University of Kasdi Merbah Ouargla
Faculty of Mathematics and Material Sciences
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Thesis for the master's degree in Modelling and numerical analysis

presented by: **Gormi Asmaa**

Study Existence for Fractional Differential Inclusions in Sobolev spaces

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In front of the jury composed of :

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Dedication

First, we thank God for givinig us the strength to achieve our goal and our accomplish our work.

To my mom, the apple of my eye and my first haven, your prayers were my path and this success is as much yours as it is mine.

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ملخص

في هذا العمل، ندرس وجود حلول لبعض مسائل الاحتوائية التفاضلية الكسرية في فضاءات با ناخ وسوبوليف، الخاضعة لشروط حدية وابتدائية، وذلك باستخدام استراتيجية قائمة على مبرهنات النقطة الثابتة.

أولاً، نعرض التعاريف الأساسية لحساب التفاضل والتكامل الكسري والتطبيقات متعددة القيم. ثم نبرهن نتائج الوجود الأساسية لكل من الحالتين المحدبة وغير المحدبة باستخدام مبرهنات النقطة الثابتة، مثل التناوب غير الخطي لمبرهنة ليراي شودر ومبرهنة كوفيتز - نادلير.

أخيراً، نناقش وجود ووحدانية الحلول للاحتوائية التفاضلية الكسرية تحت شروط ابتدائية في ميادين غير محدودة.

الكلمات المفتاحية:
تكامل ريمان - ليوفيل. مشتق ريمان - ليوفيل. التطبيق متعدد القيم. مسائل القيم الحدية والابتدائية. الاختيار القابل للقياس. مبرهنة النقطة الثابتة. الاحتوائية التفاضلي الكسري. الميادين غير المحدودة.

Abstract

In this work studies the existence of solution for some fractional differential inclusions problems in Banach and Sobolev spaces. Subject to boundary and initial conditions, using strategy of fixed point theorems.

First, we present the basic definitions of fractional calculus and multivalued map,then we prove main existence resulte for both convex and non-convex by using fixed point theorems like (nonlinear alternative of Leray-Schauder and Covitz-Nadler).

Finally,we discussed the existence and uniqueness of solutions for fractional differential inclusions under initial conditions in unbounded domains.

Key words: Riemann-Liouville integral, Riemann-Liouville derivative, set valued mapping, boundary and initial value problems, measurable selection, fixed point theorem, fractional differential inclusion, unbounded domains.

Résumé

Dans ce travail, nous étudions l'existence de solutions pour certains problèmes d'inclusions différentielles fractionnaires dans les espaces de Banach et de Sobolev, soumis à des conditions aux limites et initiales, en utilisant la stratégie des théorèmes du point fixe.

Tout d'abord, nous présentons les définitions de base du calcul fractionnaire et des applications multivoques. Ensuite, nous démontrons les principaux résultats d'existence pour les cas convexes et non convexes en utilisant des théorèmes du point fixe, tels que (l'alternative non linéaire de Leray-Schauder et le théorème de Covitz-Nadler).

Enfin, nous discutons de l'existence et de l'unicité des solutions pour les inclusions différentielles fractionnaires sous des conditions initiales dans des domaines non bornés.

Mots-clés: Intégrale de Riemann-Liouville, dérivée de Riemann-Liouville, application multivoque, problèmes aux limites et aux valeurs initiales, sélection mesurable, théorème du point fixe, inclusion différentielle fractionnaire, domaines non bornés.

Contents

Introduction

The field of **Fractional Calculus**, which generalizes the classical concepts of derivatives and integrals to non-integer orders $\alpha \in \mathbb{C}$ (with $\mathrm{Re}(\alpha) > 0$), is not merely a modern mathematical abstraction. Its theoretical inception can be traced back to September 1695, when **Leibniz** addressed a profound question to **L'Hôpital** regarding the possibility of a derivative of order $n = 1/2$. Leibniz's prophetic response-*"It will lead to a paradox, from which one day useful consequences will be drawn"*—laid the groundwork for a discipline that would eventually be refined by the foundational works of **Euler, Liouville, Riemann, and Caputo**.

For centuries, fractional calculus remained primarily within the realm of pure mathematics. However, in recent decades, it has emerged as a cornerstone of **Mathematical Modeling**. The fundamental advantage of fractional-order operators lies in their **non-local nature**. Unlike classical integer-order derivatives, which describe local rates of change, fractional operators possess an inherent **"Memory Effect"** and capture hereditary properties. This makes them exceptionally robust for modeling complex systems in anomalous diffusion, viscoelasticity, and biological dynamics, where the future state of a system depends not only on its current status but also on its historical evolution.

In this dissertation, we extend the study of fractional dynamics into a more general and sophisticated framework: **Fractional Differential Inclusions (FDIs)**. While classical differential equations rely on single-valued functions, differential inclusions involve **multi-valued maps** (or set-valued maps) of the form:

$$D^{\alpha}u(t) \in F(t, u(t)) \tag{1}$$

where $F : J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is a multifunction. This framework is the most rigorous mathematical tool for describing systems characterized by **uncertainty, structural discontinuities, or multi-state evolution**, where the governing laws are not uniquely determined.

The primary objective of this Master's thesis is to establish the **existence and uniqueness** of solutions for a specific class of semilinear fractional differential inclusions. Our analytical strategy involves transforming the FDI into an equivalent **integral inclusion** using the **Riemann–Liouville** fractional integral operator. By defining an appropriate fixed-point operator $\mathcal{N}$ acting on specific Banach spaces, we seek to prove the existence of solutions using two distinct mathematical scenarios:

- **The Convex Case:** Where the multifunction F takes convex values. We establish existence results by employing the **Leray–Schauder Nonlinear Alternative**, ensuring the compactness and **upper semi-continuity (u.s.c.)** of the associated operator.

- **The Non-Convex Case:** Where the convexity assumption is relaxed. In this context, we utilize the **Covitz–Nadler Fixed Point Theorem**, relying on the **Hausdorff metric** H_d to establish a multi-valued contraction condition.

Furthermore, we extend our investigation to **Fractional Sobolev Spaces** $W^{\alpha,p}$, providing a natural functional environment for handling the regularity of solutions. Additionally, we address the complexities of **unbounded domains** $[0, +\infty[$. In such cases, we introduce **weighted Banach spaces** to control the asymptotic behavior of the solutions at infinity and ensure the stability of the analytical approximations.

Chapter 1

Preliminary

1.1 Notations

Let $a, b \in \mathbb{R}$, $J = [a, b]$ and $(\mathrm{X}, \|.\|_{\mathrm{X}})$ be a Banach space.
$\mathcal{C}(J, \mathrm{X})$ the space of all continuous functions from J into X associated the norm

$$\|x\| = \max_{a \leq t \leq b} \|x(t)\|_{\mathrm{X}}.$$

$\mathrm{L}^1(J, \mathrm{X})$ the space of all Bochner integrable functions from J into X associated the norm

$$\|x\|_1 = \int_a^b \|x(t)\|_{\mathrm{X}} dt.$$

$\mathrm{L}^\infty(J, \mathrm{X})$ the space of all Bochner functions from J into X bounded for almost everywhere associated the norm

$$\|x\|_\infty = \inf\{M > 0; \|x(t)\|_{\mathrm{X}} \leq M \quad \text{a.e.on } J\}.$$

For a given Banach space X, let

$\mathcal{P}(\mathrm{X})$ be the set of all subsets of X.

$\mathcal{P}_b(\mathrm{X}) = \{Y \subseteq \mathrm{X}; Y \text{ bounded in X}\}$.

$\mathcal{P}_{cl}(\mathrm{X}) = \{Y \subseteq \mathrm{X}; Y \text{ closed in X}\}$.

$\mathcal{P}_{cp}(\mathrm{X}) = \{Y \subseteq \mathrm{X}; Y \text{ compact in X}\}$.

$\mathcal{P}_{c,cp}(\mathrm{X}) = \{Y \subseteq \mathrm{X}; Y \text{ convex and compact in X}\}$.

1.2 Set valued map

Definition 1.2.1. *[17, 27] Let* X *and* Y *two sets.*
A set valued map (multivalued) from X *into* $\mathcal{P}(\mathrm{Y})$ *is an application, i.e. each* $x \in \mathrm{X}$ *associate subset in* Y.

Definition 1.2.2. *[14] Let* $F : \mathrm{X} \to \mathcal{P}(\mathrm{Y})$ *be a set valued map.*

(a) The domain of F *is the set* $Dom(F)$ *where*

$$Dom(F) = \{x \in \mathrm{X}; F(x) \neq \emptyset\}.$$

(b) If $\mathrm{Y} = \mathrm{X}$, *the point* $x \in \mathrm{X}$ *is called a fixed point of* F *if* $x \in F(x)$.

(c) For all $M \subseteq \mathrm{X}$

$$F(M) = \bigcup_{x \in M} F(x).$$

(d) The graph of F *is defined to be the set*

$$G(F) = \{(x, y) \in \mathrm{X} \times \mathrm{Y}; y \in F(x)\}.$$

(e) For all $A \subseteq \mathrm{Y}$

$$F^{-1}(A) = \{x \in \mathrm{X}; F(x) \subseteq A\}.$$

Proposition 1.2.1. *[14] Let* $F : \mathrm{X} \to \mathcal{P}(\mathrm{Y})$ *be a set valued map,* $(A_i)_{i \in I} \subseteq \mathcal{P}(\mathrm{Y})$. *Then*

$$F^{-1}(\bigcap_{i \in I} A_i) = \bigcap_{i \in I} F^{-1}(A_i).$$

Definition 1.2.3. *[27] Let* X, Y *are topological spaces,* $F : \mathrm{X} \to \mathcal{P}(\mathrm{Y})$ *be a set valued map. We say that* F *is an upper semicontinuous if for every open set* O *in* Y *the set* $F^{-1}(\mathrm{O})$ *is an open set in* X.

Example 1. *Let* $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$ *continuous such that* $f_1 \leq f_2$, *then the set valued map* $F : \mathbb{R}^n \to \mathcal{P}(\mathbb{R})$ *defined by*

$$F(x) = \big[f_1(x), f_2(x)\big]$$

is upper semicontinuous.

Proposition 1.2.2. *[27] Let* X, Y *are two topological spaces and* $\mathrm{A} \subseteq \mathrm{X}$, $F : A \to \mathcal{P}(\mathrm{Y})$ *be a set valued map.*
If F *is upper semicontinuous, then for every* $x_0 \in \mathrm{A}$ *and* $\varepsilon > 0$ *there exist* $\delta > 0$ *such that*

$$F(x) \subseteq F(x_0) + B_\varepsilon(0) \quad \textit{on } \mathrm{A} \cap B_\delta(x_0),$$

where $B_\delta(x_0)$ *the ball with centered at a point* x_0 *and the radius* δ*. The converse is true if* F *has a compact values.*

Definition 1.2.4. *[7] Let* X, Y *are two normed spaces,* $F : \mathrm{X} \to \mathcal{P}(\mathrm{Y})$ *be a set valued map. We say that* F *is Lipschitzean if there exist* $L > 0$ *such that*

$$F(x) \subseteq F(y) + B(0, L\|x - y\|) \quad \textit{for each } x, y \in \mathrm{X}.$$

If the constant $L < 1$*, we say that the set valued map* F *is contraction.*

Proposition 1.2.3. *[7] Let* X, Y *are two normed spaces,* $F : \mathrm{X} \to \mathcal{P}(\mathrm{Y})$ *be a set valued map with compact values.*
If F *is Lipschitzean then* F *is upper semicontinuous.*

Proof. Let $x_0 \in \mathrm{X}$ and $\varepsilon > 0$, then there exist $\delta = \frac{\varepsilon}{L}$ such that

$$F(x) \subseteq F(x_0) + B_\varepsilon(0) \quad \text{for all} \quad x \in B_\delta(x_0).$$

$\square$

Proposition 1.2.4. *[14] Let* X, Y *are two metric spaces and* $F : \mathrm{X} \to \mathcal{P}(\mathrm{Y})$ *be a set valued map.*
F *has a closed graph if and only if for each* $(x_n) \subseteq \mathrm{X}$ *converge to* $x \in \mathrm{X}$ *and for each* (y_n) *such that* $y_n \in F(x_n)$ *converge to* y *we have* $y \in F(x)$.

Definition 1.2.5. *[14] Let* X, Y *be two topological spaces and* $F : \mathrm{X} \to \mathcal{P}(\mathrm{Y})$ *be a set valued map.*

(i) F *is called completely continuous if for each* $B \in \mathcal{P}_b(\mathrm{X})$ *the set* $F(B)$ *is relatively compact in* Y.

(ii) F *is called compact if the set* $F(\mathrm{X})$ *is relatively compact in* Y.

Proposition 1.2.5. *[25] Let $F : \mathrm{X} \to \mathcal{P}_{cp}(\mathrm{Y})$ be a set valued map such that F is completely continuous. Then F is upper semicontinuous if and only if F has a closed graph.*

Definition 1.2.6. *[9] Let (X, d) be a metric space and $A, B \in \mathcal{P}_{cl}(\mathrm{X})$.*
The Hausdorff metric is an application $H_d : \mathcal{P}_{cl}(\mathrm{X}) \times \mathcal{P}_{cl}(\mathrm{X}) \to \mathbb{R}_+ \cup \{+\infty\}$ defined by

$$H_d(A, B) = \begin{cases} \max\{\sup_{a\in A} d(a, B); \sup_{b\in B} d(A, b)\} & \textit{if} \quad \sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b) \in \mathbb{R}_+, \\ +\infty & \textit{else}. \end{cases} \tag{1.1}$$

Proposition 1.2.6. *If $A, B \in \mathcal{P}_{cp}(\mathrm{X})$, then $H_d(A, B) \in \mathbb{R}_+$.*

Proof. For all $A, B \in \mathcal{P}_{cp}(X)$, the application $a \in A \mapsto d(a, B)$ is continuous. Then by theorem 1.13-6 in [15] there exist $a_0 \in A$ and $b_0 \in B$ such that

$$\sup_{a\in A} d(a, B) = d(a_0, B) = d(a_0, b_0) \in \mathbb{R}_+.$$

Then $H_d(A, B) \in \mathbb{R}_+$. □

Proposition 1.2.7. *[7] Let X, Y are two normed spaces, and $F : \mathrm{X} \to \mathcal{P}_{cp}(\mathrm{Y})$. Then F is Lipschitzean if and only if there exist $L > 0$ such that for all $x, y \in \mathrm{X}$ we have*

$$H_d(F(x), F(y)) \leq Ld(x, y).$$

Proof. Let $x, y \in \mathrm{X}$ and F is Lipschitzean set valued map, then for each $a \in F(x)$ we have

$$d(a, F(y)) = \|a - b\| \leq L\|x - y\| \quad \textit{where } b \in F(y)$$

then

$$H_d(F(x), F(y)) \leq L\|x - y\|. \tag{1.2}$$

Conversely if F verify (1.2), then for all $a \in F(x)$ we have

$$d(a, F(y)) \leq H_d(F(x), F(y)) \leq L\|x - y\|.$$

After that there exist $b_0 \in F(y)$ such that

$$d(a, F(y)) = \|a - b_0\| \leq L\|x - y\|.$$

That means $a \in F(y) + B(0, L\|x - y\|)$, i.e. F is Lipschitzean. □

1.3 Fixed point theorem

Theorem 1.3.1. *[16] (Covitz-Nadler). Let* (X, d) *be a complete metric space. If* $T : \mathrm{X} \to \mathcal{P}_{cl}(\mathrm{X})$ *is a contraction, then* T *admits at least a fixed point, i.e., there exists* $u \in \mathrm{X}$ *such that* $u \in T(u)$*.*

Theorem 1.3.2. *[23] (nonlinear alternative of Leray-Schauder). Let* X *be a Banach space,* C *a closed convex subset of* X*,* U *an open subset of* C *with* $0 \in U$*. Suppose that* $T : \bar{U} \to \mathcal{P}_{cp,c}(C)$ *is an upper semicontinuous compact map. Then either* T *has a fixed point in* $\bar{U}$*, or there exist* $u \in \partial U$ *and* $s \in (0, 1)$ *such that* $u \in sT(u)$*.*

Theorem 1.3.3. *[34](Kakutani) Let* X *be a Banach space. Suppose that*

(i) the set valued map $T : K \subseteq \mathrm{X} \to \mathcal{P}(K)$*is upper semicontinuous,*

(ii) K *is a nonempty, compact, convex set in* X*,*

(iii) the set $T(u)$ *is closed, and convex for all* $u \in K$*.*

Then T *has a fixed point in* K*.*

1.4 Measurable multifunctions and measurable selection

Definition 1.4.1. *Let* $f : [a, b] \to \mathbb{R}$*. We say that* f *is measurable if for each open set* U *in* $\mathbb{R}$*, we have [20].*

$$f^{-1}(U) \in \mathcal{M}$$

where $\mathcal{M}$ *be a* $\sigma-$*algebra.*

Proposition 1.4.1. *[14]. Let* (t, A) *be a measurable space and* X *be a separable Banach space.*$g : t \to \mathbb{R}_+$ *be a measurable function. Then the set valued map* $t \in t \mapsto B(f(t), g(t))$ *(the closed ball with center* $f(t)$ *and radius* $g(t)$*) is a measurable.*

Proposition 1.4.2. *[14]. Let* $F_1, F_2 : t \to \mathrm{Y}$ *are set valued map with compact valued . Then the set valued map* $t \in t \mapsto F_1(t) \cap F_2(t)$ *is a measurable.*

Proposition 1.4.3. *[14]. Let $F_1, F_2 : t \to \mathrm{Y}$ are set valued map with compact valued. Then the set valued map $t \in t \mapsto F_1(t) \cup F_2(t)$ is a measurable.*

Definition 1.4.2. *[27] Let $a, b \in \mathbb{R}$ and X, Y be two separable Banach spaces.*
A set valued map $F : [a, b] \times \mathrm{X} \to \mathcal{P}_{cp}(\mathrm{Y})$ is said to be Carathéodory if

(i) for every $x \in \mathrm{X}$ the set valued map $F(., x) : [a, b] \to \mathcal{P}(\mathrm{Y})$ is measurable,

(ii) for almost $t \in [a, b]$ the set valued map $F(t, .) : \mathrm{X} \to \mathcal{P}(\mathrm{Y})$ is upper semicontinuous.

Definition 1.4.3. *[27] A Carathéodory set valued map $F : [a, b] \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R})$ is said L^1-Carathéodory if*

(i) F is Carathéodory set valued map.

(ii) for each $l > 0$ there exist $\chi_l \in \mathrm{L}^1([a, b]; \mathbb{R})$ such that

$$\|F(t, x)\| \underset{def}{=} \sup\{v; v \in F(t, x)\} \leq \chi_l(t)$$

for all $|x_i| \leq l$ $(1 \leq i \leq n)$ and almost $t \in [0, 1]$.

Definition 1.4.4. *[27] Let $a, b \in \mathbb{R}$ and Y be a Banach space.*
A function $f : [a, b] \to \mathrm{Y}$ is said to be a measurable selection of a set valued map $F : [a, b] \to \mathcal{P}_{cp}(\mathrm{Y})$ if f is measurable and

$$f(t) \in F(t) \quad a.e.\ t \in [a, b].$$

Lemma 1.4.1. *[25] Let J be a subset in $\mathbb{R}^n$, Y is a complet separable space and $F : J \to \mathcal{P}_{cl}(\mathrm{Y})$ be a measurable set valued map. Then F admits a measurable selection.*

Proposition 1.4.4. *[27]. Let $F : [0, 1] \times \mathrm{X}_0 \to \mathcal{P}_{cp}(\mathrm{X})$ be a Carathéodory set valued map and*
$f : [0, 1] \to \mathrm{X}_0$ be a measurable function. Then the set valued map $t \in [0, 1] \mapsto F(t, f(t))$ is measurable.

Lemma 1.4.2. *[13] Let $G : [0, 1] \times \mathbb{R}^n \to \mathcal{P}_{c,cp}(\mathbb{R})$ be an L^1- Carathéodory set valued map and let a linear continuous theta from $\mathrm{L}^1([0, 1]; \mathbb{R})$ to $\mathcal{C}([0, 1]; \mathbb{R})$. Then the operator $\theta \circ S_G$ defined by $\theta \circ S_G(u) = \theta(S_{G,u})$ has a closed graph, where*

$$S_{G,u} = \big\{v \in \mathrm{L}^1([0, 1]; \mathbb{R} : v(t) \in G(t, u(t)) \text{ for almost } t \in [0, 1])\big\}.$$

Definition 1.4.5. *[14] Let $G : [0,1] \times \mathbb{R}^n \to \mathcal{P}_{c,cp}(\mathbb{R})$, we say that G is integrably bounded if there exist $\phi \in \mathrm{L}^1([0,1];\mathbb{R})$ such that for any $v \in S_{G,u}$ we have*

$$|v(t)| \leq \phi(t) \quad \textit{for almost } t \in [0,1]$$

Lemma 1.4.3. *[36] Let $F : [0,1] \to \mathcal{P}(\mathbb{R})$ be a measurable set valued map and $x : [0,1] \to \mathbb{R}$ a measurable function. Then for every measurable function $y : [0,1] \to \mathbb{R}^+$ there exist a function v such that*

$$\begin{aligned} v(t) &\in F(t) \quad \textit{for all} \quad t \in [0,1], \\ |v(t) - x(t)| &\leq d(x(t), F(t)) + y(t) \quad \textit{for almost} \quad t \in [0,1]. \end{aligned}$$

1.5 Riemann-Liouville integral and derivative

Definition 1.5.1. *[31] The Euler γ function noted by $\Gamma(.)$ is defined by*

$$\Gamma(t) = \int_0^{+\infty} s^{t-1} \exp(-s) ds, \qquad t > 0$$

Remark 1.5.1. *[31] For all $t > 0$ we have $t\Gamma(t) = \Gamma(t+1)$.*

Definition 1.5.2. *[31, 32] Let $\rho > 0$ and $\varphi \in \mathrm{L}^1([0,1];\mathbb{R})$. The following integral*

$$I_{0^+}^{\rho} \varphi(t) = \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} \varphi(s) ds,$$

is named the Riemann–Liouville integral of order ρ.

Lemma 1.5.1. *[28] The fractional integration operators $I_{0^+}^{\rho}$ with $\rho > 0$ are bounded from $\mathcal{C}([0,1];\mathbb{R})$ into $\mathcal{C}([0,1];\mathbb{R})$, and for each $f \in \mathcal{C}([0,1];\mathbb{R})$ we have*

$$\|I_{0^+}^{\rho} f\| \leq \frac{1}{\Gamma(\rho+1)} \|f\|.$$

Proof. For any $f \in \mathcal{C}([0,1];\mathbb{R})$ we have

$$\begin{aligned} |I_{0^+}^{\rho} f(t)| &= \frac{1}{\Gamma(\rho)} \left| \int_0^t (t-s)^{\rho-1} f(s) ds \right| \\ &\leq \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} |f(s)| ds \\ &\leq \frac{\|f\|}{\Gamma(\rho)} \times \int_0^t (t-s)^{\rho-1} ds = \frac{\|f\|}{\Gamma(\rho)} \times \frac{t^{\rho}}{\rho}, \end{aligned}$$

then

$$\|I^{\rho}_{0+}f\| \leq \frac{\|f\|}{\Gamma(\rho+1)},$$

□

Definition 1.5.3. *[31, 32] Let φ a function given on the interval $[0,1]$. The fractional derivative of Riemann-Liouville of order $\beta > 0$ is defined by*

$$\begin{aligned}\mathcal{D}^{\beta}_{0+}\varphi(t) &= \frac{1}{\Gamma(n-\beta)}\Big(\frac{d}{dt}\Big)^{n}\int_0^t (t-s)^{n-\beta-1}\varphi(s)ds,\\ &= \Big(\frac{d}{dt}\Big)^{n} I^{n-\beta}_{0+}\varphi(t),\end{aligned}$$

where $n = [\rho]+1$ and $[\rho]$ denotes the greatest integer number less than ρ.

Corollary 1. *[31] Let $\rho > 0$ and $n = [\rho]+1$.*
The equality $\mathcal{D}^{\rho}_{0+}\varphi(t) = 0$ is valid if and only if

$$\varphi(t) = \sum_{j=1}^{n} c_j t^{\rho-j}$$

Lemma 1.5.2. *[28] Let $u \in \mathrm{L}^1(0,1)$ and $\delta_1 > \delta_2 > 0$. Then,*

(i) $I^{\delta_1}_{0+}I^{\delta_2}_{0+}u(t) = I^{\delta_1+\delta_2}_{0+}u(t)$,

(ii) $\mathcal{D}^{\delta_2}_{0+}I^{\delta_1}_{0+}u(t) = I^{\delta_1-\delta_2}_{0+}u(t)$,

(iii) $\mathcal{D}^{\delta_1}_{0+}I^{\delta_1}_{0+}u(t) = u(t)$.

Lemma 1.5.3. *[28] For a given $f \in \mathrm{L}^1([0,1];\mathbb{R})$ the solution of*

$$\mathcal{D}^{\rho}_{0+}\varphi(t) = f(t),$$

is given by

$$\varphi(t) = I^{\rho}_{0+}f(t) + \sum_{j=1}^{n} c_j t^{\rho-j},$$

where $n = [\rho]+1$.

Proof. From precedent lemma1.5.3, we have $f(t) = \mathcal{D}^{\rho}_{0+}I^{\rho}_{0+}f(t)$, then writing

$$\mathcal{D}^{\rho}_{0+}\varphi(t) = f(t)$$

is equivalent to

$$\mathcal{D}^{\rho}_{0+}(\varphi(t) - I^{\rho}_{0+}f(t)) = 0.$$

Then

$$\varphi(t) = I^{\rho}_{0+}f(t) + \sum_{j=1}^{n} c_j t^{\rho-j}.$$

□

Lemma 1.5.4. *[28] If $\rho > 0$ and $\alpha > 0$, then*

(i) $\mathcal{D}^{\rho}_{0+}t^{\alpha-1} = \begin{cases} \frac{\Gamma(\alpha)}{\Gamma(\alpha-\rho)}t^{\alpha-\rho-1}, & if \quad \rho-\alpha<0 \\ 0, \quad if \quad \alpha-\rho \in \mathbb{Z}^{-} \end{cases}$

(ii) $I^{\rho}_{0+}t^{\alpha} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\rho+1)}t^{\alpha+\rho}$.

1.6 Some properties of functional analysis

Lemma 1.6.1. *[13] Let K be a compact set in a Banach space $(\mathrm{X}, \|.\|)$. Then for each $x \in \mathrm{X}$ there exist $y_0 \in K$ such that*

$$d(x, K) = \inf_{y\in K} \|x - y\| = \|x - y_0\|.$$

Proof. Let $x \in \mathrm{X}$, by definition of inf, for each $n \geq 1$ there exist $y_n \in K$ such that

$$\inf_{y\in K} \|x - y\| \leq \|x - y_n\| < \inf_{y\in K} \|x - y\| + \frac{1}{n}$$

and by definition of the compact set, there exist a subsequence $(y_{n_k}) \in K$ such that (y_{n_k}) converge to $y_0 \in K$ and

$$\inf_{y\in K} \|x - y\| \leq \|x - y_{n_k}\| < \inf_{y\in K} \|x - y\| + \frac{1}{n_k},$$

that means

$$\inf_{y\in K} \|x - y\| = \|x - y_0\|.$$

□

Theorem 1.6.1. *[13]. Let A be a bounded set in $\mathrm{L}^1(\mathbb{R})$. Assume that*

$$\lim_{h\to 0}\|\tau_h f - f\|_1 = 0 \quad \textit{for each} \quad f \in A.$$

Then the closure of $A_{|[0,1]}$ in $\mathrm{L}^1([0,1])$ is compact in $\mathrm{L}^1([0,1])$, where $\tau_h f(t) = f(t+h)$.

Theorem 1.6.2. *[13]. Let (f_n) be a sequence in $\mathrm{L}^1([0,1])$ and let $f \in \mathrm{L}^1([0,1])$ be such that*

$$\|f_n - f\|_1 \to 0.$$

Then, there exist a subsequence (f_{n_k}) such that

$$f_{n_k}(t) \to f(t) \quad \textit{a.e. on } [0,1].$$

Definition 1.6.1. *[15] Let X be a Banach space, and let $\mathcal{A}$ be a subset in X, we called $\mathcal{A}$ is weakly compact if $\mathcal{A}$ compact set in weak topology $\sigma(\mathrm{X},\mathrm{X}')$.*

Theorem 1.6.3. *(Dunford - Pettis)[13] Let $\mathcal{A}$ be a bounded set in $\mathrm{L}^1([0,\infty[,\mathbb{R})$. Then $\mathcal{A}$ is weakly compact in $\mathrm{L}^1([0,\infty[,\mathbb{R})$ if and only if*

(i) for all $\varepsilon > 0$ there exist $\delta > 0$ such that for all $f \in \mathcal{A}$

$$\int_A |f| < \varepsilon \quad \textit{for } |A| < \delta,$$

(ii) for all $\varepsilon > 0$ there exist $A \subset [0,\infty[$ measurable such that for all $f \in \mathcal{A}$

$$|A| < \infty \quad \textit{and} \quad \int_{[0,\infty[-A} |f| < \varepsilon.$$

Theorem 1.6.4. *(Ascoli- Arzelà) [13]. Let K be a bounded subset of $\mathcal{C}([a,b],\mathbb{R})$ and equicontinuous set. Then the closure of K is compact in $\mathcal{C}([a,b],\mathbb{R})$.*

Proposition 1.6.1. *[15] Let X be a Banach space. A set $\mathcal{A}$ is weakly compact if and only if for any sequence (x_n) in $\mathcal{A}$ contains a subsequence (x_{n_k}) that weakly converge in X.*

Proposition 1.6.2. *[27]. Let $\mathcal{A} \subset \mathrm{L}^1([0,1];\mathbb{R})$ such that*

(i) $\mathcal{A}(t)$ are relatively compact for a.e $t \in [0,1]$.

(ii) There exist $h \in \mathrm{L}^1([0,1];\mathbb{R})$ such that

$$v(t) \le h(t) \qquad \textit{for } v \in \mathcal{A} \textit{ and almost } t \in [0,1].$$

Then $\mathcal{A}$ is weakly compact in $\mathrm{L}^1([0,1];\mathbb{R})$.

Chapter 2

Existence results for fractional differential inclusions in the Banach space

2.1 Existence results in the Banach space :

Let the following problem:

$$\begin{cases} D_0^{\rho}u(t) \in F(t, u(t), u'(t)), \quad a.e\ t \in [0,1] \\ u(0) = 0, \\ u'(1) = 0. \end{cases} \tag{2.1}$$

Where $1 < \rho < 2$, $F : [0,1] \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R})$ is a multi-valued and u is a function defined on $[0,1]$.

Lemma 1.1. For a given $v \in L^1([0,1], \mathbb{R})$, a function u is a solution to

$$\begin{cases} D_0^{\rho}u(t) = v(t), \quad a.e\ t \in [0,1] \\ u(0) = 0, \\ u'(1) = 0, \end{cases} \tag{2.2}$$

if and only if

$$u(t) = \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} v(s)\, ds - \frac{t^{\rho-1}}{(\rho-1)\Gamma(\rho-1)} \int_0^1 (1-s)^{\rho-2} v(s)\, ds.$$

Proof: Researching the equation integral

$$D_0^\rho u(t) = v(t), \qquad 1 < \rho < 2$$

$$u(t) = I_0^\rho v(t) + C_1 t^{\rho-1} + C_2 t^{\rho-2}$$

By boundary condition $u(0) = 0$ we obtain $C_2 = 0$

$$u(t) = I_0^\rho v(t) + C_1 t^{\rho-1},$$

$$\frac{d}{dt}(I_0^\rho v(t)) = I_0^{\rho-1} v(t),$$

$$\frac{d}{dt}\left(\frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} v(s) ds \right) = \frac{1}{\Gamma(\rho-1)} \int_0^t (t-s)^{\rho-2} v(s) ds,$$

$$\frac{d}{dt}(C_1 t^{\rho-1}) = C_1(\rho-1)t^{\rho-2}.$$

So

$$u'(t) = \frac{1}{\Gamma(\rho-1)} \int_0^t (t-s)^{\rho-2} v(s) ds + C_1(\rho-1)t^{\rho-2},$$

$$u'(1) = 0,$$

$$0 = \frac{1}{\Gamma(\rho-1)} \int_0^1 (1-s)^{\rho-2} v(s) ds + C_1(\rho-1).$$

Hence

$$C_1 = -\frac{1}{(\rho-1)\Gamma(\rho-1)} \int_0^1 (1-s)^{\rho-2} v(s) ds.$$

So

$$u(t) = \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} v(s) ds - \frac{t^{\rho-1}}{(\rho-1)\Gamma(\rho-1)} \int_0^1 (1-s)^{\rho-2} v(s) ds.$$

Let $X = \{u \in \mathcal{C}([0,1]), D^{\beta}_{0^+}u(t) \in \mathcal{C}([0,1]) \quad \text{with} \quad 0 < \beta < \rho - 1\}$ equipped with the norm

$$\|u\|_X = \|u\|_\infty + \|D^{\beta}_{0^+}u\|_\infty \tag{2.3}$$

Lemma 2.1.1. *The space X is Banach space.*

Theorem 2.1.1. *[22] Let K be a bounded set in X and equicontinuous i.e.,*

$$\forall \epsilon > 0, \exists \delta > 0, \forall f \in K, \forall x, y \in \quad [0,1] \qquad |x-y| < \delta$$
$$\Longrightarrow |f(x) - f(y)| + |D^{\beta}_{0^+}f(x) - D^{\beta_{0^+}}f(y)| < \epsilon.$$

Then $\bar{K}$ is compact in X.

Let the set

$$S_{F,u} = \left\{ v \in L^1([0,1], \mathbb{R}) \mid v(t) \in F(t, u(t), u'(t)) \text{ a.e. } t \in [0,1] \right\}.$$

2.2 The case where F is convex

Theorem 2.2.1. *Assume that the following hypothesis (A1) to (A3) are met.*

(A1) $F : [0,1] \times \mathbb{R}^2 \to \mathcal{P}_{c,cp}(\mathbb{R})$ *is Carathéodory multifunction.*

(A2) $\|F(t,x,y)\| = \sup\{|v|; v \in F(t,x,y)\} \leq |x|$.

(A3) $\frac{1}{(\rho-\beta)\Gamma(\rho-\beta)} + \frac{\Gamma(\rho-\beta)}{(\rho-1)\Gamma(\rho-1)\Gamma(\rho-1)} < 1.$

Then there exist at least one solution to problem (2.1) *in X.*

Proof:

Define the multivalued $N : X \to \mathcal{P}(X)$ by

$$N(u) = \left\{ f \in X \mid f(t) = \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} v(s)\,ds - \frac{t^{\rho-1}}{(\rho-1)\Gamma(\rho-1)} \int_0^1 (1-s)^{\rho-2} v(s)\,ds, v \in S_{F,u} \right\}$$

Step 1. ($N(u)$ is convex) From $(A1)$ the set $S_{F,u}$ is non-empty set.

For prove the set $N(u)$ is convex set, let $f_i \in N(u)$ then there exist $v_i \in S_{F,u}$ such that

$$f_i(t) = \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} v_i(s)\,ds - \frac{t^{\rho-1}}{(\rho-1)\Gamma(\rho-1)} \int_0^1 (1-s)^{\rho-2} v_i(s)\,ds, \quad i = \overline{1,2}$$

$$
\begin{aligned}
(\alpha f_1+(1-\alpha)f_2)(t) &= \frac{1}{\Gamma(\rho)}\int_0^t (t-s)^{\rho-1}\alpha v_1(s)\,ds - \frac{t^{\rho-1}}{(\rho-1)\Gamma(\rho-1)}\int_0^1 (1-s)^{\rho-2}\alpha v_1(s)\,ds \\
&+\frac{1}{\Gamma(\rho)}\int_0^t (t-s)^{\rho-1}(1-\alpha)v_2(s)\,ds \\
&-\frac{t^{\rho-1}}{(\rho-1)\Gamma(\rho-1)}\int_0^1 (1-s)^{\rho-2}(1-\alpha)v_2(s)\,ds \\
&=\frac{1}{\Gamma(\rho)}\int_0^t (t-s)^{\rho-1}(\alpha v_1(s)+(1-\alpha)v_2(s))\,ds \\
&-\frac{t^{\rho-1}}{(\rho-1)\Gamma(\rho-1)}\int_0^1 (1-s)^{\rho-2}(\alpha v_1(s)+(1-\alpha)v_2(s))\,ds.
\end{aligned}
$$

Since F has a convex values, we have $\alpha v_1+(1-\alpha)v_2 \in S_{F,u}$,
that means $\alpha f_1+(1-\alpha)f_2 \in N(u)$.

Step 2. (N maps bounded sets into bounded sets in X.) Let $B_{\mathrm{r}} = \{u \in X; \|u\|_X < \mathrm{r}\}$. We necessitate an alternative approach to establish the existence of a constant $l>0$ such that for every $u \in B_{\mathrm{r}}$, one has $\|f\|_1 \le l$ for $f \in N(u)$.
Let $u \in B_r$ and $h \in N(u)$, then there exists $v \in S_{F,u}$ with

$$
f(t) \;=\; \frac{1}{\Gamma(\rho)}\int_0^t (t-s)^{\rho-1}v(s)\,ds - \frac{t^{\rho-1}}{(\rho-1)\Gamma(\rho-1)}\int_0^1 (1-s)^{\rho-2}v(s)\,ds.
$$

By simple calculation, we have

$$
|f(t)| \le \frac{r}{\rho\Gamma(\rho)} + \frac{r}{(\rho-1)^2\Gamma(\rho-1)} \quad \text{and } |D^{\rho}_{0^+}f(t)| \le \frac{r}{(\rho-1)^2\Gamma(\rho-1)} + \frac{r}{(\rho-1)^2\Gamma(\rho-1)},
$$

then

$$
\|f\| \le \frac{r}{\rho\Gamma(\rho)} + \frac{r}{(\rho-1)^2\Gamma(\rho-1)} + \frac{r}{(\rho-1)^2\Gamma(\rho-1)} + \frac{r}{(\rho-1)^2\Gamma(\rho-1)} \le l.
$$

$$
\|f\| \le \frac{r}{\rho\Gamma(\rho)} + \frac{3r}{(\rho-1)^2\Gamma(\rho-1)} \le l.
$$

Step 3: ($N(B_r)$ is equicontinuous)

Let B_r be a bounded set in X. Let $0 \le t \le k \le 1$ and $u \in B_r$. For each $f \in N(u)$, there exists $v \in S_{F,u}$ such that:

$$
\begin{aligned}
|f(t)-f(k)| &= \left| \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} v(s)\,ds - \frac{t^{\rho-1}}{(\rho-1)\Gamma(\rho-1)} \int_0^1 (1-s)^{\rho-2} v(s)\,ds \right. \\
&\quad \left. - \frac{1}{\Gamma(\rho)} \int_0^k (k-s)^{\rho-1} v(s)\,ds + \frac{k^{\rho-1}}{(\rho-1)\Gamma(\rho-1)} \int_0^1 (1-s)^{\rho-2} v(s)\,ds \right| \\
&\le \left| \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} v(s)\,ds - \frac{1}{\Gamma(\rho)} \int_0^k (k-s)^{\rho-1} v(s)\,ds \right| \\
&\quad + \frac{|t^{\rho-1}-k^{\rho-1}|}{(\rho-1)\Gamma(\rho-1)} \int_0^1 (1-s)^{\rho-2} |v(s)|\,ds \\
&\le \frac{1}{\Gamma(\rho)} \left| \int_0^t (t-s)^{\rho-1} v(s)\,ds - \int_0^k (k-s)^{\rho-1} v(s)\,ds \right| \\
&\quad + \frac{1}{\Gamma(\rho)} \left| \int_0^t (k-s)^{\rho-1} v(s)\,ds - \int_0^t (k-s)^{\rho-1} v(s)\,ds \right| \text{ (Adding terms)} \\
&\le \frac{1}{\Gamma(\rho)} \int_0^t |(t-s)^{\rho-1} - (k-s)^{\rho-1}| |v(s)|\,ds + \frac{1}{\Gamma(\rho)} \int_t^k |(k-s)^{\rho-1}| |v(s)|\,ds.
\end{aligned}
$$

So

$$
|f(t)-f(k)| \to 0 \ \text{ as } \ t \to k, \quad \text{and we use same method for show that}
$$

$$
|D^{\rho}_{0^+} f(t) - D^{\rho}_{0^+} f(k)| \to 0 \ \text{ as } \ t \to k.
$$

Step 4: (N has a closed graph)

Let (x_n, f_n) be a sequence such that $x_n \to x$ and $f_n \in N(x_n)$ with $f_n \to f$. We must show $f \in N(x)$.

Since $f_n \in N(u_n)$, there exists $v_n \in S_{F,u_n}$ such that:

$$
f_n(t) = \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} v_n(s)\,ds - \frac{t^{\rho-1}}{(\rho-1)\Gamma(\rho-1)} \int_0^1 (1-s)^{\rho-2} v_n(s)\,ds
$$

Passing to the limit as $n \to \infty$:

$$
\implies f(t) = \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} v(s)\,ds - \frac{t^{\rho-1}}{(\rho-1)\Gamma(\rho-1)} \int_0^1 (1-s)^{\rho-2} v(s)\,ds
$$

Where $v(s) \in F(s, x(s), x'(s))$ almost everywhere.

Since F is upper semi-continuous, then $\forall \varepsilon > 0, \quad \exists n_0 \in N, \quad \forall n \ge n_0$

$$F(t, x_n(t), x_n'(t)) \subseteq F(t, x(t), x'(t)) + B_\varepsilon(0)$$

,that means

$$v_n(s) \in F(s, x(s), x'(s)) + B_\varepsilon(0),$$

there exists $v \in S_{F,x}$ such that

$$v_n(s) - v(s) \in B_\varepsilon(0) \;\text{ i.e., }\; \|v_n - v\| < \varepsilon.$$

By simple calculation, we can easily to see

$$f(t) - \Big(\frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} v(s)\, ds - \frac{t^{\rho-1}}{(\rho-1)\Gamma(\rho-1)} \int_0^1 (1-s)^{\rho-2} v(s)\Big) \to 0.$$

Step 5: (Priory poinds of solution)

By Leray-Schauder fixed point theorem 1.3.2, we have $u \in sN(u) \quad (0 \leq s < 1)$

or $u \in N(u)$.

Let $C = \{u \in X \mid \|u\|_X < r\}$, and let $s \in]0, 1[$ such that $u \in sN(u)$

$$\begin{aligned}
u(t) \;&=\; \frac{s}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} v(s) ds - s\frac{t^{\rho-1}}{(\rho-1)\Gamma(\rho-1)} \int_0^1 (1-s)^{\rho-2} v(s) ds \\
&|u(t)| \leq \left| \frac{s}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} v(s) ds - s\frac{t^{\rho-1}}{(\rho-1)\Gamma(\rho-1)} \int_0^1 (1-s)^{\rho-2} v(s) ds \right| \\
&\leq \frac{s}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} |v(s)| ds + s\frac{t^{\rho-1}}{(\rho-1)\Gamma(\rho-1)} \int_0^1 (1-s)^{\rho-2} (|v(s)|) ds \\
&\leq \frac{s\|u\|_X}{\rho\Gamma(\rho)} + \frac{s\|u\|_X}{(\rho-1)\Gamma(\rho-1)} \\
&|D_{0^+}^\beta u(t)| \leq \left| \frac{s}{\Gamma(\rho-\beta)} \int_0^t (t-s)^{\rho-\beta-1} v(s) ds - \frac{s\Gamma(\rho-\beta) t^{\rho-\beta-1}}{\Gamma(\rho-\beta)\Gamma(\rho-1)} \int_0^1 (1-s)^{\rho-2} v(s) ds \right| \\
&\leq \frac{s\|u\|_X}{(\rho-\beta)\Gamma(\rho-\beta)} + \frac{s\Gamma(\rho-\beta)\|u\|_X}{(\rho-1)\Gamma(\rho-1)\Gamma(\rho-1)}
\end{aligned}$$

From $(A3)$ and Leray-Schauder theorem 1.3.2 we have $u \in N(u)$. That means N has a fixed point theorem which represent a solution.

2.3 The case where F is nonconvex

Theorem 2.3.1. *Assume that the following hyopothesis* $(B1) - (B3)$

(B1) The set valued map $F:[0,1]\times\mathbb{R}^2\to\mathcal{P}_{cp}(\mathbb{R})$

(B2) There exists γ *>0 such that, For all* t *and* $x_1,x_2,y_1,y_2\in\mathbb{R}$*, we have :*

$$|F(t,x_1,x_2)-F(t,y_1,y_2)|\leq\gamma|x_1-y_1|.$$

(B3)

$$\frac{\gamma}{\rho\Gamma(\rho)}+\frac{\gamma}{(\rho-1)\Gamma(\rho)}<1$$

Our existence result is based on the **Covitz-nadler** fixed point theorem for multi-valued contractions. Since the problem is formulated as a differential inclusion, this theorem provides the necessary framework to ensure the existence of a fixed point for the set-valued operator in the considered functional space.By utilizing the Hausdorff metric H_d, we will show that the operator satisfies the contraction conditions required by this theorem.

Step 1. ($N(y)\in\mathcal{P}_{cl}(X)$). Let (y_n) be set of element in $\mathcal{N}(y)$ with $y_n\to\tilde{y}$ in X. Then $\tilde{y}\in X$ and there exists $v_n\in S_{F,y}$ with for $t\in[0,1]$,

$$y_n=\frac{1}{\Gamma(\rho)}\int_0^t(t-s)^{\rho-1}v_n(s)ds-\frac{t^{\rho-1}}{(\rho-1)\Gamma(\rho-1)}\int_0^1(1-s)^{\rho-2}v_n(s)ds$$

Let $\mathcal{V}=\{v_n\in S_{\mathcal{F},u};n\geq 1\}$, since $\mathcal{F}$ has a compact values, and by proposition 1.6.2 there exist a subsequence to obtain that v_{n_k} converges weakly to v in $\mathrm{L}^1([0,1];\mathbb{R})$. Therefore $v\in S_{\mathcal{F},u}$ and for $t\in[0,1]$

$$\frac{1}{\Gamma(\rho)}\int_0^t(t-s)^{\rho-1}v_{n_k}(s)ds\longrightarrow\frac{1}{\Gamma(\rho)}\int_0^t(t-s)^{\rho-1}v(s)ds$$
$$\frac{t^{\rho-1}}{(\rho-1)\Gamma(\rho-1)}\int_0^1(1-s)^{\rho-2}v_{n_k}(s)ds\longrightarrow\frac{t^{\rho-1}}{(\rho-1)\Gamma(\rho-1)}\int_0^1(1-s)^{\rho-2}v(s)ds$$

therefore $v_{n_k}\longrightarrow v$ Hence $\tilde{y}\in N(y)$.

Step 2. ($N:X\longrightarrow\mathcal{P}_{cp}(X)$ is a contraction)

We prove that there exists a constant $0<\gamma<1$ such that:

$$H_d(N(y),N(\tilde{y}))\leq\gamma\|y-\tilde{y}\|_X,\quad\forall y,\tilde{y}\in X.$$

Let $y,\tilde{y}\in X$ and $f_1\in N(y)$. By the definition of the operator $\mathcal{N}$, there exists a measurable function $v_1\in S_{F,y}$ such that for each $t\in[0,1]$:

$$f_1(t)=\frac{1}{\Gamma(\rho)}\int_0^t(t-s)^{\rho-1}v_1(s)ds-\frac{t^{\rho-1}}{(\rho-1)\Gamma(\rho-1)}\int_0^1(1-s)^{\rho-2}v_1(s)ds.$$

From the Lipschitz condition on the multi-valued map F (as shown in the hand written notes), we have:

$$H_d(F(t, y(t), y'(t)), F(t, \tilde{y}(t), \tilde{y'}(t))) \leq \gamma |y(t) - \tilde{y}(t)|.$$

Therefore, for $v_1(t) \in F(t, y(t), y'(t))$, there exists $w \in F(t, \tilde{y}(t), \tilde{y'}(t))$ such that:

$$|v_1(t) - w| \leq \gamma |y(t) - \tilde{y}(t)|.$$

Let us define $v_2(t) = w(t)$ and consider the function $f_2 \in N(\tilde{y})$ defined by:

$$f_2(t) = \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} v_2(s) ds - \frac{t^{\rho-1}}{(\rho-1)\Gamma(\rho-1)} \int_0^1 (1-s)^{\rho-2} v_2(s) ds.$$

Then, for each $t \in [0, 1]$, we have:

$$\begin{aligned} |f_1(t) - f_2(t)| \leq{}& \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} |v_1(s) - v_2(s)| ds \\ &+ \frac{t^{\rho-1}}{(\rho-1)\Gamma(\rho-1)} \int_0^1 (1-s)^{\rho-2} |v_1(s) - v_2(s)| ds. \end{aligned}$$

Substituting the inequality $|v_1(s) - v_2(s)| \leq \gamma \|y - \tilde{y}\|_X$, we get:

$$|f_1(t) - f_2(t)| \leq \left(\frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} \gamma ds + \frac{1}{(\rho-1)\Gamma(\rho-1)} \int_0^1 (1-s)^{\rho-2} \gamma ds \right) \|y - \tilde{y}\|_X.$$

By taking the supremum over $t \in [0, 1]$, we obtain:

$$\|f_1 - f_2\|_X \leq \left(\frac{\gamma}{\rho\Gamma(\rho)} + \frac{\gamma}{(\rho-1)\Gamma(\rho)} \right) \|y - \tilde{y}\|_X.$$

To prove that N is a contraction, we evaluate the norm $\|f_1 - f_2\|_X$. From the previous steps and the definition of the operator, we have:

$$|f_1(t) - f_2(t)| \leq \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} |v_1(s) - v_2(s)| ds + \frac{t^{\rho-1}}{(\rho-1)\Gamma(\rho-1)} \int_0^1 (1-s)^{\rho-2} |v_1(s) - v_2(s)| ds$$

Using the Lipschitz condition $(B2)$ where $|v_1(s) - v_2(s)| \leq \gamma \|y - \tilde{y}\|_X$, we compute the integrals as follows:

1. **First Integral:**

$$\frac{\gamma}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} ds = \frac{\gamma}{\Gamma(\rho)} \left[\frac{-(t-s)^\rho}{\rho} \right]_0^t = \frac{\gamma t^\rho}{\rho\Gamma(\rho)} = \frac{\gamma t^\rho}{\Gamma(\rho+1)}$$

For $t \in [0, 1]$, the maximum value is $\frac{\gamma}{\Gamma(\rho+1)}$.

2. **Second Integral:**

$$\frac{\gamma}{(\rho-1)\Gamma(\rho-1)} \int_0^1 (1-s)^{\rho-2} ds = \frac{\gamma}{(\rho-1)\Gamma(\rho-1)} \left[\frac{-(1-s)^{\rho-1}}{\rho-1} \right]_0^1 = \frac{\gamma}{(\rho-1)\Gamma(\rho)}$$

Combining these estimates, we obtain:

$$\|f_1 - f_2\|_X \leq \left(\frac{\gamma}{\Gamma(\rho+1)} + \frac{\gamma}{(\rho-1)\Gamma(\rho)} \right) \|y - \tilde{y}\|_X$$

Thus, $H_d(N(y), N(\tilde{y})) \leq \left(\frac{\gamma}{\rho\Gamma(\rho)} + \frac{\gamma}{(\rho-1)\Gamma(\rho)} \right) \|y - \tilde{y}\|_X$, and N has a fixed point which represent a solution.

Chapter 3

Existence and uniqueness results for fractional differential inclusions in sobolev fractional space

3.1 Existence and uniqueness results in sobolev fractional space

In this chapter we study the existence and uniqueness for BVP of fractional differential inclusion

$$\begin{cases} \mathcal{D}^{\rho}_{0^+}u(t) \in \mathcal{H}(t,u(t)), \quad \text{for almost } t \in [0,1], \\ u(0) = 0, \\ u(1) = pI^{\mu_1}_{0^+}k_1(\xi_1, u(\xi_1)) + qI^{\mu_2}_{0^+}k_2(\xi_2, u(\xi_2)), \end{cases} \tag{3.1}$$

where $1 < \rho < 2$, $p, q \geq 0$, $\mu_1, \mu_2 \geq 1$, $0 < \xi_1, \xi_2 \leq 1$, $k_j : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function for $1 \leq j \leq 2$, and $\mathcal{H} : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a set valued map.

Definition 3.1.1. *A function z is said to be a solution of the problem (3.1), if there exists* *a function* $v \in \mathrm{L}^1([0,1],\mathbb{R})$ *with* $v(t) \in \mathcal{H}(t,u(t))$ *for almost* $t \in [0,1]$ *such that*

$$\begin{cases} \mathcal{D}^{\rho}_{0^+}u(t) = v(t), \quad 1 < \rho < 2, \\ u(0) = 0, \\ u(1) = pI^{\mu_1}_{0^+}k_1(\xi_1, u(\xi_1)) + qI^{\mu_2}_{0^+}k_2(\xi_2, u(\xi_2)). \end{cases}$$

Lemma 3.1.1. *For a given* $y \in \mathrm{L}^1([0,1],\mathbb{R})$*, a function* u *is a solution to*

$$\begin{cases} \mathcal{D}^{\rho}_{0^+}u(t) = y(t), \quad 1 < \rho < 2, \quad \text{for almost } t \in [0,1], \\ u(0) = 0, \\ u(1) = pI^{\mu_1}_{0^+}k_1(\xi_1, u(\xi_1)) + qI^{\mu_2}_{0^+}k_2(\xi_2, u(\xi_2)), \end{cases} \tag{3.2}$$

if and only if

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\rho)}\int_0^t (t-s)^{\rho-1}y(s)ds - \Bigg[\frac{1}{\Gamma(\rho)}\int_0^1 (1-s)^{\rho-1}y(s)ds \\ &- \frac{p}{\Gamma(\mu_1)}\int_0^{\xi_1}(\xi_1-s)^{\mu_1-1}k_1(s,u(s))ds - \frac{q}{\Gamma(\mu_2)}\int_0^{\xi_2}(\xi_2-s)^{\mu_2-1}k_2(s,u(s))ds\Bigg]t^{\rho-1}. \end{aligned} \tag{3.3}$$

Let

$$\mathbb{X} = \{u \in \mathrm{L}^2([0,1];\mathbb{R}); D^{\beta}_{0^+}u \in \mathrm{L}^2([0,1];\mathbb{R})\} \quad \text{where } 0 < \beta < \rho - 1.$$

The space $(\mathbb{X}, \|.\|_{\mathbb{X}})$ is complete reflexive space [26], with

$$\|u\|_{\mathbb{X}} = \|u\|_2 + \|D^{\beta}_{0^+}u\|_2.$$

In this section we study the existence solution for the problem (3.1).

Theorem 3.1.1. *Assume that the following assertions* $(D1)-(D3)$ *hold*

(D1) $\mathcal{H} : [0,1]\times\mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ *is a Carathéodory set valued map and integrably bounded with* $\phi \in \mathcal{C}([0,1];\mathbb{R})$.

(D2) There are $C_1, C_2 > 0$ *such that*

$$|k_j(t,x) - k_j(t,y)| \le C_j|x-y|,$$

and

$$k_j(t,0) = 0,$$

for $t \in [0,1]$ *and* $j \in \{1,2\}$.

(D3) $\frac{C_1 p \xi_1^{\mu_1-1}}{\Gamma(\mu_1)} + \frac{C_2 q \xi_2^{\mu_2-1}}{\Gamma(\mu_2)} < 1.$

Then there exists a solution for the problem (3.1).

Proof. For each measurable function u, the set $S_{\mathcal{H},u}$ is nonempty.
We use the iterative method, let (u_n) be a sequence of measurable function with $u_0 \in \mathbb{X}$ such that

$$\begin{aligned} u_{n+1}(t) &= \frac{1}{\Gamma(\rho)}\int_0^t (t-s)^{\rho-1} v_n(s)ds - \left[\frac{1}{\Gamma(\rho)}\int_0^1 (1-s)^{\rho-1} v_n(s)ds \right. \\ &\quad -\frac{p}{\Gamma(\mu_1)}\int_0^{\xi_1} (\xi_1 - s)^{\mu_1-1} k_1(s, u_n(s))ds \\ &\quad \left. -\frac{q}{\Gamma(\mu_2)}\int_0^{\xi_2} (\xi_2 - s)^{\mu_2-1} k_2(s, u_n(s))ds \right] t^{\rho-1}, \end{aligned} \tag{3.4}$$

with $v_n \in S_{\mathcal{H},u_n}$ for each $n \in \mathbb{N}$.

Step 1 ($u_n \in \mathbb{X}$).
We use the proof by recurrence, since $\mathcal{H}$ integrably bounded, then there exist a function $\phi \in \mathrm{L}^1([0,1];\mathbb{R})$ such that

$$v_n(t) \leq \phi(t) \quad \text{for almost } t \in [0,1],$$

then $u_0 \in \mathbb{X}$ and if $u_n \in \mathbb{X}$, we get

$$|u_{n+1}(t)| \leq \frac{2\|\phi\|_1}{\Gamma(\rho)} + \frac{C_1 p \xi_1^{\mu_1-1}}{\Gamma(\mu_1)}\int_0^{\xi_1} |u_n(s)|ds + \frac{C_2 q \xi_2^{\mu_2-1}}{\Gamma(\mu_2)}\int_0^{\xi_2} |u_n(s)|ds,$$

hence $u_{n+1} \in \mathrm{L}^2([0,1];\mathbb{R})$ and

$$\begin{aligned} D_{0^+}^{\beta} u_{n+1}(t) &= \frac{1}{\Gamma(\rho-\beta)}\int_0^t (t-s)^{\rho-\beta-1} v_n(s)ds \\ &\quad -\frac{\Gamma(\rho)}{\Gamma(\rho-\beta)}\left[\frac{1}{\Gamma(\rho)}\int_0^1 (1-s)^{\rho-1} v_n(s)ds \right. \\ &\quad -\frac{p}{\Gamma(\mu_1)}\int_0^{\xi_1} (\xi_1 - s)^{\mu_1-1} k_1(s, u_n(s))ds \\ &\quad \left. -\frac{q}{\Gamma(\mu_2)}\int_0^{\xi_2} (\xi_2 - s)^{\mu_2-1} k_2(s, u_n(s))ds \right] t^{\rho-\beta-1}, \end{aligned}$$

that means

$$|D_{0^+}^{\beta} u_{n+1}(t)| \leq \frac{2\|v_n\|_1}{\Gamma(\rho-\beta)} + \frac{\Gamma(\rho)}{\Gamma(\rho-\beta)}\left(\frac{C_1 p \xi_1^{\mu_1-1}}{\Gamma(\mu_1)} + \frac{C_2 q \xi_2^{\mu_2-1}}{\Gamma(\mu_2)}\right)\|u_n\|_2,$$

that means $D^{\beta}_{0^+}u_{n+1} \in \mathrm{L}^2([0,1];\mathbb{R})$, then the sequence (u_n) belongs to $\mathbb{X}$.

Step 2 (boundedness of (u_n) in $\mathbb{X}$).

Let $n \geq 1$ and $t \in [0,1]$, then

$$|u_{n+1}(t)| \leq \frac{2}{\Gamma(\rho+1)} \int_0^1 |v_n(s)|ds + \left(\frac{C_1 p \xi_1^{\mu_1-1}}{\Gamma(\mu_1)} + \frac{C_2 q \xi_2^{\mu_2-1}}{\Gamma(\mu_2)}\right)\|u_n\|_2,$$

then by simple calculation, we have

$$|u_{n+1}(t)|^2 \leq \left(\frac{2\|\phi\|_1}{\Gamma(\rho+1)} \sum_{j=0}^{n} (\frac{C_1 p \xi_1^{\mu_1-1}}{\Gamma(\mu_1)} + \frac{C_2 q \xi_2^{\mu_2-1}}{\Gamma(\mu_2)})^j + (\frac{C_1 p \xi_1^{\mu_1-1}}{\Gamma(\mu_1)} + \frac{C_2 q \xi_2^{\mu_2-1}}{\Gamma(\mu_2)})^{n+1}\|u_0\|_2\right)^2,$$

finally we get

$$\begin{aligned} \|u_{n+1}\|_2 \quad &\leq \quad \frac{2\|\phi\|_1}{\Gamma(\rho+1)} \sum_{j=0}^{n} (\frac{C_1 p \xi_1^{\mu_1-1}}{\Gamma(\mu_1)} + \frac{C_2 q \xi_2^{\mu_2-1}}{\Gamma(\mu_2)})^j &(3.5)\\ &\quad + (\frac{C_1 p \xi_1^{\mu_1-1}}{\Gamma(\mu_1)} + \frac{C_2 q \xi_2^{\mu_2-1}}{\Gamma(\mu_2)})^{n+1}\|u_0\|_2, \quad j = \overline{1,2} &(3.6)\end{aligned}$$

i.e., the sequence (u_n) is bounded in $\mathrm{L}^2([0,1];\mathbb{R})$, and

$$\begin{aligned} D^{\beta}_{0^+}u_{n+1}(t) \quad = \quad & \frac{1}{\Gamma(\rho-\beta)} \int_0^t (t-s)^{\rho-\beta-1} v_n(s)ds - \frac{\Gamma(\rho)}{\Gamma(\rho-\beta)}\Bigg[\frac{1}{\Gamma(\rho)}\int_0^1 (1-s)^{\rho-1} v_n(s)ds \\ & - \frac{p}{\Gamma(\mu_1)} \int_0^{\xi_1} (\xi_1 - s)^{\mu_1-1} k_1(s, u_n(s))ds \\ & - \frac{q}{\Gamma(\mu_2)} \int_0^{\xi_2} (\xi_2 - s)^{\mu_2-1} k_2(s, u_n(s))ds\Bigg] t^{\rho-\beta-1}, \end{aligned}$$

so

$$|D^{\beta}_{0^+}u_{n+1}(t)|^2 \leq \left(\frac{2\|v_n\|_1}{\Gamma(\rho-\beta)} + \frac{\Gamma(\rho)}{\Gamma(\rho-\beta)}(\frac{C_1 p \xi_1^{\mu_1-1}}{\Gamma(\mu_1)} + \frac{C_2 q \xi_2^{\mu_2-1}}{\Gamma(\mu_2)})\|u_n\|_2\right)^2.$$

Since (u_n) is bounded in $\mathrm{L}^2([0,1];\mathbb{R})$ and $\mathcal{H}$ integrably bounded, we can easily to see that there exist $\gamma > 0$ such that

$$\|D^{\beta}_{0^+}u_n\|_2 \leq \gamma \quad \text{for every } n \geq 1. \tag{3.7}$$

From (3.5) and (3.7) we conclude that the sequence (u_n) is bounded in $\mathbb{X}$.

Step 3 (Passage to the limit).

Since (u_n) is bounded in $\mathbb{X}$ and $\mathbb{X}$ is reflexive Banach space, then has a subsequence (u_{n_k}) converges weakly to an element in $\mathbb{X}$ noted by $\bar{u}$. Now we show that $\bar{u}$ is solution to the problem (3.3).

Let (v_{n_k}) be a sequence in $\mathrm{L}^1([0,1];\mathbb{R})$ with

$$v_{n_k}(t) \in \mathcal{H}(t, u_{n_k}(t)) \;\text{ for almost } t \in [0,1].$$

By proposition 1.6.2 the sequence (v_{n_k}) has a subsequence converge weakly to $\bar{v}$ in $\mathrm{L}^1([0,1];\mathbb{R})$. The sequence $u_{n_k}(t)$ be a bounded sequence in $\mathbb{R}$, then has a subsequence noted by $u_{n_k}(t)$ converge to $u(t)$ and $u(t) = \bar{u}(t)$ for each $t \in [0,1]$.

The sequence $(v_{n_k}(t))$ be a bounded sequence in $\mathbb{R}$ (because ϕ is bounded), then has a subsequence noted by $v_{n_k}(t)$ converge to $w(t)$ and $w(t) = \bar{v}(t)$.

The upper semicontinuous of $\mathcal{H}$ dictates that

$$\bar{v}(t) \in \mathcal{H}(t, \bar{u}(t)) \;\text{ for almost } t \in [0,1].$$

Passage to the limit of the equation (3.4), we get

$$\begin{aligned}\bar{u}(t) &= \frac{1}{\Gamma(\rho)}\int_0^t (t-s)^{\rho-1}\bar{v}(s)ds - \Bigg[\frac{1}{\Gamma(\rho)}\int_0^1 (1-s)^{\rho-1}\bar{v}(s)ds \\ &\quad -\frac{p}{\Gamma(\mu_1)}\int_0^{\xi_1}(\xi_1-s)^{\mu_1-1}k_1(s,\bar{u}(s))ds - \frac{q}{\Gamma(\mu_2)}\int_0^{\xi_2}(\xi_2-s)^{\mu_2-1}k_2(s,\bar{u}(s))ds\Bigg]t^{\rho-1},\end{aligned}$$

that means $\bar{u}$ is solution to problem (3.1), and $\bar{u} \in \mathbb{X}$. □

Example 2. *Let us consider the following problem*

$$\begin{cases} \mathcal{D}^{1.9}_{0^+}u(t) \in \mathcal{H}(t,u(t)) \text{ for almost } t \in [0,1], \\ u(0) = 0, \\ u(1) = \dfrac{4}{\Gamma(4)}\displaystyle\int_0^1 s(1-s)^3 u(s)ds, \end{cases} \tag{3.8}$$

where

$$\mathcal{H} : (t,x) \in [0,1]\times\mathbb{R} \mapsto \left[\frac{|x\sin t|}{2}; \frac{t|x|}{2} + \arctan t\right] \in \mathcal{P}(\mathbb{R}).$$

We have

- *(i) the set valued map $\mathcal{H}$ is carathéodory set valued, and $\mathcal{H}(t,x)$ is nonempty and compact set in $\mathbb{R}$.*
- *(iii) $k_1(t,x) = tx$ and $k_2(t,x) = 0$.*

(iv) $\rho = 1.9, p = \mu_1 = 4, q = 0$ *and* $C_1 = 1$.

Since
$$\frac{C_1 p \xi_1^{\mu_1 - 1}}{\rho\Gamma(\mu_1)} + \frac{C_2 q \xi_2^{\mu_2 - 1}}{\rho\Gamma(\mu_2)} < 1.$$

Consequently, by theorem [3.1.1] the considered (3.8) *admits a solution.*

Remark 3.1.1. *If the function* ϕ *is not continuous functions, but* ϕ *is measurable and bounded on* $[0,1]$. *Then the result of theorem [3.1.1] still valid.*

Theorem 3.1.2. *Assume that the following assertions* $(D1^*), (D2)$ *and* $(D3)$ *are satisfied*

(D1)* $\mathcal{H} : [0,1] \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ *is integrably bounded with* $\phi \in \mathcal{C}([0,1]; \mathbb{R})$ *such that*

- *The set-valued map* $\mathcal{H}(t,.)$ *is* K*-lipschitzean.*
- *The set-valued map* $\mathcal{H}(.,x)$ *is measurable for* $x \in \mathbb{R}$.

(D2) *There are* $C_1, C_2 > 0$ *such that*
$$|k_j(t,x) - k_j(t,y)| \leq C_j |x - y|,$$
and
$$k_j(t,0) = 0,$$
for $t \in [0,1]$ *and* $j \in \{1,2\}$.

(D3) $\frac{2K}{\Gamma(\rho)} + \frac{C_1 p \xi_1^{\mu_1 - 1}}{\Gamma(\mu_1)} + \frac{C_2 q \xi_2^{\mu_2 - 1}}{\Gamma(\mu_2)} < 1$.

Then the problem (3.1) has one solution in $\mathbb{X}$.

Proof. From theorem [3.1.1] the problem (3.1) has a solution, now we prove this solution is unique.
Let u_1 and u_2 be two solution for the problem (3.1), then there exist $v_1 \in S_{\mathcal{H},u_1}$ and

$v_2 \in S_{\mathcal{H},u_2}$ such that

$$\begin{aligned} u_j(t) &= \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} v_j(s)ds - \Bigg[\frac{1}{\Gamma(\rho)} \int_0^1 (1-s)^{\rho-1} v_j(s)ds \\ &\quad - \frac{p}{\Gamma(\mu_1)} \int_0^{\xi_1} (\xi_1 - s)^{\mu_1 - 1} k_1(s, u_j(s))ds \\ &\quad - \frac{q}{\Gamma(\mu_2)} \int_0^{\xi_2} (\xi_2 - s)^{\mu_2 - 1} k_2(s, u_j(s))ds \Bigg] t^{\rho-1}, \end{aligned}$$

for $1 \leq j \leq 2$, then

$$\begin{aligned} |u_2(t) - u_1(t)| &\leq \frac{2K}{\Gamma(\rho)} \int_0^1 |u_2(s) - u_1(s)|ds \\ &\quad + \left(\frac{C_1 p \xi_1^{\mu_1 - 1}}{\Gamma(\mu_1)} + \frac{C_2 q \xi_2^{\mu_2 - 1}}{\Gamma(\mu_2)} \right) \int_0^1 |u_2(s) - u_1(s)|ds, \end{aligned}$$

after that

$$\|u_2 - u_1\|_2 \leq \left(\frac{2K}{\Gamma(\rho)} + \frac{C_1 p \xi_1^{\mu_1 - 1}}{\Gamma(\mu_1)} + \frac{C_2 q \xi_2^{\mu_2 - 1}}{\Gamma(\mu_2)} \right) \|u_2 - u_1\|_2,$$

while

$$\frac{2K}{\Gamma(\rho)} + \frac{C_1 p \xi_1^{\mu_1 - 1}}{\Gamma(\mu_1)} + \frac{C_2 q \xi_2^{\mu_2 - 1}}{\Gamma(\mu_2)} < 1,$$

then $u_1 = u_2$. □

Example 3. *Let us consider the following problem*

$$\begin{cases} \mathcal{D}_{0^+}^{1.9} u(t) = t + \frac{1.9 \arctan(u(t))}{2} \quad \text{for almost } t \in [0,1], \\ u(0) = 0, \\ u(1) = \dfrac{4}{\Gamma(4)} \displaystyle\int_0^1 s(1-s)^3 u(s)ds, \end{cases} \tag{3.9}$$

where $\mathcal{H}(t,x) = \{t + \frac{\arctan(x)}{2}\}$.

The set valued map $\mathcal{H}$ *is* $\frac{1.9}{2}$*-Lipschitzean.*

In this problem, we have

(i) $k_1(t,x) = 1.9tx$ *and* $k_2(t,x) = 0$,

(ii) $\rho = 1.9, p = \mu_1 = 4, q = 0$,

(iii) $C_1 = 1.9$.

Since

$$\frac{2L}{\Gamma(\rho)} + \frac{C_1 p \xi_1^{\mu_1-1}}{\Gamma(\mu_1)} + \frac{C_2 q \xi_2^{\mu_2-1}}{\Gamma(\mu_2)} \approx 0.8981 < 1.$$

Consequently, from theorem [3.1.2] the considered (3.9) *has a unique solution.*

Chapter 4

Existence and uniqueness results for IVP of nonlinear fractional inclusion on an unbounded domain

4.1 Existence and uniqueness results in unbounded domain

In this chapter we study the existence and uniqueness for IVP of fractional differential inclusion

$$\begin{cases} \mathcal{D}^{\rho}_{0^+}u(t) \in \mathcal{H}(t, u(t)), \quad \text{for almost} \quad t \in J = [0, +\infty[\\ u(0) = 0, \\ \mathcal{D}^{\rho-1}_{0^+}u(0) = I^{\beta}_{0^+}u(\xi), \end{cases} \tag{4.1}$$

where $1 < \rho < 2$, $0 < \xi < 1, \beta \geq 1$, and $\mathcal{H} : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a set valued mapping.

Definition 4.1.1. *A function u is said to be a solution of problem (4.1), if there exists a function $v \in \mathrm{L}^1([0,1], \mathbb{R})$ with $v(t) \in \mathcal{H}(t, u(t))$ for almost $t \in J$ such that*

$$\begin{cases} \mathcal{D}^{\rho}_{0^+}u(t) = v(t), \quad \textit{for almost} \quad t \in J, 1 < \rho < 2, \\ u(0) = 0, \\ \mathcal{D}^{\rho-1}_{0^+}u(0) = I^{\beta}_{0^+}u(\xi). \end{cases} \tag{4.2}$$

Lemma 4.1.1. *For a given $y \in \mathrm{L}^1(J,\mathbb{R})$, a function z is a solution to*

$$\begin{cases} \mathcal{D}^{\rho}_{0+}u(t) = y(t), & \text{for almost } t \in J, 1 < \rho < 2, \\ u(0) = 0, \\ \mathcal{D}^{\rho-1}_{0+}u(0) = I^{\beta}_{0+}u(\xi), \end{cases} \tag{4.3}$$

if and only if

$$u(t) = \frac{1}{\Gamma(\rho)}\int_0^t (t-s)^{\rho-1}y(s)ds + \frac{t^{\rho-1}}{\Gamma(\rho)\Gamma(\beta)}\int_0^\xi (\xi-s)^{\beta-1}u(s)ds. \tag{4.4}$$

Proof. Assume that u satisfies (4.3), from lemma 1.5.3 we have

$$u(t) = I^{\rho}_{0+}y(t) - c_1 t^{\rho-1} - c_2 t^{\rho-2}, \tag{4.5}$$

where $c_1, c_2 \in \mathbb{R}$.

By using first boundary condition, we get that $c_2 = 0$ and from the second boundary condition, we find

$$\mathcal{D}^{\rho-1}_{0+}u(t) = I^{\rho-1}_{0+}y(t) - c_1\Gamma(\rho),$$

this means that

$$\mathcal{D}^{\rho-1}_{0+}u(0) = -c_1\Gamma(\rho) = I^{\beta}_{0+}u(\xi),$$

that is

$$c_1 = \frac{-1}{\Gamma(\rho)\Gamma(\beta)}\int_0^\xi (\xi-s)^{\beta-1}u(s)ds.$$

By replacing c_1 in (4.5) we get the equation (4.4).

Conversely, if u satisfies (4.4) by lemma [1.5.2] and lemma [1.5.4], we have $\mathcal{D}^{\rho}_{0+}u(t) = y(t)$, and

$$\mathcal{D}^{\rho-1}_{0+}u(t) = I^{\rho-1}_{0+}y(t) + \frac{1}{\Gamma(\beta)}\int_0^\xi (\xi-s)^{\beta-1}u(s)ds,$$

by simple calculation, we get $\mathcal{D}^{\rho-1}_{0+}u(0) = I^{\beta}_{0+}u(\xi)$, and we can easily to see that $u(0) = 0$. □

4.2 Study the existence and uniqueness in weight spaces

Let $\mathcal{C}(J;\mathbb{R})$ the space of all functions defined on J and continuous on J, and let

$$\mathbb{X} = \{u \in \mathcal{C}(J,\mathbb{R}); \sup_{t\in J} \frac{|u(t)|}{(1+t^{\rho-1})^2} < \infty\},$$

equipped the norm

$$\|u\|_{\mathbb{X}} = \sup_{t\in J} \frac{|u(t)|}{(1+t^{\rho-1})^2}.$$

Lemma 4.2.1. *The space* $(\mathbb{X}, \|.\|_{\mathbb{X}})$ *is Banach space.*

Proof. Let (u_n) be be a Cauchy sequence in $\mathbb{X}$. Then for every $\varepsilon > 0$ there exist $N \in \mathbb{N}$ such that

$$\|u_m - u_n\|_{\mathbb{X}} = \sup_{t\in J} \frac{|u_m(t) - u_n(t)|}{(1+t^{\rho-1})^2} < \frac{\varepsilon}{2} \quad \text{for } m \geq n \geq N.$$

We remark that

$$\frac{|u_m(t) - u_n(t)|}{(1+t^{\rho-1})^2} \leq \|u_m - u_n\|_{\mathbb{X}} < \frac{\varepsilon}{2} \quad \text{for all } t \in J.$$

The sequence $(\frac{u_n(t)}{(1+t^{\rho-1})^2})$ be a Cauchy sequence in $\mathbb{R}$, thus it's converge to $\frac{u(t)}{(1+t^{\rho-1})^2}$ for all $t \in J$, and

$$\begin{aligned} |u(t) - u(t_0) \quad | &\leq |u_n(t) - u(t)| + |u_n(t) - u_n(t_0)| + |u_n(t_0) - u(t_0)| \\ &\leq (1+t^{\rho-1})^2 |\frac{u_n(t)}{(1+t^{\rho-1})^2} - \frac{u(t)}{(1+t^{\rho-1})^2}| + |u_n(t) - u_n(t_0)| \\ &+ (1+t^{\rho-1})^2 |\frac{u_n(t_0)}{(1+t^{\rho-1})^2} - \frac{u(t_0)}{(1+t^{\rho-1})^2}| \end{aligned}$$

as $t \to t_0$ and $n \to \infty$ we have $u(t) - u(t_0) \to 0$.

Then $u \in \mathcal{C}(J,\mathbb{R})$ and for each $t \in J$ there exist $N_1 = N_1(\varepsilon, t) \in \mathbb{N}$ such that

$$\frac{|u_n(t) - u(t)|}{(1+t^{\rho-1})^2} < \frac{\varepsilon}{2} \quad \text{for all } \ n \geq N_1.$$

Let $n \geq \max(N_1, N)$ then

$$\begin{aligned} |\frac{u(t)}{(1+t^{\rho-1})^2} \quad | &\leq \frac{|u_n(t) - u(t)|}{(1+t^{\rho-1})^2} + \frac{|u_n(t)|}{(1+t^{\rho-1})^2} \\ &\leq \frac{|u_n(t) - u(t)|}{(1+t^{\rho-1})^2} + \frac{|u_n(t) - u_n(t)|}{(1+t^{\rho-1})^2} + \frac{|u_n(t)|}{(1+t^{\rho-1})^2} \\ &< \frac{\varepsilon}{2} + \|u_n - u_n\|_{\mathbb{X}} + \|u_n\|_{\mathbb{X}} < \varepsilon + \|u_n\|_{\mathbb{X}}. \end{aligned}$$

Then $u \in \mathbb{X}$, i.e., $\mathbb{X}$ is Banach space. $\square$

Lemma 4.2.2. *Let A be a bounded set in $\mathbb{X}$. Then A is relatively compact in $\mathbb{X}$ if the following conditions hold*

- *(i) A is equicontinuous in $[0,+\infty)$.*
- *(ii) for all $\varepsilon > 0$, there exist $x > 0$, for all $u \in A$ and $t \geq x$*

$$\frac{u(t)}{(1+t^{\rho-1})^2} < \varepsilon.$$

Proof. For each $\varepsilon > 0$, by (ii) there exist $x > 0$ such that for each $u \in A$ and $t \geq x$, we have

$$\frac{u(t)}{(1+t^{\rho-1})^2} < \varepsilon.$$

Let

$$A_{|[0,x]} = \{u_{|[0,x]}, u \in A\}.$$

In view of (i) and theorem 1.6.4 we get that the closure of $A_{|[0,x]}$ is compact set in $\mathcal{C}([0,x],\mathbb{R})$, that means there exist $u_1,\ldots,u_n \in A$ such that

$$\max_{t\in[0,x]} \frac{|u(t)-u_j(t)|}{(1+t^{\rho-1})^2} < \varepsilon \quad \text{for some } 1 \leq j \leq n.$$

Thence

$$\|u-u_j\|_{\mathbb{X}} = \max\Big\{\max_{t\in[0,x]} \frac{|u(t)-u_j(t)|}{(1+t^{\rho-1})^2}, \sup_{t>x} \frac{|u(t)-u_j(t)|}{(1+t^{\rho-1})^2}\Big\} < \varepsilon.$$

Hence A is precompact,i.e., the closure of A is compact set in $\mathcal{C}(J,\mathbb{R})$. □

4.2.1 Existence result in convex case

Theorem 4.2.1. *Assume that the following assertions $(B1)-(B3)$ hold*

$(B1)$ *$\mathcal{H} : J\times\mathbb{R} \to \mathcal{P}_{c,cp}(\mathbb{R})$ such that for every measurable function $u : J \to \mathbb{R}$ we have $t \in J \mapsto \mathcal{H}(t,u(t))$ is measurable.*

$(B2)$ *There are $\varphi,\psi \in \mathrm{L}^1(J,\mathbb{R}) \cap \mathrm{L}^\infty(J,\mathbb{R})$ such that*

(i) $\|\mathcal{H}(t,x)\| = \sup\{|v| : v \in \mathcal{H}(t,x)\} \leq \varphi(t) + \frac{\psi(t)}{(1+t^{\rho-1})^2}|x|$,

(ii) $\forall\varepsilon > 0, \exists b_\varepsilon > 0$ *such that* $\int_{b_\varepsilon}^{+\infty} |\varphi(s)|ds < \frac{\varepsilon}{2}$ *and* $\int_{b_\varepsilon}^{+\infty} |\psi(s)|ds < \frac{\varepsilon}{2}$.

$(B3)$ $\frac{\|\psi\|_1}{\Gamma(\rho)} + \frac{\xi^{\beta-1}}{\Gamma(\rho)\Gamma(\beta)}(\xi + \frac{2\xi^\rho}{\rho} + \frac{\xi^{2\rho-1}}{2\rho-1}) < 1.$

Then the problem (4.1) has a least one solution.

Proof. Using the conditions $(B1), (B2)(i)$ and by applying lemma [1.4.1] we can easily to see that $S_{\mathcal{H},u} \neq \emptyset$ for all $u \in \mathbb{X}$. Define an operator $\mathcal{N} : \mathbb{X} \to \mathcal{P}(\mathbb{X})$ by

$$\mathcal{N}(u) = \Big\{ h \in \mathbb{X} : h(t) = \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} v(s)ds + \frac{t^{\rho-1}}{\Gamma(\rho)\Gamma(\beta)} \int_0^\xi (\xi - s)^{\beta-1} u(s)ds, v \in S_{\mathcal{H},u} \Big\}. \tag{4.6}$$

We will show that $\mathcal{N}$ satisfies all hypotheses of the theorem [1.3.3].

Step 1. ($\mathcal{N}$ is convex valued.) Let $u \in \mathbb{X}$, and $h_1, h_2 \in \mathcal{N}(u)$, then there exist $v_1, v_2 \in S_{\mathcal{H},u}$ such that

$$h_i(t) = \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} v_i(s)ds + \frac{t^{\rho-1}}{\Gamma(\rho)\Gamma(\beta)} \int_0^\xi (\xi - s)^{\beta-1} u(s)ds \quad \text{for } i \in \{1, 2\}.$$

Then for $s \in [0, 1]$ we have

$$\begin{aligned} sh_1(t) + (1-s)h_2(t) &= \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} \big(sv_1(s) + (1-s)v_2(s) \big) ds \\ &+ \frac{t^{\rho-1}}{\Gamma(\rho)\Gamma(\beta)} \int_0^\xi (\xi - s)^{\beta-1} u(s)ds \end{aligned}$$

Since $\mathcal{H}$ is convex set-valued, then $sh_1 + (1-s)h_2 \in \mathcal{N}(u)$.

Step 2. ($\mathcal{N}$ bounded set into bounded set.)

Let $B_r = \{u \in \mathbb{X}, \|u\|_{\mathbb{X}} < r\}$ and let $h \in \mathcal{N}(u)$ with $u \in B_r$, then there exist $v \in S_{\mathcal{H},u}$ such that

$$h(t) = \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} v(s)ds + \frac{t^{\rho-1}}{\Gamma(\rho)\Gamma(\beta)} \int_0^\xi (\xi - s)^{\beta-1} u(s)ds.$$

For all $t \in J$ we have

$$\begin{aligned} \frac{|h(t)|}{1 + t^{\rho-1}} &\leq \frac{1}{\Gamma(\rho)} \int_0^t |v(s)|ds + \frac{\xi^{\beta-1}}{\Gamma(\rho)\Gamma(\beta)} \int_0^\xi |u(s)|ds \\ &\leq \frac{\|\varphi\|_1 + \|\psi\|_1 \|u\|_{\mathbb{X}}}{\Gamma(\rho)} + \frac{\xi^{\beta-1}}{\Gamma(\rho)\Gamma(\beta)} (\xi + \frac{\xi^\rho}{\rho}) \|u\|_{\mathbb{X}} \\ &= \frac{\|\varphi\|_1}{\Gamma(\rho)} + \|u\|_{\mathbb{X}} \Big(\frac{\|\psi\|_1}{\Gamma(\rho)} + \frac{\xi^{\beta-1}}{\Gamma(\rho)\Gamma(\beta)} (\xi + \frac{2\xi^\rho}{\rho} + \frac{\xi^{2\rho-1}}{2\rho - 1}) \Big), \end{aligned}$$

remark that

$$\int_0^{\xi}|u(s)|ds=\int_0^{\xi}(1+s^{\rho-1})^2\times\frac{|u(s)|}{(1+s^{\rho-1})^2}ds \;\leq\; \|u\|_{\mathbb{X}}\int_0^{\xi}(1+s^{\rho-1})^2ds$$
$$=(\xi+\frac{2\xi^{\rho}}{\rho}+\frac{\xi^{2\rho-1}}{2\rho-1})\|u\|_{\mathbb{X}}.$$

Then

$$\|h\|_{\mathbb{X}}\leq l=\frac{\|\varphi\|_1}{\Gamma(\rho)}+r\bigg(\frac{\|\psi\|_1}{\Gamma(\rho)}+\frac{\xi^{\beta-1}}{\Gamma(\rho)\Gamma(\beta)}(\xi+\frac{2\xi^{\rho}}{\rho}+\frac{\xi^{2\rho-1}}{2\rho-1})\bigg),$$

and

$$\lim_{t\to+\infty}\frac{|h(t)|}{(1+t^{\rho-1})^2}\leq\lim_{t\to+\infty}\frac{l}{1+t^{\rho-1}}=0.$$

Step 3. ($\mathcal{N}$ bounded set into equicontinuous set.)

Let B_r the set as defined in step 2, and $t_1,t_2\in J$ where $t_1\leq t_2$, then

$$\begin{aligned}
|\frac{h(t_1)}{(1+t_1^{\rho-1})^2}-\frac{h(t_2)}{(1+t_2^{\rho-1})^2}| \;=\;& \bigg|\frac{1}{\Gamma(\rho)}\int_{t_1}^{t_2}\frac{(t_2-s)^{\rho-1}}{(1+t_2^{\rho-1})^2}v(s)ds\\
&+\frac{1}{\Gamma(\rho)}\int_0^{t_1}\bigg(\frac{(t_2-s)^{\rho-1}}{(1+t_2^{\rho-1})^2}-\frac{(t_1-s)^{\rho-1}}{(1+t_1^{\rho-1})^2}\bigg)v(s)ds\\
&+\bigg[\frac{1}{\Gamma(\rho)\Gamma(\beta)}\int_0^{\xi}(\xi-s)^{\beta-1}u(s)ds\bigg]\\
&\times(\frac{t_1^{\rho-1}}{(1+t_1^{\rho-1})^2}-\frac{t_2^{\rho-1}}{(1+t_2^{\rho-1})^2})\bigg|\\
\leq\;& \frac{1}{\Gamma(\rho)}\times\frac{(t_2-t_1)^{\rho-1}}{(1+t_2^{\rho-1})^2}\int_{t_1}^{t_2}|v(s)|ds\\
&+\frac{1}{\Gamma(\rho)}\int_0^{t_1}|v(s)|ds\times\sup_{s\in[0,t_1]}\bigg|\frac{(t_2-s)^{\rho-1}}{(1+t_2^{\rho-1})^2}-\frac{(t_1-s)^{\rho-1}}{(1+t_1^{\rho-1})^2}\bigg|\\
&+\bigg[\frac{1}{\Gamma(\rho)\Gamma(\beta)}\int_0^{\xi}(\xi-s)^{\beta-1}u(s)ds\bigg]\\
&\times|\frac{t_1^{\rho-1}}{(1+t_1^{\rho-1})^2}-\frac{t_2^{\rho-1}}{(1+t_2^{\rho-1})^2}|\\
\leq\;& \frac{1}{\Gamma(\rho)}\times\frac{(t_2-t_1)^{\rho-1}}{(1+t_2^{\rho-1})^2}\times\bigg(\|\varphi\|_1+\|\psi\|_1\|u\|_{\mathbb{X}}\bigg)\\
&+\frac{1}{\Gamma(\rho)}\times\sup_{s\in[0,t_1]}\bigg|\frac{(t_2-s)^{\rho-1}}{(1+t_2^{\rho-1})^2}-\frac{(t_1-s)^{\rho-1}}{(1+t_1^{\rho-1})^2}\bigg|\\
&\times\bigg(\|\varphi\|_1+\|\psi\|_1\|u\|_{\mathbb{X}}\bigg)
\end{aligned}$$

$$
\begin{aligned}
&+\left[\frac{1}{\Gamma(\rho)\Gamma(\beta)}\int_0^\xi(\xi-s)^{\beta-1}u(s)ds\right]\\
&\times|\frac{t_1^{\rho-1}}{(1+t_1^{\rho-1})^2}-\frac{t_2^{\rho-1}}{(1+t_2^{\rho-1})^2}|\\
&\leq\frac{1}{\Gamma(\rho)}\times\frac{(t_2-t_1)^{\rho-1}}{(1+t_2^{\rho-1})^2}\times\Big(\|\varphi\|_1+\|\psi\|_1\|u\|_{\mathbb{X}}\Big)\\
&+\frac{1}{\Gamma(\rho)}\times\sup_{s\in[0,t_1]}\left|\frac{(t_2-s)^{\rho-1}}{(1+t_2^{\rho-1})^2}-\frac{(t_1-s)^{\rho-1}}{(1+t_1^{\rho-1})^2}\right|\\
&\times\Big(\|\varphi\|_1+\|\psi\|_1\|u\|_{\mathbb{X}}\Big)\\
&+\frac{c_\xi\|u\|_{\mathbb{X}}}{\Gamma(\rho)\Gamma(\beta)}\times|\frac{t_1^{\rho-1}}{(1+t_1^{\rho-1})^2}-\frac{t_2^{\rho-1}}{(1+t_2^{\rho-1})^2}|,
\end{aligned}
$$

where $c_\xi=\int_0^\xi(1+s^{\rho-1})^2(\xi-s)^{\beta-1}ds$. As $t_2\to t_1$ we have $|\frac{h(t_1)}{(1+t_1^{\rho-1})^2}-\frac{h(t_2)}{(1+t_2^{\rho-1})^2}|\to 0$.

Step 4. ($\mathcal{N}$ has a closed graph.)

Let $u_n\to u_*$, $h_n\in\mathcal{N}(u_n)$ and $h_n\to h_*$, we need to show that $h_*\in\mathcal{N}(u_*)$. Now $h_n\in\mathcal{N}(u_n)$ implies that there exists $v_n\in S_{\mathcal{H},u_n}$ such that

$$
h_n(t) = \frac{1}{\Gamma(\rho)}\int_0^t(t-s)^{\rho-1}v_n(s)ds+\frac{t^{\rho-1}}{\Gamma(\rho)\Gamma(\beta)}\int_0^\xi(\xi-s)^{\beta-1}u_n(s)ds,
$$

we have to show that there exists $v_*\in S_{\mathcal{H},u_*}$ such that for every $t\in[0,+\infty)$,

$$
h_*(t) = \frac{1}{\Gamma(\rho)}\int_0^t(t-s)^{\rho-1}v_*(s)ds+\frac{t^{\rho-1}}{\Gamma(\rho)\Gamma(\beta)}\int_0^\xi(\xi-s)^{\beta-1}u_*(s)ds.
$$

By theorem 1.6.3, we prove that the set $\mathcal{V}=\{v_n\in S_{\mathcal{H},u_n};n\geq 1\}$ is weakly compact in $\mathrm{L}^1(J;\mathbb{R})$

$$
\begin{aligned}
\int_A|v_n(s)|ds &\leq \int_A|\varphi(s)|ds+\|u_n\|_{\mathbb{X}}\int_A|\psi(s)|ds\\
&\leq |A|\big(\|\varphi\|_\infty+\|u_n\|_{\mathbb{X}}\|\psi\|_\infty\big)<\varepsilon,
\end{aligned}
$$

we take $\delta=\frac{\varepsilon}{\|\varphi\|_\infty+\sup_{n\geq 1}\|u_n\|_{\mathbb{X}}\|\psi\|_\infty}$, then the first conditions of theorem 1.6.3 is satisfied. Now we prove the second conditions of this theorem

$$
\int_{b_\varepsilon}^{+\infty}v_n(s)ds\leq\int_{b_\varepsilon}^{+\infty}\varphi(s)ds+\|u_n\|_{\mathbb{X}}\times\int_{b_\varepsilon}^{+\infty}\psi(s)ds<\varepsilon.
$$

Then by theorem 1.6.3 we conclude that $\mathcal{V}$ is weakly compact set in $\mathrm{L}^1(J,\mathbb{R})$.

Conclusion

There exist a subsequence $v_{n_k} \in S_{\mathcal{H},u_{n_k}}$ such that $v_{n_k} \rightharpoonup v_*$ in $\mathrm{L}^1(J,\mathbb{R})$, then

$$h_*(t) = \frac{1}{\Gamma(\rho)}\int_0^t (t-s)^{\rho-1}v_*(s)ds + \frac{t^{\rho-1}}{\Gamma(\rho)\Gamma(\beta)}\int_0^\xi (\xi-s)^{\beta-1}u_*(s)ds, \tag{4.7}$$

because

$$\begin{aligned}\int_0^\xi (\xi-s)^{\beta-1}|u_n(s)-u(s)|ds &\leq \int_0^\xi |u_n(s)-u(s)|ds = \int_0^\xi (1+s^{\rho-1})^2\frac{|u_n(s)-u(s)|}{(1+s^{\rho-1})^2}ds \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{see } 0<\xi<1) \\ &\leq \|u_n - z\|_{\mathbb{X}} \times \int_0^\xi (1+s^{\rho-1})^2 ds \to 0 \quad as \quad n\to\infty.\end{aligned}$$

Step 5. (A priori bounds on solutions.)

Let

$$\begin{aligned}K = \bar{B}(0,\frac{A}{1-B}) \cap &\Big\{h\in\mathbb{X}; |\frac{h(t_1)}{(1+t_1^{\rho-1})^2} - \frac{h(t_2)}{(1+t_2^{\rho-1})^2}| \leq A(t_2-t_1)^{\rho-1} + s_1(t_1,t_2) \\ &+\frac{A}{1-B}\Big(\frac{\|\psi\|_1(t_2-t_1)^{\rho-1}}{\Gamma(\rho)} + \frac{c_\xi}{\Gamma(\rho)\Gamma(\beta)}\times|\frac{t_1^{\rho-1}}{(1+t_1^{\rho-1})^2} - \frac{t_2^{\rho-1}}{(1+t_2^{\rho-1})^2}| \\ &+s_2(t_1,t_2)\Big)\Big\},\end{aligned}$$

where

$$\begin{aligned}A = \frac{\|\varphi\|_1}{\Gamma(\rho)} \quad \text{and } B &= \frac{\|\psi\|_1}{\Gamma(\rho)} + \frac{\xi^{\beta-1}}{\Gamma(\rho)\Gamma(\beta)}(\xi + \frac{2\xi^\rho}{\rho} + \frac{\xi^{2\rho-1}}{2\rho-1}), \\ s_1(t_1,t_2) &= \frac{\|\varphi\|_1}{\Gamma(\rho)} \times \sup_{s\in[0,t_1]}\left|\frac{(t_2-s)^{\rho-1}}{(1+t_2^{\rho-1})^2} - \frac{(t_1-s)^{\rho-1}}{(1+t_1^{\rho-1})^2}\right|, \\ s_2(t_1,t_2) &= \frac{\|\psi\|_1}{\Gamma(\rho)} \times \sup_{s\in[0,t_1]}\left|\frac{(t_2-s)^{\rho-1}}{(1+t_2^{\rho-1})^2} - \frac{(t_1-s)^{\rho-1}}{(1+t_1^{\rho-1})^2}\right|.\end{aligned}$$

The set K is nonempty convex and compact set in $\mathbb{X}$, by theorem 1.3.3 the set valued map $\mathcal{N}$ has a fixed point, which represent a solution. □

Example 4. *Consider the following fractional differential inclusion*

$$\begin{cases} \mathcal{D}_{0^+}^{\frac{3}{2}} u(t) \in \mathcal{H}(t, u(t)), \text{ for almost } t \in [0, +\infty[, \\ u(0) = 0, \\ \mathcal{D}_{0^+}^{\frac{1}{2}} u(0) = \int_0^{0.6} (0.6 - s) u(s) ds, \end{cases} \tag{4.8}$$

where

$$\mathcal{H}(t, x) = \left[\varphi(t), \varphi(t) + \frac{|x|}{(1+\sqrt{t})^2} \psi(t)\right],$$

$$\psi(t) = \begin{cases} \sqrt{t} - t & t \in [0, 1] \\ 0 & t > 1 \end{cases}, \qquad \varphi(t) = \begin{cases} \cos t & t \in [0, \frac{\pi}{2}] \\ 0 & t > \frac{\pi}{2} \end{cases}$$

and

$$\frac{\|\psi\|_1}{\Gamma(\rho)} + \frac{\xi^{\beta-1}}{\Gamma(\rho)\Gamma(\beta)} (\xi + \frac{\xi^\rho}{\rho}) \approx 0.8041.$$

Then the problem (4.8) *has a least one solution.*

Corollary 2. *On suppose that the all conditions of theorem [4.2.1] hold with $\varphi = 0$. Then the function $u = 0$ is a solution for* (4.1) *.*

Theorem 4.2.2. *Assume that the conditions $(B1) - (B3)$ are hold*

$(B1)$ $\mathcal{H} : J \times \mathbb{R} \to \mathcal{P}_{c,cp}(\mathbb{R})$ *such that for every measurable function $u : J \to \mathbb{R}$ we have $t \in J \mapsto \mathcal{H}(t, u(t))$ is measurable.*

$(B2)$ *There are $m \in \mathrm{L}^1(J, \mathbb{R}) \cap \mathrm{L}^\infty(J, \mathbb{R})$ and φ continuous nondecreasing such that*

> *(i)* $\|\mathcal{H}(t, x)\| = \sup\{|v| : v \in \mathcal{H}(t, x)\} \leq m(t)\varphi(\frac{x}{(1+t^{\rho-1})^2})$
>
> *(ii)* $\forall \varepsilon > 0, \exists b_\varepsilon > 0$ *such that* $\int_{b_\varepsilon}^{+\infty} m(s) ds < \varepsilon$.

$(B3)$ *There exist $L > 0$ such that*

$$L > \frac{\|m\|_1 \varphi(L)}{\Gamma(\rho)} + \frac{\xi^{\beta-1}}{\Gamma(\rho)\Gamma(\beta)} (\xi + \frac{\xi^\rho}{\rho}) L.$$

Then the problem (4.1) *has a least one solution.*

Proof. To prove the problem (4.1) admits a solution, we have shown that $\mathcal{N}$ admits a fixed point.

From $(B1)$ and $(B2)(i)$ we can easily to show that $S_{\mathcal{H},u} \neq \emptyset$.

From step 1 in proof of theorem [4.2.1], we can see that $\mathcal{N}$ is convex valued. Now we prove that satisfied all hypothesis of theorem [1.3.2] the proof is given in four steps.

Step 1. ($\mathcal{N}$ bounded set into bounded set.)

Let $B_r = \{u \in \mathbb{X}, \|u\|_{\mathbb{X}} < r\}$ and let $h \in \mathcal{N}(u)$ with $u \in B_r$, then there exist $v \in S_{\mathcal{H},u}$ such that

$$h(t) = \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} v(s)ds + \frac{t^{\rho-1}}{\Gamma(\rho)\Gamma(\beta)} \int_0^\xi (\xi - s)^{\beta-1} u(s)ds \quad \text{for } i \in \{1,2\}.$$

For all $t \in J$ we have

$$\begin{aligned} \frac{|h(t)|}{1+t^{\rho-1}} &\leq \frac{1}{\Gamma(\rho)} \int_0^t |v(s)|ds + \frac{\xi^{\beta-1}}{\Gamma(\rho)\Gamma(\beta)} \int_0^\xi |u(s)|ds \\ &\leq l = \frac{\|m\|_1 \times \varphi(\|u\|_{\mathbb{X}})}{\Gamma(\rho)} + \frac{\xi^{\beta-1}}{\Gamma(\rho)\Gamma(\beta)} (\xi + \frac{2\xi^\rho}{\rho} + \frac{\xi^{2\rho-1}}{2\rho-1}) \|u\|_{\mathbb{X}}, \end{aligned}$$

and

$$\lim_{t \to +\infty} \frac{|h(t)|}{(1+t^{\rho-1})^2} \leq \lim_{t \to +\infty} \frac{l}{1+t^{\rho-1}} = 0.$$

Step 2 ($\mathcal{N}$ bounded set into equicontinuous set.)

$$\begin{aligned}
\left|\frac{h(t_1)}{(1+t_1^{\rho-1})^2}-\frac{h(t_2)}{(1+t_2^{\rho-1})^2}\right| &= \frac{1}{\Gamma(\rho)}\int_{t_1}^{t_2}\frac{(t_2-s)^{\rho-1}}{(1+t_2^{\rho-1})^2}v(s)ds\\
&+\frac{1}{\Gamma(\rho)}\int_0^{t_1}\left(\frac{(t_2-s)^{\rho-1}}{(1+t_2^{\rho-1})^2}-\frac{(t_1-s)^{\rho-1}}{(1+t_1^{\rho-1})^2}\right)v(s)ds\\
&+\left[\frac{1}{\Gamma(\rho)\Gamma(\beta)}\int_0^{\xi}(\xi-s)^{\beta-1}u(s)ds\right]\left|\frac{t_2^{\rho-1}}{(1+t_2^{\rho-1})^2}-\frac{t_1^{\rho-1}}{(1+t_1^{\rho-1})^2}\right|\\
&\leq \frac{1}{\Gamma(\rho)}\times\frac{(t_2-t_1)^{\rho-1}}{(1+t_2^{\rho-1})^2}\int_{t_1}^{t_2}|v(s)|ds\\
&+\frac{1}{\Gamma(\rho)}\int_0^{t_1}|v(s)|ds\times\sup_{s\in[0,t_1]}\left|\frac{(t_2-s)^{\rho-1}}{(1+t_2^{\rho-1})^2}-\frac{(t_1-s)^{\rho-1}}{(1+t_1^{\rho-1})^2}\right|\\
&+\left[\frac{1}{\Gamma(\rho)\Gamma(\beta)}\int_0^{\xi}(\xi-s)^{\beta-1}u(s)ds\right]\left|\frac{t_2^{\rho-1}}{(1+t_2^{\rho-1})^2}-\frac{t_1^{\rho-1}}{(1+t_1^{\rho-1})^2}\right|\\
&\leq \frac{\|m\|_1\times\varphi(\|u\|_{\mathbb{X}})}{\Gamma(\rho)}\times\frac{(t_2-t_1)^{\rho-1}}{(1+t_2^{\rho-1})^2}\\
&+\frac{\|m\|_1\times\varphi(\|u\|_{\mathbb{X}})}{\Gamma(\rho)}\times\sup_{s\in[0,t_1]}\left|\frac{(t_2-s)^{\rho-1}}{(1+t_2^{\rho-1})^2}-\frac{(t_1-s)^{\rho-1}}{(1+t_1^{\rho-1})^2}\right|\\
&+\left[\frac{1}{\Gamma(\rho)\Gamma(\beta)}\int_0^{\xi}(\xi-s)^{\beta-1}u(s)ds\right]\left|\frac{t_2^{\rho-1}}{(1+t_2^{\rho-1})^2}-\frac{t_1^{\rho-1}}{(1+t_1^{\rho-1})^2}\right|\\
&\leq \frac{\|m\|_1\times\varphi(\|u\|_{\mathbb{X}})}{\Gamma(\rho)}\times\frac{(t_2-t_1)^{\rho-1}}{(1+t_2^{\rho-1})^2}\\
&+\frac{\|m\|_1\times\varphi(\|u\|_{\mathbb{X}})}{\Gamma(\rho)}\times\sup_{s\in[0,t_1]}\left|\frac{(t_2-s)^{\rho-1}}{(1+t_2^{\rho-1})^2}-\frac{(t_1-s)^{\rho-1}}{(1+t_1^{\rho-1})^2}\right|\\
&+\frac{c_\xi\|u\|_{\mathbb{X}}}{\Gamma(\rho)\Gamma(\beta)}\times\left|\frac{t_2^{\rho-1}}{(1+t_2^{\rho-1})^2}-\frac{t_1^{\rho-1}}{(1+t_1^{\rho-1})^2}\right|,
\end{aligned}$$

where $c_\xi=\int_0^\xi(\xi-s)^{\beta-1}(1+s^{\rho-1})^2ds$. As $t_2\to t_1$ we have $|\frac{h(t_1)}{1+t_1^{\rho-1}}-\frac{h(t_2)}{1+t_2^{\rho-1}}|\to 0$.

Step 3. ($\mathcal{N}$ has a closed graph.)

Let $u_n\to u_*, h_n\in\mathcal{N}(u_n)$ and $h_n\to h_*$, we need to show that $h_*\in\mathcal{N}(u_*)$. Now $h_n\in\mathcal{N}(u_n)$ implies that there exists $v_n\in S_{\mathcal{H},u_n}$ such that

$$h_n(t) = \frac{1}{\Gamma(\rho)}\int_0^t(t-s)^{\rho-1}v_n(s)ds+\frac{t^{\rho-1}}{\Gamma(\rho)\Gamma(\beta)}\int_0^\xi(\xi-s)^{\beta-1}u_n(s)ds, \tag{4.9}$$

we have to show that there exists $v_*\in S_{\mathcal{H},u_*}$ such that for every $t\in[0,1]$,

$$h_*(t) = \frac{1}{\Gamma(\rho)}\int_0^t(t-s)^{\rho-1}v(s)ds+\frac{t^{\rho-1}}{\Gamma(\rho)\Gamma(\beta)}\int_0^\xi(\xi-s)^{\beta-1}u_*(s)ds. \tag{4.10}$$

By theorem 1.6.3, we prove that the set $\mathcal{V} = \{v_n \in S_{\mathcal{H},u_n}; n \geq 1\}$ is weakly compact in $\mathrm{L}^1(J;\mathbb{R})$

$$\int_A |v_n(s)|ds \leq |A|\|m\|_\infty \varphi(\|u_n\|_{\mathbb{X}}) < \varepsilon,$$

we take $\delta = \frac{\varepsilon}{\|m\|_\infty \varphi(\sup_{n\geq 1}\|u_n\|_{\mathbb{X}})}$, then the first conditions of theorem 1.6.3 is satisfied.
Now we prove the second conditions of this theorem, from (B2)(ii) there exist $b = b(\varepsilon)$ such that

$$\int_b^{+\infty} v_n(s)ds \leq \varphi(\|u_n\|_{\mathbb{X}}) \int_b^{+\infty} m(s)ds < \varepsilon.$$

Then by theorem 1.6.3 we conclude that $\mathcal{V}$ is weakly compact set in $\mathrm{L}^1(J,\mathbb{R})$.
Conclusion.
There exist a subsequence $v_{n_k} \in S_{\mathcal{H},u_{n_k}}$ such that $v_{n_k} \rightharpoonup v$ in $\mathrm{L}^1(J,\mathbb{R})$, then

$$h_*(t) = \frac{1}{\Gamma(\rho)}\int_0^t (t-s)^{\rho-1}v(s)ds + \frac{t^{\rho-1}}{\Gamma(\rho)\Gamma(\beta)}\int_0^\xi (\xi-s)^{\beta-1}u_*(s)ds. \tag{4.11}$$

because

$$\int_0^\xi (\xi-s)^{\beta-1}|u_n(s)-u(s)|ds \leq \int_0^\xi |u_n(s)-u(s)|ds = \int_0^\xi (1+s^{\rho-1})^2\frac{|u_n(s)-u(s)|}{(1+s^{\rho-1})^2}ds$$

(see $0 < \xi < 1$)

$$\leq \|u_n - u\|_{\mathbb{X}} \times \int_0^\xi (1+s^{\rho-1})^2 ds \to 0 \quad as \quad n \to \infty.$$

Step 4. (a priori bounds on solutions.) Let $u \in \mathcal{N}(u)$ for $s \in (0,1)$. Then for every $t \in J$,

$$\frac{|u(t)|}{1+t^{\rho-1}} \leq \frac{\|m\|_1 \times \varphi(\|u\|_{\mathbb{X}})}{\Gamma(\rho)} + \frac{\xi^{\beta-1}}{\Gamma(\rho)\Gamma(\beta)}(\xi + \frac{\xi^\rho}{\rho})\|u\|_{\mathbb{X}},$$

then

$$\|u\|_{\mathbb{X}} \leq \frac{\|m\|_1 \times \varphi(\|u\|_{\mathbb{X}})}{\Gamma(\rho)} + \frac{\xi^{\beta-1}}{\Gamma(\rho)\Gamma(\beta)}(\xi + \frac{\xi^\rho}{\rho})\|u\|_{\mathbb{X}}.$$

Now, Let $\mathcal{V} = \{u \in \mathbb{X} : \|u\|_{\mathbb{X}} < L\}$.
The set $\mathcal{V}$ is an open set in $\mathbb{X}$ with $0 \in \mathcal{V}$. From the steps 1, step 2, step 3 and step 4, together with theorem 1.3.2, we get that $s\mathcal{N} : \bar{\mathcal{V}} \to \mathcal{P}_{c,cp}(\mathbb{X})$ is upper semicontinuous and completely continuous. From (B3) there is no $u \in \partial\mathcal{V}$ such that $u \in s\mathcal{N}(u)$ for $s \in (0,1)$. Therefore by theorem 1.3.2, we deduce that $\mathcal{N}$ has a fixed point $u \in \bar{\mathcal{V}}$, which represents a solution of the problem 4.1. This finishes the proof. □

Example 5. *Consider the following fractional differential inclusion*

$$\begin{cases} \mathcal{D}^{\frac{3}{2}}_{0^+}u(t) \in \mathcal{H}(t,u(t)) \text{ for almost } t \in [0,+\infty[, \\ u(0) = 0, \\ \mathcal{D}^{\frac{1}{2}}_{0^+}u(0) = \int_0^{0.6}(0.6-s)u(s)ds, \end{cases} \tag{4.12}$$

where

$$\mathcal{H}(t,x) = \left[\frac{m(t)\exp\frac{x}{(1+\sqrt{t})^2}}{2}, m(t)\exp\frac{x}{(1+\sqrt{t})^2}\right],$$

$$m(t) = \begin{cases} \frac{1}{7(1+t^2)} & t \in [0,1], \\ 0 & t > 1, \end{cases}$$

we take $L = 1$*, we have* $L > \frac{\|m\|_1 \times \varphi(L)}{\Gamma(\rho)} + \frac{\xi^{\beta-1}}{\Gamma(\rho)\Gamma(\beta)}(\xi + \frac{\xi^\rho}{\rho})L \approx 0.9602.$
Then the problem (4.12) *has a least one solution in* $\mathbb{X}$.

4.2.2 Existence and uniqueness result in nonconvex case

Theorem 4.2.3. *Assume that the following assertions are met*

($C1$) $\mathcal{H} : J \times \mathbb{R} \to \mathcal{P}_{cl}(\mathbb{R})$ *such that for every measurable function* $u : J \to \mathbb{R}$ *we get* $t \mapsto \mathcal{H}(t,u(t))$ *is measurable.*

($C2$) *There is a function* $m_1 \in \mathrm{L}^\infty(J,\mathbb{R}^+)$ *such that*

- *(i)* $\|\mathcal{H}(t,x)\| \leq m_1(t)$
- *(ii)* $\forall \varepsilon > 0,\ \exists b_\varepsilon > 0$ *such that*

$$\int_{b_\varepsilon}^{+\infty} m_1(s)ds < \varepsilon.$$

($C3$) *There are* $m_2 \in \mathrm{L}^1(J,\mathbb{R}^+)$ *such that*

$$H_d(\mathcal{H}(t,x);\mathcal{H}(t,y)) \leq m_2(t)\frac{|x-y|}{(1+t^{\rho-1})^2}.$$

($C4$)

$$\frac{\|m_2\|_1}{\Gamma(\rho)} + \frac{c_\xi}{\Gamma(\rho)\Gamma(\beta)} < 1 \quad \text{with } c_\xi = \int_0^\xi (\xi-s)^{\beta-1}(1+s^{\rho-1})^2 ds.$$

Then the problem (4.1) *has a unique solution in* $\mathbb{X}$.

Proof. Let $\mathcal{N}$ the set valued map defined (4.2.1), we prove that $\mathcal{N}$ has a unique fixed point, which represent the unique solution.

Let $u \in \mathbb{X}$, from $(C1)$ and $(C2)$(i) and lemma 1.4.1 the set $S_{\mathcal{H},u} \neq \emptyset$.

1. Proof the existence.

Step 1. $(\mathcal{N}(u) \in \mathcal{P}_{cl}(\mathbb{X}))$. Let (u_n) be a set in $\mathcal{N}(u)$ with $u_n \to \tilde{u}$ in $\mathbb{X}$. After that $\tilde{u} \in \mathbb{X}$ and there exist $v_n \in S_{\mathcal{H},u}$ with for $t \in J$

$$u_n(t) = \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} v_n(s) ds + \frac{t^{\rho-1}}{\Gamma(\rho)\Gamma(\beta)} \int_0^{\xi} (\xi - s)^{\beta-1} u(s) ds.$$

By $(C2)$ the set $\mathcal{V} = \{v_n \in S_{\mathcal{H},u}; n \geq 1\}$ satisfies all the conditions of theorem 1.6.3. Then we have $\mathcal{V}$ is weakly compact in $\mathrm{L}^1(J,\mathbb{R})$, then there exist $v \in \mathcal{V}$ such that

$$\tilde{u}(t) = \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} v(s) ds + \frac{t^{\rho-1}}{\Gamma(\rho)\Gamma(\beta)} \int_0^{\xi} (\xi - s)^{\beta-1} u(s) ds.$$

Step 2. There exists $\gamma < 1$ such that

$$H_d(\mathcal{N}(u), \mathcal{N}(\tilde{u})) \leq \gamma \|u - \tilde{u}\|_{\mathbb{X}}.$$

Indeed. Let $u, \tilde{u} \in \mathbb{X}$ and $h_1 \in \mathcal{N}(u)$, then there exist $v_1 \in S_{\mathcal{H},u}$ such that for every $t \in J$

$$h_1(t) = \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} v_1(s) ds + \frac{t^{\rho-1}}{\Gamma(\rho)\Gamma(\beta)} \int_0^{\eta} (\eta - s)^{\beta-1} u(s) ds.$$

From $(C3)$, we deduce

$$H_d\Big(\mathcal{H}(t, u(t)); \mathcal{H}(t, \tilde{u}(t))\Big) \leq m_2(t) \frac{|u(t) - \tilde{u}(t)|}{(1 + t^{\rho-1})^2} \quad \text{for almost } t \in J.$$

Hence, there exists $t \in \mathcal{H}(t, \tilde{u}(t))$ such that

$$|v_1(t) - t| \leq m_2(t) \frac{|u(t) - \tilde{u}(t)|}{(1 + t^{\rho-1})^2} \quad \text{for almost } t \in J.$$

For almost $t \in J$, we consider the set-valued map defined as follows:

$$U(t) = \{t \in \mathbb{R} : |v_1(t) - t| \leq m_2(t) \frac{|u(t) - \tilde{u}(t)|}{1 + t^{\rho-1}}\} := \mathcal{B}(v_1(t), \zeta(t)),$$

where $\zeta(t) = m_2(t) \frac{|u(t) - \tilde{u}(t)|}{1 + t^{\rho-1}}$. Assumption $(C1)$ implies that the set-valued map $t \mapsto \mathcal{F}(t, \tilde{u}(t))$ is measurable. Since v_1 and ζ are measurable, it follows from proposition 1.4.1 that the multivalued U is measurable, and the set valued map $t \mapsto V(t) = U(t) \cap \mathcal{H}(t, \tilde{u}(t))$ is nonempty, measurable (see proposition 1.4.2) and closed set-valued, there are by lemma 1.4.1 an element v_2 which is a measurable selection for V. So

$$v_2(t) \in \mathcal{H}(t, \tilde{u}(t)),$$

and for almost $t \in J$, we have

$$|v_1(t) - v_2(t)| \leq m_2(t)\frac{|u(t) - \tilde{u}(t)|}{1 + t^{\rho-1}}.$$

For each $t \in J$, let us define

$$h_2(t) = \frac{1}{\Gamma(\rho)}\int_0^t (t-s)^{\rho-1} v_2(s)ds + \frac{t^{\rho-1}}{\Gamma(\rho)\Gamma(\beta)}\int_0^\xi (\xi - s)^{\beta-1}\tilde{u}(s)ds.$$

Hence

$$\begin{aligned}|h_1(t) - h_2(t)| \;\leq\; & \frac{1}{\Gamma(\rho)}\int_0^t (t-s)^{\rho-1}|v_1(s) - v_2(s)|ds \\ & + \frac{t^{\rho-1}}{\Gamma(\rho)\Gamma(\beta)}\int_0^\xi (\xi - s)^{\beta-1}|u(s) - \tilde{u}(s)|ds,\end{aligned}$$

and

$$\frac{|h_1(t) - h_2(t)|}{(1 + t^{\rho-1})^2} \leq \frac{1}{\Gamma(\rho)}\int_0^t |v_1(s) - v_2(s)|ds + \frac{c_\xi}{\Gamma(\rho)\Gamma(\beta)}\|u - \tilde{u}\|_{\mathbb{X}},$$

with $c_\xi = \int_0^\xi (\xi - s)^{\beta-1}(1 + s^{\rho-1})^2 ds$. Then

$$\|h_1 - h_2\|_{\mathbb{X}} \leq \left(\frac{\|m_2\|_1}{\Gamma(\rho)} + \frac{c_\xi}{\Gamma(\rho)\Gamma(\beta)}\right)\|u - \tilde{u}\|_{\mathbb{X}}.$$

We apply theorem 1.3.1 we get the set valued map $\mathcal{N}$ has a fixed point, which represent a solution.

2. Proof the uniqueness.

Let u_1, u_2 are two solution for (4.1), then there exist $v_1 \in S_{\mathcal{H},u_1}$ and $v_2 \in S_{\mathcal{H},u_2}$ such that

$$u_j(t) \;=\; \frac{1}{\Gamma(\rho)}\int_0^t (t-s)^{\rho-1} v_j(s)ds + \frac{t^{\rho-1}}{\Gamma(\rho)\Gamma(\beta)}\int_0^\xi (\xi - s)^{\beta-1}u_j(s)ds,$$

for $1 \leq j \leq 2$. Then

$$\|u_1 - u_2\|_{\mathbb{X}} \leq \left(\frac{\|m_2\|_1}{\Gamma(\rho)} + \frac{c_\xi}{\Gamma(\rho)\Gamma(\beta)}\right)\|u_1 - u_2\|_{\mathbb{X}}.$$

It's obvious that $u_1 = u_2$. $\square$

Example 6. *Consider the following fractional differential inclusion*

$$\begin{cases} \mathcal{D}_{0^+}^{\frac{3}{2}} u(t) \in \mathcal{H}(t, u(t)) \quad \text{for almost} \quad t \in J, \\ u(0) = 0, \\ \mathcal{D}_{0^+}^{\frac{1}{2}} u(0) = I_{0^+}^{1.7} u(\frac{1}{4}), \end{cases} \tag{4.13}$$

where

$$\mathcal{H}(t, x) = \{1\} \cup [\frac{3}{2}, \frac{3}{2} + \frac{m(t)}{(1+\sqrt{t})^2} |\sin x|],$$

and

$$m(t) = \begin{cases} \frac{1}{4} & t \in [0, 1] \\ 0 & t > 1. \end{cases}$$

It's clearly that the set valued map $\mathcal{H}$ *is measurable, and we take*

$$m_1(t) = \frac{3}{2} + \frac{m(t)}{(1+\sqrt{t})^2},$$
$$m_2(t) = m(t),$$

$c_\xi < \frac{1}{2}$ *and* $\frac{\|m_2\|_1}{\Gamma(\rho)} + \frac{c_\xi}{\Gamma(\rho)\Gamma(\beta)} < 0.9031$.
After that we conclude the problem (4.13) *has a unique solution.*

Corollary 3. *If the conditions of precedent theorem are satisfied and the function* $m_1 = 0$. *Then the unique solution for* (4.1) *is* $u = 0$.

Corollary 4. *If the conditions of precedent theorem are satisfied and* $0 \in F(t, 0)$ *for almost* $t \in J$. *Then the unique solution for* (4.1) *is* $u = 0$.

Conclusion

In this work, we have conducted a detailed study on the existence and uniqueness of solutions for a specific class of semilinear fractional differential inclusions on unbounded domains. By employing the properties of fractional calculus and various fixed point theorems, we successfully established a solid theoretical framework for these systems.

While this work provides significant insights into the qualitative theory of fractional inclusions, there are several important aspects that remained beyond the scope of this Master's research. These limitations provide fertile ground for future investigations:

- **Numerical Approximation:** Our study was primarily focused on the theoretical existence of solutions. Due to the complexity of fractional operators on the half-line, we did not implement a numerical scheme. In the future, we aim to develop efficient algorithms to simulate these models.

- **Stability Analysis:** We did not address the Ulam-Hyers stability of the considered problem. Investigating how small perturbations in the initial conditions or the fractional order affect the solution would be a crucial next step.

- **Practical Applications:** While the mathematical foundation is now established, the application of these specific inclusions to real-world problems in physics or biology remains an open area for our future work.

Moving forward, I plan to extend the results of this thesis in the following directions:

1. Developing numerical codes using software such as **FreeFem++** or **MATLAB** to visualize the behavior of fractional inclusions.

2. Exploring the **Global Controllability** of these systems, which is a key topic in control theory and engineering.

3. Extending the current results to **Stochastic Fractional Inclusions** to account for random perturbations and uncertainty in mathematical modeling.

Finally, this research has been a significant learning journey, allowing me to deepen my understanding of functional analysis and fractional operators, which I hope to apply in my future doctoral studies or professional career.

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