t_1) - \phi_2(U)(t_2)| &\leq \left| \int_a^{t_2} \frac{\partial k}{\partial t}(t_1, s, u(s), u'(s))\,ds - \int_a^{t_2} \frac{\partial k}{\partial t}(t_2, s, u(s), u'(s))\,ds \right| \\
&+ \left| \int_{t_2}^{t_1} \frac{\partial k}{\partial t}(t_1, s, u(s), u'(s))\,ds \right| + |f'(t_1) - f'(t_2)| \\
&+ |k(t_1, t_1, u(t_1), u'(t_1) - k(t_2, t_2, u(t_2), u'(t_2))|\,,
\end{aligned}$$

we also apply mean value theorem to obtain:

$$\begin{aligned}
|\phi_2(U)(t_1) - \phi_2(U)(t_2)| &\leq M\,|t_1 - t_2| \int_a^{t_2} 1\,ds + M \int_{t_2}^{t_1} 1\,ds \\
&+ o(|t_1 - t_2|) + M\,|t_1 - t_2|\,, \\
&\leq M\,|t_1 - t_2|\,(t_2 - a) + M\,|t_1 - t_2| + o(|t_1 - t_2|) \\
&+ M\,|t_1 - t_2|\,, \\
&\leq (M(b - a + 2))\,|t_1 - t_2| + o(|t_1 - t_2|), \\
&= C_2\,|t_1 - t_2| + o(|t_1 - t_2|).
\end{aligned}$$

The two preceding inequalities demonstrate equicontinuity. Since $\phi_f(F)$ is uniformly bounded and equicontinuous, then by Arzelà-Ascoli theorem, we can say that the set $\phi_f(F)$ is relatively compact. Finally, Schauder's theorem confirms that ϕ_f has a fixed point, which means that equation (2.1) admits at least one solution. □

2.1.3 Uniqueness of the solution

First, to show the uniqueness of the solution of equation (2.1), we need to add the following hypotheses:

$$(H_2)\begin{cases} 1)\exists A, B, \bar{A}, \bar{B} \in \mathbb{R}_+, \forall x, y, \bar{x}, \bar{y} \in \mathbb{R}, \forall t, s \in [0,T] \\ \quad |k(t,s,x,y) - k(t,s,\bar{x},\bar{y})| \leq A|x-\bar{x}| + B|y-\bar{y}|, \\ \quad \left|\frac{\partial k}{\partial t}(t,s,x,y) - \frac{\partial k}{\partial t}(t,s,\bar{x},\bar{y})\right| \leq \bar{A}|x-\bar{x}| + \bar{B}|y-\bar{y}| \\ 2)B < 1. \end{cases}$$

Theorem 2.1.3. *Let* (H_1) *and* (H_2) *be satisfied. Then, the solution of equation (2.1) is unique.*

Proof. Assume that there exist two solutions $u, v \in C^1([a,b],\mathbb{R})$ for equation (2.1). Let us define $\gamma(t)$, For all $t \in [a,b]$by:

$$\gamma(t) = |v(t) - u(t)| + |v'(t) - u'(t)|.$$

it is clear that:

$$\begin{cases} u(t) = \int_a^t k(t,s,u(s),u'(s))\,ds + f(t) \quad u \quad solution \\ \\ v(t) = \int_a^t k(t,s,v(s),v'(s))\,ds + f(t) \quad v \quad solution \end{cases}$$

By subtraction we obtain:

$$\begin{aligned} |u(t) - v(t)| &\leq \int_a^t |k(t,s,u(s),u'(s)) - k(t,s,v(s),v'(s))|\,ds \\ &\leq \int_a^t [A|u(s) - v(s)| + B|u'(s) - v'(s)|]\,ds \\ &\leq \max(A,B) \int_a^t \gamma(s)\,ds \end{aligned}$$

Hence

$$|u(t) - v(t)| \leq \max(A,B) \int_a^t \gamma(s)\,ds. \tag{2.3}$$

also we have:

$$\begin{cases} u'(t) = k(t,t,u(t),u'(t)) + \int_a^t \frac{\partial k}{\partial t}(t,s,u(s),u'(s))\,ds \\ \\ v'(t) = k(t,t,v(t),v'(t)) + \int_a^t \frac{\partial k}{\partial t}(t,s,u(s),v'(s))\,ds \end{cases}$$

subtraction gives :

$$\begin{aligned}|u'(t)-v'(t)| &\leq |k(t,t,u(t),u'(t))-k(t,t,v(t),v'(t))| \\ &\quad + \int_a^t \left|\frac{\partial k}{\partial t}(t,s,u(s),u'(s))-\frac{\partial k}{\partial t}(t,s,v(s),v'(s))\right| ds \\ &\leq A|u(t)-v(t)|+B|u'(t)-v'(t)|+\max(\bar{A},\bar{B})\int_a^t \gamma(s)\,ds\end{aligned}$$

Using inequality (2.3), we get:

$$\begin{aligned}|u'(t)-v'(t)| &\leq A\max(A,B)\int_a^t \gamma(s)\,ds + B|u'(t)-v'(t)| + \max(\bar{A},\bar{B})\int_a^t \gamma(s)\,ds, \\ &= B|u'(t)-v'(t)| + (A\max(A,B)+\max(\bar{A},\bar{B}))\int_a^t \gamma(s)\,ds, \\ &= B|u'(t)-v'(t)| + \alpha\int_a^t \gamma(s)\,ds,\end{aligned}$$

with

$$\alpha = A\max(A,B)+\max(\bar{A},\bar{B}).$$

Since $B<1 \Rightarrow 1-B>0$, we have:

$$(1-B)|u'(t)-v'(t)| \leq \alpha\int_a^t \gamma(s)\,ds,$$

and thus

$$|u'(t)-v'(t)| \leq \frac{\alpha}{1-B}\int_a^t \gamma(s)\,ds. \tag{2.4}$$

Adding inequalities (2.3) and (2.4):

$$|u(t)-v(t)|+|u'(t)-v'(t)| \leq \left(\frac{\alpha}{1-B}+\max(A,B)\right)\int_a^t \gamma(s)\,ds,$$

so

$$\gamma(t) \leq L\int_a^t \gamma(s)\,ds,$$

with

$$L = \frac{\alpha}{1-B}+\max(A,B).$$

Using Lemma 1.3.2, we obtain:

$$\gamma(t)=0 \implies \begin{cases} u(t)=v(t) \\ u'(t)=v'(t).\end{cases}$$

Therefore, the solution of the equation is unique. □

2.2 Numerical study of the first model

In general, we cannot determine the exact solution of the proposed equation, so we must use numerical techniques to approximate its solutions[1, 4, 3, 2]. In this chapter, we will use the method known as: the Nyström numerical method [8] based on numerical integration to obtain an approximate solution to our equation.

2.2.1 Nyström approximation

Under hypotheses $(H1)$, $(H2)$ it has been shown that equation (2.1) admits a unique solution. In this section, we use the Nyström method to approximate its solution. We define the general form of the numerical integration by: for $N \in \mathbb{N}$:

$$\int_a^b f(t)\,dt \cong h \sum_{i=0}^{N} w_i f(t_i)$$

where wi are real numbers such that there exists $w > 0$:

$$\forall N \in \mathbb{N},\ \max_{0 \leq i \leq N} |w_i| \leq w.$$

Applying this quadrature to equations (2.1) and (2.2),we obtain the following algebraic system:

- $U_0 = f(a)$,
- $V_0 = f'(a) + k(a, a, U_0, V_0)$,
- $U_n = f(t_n) + h \sum_{i=0}^{n} w_i k(t_n, t_i, U_i, V_i), 1 \leq n \leq N, \quad (2.5)$
- $V_n = f'(t_n) + k(t_n, t_n, U_n, V_n) + h \sum_{i=0}^{n} w_i \frac{\partial k}{\partial t}(t_n, t_i, U_i, V_i), \quad (2.6)$

where U_n, V_n approach $u(t_n)$, $u'(t_n)$ respectively. In the following, we will deal with the algebraic system obtained

2.2.2 Algebraic system and solvability

In order to study the existence and uniqueness of the solution of system (2.5) and (2.6), it is necessary to add the following new hypothesis:

$$(H_3): \quad A < 1.$$

Theorem 2.2.1. *Let $(H_1), (H_2), (H_3)$ be satisfied, and for sufficiently small h, then system (2.5) and (2.6) has a unique solution.*

Proof. Suppose the space $\mathbb{R}^2$ has the following usual norm:

$$\forall \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \quad \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\| = |x| + |y|.$$

For all $n \geq 1$, we define:

$$\psi_n \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(t_n) + h\omega_n k(t_n, t_n, x, y) + h \sum_{i=0}^{n-1} \omega_i k(t_n, t_i, U_i, V_i) \\ \\ f'(t_n) + k(t_n, t_n, x, y) + h\omega_n \frac{\partial k}{\partial t}(t_n, t_n, x, y) + h \sum_{i=0}^{n-1} \omega_i \frac{\partial k}{\partial t}(t_n, t_i, U_i, V_i) \end{pmatrix}$$

It is clear that:

$$\left\| \psi_n \begin{pmatrix} x \\ y \end{pmatrix} - \psi_n \begin{pmatrix} x' \\ y' \end{pmatrix} \right\| = \left\| \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \right\|$$

with

$$\begin{aligned} |\beta_1| &= h\omega_n |k(t_n, t_n, x, y) - k(t_n, t_n, x', y')|, \\ &\leq h\omega(A|x - x'| + B|y - y'|), \end{aligned}$$

and

$$\begin{aligned} |\beta_2| &= |k(t_n, t_n, x, y) - k(t_n, t_n, x', y')| + h\omega_n \left| \frac{\partial k}{\partial t}(t_n, t_n, x, y) - \frac{\partial k}{\partial t}(t_n, t_n, x', y') \right|, \\ &\leq A|x - x'| + B|y - y'| + h\omega(\bar{A}|x - x'| + \bar{B}|y - y'|), \\ &= (A + h\omega\bar{A})|x - x'| + (B + h\omega\bar{B})|y - y'|. \end{aligned}$$

Adding the two inequalities:

$$|\beta_1| + |\beta_2| \leq (\omega h A + A + \omega h \bar{A})|x - x'| + (\omega h B + B + \omega h \bar{B})|y - y'|.$$

Then

$$\left\| \psi_n \begin{pmatrix} x \\ y \end{pmatrix} - \psi_n \begin{pmatrix} x' \\ y' \end{pmatrix} \right\| \leq L \left\| \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x' \\ y' \end{pmatrix} \right\|$$

where

$$L = \max\left(\omega h A + A + \omega h \bar{A},\ \omega h B + B + \omega h \bar{B}\right).$$

Since h is small, $B < 1$ and $A < 1$, we deduce that $0 < L < 1$, therefore, by the Banach fixed point theorem, ψ_n admits a fixed point, which confirms that the solution of system (2.5) and (2.6) is unique. ☐

2.2.3 Error analysis and convergence

In this section, we will show that the numerical method constructed in the previous section is convergent. For this reason, we define:

$$\varepsilon_n = |U_n - u(t_n)| + |V_n - u'(t_n)|, \quad n \geq 0.$$

We say that the method is convergent if:

$$\lim_{h \to 0} \left(\max_{0 \leq n \leq N} \varepsilon_n \right) = 0,$$

which we will see in the following theorem:

Theorem 2.2.2. *Let* $(H_1), (H_2), (H_3)$ *be satisfied. Then,*

$$\lim_{h \to 0} \left(\max_{0 \leq n \leq N} \varepsilon_n \right) = 0.$$

Proof. Initially, the consistency of the Nyström method [8] gives us:

$$\begin{aligned} \delta(h, t_n) &= \delta_1 + \delta_2, \quad \text{where,} \\ \delta_1 &= \left| \int_a^{t_n} k(t_n, s, u(s), u'(s))ds - h \sum_{i=0}^{n} \omega_i k(t_n, t_i, u(t_i), u'(t_i)) \right| \\ \delta_2 &= \left| \int_a^{t_n} \frac{\partial k}{\partial t}(t_n, s, u(s), u'(s))ds - h \sum_{i=0}^{n} \omega_i \frac{\partial k}{\partial t}(t_n, t_i, u(t_i), u'(t_i)) \right|, \end{aligned}$$

with

$$\forall u \in C^1([a,b], \mathbb{R}), \quad \lim_{h \to 0} \left(\max_{0 \leq n \leq N} \delta(h, t_n) \right) = 0.$$

It is clear that for $n \geq 1$:

$$\begin{aligned} |u(t_n) - U_n| &= \left| \int_a^{t_n} k(t_n, s, u(s), u'(s))ds - h \sum_{i=0}^{n} \omega_i k(t_n, t_i, U_i, V_i) \right|, \\ &\leq \left| \int_a^{t_n} k(t_n, s, u(s), u'(s))ds - h \sum_{i=0}^{n} \omega_i k(t_n, t_i, u(t_i), u'(t_i)) \right|, \\ &+ \left| h \sum_{i=0}^{n} \omega_i k(t_n, t_i, u(t_i), u'(t_i)) - h \sum_{i=0}^{n} \omega_i k(t_n, t_i, U_i, V_i) \right|, \\ &\leq h\omega A |u(t_n) - U_n| + h\omega B |u'(t_n) - V_n| \\ &+ \delta_1 + h\omega \max(A, B) \sum_{i=0}^{n-1} \varepsilon_i. \quad (2.7) \end{aligned}$$

the same way, we find:

$$\begin{aligned}|u'(t_n)-V_n| &\leq |k(t_n,t_n,u(t_n),u'(t_n))-k(t_n,t_n,U_n,V_n)|\\ &+\left|\int_a^{t_n}\frac{\partial k}{\partial t}(t_n,s,u(s),u'(s))ds-h\sum_{i=0}^{n}\omega_i\frac{\partial k}{\partial t}(t_n,t_i,u(t_i),u'(t_i))\right|,\\ &+\left|h\sum_{i=0}^{n}\omega_i\frac{\partial k}{\partial t}(t_n,t_i,u(t_i),u'(t_i))-h\sum_{i=0}^{n}\omega_i\frac{\partial k}{\partial t}(t_n,t_i,U_i,V_i)\right|,\\ &\leq (A+h\omega\bar{A})|u(t_n)-U_n|+(B+h\omega\bar{B})|u'(t_n)-V_n|\\ &+\delta_2+h\omega\max(\bar{A},\bar{B})\sum_{i=0}^{n-1}\varepsilon_i. \quad (2.8)\end{aligned}$$

From (2.7) and (2.8), we obtain:

$$\begin{aligned}|u(t_n)-U_n|+|u'(t_n)-V_n| \leq{}& (A+h\omega A+h\omega\bar{A})|u(t_n)-U_n|\\ &+(B+h\omega B+h\omega\bar{B})|u'(t_n)-V_n|\\ &+\left[h\omega\max(A,B)+h\omega\max(\bar{A},\bar{B})\right]\sum_{i=0}^{n-1}\varepsilon_i\\ &+\delta_1+\delta_2,\end{aligned}$$

then,

$$\varepsilon_n\leq\alpha\varepsilon_n+h\omega\max(A+\bar{A},B+\bar{B})\sum_{i=0}^{n-1}\varepsilon_i+\delta(h,t_n),$$

where

$$\alpha=\max(A+h\omega A+h\omega\bar{A},\ B+h\omega B+h\omega\bar{B}).$$

Since h is sufficiently small, $A<1$ and $B<1$, we have:

$$0<\alpha<1\Rightarrow\varrho=1-\alpha>0,$$

then

$$\varepsilon_n\leq\frac{h\omega\max(A+\bar{A},B+\bar{B})}{\varrho}\sum_{i=0}^{n-1}\varepsilon_i+\frac{\delta(h,t_n)}{\varrho}.$$

By applying Theorem 7.1 of [21], we obtain:

$$\varepsilon_n\leq\frac{1}{\varrho}\left(1+\frac{h\omega\max(A+\bar{A},B+\bar{B})}{\varrho}\right)^{n-1}\left(\max_{1\leq i\leq n}\delta(h,t_i)+h\omega\max(A+\bar{A},B+\bar{B})\varepsilon_0\right)$$

the other hand, we have:

$$\left(1+\frac{h\omega\max(A+\bar{A},B+\bar{B})}{\varrho}\right)^{n-1}\leq\left(1+\frac{(b-a)\omega\max(A+\bar{A},B+\bar{B})}{N\varrho}\right)^{N}\xrightarrow[N\to+\infty]{}\theta<\infty.$$

Finally, we deduce that if h tends to zero, then:

$$\lim_{h\to 0}\left(\max_{0\leq n\leq N}\varepsilon_n\right)=0.$$

□

2.2.4 Numerical results

In this section, we will treat two examples in order to prove the efficiency of the proposed numerical method. We mention that the examples satisfy hypotheses (H_1), (H_2) and (H_3), then we present the error tables and the figures using the Matlab software [25].

Example 1:

We consider the following integral equation:

$$u(t) = \int_0^t \frac{st}{1 + (u(s) + u'(s))^2} ds + \left[1 - \frac{\pi}{4} + \frac{\ln(2)}{2}\right] t + \tan(t) - \frac{t}{2} \ln(t^2 + 2t + 2),$$

where the exact solution of this equation is $u(t) = t$. We use the trapezoidal quadrature formula, then apply the successive Picard iteration technique to find the approximate solution of system (2.5) and (2.6), considering a tolerance of order 10^{-7}. Next, we calculate the error obtained for each subdivision, given by:

$$e_N = \max_{0 \leq n \leq N} \{|u(t_n) - U_n|,\ |u'(t_n) - V_n|\}.$$

We also draw the figures which represent the exact and approximate solution.

N	eN
10	7.76e-03
50	1.68e-03
100	8.50e-04
250	3.42e-04
500	1.71e-04
1000	8.57e-05

Table 2.1: Numerical results for example 1.

Interpretation: The error table shows that if h approaches zero, then the error e_N also approaches zero. This confirms that the approximate solution converges to the exact solution. Moreover, it can also be seen in the graphs presented below

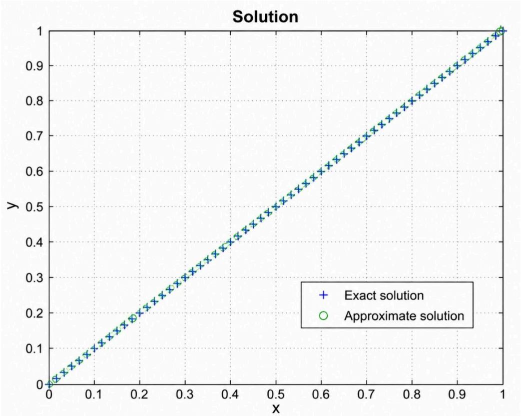


Figure 2.1: The solution (exact and approximate) N=100

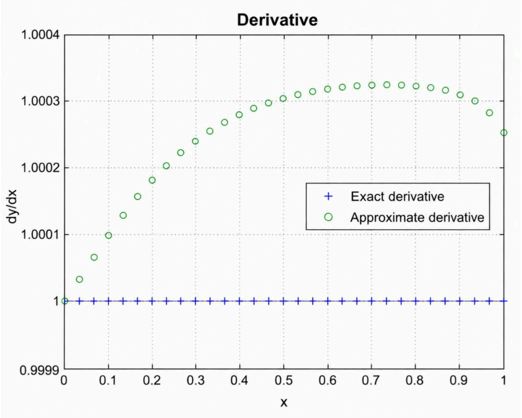


Figure 2.2: The derivative of the solution (exact and approximate)N=100

Example 2:

We consider the second integral equation:

$$u(t) = \int_0^t \frac{1}{2} \cos[s + t - \sin(2\pi s) - 2\pi \cos(2\pi s) + u(s) + u'(s)]ds + f(t),$$

where,

$$f(t) = \sin(2\pi t) - \sin(t)(\cos(t) - \frac{1}{2}).$$

The exact solution is $u(t) = \sin(2\pi t)$. We treat this equation in the same way as for studying example 1, but using the Simpson quadrature formula.

N	eN
10	4.85e-02
50	9.74e-03
100	4.90e-03
250	1.95e-03
500	9.75e-04
1000	4.87e-04

Table 2.2: Numerical results for example 2.

Interpretation: The second error table and the graphs presented below for example 2 also confirm the convergence of the approximate solution to the exact solution. Finally, from these numerical results we can judge that the Nyström method is efficient for approximating this type of Volterra integral equations.

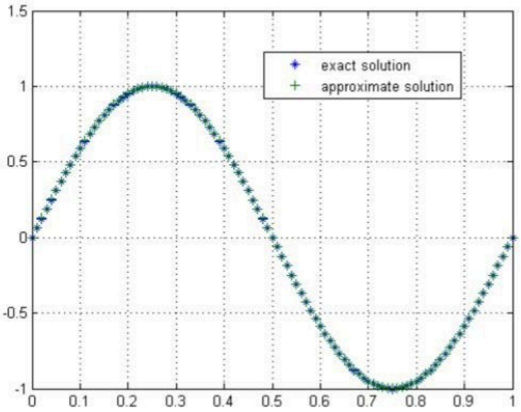


Figure 2.3: The solution (exact and approximate) N = 100

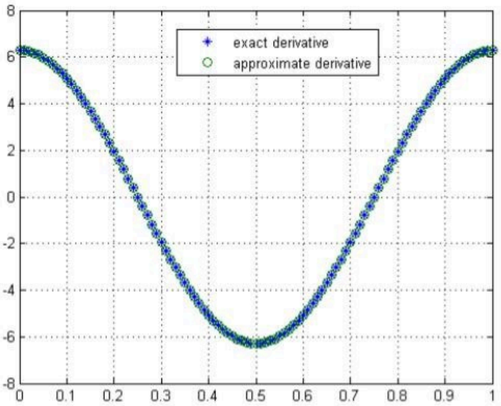


Figure 2.4: The derivative of the solution (exact and approximate) N = 100

Chapter 3

Analytical and Numerical Analysis of the second Model(with Nonlinear source Term f(t, u(t)))

3.1 Analytical Analysis of the second Model

3.2 Extended Hypotheses

The Volterra and Fredholm integral equations have a great importance in both theoretical and practical mathematics, since their presence and crucial significance in several fields, including physics, engineering, and biology[10, 16, 24, 28]. As a result, researchers have dedicated significant resources to comprehending the many kinds of these equations [7, 13, 14, 19] and improving techniques for studying them [8, 22, 18, 20].

In this chapter, we consider a new class of Volterra equation as follows:

$$u(t) = \int_0^t k(t, s, u(s), u'(s))\, ds + f(t, u(t)), \quad \forall t \in [0, 1] \tag{3.1}$$

where:

$f(t, x) \in C^1([0, 1] \times \mathbb{R}, \mathbb{R})$and $k(t, s, x, y) \in C^1([0, 1] \times \mathbb{R}^2, \mathbb{R})$are given functions and $u(t)$ is the unknown solution which is to be found in the space $C^1([0, 1])$.

We mention that the kernel k is nonlinear and contains the unknown $u(t)$ and its derivative $u'(t)$, and the source term f is also nonlinear and contains the unknown $u(t)$.

Before studying this equation, we need to derive it with respect to variable t to get an alternate equation that provides additional insight into the solution $u(t)$ as follows:

$$u'(t) = k(t,t,u(t),u'(t)) + \int_0^t \frac{\partial k}{\partial t}(t,s,u(s),u'(s))\,ds + f_1(t,u(t)) + f_2(t,u(t))\,u'(t) \tag{3.2}$$

in which,

$$f_1(t,u(t)) = \frac{\partial f(t,u(t))}{\partial t}, \qquad f_2(t,u(t)) = \frac{\partial f(t,u(t))}{\partial x}.$$

So, our main objective is to reveal this new kind of equation according to the next sections that involve conducting a thorough analysis of the equation as follows: First, we examine the system of equations (3.1)-(3.2) analytically by proving the existence and uniqueness of its solution using Krasnoselskii's fixed point theorem [15]. Second, we treat our equations numerically by applying the Nyström technique [2] in order to approach their solutions. Finally, we provide some numerical examples.

3.2.1 Existence via Krasnoselskii's fixed point theorem

In order to establish the existence of the solution of system (3.1)-(3.2), it is necessary to impose certain hypotheses $(\mathcal{H}_1)$ on functions k, f and u', which are given in the following manner:
$(\mathcal{H}_1)$: $\forall t, s \in [0,1]$, $\forall x, y \in \mathbb{R}$, there exists $M > 0$, $F > 0$, $F_1 > 0$, $F_2 > 0$, $R > 0$, $L > 0$, $L_1 > 0$, $L_2 > 0$, $\theta > 0$, such that:

- $\max(|k(t,s,x,y)|, |\frac{\partial k}{\partial t}(t,s,x,y)|) \leq M$,
- $|f(t,x)| \leq F$, $|f_1(t,x)| \leq F_1$, $|f_2(t,x)| \leq F_2$,
- $|u'(t)| \leq R$,
- $|f(t,x) - f(t,y)| \leq L|x-y|$,
- $|f_1(t,x) - f_1(t,y)| \leq L_1|x-y|$,
- $|f_2(t,x) - f_2(t,y)| \leq L_2|x-y|$,

- $\theta = \max\{L + L_1 + RL_2, F_2\} < 1$.

Theorem 3.2.1. *Let ($\mathcal{H}_1$) be verified. Then the system of equations (3.1)-(3.2) has at least one solution.*

Proof. Consider the Banach space $C^1([0,1])$ which is equipped with the following norm:

$$\|u\|_{C^1} = \|u\|_\infty + \|u'\|_\infty = \sup_{t\in[0,1]} |u(t)| + \sup_{t\in[0,1]} |u'(t)|.$$

We define the operator T from the space $C^1([0,1])$ into itself, as a decomposition into the sum of two operators T_1 and T_2 as shown below:

$$T(u)(t) = T_1(u)(t) + T_2(u)(t), \qquad \forall u \in C^1([0,1]), \qquad \forall t \in [0,1],$$

where,

$$T_1(u)(t) = \int_0^t k(t,s,u(s),u'(s))ds, \qquad T_2(u)(t) = f(t,u(t)).$$

Clearly, if the operator T has a fixed point on the space $C^1([0,1])$ then the system of equations (3.1)-(3.2) has at least one solution, which we will prove by using the Krasnoselskii fixed point theorem, when the theorem states that the following conditions are fulfilled:

1. $\forall u, v \in \mathcal{E}, T_1(u) + T_2(v) \in \mathcal{E}$, where $\mathcal{E}$ is a given non-empty, closed and convex subset,
2. T_1 is compact and continuous,
3. T_2 is a contraction.

First, we take into account the subset $\mathcal{E}$ by the following form:

$$\mathcal{E} = \{u \in C^1([0,1]), \|u\|_{C^1} \le 3M + F + F_1 + F_2R\}.$$

1. It is evident to see that the subset $\mathcal{E}$ is non-empty, closed and convex. For all $u, v \in \mathcal{E}$ we have:

   $$|T_1(u)(t)| = \left|\int_0^t k(t,s,u(s),u'(s))ds\right| \le \int_0^1 |k(t,s,u(s),u'(s))|ds \le M.$$

   Similarly, one may find:

   $$|T_1'(u)(t)| = |k(t,t,u(t),u'(t)) + \int_0^t \frac{\partial k}{\partial t}(t,s,u(s),u'(t))ds|,$$

$$\leq |k(t,t,u(t),u'(t))| + \int_0^1 \left|\frac{\partial k}{\partial t}(t,s,u(s),u(s))\right| ds \leq 2M.$$

Therefore, we obtain: $\|T_1(u)\|_{C^1} \leq 3M$. On the other hand, we have:

$$|T_2(v)(t)| = |f(t,v(t))| \leq F,$$

And

$$|T_2'(v)(t)| = |f_1(t,v(t))| + |f_2(t,v(t))v'(t))| \leq F_1 + F_2R,$$

then $\|T_2(v)\|_{C^1} \leq F + F_1 + F_2R$. Thereby: $\|T_1(u) + T_2(v)\|_{C^1} \leq 3M + F + F_1 + F_2R$. We deduce that $T_1(u) + T_2(v) \in \mathcal{E}$.

2. It is clear that T_1 is continuous. The inequality $\|T_1(u)\|_{C^1} \leq 3M$ confirms us that $T_1(\mathcal{E})$ is uniformly bounded, so to verify that T_1 is compact we just need to prove that $T_1(\mathcal{E})$ is equicontinuous.
For all $u \in \mathcal{E}$, for all $t_1, t_2 \in [0,1]$, with $t_1 > t_2$ we have:

$$\begin{aligned} |T_1(u)(t_1) - T_1(u)(t_2)| &= \left|\int_0^{t_1} k(t_1,s,u(s),u'(s))ds - \int_0^{t_2} k(t_2,s,u(s),u'(s))ds\right|, \\ &\leq \left|\int_0^{t_2} k(t_1,s,u(s),u'(s))ds\right. \\ &+ \left.\int_{t_2}^{t_1} k(t_1,s,u(s),u'(s))ds - \int_0^{t_2} k(t_2,s,u(s),u'(s))ds\right| \\ &\leq \int_0^{t_2} |k(t_1,s,u(s),u'(s))ds \\ &- k(t_2,s,u(s),u'(s))|ds + \int_{t_2}^{t_1} |k(t_1,s,u(s),u'(s))|ds. \end{aligned}$$

The mean value theorem gives us:

$$|T_1(u)(t_1) - T_1(u)(t_2)| \leq M|t_1 - t_2| \int_0^{t_2} 1ds + M \int_{t_2}^{t_1} 1ds \leq 2M|t_1 - t_2|.$$

Similarly, we can get:

$$\begin{aligned} |T_1'(u)(t_1) - T_1'(u)(t_2)| &\leq \int_0^{t_2} \left|\frac{\partial k}{\partial t}(t_1,s,u(s),u'(s))ds - \frac{\partial k}{\partial t}(t_2,s,u(s),u'(s))\right| ds, \\ &+ \int_{t_2}^{t_1} \left|\frac{\partial k}{\partial t}(t_1,s,u(s),u'(s))\right| ds, \\ &+ |k(t_1,t_1,u(t_1),u'(t_1)) - k(t_2,t_2,u(t_2),u'(t_2))|. \end{aligned}$$

Furthermore, the function $\frac{\partial k}{\partial t}(t,s,x,y)$ is continuous with respect to the variable t over the interval $[0,1]$, so it is uniformly continuous, which permits us to write:

$$\left|\frac{\partial k}{\partial t}(t_1,s,u(s),u'(s)) - \frac{\partial k}{\partial t}(t_2,s,u(s),u'(s))\right| \leq \delta(|t_1-t_2|) \to 0 \, as \, t_1 \to t_2. \tag{3.3}$$

From (3.3) and by using the mean value theorem again, we get:

$$\begin{aligned} |T_1'(u)(t_1) - T_1'(u)(t_2)| &\leq \delta(|t_1-t_2|) + M\int_{t_2}^{t_1} 1ds + M|t_1-t_2|, \\ &= \delta(|t_1-t_2|) + 2M|t_1-t_2|. \end{aligned}$$

We conclude that $|T_1(u)(t_1) - T_1(u)(t_2)| \to 0$ and $|T_1'(u)(t_1) - T_1'(u)(t_2)| \to 0$ as $t_1 \to t_2$. So, $T_1(\mathcal{E})$ is equicontinuous, and according to the Arzelà-Ascoli theorem, we can say that $T_1(\mathcal{E})$ is relatively compact, i.e. T_1 is compact.

3. Now, we go to prove T_2 is contraction mapping. For all $u, v \in \mathcal{E}$, for all $t \in [0,1]$, we have:

$$|T_2(u)(t) - T_2(v)(t)| = |f(t,u(t)) - f(t,v(t))| \leq L|u(t) - v(t)|,$$

Subsequently

$$\|T_2(u) - T_2(v)\|_\infty \leq L\|u-v\|_\infty. \tag{3.4}$$

Also

$$\begin{aligned} |T_2'(u)(t) - T_2'(v)(t)| &= |f_1(t,u(t)) + f_2(t,u(t))u'(t) - f_1(t,v(t)) \\ &\quad - f_2(t,v(t))v'(t)|, \\ &\leq |f_1(t,u(t)) - f_1(t,v(t))| + |f_2(t,u(t))u'(t) \\ &\quad - f_2(t,v(t))v'(t)|, \\ &\leq |f_1(t,u(t)) - f_1(t,v(t))| + |f_2(t,u(t)) \\ &\quad - f_2(t,v(t))|R + F_2|u'(t) - v'(t)|, \\ &\leq L_1|u(t)-v(t)| + RL_2|u(t)-v(t)| \\ &\quad + F_2|u'(t)-v'(t)|, \end{aligned}$$

Then

$$\|T_2'(u) - T_2'(v)\|_\infty \leq (L_1 + RL_2)\|u-v\|_\infty + F_2\|u'-v'\|_\infty. \tag{3.5}$$

From (3.4) and (3.5) we get:

$$\begin{aligned}\|T_2(u)-T_2(v)\|_{C^1} &\leq (L+L_1+RL_2)\|u-v\|_\infty + F_2\|u'-v'\|_\infty,\\ &\leq \max\{L+L_1+RL_2, F_2\}\|U-V\|_{c^1}\\ &= \theta\|u-v\|_{C^1}.\end{aligned}$$

since $0<\theta<1$,we conclude that T_2 is a contraction mapping. thus, the operator T has a fixed point, which means that the system (3.1)-(3.2) has at least one solution.

3.2.2 Uniqueness of The solution

In the previous theorem we have demonstrated that the hypothesis $(\mathcal{H}_\infty)$ guarantees the existence of a solution of the system of equation (3.1)-(3.2) in order to ensure the uniqueness of this solution, the following additional hypotheses $(\mathcal{H})$ is required.
$(\mathcal{H}_2)$: For all $t, s \in [0,1]$, for all $x, x', y, y' \in \mathbb{R}$, there exist $A>0, B>0, \bar{A}>0, \bar{B}>0$ such that:

- $|k(t,s,x,y)-k(t,s,x',y')| \leq A|x-x'| + B|y-y'|$,
- $|\frac{\partial k}{\partial t}(t,s,x,y)-\frac{\partial k}{\partial t}(t,s,x',y')| \leq \bar{A}|x-x'| + \bar{B}|y-y'|$,
- $B+F_2<1$.

□

Theorem 3.2.2. *Let $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ be verified. Then the solution of the system of equations (3.1)-(3.2) is unique .*

Proof. We assume that there exist two solutions $u(t), v(t) \in C^1([0,1])$for the system (3.1)-(3.2) we define the function $\gamma(t)$ for $t\in[0,1]$by:

$$\gamma(t) = |u(t)-v(t)| + |u'(t)-v'(t)|$$

. From lemma 3 in [8],if we can prove that:
$\exists C>0, \quad \text{such that } \gamma(t) \leq C\int_0^t \gamma(s)ds, \quad \text{then } \gamma(t)=0.$
which means that the solution of the system (3.1)-(3.2)is unique. For all

$t \in [0,1]$, it is apparent that:

$$\begin{aligned}|u(t)-v(t)| &\leq \int_0^t |k(t,s,u(s),u'(s)) - k(t,s,v(s),v'(s))|ds \\ &+ |f(t,u(t)) - f(t,v(t))|, \\ &\leq \int_0^t [A|u(s)-v(s)| + B|u'(s)-v'(s)|]ds + L|u(t)-v(t)|, \\ &\leq \max(A,B)\int_0^t \gamma(s)ds + L|u(t)-v(t)|.\end{aligned}$$

According to the parameter θ, we obtain that $L<1$. this is meaning that:

$$|u(t)-v(t)| \leq \frac{\max(A,B)}{1-L}\int_0^t \gamma(s)ds. \tag{3.6}$$

In a similar way we get:

$$\begin{aligned}|u'(t)-v'(t)| &\leq |k(t,t,u(t),u'(t)) - k(t,t,v(t),v'(t))| \\ &+ |f_1(t,u(t)) - f_1(t,v(t))|, \\ &+ |f_2(t,u(t)) - f_2(t,v(t))||u'(t)| + |f_2(t,v(t))||u'(t)-v'(t)| \\ &+ \int_0^t \left|\frac{\partial k}{\partial t}(t,s,u(s),u'(s)) - \frac{\partial k}{\partial t}(t,s,v(s),v'(s))\right| ds, \\ &\leq A|u(t)-v(t)| + B|u'(t)-v'(t)| + \max(\bar{A},\bar{B})\int_0^t \gamma(s)ds \\ &+ L_1|u(t)-v(t)| + RL_2|u(t)-v(t)| + F_2|u'(t)-v'(t)|, \\ &= (A+L_1+RL_2)|u(t)-v(t)| + (B+F_2)|u'(t)-v'(t)| \\ &+ \max(\bar{A},\bar{B})\int_0^t \gamma(s)ds.\end{aligned}$$

From hypothesis $(\mathcal{H}_2)$ we have $B+F_2<1$, then:

$$|u'(t)-v'(t)| \leq \frac{A+L_1+RL_2}{1-(B+F_2)}|u(t)-v(t)| + \frac{\max(\bar{A},\bar{B})}{1-(B+F_2)}\int_0^t \gamma(s)ds.$$

By using the inequality (3.6), we obtain:

$$|u'(t)-v'(t)| \leq \left(\frac{(A+L_1+RL_2)\max(A,B) + (1-L)\max(\bar{A},\bar{B})}{(1-L)(1-(B+F_2))}\right)\int_0^t \gamma(s)ds. \tag{3.7}$$

By combining the inequalities (3.6) and (3.7) we get that:

$$\exists C > 0, \quad \text{such that} \gamma(t) \leq C \int_0^t \gamma(s) ds,$$

where,

$$C = \frac{(A + L_1 + RL_2)\max(A, B) + (1 - L)\max(\bar{A}, \bar{B})}{(1 - L)(1 - (B + F_2))} + \frac{\max(A, B)}{1 - L}.$$

Thus, by applying the Lemma 3 described above, we find $\gamma(t) = 0$, i.e. $u(t) = v(t)$ and $u'(t) = v'(t)$. Therefor, the system (3.1)-(3.2) has a unique solution. □

3.3 Numerical Analysis of the second Model

In the previous section we have shown that the system of equations (3.1)-(3.2) has one solution in the space $C^1([0, 1])$. But it is very hard to find this exact solution. For this reason, we approach the solution of this system by the following strategy: First, we discretise the equations (3.1)-(3.2) by using the well- known Nyström method [2] which gives us a non linear algebraic system. Second, we solve this algebraic system by applying the successive Picard method [29].

3.3.1 Nyström Discretisation

In beginning, we define the generic of Nyström method as: For $N \in \mathbb{N}$, for $g \in C([0, 1])$, we have:

$$\int_0^1 g(t) dt \simeq h \sum_{i=0}^{N} \omega_i g(t_i), \tag{3.8}$$

Where:
$h = \frac{1}{N}$ is the step of discretisation, $t_i = ih$, $i = 0, \ldots, N$, and ω_i are real numbers (weights) which satisfy: $\max_{0 \leq i \leq N} |\omega_i| \leq \omega$.
By applying the previous formula (3.8), the equations (3.1) and (3.2) yield the following nonlinear algebraic system:

- $U_0 = f(0, U_0)$,
- $V_0 = k(0, 0, U_0, V_0) + f_1(0, U_0) + f_2(0, U_0)V_0$,
- $U_n = h \sum_{i=0}^{n} \omega_i k(t_n, t_i, U_i, V_i) + f(t_n, U_n), \quad 1 \leq n \leq N, \quad (3.9)$

- $V_n = k(t_n, t_n, U_n, V_n) + h\sum_{i=0}^{n} \omega_i \frac{\partial k}{\partial t}(t_n, t_i, U_i, V_i) + f_1(t_n, U_n) + f_2(t_n, U_n)V_n,$

  $1 \leq n \leq N, \quad (3.10)$

Where: U_n and V_n approaches $u(t_n)$ and $u'(t_n)$, respectively.

3.3.2 Existence and uniqueness of the Algebraic system solution

Now, an important question must be asked: has the algebraic system (3.9)-(3.10) a unique solution under the previous hypotheses $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$? That we will see in the next theorem.

Theorem 3.3.1. *Let* $(\mathcal{H}_1) - (\mathcal{H}_2)$ *be verified. Also we assume that:* $A + L + L_1 + RL_2 < 1$. *And when* h *is sufficiently small, then the non linear algebraic system (3.9)-(3.10) has a unique solution.*

Proof. First, from hypotheses $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ we have $L < 1$ and $B + F_2 < 1$, respectively. So, U_0 and V_0 are exists. Now, consider the Euclidean space $\mathbb{R}^2$ equipped the following usual norm:

$$\forall \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2, \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\| = |x| + |y|.$$

For $1 \leq n \leq N$, the algebraic system (3.9)-(3.10) can be represented by the following application:

$$\psi_n \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} h\omega_n k(t_n, t_n, x, y) + f(t_n, x) + s_1 \\ k(t_n, t_n, x, y) + h\omega_n \frac{\partial k}{\partial t}(t_n, t_n, x, y) + f_1(t_n, x) + f_2(t_n, x)y + s_2 \end{pmatrix}.$$

where,
$s_1 = h\sum_{i=0}^{n-1} \omega_i k(t_n, t_i, U_i, V_i)$, $s_2 = h\sum_{i=0}^{n-1} \omega_i \frac{\partial k}{\partial t}(t_n, t_i, U_i, V_i)$.
It is clear that:

$$\left\| \psi_n \begin{pmatrix} x \\ y \end{pmatrix} - \psi_n \begin{pmatrix} x' \\ y' \end{pmatrix} \right\| = \left\| \begin{pmatrix} \varrho_1 \\ \varrho_2 \end{pmatrix} \right\|$$

with

$$\begin{aligned} |\varrho_1| &\leq h|\omega_n||k(t_n, t_n, x, y) - k(t_n, t_n, x', y')| + |f(t_n, x) - f(t_n, x')| \\ &\leq h\omega(A|x - x'| + B|y - y'|) + L|x - x'| \\ &= (L + hA\omega)|x - x'|) + hB\omega|y - y'| \end{aligned}$$

.

and:

$$
\begin{aligned}
|\varrho_2| \leq{} & |k(t_n,t_n,x,y)-k(t_n,t_n,x',y')| + h\omega_n|\frac{\partial k}{\partial t}(t_n,t_n,x,y)-\frac{\partial k}{\partial t}(t_n,t_n,x',y')| \\
& + |f_1(t_n,x)-f_1(t_n,x')| + |f_2(t_n,x)-f_2(t_n,x')y'|, \\
& \leq A|x-x'| + B|y-y'| + h\omega(\bar{A}|x-x'| + \bar{}|y-y'|) \\
& + L_1|x-x'| + RL_2|x-x'| + F_2|y-y'|, \\
& = (A + h\bar{A}\omega + L_1 + RL_2)|x-x'| + (B + h\bar{B}\omega + F_2)|y'-y'|
\end{aligned}
$$

thus,
$|\varrho_1| + |\varrho_2| \leq (hA\omega + h\bar{A}\omega + A + L + L_1 + RL_2)|x-x'| + (hB\omega + h\bar{B}\omega + B + F_2)|y-y'|$.
Consequently

$$\|\psi_n\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) - \psi_n\left(\begin{pmatrix} x' \\ y' \end{pmatrix}\right)\| \leq \rho\|\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x' \\ y' \end{pmatrix}\|$$

Where:
$\varrho = \max\{hA\omega + h\bar{A}\omega + A + L + L_1 + RL_2, hB\omega + h\bar{B}\omega + B + F_2\}$. Assuming that h is very small, we obtain $\rho = \max\{A + L + L_1 + RL_2, B + F_2\} < 1$. Which confirms that ψ_n has a one fixed point according to the Banach fixed-point theorem. So, we conclude the existence and uniqueness of the solution of the algebraic system (3.9)-(3.10). □

3.4 Numerical Examples

In this section, we present two illustrative examples, where the functions of these examples satisfy all previous hypotheses. We choose the Trapezoidal rule in Nyström method to build the algebraic system (3,8)-(3,9), then we solve this system in the Matlab software by using the Picard method under a tolerance of order 10^{-7}. After getting the approximate solutions of our examples, we need to define the following error functions:

$$E_t^1 = |u(t_i) - U_t|, \quad E_t^2 = |u'(t_i) - V_t|, \quad E^1 = \max_{0\leq i\leq N}\{E_t^1\} \text{ and } E^2 = \max_{0\leq i\leq N}\{E_t^2\},$$

in order to compare between the exact and approximate solutions for a different values of N.

Example 3.4.1. *Consider the following equation:*

$$u(t) = 2\theta\int_0^t \frac{\sin(\theta t)\cos(\theta s)}{(1+\sin(\theta s)+u(s)^2+\theta^{-2}u'(s)^2)^2}ds + f(t,u(t)), \quad \forall t \in [0,1],$$

where,

$$f(t,u(t)) = \frac{2\sin(\theta t)}{2+u(t)}, \quad \textit{and} \quad \theta = \frac{\pi}{4}.$$

The exact solution of this example and its derivative are given by:

$$u(t) = \sin(\theta t), \qquad u'(t) = \theta\cos(\theta t), \qquad \forall t \in [0,1].$$

Table 3.1: The error functions E_i^1 and E_i^2 of example 1 for $N = 5$

t_i	$u(t_i)$	U_i	E_i^1	$u'(t_i)$	V_i	E_i^2	k
0	0.0000	0.0000	0.0000	0.7854	0.7854	0.0000	1
0.2	0.1564	0.1516	0.0049	0.7757	0.7553	0.0204	11
0.4	0.3090	0.3022	0.0068	0.7470	0.7342	0.0127	14
0.6	0.4540	0.4469	0.0071	0.6998	0.6917	0.0081	16
0.8	0.5878	0.5815	0.0063	0.6354	0.6304	0.0050	17
1.0	0.7071	0.7023	0.0049	0.5554	0.5526	0.0027	18

Table 3.2: The error functions E^1 and E^2 of example 1 by varying the values of N

N	E^1	E^2	Time (s)
10	3.32E-3	1.23E-2	0.237
50	6.35E-4	2.94E-3	0.221
100	3.15E-4	1.50E-3	0.285
500	6.28E-5	3.06E-4	0.510
1000	3.13E-5	1.53E-4	1.112
2000	1.56E-5	7.70E-5	3.675

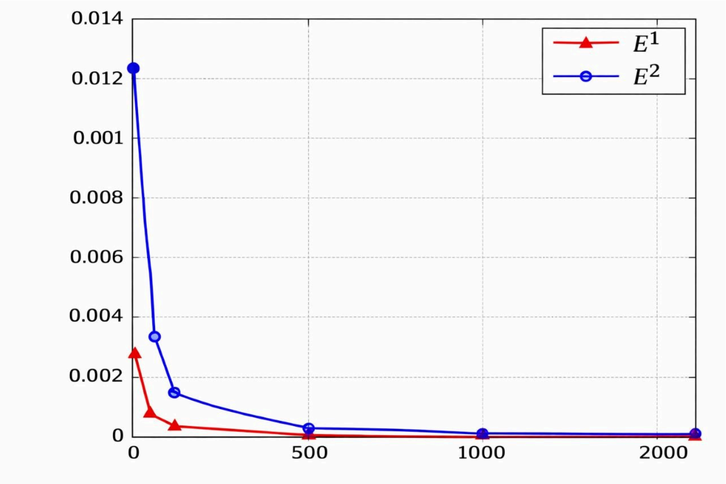
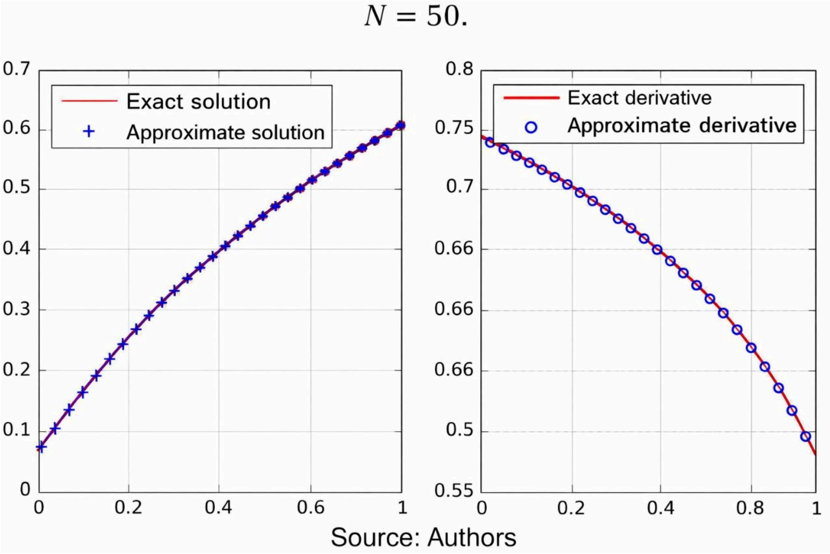


Figure 3.1: Graphical representation of the error functions E^1 and E^2 of example 1.



Source: Authors

Figure 3.2: Exact solution (derivative) versus approximate solution (derivative), of example 1 for $N = 50$.

Example 3.4.2. *Consider the following equation:*

$$u(t) = 10\int_0^t \cos(0.05(u(s)+u'(s)-s^2-0.5s+1)+\arccos(0.1t))ds+f(t,u(t)),$$

$\forall t \in [0,1]$, *where,*

$$f(t,u(t)) = 4\cos(0.1u(t) - 0.1t^2 + 0.15t - 0.05 + \arccos(-0.375t + 0.125)).$$

The exact solution of this example and its derivative are given by:

$$u(t) = \frac{1}{2}(2t-1)(t-1), \qquad u'(t) = 2t - \frac{3}{2}, \qquad \forall t \in [0,1].$$

Table 3.3: The error functions E_i^1 and E_i^2 of example 2 for $N = 5$

t_i	$u(t_i)$	U_i	E_i^1	$u'(t_i)$	V_i	E_i^2	k
0	0.5000	0.5000	0.0000	-1.5000	-1.5000	0.0000	154
0.2	0.2400	0.2257	0.0143	-1.1000	-1.1488	0.0488	158
0.4	0.0600	0.0359	0.0241	-0.7000	-0.7464	0.0464	155
0.6	-0.0400	-0.0734	0.0334	-0.3000	-0.3442	0.0442	147
0.8	-0.0600	-0.1023	0.0423	0.1000	0.0576	0.0424	119
1	0.0000	-0.0509	0.0509	0.5000	0.4592	0.0408	130

Table 3.4: The error functions E^1 and E^2 of example 2 by varying the values of N

N	E^1	E^2	Time (s)
10	2.43E-2	2.53E-2	0.245
50	4.70E-3	5.23E-3	0.341
100	2.34E-3	2.62E-3	0.507
500	4.67E-4	5.26E-4	2.258
1000	2.33E-4	2.63E-4	5.584
2000	1.16E-4	1.31E-4	15.963

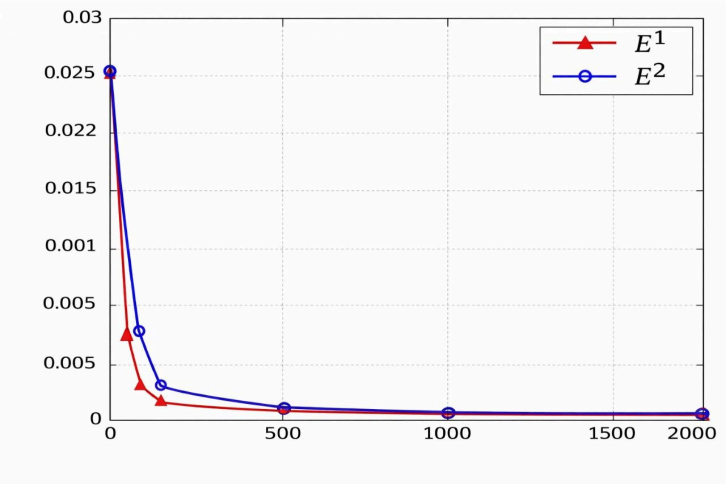


Figure 3.3: Graphical representation of the error function $E^1$and E^2 of example 2.

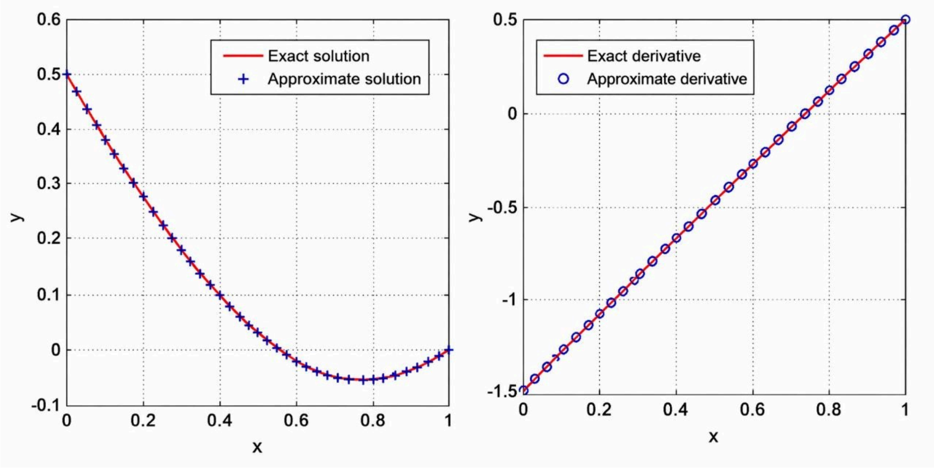


Figure 3.4: Exact solution (derivative) versus approximate solution (derivative), of example 1 for $N = 50$.

Interpretation: Tables 1 and 2, show us that the error function E^1 and $E^2$of example 1 converge to zero by increasing the values of N, likewise also regarding the example 2 from Tables 3 and 4. which implies that the approached solution and their derivatives close to the exact solution and their derivatives respectively. Moreover,in Figures 2 and 4, the graphical representation of the exact and approximate solutions with their derivatives, seem that are nearly identical for only N=50.

Conclusion :

In conclusion, this dissertation has studied a nonlinear Volterra integro-differential equation. In chapter 2, we proved the existence and uniqueness of the solution by applying Schauder's fixed-point theorem, after establishing the necessary assumptions that guarantee the well-posedness of the problem.
In chapter 3, we constructed an approximation technique based on the Nyström method, and the numerical results confirmed the effectiveness and accuracy of this approach in obtaining an approximate solution to our proposed equation.
The main objective of this work has been achieved by introducing a new class of such equations and providing both a theoretical and numerical framework for solving them.
As future perspectives, we aim to extend this study to higher-order integro-differential equations, weakly singular kernels, or to apply other analytical methods for comparative efficiency.

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