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Double Laplace–Sumudu Transform and its Applications

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إهداء

بسم الله الرحمن الرحيم

(وَآخِرُ دَعْوَاهُمْ أَنِ الْحَمْدُ لِلَّهِ رَبِّ الْعَالَمِينَ)

الحمد لله الذي بنعمته تتم الصالحات

أما بعد أهدي ثمرة جهدي إلى من رباني وكافح من أجلي، إلى المصباح الذي أنار حياتي، إلى من أحمل اسمه بكل فخر واعتزاز، إلى صاحب السيرة العطرة والفكر المستنير وإلى من شجعني طوال عمري، طاب بك العمر يا سيد الرجال وطبتَ لي عمراً (والدي العزيز).

قدوتي الأولى ومعنى الحب والحنان، إلى من كان دعاؤها سر نجاحي وحنانها بلسم جراحي، إلى جنة قلبي (أمي الغالية).

وإلى كل من يرى نجاحي نجاحه، إخوتي الأحبة كل باسمه، كنتم لي خير سند ودعماً يقويني في كل مرحلة من حياتي.

إلى صديقاتي وزملائي الذين رافقوني خلال هذا المشوار الجامعي بكل ما حمله من تعب واجتهاد وذكريات جميلة.

إلى أستاذي المشرف، الذي لم يبخل علي بالنصح والتوجيه، فله مني جزيل الشكر والتقدير والاحترام.

ملخص

تهدف هذه المذكرة إلى دراسة تحويل لابلاس--سومودو المزدوج وتطبيقه في حل المعادلات التفاضلية الجزئية الخطية، حيث تم في البداية التطرق إلى تحويلي لابلاس وسومودو وعرض أهم خصائصهما باعتبارهما الأساس الذي بُني عليه هذا التحويل المزدوج. ثم تم حل بعض المعادلات التفاضلية الجزئية باستعمال كل تحويل على حدة وإعادة حلهما باستعمال تحويل لابلاس--سومودو المزدوج من أجل مقارنة خطوات الحل وإبراز مدى فعاليته في تبسيط واختصار مراحل الوصول إلى الحل. كما تضمنت المذكرة تطبيقات على بعض المعادلات الفيزيائية.

الكلمات المفتاحية: تحويل لابلاس، تحويل سومودو، تحويل لابلاس--سومودو المزدوج، المعادلات التفاضلية الجزئية، الشروط الابتدائية والحدودية، الحلول الدقيقة.

Abstract

The objective of this dissertation is to study the double Laplace–Sumudu Transform and its application in solving linear partial differential equations. First, the Laplace transform and the Sumudu transform, along with their main properties, are introduced as the basis of this double transform. Then, some partial differential equations are solved using each transform separately and solved again using the double Laplace–Sumudu transform in order to compare the solution procedures and highlight its effectiveness in simplifying and reducing the steps required to obtain the solution. In addition, the dissertation includes applications to some physical equations.

Keywords: Laplace Transform, Sumudu Transform, Double Laplace–Sumudu Transform, Partial Differential Equations, Initial and Boundary Conditions, Exact Solutions.

Résumé

Le but de ce mémoire est d'étudier la transformée double Laplace–Sumudu et son application dans la résolution des équations aux dérivées partielles linéaires. Dans un premier temps, les transformations de Laplace et de Sumudu ainsi que leurs principales propriétés sont présentées, considérées comme la base de cette transformée double. Ensuite, quelques équations aux dérivées partielles ont été résolues en utilisant chaque transformée séparément, puis résolues à nouveau à l'aide de la transformée double Laplace–Sumudu afin de comparer les étapes de résolution et de mettre en évidence son efficacité dans la simplification et la réduction des étapes nécessaires pour obtenir la solution. Enfin, ce mémoire inclut des applications à certaines équations physiques.

Notations

$\mathscr{L}\{f(t)\}$	Laplace transform of a function with respect to time t.
$\mathscr{L}^{-1}\{F(s)\}$	Inverse Laplace transform.
$(f*g)(t)$	Convolution of two functions in one variable.
$\mathscr{S}\{f(t)\}$	Sumudu transform of a function with respect to time t.
$\mathscr{S}^{-1}\{F(u)\}$	Inverse Sumudu transform.
$\mathscr{L}_x\mathscr{S}_t\{f(x,t)\}$	Double Laplace–Sumudu transform of a function with respect to x and t.
$\mathscr{L}_x\mathscr{S}_t^{-1}\{F(s,u)\}$	Inverse double Laplace–Sumudu transform.
$f(x)**g(t)$	Double convolution of two functions in two variables.

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Introduction

The transformations defined by integrals are classified among the most important methods used in solving linear ordinary differential equations, linear partial differential equations and in solving linear integral differential equations of integer order or fractional order. Among the applications of these transformations are their use in electrical engineering and control, signal processing and communications, physics and mechanics, and pure mathematics., etc... and among these transformations that were discovered during the eighteenth and nineteenth centuries and that were used and are still used in these two important fields of science (mathematics and physics), we find the Laplace transform method [15], the Fourier transform method [7] , the Hankel transform Method [13], and the Mellin transforin method [11].

Given the importance of using single transforms in solving differential equations, interest began to focus on using them in another way, this way is represented by introducing the concept of double integrals, so they became known as double transforms. Due to this new form that includes two variables, they are only used in solving partial differential equations of integer or fractional orders, as well as integro partial differential equations of integer or fractional orders. For example, among the works that were concerned with the double Laplace transform, we find [9],[4] as well as the works that used the double Sumudu transform [9] ,[12].

The main objective of this thesis is, firstly, to present two important transformations: the Laplace transform and the Sumudu transform, as well as their combinations to obtain a new method known as " double Laplace-Sumudu transform" for solving partial differential equations.

This thesis consists of an introduction and four chapters. The first chapter is devoted to the presentation of the definition of the Laplace transform, as well as the conditions of existence and some basic properties.

Finally, we present an application to a partial differential equation in order to compare the solution method with the new method.

In the second chapter is devoted to the presentation of the definition of the Sumudu transform, as well as the conditions of existence and some basic properties. In the end, We present an application to a partial differential equation in order to compare the solution method with the new method.

In the third chapter we have presented the definition and properties of the double Laplace-

Sumudu transform, as well as the fundamental formulas of the double Laplace-Sumudu transform of derivatives.

In the fourth chapter, we applied the new method to re-solving the two examples presented in chapters one and two in order to compare and demonstrate its effectiveness. Furthermore, we applied it to solve examples of homogeneous and non-homogeneous partial differential equations such as Wave equation, the Klein–Gordon equation and the Heat equation.

Chapter 1

Laplace Transform

The Laplace transform is considered one of the most powerful and important mathematical tools for solving differential and integral equations. It transforms these equations into simpler and easier algebraic equations to solve. Therefore, it is widely used in applied mathematics and engineering to study physical models.

1.1 The Laplace transform

In this section, we introduce the definition of the Laplace transform together with its existence conditions, and we also present some of its important properties. Finally,we present an Illustrative example.

1.1.1 Definition of Laplace transform

Definition 1.1.1. [15] Let $f(t)$ be a function defined for $t > 0$. Then the Laplace transform of $f(t)$, is defined by

$$\mathscr{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt, \tag{1.1}$$

where s is a real parameter.

Example 1.1. We compute the Laplace transform of the function $f(t) = 1$ for $t > 0$, then

$$\begin{aligned}\mathscr{L}\{1\} = F(s) &= \int_0^\infty e^{-st}(1)\,dt\\ &= \int_0^\infty e^{-st}\,dt\\ &= \lim_{y\to\infty}\left[\frac{1}{-s}e^{-st}\right]_0^y\\ &= \frac{1}{s}. \quad \text{for } s > 0.\end{aligned}$$

Example 1.2. We compute the Laplace transform of the function $f(t) = t$, then

$$\mathscr{L}\{t\} = F(s) = \int_0^\infty te^{-st}\,dt.$$

Applying integration by parts, we get

$$\begin{aligned}\mathscr{L}\{t\} &= \int_0^\infty e^{-st}\cdot t\,dt\\ &= \lim_{y\to\infty}\left[\frac{-t}{s}e^{-st}\right]_0^y + \frac{1}{s}\int_0^\infty e^{-st}\,dt,\end{aligned}$$

since $\lim_{y\to\infty}\left(\frac{-y}{s}e^{-sy}\right) = 0$, then

$$\mathscr{L}\{t\} = \frac{1}{s}\mathscr{L}(1) = \frac{1}{s^2}.$$

For the function $f(t) = t^n$, we prove by recurrence that

$$\mathscr{L}\{t^n\} = \int_0^\infty e^{-st}t^n\,dt = \frac{n!}{s^{n+1}}. \tag{1.2}$$

- For $n = 0$, we have $t^0 = 1$, and

  $$\mathscr{L}\{1\} = \int_0^{+\infty} e^{-st}\,dt = \frac{1}{s}.$$

  On the other hand,

  $$\frac{0!}{s^{0+1}} = \frac{1}{s}.$$

  Thus, the formula (1.2) holds for $n = 0$.

- Let us Assume that formula (1.2) holds for some $n \in \mathbb{N}$, and let us show that the result holds for $n+1$, we obtain

  $$\mathscr{L}\{t^{n+1}\} = \int_0^{+\infty} t^{n+1}e^{-st}\,dt.$$

Using integration by parts

$$\mathscr{L}\{t^{n+1}\} = \lim_{y\to\infty}\left[-\frac{t^{n+1}}{s}e^{-st}\right]_0^y + \frac{n+1}{s}\int_0^{+\infty} t^n e^{-st}\,dt.$$

Since $\lim_{y\to\infty}\left(-\frac{y^{n+1}}{s}e^{-sy}\right) = 0$, then

$$\begin{aligned}\mathscr{L}\{t^{n+1}\} &= \frac{n+1}{s}\mathscr{L}\{t^n\}\\ &= \frac{n+1}{s}\cdot\frac{n!}{s^{n+1}}\\ &= \frac{(n+1)!}{s^{n+2}}.\end{aligned}$$

Thus, the formula (1.2) holds for all $n \in \mathbb{N}$.

Example 1.3. We compute the Laplace transform of the function $f(t) = e^{at}$, where a is a constant. Then

$$\begin{aligned}\mathscr{L}\{e^{at}\} = F(s) &= \int_0^\infty e^{-st}e^{at}\,dt\\ &= \int_0^\infty e^{-(s-a)t}\,dt\\ &= \lim_{y\to\infty}\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^y\\ &= \frac{1}{s-a}, \quad \text{for } s > a.\end{aligned}$$

1.1.2 Existence conditions for the Laplace transform

A function $f(t)$ is said to be of exponential order $a(>0)$ on $0 \le t < \infty$ if there exists a positive constant M such that for all $t > T$,

$$|f(t)| \le Me^{at}, \tag{1.3}$$

and we write this symbolically as

$$f(t) = O(e^{at}) \quad \text{as } t \to \infty. \tag{1.4}$$

Or, equivalently,

$$\lim_{t\to\infty} e^{-bt}|f(t)| \le M \lim_{t\to\infty} e^{-(b-a)t} = 0, \quad b > a. \tag{1.5}$$

Such a function $f(t)$ is simply called an exponential order as $t \to \infty$, and clearly, it does not grow faster than Me^{at} as $t \to \infty$ [8].

Theorem 1.1.2. [8] If a function $f(t)$ is continuous or piecewise continuous on every finite interval $(0,T)$, and of exponential order e^{at}, then the Laplace transform of $f(t)$ exists for all s provided that $s>a$.

Proof. We have

$$\begin{aligned}|F(s)| &= \left|\int_0^{\infty} e^{-st} f(t)\,dt\right| \\ &\leq \int_0^{\infty} e^{-st}|f(t)|\,dt.\end{aligned}$$

Using (1.3), we obtain

$$\begin{aligned}|F(s)| &\leq \int_0^{\infty} e^{-st} M e^{at}\,dt \\ &\leq M\int_0^{\infty} e^{-t(s-a)}\,dt \\ &= M\lim_{x\to\infty}\left[\frac{1}{-(s-a)}e^{-t(s-a)}\right]_0^x \\ &= \frac{M}{s-a}, \quad \text{for } s>a.\end{aligned}$$

□

1.1.3 Inverse of the Laplace transform

Definition 1.1.3. [8] Let $F(s)$ be the Laplace transform of a function $f(t)$. Then the inverse Laplace transform of $F(s)$, is defined by

$$\mathscr{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}F(s)\,ds, \quad c>0, \tag{1.6}$$

where $\mathscr{L}^{-1}$ denotes the inverse Laplace transform operator.

1.1.4 Properties

In this section, we present the main properties of the Laplace transform, presented in the following references [15] and [8].

1. Linearity property

Let $f(t)$ and $g(t)$ be functions for which the Laplace transform exists. We denote

$$\mathscr{L}\{f(t)\} = F(s), \qquad \mathscr{L}\{g(t)\} = G(s). \tag{1.7}$$

Then, for any constants $a, b \in \mathbb{R}$, it follows that

$$\mathscr{L}\{af(t)+bg(t)\} = a\mathscr{L}\{f(t)\}+b\mathscr{L}\{g(t)\} = aF(s)+bG(s). \tag{1.8}$$

Proof. Using the definition of the Laplace transform, we have

$$\mathscr{L}\{af(t)+bg(t)\} = \int_0^\infty e^{-st}\big(af(t)+bg(t)\big)\,dt.$$

By the linearity of the integral, we obtain

$$\begin{aligned}\mathscr{L}\{af(t)+bg(t)\} &= a\int_0^\infty e^{-st}f(t)\,dt + b\int_0^\infty e^{-st}g(t)\,dt\\ &= a\mathscr{L}\{f(t)\}+b\mathscr{L}\{g(t)\}\\ &= aF(s)+bG(s).\end{aligned}$$

□

2. Scaling property

If $\mathscr{L}\{f(t)\} = F(s)$, then

$$\mathscr{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right), \qquad a > 0. \tag{1.9}$$

Proof. Using the definition of the Laplace transform, we obtain

$$\mathscr{L}\{f(at)\} = \int_0^\infty e^{-st}f(at)\,dt.$$

We make the change of variable

$$u = at, \qquad t = \frac{u}{a}, \qquad dt = \frac{1}{a}du.$$

Thus,

$$\begin{aligned}\mathscr{L}\{f(at)\} &= \int_0^\infty e^{-s\left(\frac{u}{a}\right)}f(u)\frac{1}{a}\,du\\ &= \frac{1}{a}\int_0^\infty e^{-\frac{s}{a}u}f(u)\,du\\ &= \frac{1}{a}F\left(\frac{s}{a}\right).\end{aligned}$$

□

3. First shifting property

If $\mathscr{L}\{f(t)\} = F(s)$, then

$$\mathscr{L}\{e^{-at}f(t)\} = F(s+a), \tag{1.10}$$

where a is a real constant.

Proof. According to equation (1.1), we obtain

$$\begin{aligned}\mathscr{L}\{e^{-at}f(t)\} &= \int_0^\infty e^{-st}e^{-at}f(t)\,dt\\ &= \int_0^\infty e^{-(s+a)t}f(t)\,dt\\ &= F(s+a).\end{aligned}$$

□

4. Laplace transform of derivatives

Theorem 1.1.4. The Laplace transform of derivative of order $n \in \mathbb{N}^*$ of the function f is given by

$$\mathscr{L}\{f^{(n)}(t)\}(s) = s^n\mathscr{L}\{f(t)\}(s) - \sum_{k=1}^{n} s^{k-1}f^{(n-k)}(0). \tag{1.11}$$

Proof. The proof is done by recurrence.

For $n = 1$, we have

$$\begin{aligned}\mathscr{L}\{f^{(n)}(t)\}(s) &= \mathscr{L}\{f'(t)\}(s)\\ &= \int_0^{+\infty} e^{-st}f'(t)\,dt.\end{aligned}$$

We perform integration by parts with

$$u = e^{-st}, \qquad dv = f'(t)\,dt.$$

Then,

$$\mathscr{L}\{f'(t)\}(s) = \lim_{x\to\infty}\left[e^{-st}f(t)\right]_0^x + s\int_0^\infty e^{-st}f(t)\,dt \tag{1.12}$$

$$= s\mathscr{L}\{f(t)\}(s) - f(0), \tag{1.13}$$

and

$$s^n \mathscr{L}\{f(t)\}(s) - \sum_{k=1}^{n} s^{k-1} f^{(n-k)}(0) = s\mathscr{L}\{f(t)\}(s) - f(0). \tag{1.14}$$

Thus, relation (1.11) holds when $n = 1$.

Let us assume that formula (1.11) holds for a certain rank n, and let us show that it remains valid for rank $(n+1)$.

So,

$$\begin{aligned}
\mathscr{L}\{f^{(n+1)}\}(s) &= \mathscr{L}\{(f')^{(n)}\}(s) \\
&= s^n \mathscr{L}\{f'(t)\}(s) - \sum_{k=1}^{n} s^{k-1} (f')^{(n-k)}(0) \\
&= s^n \left(s\mathscr{L}\{f(t)\}(s) - f(0)\right) - \sum_{k=1}^{n} s^{k-1} f^{(n+1-k)}(0) \\
&= s^{n+1} \mathscr{L}\{f(t)\}(s) - s^n f(0) - \sum_{k=1}^{n} s^{k-1} f^{(n+1-k)}(0) \\
&= s^{n+1} \mathscr{L}\{f(t)\}(s) - \sum_{k=1}^{n+1} s^{k-1} f^{(n+1-k)}(0).
\end{aligned}$$

Finally, formula (1.11) is indeed at rank $(n+1)$.

Thus, relation (1.11) holds for all $n \in \mathbb{N}^*$.

□

Result 1

- For $n = 2$ from formula (1.11), we obtain

$$\mathscr{L}\{f''(t)\} = s^2 \mathscr{L}\{f(t)\} - sf(0) - f'(0) = s^2 F(s) - sf(0) - f'(0). \tag{1.15}$$

- For $n = 3$ from formula (1.11), we obtain

$$\mathscr{L}\{f^{(3)}(t)\} = s^3 \mathscr{L}\{f(t)\} - s^2 f(0) - sf'(0) - f''(0) \tag{1.16}$$

$$= s^3 F(s) - s^2 f(0) - sf'(0) - f''(0). \tag{1.17}$$

Result 2

Based on the Laplace transform of ordinary derivatives, we can deduce the Laplace transform of partial derivatives

$$\mathscr{L}\left\{\frac{\partial f(x,t)}{\partial t}\right\} = sF(x,s) - f(x,0). \tag{1.18}$$

$$\mathscr{L}\left\{\frac{\partial^2 f(x,t)}{\partial t^2}\right\} = s^2F(x,s) - sf(x,0) - \frac{\partial f(x,0)}{\partial t}. \tag{1.19}$$

$$\mathscr{L}\left\{\frac{\partial f(x,t)}{\partial x}\right\} = \frac{dF(x,s)}{dx}. \tag{1.20}$$

$$\mathscr{L}\left\{\frac{\partial^2 f(x,t)}{\partial x^2}\right\} = \frac{d^2F(x,s)}{dx^2}. \tag{1.21}$$

5. The Laplace transform of integral

Theorem 1.1.5. If $\mathscr{L}\{f(t)\} = F(s)$, then

$$\mathscr{L}\left\{\int_0^t f(u)\,du\right\} = \frac{F(s)}{s}. \tag{1.22}$$

Proof. Let $g(t) = \int_0^t f(u)\,du$, then $g'(t) = f(t)$ and $g(0) = 0$. Taking the Laplace transform of both sides, we have

$$\begin{aligned}\mathscr{L}\{g'(t)\} &= s\mathscr{L}\{g(t)\} - g(0)\\ &= s\mathscr{L}\{g(t)\},\end{aligned}$$

and

$$\mathscr{L}\{g'(t)\} = F(s).$$

Thus

$$\mathscr{L}\{g(t)\} = \frac{F(s)}{s},$$

therefore

$$\mathscr{L}\left\{\int_0^t f(u)\right\} = \frac{F(s)}{s}.$$

□

6. Convolution property

Theorem 1.1.6. If $\mathscr{L}\{f(t)\} = F(s)$ and $\mathscr{L}\{g(t)\} = G(s)$, then

$$\mathscr{L}\{f(t)*g(t)\} = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\} = F(s)G(s). \tag{1.23}$$

Or equivalently

$$\mathscr{L}^{-1}\{F(s)G(s)\} = f(t)*g(t). \tag{1.24}$$

Where the convolution is defined by the integral

$$f(t)*g(t) = \int_0^t f(t-u)g(u)\,du. \tag{1.25}$$

Proof. We start with the definition of the Laplace transform of the convolution product

$$\begin{aligned}\mathscr{L}\{f(t)*g(t)\} &= \int_0^\infty e^{-st}(f(t)*g(t))\,dt\\ &= \int_0^\infty e^{-st}\left(\int_0^t f(t-u)g(u)\,du\right)dt.\end{aligned}$$

By Fubini's theorem [5], we can switch the order of integration

$$\mathscr{L}\{f(t)*g(t)\} = \int_0^\infty g(u)\left(\int_u^\infty f(t-u)e^{-st}\,dt\right)du.$$

We perform the change of variable $v = t-u$, and $t = u+v$, then $dt = dv$. The inner integral becomes

$$\begin{aligned}\int_u^\infty f(t-u)e^{-st}\,dt &= \int_0^\infty f(v)e^{-s(u+v)}\,dv\\ &= e^{-su}\int_0^\infty f(v)e^{-sv}\,dv\\ &= e^{-su}\mathscr{L}\{f(t)\}.\end{aligned}$$

Thus, we have

$$\begin{aligned}\mathscr{L}\{f(t)*g(t)\} &= \int_0^\infty g(u)e^{-su}\mathscr{L}\{f(t)\}\,du\\ &= \mathscr{L}\{f(t)\}\int_0^\infty g(u)e^{-su}\,du\\ &= \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\}\\ &= F(s)G(s).\end{aligned}$$

1.1.5 Table of Laplace transform

The following table presents the Laplace transforms of some elementary functions.

Table 1.1: Table of Laplace transform

$f(t)$	$\mathscr{L}\{f(t)\} = F(s)$
1	$\frac{1}{s}, \quad s>0$
t	$\frac{1}{s^2}, \quad s>0$
$t^n \ (n=0,1,2,\dots)$	$\frac{n!}{s^{n+1}}, \quad s>0$
e^{at}	$\frac{1}{s-a}, \quad s>a$
$\cos(at)$	$\frac{s}{s^2+a^2}, \quad s>0$
$\sin(at)$	$\frac{a}{s^2+a^2}, \quad s>0$
$\cosh(at)$	$\frac{s}{s^2-a^2}, \quad s>\lvert a\rvert$
$\sinh(at)$	$\frac{a}{s^2-a^2}, \quad s>\lvert a\rvert$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}, \quad s>a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}, \quad s>a$

1.1.6 Illustrative example of the Laplace transform to solving a partial differential equation

We consider the partial differential equation

$$2\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} - u, \quad x,t \geq 0, \tag{1.26}$$

with the initial and boundary conditions

$$u(x,0) = 6e^{-3x}, \quad u(0,t) = 6e^{-2t}. \tag{1.27}$$

We apply the Laplace transform to (1.26), we obtain

$$2\mathscr{L}\left\{\frac{\partial u}{\partial t}\right\}=\mathscr{L}\left\{\frac{\partial u}{\partial x}\right\}-\mathscr{L}\{u\}, \tag{1.28}$$

let $\mathscr{L}\{u(x,t)\}=U(x,s)$, and from (1.18) and (1.20), we obtain

$$\mathscr{L}\left\{\frac{\partial u}{\partial t}\right\}=sU(x,s)-u(x,0),$$

and

$$\mathscr{L}\left\{\frac{\partial u}{\partial x}\right\}=\frac{dU}{dx}.$$

By substituting into (1.28), we get

$$2(sU(x,s)-u(x,0))=\frac{dU}{dx}-U,$$

substituting the initial condition $u(x,0)=6e^{-3x}$, we get

$$2(sU(x,s)-6e^{-3x})=\frac{dU}{dx}-U.$$

Simplify

$$2sU-12e^{-3x}=\frac{dU}{dx}-U,$$

rewriting

$$\frac{dU}{dx}-(2s+1)U=-12e^{-3x}. \tag{1.29}$$

This is a first-order linear ordinary differential equation. The integrating factor is

$$\mu(x)=e^{\int -(2s+1)\,dx}=e^{-(2s+1)x},$$

multiplying both sides of equation (1.29) by the integrating factor

$$e^{-(s+1)x}\frac{dU}{dx}-(2s+1)e^{-(2s+1)x}U=-12e^{-3x}e^{-(2s+1)x}\Rightarrow\frac{d}{dx}\left(Ue^{-(2s+1)x}\right)=-12e^{-(2s+4)x}. \tag{1.30}$$

Integrating

$$Ue^{-(2s+1)x}=\int -12e^{-(2s+4)x}\,dx=\frac{12}{2s+4}e^{-(2s+4)x}+C, \tag{1.31}$$

thus,

$$U(x,s)=\frac{12}{2s+4}e^{-(2s+4)}e^{(2s+1)x}+Ce^{(2s+1)x}. \tag{1.32}$$

Hence,

$$U(x,s)=\frac{12}{2s+1}e^{-3x}+Ce^{(2s+1)x}, \tag{1.33}$$

using the boundary condition $u(0,t) = 6e^{-2t}$, we obtain

$$U(0,s) = \mathscr{L}\{6e^{-2t}\} = 6 \cdot \frac{1}{s+2}. \tag{1.34}$$

substituting $x = 0$

$$U(0,s) = \frac{12}{2s+4} + C \Rightarrow C = 0, \tag{1.35}$$

then,

$$U(x,s) = \frac{6}{s+2} e^{-3x}. \tag{1.36}$$

Taking the inverse of the Laplace transform

$$u(x,t) = \mathscr{L}^{-1}\{U(x,s)\} = \mathscr{L}^{-1}\left\{\frac{6}{s+2} e^{-3x}\right\}. \tag{1.37}$$

Using

$$\mathscr{L}^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}.$$

Therefore, the solution to the partial differential equation is

$$u(x,t) = 6e^{-3x-2t}. \tag{1.38}$$

Chapter 2

Sumudu Transform

The Sumudu transform is considered a relatively modern integral transform, having emerged in the early 1990s as an effective tool for handling differential and integral equations. This transform is distinguished by its ability to simplify many mathematical problems, particularly due to the ”unity” property, which makes dealing with certain functions and equations easier compared to some other transforms. This feature makes it useful in physical and engineering applications.

2.1 The Sumudu transform

In this section, we present the Sumudu transform in its series form and integral form, and we also state its main properties. Finally, we present an Illustrative example.

2.1.1 Series form of the Sumudu transform

Definition 2.1.1. [17] The Sumudu transform amplifies the coefficients of the power series of the function

$$f(t) = a_0 + a_1 t + a_2 t^2 + \cdots = \sum_{n=0}^{\infty} a_n t^n, \tag{2.1}$$

by transforming it into the power series

$$F(u) = a_0 + a_1 u + 2! a_2 u^2 + 3! a_3 u^3 + \cdots = \sum_{n=0}^{\infty} n!\, a_n u^n. \tag{2.2}$$

2.1.2 Integral form of the Sumudu transform

Definition 2.1.2. [10] Let $f(t)$ be a function defined for $t > 0$. Then Sumudu transform of $f(t)$, given by

$$\mathscr{S}\{f(t)\} = F(u) = \frac{1}{u}\int_0^\infty e^{-t/u} f(t)\, dt, \quad u \in [-\tau_1, \tau_2], \tag{2.3}$$

or equivalently,

$$\mathscr{S}\{f(t)\} = F(u) = \int_0^\infty e^{-t} f(ut)\, dt, \quad u \in [-\tau_1, \tau_2], \tag{2.4}$$

on the set,

$$A = \left\{ \ f(t)/\exists M, \tau_1, \tau_2 > 0, \ |f(t)| < Me^{\frac{|t|}{\tau_j}}, \text{ if } t \in (-1)^j \times [0,\infty) \ \right\}. \tag{2.5}$$

Example 2.1. We apply Sumudu transform to function $f(t) = 1$.

Since $f(t) = 1$, we have $f(ut) = 1$. We obtain

$$\mathscr{S}\{1\} = F(u) = \int_0^\infty e^{-t}(1)\, dt = \lim_{y\to\infty} \left[-e^{-t}\right]_0^y,$$

since $\lim_{y\to\infty}(-e^{-y}) = 0$, then

$$\mathscr{S}\{1\} = 1.$$

Example 2.2. We apply Sumudu transform to function $f(t) = t$, we obtain

$$\begin{aligned} S\{t\} = F(u) &= \int_0^\infty e^{-t} ut\, dt \\ &= u\int_0^\infty te^{-t}\, dt. \end{aligned}$$

Using integration by parts, we obtain

$$\int_0^\infty te^{-t}\, dt = \lim_{y\to\infty}\left[-te^{-t}\right]_0^y + \int_0^\infty e^{-t}\, dt,$$

since $\lim_{y\to\infty}(-ye^{-y}) = 0$, then

$$\int_0^\infty te^{-t}\, dt = 1.$$

Therefore

$$S\{t\} = u \cdot 1 = u.$$

For the function $f(t) = t^n$, we prove by recurrence that

$$\mathscr{S}\{t^n\} = \int_0^\infty e^{-t}(ut)^n\, dt = n!u^n. \tag{2.6}$$

- For $n=0$, we have $t^0=1$. Then

  $$\mathscr{S}\{1\}=\int_0^\infty e^{-t}\,dt=1.$$

  On the other hand,

  $$0!\,u^0=1.$$

  Thus, the formula (2.6) holds for $n=0$.

- Let us assume that formula (2.6) holds for some $n\in\mathbb{N}$, and let us show that the result holds for $n+1$, we obtain

  $$\begin{aligned}\mathscr{S}\{t^{n+1}\}&=\int_0^\infty e^{-t}(ut)^{n+1}\,dt\\&=u^{n+1}\int_0^\infty t^{n+1}e^{-t}\,dt\\&=u^{n+1}I_{n+1}.\end{aligned}$$

  Using integration by parts

  $$I_{n+1}=\int_0^\infty t^{n+1}e^{-t}=\lim_{y\to\infty}\left[t^{n+1}(-e^{-t})\right]_0^y-\int_0^\infty(n+1)t^n(-e^{-t})\,dt.$$

  Where $\lim_{y\to\infty}\left(y^{n+1}(-e^{-y})\right)=0$, then

  $$\begin{aligned}\int_0^\infty t^{n+1}e^{-t}\,dt&=(n+1)\int_0^\infty t^ne^{-t}\,dt\\&=(n+1)I_n.\end{aligned}$$

  Using integration by parts

  $$I_n=\int_0^\infty t^ne^{-t}\,dt=n\int_0^\infty t^{n-1}e^{-t}\,dt=nI_{n-1}.$$

  Similarly, we obtain that

  $$I_{n+1}=(n+1)I_n=(n+1)nI_{n-1}=(n+1)n(n-1)I_{n-2}=\cdots=(n+1)n(n-1)(n-2)\cdots1.$$

  Since $I_0=1$, we find that

  $$I_{n+1}=(n+1)!.$$

  Thus,

  $$\mathscr{S}\{t^{n+1}\}=(n+1)!\,u^{n+1}.$$

  Therefore, the formula (2.6) holds for all $n\in\mathbb{N}$.

Example 2.3. We apply Sumudu transform to function $f(t) = e^{at}$, we obtain

$$\begin{aligned}\mathscr{S}\{e^{at}\} = F(u) &= \int_0^{\infty} e^{-t} e^{a(ut)}\, dt \\ &= \int_0^{\infty} e^{-t(1-au)}\, dt \\ &= \lim_{x\to\infty}\left[-\frac{e^{-t(1-au)}}{1-au}\right]_0^x \\ &= \frac{1}{1-au}.\end{aligned}$$

2.1.3 Inverse of Sumudu transform

Definition 2.1.3. [16] The inverse Sumudu transform of $F(u)$ is defined by

$$\mathscr{S}^{-1}\{F(u)\} = f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{ut} F\left(\frac{1}{u}\right)\frac{du}{u}, \tag{2.7}$$

where u and γ are real parameters ,and $\mathscr{S}^{-1}$ denotes the inverse Sumudu transform operator which maps $F(u)$ into $f(t)$.

2.1.4 Properties

In this part, we introduce some properties of the Sumudu transform. Presented in the following references ([17],[16],[6]).

1. Linearity property

Let $f(t)$ and $g(t)$ be functions for which the Sumudu transform exists, we denote

$$\mathscr{S}\{f(t)\} = F(u), \quad \mathscr{S}\{g(t)\} = G(u), \tag{2.8}$$

for any constants $a, b \in \mathbb{R}$, it follows that

$$\mathscr{S}\{af(t) + bg(t)\} = a\mathscr{S}\{f(t)\} + b\mathscr{S}\{g(t)\} = aF(u) + bG(u). \tag{2.9}$$

Proof. Using the formula (2.3), we obtain

$$\mathscr{S}\{af(t) + bg(t)\} = \frac{1}{u}\int_0^{\infty} e^{-\frac{t}{u}}\left(af(t) + bg(t)\right) dt,$$

by the linearity of the integral, we get

$$\begin{aligned}\mathscr{S}\{af(t)+bg(t)\} &= \frac{1}{u}\left[\int_0^\infty af(t)e^{-\frac{t}{u}}\,dt+\int_0^\infty bg(t)e^{-\frac{t}{u}}\,dt\right]\\ &= a\frac{1}{u}\int_0^\infty f(t)e^{-\frac{t}{u}}\,dt+b\frac{1}{u}\int_0^\infty g(t)e^{-\frac{t}{u}}\,dt\\ &= a\mathscr{S}\{f(t)\}+b\mathscr{S}\{g(t)\}\\ &= aF(u)+bG(u).\end{aligned}$$

□

2. Scaling property

If $\mathscr{S}\{f(t)\}=F(u)$, then

$$\mathscr{S}\{f(at)\}=F(au), \quad a>0. \tag{2.10}$$

Proof. We apply the integral formula (2.3), then

$$\mathscr{S}\{f(at)\}=\frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}f(at)\,dt.$$

We make the change of variable

$$x=at \quad t=\frac{x}{a}, \quad dt=\frac{1}{a}dx.$$

Substituting into the previous expression, we obtain

$$\begin{aligned}\mathscr{S}\{f(at)\} &= \frac{1}{u}\int_0^\infty e^{-\frac{x}{au}}f(x)\frac{1}{a}\,dx\\ &= \frac{1}{au}\int_0^\infty e^{-\frac{x}{au}}f(x)\,dx\\ &= F(au).\end{aligned}$$

□

3. First shifting property

If $\mathscr{S}\{f(t)\}=F(u)$, then

$$\mathscr{S}\{e^{at}f(t)\}=\frac{1}{1-au}F\left(\frac{u}{1-au}\right). \tag{2.11}$$

Proof. From (2.4), we have

$$\begin{aligned}\mathscr{S}\{e^{at}f(t)\} &= \int_0^\infty e^{-t}e^{aut}f(ut)\,dt\\ &= \int_0^\infty e^{-(1-au)t}f(ut)\,dt.\end{aligned}$$

We perform a change of variable, let $w=(1-au)t$. Then

$$\begin{aligned}\mathscr{S}\{e^{at}f(t)\} &= \int_0^\infty e^{-w}f\left(\frac{uw}{1-au}\right)\frac{dw}{1-au}\\ &= \frac{1}{1-au}\int_0^\infty e^{-w}f\left(\frac{u}{1-au}w\right)dw\\ &= \frac{1}{1-au}\mathscr{S}\left(\frac{u}{1-au}\right).\end{aligned}$$

□

4. Sumudu transform of integral

Theorem 2.1.4. If $\mathscr{S}\{f(t)\}=F(u)$, then

$$\mathscr{S}\left\{\int_0^t f(\tau)\,d\tau\right\} = uF(u). \tag{2.12}$$

Proof. We have

$$f(\tau) = a_0 + a_1\tau + a_2\tau^2 + \cdots + a_n\tau^n + \cdots$$

Integrating, we obtain

$$\int_0^t f(\tau)\,d\tau = a_0t + \frac{1}{2}a_1t^2 + \frac{1}{3}a_2t^3 + \cdots + \frac{1}{n+1}a_nt^{n+1} + \cdots$$

Applying the Sumudu transform term by term, we get

$$\begin{aligned}\mathscr{S}\left\{\int_0^t f(\tau)\,d\tau\right\}(u) &= a_0u + a_1u^2 + 2!a_2u^3 + \cdots + n!a_nu^{n+1} + \cdots\\ &= uF(u).\end{aligned}$$

□

5. Sumudu transform of derivatives

Theorem 2.1.5. If $\mathscr{S}\{f(t)\} = F(u)$, then the Sumudu transform of $f'(t)$ is given by

$$\mathscr{S}\{f'(t)\} = \frac{F(u) - f(0)}{u} = \frac{F(u) - F(0)}{u}. \tag{2.13}$$

Proof. Consider functions which can be expressed as a power series or as a polynomial

$$f(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n + \cdots$$

By the definition of the Sumudu transform

$$\mathscr{S}\{f(t)\} = F(u) = a_0 + a_1 u + 2! a_2 u^2 + \cdots + n! a_n u^n + \cdots$$

The derivative of $f(t)$ and its transform are

$$f'(t) = a_1 + 2a_2 t + 3a_2 t^2 + \cdots + n a_n t^{n-1} + \cdots$$

$$\begin{aligned}\mathscr{S}\{f'(t)\} &= a_1 + 2a_2 u + 3! a_3 u^2 + \cdots + n! a_n u^{n-1} + \cdots \\ &= \frac{F(u) - a_0}{u} = \frac{F(u) - F(0)}{u} = \frac{F(u) - f(0)}{u}.\end{aligned}$$

□

Result

Based on the Sumudu transform of ordinary derivatives, we can deduce the Sumudu transform of partial derivatives

$$\mathscr{S}\left\{\frac{\partial f(x,t)}{\partial t}\right\} = \frac{1}{u}\big[F(x,u) - f(x,0)\big]. \tag{2.14}$$

$$\mathscr{S}\left\{\frac{\partial^2 f(x,t)}{\partial t^2}\right\} = \frac{1}{u^2}F(x,u) - \frac{1}{u^2}f(x,0) - \frac{1}{u}\frac{\partial f(x,0)}{\partial t}. \tag{2.15}$$

$$\mathscr{S}\left\{\frac{\partial f(x,t)}{\partial x}\right\} = \frac{dF(x,u)}{dx}. \tag{2.16}$$

$$\mathscr{S}\left\{\frac{\partial^2 f(x,t)}{\partial x^2}\right\} = \frac{d^2 F(x,u)}{dx^2}. \tag{2.17}$$

6. Duality with Laplace transform

If $\mathscr{S}\{f(t)\} = G(u)$ and $\mathscr{L}\{f(t)\} = F(s)$. Then

$$G(u) = \frac{F\left(\frac{1}{u}\right)}{u}, \tag{2.18}$$

and

$$F(s) = \frac{G\left(\frac{1}{s}\right)}{s}. \tag{2.19}$$

7. Convolution property

Theorem 2.1.6. If $\mathscr{S}\{f(t)\} = F(u)$ and $\mathscr{S}\{g(t)\} = G(u)$, then

$$\mathscr{S}\{f(t) * g(t)\} = u\mathscr{S}\{f(t)\}\mathscr{S}\{g(t)\} = uF(u)G(u). \tag{2.20}$$

Where the convolution is defined by the integral

$$f(t) * g(t) = \int_0^t f(\tau)\,g(t-\tau)\,d\tau. \tag{2.21}$$

Proof. First, recall that the Laplace transform of the convolution $(f * g)$ is given by

$$\mathscr{L}\{(f * g)(t)\} = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\}.$$

Now, by the duality relation (2.18), we get

$$\mathscr{S}\{(f * g)(t)\} = \frac{1}{u}\mathscr{L}\{(f * g)(t)\}\left(\frac{1}{u}\right),$$

since

$$\mathscr{S}\{f(t)\} = \frac{1}{u}\mathscr{L}\{f(t)\}\left(\frac{1}{u}\right), \quad \mathscr{S}\{g(t)\} = \frac{1}{u}\mathscr{L}\{g(t)\}\left(\frac{1}{u}\right).$$

Thus, we obtain

$$\mathscr{S}\{(f * g)(t)\} = \frac{1}{u}\left[\mathscr{L}\{f(t)\}\left(\frac{1}{u}\right)\mathscr{L}\{g(t)\}\left(\frac{1}{u}\right)\right].$$

Hence

$$\mathscr{S}\{(f * g)(t)\} = \frac{1}{u}\left[u^2\mathscr{S}\{f(t)\}\mathscr{S}\{g(t)\}\right] = u\mathscr{S}\{f(t)\}\mathscr{S}\{g(t)\}.$$

□

2.1.5 Table of Sumudu transform

The following table lists some basic functions along with their Sumudu transforms.

Table 2.1: Table of Sumudu transform

$f(t)$	$F(u)$
1	1
t	u
$t^n \ (n=0,1,2,\dots)$	$n!u^n$
e^{at}	$\frac{1}{1-au}, \quad au<1$
te^{at}	$\frac{u}{(1-au)^2}, \quad au<1$
$\sin(at)$	$\frac{au}{1+(au)^2}$
$\cos(at)$	$\frac{1}{1+(au)^2}$
$\sinh(at)$	$\frac{au}{1-a^2u^2}, \quad \lvert au\rvert<1$
$\cosh(at)$	$\frac{1}{1-a^2u^2}, \quad \lvert au\rvert<1$

2.1.6 Illustrative example of the Sumudu transform to solving a partial differential equation

We consider the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}, \quad x,t \geq 0, \tag{2.22}$$

with the initial and boundary conditions

$$u(x,0) = x, \quad u(0,t) = t. \tag{2.23}$$

We apply the Sumudu transform to (2.22), we obtain

$$\mathscr{S}\left\{\frac{\partial u}{\partial t}\right\}=\mathscr{S}\left\{\frac{\partial u}{\partial x}\right\}, \tag{2.24}$$

let $\mathscr{S}\{u(x,t)\}=U(x,u)$, and from (2.14) and (2.16), we obtain

$$\mathscr{S}\left\{\frac{\partial u}{\partial t}\right\}=\frac{U(x,u)-u(x,0)}{u},$$

and

$$\mathscr{S}\left\{\frac{\partial u}{\partial x}\right\}=\frac{dU}{dx}.$$

By substituting into (2.24), we get

$$\frac{U(x,u)-u(x,0)}{u}=\frac{dU}{dx},$$

substituting the initial condition $u(x,0)=x$, we obtain

$$\frac{U(x,u)-x}{u}=\frac{dU}{dx}.$$

Rewriting

$$\frac{dU}{dx}-\frac{1}{u}U(x,u)=-\frac{x}{u}, \tag{2.25}$$

this is a first-order linear ordinary differential equation. The integrating factor is

$$\mu(x)=e^{\int -\frac{1}{u}\,dx}=e^{-\frac{x}{u}}.$$

Multiplying both sides of equation (2.25) by the integrating factor

$$e^{-x/u}\frac{dU}{dx}-\frac{1}{u}e^{-\frac{x}{u}}U=-\frac{x}{u}e^{-x/u}\Rightarrow\frac{d}{dx}\left(e^{-x/u}U\right)=-\frac{x}{u}e^{-x/u},$$

integrating

$$e^{-x/u}U(x,u)=\int -\frac{x}{u}e^{-x/u}\,dx.$$

Using integration by parts

$$-\frac{1}{u}\int xe^{-x/u}\,dx=-\frac{1}{u}\left(-xue^{-x/u}-u^2e^{-x/u}\right),$$

thus,

$$e^{-x/u}U(x,u)=xe^{-x/u}+ue^{-x/u}+C.$$

Hence,

$$U(x,u) = x + u + Ce^{x/u},$$

using the boundary condition $u(0,t) = t$, we obtain

$$U(0,u) = \mathscr{S}\{t\} = u.$$

Substituting $x = 0$

$$U(0,u) = u + C \Rightarrow C = 0,$$

then,

$$U(x,u) = x + u.$$

Taking inverse Sumudu transform

$$u(x,t) = \mathscr{S}^{-1}\{x + u\},$$

therefore, the solution to the partial differential equation is

$$u(x,t) = x + t.$$

Chapter 3

Double Laplace–Sumudu Transform

The double Laplace–Sumudu transform is an efficient integral transform that combines the properties of both the Laplace transform and the Sumudu transform. It also contributes to reducing computational complexity and facilitates finding solutions in a systematic and concise way. This transform is considered an important tool in mathematical analysis and in physical and engineering applications.

3.1 The double Laplace–Sumudu transform

3.1.1 Definition of double Laplace–Sumudu transform

Definition 3.1.1. [1] The Double Laplace-Sumudu Tranform of the function $f(x,t)$ of two variable $x > 0$ and $t > 0$, denoted by

$$\mathscr{L}_x\mathscr{S}_t\{f(x,t)\} = F(s,u), \tag{3.1}$$

is defined as follows

$$\mathscr{L}_x\mathscr{S}_t\{f(x,t)\} = F(s,u) = \frac{1}{u}\int_0^\infty \int_0^\infty e^{-sx-\frac{t}{u}} f(x,t)\,dx\,dt, \tag{3.2}$$

where $s > 0$ and $u > 0$.

3.1.2 Inverse of double Laplace–Sumudu transform

Definition 3.1.2. [1] The inverse of the double Laplace–Sumudu transform it is written in the form

$$\mathscr{L}_x^{-1}\mathscr{S}_t^{-1}\{F(s,u)\} = f(x,t), \tag{3.3}$$

is defined by

$$\mathscr{L}_x^{-1}\mathscr{S}_t^{-1}\{F(s,u)\} = f(x,t) = \frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty} e^{sx}\,ds \frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} \frac{1}{u} e^{\frac{t}{u}} F(s,u)\,du. \tag{3.4}$$

3.1.3 Existence and uniqueness of double Laplace–Sumudu transform

Theorem 3.1.3. [4] Let $f(x,t)$ be a continuous function on every finite intervals $(0,X)$ and $(0,T)$ and of exponential order, that is for some $a,b \in \mathbb{R}$

$$\sup_{x,t>0} \frac{|f(x,t)|}{e^{(ax+bt)}} < \infty, \tag{3.5}$$

then the double Laplace–Sumudu transform of $f(x,t)$ exists.

Proof. Using definition of the double Laplace–Sumudu transform

$$\begin{aligned}
|\mathscr{L}_x\mathscr{S}_t\{f(x,t)\}| &= \left| \frac{1}{u}\int_0^\infty \int_0^\infty e^{-sx-\frac{t}{u}} f(x,t)\,dx\,dt \right| \\
&\le \frac{1}{u}\int_0^\infty \int_0^\infty e^{-sx-\frac{t}{u}} |f(x,t)|\,dx\,dt \\
&\le \frac{k}{u}\int_0^\infty \int_0^\infty e^{-(sx-ax)-(\frac{t}{u}-bt)}\,dx\,dt \\
&= \frac{k}{u}\int_0^\infty e^{-(s-a)x}\,dx \int_0^\infty e^{-(\frac{1}{u}-b)t}\,dt \\
&= \frac{k}{(s-a)(1-bu)}.
\end{aligned}$$

□

Theorem 3.1.4. [4] Let $g(x,t)$ and $h(x,t)$ be a continuous functions and having the double Laplace–Sumudu transform $\mathscr{L}_x\mathscr{S}_t\{g(x,t)\}$ and $\mathscr{L}_x\mathscr{S}_t\{h(x,t)\}$. If $\mathscr{L}_x\mathscr{S}_t\{g(x,t)\} = \mathscr{L}_x\mathscr{S}\{h(x,t)\}$ then $g(x,t) = h(x,t)$.

Proof. Assume η and μ to be sufficiently large, then since

$$f(x,t) = \mathscr{L}_x^{-1}\mathscr{S}_t^{-1}\{F(s,u)\} = \frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty} e^{sx}\,ds \frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} \frac{1}{u} e^{\frac{t}{u}} F(s,u)\,du,$$

we deduce that

$$\begin{aligned} g(x,t) &= \frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty} e^{sx}\,ds \frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} \frac{1}{u} e^{\frac{t}{u}} G(s,u)\,du \\ &= \frac{1}{2\pi i}\int_{\eta-i\infty}^{\eta+i\infty} e^{sx}\,ds \frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} \frac{1}{u} e^{\frac{t}{u}} H(s,u)\,du \\ &= h(x,t), \end{aligned}$$

and the theorem is established. □

3.1.4 Properties

In this section, we present the main properties of the double Laplace–Sumudu transform. Presented in the following references ([1],[3]).

1. Linearity property

Let $f(x,t)$ and $g(x,t)$ be functions for which the double Laplace–Sumudu transform exists, we denote

$$\mathscr{L}_x\mathscr{S}_t\{f(x,t)\} = F(s,u), \quad \mathscr{L}_x\mathscr{S}_t\{g(x,t)\} = G(s,u). \tag{3.6}$$

For any constants γ and δ, it follows that

$$\begin{aligned} \mathscr{L}_x\mathscr{S}_t\{\gamma f(x,t)+\delta g(x,t)\} &= \gamma\mathscr{L}_x\mathscr{S}_t\{f(x,t)\}+\delta\mathscr{L}_x\mathscr{S}_t\{g(x,t)\} \qquad (3.7)\\ &= \gamma F(s,u)+\delta G(s,u). \qquad (3.8) \end{aligned}$$

Proof. From (3.2), we obtain

$$\begin{aligned} \mathscr{L}_x\mathscr{S}_t\{\gamma f(x,t)+\delta g(x,t)\} &= \frac{1}{u}\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{u}}\{\gamma f(x,t)+\delta g(x,t)\}\,dx\,dt \\ &= \frac{1}{u}\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{u}}\gamma f(x,t)+\frac{1}{u}\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{u}}\delta g(x,t)\,dx\,dt \\ &= \gamma\frac{1}{u}\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{u}} f(x,t)\,dx\,dt+\delta\frac{1}{u}\int_0^\infty\int_0^\infty g(x,t)\,dx\,dt \\ &= \gamma\mathscr{L}_x\mathscr{S}_t\{f(x,t)\}+\delta\mathscr{L}_x\mathscr{S}_t\{g(x,t)\} \\ &= \gamma F(s,u)+\delta G(s,u). \end{aligned}$$

□

2. Shifting property

If $\mathscr{L}_x\mathscr{S}_t\{f(x,t)\} = F(s,u)$, then

$$\mathscr{L}_x\mathscr{S}_t\{e^{(cx+dt)}f(x,t)\} = \frac{1}{1-du}F\left(s-c,\ \frac{u}{1-du}\right). \tag{3.9}$$

Proof. From (3.2), we obtain

$$\begin{aligned}\mathscr{L}_x\mathscr{S}_t\{e^{(cx+dt)}f(x,t)\} &= \frac{1}{u}\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{u}}e^{cx+dt}f(x,t)\,dx\,dt\\ &= \frac{1}{u}\int_0^\infty\int_0^\infty e^{-(s-c)x}e^{-\left(\frac{1}{u}-d\right)t}f(x,t)\,dx\,dt.\end{aligned}$$

We perform a change of variable $w = \frac{u}{1-du}$, thus $u = w(1-du)$, then

$$\begin{aligned}\mathscr{L}_x\mathscr{S}_t\{e^{(cx+dt)}f(x,t)\} &= \frac{1}{w(1-du)}\int_0^\infty\int_0^\infty e^{-(s-c)x}e^{-\frac{t}{w}}f(x,t)\,dx\,dt\\ &= \frac{1}{1-du}\frac{1}{w}\int_0^\infty\int_0^\infty e^{-(s-c)x-\frac{t}{w}}f(x,t)\,dx\,dt\\ &= \frac{1}{1-du}F(s-c,w)\\ &= \frac{1}{1-du}F\left(s-c,\frac{u}{1-du}\right).\end{aligned}$$

□

3. Product of two separate functions

Let $f(x,t) = f(x)g(t)$, then

$$F(s,u) = \mathscr{L}_x\{f(x)\}\mathscr{S}_t\{g(t)\}. \tag{3.10}$$

Proof. From (3.2), we have

$$\begin{aligned}\mathscr{L}_x\mathscr{S}_t\{f(x)g(t)\} = F(s,u) &= \frac{1}{u}\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{u}}f(x)g(t)\,dx\,dt\\ &= \left(\int_0^\infty e^{-sx}f(x)\,dx\right)\left(\int_0^\infty \frac{1}{u}e^{-\frac{t}{u}}g(t)\,dt\right)\\ &= \mathscr{L}_x\{f(x)\}\mathscr{S}_t\{g(t)\}.\end{aligned}$$

□

4. Double Laplace–Sumudu transform of derivatives

In this property, we study the partial derivatives in the double Laplace–Sumudu transform, if $\mathscr{L}_x\mathscr{S}_t\{f(x,t)\} = F(s,u)$, then

$$\mathscr{L}_x\mathscr{S}_t\left\{\frac{\partial f(x,t)}{\partial x}\right\} = sF(s,u) - \mathscr{S}\{f(0,t)\}. \tag{3.11}$$

$$\mathscr{L}_x\mathscr{S}_t\left\{\frac{\partial f(x,t)}{\partial t}\right\} = \frac{1}{u}F(s,u) - \frac{1}{u}\mathscr{L}\{f(x,0)\}. \tag{3.12}$$

$$\mathscr{L}_x\mathscr{S}_t\left\{\frac{\partial^2 f(x,t)}{\partial x^2}\right\} = s^2F(s,u) - s\mathscr{S}\{f(0,t)\} - \mathscr{S}\{f_x(0,t)\}. \tag{3.13}$$

$$\mathscr{L}_x\mathscr{S}_t\left\{\frac{\partial^2 f(x,t)}{\partial t^2}\right\} = \frac{1}{u^2}F(s,u) - \frac{1}{u^2}\mathscr{L}\{f(x,0)\} - \frac{1}{u}\mathscr{L}\{f_t(x,0)\}. \tag{3.14}$$

$$\mathscr{L}_x\mathscr{S}_t\left\{\frac{\partial^2 f(x,t)}{\partial x\partial t}\right\} = \frac{s}{u}F(s,u) - \frac{s}{u}\mathscr{L}\{f(x,0)\} - \mathscr{S}\{f_t(0,t)\}. \tag{3.15}$$

Proof. According to (3.2), we obtain

- For (3.11)

$$\begin{aligned}\mathscr{L}_x\mathscr{S}_t\left\{\frac{\partial f(x,t)}{\partial x}\right\} &= \frac{1}{u}\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{u}}\frac{\partial f(x,t)}{\partial x}\,dx\,dt\\ &= \frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}\,dt\int_0^\infty e^{-sx}\frac{\partial f(x,t)}{\partial x}\,dx.\end{aligned}$$

Using integration by parts, let $u = e^{-sx}$, $dv = \frac{\partial f(x,t)}{\partial x}$, then

$$\begin{aligned}\mathscr{L}_x\mathscr{S}_t\left\{\frac{\partial f(x,t)}{\partial x}\right\} &= \frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}\,dt\left\{e^{-sx}f(x,t)\Big|_0^\infty + s\int_0^\infty e^{-sx}f(x,t)\,dx\right\}\\ &= \frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}\,dt\left\{-f(0,t) + s\int_0^\infty e^{-sx}f(x,t)\,dx\right\}\\ &= sF(s,u) - \mathscr{S}\{f(0,t)\}.\end{aligned}$$

- For (3.12)

$$\begin{aligned}\mathscr{L}_x\mathscr{S}_t\left\{\frac{\partial f(x,t)}{\partial t}\right\} &= \frac{1}{u}\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{u}}\frac{\partial f(x,t)}{\partial t}\,dx\,dt\\ &= \frac{1}{u}\int_0^\infty e^{-sx}\,dx\int_0^\infty e^{-\frac{t}{u}}\frac{\partial f(x,t)}{\partial t}\,dt.\end{aligned}$$

Using integration by parts, let $u = e^{-\frac{t}{u}}$, $dv = \frac{\partial f(x,t)}{\partial t}$, then

$$\begin{aligned}\mathscr{L}_x\mathscr{S}_t\left\{\frac{\partial f(x,t)}{\partial t}\right\} &= \frac{1}{u}\int_0^\infty e^{-sx}\,dx\left\{e^{-\frac{t}{u}}f(x,t)\Big|_0^\infty + \frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}f(x,t)\,dt\right\}\\ &= \frac{1}{u}\int_0^\infty e^{-sx}\,dx\left\{-f(x,0) + \frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}f(x,t)\,dt\right\}\\ &= \frac{1}{u}F(s,u) - \frac{1}{u}\mathscr{L}\{f(x,0)\}.\end{aligned}$$

- For (3.13)

$$\begin{aligned}\mathscr{L}_x\mathscr{S}_t\left\{\frac{\partial^2 f(x,t)}{\partial x^2}\right\} =& \frac{1}{u}\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{u}}\frac{\partial^2 f(x,t)}{\partial x^2}\,dx\,dt\\ &= \frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}\,dt\int_0^\infty e^{-sx}\frac{\partial^2 f(x,t)}{\partial x^2}\,dx.\end{aligned}$$

Using integration by parts, let $u = e^{-sx}$, $dv = \frac{\partial^2 f(x,t)}{\partial x^2}$, then

$$\begin{aligned}\mathscr{L}_x\mathscr{S}_t\left\{\frac{\partial^2 f(x,t)}{\partial x^2}\right\} &= \frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}\,dt\left\{e^{-sx}\frac{\partial f(x,t)}{\partial x}\Big|_0^\infty + s\int_0^\infty e^{-sx}\frac{\partial f(x,t)}{\partial x}\,dx\right\}\\ &= \frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}\,dt\left\{-\frac{\partial f(0,t)}{\partial x} + s\int_0^\infty e^{-sx}\frac{\partial f(x,t)}{\partial x}\,dx\right\}.\end{aligned}$$

Similarly, using integration by parts, let $u = e^{-sx}$, $dv = \frac{\partial f(x,t)}{\partial x}$, then

$$\begin{aligned}\mathscr{L}_x\mathscr{S}_t\left\{\frac{\partial^2 f(x,t)}{\partial x^2}\right\} &= \frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}\,dt\left\{-\frac{\partial f(0,t)}{\partial x} + s\left(-f(0,t) + s\int_0^\infty e^{-sx}f(x,t)\,dx\right)\right\}\\ &= s^2F(s,u) - s\mathscr{S}\{f(0,t)\} - \mathscr{S}\{f_x(0,t)\}.\end{aligned}$$

- For (3.14)

$$\begin{aligned}\mathscr{L}_x\mathscr{S}_t\left\{\frac{\partial^2 f(x,t)}{\partial t^2}\right\} &= \frac{1}{u}\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{u}}\frac{\partial^2 f(x,t)}{\partial t^2}\,dx\,dt\\ &= \frac{1}{u}\int_0^\infty e^{-sx}\,dx\int_0^\infty e^{-\frac{t}{u}}\frac{\partial^2 f(x,t)}{\partial t^2}\,dt.\end{aligned}$$

Using integration by parts, let $u = e^{-\frac{t}{u}}$, $dv = \frac{\partial^2 f(x,t)}{\partial t^2}$, then

$$\begin{aligned}\mathscr{L}_x\mathscr{S}_t\left\{\frac{\partial^2 f(x,t)}{\partial t^2}\right\} &= \frac{1}{u}\int_0^\infty e^{-sx}\,dx\left\{e^{-\frac{t}{u}}\frac{\partial f(x,t)}{\partial t}\Big|_0^\infty + \frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}\frac{\partial f(x,t)}{\partial t}\,dt\right\}\\ &= \frac{1}{u}\int_0^\infty e^{-sx}\,dx\left\{-\frac{\partial f(x,0)}{\partial t} + \frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}\frac{\partial f(x,t)}{\partial t}\,dt\right\}.\end{aligned}$$

Similarly, using integration by parts, let $u = e^{-\frac{t}{u}}$, $dv = \frac{\partial f(x,t)}{\partial t}$, then

$$\begin{aligned}\mathscr{L}_x\mathscr{S}_t\left\{\frac{\partial^2 f(x,t)}{\partial t^2}\right\} &= \frac{1}{u}\int_0^\infty e^{-sx}dx\left\{-\frac{\partial f(x,0)}{\partial t} + \frac{1}{u}\left(-f(x,0) + \frac{1}{u}\int_0^\infty f(x,t)e^{-\frac{t}{u}}dt\right)\right\}\\ &= \frac{1}{u^2}F(s,u) - \frac{1}{u^2}\mathscr{L}\{f(x,0)\} - \frac{1}{u}\mathscr{L}\{f_t(x,0)\}.\end{aligned}$$

- For (3.15)

$$\begin{aligned}\mathscr{L}_x\mathscr{S}_t\left\{\frac{\partial^2 f(x,t)}{\partial x\partial t}\right\} &= \frac{1}{u}\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{u}}\frac{\partial^2 f(x,t)}{\partial x\partial t}dxdt\\ &= \frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}\left\{\int_0^\infty e^{-sx}\frac{\partial^2 f(x,t)}{\partial x\partial t}dx\right\}dt.\end{aligned}$$

Using integration by parts with respect to x, let $u = e^{-sx}$, $dv = \frac{\partial^2 f(x,t)}{\partial x\partial t}$, then

$$\begin{aligned}\mathscr{L}_x\mathscr{S}_t\left\{\frac{\partial^2 f(x,t)}{\partial x\partial t}\right\} &= \frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}dt\left\{e^{-sx}\frac{\partial f(x,t)}{\partial t}\Big|_0^\infty + s\int_0^\infty \frac{\partial f(x,t)}{\partial t}e^{-sx}dx\right\}\\ &= \frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}dt\left\{-\frac{\partial f(0,t)}{\partial t} + s\int_0^\infty \frac{\partial f(x,t)}{\partial t}e^{-sx}dx\right\}\\ &= -\frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}\frac{\partial f(0,t)}{\partial t}dt + \frac{s}{u}\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{u}}\frac{\partial f(x,t)}{\partial t}dxdt\\ &= -\mathscr{S}\{f_t(0,t)\} + s\mathscr{L}_x\mathscr{S}_t\left\{\frac{\partial f(x,t)}{\partial t}\right\}\\ &= -\mathscr{S}\{f_t(0,t)\} + s\left\{\frac{1}{u}F(s,u) - \frac{1}{u}\mathscr{L}\{f(x,0)\}\right\}\\ &= \frac{s}{u}F(s,u) - \frac{s}{u}\mathscr{L}\{f(x,0)\} - \mathscr{S}\{f_t(0,t)\}.\end{aligned}$$

□

5. Heaviside step function

Theorem 3.1.5. If $\mathscr{L}_x\mathscr{S}_t\{f(x,t)\} = F(s,u)$, then

$$\mathscr{L}_x\mathscr{S}_t\{f(x-\delta,t-\varepsilon)H(x-\delta,t-\varepsilon)\} = e^{-s\delta-\frac{\varepsilon}{u}}F(s,u), \tag{3.16}$$

where $H(x,t)$ is the Heaviside unit step function defined by

$$H(x-\delta,t-\varepsilon) = \left\{\begin{array}{ll} 1, & x>\delta,\ t>\varepsilon \\ 0, & \text{otherwise.}\end{array}\right\} \tag{3.17}$$

Proof. From (3.2), we have

$$\begin{aligned}\mathscr{L}_x\mathscr{S}_t\{f(x-\delta,t-\varepsilon)H(x-\delta,t-\varepsilon)\} &= \frac{1}{u}\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{u}}f(x-\delta,t-\varepsilon)H(x-\delta,t-\varepsilon)\,dx\,dt\\ &= \frac{1}{u}\int_\delta^\infty\int_\varepsilon^\infty e^{-sx-\frac{t}{u}}f(x-\delta,t-\varepsilon)\,dx\,dt,\end{aligned}$$

that is, by putting $x-\delta=q$, $t-\varepsilon=w$, thus $x=q+\delta$, $t=w+\varepsilon$, we obtain

$$\begin{aligned}\mathscr{L}_x\mathscr{S}_t\{f(x-\delta,t-\varepsilon)H(x-\delta,t-\varepsilon)\} &= \frac{1}{u}\int_\delta^\infty\int_\varepsilon^\infty e^{-s(q+\delta)-\frac{w+\varepsilon}{u}}f(q,w)\,dq\,dw\\ &= e^{-s\delta-\frac{\varepsilon}{u}}\frac{1}{u}\int_0^\infty\int_0^\infty e^{-sq-\frac{w}{u}}f(q,w)\,dq\,dw\\ &= e^{-s\delta-\frac{\varepsilon}{u}}F(s,u).\end{aligned}$$

□

3.1.5 Convolution theorem of double Laplace–Sumudu transform

Definition 3.1.6. [1] The convolution of two functions $f(x,t)$ and $g(x,t)$ is defined by

$$(f**g)(x,t)=\int_0^x\int_0^t f(x-\delta,t-\varepsilon)\,g(\delta,\varepsilon)\,d\delta\,d\varepsilon. \tag{3.18}$$

Theorem 3.1.7. If $\mathscr{L}_x\mathscr{S}_t\{f(x,t)\}=F(s,u)$ and $\mathscr{L}_x\mathscr{S}_t\{g(x,t)\}=G(s,u)$, then

$$\mathscr{L}_x\mathscr{S}_t\{(f**g)(x,t)\}=u\,F(s,u)\,G(s,u). \tag{3.19}$$

Proof. From (3.2), we obtain

$$\begin{aligned}\mathscr{L}_x\mathscr{S}_t\{(f**g)(x,t)\} &= \frac{1}{u}\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{u}}(f**g)(x,t)\,dx\,dt\\ &= \frac{1}{u}\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{u}}\left\{\int_0^x\int_0^t f(x-\delta,t-\varepsilon)\,g(\delta,\varepsilon)\,d\delta\,d\varepsilon\right\}dx\,dt,\end{aligned}$$

which is, using the Heaviside unit step function

$$\begin{aligned}&= \frac{1}{u}\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{u}}\left\{\int_0^\infty\int_0^\infty f(x-\delta,t-\varepsilon)\,H(x-\delta,t-\varepsilon)\,g(\delta,\varepsilon)\,d\delta\,d\varepsilon\right\}dx\,dt\\ &= \int_0^\infty\int_0^\infty g(\delta,\varepsilon)\,d\delta\,d\varepsilon\left\{\frac{1}{u}\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{u}}f(x-\delta,t-\varepsilon)\,H(x-\delta,t-\varepsilon)\,dx\,dt\right\},\end{aligned}$$

by Theorem 3.1.5, we get

$$\begin{aligned}\mathscr{L}_x\mathscr{S}_t\{(f**g)(x,t)\} &= \int_0^\infty\int_0^\infty g(\delta,\varepsilon)\,d\delta\,d\varepsilon\left\{e^{-s\delta-\frac{\varepsilon}{u}}F(s,u)\right\}\\ &= F(s,u)\int_0^\infty\int_0^\infty e^{-s\delta-\frac{\varepsilon}{u}}g(\delta,\varepsilon)\,d\delta\,d\varepsilon\\ &= uF(s,u)G(s,u).\end{aligned}$$

□

3.1.6 Double Laplace–Sumudu transform of basic functions

In this part, we study the double Laplace–Sumudu transform of basic functions by relying on its main properties to simplify computations and obtain results in a direct way:

(1) We compute the double Laplace–Sumudu transform of the function $f(x,t)=1$, for $x>0$ and $t>0$, then

$$\begin{aligned}\mathscr{L}_x\mathscr{S}_t\{1\} &= \frac{1}{u}\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{u}}(1)\,dx\,dt\\ &= \left(\int_0^\infty e^{-sx}\,dx\right)\left(\frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}\,dt\right)\\ &= \mathscr{L}_x\{1\}\mathscr{S}_t\{1\}=\frac{1}{s}.\end{aligned}$$

(2) We compute the double Laplace–Sumudu transform of the function $f(x,t)=x$, then

$$\begin{aligned}\mathscr{L}_x\mathscr{S}_t\{x\} &= \frac{1}{u}\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{u}}x\,dx\,dt\\ &= \left(\int_0^\infty xe^{-sx}\,dx\right)\left(\frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}\,dt\right)\\ &= \mathscr{L}_x\{x\}\mathscr{S}_t\{1\}=\frac{1}{s^2}.\end{aligned}$$

(3) We compute the double Laplace–Sumudu transform of the function $f(x,t)=t$, then

$$\begin{aligned}\mathscr{L}_x\mathscr{S}_t\{t\} &= \frac{1}{u}\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{u}}t\,dx\,dt\\ &= \left(\int_0^\infty e^{-sx}\,dx\right)\left(\frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}t\,dt\right)\\ &= \mathscr{L}_x\{1\}\mathscr{S}_t\{t\}=\frac{u}{s}.\end{aligned}$$

(4) We compute the double Laplace–Sumudu transform of the function $f(x,t)=xt$, then

$$\begin{aligned}
\mathscr{L}_x\mathscr{S}_t\{xt\} &= \frac{1}{u}\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{u}}\,xt\,dx\,dt\\
&= \left(\int_0^\infty e^{-sx}x\,dx\right)\left(\frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}\,t\,dt\right)\\
&= \mathscr{L}_x\{x\}\mathscr{S}_t\{t\} = \frac{u}{s^2}.
\end{aligned}$$

(5) We compute the double Laplace–Sumudu transform of the function $f(x,t)=e^{cx+dt}$, then

$$\begin{aligned}
\mathscr{L}_x\mathscr{S}_t\{e^{cx+dt}\} &= \frac{1}{u}\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{u}}e^{cx+dt}\,dx\,dt\\
&= \frac{1}{u}\left(\int_0^\infty e^{-(s-c)x}\,dx\right)\left(\int_0^\infty e^{-(\frac{1}{u}-d)t}\,dt\right)\\
&= \frac{1}{u}\left(-\frac{1}{s-c}e^{-(s-c)x}\Big|_0^\infty\right)\left(-\frac{1}{\frac{1}{u}-d}e^{-(\frac{1}{u}-d)t}\Big|_0^\infty\right)\\
&= \frac{1}{(s-c)(1-du)}.
\end{aligned}$$

Similarly, for the function $f(x,t)=e^{-(cx+dt)}$, we obtain the following result

$$\mathscr{L}_x\mathscr{S}_t\{e^{-(cx+dt)}\} = \frac{1}{(s+c)(1+du)}.$$

(6) We compute the double Laplace–Sumudu transform of the function $f(x,t)=e^{i(cx+dt)}$, then

$$\begin{aligned}
\mathscr{L}_x\mathscr{S}_t\{e^{i(cx+dt)}\} &= \frac{1}{u}\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{u}}e^{i(cx+dt)}\,dx\,dt\\
&= \frac{1}{u}\int_0^\infty e^{-(s-ic)x}\,dx\int_0^\infty e^{-(\frac{1}{u}-id)t}\,dt\\
&= \frac{1}{u}\left(-\frac{1}{s-ic}e^{-(s-ic)x}\Big|_0^\infty\right)\left(-\frac{1}{\frac{1}{u}-id}e^{-(\frac{1}{u}-id)t}\Big|_0^\infty\right)\\
&= \frac{1}{(s-ic)(1-idu)}\\
&= \frac{(s+ic)}{(s-ic)(s+ic)}\cdot\frac{(1+idu)}{(1-idu)(1+idu)}\\
&= \frac{(s-cdu)+i(c+sdu)}{(s^2+c^2)(1+d^2u^2)}.
\end{aligned}$$

Similarly, for the function $f(x,t)=e^{-i(cx+dt)}$, we obtain the following result

$$\mathscr{L}_x\mathscr{S}_t\{e^{-i(cx+dt)}\} = \frac{(s-cdu)-i(c+sdu)}{(s^2+c^2)(1+d^2u^2)}.$$

(7) We compute the double Laplace–Sumudu transform of the function $f(x,t) = \sin(cx+dt)$, then

using

$$\sin(cx+dt) = \frac{e^{i(cx+dt)} - e^{-i(cx+dt)}}{2i},$$

applying the linearity property of the double Laplace–Sumudu transform, we obtain

$$\begin{aligned}
\mathscr{L}_x\mathscr{S}_t\{\sin(cx+dt)\} &= \frac{1}{2i}\left(\mathscr{L}_x\mathscr{S}_t\{e^{i(cx+dt)}\} - \mathscr{L}_x\mathscr{S}_t\{e^{-i(cx+dt)}\}\right) \\
&= \frac{1}{2i}\left(\frac{(s-cdu)+i(c+sdu)}{(s^2+c^2)(1+d^2u^2)} - \frac{(s-cdu)-i(c+sdu)}{(s^2+c^2)(1+d^2u^2)}\right) \\
&= \frac{1}{2i}\left(\frac{2i(c+sdu)}{(s^2+c^2)(1+d^2u^2)}\right) \\
&= \frac{c+sdu}{(s^2+c^2)(1+d^2u^2)}.
\end{aligned}$$

Similarly, for the function $f(x,t) = \cos(cx+dt)$, since

$$\cos(cx+dt) = \frac{e^{i(cx+dt)} + e^{-i(cx+dt)}}{2},$$

we obtain the following result

$$\mathscr{L}_x\mathscr{S}_t\{\cos(cx+dt)\} = \frac{s-cdu}{(s^2+c^2)(1+d^2u^2)}.$$

(8) We compute the double Laplace–Sumudu transform of the function $f(x,t) = \sinh(cx+dt)$, then using

$$\sinh(cx+dt) = \frac{e^{cx+dt} - e^{-(cx+dt)}}{2},$$

applying the linearity property of the double Laplace–Sumudu transform, we obtain

$$\begin{aligned}
\mathscr{L}_x\mathscr{S}_t\{\sinh(cx+dt)\} &= \frac{1}{2}\left(\mathscr{L}_x\mathscr{S}_t\{e^{cx+dt}\} - \mathscr{L}_x\mathscr{S}_t\{e^{-(cx+dt)}\}\right) \\
&= \frac{1}{2}\left(\frac{1}{(s-c)(1-du)} - \frac{1}{(s+c)(1+du)}\right) \\
&= \frac{c+dus}{(s^2-c^2)(1-d^2u^2)}.
\end{aligned}$$

Similarly, for the function $f(x,t) = \cosh(cx+dt)$, since

$$\cosh(cx+dt) = \frac{e^{cx+dt} + e^{-(cx+dt)}}{2},$$

we obtain the following result

$$\mathscr{L}_x\mathscr{S}_t\{\cosh(cx+dt)\} = \frac{s+cdu}{(s^2-c^2)(1-d^2u^2)}.$$

3.1.7 Table of double Laplace–Sumudu transform

The following table presents the double Laplace–Sumudu transforms of some basic functions.

Table 3.1: Table of double Laplace–Sumudu transform

$f(x,t)$	$F(s,u)$
1	$\frac{1}{s}, \quad s>0,\ u>0$
x	$\frac{1}{s^2}, \quad s>0,\ u>0$
x^n	$\frac{n!}{s^{n+1}}, \quad s>0,\ u>0$
t	$\frac{u}{s}, \quad s>0,\ u>0$
t^n	$\frac{n!u^n}{s}, \quad s>0,\ u>0$
xt	$\frac{u}{s^2}, \quad s>0,\ u>0$
e^{cx+dt}	$\frac{1}{(s-c)(1-du)}, \quad s>c,\ \frac{1}{u}>d$
$e^{i(cx+dt)}$	$\frac{(s-cdu)+i(c+dus)}{(s^2+c^2)(1+d^2u^2)}, \quad s>0,\ u>0$
$\sin(cx+dt)$	$\frac{c+dus}{(s^2+c^2)(1+d^2u^2)}, \quad s>0,\ u>0$
$\cos(cx+dt)$	$\frac{s-cdu}{(s^2+c^2)(1+d^2u^2)}, \quad s>0,\ u>0$
$\sinh(cx+dt)$	$\frac{c+dus}{(s^2-c^2)(1-d^2u^2)}, \quad s>\lvert c\rvert,\ \lvert du\rvert<1$
$\cosh(cx+dt)$	$\frac{s+cdu}{(s^2-c^2)(1-d^2u^2)}, \quad s>\lvert c\rvert,\ \lvert du\rvert<1$
$\sin(ax)\cos(bt)$	$\frac{a}{(s^2+a^2)(1+b^2u^2)}, \quad s>0,\ u>0$
$\cos(ax)e^{bt}$	$\frac{s}{(s^2+a^2)(1-bu)}, \quad s>0,\ bu<1$
$\sin(ax)e^{bt}$	$\frac{a}{(s^2+a^2)(1-bu)}, \quad s>0,\ bu<1$

3.1.8 Principle of double Laplace–Sumudu transform method

We consider linear one dimensional, time dependent partial differential equation (PDE) of the form [2]

$$\sum_{n=0}^{N} c_n \frac{\partial^n f(x,t)}{\partial t^n} = \sum_{m=1}^{M} d_m \frac{\partial^m f(x,t)}{\partial x^m} + g(x,t), \quad (x,t) \in \mathbb{R}_+^2, \tag{3.20}$$

with the initial conditions

$$\frac{\partial^n f(x,0)}{\partial t^n} = f_n(x), \quad n = 0,1,\ldots,N-1, \quad x \in \mathbb{R}_+, \tag{3.21}$$

and boundary conditions

$$\frac{\partial^m f(0,t)}{\partial x^m} = h_m(t), \quad m = 0,1,\ldots,M-1, \quad t \in \mathbb{R}_+, \tag{3.22}$$

where c_n, $0 \leq n \leq N$; d_m, $1 \leq m \leq M$ are given coefficients and N,M are positive integers and $g(x,t)$ is the source term.
Applying the double Laplace–Sumudu transform on both sides of (3.20), we get

$$\sum_{n=0}^{N} c_n \left[u^{-n} F(s,u) - \sum_{j=0}^{n-1} u^{-n+j} \mathscr{L}_x \left\{ \frac{\partial^j f(x,0)}{\partial t^j} \right\} \right] \tag{3.23}$$

$$= \sum_{m=1}^{M} d_m \left[s^m F(s,u) - \sum_{k=0}^{m-1} s^{m-1-k} \mathscr{S}_t \left\{ \frac{\partial^k f(0,t)}{\partial x^k} \right\} \right] + G(s,u). \tag{3.24}$$

Further, applying single Laplace transform (LT) to initial (3.21) and single Sumudu transform (ST) to boundary (3.22), we get

$$\mathscr{L}_x \left\{ \frac{\partial^n f(x,0)}{\partial t^n} \right\} = F_n(s), \quad n = 0,1,\ldots,N-1, \tag{3.25}$$

$$\mathscr{S}_t \left\{ \frac{\partial^m f(0,t)}{\partial x^m} \right\} = H_m(u), \quad m = 0,1,\ldots,M-1, \tag{3.26}$$

substituting (3.25) and (3.26) in (3.23) and simplifying, we obtain

$$\sum_{n=0}^{N} c_n \left[u^{-n} F(s,u) - \sum_{j=0}^{n-1} u^{-n+j} F_n(s) \right] \tag{3.27}$$

$$= \sum_{m=1}^{M} d_m \left[s^m F(s,u) - \sum_{k=0}^{m-1} s^{m-1-k} H_m(u) \right] + G(s,u), \tag{3.28}$$

equation (3.27) is an algebraic equation in $F(s,u)$. Solving this algebraic equation and taking $\mathscr{L}_x^{-1} \mathscr{S}_t^{-1} \{F(s,u)\} = f(x,t)$, we obtain an exact solution $f(x,t)$ of equation (3.20).

Chapter 4

Applications

In this chapter, we apply the double Laplace–Sumudu transform to some partial differential equations previously solved using the Laplace and Sumudu transforms, in order to highlight its role in simplifying the solution steps and reducing the time required to obtain the solution. We also study the solution of some non-homogeneous physical equations to demonstrate its effectiveness in dealing with this type of equations.

4.1 Double Laplace–Sumudu transform for homogeneous partial differential equations

Example 1:

We consider the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}, \quad x, t \geq 0, \tag{4.1}$$

with the initial and boundary conditions

$$u(x,0) = x, \quad u(0,t) = t. \tag{4.2}$$

We apply the double Laplace–Sumudu Transform to (4.1), we obtain

$$\mathscr{L}_x \mathscr{S}_t \left\{ \frac{\partial u}{\partial t} \right\} = \mathscr{L}_x \mathscr{S}_t \left\{ \frac{\partial u}{\partial x} \right\},$$

let $\mathscr{L}_x \mathscr{S}_t \{u(x,t)\} = U(s,u)$, and from (3.11) and (3.12), we obtain

$$\frac{1}{u} U(s,u) - \frac{1}{u} \mathscr{L}\{u(x,0)\} = sU(s,u) - \mathscr{S}\{u(0,t)\}. \tag{4.3}$$

Substituting the initial and boundary conditions

$$\mathscr{L}\{u(x,0)\} = \mathscr{L}\{x\} = \frac{1}{s^2}, \quad \mathscr{S}\{u(0,t)\} = \mathscr{S}\{t\} = u,$$

from equation (4.3), we get

$$\frac{1}{u}U(s,u) - \frac{1}{us^2} = sU(s,u) - u.$$

Multiplying by u, we obtain

$$(1-su)U(s,u) = \frac{1}{s^2} - u^2,$$

and

$$U(s,u) = \frac{\frac{1}{s^2} - u^2}{1-su}.$$

Noting that

$$1 - s^2u^2 = (1-su)(1+su),$$

by simplification, we find that

$$U(s,u) = \frac{1}{s^2}(1+su).$$

Thus

$$U(s,u) = \frac{1}{s^2} + \frac{u}{s},$$

taking inverse of the double Laplace–Sumudu transform

$$u(x,t) = \mathscr{L}_x^{-1}\mathscr{S}_t^{-1}\{U(s,u)\}.$$

The solution to the partial differential equation is

$$u(x,t) = x + t.$$

Example 2:

We consider the partial differential equation

$$2\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} - u, \quad x,t \geq 0, \tag{4.4}$$

with initial and boundary conditions

$$u(x,0) = 6e^{-3x}, \quad u(0,t) = 6e^{-2t}. \tag{4.5}$$

We apply the Double Laplace–Sumudu Transform to (4.4), we obtain

$$2\mathscr{L}_x\mathscr{S}_t\left\{\frac{\partial u}{\partial t}\right\}=\mathscr{L}_x\mathscr{S}_t\left\{\frac{\partial u}{\partial x}\right\}-\mathscr{L}_x\mathscr{S}_t\{u\},$$

let $\mathscr{L}_x\mathscr{S}_t\{u(x,t)\}=U(s,u)$, and from (3.11) and (3.12). We obtain

$$\frac{2}{u}\left(U(s,u)-\mathscr{L}\{u(x,0)\}\right)=sU(s,u)-\mathscr{S}\{u(0,t)\}-U(s,u). \tag{4.6}$$

Substituting the initial and boundary conditions

$$\mathscr{L}\{u(x,0)\}=\mathscr{L}\{6e^{-3x}\}=\frac{6}{s+3},\quad \mathscr{S}\{u(0,t)\}=\mathscr{S}\{6e^{-2t}\}=\frac{6}{1+2u},$$

from equation (4.6), we get

$$\frac{2}{u}\left(U-\frac{6}{s+3}\right)=sU-\frac{6}{1+2u}-U.$$

Multiplying both sides by u

$$2U-\frac{12}{s+3}=usU-\frac{6u}{1+2u}-uU,$$

rearranging

$$\begin{aligned}(2-us+u)U&=\frac{12}{s+3}-\frac{6u}{1+2u}\\&=\frac{12(1+2u)-6u(s+3)}{(s+3)(1+2u)}\\&=\frac{6(2+u-us)}{(s+3)(1+2u)}.\end{aligned}$$

Thus

$$(2-us+u)U=\frac{6(2-us+u)}{(s+3)(1+2u)},$$

by simplification, we find that

$$U(s,u)=\frac{6}{(s+3)(1+2u)}.$$

Taking inverse of the double Laplace–Sumudu transform

$$u(x,t)=\mathscr{L}_x^{-1}\mathscr{S}_t^{-1}\{U(s,u)\},$$

the solution of the partial differential equation is

$$u(x,t)=6e^{-3x-2t}.$$

Example 3: The Wave Equation

The wave equation is used to describe the propagation of waves and vibrations in many physical phenomena such as sound, water waves, and electromagnetic waves. In this example, we study the linear homogeneous wave equation with initial and boundary conditions using the double Laplace–Sumudu transform method in order to obtain the exact and concise solution of the equation, which demonstrates the effectiveness of this method.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad c \in \mathbb{R}. \tag{4.7}$$

With initial and boundary conditions

$$u(x,0) = \sin x, \quad u(0,t) = 2t, \quad u_t(x,0) = 2, \quad u_x(0,t) = \cos ct. \tag{4.8}$$

We apply the double Laplace–Sumudu transform to (4.7), we obtain

$$\mathscr{L}_x \mathscr{S}_t \left\{ \frac{\partial^2 u}{\partial t^2} \right\} = c^2 \mathscr{L}_x \mathscr{S}_t \left\{ \frac{\partial^2 u}{\partial x^2} \right\},$$

let $\mathscr{L}_x \mathscr{S}_t \{u(x,t\} = U(s,u)$, and from (3.13) and (3.14), we find

$$\frac{1}{u^2} U(s,u) - \frac{1}{u^2} \mathscr{L}\{u(x,0)\} - \frac{1}{u} \mathscr{L}\{u_t(x,0)\} = c^2 \left(s^2 U(s,u) - s\mathscr{S}\{u(0,t)\} - \mathscr{S}\{u_x(0,t)\} \right), \tag{4.9}$$

substituting the initial and boundary conditions

$$\mathscr{L}\{u(x,0)\} = \mathscr{L}\{\sin x\} = \frac{1}{s^2+1}, \quad \mathscr{L}\{u_t(x,0)\} = \mathscr{L}\{2\} = \frac{2}{s},$$

$$\mathscr{S}\{u(0,t)\} = \mathscr{S}\{2t\} = 2u, \quad \mathscr{S}\{u_x(0,t)\} = \mathscr{S}\{\cos ct\} = \frac{1}{1+(cu)^2}.$$

From equation (4.9), we get

$$\frac{1}{u^2} U(s,u) - \frac{1}{u^2(s^2+1)} - \frac{2}{us} = c^2 s^2 U(s,u) - c^2 s 2u - \frac{c^2}{1+(cu)^2},$$

then

$$\begin{aligned} \left(\frac{1 - c^2 s^2 u^2}{u^2} \right) U(s,u) &= \frac{1}{u^2(s^2+1)} + \frac{2}{us} - 2c^2 su - \frac{c^2}{1+(cu)^2} \\ &= \frac{2 - 2c^2 s^2 u^2}{us} + \frac{(1+c^2u^2) - c^2u^2(s^2+1)}{u^2(s^2+1)(1+c^2u^2)} \\ &= \frac{2(1 - c^2 s^2 u^2)}{us} + \frac{1 - c^2 s^2 u^2}{u^2(s^2+1)(1+c^2u^2)}, \end{aligned}$$

by simplification, we obtain

$$U(s,u) = \frac{2u}{s} + \frac{1}{(s^2+1)(1+c^2u^2)},$$

taking inverse of the double Laplace–Sumudu transform

$$u(x,t) = \mathscr{L}_x^{-1}\mathscr{S}_t^{-1}\{U(s,u)\} = \mathscr{L}_x^{-1}\mathscr{S}_t^{-1}\left\{\frac{2u}{s} + \frac{1}{(s^2+1)(1+c^2u^2)}\right\},$$

therefore, the solution of the equation is

$$u(x,t) = 2t + \sin(x)\cos(ct).$$

This result is presented in the following reference [14].

Example 4: The Klein–Gordon Equation

The Klein–Gordon equation is used in physics and quantum mechanics to describe the behavior of relativistic particles and waves and their propagation in spacetime. In this example, we study the linear homogeneous Klein–Gordon equation with initial and boundary conditions using the double Laplace–Sumudu transform method in order to obtain the exact and concise solution of the equation, which demonstrates the effectiveness of this method.

$$\frac{\partial^2 u}{\partial t^2} - u = \frac{\partial^2 u}{\partial x^2}. \tag{4.10}$$

With initial and boundary conditions

$$u(x,0) = 1+\sin x, \quad u(0,t) = \cosh t, \quad u_t(x,0) = 0, \quad u_x(0,t) = 1. \tag{4.11}$$

We apply the double Laplace–Sumudu transform to (4.10), we obtain

$$\mathscr{L}_x\mathscr{S}_t\left\{\frac{\partial^2 u}{\partial t^2}\right\} - \mathscr{L}_x\mathscr{S}_t\{u\} = \mathscr{L}_x\mathscr{S}_t\left\{\frac{\partial^2 u}{\partial x^2}\right\}, \tag{4.12}$$

let $\mathscr{L}_x\mathscr{S}_t\{u(x,t)\} = U(s,u)$, and from (3.13) and (3.14), we obtain

$$\frac{1}{u^2}U(s,u) - \frac{1}{u^2}\mathscr{L}\{u(x,0)\} - \frac{1}{u}\mathscr{L}\{u_t(x,0)\} - U(s,u) = s^2U(s,u) - s\mathscr{S}\{u(0,t)\} - \mathscr{S}\{u_x(0,t)\}, \tag{4.13}$$

substituting the initial and boundary conditions

$$\mathscr{L}\{u(x,0)\} = \mathscr{L}\{1+\sin x\} = \frac{1}{s} + \frac{1}{s^2+1}, \quad \mathscr{L}\{u_t(x,0)\} = \mathscr{L}\{0\} = 0,$$

$$\mathscr{S}\{u(0,t)\}=\mathscr{S}\{\cosh t\}=\frac{1}{1-u^2},\quad \mathscr{S}\{u_x(0,t)\}=\mathscr{S}\{1\}=1.$$

From equation (4.13), we get

$$\frac{1}{u^2}U(s,u)-\frac{1}{u^2}\left(\frac{1}{s}+\frac{1}{s^2+1}\right)-U(s,u)=s^2U(s,u)-\frac{s}{1-u^2}-1,$$

then

$$\begin{aligned}\left(\frac{1}{u^2}-1-s^2\right)U(s,u)&=\frac{1}{u^2}\left(\frac{1}{s}+\frac{1}{s^2+1}\right)-\frac{s}{1-u^2}-1\\&=\frac{1}{su^2}+\frac{1}{u^2(s^2+1)}-\frac{s}{1-u^2}-1\\&=\frac{1-u^2-s^2u^2}{su^2(1-u^2)}+\frac{1-u^2(s^2+1)}{u^2(s^2+1)},\end{aligned}$$

therefore

$$\left(\frac{1-u^2-s^2u^2}{u^2}\right)U(s,u)=\frac{1-u^2-s^2u^2}{su^2(1-u^2)}+\frac{1-u^2s^2-u^2}{u^2(s^2+1)}.$$

By simplification, we obtain

$$U(s,u)=\frac{1}{s(1-u^2)}+\frac{1}{s^2+1}.$$

Taking inverse of the double Laplace–Sumudu transform

$$u(x,t)=\mathscr{L}_x^{-1}\mathscr{S}_t^{-1}\{U(s,u)\}=\mathscr{L}_x^{-1}\mathscr{S}_t^{-1}\left\{\frac{1}{s(1-u^2)}+\frac{1}{s^2+1}\right\},$$

thus, the solution of the equation is

$$u(x,t)=\sin x+\cosh t.$$

This result is presented in the following reference [9].

4.2 Double Laplace–Sumudu transform for nonhomogeneous partial differential equations

Example 5: The Linear Dissipative Wave Equation

The linear dissipative wave equation is used to study waves that weaken over time and eventually disappear due to effects such as friction or resistance of the medium. In this example, we study a linear and non-homogeneous dissipative wave equation with initial and boundary conditions, using the double Laplace–Sumudu transform method in order to obtain an exact and concise solution of the equation, which highlights the effectiveness of this method.

$$\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+2(t-x), \tag{4.14}$$

with initial and boundary conditions

$$u(x,0)=x^2, \quad u(0,t)=t^2, \quad u_t(x,0)=0, \quad u_x(0,t)=0. \tag{4.15}$$

We apply the double Laplace–Sumudu transform to (4.14), we obtain

$$\mathscr{L}_x\mathscr{S}_t\left\{\frac{\partial u}{\partial t}\right\}+\mathscr{L}_x\mathscr{S}_t\left\{\frac{\partial^2 u}{\partial t^2}\right\}=\mathscr{L}_x\mathscr{S}_t\left\{\frac{\partial^2 u}{\partial x^2}\right\}+\mathscr{L}_x\mathscr{S}_t\left\{\frac{\partial u}{\partial x}\right\}+\mathscr{L}_x\mathscr{S}_t\{2(t-x)\}, \tag{4.16}$$

let $\mathscr{L}_x\mathscr{S}_t\{u(x,t)\}=U(s,u)$, and from (3.11), (3.12),(3.13) and (3.14), we obtain
since

$$\mathscr{L}_x\mathscr{S}_t\{2(t-x)\}=2\left(\frac{u}{s}-\frac{1}{s^2}\right),$$

from equation (4.16), we get

$$\frac{1}{u}U(s,u)-\frac{1}{u}\mathscr{L}\{u(x,0)\}+\frac{1}{u^2}U(s,u)-\frac{1}{u^2}\mathscr{L}\{u(x,0)\}-\frac{1}{u}\mathscr{L}\{u_t(x,0)\} \tag{4.17}$$

$$=s^2U(s,u)-s\mathscr{S}\{u(0,t)\}-\mathscr{S}\{u_x(0,t)\}+sU(s,u)-\mathscr{S}\{u(0,t)\}+2\left(\frac{u}{s}-\frac{1}{s^2}\right), \tag{4.18}$$

substituting the initial and boundary conditions

$$\mathscr{L}\{u(x,0)\}=\mathscr{L}\{x^2\}=\frac{2}{s^3}, \quad \mathscr{S}\{u(0,t)\}=\mathscr{S}\{t^2\}=2u^2,$$

$$\mathscr{L}\{u_t(x,0)\}=\mathscr{L}\{0\}=0, \quad \mathscr{S}\{u_x(0,t)\}=\mathscr{S}\{0\}=0.$$

Substituting into (4.17), we obtain

$$\frac{1}{u}U(s,u)-\frac{1}{u}\frac{2}{s^3}+\frac{1}{u^2}U(s,u)-\frac{1}{u^2}\frac{2}{s^3}=s^2U(s,u)-2su^2+sU(s,u)-2u^2+\frac{2u}{s}-\frac{2}{s^2},$$

then

$$\left(\frac{1}{u}+\frac{1}{u^2}-s^2-s\right)U(s,u)=\frac{2}{us^3}+\frac{2}{u^2s^3}-2su^2-2u^2+\frac{2u}{s}-\frac{2}{s^2},$$

$$\left(\frac{u+1-s^2u^2-su^2}{u^2}\right)U(s,u)=\frac{2}{us^3}+\frac{2}{u^2s^3}-2su^2-2u^2+\frac{2u}{s}-\frac{2}{s^2},$$

therefore

$$\begin{aligned}(u+1-s^2u^2-su^2)U(s,u) &= \frac{2u}{s^3}+\frac{2}{s^3}-2su^4-2u^4+\frac{2u^3}{s}-\frac{2u^2}{s^2} \\ &= \frac{2u}{s^3}+\frac{2}{s^3}-2su^4-2u^4+\frac{2u^3}{s}-\frac{2u^2}{s^2}-\frac{2u^2}{s}+\frac{2u^2}{s} \\ &= (u+1-s^2u^2-su^2)\left(\frac{2}{s^3}+\frac{2u^2}{s}\right).\end{aligned}$$

By simplification, we obtain

$$U(s,u)=\frac{2}{s^3}+\frac{2u^2}{s}.$$

Taking inverse of the double Laplace–Sumudu transform

$$u(x,t)=\mathscr{L}_x^{-1}\mathscr{S}_t^{-1}\{U(s,u)\}=\mathscr{L}_x^{-1}\mathscr{S}_t^{-1}\left\{\frac{2}{s^3}+\frac{2u^2}{s}\right\},$$

thus, the solution of the equation is

$$u(x,t)=x^2+t^2.$$

This result is presented in the following reference [9].

Example 6: The Heat Equation

The heat equation is used to describe how heat is transferred and distributed within a given medium over time, where heat flows from hotter regions to colder ones until thermal equilibrium is reached. In this example, we study a linear and non-homogeneous heat equation with initial and boundary conditions, using the double Laplace–Sumudu transform method in order to obtain an exact and concise solution of the equation, which highlights the effectiveness of this method.

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\sin x, \tag{4.19}$$

with initial and boundary conditions

$$u(0,t)=e^{-t}, \quad u(x,0)=\cos x, \quad u_x(0,t)=1-e^{-t}. \tag{4.20}$$

We apply the double Laplace–Sumudu transform to (4.19), we obtain

$$\mathscr{L}_x\mathscr{S}_t\left\{\frac{\partial u}{\partial t}\right\}=\mathscr{L}_x\mathscr{S}_t\left\{\frac{\partial^2 u}{\partial x^2}\right\}+\mathscr{L}_x\mathscr{S}_t\{\sin x\}, \tag{4.21}$$

let $\mathscr{L}_x\mathscr{S}_t\{u(x,t)\} = U(s,u)$, and from (3.12) and (3.13), we obtain

$$\frac{1}{u}U(s,u) - \frac{1}{u}\mathscr{L}\{u(x,0)\} = s^2U(s,u) - s\mathscr{S}\{u(0,t)\} - \mathscr{S}\{u_x(0,t)\} + \frac{1}{s^2+1}. \tag{4.22}$$

Substituting the initial and boundary conditions

$$\mathscr{L}\{u(x,0)\} = \mathscr{L}\{\cos x\} = \frac{s}{s^2+1}, \quad \mathscr{S}\{u(0,t)\} = \mathscr{S}\{e^{-t}\} = \frac{1}{1+u},$$

$$\mathscr{S}\{u_x(0,t)\} = \mathscr{S}\{1-e^{-t}\} = 1 - \frac{1}{1+u},$$

from equation (4.22), we get

$$\frac{1}{u}U(s,u) - \frac{s}{u(s^2+1)} = s^2U(s,u) - \frac{s}{1+u} - \left(1 - \frac{1}{1+u}\right) + \frac{1}{s^2+1},$$

then

$$\left(\frac{1}{u} - s^2\right)U(s,u) = \frac{s}{u(s^2+1)} - \frac{s}{1+u} - \frac{u}{1+u} + \frac{1}{s^2+1},$$

$$\left(\frac{1-s^2u}{u}\right)U(s,u) = \frac{s}{u(s^2+1)} - \frac{(s+u)}{1+u} + \frac{1}{s^2+1},$$

therefore

$$\begin{aligned}(1-s^2u)U(s,u) &= \frac{s}{s^2+1} - \frac{(s+u)u}{1+u} + \frac{u}{s^2+1}\\ &= \frac{s(1+u)}{(s^2+1)(1+u)} - \frac{(su+u^2)(s^2+1)}{(1+u)(s^2+1)} + \frac{u(1+u)}{(s^2+1)(1+u)}\\ &= \frac{s(1-s^2u)}{(1+u)(s^2+1)} + \frac{u(1-s^2u)}{(1+u)(s^2+1)}.\end{aligned}$$

By simplification, we obtain

$$U(s,u) = \frac{s}{(1+u)(s^2+1)} + \frac{1}{s^2+1}\left(1 - \frac{1}{1+u}\right).$$

Taking inverse of the double Laplace–Sumudu transform

$$u(x,t) = \mathscr{L}_x^{-1}\mathscr{S}_t^{-1}\{U(s,u)\} = \mathscr{L}_x^{-1}\mathscr{S}_t^{-1}\left\{\frac{s}{(1+u)(s^2+1)} + \frac{1}{s^2+1}\left(1 - \frac{1}{1+u}\right)\right\},$$

thus, the solution of the equation

$$u(x,t) = \cos x e^{-t} + \sin x(1-e^{-t}).$$

This result is presented in the following reference [12].

Conclusion

In this work, we have presented the double Laplace–Sumudu transform method for solving linear partial differential equations. The applications presented in Chapter Four have shown that the double Laplace–Sumudu transform method is an excellent alternative compared to classical transforms in handling many partial differential equations in physics and mathematics.

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