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On the Well-Posedness and Stability of Thermoelastic Timoshenko Systems without the Second Spectrum

Presented by : ABOUB Zakaria

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In front of the jury composed of:

Ben Attia Messaouda	M.C.B	Univ. Ouargla	President
Ahmima Afaf	M.C.B	Univ. Ouargla	Supervisor
Lachheb Ilyes	M.C.B	Univ. Ouargla	Examiner

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Dedication

In the name of Allah, the Most Gracious, the Most Merciful

Dedication

”All praise is for Allah Who has guided us to this, and we would never have been guided if Allah had not guided us” (Al-A’raf: 43)

All praise is due to Allah who granted me strength and success to complete this work.

Peace and blessings be upon Prophet Muhammad (PBUH).

I dedicate this dissertation to the dearest people to me:

To my beloved mother and father, who have been the greatest source of support throughout my academic journey. Their love, encouragement, patience, and sacrifices are priceless, and I am forever grateful to them for everything they have given me.

To my sister and brothers, for their continuous support and encouragement, and for the touch of loyalty they added to my journey while accomplishing this project.

This achievement is for you before it is for me.

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Abstract

In this memoir, we study the long-time behavior of certain truncated Timoshenko thermoelastic systems, which are free from the adverse effects associated with the second spectrum of the frequencies. We examine tow problems involving different heat effect conduction laws, and prove exponential stability result for each system.

In the first problem, the thermal disturbance is described by the Cattaneo law, leading to a problem characterized by second-sound heat conduction. The second problem consists hereditary heat conduction governed by the Gurtin-Pipkin law.

For all the problems, well-posedness is established using a semigroup approach along with the Hille-Yosida and Lax-Milgram theorems. Additionally, stability is achieved by employing the multiplier method and Lyapunov functionals.

Keywords: Timoshenko system, thermoelasticity, well-posedness, exponential stability, second spectrum, truncated equation.

Résumé

Dans cette memoir, nous étudions le comportement à long terme de certains systèmes thermoélastiques Timoshenko tronqués, non affectés par les effets indésirables liés au second spectre de fréquences. Nous analysons deux problèmes présentant des effets thermiques distincts et démontrons un résultat de stabilité exponentielle pour chacun de ces systèmes.

Dans le premier problème, la perturbation thermique est décrite par la loi de Cattaneo, aboutissant à un modèle caractérisé par une conduction de chaleur du second son. Le deuxième problème carractérisé par la conduction thermique héréditaire, régie par la loi de Gurtin-Pipkin.

Pour tous ces problèmes, l'existence et l'unicité ont été établies grâce à une approche par semi-groupes ainsi qu'aux théorèmes de Hille-Yosida et de Lax-Milgram. De plus, la stabilité a été démontrée par la méthode des multiplicateurs et les fonctionnelles de Lyapunov.

Mots clés: Système Timochenko, thermo-élasticité, second spectre, existence et unicité, décroissance exponentielle, équation tronquée.

ملخص

تتناول هذه المذكرة دراسة السلوك بعيد المدى لبعض أنظمة المرونة الحرارية لتيموشنكو المبتورة، والتي تخلو من التأثيرات السلبية المرتبطة بالطيف الثاني للترددات. تم تحليل مسألتين تختلف في قوانين التأثيرات الحرارية، مع اثبات خاصية الاستقرار الأسي لكل نظام على حدة.

في المسألة الأولى، تخضع عملية انتقال الحرارةلقانون كاتانيو (law Cattaneo) ، مما ينتج نموذجا يمثل ظاهرة الصوت الثاني في المرونة الحرارية. وتتناول المسألة الثانية انتقال الحرارة بتأثير الذاكرة المحكوم بقانون غورتن- بيبكين (Gurtin-Pipkin). تم إثبات وجود ووحدانية الحلول لجميع الأنظمة المدروسة باستخدام منهج أنصاف الزمر، بالاستعانة بنظريتي(Hille-Yocida) و (Lax-Milgram). كما تم تحقيق نتائج الاستقرار الأسي باستعمال طريقة المضاعف ودوال ليابونوف (functionals Lyapunov).

الكلمات المفتاحية: نظام تيموشنكو، المرونة الحرارية، وجود ووحدانية، الإستقرار الأسي، الطيف الثاني، المبتورة.

Notation

$C_0^\infty(0,L)$	the test functions space,
$L^p(0,L),\ L^\infty(0,L)$	the Lebesgue space,
$L^q(0,L)=(L^p(0,L))'$	the dual space of $L^p(0,L),\ \dfrac{1}{p}+\dfrac{1}{q}=1$,
H^m	Sobolev Spaces,
$H_0^1(0,L)$	the closure of $C_0^\infty(0,L)$ in $H^1(0,L)$,
$H^{-1}(0,L)$	$=\mathscr{L}\left(H_0^1(0,L),\mathbb{R}\right)$ the dual space of $H_0^1(0,L)$,
$D\left(\mathcal{A}\right)$	the domain of the operator $\mathcal{A}$,
$\sigma\left(\mathcal{A}\right)$	The spectrum of operator $\mathcal{A}$,
$\rho\left(\mathcal{A}\right)$	The resolvent set of the operator $\mathcal{A}$,
$N\left(\mathcal{A}\right)=ker(\mathcal{A})$	The kernel of the operator $\mathcal{A}$,
$R\left(\mathcal{A}\right)$	The range of the operator $\mathcal{A}$,
$\lvert\cdot\rvert$	The euclidean norm on $\mathbb{R}^d$,
$\lVert\cdot\rVert_H$	The norm on a normed space H,
$\langle\cdot,\cdot\rangle$	the inner Product in a Hilbert space,
∂	The operator of partial differentiation,
$\mathcal{L}(H)$	The space of bounded linear operators from H into H,
H'	The dual space of H,
$Re\langle.,.\rangle$	The real part of the inner product,
$\left(T(t)\right)_{t\geqslant 0}$	A semigroup of linear operators,
$C\left(X,Y\right)$	The space of all continuous functions from X into Y,
$a.e.$	almost everywhere (except on a negligible set).

Contents

Introduction

The Timoshenko beam theory, introduced in 1921 by Stephen Timoshenko [25], is a fundamental model in structural mechanics for describing the transverse vibrations of elastic beams. In contrast to the Euler–Bernoulli and Rayleigh theories, it accounts for both shear deformation and rotational inertia effects, leading to a more realistic description of beam behavior. The model is governed by the following system

$$\begin{cases} \rho_1 \varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) = 0, \end{cases}$$

where $\rho_1 = \rho A$, $\rho_2 = \rho I$, $b = EI$, and $\kappa = \kappa_0 GA$ are positive constants. Here, ρ denotes the mass density of the material, A the cross-sectional area, I the second moment of area, E Young's modulus of elasticity, G the shear modulus, and κ_0 the shear correction factor. The unknown functions φ and ψ represent the transverse displacement and the angular rotation of the beam, respectively.

The asymptotic behavior of the Timoshenko system has attracted considerable attention over the past decades. Various damping mechanisms have been introduced to stabilize the system, leading to numerous uniform and non-uniform decay results. See, for example, [12, 15, 20, 21].

A distinctive feature of the classical Timoshenko beam model is the presence of two natural wave propagation speeds, which give rise to a non-physical phenomenon known as the second spectrum. Although this paradox was not observed in the original work of Timoshenko, it was subsequently identified through several analytical investigations [2, 9].

To overcome this drawback, Elishakoff [11] proposed, in 2010, a truncated version of the classical Timoshenko system, which eliminates the second spectrum while preserving the essential mechanical characteristics of the model. The modified system is given by

$$\begin{cases} \rho_1 \varphi_{tt} - \kappa\left(\varphi_x + \psi\right)_x = 0, \\ -\rho_2 \varphi_{ttx} - b\psi_{xx} + \kappa\left(\varphi_x + \psi\right) = 0, \end{cases} \tag{1}$$

Recently, Almeida Júnior et al [7]. studied system (1) with a linear frictional damping mechanism $\mu\varphi_t$ acting on the rotational variable and demonstrated that the corresponding energy decays exponentially, regardless of the positive values of the system parameters. A similar exponential stability result was established in [5] when the damping term was incorporated into the transverse-displacement equation.

Also, similar results were obtained in [3, 4].

The effect of thermal dissipation on system (1) was examined by Apalara et al. [8], who studied the following Timoshenko-type system

$$\begin{cases} \rho_1 \varphi_{tt} - \kappa \left(\varphi_x + \psi\right)_x = 0, \\ -\rho_2 \varphi_{ttx} - b\psi_{xx} + \kappa \left(\varphi_x + \psi\right) + \gamma \theta_x = 0, \\ \rho_3 \theta_t - \beta \theta_{xx} + \gamma \psi_{xt} = 0, \end{cases} \tag{2}$$

Here, θ is the temperature difference and the parameters $\rho_3, \beta > 0$ and $\gamma \neq 0$ represent the capacity, diffusivity, and coupling constants, respectively. The authors proved an exponential decay result irrespective of the parameters of the system.

Note that the heat conduction described by Fourier's law

$$q = -\kappa \theta_x, \tag{3}$$

leads to a parabolic equation. Consequently, the heat propagates at an infinite speed, which is unrealistic. To overcome this physical paradox, many alternative theories have been developed. Lord and Shulman [19] replaced Fourier's law (3) with Cattaneo's law

$$\tau_0 q_t + q + \kappa \theta_x = 0, \tag{4}$$

where the positive constant τ_0 represents the time lag in the response of the heat flux to the temperature gradient. According to this theory, called second sound thermoelasticity, the system becomes fully hyperbolic, and heat propagates as a wave with finite speed. Consequently, the speed of the heat equation is involved in the exponential stability condition.

Moreover, the theory of thermoelasticity with second sound is incapable of depicting the memory effect that prevails in some materials, particularly at low temperature. This fact leads to the search for a more general constitutive hypothesis that links the heat flux to the thermal memory. Gurtin and Pipkin [16] proposed that the heat flux depends on the cumulative history of the temperature gradient weighted by a relaxation function called the heat flux kernel. They formulated a general nonlinear theory in which thermal perturbations disseminate with finite speed. According to this theory, the linearized constitutive equation for q is written as follows

$$q = -\int_{-\infty}^{t} g\,(t-s)\,\theta_x\,(x,s)\,ds, \tag{5}$$

where $g(s)$ is the heat conductivity relaxation kernel.

This memoir is divided into three chapters and is organized as follows: In the first chapter, we give some functional preliminaries and tools that we used in the subsequent chapters. The second chapter is devoted to the study of a thermoelastic Timoshenko system free of second spectrum, where the heat conduction is given by Cattaneo's law. In the third chapter, we investigate a a thermoelastic Gurtin–Pipkin–Timoshenko system without the second spectrum.

chapter 1

Preliminear

In this chapter, we review a few mathematical ideas that will be needed in the subsequent chapters of this memoir.

Throughout this chapter, Ω denotes a bounded domain in $\mathbb{R}^n$ and X denotes a Hilbert space.

1.1 Sobolev Spaces

Definition 1.1. *[10] Let $p \in [1, +\infty[$, we define the space*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R}; f \quad \textit{is measurable and} \int_\Omega |f(x)|^p \, dx < +\infty \right\},$$

which is a Banach space with respect to the norm.

$$\|f\|_{L^p} = \|f\|_p = \left[\int_\Omega |f(x)|^p \, dx \right]^{1/p}, \quad 1 \le p < +\infty.$$

For $p = \infty$,

$$L^\infty(\Omega) = \left\{ \begin{array}{c} f : \Omega \to \mathbb{R};\ f \textit{ is measurable and } \exists \ M > 0 \textit{ suth that} \\ |f(x)| \le M \ \textit{a.e. on } \Omega \end{array} \right\},$$

which is a Banach space with respect to the norm

$$\|f\|_{L^\infty} = \|f\|_\infty = \inf \{ M; \ |f(x)| \le M \quad \textit{a.e. on } \Omega \}.$$

Remark 1.1. *For $p = 2$, $L^2(\Omega)$ is a Hilbert space with respect to the inner product*

$$\langle f, g \rangle_{L^2} = \int_\Omega f(x) g(x) \, dx.$$

Lemma 1.1. *The subspace $L^2_*(\Omega)$ of $L^2(\Omega)$, defined by*

$$L^2_*(\Omega) = \left\{ f \in L^2(\Omega) \ ; \ \int_\Omega f(x) \, dx = 0 \right\},$$

is a Hilbert space.

Proof. Let $(\phi_n) \subset L_*^2(\Omega)$ be a convergent sequence to ϕ in $L^2(\Omega)$, then

$$\begin{aligned}\left|\int_\Omega \phi(x)\,dx\right| &= \left|\int_\Omega \phi(x)\,dx - \int_\Omega \phi_n(x)\,dx\right| \\ &= \left|\int_\Omega [\phi(x) - \phi_n(x)]\,dx\right|.\end{aligned}$$

Using the Cauchy-Schwarz inequality, we obtain

$$\left|\int_\Omega \phi(x)\,dx\right| \le mes(\Omega)\left[\int_\Omega |\phi(x) - \phi_n(x)|^2\,dx\right]^{1/2}.$$

Since $\lim_{n\to\infty}\int_\Omega |\phi(x) - \phi_n(x)|^2\,dx = 0$, we get

$$\int_\Omega \phi(x)\,dx = 0,$$

which implies that $\phi \in L_*^2(\Omega)$. Consequently $L_*^2(\Omega)$ is closed in $L^2(\Omega)$, which completes the proof. □

Definition 1.2. *[26] For $k \in \mathbb{N}$ and $1 \le p \le \infty$, we define the Sobolev space $W^{k,p}(\Omega)$ by*

$$W^{k,p}(\Omega) = \left\{u \in L^p(\Omega);\ D^\alpha u \in L^p(\Omega),\ \forall\alpha \in \mathbb{N}^n \text{ with } |\alpha| \le k\right\},$$

where $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and $D^\alpha u$ is the α-th weak derivative of u which is defined as

$$\int_\Omega u(x)D^\alpha v(x)dx = (-1)^\alpha \int_\Omega \varphi(x)v(x)dx, \quad \forall v \in C_c^\infty(\Omega)$$

and

$$\varphi = D^\alpha u = \frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}.$$

The space $W^{k,p}(\Omega)$, equipped with the norm

$$\|u\|_{k,p} = \left(\sum_{|\alpha|\le k}\|D^\alpha u\|_p^p\right)^{\frac{1}{p}}, \quad 1 \le p < \infty$$

and

$$\|u\|_{k,\infty} = \max_{|\alpha|\le k}\|D^\alpha u\|_\infty$$

are Banach spaces. The space $W^{k,2}(\Omega)$ is denoted by $H^k(\Omega)$ and it is a Hilbert space with respect to the inner product

$$\langle u, v\rangle_{H^k} = \sum_{|\alpha|\le k}\int_\Omega D^\alpha u(x)D^\alpha v(x)dx, \quad \forall u, v \in H^k(\Omega).$$

Remark 1.2. *We denote by $H^1_*(\Omega)$, $H^2_*(\Omega)$ and $H^3_*(\Omega)$ the Hilbert spaces*

$$H^1_*(\Omega) = H^1(\Omega) \cap L^2_*(\Omega) = \left\{ u \in H^1(\Omega) \; ; \; \int_\Omega u(x)\, dx = 0 \right\},$$

$$H^2_*(\Omega) = \left\{ u \in H^2(\Omega) \; ; \; \frac{\partial u}{\partial \eta} = 0 \text{ on } \partial\Omega \right\} = \left\{ \nabla u \in (H^1(\Omega))^n \; ; \; \frac{\partial u}{\partial \eta} = 0 \text{ on } \partial\Omega \right\},$$

$$H^3_*(\Omega) = \left\{ u \in H^3(\Omega) \cap H^1_0(\Omega); \quad \frac{\partial^2 u}{\partial \eta^2} = 0 \text{ on } \partial\Omega \right\},$$

where η is the unit outward vector on $\partial\Omega$.

Definition 1.3. *[10] Given $1 \le p < \infty$, we denote by $W^{k,p}_0(\Omega)$ the closure of $C^\infty_c(\Omega)$ in*

$$W^{k,p}(\Omega).$$

For $p = 2$, we note

$$H^k_0(\Omega) = W^{k,2}_0(\Omega).$$

Notation 1.1. *For $1 \le p < \infty$ and $1/p + 1/q = 1$. The dual space of $W^{k,p}_0(\Omega)$ is denoted by $W^{-k,q}(\Omega)$ and the dual space of $H^k_0(\Omega)$ is denoted by $H^{-k}(\Omega)$.*

Theorem 1.2. ***(Rellich-Kondrachov)****[1] Suppose that Ω is bounded and of class C^1. Then we have the following compact embedding:*

$$H^m(\Omega) \subset H^j(\Omega), \; \forall j < m.$$

Definition 1.4. ***(Bilinear form)*** *Let H be a Hilbert space over $\mathbb{R}$.*

$$A : H \times H \to \mathbb{R}.$$

is called a bilinear form if it is linear in each component separately, i.e.,

$$A(\phi + \psi, \varphi) = A(\phi, \varphi) + A(\psi, \varphi), \quad A(\mu\phi, \psi) = \mu\, A(\phi, \psi),$$

and similarly in the second argument.

Definition 1.5. ***(Continuity)*** *A bilinear form $A : H \times H \to \mathbb{R}$ is said to be continuous (or bounded) if there exists a constant $M > 0$ such that*

$$|A(\phi, \psi)| \le M\, \|\phi\|\, \|\psi\|, \quad \forall \phi, \psi \in H.$$

Definition 1.6. ***(Coercivity)*** *A bilinear form $A : H \times H \to \mathbb{R}$ is said to be coercive if there exists a constant $\alpha > 0$ such that*

$$A(\phi, \phi) \geq \alpha \|\phi\|^2, \quad \forall \phi \in H.$$

Theorem 1.3. *[10]* ***(Lax Milgram)***
Assume that $b(x, y)$ is a continuous and coercive bilinear form on X. Then, given any $\phi \in X'$, there exists a unique element $x \in X$ such that

$$b(x, y) = \langle \phi, y \rangle, \quad \forall y \in X.$$

Moreover, if b is symmetric, then x is characterized by the property

$$\frac{1}{2} b(x, x) - \langle \phi, x \rangle = \min_{y \in X} \left\{ \frac{1}{2} b(y, y) - \langle \phi, y \rangle \right\}.$$

1.2 Some inequalities

Theorem 1.4. *[10]****(Young's inequality)*** *Let p, q be real conjugates, then*

$$\forall a, b \in \mathbb{R}, \quad |ab| \leq \frac{|a|^p}{p} + \frac{|b|^q}{q}.$$

In particular if $g, h \in L^2(0, L)$ we have, for any $\varepsilon > 0$,

$$\int_0^L |gh| dx \leq \varepsilon \int_0^L |g|^2 dx + \frac{1}{4\varepsilon} \int_0^L |h|^2 dx.$$

Theorem 1.5. *[10]****(Hölder's inequality)*** *Suppose that $g \in L^p$ and $h \in L^q$ where $1 \leq p \leq \infty$ and*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then, $gh \in L^1$ and

$$\int |gh| \leq \|g\|_p \|h\|_q.$$

Theorem 1.6. *[10]****(Cauchy-Schwarz inequality)*** *Let X be a Hilbert space and $\langle \cdot, \cdot \rangle$ be its inner product, then*

$$|\langle g, h \rangle| \leq \langle g, g \rangle^{\frac{1}{2}} \langle h, h \rangle^{\frac{1}{2}}, \quad \forall f, g \in X.$$

Theorem 1.7. ***(Poincaré's inequality)*** *Suppose that* $\phi \in H_0^1(0,L)$*. Then there exists a constant* $M > 0$*, depending only on* L*, such that*

$$\|\phi\|_{L^2(0,L)} \leq M\|\phi_x\|_{L^2(0,L)}, \quad \forall \phi \in H_0^1(0,L).$$

Remark 1.3. *Poincaré's inequality also holds for all* $\phi \in H^1(0,L)$ *that satisfy*

$$\int_0^L \phi(x)\, dx = 0.$$

1.3 Spectral Theory of Operators

Definition 1.7. *Let* H *be a Hilbert space and* $\mathcal{A} : H \to H$ *an operator,*

1. *The operator* $\mathcal{A}$ *is said to be positive, if*

$$\langle \mathcal{A}\phi, \phi \rangle \geq 0, \quad \forall \phi \in H.$$

2. *The operator* $\mathcal{A}$ *is said to be self-adjoint, if*

$$\langle \mathcal{A}\phi, \psi \rangle = \langle \phi, \mathcal{A}\psi \rangle, \quad \forall \phi, \psi \in H.$$

Theorem 1.8. *[24]* ***(Invertibility of positive self-adjoint operators)***
Let $\mathcal{A} : H \to H$ *be a bounded, self-adjoint operator. Suppose that there exists* $c > 0$ *such that*

$$\langle \mathcal{A}\phi, \phi \rangle \geq c\|\phi\|^2, \quad \forall \phi \in H,$$

then $\mathcal{A}$ *is invertible and*

$$\|A^{-1}\| \leq \frac{1}{c}.$$

1.4 Some Semigroup arguments

Definition 1.8. *[23] Let* H *be a Banach space. A semigroup of bounded linear operators is a family of linear operators* $T(t) \in \mathcal{L}(H)$*, which depend on a parameter* $0 \leq t < \infty$ *and that fulfills the following characteristics.*
(i) $T(0) = I$*, (*I *is the identity operator on* H*).*

(ii) $T(t+s) = T(t)T(s)$ for every $t, s \geq 0$ (the semigroup property).

A semigroup of bounded linear operators $T(t)$ is uniformly continuous if

$$\lim_{t \to 0^+} \|T(t) - I\| = 0.$$

The infinitesimal generator of a semigroup $T(t)$ is the operator A defined on

$$D(A) = \left\{ x \in H : \lim_{t \to 0^+} \frac{T(t)x - x}{t} \text{ exists} \right\}$$

by

$$Ax = \lim_{t \to 0^+} \frac{T(t)x - x}{t}, \quad \text{for } x \in D(A).$$

Definition 1.9. *[10] **(Maximal Monotone Operators)** Let H be a Hilbert space, an unbounded linear operator $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ is said to be monotone (accretive) if it satisfies*

$$\operatorname{Re} \langle \mathcal{A}u, u \rangle \geq 0, \quad \forall u \in D(\mathcal{A}).$$

In addition, the operator is said to be maximal monotone, if $R(I + \mathcal{A}) = H$ i.e.,

$$\forall g \in H, \exists v \in D(\mathcal{A}), \text{ such that } v + \mathcal{A}v = g.$$

Remark 1.4. *If $-\mathcal{A}$ is monotone, we say that $\mathcal{A}$ is dissipative.*

Proposition 1.1. *[10] Let $\mathcal{A}$ be a maximal monotone operator on a Hilbert space. Then*
(i) $D(\mathcal{A})$ is dense in H.
(ii) $\mathcal{A}$ is a closed operator.
(iii) For all $\alpha > 0$, $(I + \alpha\mathcal{A})$ is bijective from $D(\mathcal{A})$ into H, $(I + \alpha\mathcal{A})^{-1}$ is a bounded operator, and $\|(I + \alpha\mathcal{A})^{-1}\|_{\mathcal{L}(H)} \leq 1$.

Remark 1.5. *If the operator A is coercive **(strictly monotone)** and maximal then*

$$\|(I + \alpha A)^{-1}\|_{\mathcal{L}(H)} < 1.$$

Theorem 1.9. *[10](Hille-Yosida) Let A be a maximal monotone operator. Then, given any $v_0 \in D(A)$ there exists a unique function*

$$v \in C^1([0, +\infty); H) \cap C([0, +\infty); D(A))$$

satisfying

$$\begin{cases} \dfrac{dv}{dt} + Av = 0 \quad on\ [0,+\infty), \\ v(0) = v_0. \end{cases} \tag{1.1}$$

Theorem 1.10. *[23],[26]* ***(Lumer-Phillips)*** *Let $A : D(A) \subset H \to H$ be a densely defined operator on a Hilbert space H. Then A generates a C_0-semigroup of contractions on H if and only if*
(i) A is dissipative.
(ii) there exists $\alpha > 0$ such that $\alpha I - A$ is surjective.

Remark 1.6. *Suppose that assumptions (i) and (ii) of Theorem 1.10 are satisfied, then $D(A)$ is dense in H.*

Theorem 1.11. *[18] Let $A : D(A) \subset H \to H$, be a linear operator and H Hilbert space. Suppose that $D(A)$ is dense in H, A is dissipative and $0 \in \rho(A)$. Then, A is the infinitesimal generator of a C_0-semi-group of contractions on H.*

Theorem 1.12. *[26] Let H be a Banach space and $A : D(A) \subset H \to H$ be the infinitesimal generator of a C_0-semigroup $\{S(t); t \geq 0\}$ on H. Then, for each $\xi \in D(A)$ and each $t \geq 0$, we have $S(t)\xi \in D(A)$, and the mapping*

$$t \to S(t)\xi$$

is of class C^1 on $[0,+\infty)$ and satisfies

$$\frac{d}{dt}(S(t)\xi) = AS(t)\xi = S(t)A\xi. \tag{1.2}$$

1.5 Exponential stability notions

Definition 1.10. *The solution $U(t) = e^{At}U_0$ of (1.1) is said to be exponentially stable if there exist two positive constants λ and $M > 1$ such that*

$$\|U(t)\| \leq Me^{-\lambda t}, \quad \forall t \geq 0.$$

Theorem 1.13. *[13],[18]* ***(Gerhard-Bruss)*** *A C_0-semigroup of contractions $S(t) = e^{At}$ generated by an operator A in a Hilbert space H is exponentially stable if and only if*
i) $i\mathbb{R} = \{i\lambda, \lambda \in \mathbb{R}\} \subset \rho(A)$,
ii) $\lim_{|\lambda|\to\infty} \|(i\lambda I - A)^{-1}\| < \infty$.

Theorem 1.14. *[6] Let A (unbounded operator) be the infinitesimal generator of a semigroup of contractions $S(t) = e^{At}$. Then, $S(t)$ is exponentially stable if and only if there exists a positive constant c such that*

$$\inf_{\lambda\in\mathbb{R}} \|(i\lambda I - A)U\| \geq c\|U\|, \quad \forall U \in D(A). \tag{1.3}$$

Theorem 1.15. *[13],[18] A C_0-semigroup of contractions $S(t) = e^{At}$, generated by an operator A in a Hilbert space H, is analytic if and only if*
i) $i\mathbb{R} \subset \rho(A)$,
ii) $\lim_{|\lambda|\to\infty} \|\lambda(i\lambda I - A)^{-1}\| < \infty$.

chapter

2 Exponential Stability of a Truncated Timoshenko system with second sound thermoelasticity

2.1 Introduction

In this chapter, we are concerned a thermoelastic Timoshenko system which has only one spectrum and where the heat conduction is given by Cattaneo's law. We study the well-posedness and an exponential decay result. More specifically, we consider

$$\begin{cases} \rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0, & \text{in } (0,1)\times(0,\infty), \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \kappa(\varphi_x + \psi) + \delta\theta_x = 0, & \text{in } (0,1)\times(0,\infty), \\ c\theta_t + q_x + \delta\psi_{xt} = 0, & \text{in } (0,1)\times(0,\infty), \\ \tau q_t + \beta q + \theta_x = 0, & \text{in } (0,1)\times(0,\infty), \end{cases} \tag{2.1}$$

where the positive constants c and δ represent respectively, the specific heat of the material and the coupling constant; and q is the heat flux.

It is associated with the boundary conditions of Dirichlet-Neumann-Dirichlet type:

We also assume that the unknown functions φ, ψ, θ and q satisfy the following initial and boundary conditions

$$\varphi(0,t) = \varphi(1,t) = \psi_x(0,t) = \psi_x(1,t) = \theta(0,t) = \theta(1,t) = 0, \quad \forall t \geq 0, \tag{2.2}$$

$$\begin{gathered} \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \quad \psi(x,0) = \psi_0(x), \\ \theta(x,0) = \theta_0(x), \quad q(x,0) = q_0(x), \qquad \forall x \in (0,1). \end{gathered} \tag{2.3}$$

Since the boundary conditions on ψ are of Neumann type, then they prevent the application of Poincaré's inequality. To overcome this inconvenience, we use the second equation of (2.1) and the boundary conditions (2.2) to obtain

$$\int_0^1 \psi dx = 0.$$

2.2 Well-possednes

In this section, we start by applying a series of transformations to the system (2.1)-(2.3) to recast it within the framework of the Hille-Yosida theorem. This approach enables us to establish the existence and uniqueness of the solution.

First, multiply the second equation of (2.1) by κ and differentiable with respect to x. We then find

$$\begin{cases} \kappa\psi_x = \rho_1\varphi_{tt} - \kappa\varphi_{xx}, \\ -\rho_2\kappa\varphi_{ttxx} - b\kappa\psi_{xxx} + \kappa^2(\varphi_x + \psi)_x + \delta\kappa\theta_{xx} = 0, \\ c\theta_t + q_x + \delta\psi_{xt} = 0, \\ \tau q_t + \beta q + \theta_x = 0, \end{cases} \tag{2.4}$$

we substitute the first equation of (2.4) into the second and the third equation of (2.4) to arrive that

$$\begin{cases} -\rho_2\kappa\varphi_{ttxx} - b(\rho_1\varphi_{tt} - \kappa\varphi_{xx})_{xx} + \kappa\rho_1\varphi_{tt} + \delta\kappa\theta_{xx} = 0, \\ c\theta_t + q_x + \dfrac{\delta}{\kappa}(\rho_1\varphi_{tt} - \kappa\varphi_{xx})_t = 0, \\ \tau q_t + \beta q + \theta_x = 0, \end{cases} \tag{2.5}$$

therefore,

$$\begin{cases} (\kappa\rho_1 I - (\rho_2\kappa + b\rho_1)\partial_{xx})\varphi_{tt} + b\kappa\varphi_{xxxx} + \delta\kappa\theta_{xx} = 0, \\ c\theta_t + q_x + \dfrac{\delta}{\kappa}(\rho_1\varphi_{ttt} - \kappa\varphi_{xxt}) = 0, \\ \tau q_t + \beta q + \theta_x = 0, \end{cases}$$

Putting $B = \kappa\rho_1 I - (\rho_2\kappa + b\rho_1)\partial_{xx}$ with domain $D(B) = H^2(0,1) \cap H_0^1(0,1)$, we obtain

$$\begin{cases} B\varphi_{tt} + b\kappa\varphi_{xxxx} + \delta\kappa\theta_{xx} = 0, \\ c\theta_t + q_x + \dfrac{\delta}{\kappa}(\rho_1\varphi_{ttt} - \kappa\varphi_{xxt}) = 0, \\ \tau q_t + \beta q + \theta_x = 0 \end{cases} \tag{2.6}$$

and

$$\partial_{xx} = \frac{1}{\rho_2\kappa + b\rho_1}(\kappa\rho_1 I - B). \tag{2.7}$$

Remark 2.1. *The operator $B = \kappa\rho_1 I - (\kappa\rho_2 + b\rho_1)\partial_{xx}$ is invertible.*

Indeed:

By applying the Lax-Milgram Theorem, for any $g \in L^2([0,1])$, the problem

$$\begin{cases} -(\kappa\rho_2 + b\rho_1)\varphi_{xx} + \kappa\rho_1\varphi = g, & \text{in } (0,1), \\ \varphi(0) = \varphi(1) = 0, \end{cases} \tag{2.8}$$

admits a unique solution $\varphi \in H_0^1([0,1])$ that satisfies the inequality

$$\|\varphi\|_{H_0^1(0,1)} \leq C\|g\|_{L^2(0,1)}. \tag{2.9}$$

Next, by using standard elliptic regularity theory, we infer that the solution φ belongs to $\big(H^2(0,1) \cap H_0^1(0,1)\big)$. From the equation in (2.8), we have

$$\varphi_{xx} = \frac{1}{\kappa\rho_2 + b\rho_1}(\kappa\rho_1\varphi - g).$$

Therefore,

$$\|\varphi_{xx}\|_{L^2(0,1)} \leq \widehat{C}\big(\|\varphi\|_{L^2(0,1)} + \|g\|_{L^2(0,1)}\big).$$

By virtue of (2.9), we obtain

$$\|\varphi_{xx}\|_{L^2(0,1)} \leq C'\|g\|_{L^2(0,1)}.$$

Consequently,

$$\|\varphi\|_{H^2(0,1)} \leq C'\|g\|_{L^2(0,1)}.$$

Thus, for every $g \in L^2(0,1)$, there exists a unique $\varphi \in H^2(0,1) \cap H_0^1(0,1)$ such that $B\varphi = g$. Hence,

$$B^{-1} : L^2(0,1) \longrightarrow H^2(0,1) \cap H_0^1(0,1)$$

is a bounded operator, and therefore B is invertible.

In addition B^{-1} is positive and commutes with ∂_{xx} since B does. □

Now, By differentiating the first equation of (2.6), we get

$$\varphi_{ttt} = -bB^{-1}\kappa\varphi_{xxxxt} - \delta\kappa B^{-1}\theta_{xxt}, \tag{2.10}$$

substituting (2.10) into the second equation of (2.6), we find

$$c\theta_t - \delta^2\rho_1 B^{-1}\theta_{xxt} - \delta\varphi_{xxt} - \delta b\rho_1 B^{-1}\varphi_{xxxxt} + q_x = 0,$$

this is equivalent to

$$(cI - \delta^2\rho_1 B^{-1}\circ\partial_{xx})\theta_t - \delta(\partial_{xx} + b\rho_1 B^{-1}\circ\partial_{xxxx})\varphi_t + q_x = 0. \tag{2.11}$$

On pose $S = cI - \delta^2\rho_1 B^{-1}\circ\partial_{xx}$ and $T = -(\partial_{xx} + b\rho_1 B^{-1}\circ\partial_{xxxx})$, we obtain

$$S\theta_t + q_x - \delta T\varphi_t = 0. \tag{2.12}$$

In addition, from (2.7) and since $B^{-1} : L^2(0,1) \longrightarrow H^2(0,1)\cap H_0^1(0,1)$, we have

$$S = cI - \frac{\delta^2\rho_1}{\rho_2\kappa + b\rho_1}(\kappa\rho_1 B^{-1} - I) : L^2(0,1) \longrightarrow L^2(0,1)$$

and

$$\begin{aligned}
T &= -(\partial_{xx} + b\rho_1 B^{-1}\circ\partial_{xxxx})\\
&= -(I + b\rho_1 B^{-1}\circ\partial_{xx})\circ\partial_{xx}\\
&= -\frac{1}{\rho_2\kappa + b\rho_1}(I + b\rho_1 B^{-1}\circ\partial_{xx})\circ(\kappa\rho_1 I - B)\\
&= -\frac{1}{\rho_2\kappa + b\rho_1}(\kappa\rho_1 I - B + \kappa\rho_1^2 bB^{-1}\circ\partial_{xx} - b\rho_1\partial_{xx})\\
&= -\frac{1}{\rho_2\kappa + b\rho_1}(\kappa\rho_1 I - \kappa\rho_1 I + (\rho_2\kappa + b\rho_1)\partial_{xx} + \kappa\rho_1^2 bB^{-1}\circ\partial_{xx} - b\rho_1\partial_{xx})\\
&= -\frac{1}{\rho_2\kappa + b\rho_1}(\rho_2\kappa\partial_{xx} + \kappa\rho_1^2 bB^{-1}\circ\partial_{xx})\\
&= -\frac{\kappa}{\rho_2\kappa + b\rho_1}(\rho_2 I + \rho_1^2 bB^{-1})\circ\partial_{xx},
\end{aligned}$$

this equivalent to

$$T = -\frac{\kappa}{\rho_2\kappa + b\rho_1}(\rho_2 I + \rho_1^2 bB^{-1})\circ\partial_{xx} : L^2(0,1) \longrightarrow L^2(0,1).$$

On the other hand, we multiply the first equation of (2.6) by $\frac{1}{\rho_2\kappa + b\rho_1}(\rho_2 I + \rho_1^2 bB^{-1})$ to obtain

$$\frac{1}{\rho_2\kappa + b\rho_1}[(\rho_2 I + \rho_1^2 bB^{-1})\circ B]\varphi_{tt} - bT\varphi_{xx} - \delta T\theta = 0,$$

we put $R = \frac{1}{\rho_2\kappa + b\rho_1}[(\rho_2 I + \rho_1^2 bB^{-1})\circ B]$ to find

$$R\varphi_{tt} - bT\varphi_{xx} - \delta T\theta = 0. \tag{2.13}$$

In addition,

$$\begin{aligned} R &= \frac{1}{\rho_2\kappa + b\rho_1}[(\rho_2 I + \rho_1^2 bB^{-1}) \circ B] \\ &= \frac{1}{\rho_2\kappa + b\rho_1}[\rho_2 B + \rho_1^2 bI]. \end{aligned}$$

Substituting B, we arrive at

$$\begin{aligned} R &= \frac{1}{\rho_2\kappa + b\rho_1}[\rho_1(\kappa\rho_2 + \rho_1 b)I - \rho_2(\rho_2\kappa + b\rho_1)\partial_{xx}] \\ &= \rho_1 I - \rho_2\partial_{xx} : L^2(0,1) \longrightarrow L^2(0,1). \end{aligned}$$

Finally, from (2.12) and (2.13), problem (2.6) can be written as the following auxiliary problem

$$\begin{cases} R\varphi_{tt} - bT\varphi_{xx} - \delta T\theta = 0, \\ S\theta_t + q_x + \delta T\phi_t = 0, \\ \tau q_t + \beta q + \theta_x = 0, \end{cases} \tag{2.14}$$

Remark 2.2. *The operator S is invertible.*

Indeed: *we have*

$$\langle Sv, v\rangle = c\|v\|_{L^2}^2 + \delta^2\rho_1\langle B^{-1}v_x, v_x\rangle.$$

Since B^{-1} is positive self-adjoint, the second term is nonnegative, so

$$\langle Sv, v\rangle \geq c\|v\|_{L^2}^2, \ \forall v \in L^2(0,1).$$

Therefore, S is coercive and consequently, is invertible. □

In fact, $R = \rho_1 I - \rho_2\partial_{xx}$ is clearly positive definite. Moreover, since both B and $-\partial_{xx}$ are positive definite operators, we deduce that $-B^{-1}\partial_{xx}$ is also positive definite, and consequently $S = cI - \delta^2\rho_1 B^{-1} \circ \partial_{xx}$ is also positive definite. In addition, we have $T = -\kappa(\rho_1 I - \rho_2\partial_{xx})B^{-1} \circ \partial_{xx}$ on $H^2(0,1) \cap H_0^1(0,1)$. Then, for any $\phi \in D(T)$, recalling that B^{-1} is positive, we have

$$\begin{aligned} \langle T\phi, \phi\rangle &= -\kappa \left\langle (\rho_1 I - \rho_2\partial_{xx})B^{-1}\phi_{xx}, \phi\right\rangle \\ &= \kappa\rho_1\langle B^{-1}\phi_x, \phi_x\rangle + \kappa\rho_2\langle B^{-1}\phi_{xx}, \phi_{xx}\rangle \geq 0. \end{aligned}$$

Moreover, if $\langle T\phi, \phi\rangle = 0$, then $\langle B^{-1}\phi_x, \phi_x\rangle = 0$. Since B^{-1} is positive definite, this implies that $\phi_x = 0$. Because $\phi \in H_0^1(0,1)$, it follows that $\phi = 0$. Therefore, T is definite.

Consequently, R, S, and T admit well-defined square roots, denoted by $R^{1/2}$, $S^{1/2}$, and $T^{1/2}$, with domains $D(R^{1/2}) = D(T^{1/2}) = H_0^1(0,1)$ and $D(S^{1/2}) = L^2(0,1)$.

Remark 2.3. *The operator T is self-adjoint.*

Indeed: *we have*

$$T = -\kappa R\left(B^{-1} \circ \partial_{xx}\right) = -\kappa(\rho_1 I - \rho_2 \partial_{xx}) B^{-1} \circ \partial_{xx} = -\kappa(\rho_1 B^{-1} \partial_{xx} - \rho_2 \partial_{xx} B^{-1} \partial_{xx}).$$

Taking into account the commutativity of B^{-1} and ∂_{xx} together with

$$D(R) = D(B^{-1} \circ \partial_{xx}) = H^2(0,1) \cap H_0^1(0,1),$$

we deduce that

$$\begin{aligned} T &= -\kappa(\rho_1 \partial_{xx} B^{-1} - \rho_2 \partial_{xx} B^{-1} \partial_{xx}) = -\kappa \partial_{xx} B^{-1}(\rho_1 I - \rho_2 \partial_{xx}) \\ &= -\kappa B^{-1} \partial_{xx}(\rho_1 I - \rho_2 \partial_{xx}) = -\kappa\left(B^{-1} \partial_{xx}\right) R. \end{aligned}$$

Thus, R and $B^{-1}\partial_{xx}$ commute therefore T is symmetric.

Since R and $B^{-1}\partial_{xx}$ are self-adjoint operators and commute, the product theorem for self-adjoint operators yields that T is self-adjoint. □

Next, in order to apply the Hille-Yosida Theorem **??**, we introduce the new variables $\phi = \varphi_t$ and defined the Hilbert space

$$\mathcal{H} = \left(H^2(0,1) \cap H_0^1(0,1)\right) \times H_0^1(0,1) \times L^2(0,1) \times L_*^2(0,1).$$

The space $\mathcal{H}$ is equipped with the inner product

$$\langle \Phi, \Phi^* \rangle_{\mathcal{H}} = b(T\varphi_x, \varphi_x^*) + \langle R\phi, \phi^* \rangle + \langle S\theta, \theta^* \rangle + \tau \langle q, q^* \rangle,$$

where $\Phi = (\varphi, \phi, \theta, q)^T$, $\Phi^* = (\varphi^*, \phi^*, \theta^*, q^*)^T \in \mathcal{H}$.

Hence, the problem (2.14) can be written as follows

$$\begin{cases} \Phi'(t) + \mathcal{B}\Phi(t) = 0, & \forall t \geq 0, \\ \Phi(0) = (\varphi_0, \varphi_1, \theta_0, q_0)^T, \end{cases} \tag{2.15}$$

where the operator $\mathcal{B} : \mathcal{H} \longrightarrow \mathcal{H}$ is defined by

$$\mathcal{B}\Phi = \begin{pmatrix} -\phi \\ -bR^{-1}T\varphi_{xx} - \delta R^{-1}T\theta \\ S^{-1}q_x + \delta S^{-1}T\phi \\ \frac{\beta}{\tau}q + \frac{1}{\tau}\theta_x \end{pmatrix}, \tag{2.16}$$

with domain

$$D(\mathcal{B}) = \left\{ \begin{array}{l} \Phi = (\varphi, \phi, \theta, q)^T \in \mathcal{H} : \varphi \in H^3_*(0,1); \quad \phi \in H^2(0,1) \cap H^1_0(0,1); \\ \theta \in H^1_0(0,1); \quad q \in H^1(0,1) \cap L^2_*(0,1) \end{array} \right\}.$$

Our well-posedness result for the auxiliary problem (2.14) reads as follows:

Theorem 2.1. *Let $\Phi_0 \in \mathcal{H}$. Then there exists a unique solution $\Phi \in C(\mathbb{R}^+, \mathcal{H})$ of problem (2.15). Moreover, if $\Phi_0 \in D(\mathcal{B})$, then $\Phi \in C(\mathbb{R}^+, D(\mathcal{B})) \cap C^1(\mathbb{R}^+, \mathcal{H})$.*

The proof of Theorem 2.1 is based on the Hille-Yosida Theorem 1.9 and the Lax-Milgram Theorem 1.3, and it will be established through several lemmas.

Lemma 2.1. *The operator $\mathcal{B}$ defined by (2.16) is monotone*

Proof. A direct calculation, yields, for $\Phi \in D(\mathcal{B})$

$$\langle \mathcal{B}\Phi, \Phi \rangle_{\mathcal{H}} = -b\langle T\phi_x, \varphi_x \rangle - b\langle T\varphi_{xx}, \phi \rangle - \delta\langle T\theta, \phi \rangle + \langle q_x, \theta \rangle + \delta\langle T\phi, \theta \rangle + \beta\langle q, q \rangle + \langle \theta_x, q \rangle.$$

Since T is a self-adjoint operator, we obtain

$$\langle \mathcal{B}\Phi, \Phi \rangle_{\mathcal{H}} = -b\langle T\phi_x, \varphi_x \rangle - b\langle \varphi_{xx}, T\phi \rangle - \delta\langle \theta, T\phi \rangle + \langle q_x, \theta \rangle + \delta\langle T\phi, \theta \rangle + \beta\langle q, q \rangle + \langle \theta_x, q \rangle.$$

Using integration by parts and the boundary conditions, we obtain

$$\begin{aligned} \langle \mathcal{B}\Phi, \Phi \rangle_{\mathcal{H}} &= -b\langle T\phi_x, \varphi_x \rangle + b\langle \varphi_x, (T\phi)_x \rangle - \delta\langle \theta, T\phi \rangle - \langle q, \theta_x \rangle + \delta\langle T\phi, \theta \rangle + \beta\langle q, q \rangle + \langle \theta_x, q \rangle \\ &= -b\langle T\phi_x, \varphi_x \rangle + b\langle \varphi_x, (T\phi)_x \rangle + \beta\langle q, q \rangle. \end{aligned}$$

Now, as T commutes with differentiation, that is,

$$\langle \mathcal{B}\Phi, \Phi \rangle_{\mathcal{H}} = \beta\langle q, q \rangle = \beta\|q\|^2_{L^2} \geq 0.$$

Therefore, the operator $\mathcal{B}$ is monotone. □

Lemma 2.2. *The operator $\mathcal{B}$ defined by (2.16) is maximal.*

Proof. Let $G = (g_1, g_2, g_3, g_4)^T \in \mathcal{H}$, we seek $\Phi \in D(\mathcal{B})$ such that

$$(\mathrm{I} - \mathcal{B})\Phi = G,$$

this means

$$\begin{cases} \varphi - \phi = g_1, \\ \phi - bR^{-1}T\varphi_{xx} - \delta R^{-1}T\theta = g_2, \\ \theta - S^{-1}(-q_x - \delta T\phi) = g_3, \\ q + \frac{1}{\tau}(\beta q + \theta_x) = g_4. \end{cases}$$

Therefore,

$$\begin{cases} \varphi - \phi = g_1, \\ R\phi - bT\varphi_{xx} - \delta T\theta = Rg_2, \\ S\theta + q_x + \delta T\phi = Sg_3, \\ (\tau + \beta)q + \theta_x = \tau g_4. \end{cases} \tag{2.17}$$

From the first and fourth equations of (2.17), we deduce that

$$\begin{aligned} &\phi = \varphi - g_1, \\ &\theta_x = \tau g_4 - (\tau + \beta)q. \end{aligned} \tag{2.18}$$

The integration of the second equation in (2.18) yields

$$\theta(x) = \tau \int_0^x g_4(y)dy - (\tau + \beta)\int_0^x q(y)dy. \tag{2.19}$$

Substitute equation (2.18) and equation (2.19) into equation (2.17), we find

$$\begin{cases} R\varphi - bT\varphi_{xx} + \delta(\tau + \beta)T\left(\int_0^x q(y)dy\right) = R(g_1 + g_2) + \delta\tau T\left(\int_0^x g_4(y)dy\right), \\ q_x - (\tau + \beta)S\left(\int_0^x q(y)dy\right) + \delta T\varphi = S\left(g_3 - \tau\int_0^x g_4(y)dy\right) + \delta T g_1. \end{cases} \tag{2.20}$$

At this point, we define the space

$$\mathcal{W} = (H^2(0,1) \cap H_0^1(0,1)) \times L_*^2(0,1),$$

endowed with the norm

$$\|(\varphi, q)\|_{\mathcal{W}}^2 = \|\varphi_x\|^2 + \|\varphi_{xx}\|^2 + \|q\|^2.$$

We then consider the following variational problem associated with the system (2.20)

$$A((\varphi, q), (\tilde{\varphi}, \tilde{q})) = L(\tilde{\varphi}, \tilde{q}), \qquad \forall (\tilde{\varphi}, \tilde{q}) \in \mathcal{W}, \tag{2.21}$$

where A and L are the bilinear and linear form defined on $\mathcal{W}$ by

$$\begin{aligned} A((\varphi, q), (\tilde{\varphi}, \tilde{q})) =& \langle R^{\frac{1}{2}}\varphi, R^{\frac{1}{2}}\tilde{\varphi}\rangle + b\langle T^{\frac{1}{2}}\varphi_x, T^{\frac{1}{2}}\tilde{\varphi}_x\rangle \\ &+ \delta(\tau+\beta)\left\langle T^{\frac{1}{2}}\left(\int_0^x q(y)\,dy\right), T^{\frac{1}{2}}\tilde{\varphi}\right\rangle + (\tau+\beta)\langle q, \tilde{q}\rangle \\ &+ (\tau+\beta)^2\left\langle S^{\frac{1}{2}}\left(\int_0^x q(y)\,dy\right), S^{\frac{1}{2}}\left(\int_0^x \tilde{q}(y)\,dy\right)\right\rangle \\ &- \delta(\tau+\beta)\left\langle T^{\frac{1}{2}}\varphi, T^{\frac{1}{2}}\left(\int_0^x \tilde{q}(y)\,dy\right)\right\rangle \end{aligned}$$

and

$$\begin{aligned} L(\tilde{\varphi}, \tilde{q}) =& \langle R(g_1+g_2), \tilde{\varphi}\rangle + \delta\tau\left\langle T\left(\int_0^x g_4(y)\,dy\right), \tilde{\varphi}\right\rangle \\ &- (\tau+\beta)\left\langle S\left(g_3 - \tau\int_0^x g_4(y)\,dy\right), \int_0^x \tilde{q}(y)\,dy\right\rangle \\ &- \delta(\tau+\beta)\left\langle Tg_1, \int_0^x \tilde{q}(y)\,dy\right\rangle. \end{aligned}$$

Clearly, A and L are bounded. Moreover,

$$\begin{aligned} A((\varphi, q), (\varphi, q)) =& \langle R\varphi, \varphi\rangle + b\langle T\varphi_x, \varphi_x\rangle + \delta(\tau+\beta)\left\langle T\left(\int_0^x q(y)\,dy\right), \varphi\right\rangle \\ &+ (\tau+\beta)\langle q, q\rangle + (\tau+\beta)^2\left\langle S\left(\int_0^x q(y)\,dy\right), \left(\int_0^x q(y)\,dy\right)\right\rangle \\ &- \delta(\tau+\beta)\left\langle T\varphi, \left(\int_0^x q(y)\,dy\right)\right\rangle \\ =& \langle R\varphi, \varphi\rangle + b\langle T\varphi_x, \varphi_x\rangle + (\tau+\beta)\langle q, q\rangle \\ &+ (\tau+\beta)^2\left\langle S\left(\int_0^x q(y)\,dy\right), \left(\int_0^x q(y)\,dy\right)\right\rangle. \end{aligned}$$

The definition of T and R, yield

$$\begin{aligned} A((\varphi, q), (\varphi, q)) =& \langle \rho_1\varphi, \varphi\rangle - \langle \rho_2\varphi_{xx}, \varphi\rangle - \frac{\kappa b\rho_2}{b\rho_1+\kappa\rho_2}\langle \varphi_{xxx}, \varphi_x\rangle - \frac{\kappa b^2\rho_1^2}{b\rho_1+\kappa\rho_2}\langle B^{-1}\varphi_{xxx}, \varphi_x\rangle \\ &+ (\tau+\beta)\langle q, q\rangle + (\tau+\beta)^2\left\langle S\left(\int_0^x q(y)\,dy\right), \left(\int_0^x q(y)\,dy\right)\right\rangle. \end{aligned}$$

By integration by parts and the boundary condition, we obtain

$$\begin{aligned} A((\varphi,q),(\varphi,q)) =& \rho_1\|\varphi\|^2+\rho_2\|\varphi_x\|^2+\frac{\kappa b\rho_2}{b\rho_1+\kappa\rho_2}\|\varphi_{xx}\|^2+(\tau+\beta)\|q\|^2+\frac{\kappa b^2\rho_1^2}{b\rho_1+\kappa\rho_2}\langle B^{-1}\varphi_{xx},\varphi_{xx}\rangle \\ &+(\tau+\beta)^2\left\langle S\left(\int_0^x q(y)\,dy\right),\left(\int_0^x q(y)\,dy\right)\right\rangle. \end{aligned}$$

The positivity of B^{-1} implies that

$$\begin{aligned} A((\varphi,q),(\varphi,q)) \geq& \rho_2\|\varphi_x\|^2+\frac{\kappa b\rho_2}{b\rho_1+\kappa\rho_2}\|\varphi_{xx}\|^2+(\tau+\beta)\|q\|^2 \\ \geq& C\left(\|\varphi_x\|^2+\|\varphi_{xx}\|^2+\|q\|^2\right), \end{aligned}$$

for some $C>0$; hence, A is coercive. Consequently, Lax-Milgram theorem guarantees the existence of a unique solution $(\varphi,q)\in\mathcal{W}$ satisfying (2.21).

By substituting φ and q into the equation (2.18), we infer that

$$\phi\in H^2(0,1)\cap H_0^1(0,1)$$

and

$$\theta\in H_0^1(0,1).$$

Next, we take $(\tilde{\varphi},\tilde{q})=(0,\tilde{q})$ in (2.21) to get

$$\begin{aligned} &-\delta\left\langle T\varphi,\int_0^x\tilde{q}(y)dy\right\rangle+(\tau+\beta)\left\langle S\int_0^x q(y)dy,\int_0^x\tilde{q}(y)dy\right\rangle+\langle q,\tilde{q}\rangle \\ =&-\left\langle S\left(g_3-\tau\int_0^x g_4(y)dy\right),\int_0^x\tilde{q}(y)dy\right\rangle-\delta\left\langle Tg_1,\int_0^x\tilde{q}(y)dy\right\rangle, \end{aligned}$$

for all $\tilde{q}\in L_*^2(0,1)$. In particular, for $\tilde{q}=v_x$, with $v\in C_0^1(0,1)$, we get

$$\langle q,v_x\rangle=-\left\langle(\tau+\beta)S\int_0^x q(y)dy-\delta T\varphi+S\left(g_3-\tau\int_0^x g_4(y)dy\right)+\delta Tg_1,v\right\rangle, \quad (2.22)$$

for all $v\in C_0^1(0,1)$. From the definitions of S and T we have that

$$\delta T\varphi-(\tau+\beta)S\int_0^x q(y)dy-S\left(g_3-\tau\int_0^x g_4(y)dy\right)-\delta Tg_1\in L^2(0,1),$$

this shows that $q\in H_*^1(0,1)$. In addition, an integration by parts in (2.22) shows that

$$\delta T\varphi-(\tau+\beta)S\int_0^x q(y)dy-S\left(g_3-\tau\int_0^x g_4(y)dy\right)+q_x=\delta Tg_1.$$

By using (2.18), we obtain

$$q_x = Sg_3 - \delta T\phi - S\theta \in L^2(0,1),$$

then,

$$q \in H_0^1(0,1) \cap L_*^2(0,1)$$

and

$$\delta T\phi + S\theta + q_x = Sg_3,$$

which shows that q solves the third equation of (2.17).

Similarly, by taking $(\tilde{\varphi}, \tilde{q}) = (\tilde{\varphi}, 0)$ in (2.21), we obtain

$$\begin{aligned} & b\langle T\varphi_x, \tilde{\varphi}_x\rangle + \langle R\varphi, \tilde{\varphi}\rangle + \delta(\tau+\beta)\left\langle T\int_0^x q(y)dy, \tilde{\varphi}\right\rangle \\ & = \langle R(g_1+g_2), \tilde{\varphi}\rangle + \delta\tau\left\langle T\int_0^x g_4(y)dy, \tilde{\varphi}\right\rangle, \quad \forall\tilde{\varphi} \in H^2(0,1)\cap H_0^1(0,1). \end{aligned}$$

Consequently

$$b\langle T\varphi_x, \tilde{\varphi}_x\rangle = -\left\langle R\varphi + \delta(\tau+\beta)T\int_0^x q(y)dy - R(g_1+g_2) - \delta\tau T\int_0^x g_4(y)dy, \tilde{\varphi}\right\rangle, \ \forall\tilde{\varphi} \in C_0^1(0,1),$$

which implies that

$$bT\varphi_{xx} = R\varphi + \delta(\tau+\beta)T\int_0^x q(y)dy - R(g_1+g_2) - \delta\tau T\int_0^x g_4(y)dy. \tag{2.23}$$

Replacing (2.18) into (2.23) we find

$$bT\varphi_{xx} = R(\phi - g_2) - \delta T\theta \in H^{-1}(0,1).$$

Remark 2.4. *$Rg_2 \in H^{-1}(0,1)$ and $T\theta \in H^{-1}(0,1)$.*

Indeed:

Since $g_2 \in H_0^1(0,1)$, then, the functional $-\partial_{xx}g_2 : H_0^1(0,1) \to \mathbb{R}$ operates on $H_0^1(0,1)$ through the bilinear pairing

$$\langle -\partial_{xx}g_2, u\rangle = \int_0^1 \frac{\partial g_2}{\partial x}\frac{\partial u}{\partial x}dx.$$

By the Cauchy–Schwarz inequality, we obtain

$$\left|\langle -\partial_{xx}g_2, u\rangle\right| \leq \|g_{2x}\|_{L^2(0,1)}\|u_x\|_{L^2(0,1)}.$$

Therefore, this operator is bounded on $H_0^1(0,1)$*, which implies that* $-\partial_{xx} g_2 \in H^{-1}(0,1)$.

Furthermore, by applying the Lax–Milgram theorem, it can be rigorously established that the operator $-\partial_{xx}$ *defines an isomorphism between* $H_0^1(0,1)$ *into* $H^{-1}(0,1)$*. Moreover,*

$$Rg_2 = (\rho_1 I - \rho_2 \partial_{xx})\, g_2 \in H^{-1}(0,1).$$

On the other hand, recalling that the identity $T \equiv -\left(I + B^{-1}\right)\partial_{xx}$*, and using the fact that*

$$-\partial_{xx} : H_0^1(0,1) \longrightarrow H^{-1}(0,1)$$

and

$$D\left(I + B^{-1}\right) = L^2(0,1),$$

it follows that

$$T\theta \in H^{-1}(0,1) \qquad \text{and} \qquad \forall\, \theta \in H_0^1(0,1).$$

Consequently,

$$\alpha\mu T \varphi_{xx} \in H^{-1}(0,1).$$

Based on the preceding analysis and the definition of the operator T, we can deduce that

$$\varphi_{xx} \in H_0^1(0,1),$$

which implies that $\varphi_{xx}(0) = \varphi_{xx}(1) = 0$. Consequently,

$$\varphi \in H_*^3(0.1)$$

and

$$R\phi - bT\varphi_{xx} - \delta T\theta = Rg_2.$$

As a result, $(\varphi, \phi, \theta, q) \in D(\mathcal{B})$ and it satisfies system (2.17), demonstrating that $\mathcal{B}$ is maximal. □

Thanks to the Hille–Yosida Theorem, problem (2.15) has a unique solution. This completes the proof of Theorem 2.1.

At this stage, we denote by H the Hilbert space

$$H = \left(H^2(0,1) \cap H_0^1(0,1)\right) \times H^1(0,1) \times L^2(0,1) \times L_*^2(0,1),$$

and define the set

$$D=\left\{\begin{array}{c}\Phi=(\varphi,\phi,\psi,\theta,q)^{T}\in H:\varphi\in H_{*}^{3}\left(0,1\right);\ \phi\in H^{2}\left(0,1\right)\cap H_{0}^{1}(0,1);\\ \psi\in H_{*}^{2}\left(0,1\right);\ \theta\in H_{0}^{1}\left(0,1\right);\ q\in H^{1}\left(0,1\right)\cap L_{*}^{2}\left(0,1\right)\end{array}\right\}.$$

The well-posedness result of problem (2.1)-(2.3) is given by the following theorem:

Theorem 2.2. *Let* $(\varphi_0,\varphi_1,\psi_0,\theta_0,q_0)\in D$*. Then there exists a unique solution* $(\varphi,\varphi_t,\psi,\theta,q)\in C\left(\mathbb{R}^+,D\right)\cap C^1\left(\mathbb{R}^+,H\right)$ *of problem* $(2.1)-(2.3)$.

Proof of Theorem 2.2

Based on Theorem 2.1, there exists a unique solution $(\varphi,\varphi_t,\theta,q)$ in the space $C\left(\mathbb{R}^+;D(\mathcal{B})\right)\cap C^1\left(\mathbb{R}^+;\mathcal{H}\right)$ to the problem (2.14). Thus, the problem (2.1)-(2.3) can be reduced to finding a solution to the following problem:

$$\begin{cases}-b\psi_{xx}+\kappa\psi=f\ \text{ in }(0.1),\\ \psi_x(0)=\psi_x(1)=0,\end{cases}\tag{2.24}$$

where

$$f=\rho_2\varphi_{xtt}-\kappa\varphi_x-\delta\theta_x.$$

From Theorem 2.1, we have

$$\varphi\in C\left(\mathbb{R}^+;H_*^3(0,1)\cap H_0^1(0,1)\right)\cap C^1\left(\mathbb{R}^+,H^2(0,1)\cap H_0^1(0,1)\right)\cap C^2\left(\mathbb{R}^+,H_0^1(0,1)\right),$$

and

$$\theta\in C\left(\mathbb{R}^+;H_0^1(0,1)\right)\cap C^1\left(\mathbb{R}^+;L^2(0,1)\right).$$

On the other hand, from (2.10), we have

$$\varphi_{ttt}\in C\left(\mathbb{R}^+;L^2(0,1)\right).$$

Therefore, $\varphi\in C^3\left(\mathbb{R}^+,L^2(0,1)\right)$.

The function f is formed by summing several terms that are continuous over time and take values in the space $L^2(0,1)$. Since the term φ_{xtt},φ_x and θ_x each possess the required smoothness and are in $L^2(0,1)$, and because multiplication by constants does not affect their smoothness, it follows that:

$$f = \rho_2 \varphi_{xtt} - \kappa\varphi - \delta\theta_x \in C^1(\mathbb{R}^+, L^2(0,1)).$$

To prove that the problem (2.24) admits a unique solution, we apply the Lax-Milgram theorem. Let be consider the following variational formulation

$$\widetilde{A}(\psi, u) = \widetilde{L}(u), \ \forall u \in H^1_*(0,1), \tag{2.25}$$

where

$$\widetilde{A}(\psi, u) = b\int_0^1 \psi_x u_x dx + \kappa \int_0^1 \psi u dx$$

and

$$\widetilde{L}(u) = \int_0^1 f u dx.$$

It is clear that $\widetilde{A}$ is a bilinear form and $\widetilde{L}$ is linear form. Both $\widetilde{A}$ and $\widetilde{L}$ are bounded. Moreover

$$\begin{aligned}\widetilde{A}(\psi, \psi) &= b\int_0^1 \psi_x^2\, dx + \kappa \int_0^1 \psi^2\, dx \\ &\geq C\left(\int_0^1 \psi_x^2\, dx + \int_0^1 \psi^2\, dx\right).\end{aligned}$$

By choosing $C = \min(b, \kappa)$, we get $\widetilde{A}(\psi,\psi) \geq C\|\psi\|^2_{H^1_*}$. We conclude that, the bilinear form $\widetilde{A}$ is coercive.

Thanks to the Lax-Milgram theorem, the problem (2.25) has a unique solution $\psi \in H^1_*(0,1)$. Furthermore,

$$b\int_0^1 \psi_x u_x dx = \int_0^1 (f - \kappa\psi) u dx, \quad \forall u \in H^1_*(0,1). \tag{2.26}$$

Here, we cannot apply the elliptic regularity directly. To do so, we proceed as follows: Let $v \in H^1_0(0,1)$ and set $u = v - \int_0^1 v(x)dx$. We easily check that $u \in H^1_*(0,1)$. Plugging u in (2.26), taking into account that

$$\kappa\psi - f \in L^2_*(0,1),$$

we obtain

$$b\int_0^1 \psi_x u_x dx = \int_0^1 (f - \kappa\psi) u dx, \quad \forall u \in H^1_0(0,1).$$

This shows that

$$\psi \in H^2(0,1) \cap H^1_*(0,1).$$

An integration by parts leads to

$$b\psi_{xx} = \kappa\psi - f = h, \ \in L^2(0,1).$$

On the other hand, since $b\psi_{xx} = h \in L^2(0,1)$ then

$$b\int_0^1 \psi_{xx}\Psi dx = \int_0^1 h\Psi dx \quad \forall \Psi \in H^1(0,1).$$

Integration by parts gives

$$b\psi_x\Psi|_0^1 - b\int_0^1 \psi_x\Psi_x dx = \int_0^1 h\Psi dx \quad \forall \Psi \in H^1(0,1).$$

As $H^1_*(0,1) \subset H^1(0,1)$, we get

$$b\psi_x\Psi|_0^1 - b\int_0^1 \psi_x\Psi_x dx = \int_0^1 h\Psi dx \quad \forall \Psi \in H^1_*(0,1).$$

The use of (2.26) leads to

$$b\psi_x(1)\Psi(1) - b\psi_x(0)\Psi(0) = 0,$$

since Ψ is arbitrary, we deduce that:

$$\psi_x(0) = \psi_x(1) = 0. \tag{2.27}$$

Consequently, $\psi \in H^2_*(0,1)$, and it satisfies the equation given in (2.24).

Therefore, $(\varphi, \varphi_t, \psi, \theta, q)$ belongs to the domain D. Moreover, since $(\varphi, \varphi_t, \theta, q)$ is a solution of equation (2.14), we conclude that $(\varphi, \varphi_t, \psi, \theta, q)$ is the unique solution to problem (2.1) with the initial and boundary conditions given by (2.2) and (2.3). This completes the proof of Theorem 2.2.

2.3 Exponential stability

In this section, we prove the exponential stability of the solution of system (2.1). We use the multiplier method to achieve our goal.

First, we define the energy associated to the solution of (2.1)-(2.3) by

$$E(t) = \frac{1}{2}\int_0^1 \left(\rho_1\varphi_t^2 + \frac{\rho_2\rho_1}{\kappa}\varphi_{tt}^2 + \rho_2\varphi_{xt}^2 + b\psi_x^2 + k(\varphi_x+\psi)^2 + c\theta^2 + \tau q^2\right) dx. \tag{2.28}$$

We have the following result

Lemma 2.3. *The energy $E(t)$, defined by (2.28), satisfies, along the solution $(\varphi, \psi, \theta, q)$ of (2.1), the estimate*

$$E'(t) = -\beta \int_0^1 q^2 dx \leq 0. \tag{2.29}$$

Proof. Taking the L^2-inner product of the equations of (2.1) by φ_t, ψ_t, θ and q, respectively, and applying integration by parts, the boundary conditions , we obtain

$$\frac{\rho_1}{2}\frac{d}{dt}\int_0^1 \varphi_t^2\, dx - \kappa\int_0^1 (\varphi_x+\psi)_x\varphi_t\, dx = 0, \tag{2.30}$$

$$-\rho_2\int_0^1 \varphi_{ttx}\psi_t\, dx + \frac{b}{2}\frac{d}{dt}\int_0^1 \psi_x^2\, dx + \kappa\int_0^1 (\varphi_x+\psi)\psi_t\, dx + \delta\int_0^1 \theta_x\psi_t\, dx = 0, \tag{2.31}$$

$$\frac{c}{2}\frac{d}{dt}\int_0^1 \theta^2\, dx + \int_0^1 q_x\theta\, dx + \delta\int_0^1 \psi_{xt}\theta\, dx = 0, \tag{2.32}$$

$$\frac{\tau}{2}\frac{d}{dt}\int_0^1 q^2\, dx + \beta\int_0^1 q^2\, dx + \int_0^1 \theta_x q\, dx = 0, \tag{2.33}$$

The addition of (2.30)-(2.33) and using integration by parts, we find

$$\frac{1}{2}\frac{d}{dt}\left[\rho_1\int_0^1 \varphi_t^2\, dx + b\int_0^1 \psi_x^2\, dx + c\int_0^1 \theta^2\, dx + \tau\int_0^1 q^2\, dx\right]$$
$$+\kappa\int_0^1 (\varphi_x+\psi)(\varphi_x+\psi)_t\, dx - \rho_2\int_0^1 \varphi_{ttx}\psi_t\, dx + \beta\int_0^1 q^2\, dx = 0,$$

therefore,

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \left[\rho_1\varphi_t^2 + b\psi_x^2 + c\theta^2 + \tau q^2 + \kappa(\varphi_x+\psi)^2\right] dx - \rho_2\int_0^1 \varphi_{ttx}\psi_t\, dx = -\beta\int_0^1 q^2\, dx. \tag{2.34}$$

On the other hand, applying integration by parts and using the first equation of (2.1), we get

$$\begin{aligned}
-\rho_2 \int_0^1 \varphi_{ttx}\psi_t \, dx &= \rho_2 \int_0^1 \varphi_{tt}\psi_{xt} \, dx = \rho_2 \int_0^1 \varphi_{tt} \left[\frac{\rho_1}{\kappa}\varphi_{ttt} - \varphi_{xxt}\right] \, dx \\
&= \frac{\rho_1\rho_2}{\kappa} \int_0^1 \varphi_{tt}\varphi_{ttt} \, dx - \rho_2 \int_0^1 \varphi_{tt}\varphi_{xxt} \, dx \\
&= \frac{\rho_1\rho_2}{2\kappa} \frac{d}{dt} \int_0^1 \varphi_{tt}^2 \, dx + \frac{\rho_2}{2} \frac{d}{dt} \int_0^1 \varphi_{xt}^2 \, dx, \qquad (2.35)
\end{aligned}$$

substituting equation (2.35) into equation(2.34), we obtain

$$\frac{1}{2}\frac{d}{dt} \int_0^1 \left[\rho_1\varphi_t^2 + b\psi_x^2 + c\theta^2 + \tau q^2 + \kappa(\varphi_x + \psi)^2 + \frac{\rho_1\rho_2}{\kappa}\varphi_{tt}^2 + \rho_2\varphi_{xt}^2\right] dx = -\beta \int_0^1 q^2 \, dx.$$

Consequently,

$$E'(t) = -\beta \int_0^1 q^2 \, dx \leq 0.$$

which complete the proof. □

Our stability result reads as follows:

Theorem 2.3. *The energy functional of system* (2.1) − (2.3) *satisfies, for two positive constants* λ *and* ξ*,*

$$E(t) \leq \lambda e^{-\xi t}, \forall t \geq 0.$$

The proof of Theorem 2.3 will be established through several lemmas.

Lemma 2.4. *Let*

$$J_1(t) = \tau c \int_0^1 \theta \int_0^x q(s)ds \, dx - \delta\tau \int_0^1 q(\varphi_x + \psi) \, dx + \frac{\delta\rho_1}{\kappa} \int_0^1 \theta\varphi_t \, dx,$$

$$J_2(t) = -\rho_2 \int_0^1 \varphi_{tx}(\varphi_x + \psi) \, dx + \frac{\rho_1 b}{\kappa} \int_0^1 \psi_x\varphi_t \, dx - \frac{\delta\rho_1}{\kappa} \int_0^1 \theta\varphi_t \, dx,$$

and

$$F = \mu_1 J_1 + \mu_0 J_2.$$

The functional F *satisfies, along the solution of (2.1)-(2.3), the estimate*

$$F'(t) \leq -\frac{c\mu_1}{2} \int_0^1 \theta^2 \, dx - \frac{\rho_2\mu_0}{2} \int_0^1 \varphi_{xt}^2 \, dx - \frac{\kappa\mu_0}{2} \int_0^1 (\varphi_x + \psi)^2 \, dx + m \int_0^1 q^2 \, dx, \qquad (2.36)$$

where $\mu_0 = \frac{\delta^2\rho_1}{\kappa c} > 0$*,* $\mu_1 = \mu_0 + \frac{\rho_2\kappa + \rho_1 b}{\kappa} > 0$ *and* $m > 0$ *is a positive constant.*

Proof. By differentiating J_1, we obtain

$$J_1'(t) =\tau c\int_0^1 \theta_t \int_0^x q(s)ds\,dx + \tau c\int_0^1 \theta \int_0^x q_t(s)ds\,dx - \delta\tau\int_0^1 q_t(\varphi_x+\psi)\,dx - \delta\tau\int_0^1 q(\varphi_x+\psi)_t\,dx$$
$$+ \frac{\delta\rho_1}{\kappa}\int_0^1 \theta_t\varphi_t\,dx, + \frac{\delta\rho_1}{\kappa}\int_0^1 \theta\varphi_{tt}\,dx.$$

From equations three and four in (2.1), we infer

$$J_1'(t) = -\tau\int_0^1 q_x\int_0^x q(s)ds\,dx - \delta\tau\int_0^1 \psi_{xt}\int_0^x q(s)ds\,dx - c\beta\int_0^1 \theta\int_0^x q(s)ds\,dx - c\int_0^1 \theta^2\,dx$$
$$+\delta\beta\int_0^1 q(\varphi_x+\psi)\,dx + \delta\int_0^1 \theta_x(\varphi_x+\psi)\,dx - \delta\tau\int_0^1 q\varphi_{xt}\,dx - \delta\tau\int_0^1 q\psi_t\,dx - \frac{\delta\rho_1}{c\kappa}\int_0^1 q_x\varphi_t\,dx$$
$$-\frac{\delta^2\rho_1}{c\kappa}\int_0^1 \psi_{xt}\varphi_t\,dx + \delta\int_0^1 \theta(\varphi_x+\psi)_x\,dx.$$

Using integration by parts and the boundary conditions, we get

$$\begin{aligned} J_1'(t) =& -c\int_0^1 \theta^2dx - c\beta\int_0^1\theta\int_0^x q\left(s\right)dsdx + \tau\int_0^1 q^2dx \\ &+\delta\beta\int_0^1 q\left(\varphi_x+\psi\right)dx - \delta\int_0^1\theta\left(\varphi_x+\psi\right)_x dx - \delta\tau\int_0^1 q\varphi_{xt}dx \\ &+\frac{\delta\rho_1}{c\kappa}\int_0^1 q\varphi_{xt}dx - \frac{\delta^2\rho_1}{c\kappa}\int_0^1\psi_{xt}\varphi_t dx + \frac{\delta\rho_1}{\kappa}\int_0^1\theta\varphi_{tt}dx. \end{aligned}$$

Using the first equation (2.1), we infer that

$$\frac{\rho_1}{\kappa}\varphi_{tt} = \left(\varphi_x+\psi\right)_x,$$

which yields

$$\begin{aligned} J_1'(t) =& -c\int_0^1 \theta^2dx + \tau\int_0^1 q^2dx - c\beta\int_0^1\theta\int_0^x q\left(s\right)dsdx + \delta\beta\int_0^1 q\left(\varphi_x+\psi\right)dx \\ &+\delta\left(\frac{\rho_1}{c\kappa}-\tau\right)\int_0^1 q\varphi_{xt}dx - \mu_0\int_0^1\psi_{xt}\varphi_t dx. \end{aligned} \tag{2.37}$$

Furthermore, differentiation of J_2 gives

$$J_2'(t) = -\rho_2\int_0^1 \varphi_{ttx}(\varphi_x+\psi)\,dx - \rho_2\int_0^1 \varphi_{tx}(\varphi_x+\psi)_t\,dx + \frac{\rho_1 b}{\kappa}\int_0^1 \psi_{xt}\varphi_t\,dx + \frac{\rho_1 b}{\kappa}\int_0^1 \psi_x\varphi_{tt}\,dx$$
$$-\frac{\delta\rho_1}{\kappa}\int_0^1 \theta_t\varphi_t\,dx - \frac{\delta\rho_1}{\kappa}\int_0^1 \theta\varphi_{tt}\,dx,$$

using the second and third equation of (2.1), we arrive that

$$\begin{aligned} J_2'(t) = & \, b\int_0^1 \psi_{xx}\left(\varphi_x+\psi\right)dx - \kappa\int_0^1 \left(\varphi_x+\psi\right)^2 dx - \delta\int_0^1 \theta_x\left(\varphi_x+\psi\right)dx \\ & - \rho_2\int_0^1 \varphi_{tx}^2 dx - \rho_2\int_0^1 \varphi_{tx}\psi_t dx + \frac{\rho_1 b}{\kappa}\int_0^1 \psi_{xt}\varphi_t dx + \frac{\rho_1 b}{\kappa}\int_0^1 \psi_x\varphi_{tt}dx \\ & + \frac{\delta\rho_1}{c\kappa}\int_0^1 q_x\varphi_t dx + \frac{\delta^2\rho_1}{c\kappa}\int_0^1 \psi_{xt}\varphi_t dx - \frac{\delta\rho_1}{\kappa}\int_0^1 \theta\varphi_{tt}dx. \end{aligned}$$

Integration by parts together with the boundary conditions yields

$$\begin{aligned} J_2'(t) = & = -b\int_0^1 \psi_x\left(\varphi_x+\psi\right)_x dx - \kappa\int_0^1 \left(\varphi_x+\psi\right)^2 dx + \delta\int_0^1 \theta\left(\varphi_x+\psi\right)_x dx \\ & - \rho_2\int_0^1 \varphi_{tx}^2 dx + \rho_2\int_0^1 \varphi_t\psi_{xt}dx + \frac{\rho_1 b}{\kappa}\int_0^1 \psi_{xt}\varphi_t dx + \frac{\rho_1 b}{\kappa}\int_0^1 \psi_x\varphi_{tt}dx \\ & - \frac{\delta\rho_1}{c\kappa}\int_0^1 q\varphi_{xt}dx + \frac{\delta^2\rho_1}{c\kappa}\int_0^1 \psi_{xt}\varphi_t dx - \frac{\delta\rho_1}{\kappa}\int_0^1 \theta\varphi_{tt}dx. \end{aligned} \tag{2.38}$$

By substituting $\frac{\rho_1}{\kappa}\varphi_{tt} = \left(\varphi_x+\psi\right)_x$ into (2.38), we obtain

$$\begin{aligned} J_2'(t) = & - b\int_0^1 \psi_x\left(\varphi_x+\psi\right)_x dx - \kappa\int_0^1 \left(\varphi_x+\psi\right)^2 dx + \delta\int_0^1 \theta\left(\varphi_x+\psi\right)_x dx \\ & - \rho_2\int_0^1 \varphi_{tx}^2 dx + \rho_2\int_0^1 \varphi_t\psi_{xt}dx + \frac{\rho_1 b}{\kappa}\int_0^1 \psi_{xt}\varphi_t dx + b\int_0^1 \psi_x(\varphi_x+\psi)_x dx \\ & - \frac{\delta\rho_1}{c\kappa}\int_0^1 q\varphi_{xt}dx + \frac{\delta^2\rho_1}{c\kappa}\int_0^1 \psi_{xt}\varphi_t dx - \delta\int_0^1 \theta(\varphi_x+\psi)_x dx, \end{aligned}$$

hence,

$$J_2'(t) = -\kappa\int_0^1 \left(\varphi_x+\psi\right)^2 dx - \rho_2\int_0^1 \varphi_{tx}^2 dx - \frac{\delta\rho_1}{c\kappa}\int_0^1 q\varphi_{xt}dx. + \mu_1\int_0^1 \psi_{xt}\varphi_t dx. \tag{2.39}$$

Combining (2.37) and (2.39) with (2.36), we find

$$\begin{aligned} F' = & - c\mu_1\int_0^1 \theta^2 dx - \kappa\mu_0\int_0^1 \left(\varphi_x+\psi\right)^2 dx - \mu_0\rho_2\int_0^1 \varphi_{tx}^2 dx + \mu_1\tau\int_0^1 q^2 dx \\ & - c\mu_1\beta\int_0^1 \theta\int_0^x q\left(s\right)ds dx + \mu_1\delta\beta\int_0^1 q\left(\varphi_x+\psi\right)dx \\ & + \left(\mu_1\delta\left(\frac{\rho_1}{c\kappa}-\tau\right) - \mu_0\frac{\delta\rho_1}{c\kappa}\right)\int_0^1 q\varphi_{xt}dx. \end{aligned} \tag{2.40}$$

At this point, we apply Cauchy–Schwarz and Young's inequalities to obtain

$$\begin{aligned}
-c\mu_1\beta\int_0^1\theta\int_0^x q(s)dsdx &\le c\mu_1\beta\left(\int_0^1\theta^2dx\right)^{\frac{1}{2}}\left(\int_0^1\left(\int_0^x q(s)ds\right)^2dx\right)^{\frac{1}{2}}\\
&\le \frac{c\mu_1}{2}\int_0^1\theta^2dx+\frac{c\mu_1\beta^2}{2}\int_0^1\left(\int_0^x q(s)ds\right)^2dx\\
&\le \frac{c\mu_1}{2}\int_0^1\theta^2dx+\frac{c\mu_1\beta^2}{2}\int_0^1\left(\int_0^1 q(s)ds\right)^2dx\\
&\le \frac{c\mu_1}{2}\int_0^1\theta^2dx+\frac{c\mu_1\beta^2}{2}\int_0^1 q^2dx \qquad (2.41)
\end{aligned}$$

and

$$\mu_1\delta\beta\int_0^1 q\left(\varphi_x+\psi\right)dx\le\frac{\kappa\mu_0}{2}\int_0^1\left(\varphi_x+\psi\right)^2dx+\frac{\mu_1^2\delta^2\beta^2}{2\kappa\mu_0}\int_0^1 q^2dx. \qquad (2.42)$$

Similarly, we have

$$\left(\mu_1\delta\left(\frac{\rho_1}{c\kappa}-\tau\right)-\mu_0\frac{\delta\rho_1}{c\kappa}\right)\int_0^1 q\varphi_{xt}dx\le\frac{\mu_0\rho_2}{2}\int_0^1\varphi_{xt}^2dx+\frac{1}{2\mu_0\rho_2}\left(\mu_1\delta\left(\frac{\rho_1}{c\kappa}-\tau\right)-\mu_0\frac{\delta\rho_1}{c\kappa}\right)^2\int_0^1 q^2dx. \qquad (2.43)$$

The substitution of (2.41), (2.42 and (2.43) into (2.40) yields (2.36). □

Lemma 2.5. *The functional*

$$G(t)=\kappa\rho_2\int_0^1\varphi_x\varphi_{xt}\,dx+\kappa\delta\tau\int_0^1 q\varphi_x\,dx$$

satisfies, along the solution of (2.1)-(2.3), the estimate

$$\begin{aligned}
G'(t)\le&-\frac{\rho_1\rho_2}{2}\int_0^1\varphi_t^2\,dx-\frac{b\kappa}{2}\int_0^1\psi_x^2\,dx+m\int_0^1\varphi_x^2\,dx\\
&+m\int_0^1(\varphi_x+\psi)^2\,dx+m\int_0^1\theta^2\,dx+m\int_0^1 q^2\,dx, \qquad (2.44)
\end{aligned}$$

for some constant $m>0$.

Proof. Multiplying the equation $(2.1)_1$ by $\rho_2\varphi_{tt}$ yields

$$\rho_1\rho_2\int_0^1\varphi_{tt}^2dx-\kappa\rho_2\int_0^1\left(\varphi_x+\psi\right)_x\varphi_{tt}dx=0.$$

An integration by parts over $(0,1)$, together with the boundary conditions, gives

$$\rho_1\rho_2\int_0^1\varphi_{tt}^2dx+\kappa\rho_2\int_0^1\left(\varphi_x+\psi\right)\varphi_{ttx}dx=0.$$

By using equation $(2.1)_2$, we obtain

$$\rho_1\rho_2\int_0^1 \varphi_{tt}^2 dx - b\kappa\int_0^1 \psi\psi_{xx}dx - b\kappa\int_0^1 \varphi_x\psi_{xx}dx + \kappa^2\int_0^1 (\varphi_x+\psi)^2 dx - \delta\kappa\int_0^1 (\varphi_x+\psi)_x \theta dx = 0.$$

Applying integration by parts and using the boundary conditions, we find

$$\rho_1\rho_2\int_0^1 \varphi_{tt}^2 dx + b\kappa\int_0^1 \psi_x^2 dx - b\kappa\int_0^1 \varphi_x\psi_{xx}dx + \kappa^2\int_0^1 (\varphi_x+\psi)^2 dx - \delta\kappa\int_0^1 (\varphi_x+\psi)_x \theta dx = 0.$$

Replacing $b\psi_{xx}$ by $-\rho_2\varphi_{ttx} + \kappa(\varphi_x+\psi) + \delta\theta_x$, we obtain

$$\begin{aligned}&\rho_1\rho_2\int_0^1 \varphi_{tt}^2 dx + b\kappa\int_0^1 \psi_x^2 dx + \kappa\rho_2\int_0^1 \varphi_x\varphi_{ttx}dx - \kappa^2\int_0^1 \varphi_x(\varphi_x+\psi)dx\\ &- \kappa\delta\int_0^1 \varphi_x\theta_x dx + \kappa^2\int_0^1 (\varphi_x+\psi)^2 dx - \delta\kappa\int_0^1 (\varphi_x+\psi)_x \theta dx = 0.\end{aligned}$$

Again replacing $\kappa(\varphi_x+\psi)_x$ by $\rho_1\varphi_{tt}$, we get

$$\begin{aligned}&\rho_1\rho_2\int_0^1 \varphi_{tt}^2 dx + b\kappa\int_0^1 \psi_x^2 dx + \kappa\rho_2\int_0^1 \varphi_x\varphi_{ttx}dx - \kappa^2\int_0^1 \varphi_x(\varphi_x+\psi)dx\\ &- \kappa\delta\int_0^1 \varphi_x\theta_x dx + \kappa^2\int_0^1 (\varphi_x+\psi)^2 dx - \delta\rho_1\int_0^1 \varphi_{tt}\theta dx = 0,\end{aligned}$$

consequently,

$$\begin{aligned}&\rho_1\rho_2\int_0^1 \varphi_{tt}^2 dx + b\kappa\int_0^1 \psi_x^2 dx + \kappa\rho_2\int_0^1 \varphi_x\varphi_{ttx}dx + \kappa^2\int_0^1 (\varphi_x+\psi)\psi dx\\ &- \kappa\delta\int_0^1 \varphi_x\theta_x dx - \delta\rho_1\int_0^1 \varphi_{tt}\theta dx = 0.\end{aligned}$$

By virtue of the fourth equation in (2.1), we infer that

$$\begin{aligned}&\rho_1\rho_2\int_0^1 \varphi_{tt}^2 dx + b\kappa\int_0^1 \psi_x^2 dx + \kappa\rho_2\int_0^1 \varphi_x\varphi_{ttx}dx + \kappa^2\int_0^1 (\varphi_x+\psi)\psi dx\\ &+ \kappa\delta\tau\int_0^1 q_t\varphi_x dx + \kappa\delta\beta\int_0^1 q\varphi_x dx - \delta\rho_1\int_0^1 \varphi_{tt}\theta dx = 0.\end{aligned} \tag{2.45}$$

On the other hands, by differentiating $G(t)$, we get

$$G'(t) = \kappa\rho_2\int_0^1 \varphi_{xt}^2 dx + \kappa\rho_2\int_0^1 \varphi_x\varphi_{ttx}dx + \kappa\delta\tau\int_0^1 q\varphi_{xt}dx + \kappa\delta\tau\int_0^1 q_t\varphi_x dx, \tag{2.46}$$

substituting the terms $\kappa\rho_2\int_0^1 \varphi_x\varphi_{ttx}dx + \kappa\delta\tau\int_0^1 q_t\varphi_x dx$ from (2.45) into (2.46), we arrive that

$$\begin{aligned}G'(t) = &-\rho_1\rho_2\int_0^1 \varphi_{tt}^2 dx - b\kappa\int_0^1 \psi_x^2 dx + \kappa\rho_2\int_0^1 \varphi_{xt}^2 dx - \kappa^2\int_0^1 (\varphi_x+\psi)\psi dx\\ &+ \kappa\delta\tau\int_0^1 q\varphi_{xt}dx - \kappa\delta\beta\int_0^1 q(\varphi_x+\psi)dx + \kappa\delta\beta\int_0^1 q\psi dx + \delta\rho_1\int_0^1 \varphi_{tt}\theta dx.\end{aligned}$$

Cauchy-Schwarz and Young's inequalities yield

$$\kappa\delta\tau\int_0^1 q\varphi_{xt}dx \le \kappa\rho_2\int_0^1 \varphi_{xt}^2dx + \frac{\kappa\delta^2\tau^2}{4\rho_2}\int_0^1 q^2dx, \tag{2.47}$$

$$\kappa\delta\beta\int_0^1 q\left(\varphi_x+\psi\right)dx \le \varepsilon\int_0^1 \left(\varphi_x+\psi\right)^2 dx + \frac{\kappa^2\delta^2\beta^2}{4\varepsilon}\int_0^1 q^2dx \tag{2.48}$$

and

$$\delta\rho_1\int_0^1 \varphi_{tt}\theta dx \le \frac{\rho_1\rho_2}{2}\int_0^1 \varphi_{tt}^2dx + \frac{\delta^2\rho_1}{2\rho_2}\int_0^1 \theta^2dx. \tag{2.49}$$

Similarly, using Poincaré inequality, we obtain

$$\kappa\delta\beta\int_0^1 q\psi dx \le \frac{b\kappa}{4}\int_0^1 \psi_x^2dx + \frac{\kappa\delta^2\beta^2C_p}{b}\int_0^1 q^2dx \tag{2.50}$$

and

$$\kappa^2\int_0^1 \left(\varphi_x+\psi\right)\psi dx \le \frac{b\kappa}{4}\int_0^1 \psi_x^2dx + \frac{\kappa^3C_p}{b}\int_0^1 \left(\varphi_x+\psi\right)^2 dx. \tag{2.51}$$

Therefore, (2.44) follows from (2.47)-(2.51). □

Lemma 2.6. *The functional*

$$K(t) = -\rho_1\int_0^1 \varphi_t\varphi\,dx$$

satisfies, along the solution of (2.1)-(2.3), the estimate

$$K'(t) \le -\rho_1\int_0^1 \varphi_t^2\,dx + m\int_0^1 (\varphi_x+\psi)^2\,dx + m\int_0^1 \psi_x^2\,dx, \tag{2.52}$$

for some constant $m>0$.

Proof. Direct differentiation of K yields

$$K'(t) = -\rho_1\int_0^1 \varphi_t^2\,dx - \rho_1\int_0^1 \varphi_{tt}\varphi\,dx.$$

The use of the first equation of (2.1) gives

$$K'(t) = -\rho_1\int_0^1 \varphi_t^2\,dx - \kappa\int_0^1 (\varphi_x+\psi)_x\varphi\,dx,$$

using integration by parts and the boundary conditions, we get

$$K'(t) = -\rho_1 \int_0^1 \varphi_t^2\, dx + \kappa \int_0^1 (\varphi_x + \psi)\varphi_x\, dx,$$

consequently,

$$\begin{aligned} K'(t) =& -\rho_1 \int_0^1 \varphi_t^2\, dx + \kappa \int_0^1 (\varphi_x + \psi)((\varphi_x + \psi) - \psi)\, dx \\ =& -\rho_1 \int_0^1 \varphi_t^2\, dx + \kappa \int_0^1 (\varphi_x + \psi)^2\, dx - \kappa \int_0^1 (\varphi_x + \psi)\psi\, dx. \end{aligned} \tag{2.53}$$

Cauchy Schwarz, Poincaré's and Young's inequalities yield

$$\kappa \int_0^1 (\varphi_x + \psi)\, \psi dx \leq \frac{\kappa C_p}{2} \int_0^1 \psi_x^2 dx + \frac{\kappa}{2} \int_0^1 (\varphi_x + \psi)^2\, dx. \tag{2.54}$$

The substitution of (2.54) into (2.53) yields (2.52). □

Now, we define the Lyapunov functional $\mathcal{L}$ by

$$\mathcal{L}(t) := NE(t) + nF(t) + G(t) + \varepsilon K(t),$$

where N, n, ε are positive constants to be fixed later.

Lemma 2.7. *There exists a positive constant $\tilde{m}$ such that*

$$(N - \tilde{m})\, E(t) \leq \mathcal{L}(t) \leq (N + \tilde{m})\, E(t). \tag{2.55}$$

Proof. We have

$$\begin{aligned} |\mathcal{L}(t) - NE(t)| \leq & \, n \int_0^1 \left(\tau c \mu_1 \left| \theta \int_0^x q(s)\, ds \right| + \delta\tau\mu_1 |q(\varphi_x + \psi)| + \rho_2\mu_0 |\varphi_{tx}(\varphi_x + \psi)| \right) dx \\ & + n \int_0^1 \left(\frac{\rho_1 b}{\kappa}\mu_0 |\psi_x \varphi_t| + \frac{\delta\rho_1}{\kappa}(\mu_1 - \mu_0)|\theta\varphi_t| \right) dx \\ & + \int_0^1 (\kappa\rho_2 |\varphi_x\varphi_{tx}| + \kappa\delta\tau |q\varphi_x|)\, dx + \rho_1\varepsilon \int_0^1 |\varphi_t\varphi|\, dx. \end{aligned}$$

By using the Young, Poincaré, Cauchy–Schwarz inequalities and that

$$\varphi_x^2 \leq 2(\varphi_x + \psi)^2 + 2\psi^2,$$

we have, for a positive constant m_0,

$$|\mathcal{L}(t) - NE(t)| \leq m_0 \int_0^1 \left(\varphi_t^2 + \varphi_{xt}^2 + \psi_x^2 + (\varphi_x + \psi)^2 + \theta^2 + q^2 \right) dx.$$

Then, from the definition of E, we conclude, for some constant $\tilde{m} > 0$,

$$|\mathcal{L}(t) - NE(t)| \leq \tilde{m}E(t).$$

Therefore,

$$-\tilde{m}E(t) \leq \mathcal{L}(t) - NE(t) \leq \tilde{m}E(t).$$

Thus,

$$(N - \tilde{m})E(t) \leq \mathcal{L}(t) \leq (N + \tilde{m})E(t). \tag{2.56}$$

□

Proof of theorem 2.3

By differentiating $\mathcal{L}$ and using (2.29),(2.36),(2.44) and (2.52), we get

$$\begin{aligned}\mathcal{L}'(t) \leq & -(\beta N - m(n+1)) \int_0^1 q^2 dx - \left(\frac{c\mu_1}{2}n - m\right) \int_0^1 \theta^2 dx \\ & - \left(\frac{\rho_2\mu_0}{2}n - m\right) \int_0^1 \varphi_{xt}^2 dx - \left(\frac{\kappa\mu_0}{2}n - m(1+\varepsilon)\right) \int_0^1 (\varphi_x + \psi)^2 dx \\ & - \frac{\rho_1\rho_2}{2} \int_0^1 \varphi_{tt}^2 dx - \left(\frac{b\kappa}{2} - m\varepsilon\right) \int_0^1 \psi_x^2 dx - \rho_1\varepsilon \int_0^1 \varphi_t^2 dx.\end{aligned}$$

Now, we choose carefully the constants.

First, we take ε small enough so that

$$\frac{b\kappa}{2} - m\varepsilon > 0.$$

Next, we pick n large enough so that

$$n > 2\max\left\{\frac{m(1+\varepsilon)}{\kappa\mu_0}, \frac{m}{\rho_2\mu_0}, \frac{m}{c\mu_1}\right\}.$$

Finally, we select N large enough so that

$$N > \max\left\{\tilde{m}, \frac{m(n+1)}{\beta}\right\}.$$

Therefore, there exist three positive constants ω, α_1 and α_2 such that

$$\mathcal{L}'(t) \leq -\omega E(t)$$

and

$$\alpha_1 E(t) \leq \mathcal{L}(t) \leq \alpha_2 E(t), \quad \forall t \geq 0. \tag{2.57}$$

consequently, we get

$$\mathcal{L}'(t) \leq -\xi \mathcal{L}(t),$$

for some $\xi > 0$.

An integration over $(0;t)$ gives

$$\mathcal{L}(t) \leq \mathcal{L}(0)\, e^{-\xi t}, \forall t \geq 0.$$

Again, the equivalence (2.57) yields

$$E(t) \leq \lambda e^{-\xi t}, \forall t \geq 0,$$

for some constant $\lambda > 0$; which completes the proof of Theorem 2.3.

chapter

3 Exponential stability of a thermoelastic Gurtin–Pipkin–Timoshenko system without the second spectrum

3.1 Introduction

In this chapter we study the well-posedness and asymptotic stability of a thermoelastic Timoshenko system with one spectrum, utilizing the Gurtin-Pipkin thermal law for heat conduction and the rotation angle equation for coupling. Specifically, we consider the following system

$$\begin{cases} \rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0, & \text{in } (0,1)\times(0,\infty), \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \kappa(\varphi_x + \psi) + \delta\theta_x = 0, & \text{in } (0,1)\times(0,\infty), \\ c\theta_t - \dfrac{1}{\beta}\displaystyle\int_0^{+\infty} g(s)\theta_{xx}(t-s)\,ds + \delta\psi_{xt} = 0, & \text{in } (0,1)\times(0,\infty), \end{cases} \tag{3.1}$$

where φ as the transverse displacement, ψ as the volume fraction, and θ as the temperature difference from a reference configuration of a porous material and The coefficients ρ_1,ρ_2 ,κ ,β, b,δ , c are positive constants .

Supplemented with the following boundary conditions of Dirichlet–Neumann–Dirichlet type:

$$\begin{cases} \varphi(0,t) = \psi_x(0,t) = \theta(0,t) = 0, \\ \varphi(1,t) = \psi_x(1,t) = \theta(1,t) = 0, \end{cases} \quad \forall t \geq 0, \tag{3.2}$$

and initial conditions

$$\begin{cases} \varphi(x,0)=\varphi_0(x), \quad \varphi_t(x,0)=\varphi_1(x), \\ \psi(x,0)=\psi_0(x), \quad \theta(x,0)=\theta_0(x), \end{cases} \qquad \forall x \in (0,1). \tag{3.3}$$

3.2 Preliminaries

In order to formulate problem (3.1) in the semigroup setting, we adopt the methodology developed by Giorgi et al. [14] and define the new variable

$$\theta^t(x,s) := \theta(x,t-s), \quad s \geq 0$$

and

$$\eta(x,s) := \eta^t(x,s) := \int_0^s \theta^t(x,\tau)d\tau, \ s \geq 0, \tag{3.4}$$

which represent the past history and the integrated past history of θ up to t, respectively. Clearly, we have

$$\eta_t(x,s) + \eta_s(x,s) = \theta(x,t).$$

The system (3.1) is supplemented with the following initial and boundary conditions with boundary conditions

$$\begin{cases} \varphi(x,0)=\varphi_0(x), \varphi_t(x,0)=\varphi_1(x), \psi(x,0)=\psi_0(x), \theta(x,0)=\theta_0(x), \ x \in (0,1), \\ \theta(x,-s)=h(x,s), s>0, \ \eta^0(x,s)=\displaystyle\int_0^s h(x,\tau)d\tau=\eta_0(x,s), \\ \eta(x,0)=\displaystyle\lim_{s\longrightarrow 0}\eta^t(x,s)=0. \end{cases} \tag{3.5}$$

$$\begin{cases} \varphi(0,t)=\varphi(1,t)=\psi_x(0,t)=\psi_x(1,t)=\theta(0,t)=\theta(1,t)=0 \\ \eta^t(0,s)=\eta^t(1,s)=0, \ t\in(0,+\infty), \ s>0, \end{cases} \tag{3.6}$$

where, $h \in C\left((0,+\infty); H^1(0,1)\right)$ expresses the history of θ.

Since ψ satisfies Neumann boundary conditions, the classical Poincaré inequality cannot be applied directly. However, combining the second equation of (3.1) with the boundary conditions (3.6) yields

$$\int_0^1 \psi(x,t)dx = 0.$$

which enables us to apply Poincaré's inequality.

Regarding the memory kernel g, we assume that $\lim_{s\to\infty} g(s) = 0$, and that there exists a function μ such that $g'(s) = -\mu(s)$, with the following hypotheses:

(h1) $\mu \in C(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$,

(h2) $\mu(s) > 0,\ \mu'(s) \le 0,\ \forall s \ge 0$,

(h3) $\int_0^\infty \mu(s)ds = g(0)$,

(h4) there exists $\lambda > 0$ such that $\mu'(s) \le -\lambda\mu(s),\ \forall s \ge 0$.

A formal integration by part yields

$$q = -\int_{-\infty}^{t} g(t-s)\,\theta_x(x,s)\,ds = -\int_0^{+\infty} \mu(s)\,\eta_x^t(x,s)\,ds.$$

Thus, the system (3.1) becomes

$$\begin{cases} \rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0, \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \kappa(\varphi_x + \psi) + \delta\theta_x = 0, \\ c\theta_t - \frac{1}{\beta}\int_0^{+\infty} \mu(s)\eta_{xx}(s)\,ds + \delta\psi_{xt} = 0, \\ \eta_t + \eta_s = \theta. \end{cases} \tag{3.7}$$

Let define the weighted Hilbert space

$$\mathcal{M} = L^2_\mu(\mathbb{R}^+, H_0^1(0,1)) = \left\{\eta : \mathbb{R}^+ \to H_0^1(0,1)\ ;\ \int_0^{+\infty} \mu(s)\|\eta_x(s)\|_2^2 ds < +\infty\right\},$$

and the inner product

$$\langle \eta, \theta\rangle_{\mathcal{M}} = \int_0^{+\infty} \mu(s)\langle \eta_x, \theta_x\rangle ds,$$

with the associated norm

$$\|\eta\|^2_{\mathcal{M}} = \int_0^{+\infty} \mu(s)\|\eta_x(s)\|_2^2 ds,$$

where $\langle\cdot,\cdot\rangle$ denotes the L^2-inner product

$$\langle \varphi, \tilde{\varphi}\rangle = \int_0^1 \varphi(x)\tilde{\varphi}(x)dx,$$

and $\|\cdot\|_2$ is the associated L^2-norm defined by

$$\|\varphi\|_2^2 = \int_0^1 |\varphi|^2 dx.$$

and

$$\mathcal{N} := \{\eta \in \mathcal{M} :\ \eta_s \in \mathcal{M},\ \eta(0) = 0\}.$$

3.3 The well-posedness of the problem

In this section, we subject system (3.7)–(3.3) to a series of transformations, thereby recasting within the framework of the Hille Yosida theorem. This approach enables us to establish the existence and uniqueness of solutions.

First, we multiply the second equations in (3.7) by κ, and then differentiate it with respect to x, to obtain

$$\begin{cases} \kappa\psi_x = \rho_1\varphi_{tt} - \kappa\varphi_{xx}, \\ -\rho_2\kappa\varphi_{ttxx} - b\kappa\psi_{xxx} + \kappa^2(\varphi_x + \psi)_x + \delta\kappa\theta_{xx} = 0, \\ c\theta_t - \frac{1}{\beta}\int_0^{+\infty}\mu(s)\eta_{xx}(s)\,ds + \delta\psi_{xt} = 0, \\ \eta_t + \eta_s = \theta, \end{cases} \tag{3.8}$$

by substitute the first equation of (3.8) into the second and the third equation, we get

$$\begin{cases} -\rho_2\kappa\varphi_{ttxx} - b\rho_1\varphi_{ttxx} + b\kappa\varphi_{xxxx} + \kappa\rho_1\varphi_{tt} + \delta\kappa\theta_{xx} = 0, \\ c\theta_t - \frac{1}{\beta}\int_0^{+\infty}\mu(s)\eta_{xx}(s)\,ds + \dfrac{\delta}{\kappa}(\rho_1\varphi_{ttt} + \kappa\varphi_{xxt}) = 0, \\ \eta_t + \eta_s = \theta, \end{cases} \tag{3.9}$$

consequently,

$$\begin{cases} B\varphi_{tt} + b\kappa\varphi_{xxxx} + \delta\kappa\theta_{xx} = 0, \\ c\theta_t - \frac{1}{\beta}\int_0^{+\infty}\mu(s)\eta_{xx}(s)\,ds + \dfrac{\delta}{\kappa}(\rho_1\varphi_{ttt} + \kappa\varphi_{xxt}) = 0, \\ \eta_t + \eta_s = \theta, \end{cases} \tag{3.10}$$

where B is a self-adjoint, positive operator defined on $L^2(0,1)$ with domain $H^2(0,1) \cap H_0^1(0,1)$, defined by $B = \kappa\rho_1 I - (\rho_2\kappa + b\rho_1)\partial_{xx}$

Clearly, we have

$$\partial_{xx} = \frac{1}{\rho_2\kappa + b\rho_1}(\kappa\rho_1 I - B). \tag{3.11}$$

Not that B is invertible because it is coercive.

Next, applying B^{-1} to the first equation of (3.10) and differentiating with respect to t, we obtain

$$\varphi_{ttt} = -\kappa B^{-1}\left(b\varphi_{xxxxt} - \delta\theta_{xxt}\right), \tag{3.12}$$

substitution of (3.12) into the second equation of (3.10) gives

$$\begin{cases} B\varphi_{tt} + b\kappa\varphi_{xxxx} + \delta\kappa\theta_{xx} = 0, \\ (cI - \delta^2\rho_1 B^{-1}\partial_{xx})\theta_t - \frac{1}{\beta}\int_0^{+\infty} \mu(s)\eta_{xx}(s)\,ds - \delta(\rho_1 B^{-1} b\partial_{xxxx} + \partial_{xx})\varphi_t = 0, \\ \eta_t + \eta_s - \theta = 0, \end{cases} \tag{3.13}$$

therefore,

$$\begin{cases} B\varphi_{tt} + b\kappa\varphi_{xxxx} + \delta\kappa\theta_{xx} = 0 \\ S\theta_t - \frac{1}{\beta}\int_0^{+\infty} \mu(s)\eta_{xx}(s)\,ds - \delta T\varphi_t = 0, \\ \eta_t + \eta_s - \theta = 0, \end{cases} \tag{3.14}$$

where S and $T : L^2(0,1) \to L^2(0,1)$, are operators defined by $S = cI - \delta^2\rho_1 B^{-1}\partial_{xx}$ and

$$\begin{aligned} T &= \left[I + b\rho_1 B^{-1}\left(\frac{1}{\rho_2\kappa + b\rho_1}(\kappa\rho_1 I - B)\right)\right]\partial_{xx}, \\ &= [\mathrm{I} + \frac{b\rho_1^2 B^{-1}\kappa}{b\rho_1 + \kappa\rho_2} - \frac{b\rho_1 \mathrm{I}}{b\rho_1 + \kappa\rho_2}]\partial_{xx}, \\ &= \frac{\kappa}{b\rho_1 + \kappa\rho_2}[(\rho_2\mathrm{I} + b\rho_1^2 B^{-1}) \circ \partial_{xx}], \end{aligned}$$

On the other hand, multiply the first in equation (3.14) by $\frac{1}{b\rho_1+\kappa\rho_2}[\rho_2\mathrm{I} + b\rho_1^2 B^{-1}]$, we obtain

$$\frac{1}{b\rho_1 + \kappa\rho_2}[\rho_2 B + b\rho_1^2\mathrm{I}]\varphi_{tt} - bT\varphi_{xx} - \delta T\theta = 0.$$

Setting $R = \frac{1}{b\rho_1+\kappa\rho_2}[\rho_2 B + b\rho_1^2\mathrm{I}]$, we get

$$\varphi_{tt} - bT\varphi_{xx} - \delta T\theta = 0.$$

Moreover substituting B, we arrive at

$$\begin{aligned} R &= \frac{1}{b\rho_1 + \kappa\rho_2}[\rho_2[\rho_1\kappa\mathrm{I} - (\rho_2\kappa + b\rho_1)\partial_{xx}] + b\rho_1^2\mathrm{I}] \\ &= \frac{1}{b\rho_1 + \kappa\rho_2}[\rho_2[\rho_1\kappa\mathrm{I} - \rho_2\kappa\partial_{xx} - b\rho_1\partial_{xx}] + b\rho_1^2\mathrm{I}] \\ &= \frac{1}{b\rho_1 + \kappa\rho_2}[\rho_2\rho_1\kappa\mathrm{I} - \rho_2^2\kappa\partial_{xx} - b\rho_2\rho_1\partial_{xx} + b\rho_1^2\mathrm{I}] \\ &= \frac{1}{b\rho_1 + \kappa\rho_2}[\rho_1(b\rho_1 + \rho_2\kappa)\mathrm{I} - \rho_2(\rho_2\kappa + b\rho_1)\partial_{xx}] \\ &= \rho_1\mathrm{I} - \rho_2\partial_{xx}. \end{aligned}$$

Finally, we can reformulate the system (3.8) as the following auxiliary problem:

$$\begin{cases} R\varphi_{tt} + bT\varphi_{xx} + \delta T\theta = 0 \\ S\theta_t - \frac{1}{\beta}\int_0^{+\infty} \mu(s)\eta_{xx}(s)\,ds - \delta T\varphi_t = 0, \\ \eta_t + \eta_s - \theta = 0, \end{cases} \tag{3.15}$$

In order to formulate the auxiliary problem within the semigroup framework, we introduce the Hilbert space

$$\mathcal{X} = (H^2(0,1) \cap H_0^1(0,1)) \times H_0^1(0,1) \times L^2(0,1) \times \mathcal{M},$$

The space $\mathcal{X}$ is equipped with the inner product

$$\langle \Psi, \Psi^* \rangle_{\mathcal{X}} = b(T\varphi, \varphi_{xx}^*) + \langle R\phi, \phi^* \rangle + \langle S\theta, \theta^* \rangle + \frac{1}{\beta}\langle \eta, \eta^* \rangle_{\mathcal{M}},$$

for $\Psi = (\varphi, \phi, \theta, \eta)^T, \Psi^* = (\varphi^*, \phi^*, \theta^*, \eta^*)^T$ in $\mathcal{X}$.
Moreover, we define a new independent variable $\phi = \varphi_t$,
then, the system (3.15) can be written as follows:

$$\begin{cases} \Psi'(t) + \mathcal{A}\Psi(t) = 0, \quad \forall t \geq 0, \\ \Psi(0) = \Psi_0 = (\varphi_0, \varphi_1, \theta_0, \eta_0)^T, \end{cases} \tag{3.16}$$

where $\mathcal{A}$ is the operator defined on $\mathcal{X}$ by

$$\mathcal{A}\Psi = \begin{pmatrix} -\phi \\ bR^{-1}T\varphi_{xx} + \delta R^{-1}T\theta \\ -S^{-1}\left(\frac{1}{\beta}\int_0^{+\infty} \mu(s)\eta_{xx}(s)ds - \delta T\phi\right) \\ \eta_s - \theta \end{pmatrix},$$

with domain

$$D(\mathcal{A}) = \left\{ \begin{array}{c} \Psi = (\varphi, \phi, \theta, \eta)^T \in \mathcal{X} : \varphi \in H_*^3(0,1), \phi \in H^2(0,1) \cap H_0^1(0,1), \\ \theta \in H_0^1(0,1), \eta \in \mathcal{N}, \displaystyle\int_0^{+\infty} \mu(s)\eta_{xx}(s)ds \in L^2(0,1) \end{array} \right\}. \tag{3.17}$$

Lemma 3.1. *[14] For any $\eta \in \mathcal{N}$, we have*

$$\langle \eta_s, \eta \rangle_{\mathcal{M}} = -\frac{1}{2}\int_0^{+\infty} \mu'(s)\|\eta_x(s)\|_2^2 ds. \tag{3.18}$$

Proof. By the use of integration by parts, we have

$$\begin{aligned}\langle \eta_s, \eta \rangle_{\mathcal{M}} &= \int_0^{+\infty} \mu(s) \langle \eta_{sx}(s), \eta_x(s) \rangle ds \\ &= \frac{1}{2} \int_0^{+\infty} \mu(s) \frac{d}{ds} \|\eta_x(s)\|^2 ds \\ &= \frac{1}{2}\, \mu(s) \|\eta_x(s)\|^2 \Big|_0^{+\infty} - \frac{1}{2} \int_0^{+\infty} \mu'(s) \|\eta_x(s)\|^2 ds,\end{aligned}$$

therefore,

$$2\langle \eta_s, \eta \rangle_{\mathcal{M}} = \lim_{s\to\infty} \mu(s)\|\eta_x(s)\|^2 - \lim_{s\to 0} \mu(s)\|\eta_x(s)\|^2 - \int_0^{+\infty} \mu'(s)\|\eta_x(s)\|^2 ds. \tag{3.19}$$

Since $\mu(s)\|\eta_x(s)\|$ and $\mu(s)\|\eta_{xs}(s)\|$ belong to $L^1(\mathbb{R}^+)$ and $\eta_x(0) = 0$, then, the second term in the right hand side of (3.19) worth

$$\begin{aligned}\lim_{s\to 0} \mu(s)\|\eta_x(s)\|^2 &= \lim_{s\to 0} \mu(s) \left\| \int_0^s \eta_{xs}(\tau) d\tau \right\|^2 \\ &\leq \limsup_{s\to 0} \left(\int_0^s \mu(s)^{1/2} \|\eta_{xs}(\tau)\| \, d\tau \right)^2.\end{aligned}$$

The use of Cauchy-Schwarz inequality, leads to

$$\lim_{s\to 0} \mu(s)\|\eta_x(s)\|^2 \leq \limsup_{s\to 0} s \int_0^s \mu(\tau) \|\eta_{xs}(\tau)\|^2 \, d\tau = 0.$$

Therefore,

$$2\langle \eta_s, \eta \rangle_{\mathcal{M}} = \lim_{s\to\infty} \mu(s)\|\eta_x(s)\|^2 - \int_0^{+\infty} \mu'(s)\|\eta_x(s)\|^2 ds.$$

The left-hand side of the last equation is bounded, and from $h1$ both terms on the right-hand side are non-positive. Therefore, we infer that the above limit exists and is finite. Consequently, it is equals zero. Thus

$$\langle \eta_s, \eta \rangle_{\mathcal{M}} = -\frac{1}{2} \int_0^{\infty} \mu'(s)\|\eta_x(s)\|^2 ds.$$

Moreover, the use of the assumption ($h4$) leads to (3.18), which completes the proof. □

The following theorem establishes the well-posedness of the auxiliary problem (3.15).

Theorem 3.1. *For any $\Psi_0 \in \mathcal{X}$, the problem (4.11) has a weak unique solution $\Psi \in C(\mathbb{R}^+, \mathcal{X})$. Moreover, if $\Psi_0 \in D(\mathcal{A})$, then $\Psi \in C(\mathbb{R}^+, D(\mathcal{A})) \cap C^1(\mathbb{R}^+, \mathcal{X})$.*

Proof. The proof will be based on the semigroup theory. According to the Hille-Yosida theorem 1.9 and the Lax-Milgram Theorem 1.3, it suffices to prove that $\mathcal{A}$ is maximal and monotone.

First, we prove that $\mathcal{A}$ is monotone. Let $\Psi \in D(\mathcal{A})$, we have

$$\begin{aligned}\langle \mathcal{A}\Psi, \Psi\rangle_{\mathcal{X}} = -b\langle T\phi, \varphi_{xx}\rangle + b\langle T\varphi_{xx}, \phi\rangle + \delta\langle T\theta, \phi\rangle - \frac{1}{\beta}\int_0^{+\infty} \mu(s)\langle \eta_{xx}(s), \theta\rangle ds - \delta\langle T\phi, \theta\rangle \\ + \frac{1}{\beta}\langle \eta_s, \eta\rangle_{\mathcal{M}} - \frac{1}{\beta}\langle \theta, \eta\rangle_{\mathcal{M}},\end{aligned}$$

since T is a self-adjoint operator, we obtain

$$\begin{aligned}\langle \mathcal{A}\Psi, \Psi\rangle_{\mathcal{X}} &= -b\langle \phi, T\varphi_{xx}\rangle + b\langle T\varphi_{xx}, \phi\rangle + \delta\langle \theta, T\phi\rangle - \frac{1}{\beta}\int_0^{+\infty} \mu(s)\langle \eta_{xx}(s), \theta\rangle ds - \delta\langle T\phi, \theta\rangle \\ &\quad + \frac{1}{\beta}\langle \eta_s, \eta\rangle_{\mathcal{M}} - \frac{1}{\beta}\langle \theta, \eta\rangle_{\mathcal{M}} \\ &= -\frac{1}{\beta}\int_0^{+\infty} \mu(s)\langle \eta_{xx}(s), \theta\rangle ds + \frac{1}{\beta}\langle \eta_s, \eta\rangle_{\mathcal{M}} - \frac{1}{\beta}\langle \theta, \eta\rangle_{\mathcal{M}}.\end{aligned}$$

Using integration by parts and the boundary conditions, we find that

$$\begin{aligned}\langle \mathcal{A}\Psi, \Psi\rangle_{\mathcal{X}} &= \frac{1}{\beta}\int_0^{+\infty} \mu(s)\langle \eta_x(s), \theta_x\rangle ds + \frac{1}{\beta}\langle \eta_s, \eta\rangle_{\mathcal{M}} - \frac{1}{\beta}\langle \theta, \eta\rangle_{\mathcal{M}} \\ &= \frac{1}{\beta}\langle \eta, \theta\rangle_{\mathcal{M}} + \frac{1}{\beta}\langle \eta_s, \eta\rangle_{\mathcal{M}} - \frac{1}{\beta}\langle \theta, \eta\rangle_{\mathcal{M}} \\ &= \frac{1}{\beta}\langle \eta_s, \eta\rangle_{\mathcal{M}}.\end{aligned}$$

From Lemma 3.1 and the hypothesis $(h4)$, we infer that

$$\langle \mathcal{A}\Psi, \Psi\rangle_{\mathcal{X}} = -\frac{1}{2\beta}\int_0^{+\infty} \mu'(s)\|\eta_x(s)\|_2^2 ds \geq 0,$$

and hence, $\mathcal{A}$ is monotone. □

Secondly, we prove that the operator $(I - \mathcal{A})$ is maximal. Let $\mathcal{K} = (k_1, k_2, k_3, k_4)^T \in \mathcal{X}$, and we seek $\Psi \in D(\mathcal{A})$ that satisfies

$$(I - \mathcal{A})\Psi = \mathcal{K}, \tag{3.20}$$

that is,

$$\begin{cases} \varphi - \phi = k_1, \\ R\phi + bT\varphi_{xx} + \delta T\theta = Rk_2, \\ S\theta - \frac{1}{\beta}\int_0^{+\infty} \mu(s)\eta_{xx}(s)ds - \delta T\phi = Sk_3, \\ \eta + \eta_s - \theta = k_4. \end{cases} \tag{3.21}$$

From the first and fourth equations of (3.21), we have

$$\phi = \varphi - k_1, \tag{3.22}$$

and

$$\frac{d\eta}{ds} + \eta = \theta + k_4(s),$$

by multiplying both sides of the equation by e^s, we obtain:

$$e^s \frac{d\eta}{ds} + e^s\eta = e^s\theta + e^s k_4(s),$$

since the left-hand side is the derivative of a product, we get

$$\frac{d}{ds}(e^s\eta) = e^s\theta + e^s k_4(s).$$

An integration of both sides over [0,s] gives:

$$e^s\eta(s) - \eta(0) = \int_0^s e^r\theta\, dr + \int_0^s e^r k_4(r)\, dr.$$

Since θ is independent of r, and $\eta(0) = 0$ we have:

$$e^s\eta(s) = (e^s - 1)\theta + \int_0^s e^r k_4(r)\, dr,$$

dividing both sides by e^{-s}, we find that

$$e^{-s}\eta(s) = e^{-s}(1 - e^{-s})\theta + e^{-s}\int_0^s e^r k_4(r)\, dr,$$

hence, the solution simplifies to:

$$\eta(s) = (1 - e^{-s})\theta + \int_0^s e^{r-s} k_4(r)\, dr. \tag{3.23}$$

Replacing ϕ and η from (3.22) and (3.23) into the second and third equations of (3.21), we get

$$\begin{cases} R\varphi + bT\varphi_{xx} + \delta T\theta = R(k_1 + k_2) \in H^{-1}(0,1), \\ S\theta - \dfrac{1}{\beta}c_\mu\theta_{xx} - \delta T\varphi = Sk_3 - \delta Tk_1 + \dfrac{1}{\beta}\displaystyle\int_0^{+\infty}\mu(s)\left(\int_0^s e^{r-s}(k_4)_{xx}(r)dr\right)ds \in H^{-1}(0,1), \end{cases} \tag{3.24}$$

where

$$c_\mu = \int_0^{+\infty}\mu(s)(1-e^{-s})\,ds \le g(0).$$

Let $\mathcal{W} = (H^2(0,1)\cap H_0^1(0,1)) \times H_0^1(0,1)$. By 'formally' multiplying the equations of (3.24) by $\tilde{\varphi} \in (H^2(0,1)\cap H_0^1(0,1))$ and $\tilde{\theta} \in H_0^1(0,1)$, respectively, we arrive at the following variational formulation

$$A((\varphi,\theta),(\tilde{\varphi},\tilde{\theta})) = L(\tilde{\varphi},\tilde{\theta}), \qquad \forall(\tilde{\varphi},\tilde{\theta}) \in \mathcal{W}, \tag{3.25}$$

where $A(\,\cdot\,,\,\cdot\,)$ and $L(\,\cdot\,)$ are the bilinear and linear forms defined over $\mathcal{W}$ by

$$\begin{aligned} A((\varphi,\theta),(\tilde{\varphi},\tilde{\theta})) &= \langle R^{\frac{1}{2}}\varphi, R^{\frac{1}{2}}\tilde{\varphi}\rangle - b\langle T^{\frac{1}{2}}\varphi_x, T^{\frac{1}{2}}\tilde{\varphi}_x\rangle + \delta\langle T^{\frac{1}{2}}\theta, T^{\frac{1}{2}}\tilde{\varphi}\rangle + \langle S^{\frac{1}{2}}\theta, S^{\frac{1}{2}}\tilde{\theta}\rangle \\ &\quad + \frac{1}{\beta}c_\mu\langle\theta_x,\tilde{\theta}_x\rangle - \delta\langle T^{\frac{1}{2}}\varphi, T^{\frac{1}{2}}\tilde{\theta}\rangle, \end{aligned} \tag{3.26}$$

and

$$L(\tilde{\varphi},\tilde{\theta}) = \langle R(k_1+k_2),\tilde{\varphi}\rangle + \langle Sk_3,\tilde{\theta}\rangle - \delta\langle Tk_1,\tilde{\theta}\rangle - \frac{1}{\beta}\int_0^{+\infty}\mu(s)\langle\int_0^s e^{r-s}(k_4)_x(r)\,dr,\tilde{\theta}_x\rangle\,ds. \tag{3.27}$$

A straightforward calculation shows that A and L are bounded. In addition, we have

$$\begin{aligned} A((\varphi,\theta),(\varphi,\theta)) &= \langle R\varphi,\varphi\rangle - b\langle T\varphi_x,\varphi_x\rangle + \delta\langle T\theta,\varphi\rangle \\ &\quad + \langle S\theta,\theta\rangle + \frac{1}{\beta}c_\mu\langle\theta_x,\theta_x\rangle - \delta\langle T\varphi,\theta\rangle \\ &= \langle R\varphi,\varphi\rangle - b\langle T\varphi_x,\varphi_x\rangle + \langle S\theta,\theta\rangle + \frac{1}{\beta}c_\mu\langle\theta_x,\theta_x\rangle, \end{aligned} \tag{3.28}$$

from the definition of R, S and T, we obtain

$$\begin{aligned} A((\varphi,\theta),(\varphi,\theta)) &= \rho_1\langle\varphi,\varphi\rangle + c\langle\theta,\theta\rangle + \rho_2\langle\varphi_x,\varphi_x\rangle + \frac{\kappa b\rho_2}{b\rho_1+\kappa\rho_2}\langle\varphi_{xx},\varphi_{xx}\rangle \\ &\quad + \frac{\kappa b^2\rho_1^2}{b\rho_1+\kappa\rho_2}\langle B^{-1}\varphi_{xx},\varphi_{xx}\rangle + \delta^2\rho_1\langle B^{-1}\theta_x,\theta_x\rangle + \frac{1}{\beta}c_\mu\langle\theta_x,\theta_x\rangle. \end{aligned} \tag{3.29}$$

Then, the operator B^{-1} define positive, we deduce that

$$\begin{aligned} A((\varphi,\theta),(\varphi,\theta)) &\geq \rho_2\langle\varphi_x,\varphi_x\rangle + \frac{\kappa b\rho_2}{b\rho_1+\kappa\rho_2}\langle\varphi_{xx},\varphi_{xx}\rangle + \frac{1}{\beta}c_\mu\langle\theta_x,\theta_x\rangle \\ &\geq \rho_2\|\varphi_x\|_2^2 + \frac{\kappa b\rho_2}{b\rho_1+\kappa\rho_2}\|\varphi_{xx}\|_2^2 + \frac{1}{\beta}c_\mu\|\theta_x\|_2^2 \\ &\geq M(\|\varphi_x\|_2^2 + \|\varphi_{xx}\|_2^2 + \|\theta_x\|_2^2) = M\|(\varphi,\theta)\|_{\mathcal{W}}^2, \end{aligned} \tag{3.30}$$

hence, A is coercive. Consequently, Lax-Milgram theorem guarantees that the equation (3.25) has a unique solution $(\varphi,\theta) \in \mathcal{W}$.

Next, by substituting φ into (3.22), we infer that

$$\phi \in (H^2(0,1) \cap H_0^1(0,1)).$$

On the other hand, using (3.23) together with the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} \int_0^{+\infty} \mu(s)\|\eta_x\|_2^2 ds &\leq 2\int_0^\infty \mu(s)(1-e^{-s})^2\|\theta_x\|_2^2 ds \\ &+ 2\int_0^\infty \mu(s)\left(\int_0^s e^{2(r-s)}dr\right)\left(\int_0^s \|k_{4x}(r)\|_2^2 dr\right)ds \\ &\leq 2g(0)\|\theta_x\|_2^2 + 2\int_0^\infty \mu(s)\left(\frac{1}{2}-\frac{e^{-2s}}{2}\right)\left(\int_0^s \|k_{4x}\|_2^2 dr\right)ds \\ &\leq 2g(0)\|\theta_x\|_2^2 + \int_0^\infty \mu(s)\left(\int_0^s \|k_{4x}\|_2^2 dr\right)ds \\ &\leq 2g(0)\|\theta_x\|_2^2 + \int_0^\infty \|k_{4x}\|_2^2 \int_r^{+\infty} \mu(s) ds dr. \end{aligned}$$

Applying hypothesis (h4), we arrive at

$$\begin{aligned} \int_0^{+\infty} \mu(s)\|\eta_x\|_2^2 ds &\leq 2g(0)\|\theta_x\|_2^2 - \frac{1}{\lambda}\int_0^\infty \|k_{4x}\|_2^2 \int_r^{+\infty} \mu'(s) ds dr \\ &\leq 2g(0)\|\theta_x\|_2^2 + \frac{1}{\lambda}\int_0^\infty \mu(r)\|k_{4x}\|_2^2 dr < +\infty. \end{aligned}$$

Which shows that $\eta \in \mathcal{M}$.

From (3.23) we have $\eta(0) = 0$.

$$\eta_s(s) = e^{-s}\theta + k_4(s) - \int_0^s e^{r-s}k_4(r)dr = \theta + k_4 - \eta(s) \in \mathcal{M}.$$

Therefore, $\eta \in \mathcal{N}$.

At this stage, taking $\tilde{\theta} = 0$ in (3.25), we get

$$b\langle T\varphi_x, \tilde{\varphi}_x\rangle = -\langle R(k_1 + k_2) - R\varphi - \delta T\theta, \tilde{\varphi}\rangle, \qquad \forall\varphi \in C_0^1(0,1),$$

which gives

$$bT\varphi_{xx} = R(k_1 + k_2) - \delta T\theta - R\varphi, \tag{3.31}$$

Replacing (3.22) into (3.31) we find

$$bT\varphi_{xx} = Rk_2 - \delta T\theta - R\phi, \tag{3.32}$$

Using elliptic regularity theory together with the definition of the operator T, we deduce

$$\varphi_{xx} \in H_0^1(0,1),$$

which implies that

$$\varphi_{xx}(0) = \varphi_{xx}(1) = 0,$$

consequently

$$\varphi \in H_*^3(0,1).$$

Similarly, by taking $\tilde{\varphi} = 0$ in (3.25), we obtain for any $\tilde{\theta} \in C_0^1(0,1)$:

$$\frac{1}{\beta}\langle c_\mu\theta_x - \int_0^{+\infty} \mu(s) \int_0^s e^{r-s}(k_4)_x(r)\,dr\,ds, \tilde{\theta}_x\rangle = \langle -S\theta + \delta T\varphi + Sk_3 - \delta Tk_1, \tilde{\theta}\rangle.$$

Note that, using the definitions of the operators S and T, together with the facts that $\varphi, k_1 \in H^2(0,1) \cap H^1(0,1)$, $\theta \in H_0^1(0,1)$, and $k_3 \in L^2(0,1)$, we deduce that:

$$S\theta - \delta T\varphi - Sk_3 + \delta Tk_1 \in L^2(0,1),$$

which gives,

$$c_\mu\theta_x + \int_0^{+\infty} \mu(s) \left(\int_0^s e^{r-s}k_{4x}(r)dr\right) ds \in H^{-1}(0,1)$$

and therefore

$$\frac{1}{\beta}c_\mu\theta_{xx} + \frac{1}{\beta}\int_0^{+\infty} \mu(s) \left(\int_0^s e^{r-s}(k_4)_{xx}(r)\,dr\right) ds = S\theta - \delta T\varphi - Sk_3 + \delta Tk_1,$$

By virtue of (3.22) and (3.23), we arrive that

$$\frac{1}{\beta}\int_0^{+\infty}\mu(s)\eta_{xx}(s)\,ds = S\theta - \delta T\phi - Sk_3 \in L^2(0,1).$$

Therefore, the solution $(\varphi,\phi,\theta,\eta)$ of system (3.16) belongs to $D(\mathcal{A})$and hence $\mathcal{A}$ is maximal. This completes the proof of Theorem 3.1.

At this point, we put

$$\mathcal{H} = \left(H^2(0,1)\cap H_0^1(0,1)\right)\times H_0^1(0,1)\times H^1(0,1)\times L^2(0,1)\times\mathcal{M},$$

and

$$D = \left\{ \Phi = (\varphi,\phi,\psi,\theta,\eta)^T \in \mathcal{H} \;\middle|\; \begin{array}{c} \varphi\in H_*^3(0,1),\;\; \phi\in H^2(0,1)\cap H_0^1(0,1),\;\; \psi\in H_*^2(0,1), \\ \theta\in H_0^1(0,1),\;\; \eta\in\mathcal{N},\;\; \displaystyle\int_0^{+\infty}\mu(s)\eta_{xx}(s)ds\in L^2(0,1) \end{array} \right\},$$

where

$$H_*^2(0,1) = \left\{\psi\in H^2(0,1) : \psi_x(0)=\psi_x(1)=0\right\}$$

and

$$H_*^3(0,1) = \left\{\varphi\in H^3(0,1)\cap H_0^1(0,1) : \varphi_{xx}\in H_0^1(0,1)\right\}.$$

The well-posedness of problem (3.5)–(3.7) is established in the following theorem.

Theorem 3.2. *Let $(\varphi_0,\varphi_1,\psi_0,\theta_0,\eta_0)\in D$, where the following compatibility condition:*

$$B(\varphi_{0x}+\psi_0) = -\rho_1(b\varphi_{0xxx}+\delta\theta_{0x})$$

is satisfied. Then, there exists a unique solution $(\varphi,\varphi_t,\psi,\theta,\eta)\in C(\mathbb{R}^+,D)\cap C^1(\mathbb{R}^+,\mathcal{H})$ of problem (3.5)–(3.7).

Proof. Let $(\varphi_0,\varphi_1,\theta_0,\eta_0)\in D(\mathcal{A})$, then from theorem 3.1, there exists a unique solution $(\varphi,\varphi_t,\theta,\eta)\in C\left(\mathbb{R}^+;D(\mathcal{A})\right)\cap C^1\left(\mathbb{R}^+;\mathcal{H}\right)$ to the problem (3.15). As a result, we have

$$\varphi\in C\left(\mathbb{R}^+;H_*^3(0,1)\cap H_0^1(0,1)\right)\cap C^1\left(\mathbb{R}^+;H^2(0,1)\cap H_0^1(0,1)\right)\cap C^2\left(\mathbb{R}^+;H_0^1(0,1)\right).$$

Thus, (3.12) obtains

$$\varphi_{ttt} = -\kappa B^{-1}\partial_{xx}\left(b\varphi_{xxt}-\delta\theta_t\right)\in C\left(\mathbb{R}^+;L^2(0,1)\right).$$

Next, let define ψ by

$$\psi(x,t) = -\varphi_x(x,t) + \frac{\rho_1}{\kappa}\int_0^x \varphi_{tt}(y,t)dy, \tag{3.33}$$

then we get

$$\rho_1\varphi_{tt} - \kappa\left(\varphi_x + \psi\right)_x = 0. \tag{3.34}$$

Replacing (3.34) into the second equation of (3.10), we arrive at

$$c\theta_t - \frac{1}{\beta}\int_0^s \kappa(s)\eta_{xx}(s)ds + \delta\psi_{tx} = 0. \tag{3.35}$$

Consequently,

$$\psi_{xt} \in C\left(\mathbb{R}^+, L^2(0,1)\right),$$

which implies that

$$\psi \in C^1(\mathbb{R}^+, H_0^1(0,1)).$$

Moreover, from (3.33), we have

$$\psi_x = -\frac{\kappa}{b}\varphi_{xx} + \frac{\rho_1}{b}\varphi_{tt}.$$

Bearing in mind that $\varphi \in H_*^3(0,1)$, we infer that $\psi \in H_*^2(0,1)$. Therefore, $(\varphi,\psi,\theta,\eta)$ solves the problem (3.7) with the initial and boundary conditions (3.5) and (3.6). which completes the proof of Theorem 3.2. □

3.4 Exponential stability

In this section, we analyze the time-behavior of the solution to problem (3.5)-(3.7). We demonstrate that the solution exhibits exponential decay without imposing any restrictions on the coefficients of the system (3.7).

First, we introduce the energy associated with the solution (φ, ψ, θ,η) of problem (3.5)-(3.7), defined as:

$$E(t) = \frac{1}{2}\int_0^1 \left(\rho_1\varphi_t^2 + \frac{\rho_2\rho_1}{\kappa}\varphi_{tt}^2 + \rho_2\varphi_{xt}^2 + b\psi_x^2 + k(\varphi_x+\psi)^2 + c\theta^2\right)dx$$
$$+ \frac{1}{2\beta}\int_0^{+\infty} \mu(s)\|\eta_x(s)\|_2^2 ds. \tag{3.36}$$

The main result reads as follow:

Theorem 3.3. *The energy functional $E(t)$ satisfies, along the solution of (3.5)-(3.7), the estimate*

$$E(t) \leq \sigma e^{-\zeta t}, \qquad \forall t \geq 0, \tag{3.37}$$

where σ and ζ are two positive constants.

The proof of Theorem 3.3 will be established by the multipliers method through several lemmas.

Lemma 3.2. *Let $(\varphi, \psi, \theta, \eta)$ be a solution to (3.5)-(3.7). Then , the energy functional $E(t)$ satisfies*

$$E'(t) = \frac{1}{2\beta} \int_0^{+\infty} \mu'(s) \|\eta_x(s)\|_2^2 ds \leq 0. \tag{3.38}$$

Proof. Taking the L^2-inner product of the first three equations of (3.7) with φ_t, ψ_t, and θ, respectively, and then applying integration by parts, we arrive at

$$\frac{\rho_1}{2} \frac{d}{dt} \int_0^1 \varphi_t^2 dx + \kappa \int_0^1 (\varphi_x + \psi) \varphi_{xt} dx = 0, \tag{3.39}$$

$$\rho_2 \int_0^1 \varphi_{tt} \psi_{xt} dx + \frac{b}{2} \frac{d}{dt} \int_0^1 \psi_x^2 dx + \kappa \int_0^1 (\varphi_x + \psi) \psi_t + \delta \int_0^1 \theta_x \psi_t dx = 0 \tag{3.40}$$

and

$$\frac{c}{2} \frac{d}{dt} \int_0^1 \theta^2 dx - \frac{1}{\beta} \int_0^1 \theta \left(\int_0^{+\infty} \mu(s) \eta_{xx}(s) ds \right) dx - \delta \int_0^1 \psi_t \theta_x dx = 0. \tag{3.41}$$

On the other hand, applying integration by parts and using the first equation of (3.7), we get

$$\begin{aligned}
\rho_2 \int_0^1 \varphi_{tt} \psi_{xt} \, dx &= \rho_2 \int_0^1 \varphi_{tt} \left[\frac{\rho_1}{\kappa} \varphi_{ttt} - \varphi_{xxt} \right] \, dx \\
&= \frac{\rho_1 \rho_2}{\kappa} \int_0^1 \varphi_{tt} \varphi_{ttt} \, dx - \rho_2 \int_0^1 \varphi_{tt} \varphi_{xxt} \, dx \\
&= \frac{\rho_1 \rho_2}{2\kappa} \frac{d}{dt} \int_0^1 \varphi_{tt}^2 \, dx + \frac{\rho_2}{2} \frac{d}{dt} \int_0^1 \varphi_{xt}^2 \, dx,
\end{aligned} \tag{3.42}$$

substituting 3.42 into (3.40), we find

$$\frac{\rho_1 \rho_2}{2\kappa} \frac{d}{dt} \int_0^1 \varphi_{tt}^2 \, dx + \frac{\rho_2}{2} \frac{d}{dt} \int_0^1 \varphi_{xt}^2 \, dx + \frac{b}{2} \frac{d}{dt} \int_0^1 \psi_x^2 dx + \kappa \int_0^1 (\varphi_x + \psi) \psi_t + \delta \int_0^1 \theta_x \psi_t dx = 0. \tag{3.43}$$

Moreover, integration by parts and the boundary conditions (3.6), we obtain

$$\int_0^1 \theta \left(\int_0^{+\infty} \mu(s)\eta_{xx}(s)ds \right) dx = -\int_0^{+\infty}\int_0^1 \mu(s)\theta_x\eta_x(s)dxds.$$

From the fourth equation of (3.7), we get

$$\begin{aligned} -\int_0^{+\infty}\int_0^1 \mu(s)\theta_x\eta_x(s)dxds &= -\int_0^{+\infty}\int_0^1 \mu(s)\left(\eta_t(s)+\eta_s(s)\right)_x \eta_x(s)dxds \\ &= -\frac{1}{2}\frac{d}{dt}\int_0^{+\infty}\mu(s)\|\eta_x(s)\|_2^2 ds - \frac{1}{2}\int_0^{+\infty}\mu(s)\frac{d}{ds}\|\eta_x(s)\|_2^2 ds. \end{aligned}$$

Using (3.18), we obtain

$$\int_0^1 \theta \left(\int_0^{+\infty} \mu(s)\eta_{xx}(s)ds \right) dx = -\frac{1}{2}\frac{d}{dt}\int_0^{+\infty}\mu(s)\|\eta_x(s)\|_2^2 ds + \frac{1}{2}\int_0^{+\infty}\mu'(s)\|\eta_x(s)\|_2^2 ds. \tag{3.44}$$

substituting(3.44) into (3.41), we arrive at

$$\frac{c}{2}\frac{d}{dt}\int_0^1 \theta^2 dx + \frac{1}{2\beta}\frac{d}{dt}\int_0^{+\infty}\mu(s)\|\eta_x(s)\|_2^2 ds + \frac{1}{2\beta}\int_0^{+\infty}\mu'(s)\|\eta_x(s)\|_2^2 ds - \delta\int_0^1 \varphi_t\theta_x dx = 0. \tag{3.45}$$

Finally, adding (3.39), (3.43), and (3.45) yields the desired estimate (3.38). □

Lemma 3.3. *For any $\varphi \in L^2(0,1)$ and any ε positive, we have*

$$-\int_0^1 \varphi \int_0^{\infty} \mu'(s)\eta(s)dsdx \le \varepsilon \int_0^L \varphi^2 dx - \frac{\lambda c_p g(0)}{4\varepsilon}\int_0^{\infty}\mu'(s)\|\eta_x(s)\|^2 ds. \tag{3.46}$$

and

$$\int_0^1 \left(\int_0^{+\infty}\mu(s)\eta_x(s)ds \right)^2 dx \le -\frac{g(0)}{\lambda}\int_0^{\infty}\mu'(s)\|\eta_x(s)\|_2^2 ds. \tag{3.47}$$

Now, we consider the following two functional:

$$\begin{aligned} F_1(t) &= -\frac{c}{g(0)}\int_0^{+\infty}\mu(s)\int_0^1 \theta\eta(s)dxds + \frac{\delta\rho_1}{\kappa}\int_0^1 \varphi_t\theta dx \\ &\quad - \frac{\delta\rho_1}{\kappa g(0)}\int_0^1 \varphi_{tt}\int_0^{+\infty}\mu(s)\eta(s)dsdx, \\ F_2(t) &= b\int_0^1 \psi_x\varphi_t dx - \delta\int_0^1 \theta\varphi_t dx, \end{aligned}$$

and the two positive constants

$$\mu_2 = \frac{bc+\delta^2}{c},$$
$$\mu_1 = \frac{\delta^2\rho_1}{c\kappa}.$$

Lemma 3.4. *Let $(\varphi, \psi, \theta, \eta)$ be the solution of problem (3.5)– (3.7). Then, the functional*

$$\mathcal{F} = \mu_2 F_1 + \mu_1 F_2$$

satisfies the estimate

$$\begin{aligned}\mathcal{F}'(t) \leq & -\frac{c\mu_2}{2}\int_0^1 \theta^2 dx - \frac{\rho_2\mu_1}{2}\int_0^1 \varphi_{tt}^2 dx - \frac{\kappa^2\mu_1}{\rho_1}\int_0^1 (\varphi_x+\psi)^2 dx \\ & + \varepsilon_1 \int_0^1 \varphi_{xt}^2 dx - m\left(1+\frac{1}{\varepsilon_1}\right)\int_0^{+\infty} \mu'(s)\|\eta_x(s)\|_2^2 ds \end{aligned} \tag{3.48}$$

for any $\varepsilon_1 > 0$ and for some constant $m > 0$ independent of ε_1.

Proof. we differentiation the equation $F_1(t)$, we get

$$\begin{aligned}F_1'(t) = & -\frac{c}{g(0)}\int_0^{+\infty}\mu(s)\int_0^1 \theta_t \eta(s)\,dx\,ds - \frac{c}{g(0)}\int_0^{+\infty}\mu(s)\int_0^1 \theta\eta_t(s)\,dx\,ds \\ & + \frac{\delta\rho_1}{\kappa}\int_0^1 \varphi_{tt}\theta\,dx + \frac{\delta\rho_1}{\kappa}\int_0^1 \varphi_t\theta_t\,dx - \frac{\delta\rho_1}{\kappa g(0)}\int_0^1 \varphi_{ttt}\int_0^{+\infty}\mu(s)\eta(s)\,ds\,dx \\ & - \frac{\delta\rho_1}{\kappa g(0)}\int_0^1 \varphi_{tt}\int_0^{+\infty}\mu(s)\eta_t(s)\,ds\,dx,\end{aligned}$$

using $(3.7)_3$, and $(3.7)_4$, leads to

$$\begin{aligned}F_1'(t) = & -\frac{c}{g(0)}\frac{1}{c\beta}\int_0^{+\infty}\mu(s)\int_0^1\int_0^{+\infty}\mu(s)\eta_{xx}(s)\eta(s)\,ds\,dx\,ds + \frac{c}{g(0)}\frac{\delta}{c}\int_0^{+\infty}\mu(s)\int_0^1 \psi_{xt}\eta(s)\,dx\,ds \\ & - \frac{c}{g(0)}\int_0^{+\infty}\mu(s)\int_0^1 \theta^2\,dx\,ds + \frac{c}{g(0)}\int_0^{+\infty}\mu(s)\int_0^1 \theta\eta_s\,dx\,ds + \frac{\delta\rho_1}{\kappa}\int_0^1 \varphi_{tt}\theta\,dx \\ & + \frac{\delta\rho_1}{\kappa}\frac{1}{c\beta}\int_0^1 \varphi_t\int_0^{+\infty}\mu(s)\eta_{xx}(s)\,ds\,dx - \frac{\delta\rho_1}{\kappa}\frac{\delta}{c}\int_0^1 \varphi_t\psi_{xt}\,dx \\ & - \frac{\delta\rho_1}{\kappa g(0)}\int_0^1 \varphi_{ttt}\int_0^{+\infty}\mu(s)\eta(s)\,ds\,dx - \frac{\delta\rho_1}{\kappa g(0)}\int_0^1 \varphi_{tt}\theta\int_0^{+\infty}\mu(s)\,ds\,dx \\ & + \frac{\delta\rho_1}{\kappa g(0)}\int_0^1 \varphi_{tt}\eta_s\int_0^{+\infty}\mu(s)\,ds\,dx,\end{aligned}$$

Integration by parts and the boundary conditions , we get

$$\begin{aligned}F_1'(t) = & \frac{1}{g(0)\beta}\int_0^1\left(\int_0^{+\infty}\mu(s)\eta_x(s)\,ds\right)^2 dx + \frac{\delta}{g(0)}\int_0^1 \psi_{xt}\int_0^{+\infty}\mu(s)\eta(s)\,ds\,dx \\ & - c\int_0^1 \theta^2\,dx + \frac{c}{g(0)}\int_0^1 \theta\int_0^{+\infty}\mu(s)\eta_s(s)\,ds\,dx \\ & + \frac{\delta\rho_1}{\kappa c\beta}\int_0^1 \varphi_t\int_0^{+\infty}\mu(s)\eta_{xx}(s)\,ds\,dx - \frac{\delta^2\rho_1}{\kappa c}\int_0^1 \varphi_t\psi_{xt}\,dx \\ & - \frac{\delta\rho_1}{\kappa g(0)}\int_0^1 \varphi_{ttt}\int_0^{+\infty}\mu(s)\eta(s)\,ds\,dx + \frac{\delta\rho_1}{\kappa g(0)}\int_0^1 \varphi_{tt}\int_0^{+\infty}\mu(s)\eta_s(s)\,ds\,dx,\end{aligned}$$

using the first equation in (3.7) , we find that

$$
\begin{aligned}
F_1'(t) = &\frac{1}{g(0)\beta}\int_0^1\left(\int_0^{+\infty}\mu(s)\eta_x(s)\,ds\right)^2 dx - c\int_0^1\theta^2\,dx \\
&+\frac{c}{g(0)}\int_0^1\theta\int_0^{+\infty}\mu(s)\eta_s\,ds\,dx + \frac{\delta\rho_1}{\kappa c\beta}\int_0^1\varphi_t\int_0^{+\infty}\mu(s)\eta_{xx}(s)\,ds\,dx \\
&-\frac{\delta^2\rho_1}{\kappa c}\int_0^1\varphi_t\psi_{xt}\,dx - \frac{\delta}{g(0)}\int_0^{+\infty}\mu(s)\int_0^1\varphi_{xxt}\eta(s)\,dx\,ds \\
&+\frac{\delta\rho_1}{\kappa g(0)}\int_0^1\varphi_{tt}\int_0^{+\infty}\mu(s)\eta_s(s)\,ds\,dx,
\end{aligned}
$$

Applying integration by parts with respect to s and x, we obtain

$$
\begin{aligned}
F_1'(t) = &\frac{1}{g(0)\beta}\int_0^1\left(\int_0^{+\infty}\mu(s)\eta_x(s)\,ds\right)^2 dx - c\int_0^1\theta^2\,dx \\
&-\frac{c}{g(0)}\int_0^{+\infty}\mu'(s)\int_0^1\theta\eta\,dx\,ds - \frac{\delta\rho_1}{\kappa c\beta}\int_0^{+\infty}\mu(s)\int_0^1\varphi_{xt}\eta_x(s)\,dx\,ds \\
&-\frac{\delta^2\rho_1}{\kappa c}\int_0^1\varphi_t\psi_{xt}\,dx + \frac{\delta}{g(0)}\int_0^{+\infty}\mu(s)\int_0^1\varphi_{xt}\eta(s)_x\,dx\,ds \\
&-\frac{\delta\rho_1}{\kappa g(0)}\int_0^{+\infty}\mu'(s)\int_0^1\varphi_{tt}\eta(s)\,dx\,ds.
\end{aligned}
$$

By virtue of (3.47), we get

$$
\begin{aligned}
F_1'(t) \le &-\frac{1}{\beta\lambda}\int_0^{+\infty}\mu'(s)\|\eta_x(s)\|_2^2\,ds - c\int_0^1\theta^2\,dx \\
&-\frac{c}{g(0)}\int_0^{+\infty}\mu'(s)\int_0^1\theta\eta\,dx\,ds - \frac{\delta^2\rho_1}{\kappa c}\int_0^1\varphi_t\psi_{xt}\,dx \\
&+\frac{\delta}{g(0)}\left(1-\frac{\rho_1 g(0)}{c\kappa\beta}\right)\int_0^{+\infty}\mu(s)\int_0^1\varphi_{xt}\eta_x(s)\,dx\,ds \\
&-\frac{\delta\rho_1}{\kappa g(0)}\int_0^{+\infty}\mu'(s)\int_0^1\varphi_{tt}\eta(s)\,dx\,ds. \qquad (3.49)
\end{aligned}
$$

Next, differentiating $F_2(t)$ with respect to t, we get

$$
F_2'(t) = b\int_0^1\psi_{xt}\varphi_t\,dx + b\int_0^1\psi_x\varphi_{tt}\,dx - \delta\int_0^1\theta_t\varphi_t\,dx, -\delta\int_0^1\theta\varphi_{tt}\,dx,
$$

using the first and third equation in (3.7) leads to

$$
\begin{aligned}
F_2'(t) = b&\int_0^1\psi_{xt}\varphi_t\,dx + \frac{b\kappa}{\rho_1}\int_0^1\psi_x(\varphi_x+\psi)_x\,dx - \frac{\delta}{c\beta}\int_0^1\varphi_t\int_0^{\infty}\mu(s)\eta_{xx}(s)\,ds\,dx \\
&+\frac{\delta^2}{c}\int_0^1\psi_{xt}\varphi_t\,dx - \frac{\delta\kappa}{\rho_1}\int_0^1\theta(\varphi_x+\psi)_x\,dx,
\end{aligned}
$$

thanks to integration by parts, we obtain

$$\begin{aligned}F_2'(t) &= \left(b+\frac{\delta^2}{c}\right)\int_0^1 \psi_{xt}\varphi_t\,dx + \frac{\delta}{c\beta}\int_0^\infty \mu(s)\int_0^1 \varphi_{xt}\eta_x(s)\,dx\,ds \\ &\quad - \frac{b\kappa}{\rho_1}\int_0^1 \psi_{xx}(\varphi_x+\psi)\,dx + \frac{\delta\kappa}{\rho_1}\int_0^1 \theta_x(\varphi_x+\psi)\,dx.\end{aligned}$$

From the second equation in (3.7), it follows that

$$\begin{aligned}F_2'(t) &= \left(b+\frac{\delta^2}{c}\right)\int_0^1 \psi_{xt}\varphi_t\,dx + \frac{\delta}{c\beta}\int_0^\infty \mu(s)\int_0^1 \varphi_{xt}\eta_x(s)\,dx\,ds \\ &\quad + \frac{\kappa\rho_2}{\rho_1}\int_0^1 \varphi_{ttx}(\varphi_x+\psi)\,dx - \frac{\kappa^2}{\rho_1}\int_0^1 (\varphi_x+\psi)^2\,dx,\end{aligned}$$

Applying integration by parts and using the first equation in (3.7), we obtain

$$\begin{aligned}F_2'(t) &= \left(b+\frac{\delta^2}{c}\right)\int_0^1 \psi_{xt}\varphi_t\,dx + \frac{\delta}{c\beta}\int_0^\infty \mu(s)\int_0^1 \varphi_{xt}\eta_x(s)\,dx\,ds \\ &\quad - \frac{\kappa\rho_2}{\rho_1}\int_0^1 \varphi_{tt}(\varphi_x+\psi)_x\,dx - \frac{\kappa^2}{\rho_1}\int_0^1 (\varphi_x+\psi)^2\,dx, \\ &= -\rho_2\int_0^1 \varphi_{tt}^2\,dx - \frac{\kappa^2}{\rho_1}\int_0^1 (\varphi_x+\psi)^2\,dx + \left(b+\frac{\delta^2}{c}\right)\int_0^1 \psi_{xt}\varphi_t\,dx \\ &\quad + \frac{\delta}{c\beta}\int_0^\infty \mu(s)\int_0^1 \varphi_{xt}\eta_x(s)\,dx\,ds.\end{aligned} \tag{3.50}$$

Substituting (3.49) and (3.50) into $\mathcal{F}'(t) = \mu_2 F_1'(t) + \mu_1 F_2'(t)$, we arrive at

$$\begin{aligned}\mathcal{F}'(t) &\le -c\mu_2\int_0^1 \theta^2\,dx - \rho_2\mu_1\int_0^1 \varphi_{tt}^2\,dx - \frac{\kappa^2\mu_1}{\rho_1}\int_0^1 (\varphi_x+\psi)^2\,dx \\ &\quad - \frac{\mu_2 c}{g(0)}\int_0^{+\infty} \mu'(s)\int_0^1 \theta\eta(s)\,dx\,ds - \frac{\mu_2}{\beta g(0)}\int_0^{+\infty} \mu'(s)\|\eta_x(s)\|_2^2\,ds \\ &\quad - \frac{\delta\rho_1\mu_2}{\kappa g(0)}\int_0^{+\infty} \mu'(s)\int_0^1 \varphi_{tt}\eta(s)\,dx\,ds + \xi\int_0^{+\infty} \mu(s)\int_0^1 \varphi_{xt}\eta_x(s)\,dx\,ds,\end{aligned} \tag{3.51}$$

where

$$\xi = \frac{\delta\mu_2}{g(0)}\left(1 - \frac{\rho_1 g(0)}{c\kappa\beta}\right) + \frac{\delta}{c\beta}\mu_1.$$

Applying the Cauchy–Schwarz and Young inequalities yields

$$\begin{aligned}\xi\int_0^{+\infty} \mu(s)\int_0^1 \varphi_{xt}\eta_x(s)\,dx\,ds &\le \left[\int_0^1 \varphi_{xt}^2\,dx\right]^{\frac{1}{2}}\left[\left(\int_0^{+\infty}\mu(s)\right)^2\int_0^1 \xi^2\eta_x(s)^2\,dx\right]^{\frac{1}{2}} ds \\ &\le \varepsilon\int_0^1 \varphi_{xt}^2\,dx + \frac{\xi^2 g(0)}{4\varepsilon}\int_0^{+\infty}\mu(s)\int_0^1 \|\eta_x(s)\|^2\,ds.\end{aligned} \tag{3.52}$$

Similarly, using poincaré's inequality and (**??**), we obtain

$$-\frac{\mu_2 c}{g(0)}\int_0^{+\infty}\mu'(s)\int_0^1 \theta\eta(s)\,dx\,ds \leq \frac{\mu_2 c}{g(0)}\left[\int_0^1 \theta^2\,dx\right]^{\frac{1}{2}}\left[\int_0^1\left(\int_0^{+\infty}\mu'(s)\eta(s)\,ds\right)^2 dx\right]^{\frac{1}{2}}$$
$$\leq \varepsilon\int_0^1 \theta^2\,dx + \frac{\mu_2^2 c^2 C_p}{4\varepsilon g(0)^2}\int_0^{+\infty}\mu'(s)\,ds\int_0^{+\infty}\mu'(s)\|\eta_x(s)\|_2^2\,ds$$
$$\leq \frac{\mu_2 c}{2}\int_0^1 \theta^2\,dx - \frac{c\mu_2\lambda C_p}{2g(0)}\int_0^{+\infty}\mu'(s)\|\eta_x(s)\|_2^2\,ds \tag{3.53}$$

and

$$-\frac{\delta\rho_1\mu_2}{\kappa g(0)}\int_0^{+\infty}\mu'(s)\int_0^1 \varphi_{tt}\eta(s)dxds \leq \frac{\rho_2\mu_1}{2}\int_0^1 \varphi_{tt}^2 dx - \frac{(\delta\rho_1\mu_2)^2\lambda c_p}{2\kappa^2 g(0)\rho_2\mu_1}\int_0^{+\infty}\mu'(s)\|\eta_x(s)\|_2^2 ds. \tag{3.54}$$

substituting the equations(3.52)-(3.54) into (3.51), we find that

$$\mathcal{F}'(t) \leq -\frac{c\mu_2}{2}\int_0^1 \theta^2\,dx - \frac{\rho_2\mu_1}{2}\int_0^1 \varphi_{tt}^2\,dx - \frac{\kappa^2\mu_1}{\rho_1}\int_0^1(\varphi_x+\psi)^2\,dx$$
$$+\varepsilon_1\int_0^1 \varphi_{xt}^2\,dx - m\left(1+\frac{1}{\varepsilon_1}\right)\int_0^{+\infty}\mu'(s)\|\eta_x(s)\|_2^2\,ds, \tag{3.55}$$

which completes the proof. □

Lemma 3.5. *Let $(\varphi,\psi,\theta,\eta)$ be the solution of problem (3.7)– (3.3). Then, the functional*

$$\mathcal{G}(t) = -\kappa\int_0^1 \varphi_x\varphi_{xt}\,dx$$

satisfies the estimate

$$\mathcal{G}'(t) \leq -\kappa\int_0^1 \varphi_{xt}^2 dx + \varepsilon_2\int_0^1 \psi_x^2 dx + m\left(1+\frac{1}{\varepsilon_2}\right)\int_0^1 \varphi_{tt}^2 dx \tag{3.56}$$

for any $\varepsilon_2 > 0$ and for some constant $m > 0$ independent of ε_2.

Proof. By differentiating $\mathcal{G}(t)$ and applying integration by parts, we obtain

$$\mathcal{G}'(t) = -\kappa\int_0^1 \varphi_{xt}\varphi_{xt}dx - \kappa\int_0^1 \varphi_x\varphi_{xtt}dx$$
$$= -\kappa\int_0^1 \varphi_{xt}^2 dx\kappa\int_0^1 \varphi_{xx}\varphi_{tt}dx,$$

By virtue of the first equation in (3.7), we get

$$\mathcal{G}'(t) = -\kappa\int_0^1 \varphi_{xt}^2 dx + \rho_1\int_0^1 \varphi_{tt}^2 dx - \kappa\int_0^1 \psi_x\varphi_{tt}dx,$$

using Young's inequality, we obtain

$$\begin{aligned}\mathcal{G}'(t) &\leq -\kappa \int_0^1 \varphi_{xt}^2 dx + \rho_1 \int_0^1 \varphi_{tt}^2 dx + \frac{\kappa\varepsilon_2}{2}\int_0^1 \psi_x^2\, dx + \frac{\kappa}{2\varepsilon_2}\int_0^1 \varphi_{tt}^2 dx \\ &\leq -\kappa \int_0^1 \varphi_{xt}^2 dx + \varepsilon_2 \int_0^1 \psi_x^2 dx + m\left(1+\frac{1}{\varepsilon_2}\right)\int_0^1 \varphi_{tt}^2 dx. \qquad (3.57)\end{aligned}$$

Hence, the proof is complete. □

Lemma 3.6. *Let* $(\varphi, \psi, \theta, \eta)$ *be the solution of problem (3.5)– (3.7). Then, the functional*

$$\mathcal{H}(t) = \rho_2\kappa \int_0^1 \varphi_{xt}\varphi_x dx$$

satisfies, for some $m > 0$*, the estimate*

$$\mathcal{H}'(t) \leq -\frac{b\kappa}{2}\int_0^1 \psi_x^2 dx + \rho_2\kappa\int_0^1 \varphi_{xt}^2 dx + m\int_0^1 \varphi_{tt}^2 dx + m\int_0^1 (\varphi_x+\psi)^2 dx + m\int_0^1 \theta^2 dx. \qquad (3.58)$$

Proof. We differentiation $\mathcal{H}(t)$ and using the third equations in (3.7), we get

$$\begin{aligned}\mathcal{H}'(t) &= \rho_2\kappa \int_0^1 \varphi_{xtt}\varphi_x\, dx + \rho_2\kappa\int_0^1 \varphi_{xt}\varphi_{xt}\, dx \\ &= \rho_2\kappa\int_0^1 \varphi_{xt}^2\, dx - b\kappa\int_0^1 \psi_{xx}\varphi_x\, dx + \kappa^2\int_0^1 (\varphi_x+\psi)\varphi_x\, dx + \delta\kappa\int_0^1 \theta_x\varphi_x\, dx.\end{aligned}$$

Applying integrating by parts and using $\varphi_x = ((\varphi_x+\psi)-\psi)$, we find that

$$\begin{aligned}\mathcal{H}'(t) = {} & \rho_2\kappa\int_0^1 \varphi_{xt}^2\, dx + b\kappa\int_0^1 \psi_x\varphi_{xx}\, dx + \kappa^2\int_0^1 (\varphi_x+\psi)^2\, dx \\ & - \kappa^2\int_0^1 (\varphi_x+\psi)\psi\, dx - \delta\kappa\int_0^1 \theta\varphi_{xx}\, dx,\end{aligned}$$

the use of the first equation in (3.7) yields

$$\begin{aligned}\mathcal{H}'(t) = {} & \rho_2\kappa\int_0^1 \varphi_{xt}^2\, dx - b\kappa\int_0^1 \psi_x^2\, dx + b\rho_1\int_0^1 \psi_x\varphi_{tt}\, dx + \kappa^2\int_0^1 (\varphi_x+\psi)^2\, dx \\ & - \kappa^2\int_0^1 (\varphi_x+\psi)\psi\, dx + \delta\int_0^1 \theta\varphi_{tt}\, dx - \delta\kappa\int_0^1 \theta\psi_x\, dx.\end{aligned}$$

An application of Young's inequality yields

$$\begin{aligned}\mathcal{H}'(t) \leq {} & \rho_2\kappa\int_0^1 \varphi_{xt}^2\, dx - b\kappa\int_0^1 \psi_x^2\, dx + \frac{b\rho_1\varepsilon}{2}\int_0^1 \psi_x^2\, dx + \frac{b\rho_1}{2\varepsilon}\int_0^1 \varphi_{tt}^2\, dx + \kappa^2\int_0^1 (\varphi_x+\psi)^2\, dx \\ & + \frac{\kappa^2\varepsilon}{2}\int_0^1 \psi^2\, dx + \frac{\kappa^2}{2\varepsilon}\int_0^1 (\varphi_x+\psi)^2\, dx + \frac{\delta\varepsilon}{2}\int_0^1 \varphi_{tt}^2\, dx + \frac{\delta}{2\varepsilon}\int_0^1 \theta^2\, dx \\ & + \frac{\delta\kappa\varepsilon}{2}\int_0^1 \psi_x^2\, dx + \frac{\delta\kappa}{2\varepsilon}\int_0^1 \theta^2\, dx,\end{aligned}$$

An application of Poincaré's inequality yields

$$\mathcal{H}'(t) \leq -\frac{b\kappa}{2}\int_0^1 \psi_x^2 dx + \rho_2\kappa\int_0^1 \varphi_{xt}^2 dx + m\int_0^1 \varphi_{tt}^2 dx + m\int_0^1 (\varphi_x+\psi)^2 dx + m\int_0^1 \theta^2 dx. \tag{3.59}$$

□

Lemma 3.7. *Let $(\varphi,\psi,\theta,\eta)$ be the solution of problem (3.5)– (3.7). Then, the functional*

$$\mathcal{K}(t) = -\rho_1\int_0^1 \varphi_t\varphi\, dx$$

satisfies, for some $m>0$, the estimate

$$\mathcal{K}'(t) \leq -\rho_1\int_0^1 \varphi_t^2\, dx + m\int_0^1 (\varphi_x+\psi)^2\, dx + m\int_0^1 \psi_x^2\, dx. \tag{3.60}$$

Proof. Differentiating $\mathcal{K}$ and using the first equation in (3.7), we obtain

$$\begin{aligned}\mathcal{K}'(t) &= -\rho_1\int_0^1 \varphi_t\varphi_t\, dx - \rho_1\int_0^1 \varphi_{tt}\varphi\, dx\\ &= -\rho_1\int_0^1 \varphi_t^2\, dx - \int_0^1 \kappa(\varphi_x+\psi)_x\varphi\, dx,\end{aligned}$$

by integration by parts and $\varphi_x = ((\varphi_x+\psi)-\psi)$, we find

$$\mathcal{K}'(t) = -\rho_1\int_0^1 \varphi_t^2\, dx + \kappa\int_0^1 (\varphi_x+\psi)^2\, dx - \kappa\int_0^1 (\varphi_x+\psi)\psi\, dx.$$

An application of the Cauchy–Schwarz, Young's, and Poincaré's inequalities yields

$$\begin{aligned}\mathcal{K}'(t) &\leq -\rho_1\int_0^1 \varphi_t^2\, dx + \kappa\int_0^1 (\varphi_x+\psi)^2\, dx + \frac{\kappa}{2\varepsilon}\int_0^1 (\varphi_x+\psi)^2\, dx + \frac{\kappa\varepsilon}{2}\int_0^1 \psi^2\, dx\\ &\leq -\rho_1\int_0^1 \varphi_t^2\, dx + m\int_0^1 (\varphi_x+\psi)^2\, dx + m\int_0^1 \psi_x^2\, dx.\end{aligned} \tag{3.61}$$

□

Now, we introduce the following Lyapunov functional:

$$\mathcal{L}(t) = NE(t) + n_1\mathcal{F}(t) + n_2\mathcal{G}(t) + n_3\mathcal{H}(t) + \mathcal{K}(t),$$

where n_1, n_2, n_3 and N are positive constants to be determined later.

Lemma 3.8. *The functional $E(t)$ and $\mathcal{L}(t)$ satisfy*

$$(N-\chi)E(t) \leq \mathcal{L}(t) \leq (N+\chi)E(t), \qquad \forall t > 0, \tag{3.62}$$

for a positive constant χ.

Proof. Simple calculation shows that

$$\begin{aligned}|\mathcal{L}(t) - NE(t)| \leq \frac{c\mu_2}{g(0)} \left| \int_0^{+\infty} \mu(s) \int_0^1 \theta\eta(s)\, dx\, ds \right| + \frac{\delta\rho_1\mu_2}{\kappa} \left| \int_0^1 \varphi_t \theta\, dx \right| \\ + \frac{\delta\rho_1\mu_2}{\kappa g(0)} \left| \int_0^1 \varphi_{tt} \int_0^{+\infty} \mu(s)\eta(s)\, ds\, dx \right| + b\mu_1 \left| \int_0^1 \psi_x \varphi_t dx \right| \\ + \delta\mu_1 \left| \int_0^1 \theta\varphi_t dx \right| + \kappa(\rho_2+1) \left| \int_0^1 \varphi_x \varphi_{xt} dx \right| + \rho_1 \left| \int_0^1 \varphi_t \varphi\, dx \right|.\end{aligned}$$

Therefore, Cauchy-Schwarz and Young's inequalities yield

$$|\mathcal{L}(t) - NE(t)| \leq \chi E(t).$$

Consequently,

$$(N-\chi)E(t) \leq \mathcal{L}(t) \leq (\chi + NE(t))E(t).$$

□

Proof of Theorem ??

By differentiating $\mathcal{L}(t)$ and using (3.38), (3.55), (3.57), (3.59), (**??**), we get

$$\begin{aligned}\mathcal{L}'(t) \leq & -\rho_1 \int_0^1 \varphi_t^2 dx - \left(\frac{b\kappa}{2} n_3 - m - 1 \right) \int_0^1 \psi_x^2 dx \\ & - (\kappa n_2 - \rho_2 \kappa n_3 - 1) \int_0^1 \varphi_{xt}^2 dx \\ & - \left(\frac{\kappa^2 \mu_1}{\rho_1} n_1 - m n_3 - m \right) \int_0^1 (\varphi_x + \psi)^2 dx \\ & - \left[\frac{\rho_2 \mu_1}{2} n_1 - m n_2 (1+n_2) - m n_3 \right] \int_0^1 \varphi_{tt}^2 dx \\ & - \left(\frac{c\mu_2}{2} n_1 - m n_3 \right) \int_0^1 \theta^2 dx \\ & + \left[\frac{1}{2\beta} N - m n_1 (1+n_1) \right] \int_0^{+\infty} \mu'(s) \|\eta_x(s)\|_2^2 ds.\end{aligned} \tag{3.63}$$

First, we choose $n_3 > 0$, Large enough, such that

$$n_3 > \frac{2m+2}{b\kappa}.$$

Then, we select n_2

$$n_2 > \frac{\rho_2 \kappa n_3 + 1}{\kappa}.$$

After fixing n_2 and n_3, we select n_1 such that

$$n_1 > \max\left\{\frac{m\rho_1(n_3+1)}{\kappa^2\mu_1}, \frac{2mn_2((1+n_2)+n_3)}{\rho_1\mu_1}, \frac{2mn_3}{c\mu_2}\right\}.$$

Finally, we choose for N to be large enough such that

$$N = \max\left\{\frac{1}{2\beta}mn_1(1+n_1), \chi\right\}.$$

Therefore, there exist constant $\gamma \geq 0$ and $m_0 > 0$ such that

$$m_0 E(t) \leq \mathcal{L}(t) \leq m_0 E(t) \tag{3.64}$$

and

$$\mathcal{L}'(t) \leq \gamma E(t). \tag{3.65}$$

Using (3.64), we infer that there exists $\zeta > 0$, such that

$$\mathcal{L}'(t) \leq -\zeta \mathcal{L}(t) \quad \forall t \geq 0.$$

An integration over $(0;t)$ leads to

$$\mathcal{L}(t) \leq \mathcal{L}(0)e^{-\zeta t} \quad \forall t \geq 0.$$

Using (3.64) again, we get

$$E(t) \leq \sigma e^{-\zeta t} \quad \forall t \geq 0.$$

This completes the proof of Theorem 3.3.

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