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الحمد لله عند البدء وعن الختام ,فما تناهى درب ولا ختم جيد ولا تم السعي إلا بفضله ,فاللهم لك الحمد إذا رضيت ولك الحمد بعد الرضا.

أهدي ثمرة جهدي المتواضع بكل حب وإمتنان

إلى من أحمل إسمه بكل فخر, معراج روحي وحبيبي الأول وظلي الخفي الذي مهد لي الطريق بكل ما عنده , ولم يبخل عني يوما بدعمه , أبي الغالي

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وإلى ريحانة قلبي وبلسمي الشافي إلى من حاوطتني دعواتها الخفية والحانية ومن أفرشت لي الأرض ذهبا , والصوت الذي يدفعني إلى الأمام حين يتسلل إلي اليأس , أمي الحبيبة

"فنو"

ولمن شد عضدي بهم مصابيح عتمتي ونجوم سمائي وصفوة أيامي وبهجة الحياة أخواتي وأخي

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وأخيرا

إلى كل من دعمني بكلمة طيبة,أو دعوة صادقة,أهديكم هذا العمل المتواضع الذي هو بداية حلم أكبر أسعى لتحقيقه

إبنتكم رانيا

Abbreviations and Notations

The different abbreviations and notations used throughout this dissertation are explained below:

SDE	Stochastic Differential Equations.
$BSDE$	Backward Stochastic Differential Equation.
$GBSDE$	Generalized Backward Stochastic Differential Equation.
$\mathbb{E}[X]$	Expectation at X.
$\mathbb{E}\left[X \vert \mathcal{B}\right]$	Conditional expectation of X given the σ-field $\mathcal{B}$.
Ω	Fundamental space of a random experiment.
$\mathbb{P}$	Propability.
$(\Omega, \mathcal{F}, \mathbb{P})$	Propability space.
$C = C\left(\mu, \alpha, \beta, K, T\right)$	A generic positive constant that may vary from line to line depending only on the parameters $\left(\mu, \alpha, \beta, K, T\right)$.
$a.s.$	Almost surely.
$i.e$	That is.
$\mathbb{P}{-}a.s$	Almost surely in propability.
$a.e.$	Almost everywhere.
$r.v$	Random variable.
$r.c.l.l.$	Right continuous with left limits.
$\Vert f \Vert_p$	$\mathbb{L}^p$-norm of the function f.

Contents

Introduction

In the modern and intricate domain of stochastic analysis, backward stochastic differential equations, commonly referred to as BSDEs, have undeniably emerged as a crucial and transformative mathematical framework that is instrumental in various applications. In contrast to the more traditional and widely used forward equations, BSDEs possess a distinctive design that allows them to ascertain not only the current state of a system but also to formulate the necessary hedging strategy that is imperative for achieving a specified objective in the future. This unique feature makes BSDEs particularly valuable in fields such as mathematical finance, stochastic control, and risk management, where future targets and current decision-making are intrinsically linked. Since their introduction by Pardoux and Peng [8], the theory of BSDEs has witnessed remarkable growth, evolving from linear to nonlinear formulations and extending into generalized frameworks that accommodate more realistic and complex system behaviors.

This dissertation delves into an advanced and intricate class of generalized backward stochastic differential equations (see Pardoux and Zhang [7]), which are characterized by their complexity and their capacity to extend the conventional understanding of BSDEs. Unlike standard BSDEs that rely solely on a Brownian motion driven term, the generalized formulation considered here incorporates an additional stochastic integral with respect to a continuous increasing process. The dynamics governing this sophisticated system are fundamentally driven by the following principal equation, which serves as the cornerstone for further exploration and analysis, setting the stage for a comprehensive investigation into the implications and applications of these equations in various theoretical and practical contexts:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds + \int_t^T g(s, Y_s)\, dA_s - \int_t^T Z_s dB_s,\ t \in [0, T].$$

In this study, the addition of the integral with respect to a continuous increasing process A_t is not merely a mathematical abstraction; it represents the inclusion of cumulative, irreversible factors such as continuous operational costs, asset depreciation, or systemic friction that persistently influence the trajectory of the system. This term, governed by the driver $g(s, Y_s)$, fundamentally alters the structure of the equation, introducing a path-dependent and non-reversible component. While the process W_t captures the random, diffusive noise (hedged by Z_t), the process A_t models a deterministic or predictable drift that accumulates over time, reflecting real-world phenomena where certain effects (like wear, tear, or cost accrual) cannot be reversed by future randomness.

Despite the extensive literature on standard BSDEs, the class of generalized BSDEs driven by a continuous non-decreasing process A_t remains relatively less explored, particularly when the driver g depends nonlinearly on the solution variable Y. The central objective of this study is therefore to establish the well-posedness of this class of Generalized BSDEs. We address the complexities arising from the nonlinear driver $g(s, Y_s)$ and the associated integral $\int_t^T g(s, Y_s)\, dA_s$, which does not fit into the classical Itô framework. A key contribution of this work is determining whether classical $\mathbb{L}^p$ estimates (e.g., energy estimates, Burkholder-Davis-Gundy inequalities) and stability results (e.g., continuity with respect to terminal data and coefficients) remain robust under the influence of this additional drift term. More precisely, we investigate the conditions under which existence and uniqueness of a solution pair (Y, Z) can be guaranteed, and whether the presence of A_t introduces new technical challenges, such as a breakdown of standard comparison theorems or a need for refined a priori bounds.

To address the problem statement and test the above hypotheses, this research adopts an analytical approach grounded in stochastic calculus and functional analysis. The main steps include:

1. Defining appropriate function spaces (e.g.,$\mathbb{L}^2$ or $\mathbb{L}^p$ spaces) of progressively measurable

processes with norms adapted to the presence of both W_t and A_t.

2. Proving a priori estimates using Itô's formula, Gronwall-type lemmas, and careful handling of the integral with respect to A_t (viewed as a Stieltjes integral).

3. Establishing existence and uniqueness via a contraction mapping argument.

4. Investigating stability by examining the continuous dependence of (Y, Z) on the parameters $(\xi, f; g)$.

5. Exploring potential extensions, such as allowing g to have quadratic growth or relaxing the Lipschitz condition.

The remainder of this dissertation is organized as follows: In the first chapter, we present all the concepts related to stochastic calcul that we need in this work. Chapter 2 presents the existence and uniquesess result for SDEs and BSDEs. Chapter 3 contains the main theoretical results: existence and uniqueness theorems for the generalized BSDE under Lipschitz conditions, along with detailed proofs.

Chapter 1

Background on stochastic calculus

1.1 Conditional expectation

Definition 1.1 *A probability space is defined as a triple* $(\Omega, \mathcal{F}, \mathbb{P})$*, where:*

1. A non-empty set Ω, called the sample space.

2. A $\sigma-$algebra $\mathcal{F}$ of subsets of Ω with the following properties:

   - $\varnothing \in \mathcal{F}$.

   - $A \in \mathcal{F} \implies A^c \in \mathcal{F}$, where A^c is the complement of A in Ω.

   - $A_1, A_2, \ldots \in \mathcal{F}$ implies that $\bigcup\limits_{i=1}^{\infty} A_i \in \mathcal{F}$.

3. A probability measure $\mathbb{P} : \mathcal{F} \to [0, 1]$ on measurable space $(\Omega, \mathcal{F})$, that satisfies the following axioms:

   - **Non-negativity**: $\mathbb{P}(A) \geq 0$, for all $A \in \mathcal{F}$.

   - **Normalization**: $\mathbb{P}(\Omega) = 1$, $\mathbb{P}(\varnothing) = 0$.

   - $\sigma-$**additivity**: For any sequence of disjoint events $\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{F}$, we have:

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

Definition 1.2 *(**Random Variable** [1]) Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a random variable X is a measurable function $X\colon \Omega \longrightarrow \mathbb{R}$ such that for every Borel set $B \in \mathcal{B}(\mathbb{R})$, the preimage in the σ-algebra $\mathcal{F}$:*

$$X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F}.$$

Remark 1.1 *The Borel $\sigma-$algebra on $\mathbb{R}$, denoted by $\mathcal{B}(\mathbb{R})$, is the smallest $\sigma-$algebra containing all open intervals of the form (a, b) for any $(a, b) \in \mathbb{R}^2$.*

Definition 1.3 *(**Conditional Expectation)** Let $X \in \mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathcal{G}$ be sub $\sigma-$algebra of $\mathcal{F}$ $(\mathcal{G}\subset\mathcal{F})$, the conditional expectation of X given $\mathcal{G}$ denoted by $Y = \mathbb{E}[X|\mathcal{G}]$, is the unique r.v satisfying the following two conditions:*

1. $\mathcal{G}-$**Measurability:** Y is measurable with respect to the sub $\sigma-$algebra $\mathcal{G}$.

2. **Integration Property**: For every set $A \in \mathcal{G}$, we have:

$$\int_A Y(\omega)\, d\mathbb{P}(\omega) = \int_A X(\omega)\, d\mathbb{P}(\omega).$$

1.1.1 Properities of conditional expectation

Proposition 1.1 *Let $X, Y \in \mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G}$ sub $\sigma-$algebra of $\mathcal{F}$, then, we have $\mathbb{P}-$almost surely the following properities:*

1. Linearity: For any constants $a, b \in \mathbb{R} : \mathbb{E}[aX + bY|\mathcal{G}] = a\mathbb{E}[X|\mathcal{G}] + b\mathbb{E}[Y|\mathcal{G}]$.

2. Monotonicity: If $X \leq Y$ almost surely, then: $\mathbb{E}[X|\mathcal{G}] \leq \mathbb{E}[Y|\mathcal{G}]$.

3. Independence: If X is idependence of the σ-algebra of $\mathcal{G}$, we have $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X]$.

4. If X is $\mathcal{G}-$measurable, then $\mathbb{E}[X|\mathcal{G}] = X$.

5. If $\mathcal{G}_2 \subset \mathcal{G}_1 \subset \mathcal{F}$, then, we get $\mathbb{E}[\mathbb{E}[X|\mathcal{G}_1]|\mathcal{G}_2] = \mathbb{E}[X|\mathcal{G}_2]$.

6. If Y is $\mathcal{G}-$measurable and bounded r.v, we have $\mathbb{E}[YX|\mathcal{G}] = Y\mathbb{E}[X]$.

7. Jensen's inequality: If $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is a convex function and $\mathbb{E}[|\phi(X)|] < \infty$, then

$$\phi(\mathbb{E}[X|\mathcal{G}]) \leq \mathbb{E}[\phi(X)|\mathcal{G}].$$

Following the development of the static framework of random variables and conditional expectations, the introduction of time becomes essential. In the framework of backward stochastic differential equations, uncertainty evolves dynamically over a fixed time horizon $[0, T]$. This temporal evolution is rigorously formalized through filtrations and stochastic processes.

1.2 Stochastic process and filtration

Definition 1.4 *A stochastic process is a family of random varibales* $X = (X_t)_{t\geq 0}$,*defined on* $(\Omega, \mathcal{F}, \mathbb{P})$ *and taking values in* $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$:

- For each fixed $\omega \in \Omega$ the mapping: $t \longmapsto X_t(\omega)$, is called a sample path (or trajectory) of the process.
- For each fixed $t \in T$ the mapping: $\omega \longmapsto X_t(\omega)$, is a r.v.

Remark 1.2 *Equivalently, the process may be viewed as a measurable mapping*

$$\begin{aligned} X: \quad (T \times \Omega) &\longrightarrow (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \\ (t, \omega) &\longrightarrow X(t, \omega). \end{aligned}$$

Definition 1.5 *A filtration on* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a sequence of sub* $\sigma-$*algebra* $(\mathcal{F}_t)_{t\geq 0}$ *of* $\mathcal{F}$ *that is increasing with respect to inclusion meaning, for all* $0 \leq s \leq t \leq T$ *such that* $\mathcal{F}_s \subset \mathcal{F}_t$.

Remark 1.3 *We said a filltration* $(\mathcal{F}_t)_{t\geq 0}$ *satisfy the usual conditions if:*

- $\mathcal{F}_0$ contains all $\mathbb{P}-$null sets of $\mathcal{F}$.
- $(\mathcal{F}_t)_{t\geq 0}$ is right-continuity**:** For all $t \geq 0$, we have

$$\mathcal{F}_t = \mathcal{F}_{t^+} \triangleq \bigcap_{s>t} \mathcal{F}_s.$$

Definition 1.6 ***(Adaptivity)*** *A process X is $(\mathcal{F}_t)_{t\geq 0}-$adapted if the r.v X_t is $\mathcal{F}_t-$measurable for each $t \geq 0$.*

Definition 1.7 *The stochastic process $(X_t)_{t\geq 0}$ is progressevly measurable with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ if the mapping*

$$\begin{array}{llll} X: & ([0,t]\times\Omega, \mathcal{B}([0,t])\otimes\mathcal{F}_t) & \longmapsto & \left(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right)\right) \\ & (\omega, s) & \longmapsto & X(\omega, s), \end{array}$$

is mesurable for each $t \geq 0$.

Theorem 1.1 ***(Relation between Adaptivity and Progressive Measurability** [2]) *Let $(X_t)_{t\geq 0}$ be an adapted process with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$. If the sample paths $t \longmapsto X_t(w)$ are right-continuous (or left-continuous) for almost all $\omega \in \Omega$, then $(X_t)_{t\geq 0}$ is progressevely measurable with respect to $(\mathcal{F}_t)_{t\geq 0}$.*

Definition 1.8 ***(Continuous Increasing Process)*** *A process $(A_t)_{t\geq 0}$ is called a continuous increasing process if it 's $(\mathcal{F}_t)_{t\geq 0}-$adapted, $A_0 = 0$ a.s, and the mapping $t \rightarrow A_t(\omega)$ is continuous and non-decreasing for almost all ω.*

Theorem 1.2 *(**Doob-Meyer Decomposition**) Let Y_t be a right-continuous supermartingale. Then there exists a unique decomposition:*

$$Y_t = M_t - A_t$$

where M_t is a martingale and A_t is a predictable increasing process.

Definition 1.9 *A process X defined a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is called a Markov process if it satisfies the following conditions:*

1. **Adaptivity:** X_t is adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$.

2. **Independence of the future:** For any $s > t$, the future state X_s is independent of the past information $\mathcal{F}_t$, given the present state X_t.

1.3 Martingale and Brownian motion

Definition 1.10 ***(Martingales)*** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathcal{F}_t)_{t\geq 0}$ a filtration. An stochastic process $M = (M_t)_{t\geq 0}$ is a martingale if:*

1. M_t is $(\mathcal{F}_t)_{t\geq 0}$ −adapted.

2. For any $t \in [0, T]$, M_t−is itegrable, **i.e.**$(\mathbb{E}[| X |] < \infty)$.

3. For all $0 \leq s \leq t$, $\mathbb{E}[M_t | \mathcal{F}_s] = M_s$.

And is a called (Supermartingales, resp Submartingales) when:

$$\text{For all } 0 \leq s \leq t,\ \mathbb{E}[M_t | \mathcal{F}_s] \geq M_s,\ \text{resp } \mathbb{E}[M_t | \mathcal{F}_s] \leq M_s.$$

Definition 1.11 ***(Stoping Time)*** *A random variable $\tau : \Omega \to [0, T] \cup \{+\infty\}$ on a filtered probability space $\left(\Omega\ , \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ is called a stoping time if for every $t \geq 0$, we have*

$$\{\tau \leq t\} \in \mathcal{F}_t.$$

Definition 1.12 ***(Local Martingale [4])*** *A continuous stochastic process M on $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ is called local martingal if there exists a sequence of stoping times $\{\tau\}_{n=1}^{\infty}$ such that:*

1. The sequence is non-decreasing, i.e.$\tau_n \leq \tau_{n+1}$, and $\lim_{n\to\infty} \tau_n = \infty$ a.s.

2. For each $n \geq 1$, the stopped process$M_{t\wedge\tau_n}$ is an $(\mathcal{F}_t)_{t\geq 0}$ −martingale.

1.3.1 Brownian motion

Definition 1.13 *A standard, one-dimensional Brownian motion (also known as a Wiener process) is a stochastic process $W = (W_t(\omega))_{t\geq 0}$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, that satisfies the following fundamental properties:*

- **Initial Value**: $\mathbb{P}\{\omega,\ W_0(\omega) = 0\} = 1$, such that $W_0(\omega) = 0$.

- **Independent Increments**: For any times $0 \leq s < t$, the increment $W_t - W_s$ is independent of the information (filtration) available up to time s, denoted by $\mathcal{F}_s$.

- **Stationary Gaussian Distribution**: Each increment $(W_t - W_s) \rightsquigarrow \mathcal{N}(0, t-s)$.

- **Path Continuity**: The sample paths $t \longmapsto W_t(\omega)$ are continuous functions of time almost everywhere, meaning the motion has no jumps.

1.3.2 The Browian sample paths

Definition 1.14 *A stochastic process $(X_t)_{t\geq 0}$ is defined as Gaussian if, for any finite set of times $0 \leq t_1 \leq t_2 \leq \cdots \leq t_k < \infty$, the random vector $\{X_{t_1}, \cdots, X_{t_k}\}$ follows a joint normal distribution.*

- **Characterization**: The distribution is entirely determined by its mean function $m(t) = \mathbb{E}[X_t]$ and its covariance function $\rho(s,t) = \mathbb{E}[(X_s - m(s))(X_t - m(t))^\top]$.

- **Application to Brownian Motion**: Standard Brownian motion W, is a zero-mean Gaussian process $m(t) = 0$ with a covariance function $\rho(s,t) = s \wedge t$.

Lemma 1.1 *Brownian motion possesses remarkable structural flexibility. The following transformations of a standard Brownian motion W result in a new process that is also a standard Brownian motion:*

1. *Scaling:* For $c > 0$ X_t defined as

$$X_t = \frac{1}{\sqrt{c}} W_{ct} \qquad , 0 \leq t < \infty.$$

2. *Time iversion:* The process$Y = (Y_t)_{t\leq 0}$ defined by

$$Y_t = \begin{cases} tW_{\frac{1}{t}}, & 0 < t < \infty \\ 0, & t = 0. \end{cases}$$

3. *Time reversel:* For $T > 0$ $Z = (Z_t)_{t \leq 0}$ defined by

$$Z_t = W_T - W_{T-1} \qquad , 0 \leq t < \infty.$$

4. *Symmetry:* $-W = (-W_t)_{t \leq 0}$.

Nowhere Differentiability

Theorem 1.3 *The Brownian motion $W_t(\omega)$ is nowhere differentiable for almost $\omega \in \Omega$.*

Proof. See [4, page 110]. ■

Law of iterated logarithm

While Brownian motion is characterized by its high volatility and non-differentiability, its long-term and short-term fluctuations are not completely arbitrary. The Law of the iterated logarithm provides the precise growth envelopes that bound the sample paths with probability one.

- **The Growth Envelopes:** The theorem identifies a specific function, $\phi(t)$, which acts as a stochastic boundary for the process. This function is defined as

$$\phi(t) = \sqrt{2t \log \log (t)}.$$

  Theorem 1.4 ***(Khinchin's Law)*** *For a standard Brownian motion $W_t(\omega)$, the sample paths satisfy the following limits a.s:*

$$\limsup_{t \to \infty} \frac{W_t(\omega)}{\sqrt{2t \log \log (t)}} = 1, \qquad \liminf_{t \to \infty} \frac{W_t(\omega)}{\sqrt{2t \log \log (t)}} = -1$$

- **Local Behavior at the Origin:** Due to the Time-Inversion property, a similar law applies as $t \to 0$. This describes how the process "starts" its fluctuations immediately after leaving the origin:

$$\limsup_{t\downarrow\infty}\frac{W_t(\omega)}{\sqrt{2t\log\log(t)}}=1$$

Modulus of continuity

A function $g(\cdot)$ is called a modulus of continuity for a function $f{:}[0,T]\rightarrow\mathbb{R}$ if, for all sufficiently small positive δ, we have:

$$|f(t)-f(s)|\leq g(\delta),\qquad \text{whenever } 0\leq s<t\leq T \text{ and } |t-s|\leq\delta.$$

Theorem 1.5 ***(Lévy's Exact Modulus)*** *Let $g(\delta)=\sqrt{2\delta\log(1/\delta)}$ for $\delta>0$. Then, for a Brownian motion W, we have*

$$\mathbb{P}\left[\limsup_{\substack{\delta\downarrow 0,\ 0\leq s<t\leq 1\\ |t-s|\leq\delta}}\frac{|W_t-W_s|}{g(\delta)}=1\right]=1.$$

Proof. See [4]. ■

1.4 The stochastic integral

Stochastic integral are a central in stochastic calculs,especially in the study of Brownian motion and Itô processes. Since classical integration is not applicable in this setting, a rigorous construction is reauired to define the Itô integral properly. This construction is carried out in a stepwise manner, starting from simple processes and extanding to more general cases in order to ensure mathimatical consistency.

1.4.1 Construction of the Itô integral

The construction follows a three-step approximation procedure (See [6] for more details):

1. Integration of Elementary Functions: We begin by defining the integral for elementary

functions $\phi \in \mathbb{L}^2$, which are step functions of the form:

$$\phi(t,\omega) = \sum_j e_j(\omega)\, \mathbf{1}_{[t_j,t_{j+1})}(t).$$

The Itô integral for such a function is defined as:

$$\int_t^T \phi(t,\omega)\, dW_t = \sum_{j\geq 0} e_j(\omega)\, [W_{t+j} - W_t](\omega).$$

Crucially, the evaluation point $e_j(\omega)$ is taken at the left endpoint of each sub-interval.

2. The Itô Isometry provides a link between the stochastic world and the deterministic $\mathbb{L}^2$ space:

$$\mathbb{E}\left[\left(\int_0^t \phi(t,\omega)\, dW_t\right)^2\right] = \mathbb{E}\left[\left(\int_0^t \phi(t,\omega)^2\, dt\right)\right].$$

3. Extension to General Functions: Since the set of elementary functions is dense in $\mathbb{L}^2$, for any $X \in \mathbb{L}^2$ there exists a sequence of elementary functions $\{\phi_n\}$ such that:

$$\mathbb{E}\left[\int_0^t X\phi_n dt\right] \to 0 \qquad \text{as } n \to \infty.$$

The Itô integral of f is then defined as the limit in $\mathbb{L}^2$:

$$\int_0^t X(t,\omega)\, dW_t = \lim_{n\to\infty} \int_0^t \phi_n(t,\omega)\, dW_t.$$

1.4.2 Fundamental properties

1. **Linearity:** For $a, b \in \mathbb{R}$ and two process $X, Y \in \mathbb{L}^2$:

$$\int (aX_t + bY_t)\, dW_t = a\int X_t dW_t + b\int Y_t dW_t.$$

2. **Martingale property:** The process $J_t = \int_0^t X dW_s$ is an $\mathcal{F}_t-$martingale, this implies

$$\mathbb{E}\left[J_t \middle| \mathcal{F}_s \right] = J_s, \qquad \text{for all } s < t.$$

3. $\mathbb{E}\left[J_t\right] = 0.$

Lemma 1.2 *Assume Y and Z are two progressively measurable processes such that*

$$\mathbb{E}\left(\sup_{0\le t\le T} |Y_t|^2 + \int_0^T \|Z_s\|^2 dt \right) < \infty.$$

Then, the process defined by

$$M_t = \int_0^t Y_s Z_s dW_s,$$

constitutes a uniformly integrable martingale on $[0,T]$.

Proof. We aim to show that the expectation of the supremum of the stochastic integral is bounded. Using the BDG inequality, there exists a constant $C > 0$ such that:

$$\mathbb{E}\left[\sup_{0\le t\le T} |M_t| \right] \le C\mathbb{E}\left[\langle M_t \rangle_T^{\frac{1}{2}} \right] = C\mathbb{E}\left[\left(\int_0^T |Y_s|^2 \|Z_s\|^2 ds \right)^{\frac{1}{2}} \right].$$

Applying the Cauchy-Schwarz inequality within the integral, we can bound the term as follows:

$$\mathbb{E}\left[\sup_{0\le t\le T} |M_t| \right] \le C\mathbb{E}\left[\sup_{0\le t\le T} |Y_t| \left(\int_0^T \|Z_s\|^2 ds \right)^{\frac{1}{2}} \right].$$

Next, we invoke Young's inequality for the product of two terms. This yields:

$$\mathbb{E}\left[\sup_{0\le t\le T} |M_t| \right] \le \frac{C}{2} \left(\mathbb{E}\left[\sup_{0\le t\le T} |Y_t|^2 \right] + \mathbb{E}\left[\int_0^T \|Z_s\|^2 ds \right] \right).$$

Since both $Y \in \mathbb{S}^2\left(\mathbb{R}^k\right)$ and $Z \in \mathbb{M}^2\left(\mathbb{R}^{k\times d}\right)$, the expectation is finite. Thus M_t is a uniformly integrable martingale. ■

1.4.3 Itô's processes and Itô's formula

Definition 1.15 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtred probability space. An Itô process is a stochastic process X_t on $[0,T]$ that can be represented in the form:*

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s,$$

where X_0 is $\mathcal{F}_0$-measurable, b_s is an adapted drift process, and σ_s is a diffusion process such that $\int_0^t (|b_s| + \sigma_s^2) ds < \infty$ a.s.

Theorem 1.6 ***(One-dimensional Itô formula)*** *Let $\{X_t\}_{t\in[0,T]}$ be an Itô process and $f : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^2$ function. The process $f(X_t)$ is also an Itô process, and its dynamics is given by:*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\, dX_s + \frac{1}{2}\int_0^t f''(X_s)\, d\langle X, X\rangle_s . \tag{1.1}$$

Theorem 1.7 ***(Time-dependent Itô formula)*** *Let X_t be an Itô process and $f(t,x) \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R})$. Then $f(t, X_t)$ is also an Itô process, and its integral form is*

$$f(t, X_t) = f(0, X_0) + \int_0^t f'(s, X_s)\, ds + \int_0^t f'(s, X_s)\, dX_s + \frac{1}{2}\int_0^t f^{\prime\prime}(s, X_s)\, d\langle X, X\rangle_s. \tag{1.2}$$

Theorem 1.8 ***(Multidimensional Itô formula)*** *[9] Let $X = (X^1, \ldots, X^d)$ be a vector of Itô processes and $f \in C^{1,2}([0,T]\times\mathbb{R}^d)$. Then*

$$df(t, X_t) = \frac{\partial f}{\partial t}dt + \sum_{i=1}^d \frac{\partial f}{\partial x_i} dX_t^i + \frac{1}{2}\sum_{i,j=1}^d \frac{\partial^2 f}{\partial x_i \partial x_j} d\left\langle X^i, X^j\right\rangle_t .$$

Remark 1.4 *This general version is essential for analyzing systems of BSDEs where the solution Y_t takes values in $\mathbb{R}^d$.*

Proposition 1.2 ***(Multiple Processes)*** *Let $X_1(t)$ and $X_2(t)$ two Itô processes.then:*

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \langle X, Y\rangle_t. \tag{1.3}$$

Notation 1 *The term* $d\langle X_{1,}X_2\rangle_s$ *represents the quadratic covariation between the two processes, which is calculated using the heuristic rules:*

$\times$	dt	dW_t
dt	0	0
dW_t	0	t

Chapter 2

Forward and backward stochastic differential equations

2.1 Stochastic differential equation

Stochastic differential equations (SDEs) are introduced as differential equations with random coefficients, which are interpreted as Itô integral equations. The general form is given as:

$$\begin{cases} dX_t = b\,(t, X_t)\,dt + \sigma\,(t, X_t)\,dW_t, \\ X_0 = x, \end{cases} \tag{2.1}$$

where:

- W_t represents a standard m-dimensional Brownian motion defined on a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}\ \ , \mathbb{P}\right)$.
- The function $b\,(t, X_t)$ denotes the drift coefficient, which dictates the average rate of change of the process.
- $\sigma\,(t, X_t)$ represents the diffusion coefficient, measuring the intensity of the stochastic noise.

The existence of a solution to differetial equation (2.1) is not guaranteed without imposing regularity constraints on the coefficients. For the system to be well-defined, we assume that b and σ satisfy the following two fundamental conditions for a constant $C > 0$:

1. **The Lipschitz Continuity Condition:** For $x, y \in \mathbb{R}^n$ and $t \in [0, T]$:

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq C|x - y|.$$

   This condition ensures that the paths of the solution do not deviate too abruptly, which is the key to proving the uniqueness of the solution.

2. **Linear growth condition:** For $x \in \mathbb{R}^n$ and $t \in [0, T]$

$$|b(t,x)| + |\sigma(t,x)| \leq C(1 + |x|).$$

   This condition guarantees that the solution process (X_t) does not "explode" and remains within the space of square-integrable processes, i.e., $\mathbb{E}\left[\int_0^T |X_t|\, dt\right] < \infty$.

Theorem 2.1 *Under the above assumption, the SDEs (2.1) has a unique solution X which is an a progressively measurable process such that*

$$\mathbb{E}\left(\sup_{0\leq t\leq T} |X_t|^2\right) < \infty.$$

Remark 2.1 *While SDEs are driven by an initial condition X_0, BSDEs arise when we specify a terminal condition ξ at time T. In the following section, we introduce the framework of BSDEs*

2.2 Backward stochastic differential equation

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space and ξ be a random variable measurable in relation to $(\mathcal{F}_t)_{t\geq 0}$. The following fundamental stochastic equation could be solved:

$$-\frac{dy_t}{dt} = f(y_t), \qquad t \in [0, T], \qquad \text{with} \quad Y_T = \xi.$$

This means that the process Y must be adjusted to the filtering $(\mathcal{F}_t)_{t\geq 0}$ in order for the solution at time t to depend only on the past. Let's start with the case $f \geq 0$. It is attempted to provide the solution $Y_T = \xi$, which is only applicable if ξ is deterministic.We have just an adapted approximation in $\mathbb{L}^2$ space, which is the martingale $Y_t = \mathbb{E}(\xi | \mathcal{F}_t)$. If $(\mathcal{F}_t)_{t\geq 0}$ is the Brownien movement's filtration, the theory of martingale representation can be used to create a $Z-$adapted process with an intractable characteristic like

$$Y_t = \mathbb{E}(\xi | \mathcal{F}_t) = \mathbb{E}(\xi) + \int_0^t Z_s dW_s.$$

Given that $\mathbb{E}(\xi | \mathcal{F}_t)$ is a martingale, it follows that:

$$Y_t = \mathbb{E}(\xi) + \int_0^t Z_s dW_s, \qquad \forall t \in [0, T],$$

therefore

$$Y_T = \mathbb{E}(\xi) + \int_0^T Z_s dW_s.$$

Then, we get

$$\begin{aligned} \xi &= \mathbb{E}(\xi) + \int_0^t Z_s dW_s + \int_t^T Z_s dW_s \\ &= Y_t + \int_t^T Z_s dW_s, \end{aligned}$$

then we have

$$Y_t = \xi - \int_t^T Z_s dW_s \quad i.e. - dY_t = -Z_t dW_t \quad Y_T = \xi.$$

As a result, we notice the formation of the process Z, whose function is to guarantee that the process Y is modified. As a result, we let the function f to depend on the process Z in order to get the highest degree of generality because a second variable appears. As a result, the equation becomes

$$-dY_t = f(t, Y_t, Z_t) - Z_t dW_t, \quad Y_T = \xi.$$

- Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a complete probability space, and let W be a d-dimensional

Brownian motion on this space.

- $(\mathcal{F}_t)_{t\geq 0}$ denotes the natural filtration of the Brownian motion.
- $\mathbb{S}^2\left(\mathbb{R}^k\right)$ denotes the vector space of progressively measurable processes Y taking values in $\mathbb{R}^k$such that
$$\|Y_t\|^2_{\mathbb{S}^2} = \mathbb{E}\left(\sup_{0\leq t\leq T}|Y_t|^2\right) < \infty.$$
- $\mathbb{S}^2_c\left(\mathbb{R}^k\right)$ is the subspace consisting of continuous processes.
- $\mathbb{M}^2\left(\mathbb{R}^{k\times d}\right)$ denotes the vector space of progressively measurable processes Z taking values in $\mathbb{R}^{k\times d}$such that
$$\|Z_t\|^2_{\mathbb{M}^2} = \mathbb{E}\left(\int_0^T \|Z_s\|^2 dt\right) < \infty.$$
and $z \in \mathbb{R}^{k\times d}$, $\|z\|^2 = \mathrm{trace}\left(zz^\star\right).$
- $\mathbb{B}^2 = \mathbb{S}^2_c\left(\mathbb{R}^k\right) \times \mathbb{M}^2\left(\mathbb{R}^{k\times d}\right)$ is a Banach space.

Now, we assume

- $f : \Omega \times [0,T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^k$ such that $\forall\,(y,z) \in \mathbb{R}^k \times \mathbb{R}^{k\times d}$, $\left(f\left(t,y,z\right)\right)_{0\leq t\leq T}$ is progressively measurable.
- ξ is an $\mathbb{R}^k-$valued random vector that is $\mathcal{F}_t-$measurable.

Consider the backward stochastic differential equation:

$$\begin{cases} -dY_t = f\left(t, Y_t, Z_t\right)dt - Z_t dW_t, \quad 0 \leq t \leq T, \\ Y_T = \xi. \end{cases}$$

Rewriting in the integrale form:

$$Y_t = \xi + \int_t^T f\left(s, Y_s, Z_s\right)ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T. \tag{2.2}$$

Definition 2.1 *A solution to the BSDEs associated with the terminal conditio ξ and the driver f is pair processes $(Y_t, Z_t)_{0\leq t\leq T}$ taking values in $\mathbb{R}^k \times \mathbb{R}^{k\times d}$ such that:*

1. $\mathbb{E}\left(\int_t^T |f(s, Y_s, Z_s)|\, ds - \int_t^T Z_s^2 ds\right) < \infty$

2. The equation holds $\mathbb{P} - a.s.$for all $t \in [0, T]$ i.e.

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds - \int_t^T Z_s dW_s.$$

The next proposition ensures that $Y \in \mathbb{S}^2$ under a weak assumptions to the ceofficient f.

Proposition 2.1 *Suppose there exists a non-negative process $(f_t)_{0\leq t\leq T}$ belonging to $\mathbb{M}^2\left(\mathbb{R}^k\right)$ and a positive constant $\lambda > 0$ such that for all $\forall (t, y, z) \in [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d}$ the driver f satisfies the following growth condition:*

$$|f(t, y, z)| \leq f_t + \lambda (|y| + \|z\|),$$

if $\{(Y_t, Z_t)\}_{0\leq t\leq T}$ is a solution to the BSDE (2.2) such that $Z \in \mathbb{M}^2\left(\mathbb{R}^k\right)$, then the process Y belongs to the space $\mathbb{S}_c^2\left(\mathbb{R}^k\right)$, i.e. $\mathbb{E}\left(\sup_{0\leq t\leq T} |Y_t|^2\right) < \infty$.

Proof. We have for all $t \in [0, T]$:

$$Y_t = Y_0 - \int_0^t f(s, Y_s, Z_s)\, ds + \int_0^t Z_s dW_s,$$

Under a relatively weak hypothesis on f

$$\begin{aligned}
|Y_t| &\leq |Y_0| + \int_0^t |f(s, Y_s, Z_s)|\, ds + \int_0^t |Z_s|\, dW_s \\
&\leq |Y_0| + \int_0^T (f_s + \lambda \|Z_s\|)\, ds + \sup_{0\leq t\leq T} \left|\int_0^t Z_s dW_s\right| + \lambda \int_0^t |Y_s|\, ds.
\end{aligned}$$

Set

$$\xi = |Y_0| + \int_0^T (f_s + \lambda \|Z_s\|)\, ds + \sup_{0\leq t\leq T} \left|\int_0^t Z_s dW_s\right|.$$

Considering the integrability properties of the coefficients, we note that Z belongs to the space $Z \in \mathbb{M}^2\left(\mathbb{R}^k\right)$. Applying Doob's inequality ensures that the third term in the equation is of integrable square. Given that $f \in \mathbb{M}^2\left(\mathbb{R}^k\right)$ and Y_0 is a constant, we conclude that ξ is a square integrable random variable. Accordingly, the continuity of the process Y is maintained under the condition:

$$|Y_t| \leq \xi + \lambda \int_0^t |Y_s|\, ds, \quad \forall t \in [0,T].$$

By applying Gronwall's Lemma, we obtain:

$$|Y_t| \leq \xi \exp(\lambda T).$$

Finally, since ξ is sequare-integrable $\mathbb{E}(\xi^2) < \infty$, it follows that:

$$\mathbb{E}\left(\sup_{0\leq t\leq T} |Y_t|^2\right) \leq \exp(\lambda T)\, \mathbb{E}\left(\xi^2\right) < \infty.$$

This confirms that the process Y belongs to the space $\mathbb{S}^2$. ■

2.2.1 The Lipschitz case

In this section, we present the fundamental result concerning the existence and uniqueness of solutions for backward stochastic differential equations. This pioneering work, established by E. Pardoux and S. Peng (1990) [**?**], provided the first proof for BSDEs with non-linear generators.To esure the existence of a unique solution pair (Y, Z), we impose the following conditions on the driver f:

1. **(L1) Lipschitz Continuity:** There exists a constant $\lambda > 0$ such that $\mathbb{P} - a.s.$for all $t \in [0,T]$ and $(y,z), (y',z') \in \mathbb{R}^k \times \mathbb{R}^{k\times d}$:

$$|f(t,y,z) - f(t,y',z')| \leq \lambda\left(|y-y'| + \|z-z'\|\right).$$

2. **(L2) Integrability Conditions:**

$$\mathbb{E}\left(|\xi|^2 + \int_0^T |f(t,0,0)|^2 \, ds\right) < \infty.$$

To begin, we examine a simplified framework in which the generator f does not depend on the solution components (y, z). Given $\xi \in \mathbb{L}^2$ and a process $(\mathcal{F}_t)_{t\geq 0}$ in the space $\mathbb{M}^2\left(\mathbb{R}^k\right)$, we seek a pair of processes $(Y_t, Z_t)_{0\leq t\leq T}$ that satisfies the equation:

$$Y_t = \xi + \int_t^T F_s ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T. \tag{2.3}$$

Lemma 2.1 *Let $\xi \in \mathbb{L}^2$, $\mathcal{F}_T-$measurable and $(\mathcal{F}_t)_{t\geq 0}$ in the space $\mathbb{M}^2\left(\mathbb{R}^k\right)$. The BSDE* (**2.3**) *have a unique solution (Y_t, Z_t), such that $Z \in \mathbb{M}^2\left(\mathbb{R}^k\right)$.*

Proof. To establish the existence of a solution for the linear BSDE (**2.3**) we take the conditional expectation with respect to $\mathcal{F}_t$ on both sides. This directly defines the process Y_t as:

$$Y_t = \mathbb{E}\left(\xi + \int_t^T F_s ds \middle| \mathcal{F}_t\right).$$

Since Y_t is well-defined and square-integrable, we have

$$\begin{aligned} Y_t &= \mathbb{E}\left(\xi + \int_t^T F_s ds \middle| \mathcal{F}_t\right) = \mathbb{E}\left(\xi + \int_0^T F_s ds \middle| \mathcal{F}_t\right) - \int_0^t F_s ds \\ &:= M_t - \int_0^t F_s ds. \end{aligned}$$

Applying the Brownian martingale representation theorem **??** to M_t, there exists a unique process $Z \in \mathbb{M}^2\left(\mathbb{R}^k\right)$, such that:

$$M_t = M_0 + \int_0^t Z_s dW_s.$$

Thus, the process Y_t is expressed as:

$$Y_t = M_t - \int_0^t F_s ds = M_0 + \int_0^t Z_s dW_s - \int_0^t F_s ds.$$

To verify that the pair (Y, Z) satisfies the original BSDE**,** we use the fact that $Y_T = \xi$. By calculating the difference $Y_T - \xi$, we obtain:

$$Y_t - \xi = M_0 + \int_0^t Z_s dW_s - \int_0^t F_s ds \\ - \left(M_0 + \int_0^T Z_s dW_s - \int_0^T F_s ds \right).$$

Simplifyig the terms yields:

$$Y_t = \xi + \int_t^T F_s ds - \int_t^T Z_s dW_s.$$

This confirms that constructed pair is indeed a solution to eaqution (2.3). Finally, the uniqueness of the solution in the space $\mathbb{M}^2$ is established. ■

Theorem 2.2 *Under the assumptions* ***(L1)*** *and* ***(L2)****, the BSDE* (**2.2**) *admit a unique solution* (Y, Z) *with* $Z \in \mathbb{M}^2$*.*

Proof. By defining a mapping $\Psi \colon \mathbb{B}^2 \to \mathbb{B}^2$ we seek a solution to the BSDE (**2.2**) as a fixed point in the Banach space $\mathbb{B}^2$. Specifically, for an element $(U, V) \in \mathbb{B}^2$, the image $(Y, Z) = \Psi(U, V)$ is given by the solution to:

$$Y_t = \xi + \int_t^T f(s, U_s, V_s)\, ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T.$$

Note that the aforementioned BSDE admits a unique solution within the space $\mathbb{B}^2$. To see this, we define the process $|F_s|^2 :=| \, |f(s, U_s, V_s)|^2$. Under the Lipschitz continuity of f, we can establish the following estimate:

$$|F_s|^2 \leq | \, f(s, 0, 0) \, | + \lambda^2 \left(|U_s|^2 + \|V_s\|^2 \right),$$

where these three processes are square-integrable.it follows that $F \in \mathbb{M}^2$. By virtue of Lemma 2.1, there exists a unique pair (Y, Z) where $Z \in \mathbb{M}^2$ and $(Y, Z) \in \mathbb{B}^2$. Specifically, the integrability of Z is guaranteed by the construction of the solution, while the fact that

Y belongs to $\mathbb{S}^2$ is a direct consequence of Proposition 2.1. Consequently, we conclude that the operator Ψ is a well-defined mapping from $\mathbb{B}^2$ to itself.consider (U,V), $(U',V')\in\mathbb{B}^2$ and thier respective images (Y,Z), denoting the differences by $y:=Y-Y'$ and $z:=Z-Z'$ we get:

$$dy_t=-\left\{f\left(t,U_t,V_t\right)-f\left(t,U_t',V_t'\right)\right\}dt-z_t dW_t.$$

By invoking Itô's calculus (1.2) on the process $\exp\left(\alpha t\right)\left|y_t\right|^2$,with $\bar{f}\left(t\right)=\left[f\left(t,U_t,V_t\right)-f\left(t,U_t',V_t'\right)\right]$, yields:

$$\begin{aligned}d\left(\exp\left(\alpha t\right)\left|y_t\right|^2\right)&=\alpha\exp\left(\alpha t\right)\left|y_t\right|^2dt-2\exp\left(\alpha t\right)\left\langle y_t,\bar{f}\left(t\right)\right\rangle\\&\quad+2\exp\left(\alpha t\right)\left\langle y_t,z_t dW_t\right\rangle+\exp\left(\alpha t\right)\|z_t\|^2dt,\end{aligned}$$

integration over $[0,T]$ then yields

$$\begin{aligned}\int_t^T d\left(\exp\left(\alpha s\right)\left|y_s\right|^2\right)&=\alpha\int_t^T\exp\left(\alpha s\right)\left|y_s\right|^2ds-2\int_t^T\exp\left(\alpha s\right)\left\langle y_s,\bar{f}\left(s\right)\right\rangle ds\\&\quad+2\int_t^T\exp\left(\alpha s\right)\left\langle y_s,z_s dW_s\right\rangle+\int_t^T\exp\left(\alpha s\right)\|z_s\|^2ds.\end{aligned}$$

In addition, we observe that

$$\int_t^T d\left(\exp\left(\alpha s\right)\left|y_s\right|^2\right)=\exp\left(\alpha T\right)\left|y_T\right|^2-\exp\left(\alpha t\right)\left|y_t\right|^2,$$

with $y_T=0$, we can get:

$$\begin{aligned}\exp\left(\alpha t\right)\left|y_T\right|^2+\int_t^T\exp\left(\alpha s\right)\|z_s\|^2ds&=-\alpha\int_t^T\exp\left(\alpha s\right)\left|y_s\right|^2ds+2\int_t^T\exp\left(\alpha s\right)\left\langle y_s,\bar{f}\left(s\right)\right\rangle ds\\&\quad-2\int_t^T\exp\left(\alpha s\right)\left\langle y_s,z_s dW_s\right\rangle,\end{aligned}$$

Given the Lipschitz continuity of f, and defining $u = U - U$' and $v = V - V\prime$, we obtain:

$$
\begin{aligned}
&\exp(\alpha t)\left|y_t\right|^2+\int_t^T \exp(\alpha s)\left\|z_s\right\|^2 ds \\
&\leq -\alpha \int_t^T \exp(\alpha s)\left|y_s\right|^2 ds+2\lambda \int_t^T \exp(\alpha s)\, y_s,\left(\left|u_s\right|+\left\|v_s\right\|\right) ds \\
&-2 \int_t^T \exp(\alpha s)\left\langle y_s, z_s dW_s\right\rangle .
\end{aligned}
$$

Using, for any $\epsilon > 0$, the following inequality

$$
2ab \leq \frac{a^2}{\epsilon}+\epsilon b^2,
$$

we have

$$
\begin{aligned}
&\exp(\alpha t)\left|y_t\right|^2+\int_t^T \exp(\alpha s)\left\|z_s\right\|^2 ds \\
&\leq -\alpha \int_t^T \exp(\alpha s)\left|y_s\right|^2 ds+\frac{\lambda^2}{\epsilon} \int_t^T \exp(\alpha s)\left|y_s\right|^2 ds \\
&+\epsilon \int_t^T \exp(\alpha s)\left|u_s\right|^2 ds+\frac{\lambda^2}{\epsilon} \int_t^T \exp(\alpha s)\left|y_s\right|^2 ds \\
&+\epsilon \int_t^T \exp(\alpha s)\left\|v_s\right\|^2 ds-2 \int_t^T \exp(\alpha s)\left\langle y_s, z_s dW_s\right\rangle ,
\end{aligned}
$$

taking $\alpha = \frac{2\lambda^2}{\epsilon}$, leads to

$$
\begin{aligned}
&\exp(\alpha t)\left|y_t\right|^2+\int_t^T \exp(\alpha s)\left\|z_s\right\|^2 ds \\
&\leq \epsilon \int_t^T \exp(\alpha s)\left(\left|u_s\right|^2+\left\|v_s\right\|^2\right) ds-2 \int_t^T \exp(\alpha s)\left\langle y_s, z_s dW_s\right\rangle .
\end{aligned} \tag{2.4}
$$

In view of Lemma 2.1, the integrability properties $Y, Y' \in \mathbb{S}^2$ and $Z, Z' \in \mathbb{M}^2$ guarantee that the local martingale defined by $\left\{\int_t^T \exp(\alpha s)\left\langle y_s, z_s dW_s\right\rangle\right\}$ is a true martingale starting from the origin. Consequently, its expectation vanishes. By taking the expectation on both sides of the identity and setting $t = 0$, we straightforwardly obtain:

$$\mathbb{E}\left(\int_t^T \exp\left(\alpha s\right) \|z_s\|^2 ds\right) \leq \epsilon \mathbb{E}\left(\int_t^T \exp\left(\alpha s\right)\left(|u_s|^2 + \|v_s\|^2\right) ds\right). \tag{2.5}$$

In view of inequality (2.4), Burkhölder-Davis-Gundy inequalities ($A.5$), yield

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq t\leq T} \exp\left(\alpha t\right)|y_t|^2\right] \\
&\leq \epsilon \mathbb{E}\left[\int_t^T \exp\left(\alpha s\right)\left(|u_s|^2 + \|v_s\|^2\right) ds\right] + 2\mathbb{E}\left[\sup_{0\leq t\leq T}\left(\int_t^T \exp\left(\alpha s\right)\langle y_s, z_s dW_s\rangle\right)\right] \\
&\leq \epsilon \mathbb{E}\left[\int_t^T \exp\left(\alpha s\right)\left(|u_s|^2 + \|v_s\|^2\right) ds\right] + 2C_{\mathbf{BDG}}\mathbb{E}\left[\left(\int_t^T \exp\left(2\alpha s\right)|y_s|^2 \|z_s\|^2 ds\right)^{1/2}\right] \\
&\leq \epsilon \mathbb{E}\left[\int_t^T \exp\left(\alpha s\right)\left(|u_s|^2 + \|v_s\|^2\right) ds\right] + 2C_{\mathbf{BDG}}\mathbb{E}\left[\sup_{0\leq t\leq T} \exp\left(\alpha t\right)|y_t|\left(\int_t^T \exp\left(\alpha s\right)\|z_s\|^2 ds\right)^{1/2}\right].
\end{aligned}$$

Moreover, with $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$, we get

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq t\leq T} \exp\left(\alpha t\right)|y_t|^2\right] \\
&\leq \epsilon \mathbb{E}\left[\int_t^T \exp\left(\alpha s\right)\left(|u_s|^2 + \|v_s\|^2\right) ds\right] + \frac{1}{2}\mathbb{E}\left[\sup_{0\leq t\leq T} \exp\left(\alpha t\right)|y_t|^2\right] \\
&+ 2C_{\mathbf{BDG}}^2 \mathbb{E}\left[\int_t^T \exp\left(\alpha s\right)\|z_s\|^2 ds\right].
\end{aligned}$$

After grouping the terms associated with (ds), multiplying by 2 leads to:

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\leq t\leq T} \exp\left(\alpha t\right)|y_t|^2\right] &\leq 2\epsilon \mathbb{E}\left[\int_t^T \exp\left(\alpha s\right)\left(|u_s|^2 + \|v_s\|^2\right) ds\right] \\
&+ 4C_{\mathbf{BDG}}^2 \mathbb{E}\left[\int_t^T \exp\left(\alpha s\right)\|z_s\|^2 ds\right],
\end{aligned} \tag{2.6}$$

By gathering (2.5) and (2.6), it follows that:

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq t\leq T}\exp(\alpha t)\left|y_t\right|^2+\int_t^T\exp(\alpha s)\left\|z_s\right\|^2ds\right]\\
&\leq 3\epsilon\mathbb{E}\left[\int_t^T\exp(\alpha s)\left(\left|u_s\right|^2+\left\|v_s\right\|^2\right)ds\right]+4C_{\mathbf{BDG}}^2\mathbb{E}\left[\int_t^T\exp(\alpha s)\left(\left|u_s\right|^2+\left\|v_s\right\|^2\right)ds\right]\\
&\leq \epsilon\left(3+4C_{\mathbf{BDG}}^2\right)\mathbb{E}\left(\int_t^T\sup_{0\leq t\leq T}\exp(\alpha s)\left|u_s\right|^2ds\right)+\epsilon\left(3+4C_{\mathbf{BDG}}^2\right)\mathbb{E}\left(\int_t^T\exp(\alpha s)\left\|v_s\right\|^2ds\right)\\
&\leq \epsilon\left(3+4C_{\mathbf{BDG}}^2\right)\left[\mathbb{E}\left(\sup_{0\leq t\leq T}\exp(\alpha t)\left|u_s\right|^2\right)\int_t^T ds\right]+\epsilon\left(3+4C_{\mathbf{BDG}}^2\right)\mathbb{E}\left(\int_t^T\exp(\alpha s)\left\|v_s\right\|^2ds\right).
\end{aligned}
$$

we get

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq t\leq T}\exp(\alpha t)\left|y_t\right|^2+\int_t^T\exp(\alpha s)\left\|z_s\right\|^2ds\right]\\
&\leq \epsilon\sup(1,T)\left(3+4C_{\mathbf{BDG}}^2\right)\mathbb{E}\left(\sup_{0\leq t\leq T}\exp(\alpha t)\left|u_t\right|^2\right)\\
&+\epsilon\sup(1,T)\left(3+4C_{\mathbf{BDG}}^2\right)\mathbb{E}\left(\int_t^T\exp(\alpha s)\left\|v_s\right\|^2ds\right).
\end{aligned}
$$

Under the condition $\epsilon\left(3+4C_{\mathbf{BDG}}^2\right)(1\vee T)=\frac{3}{4}$, Ψ defines a strict contraction on $\mathbb{B}^2$, ensuring that the mapping possesses a unique fixed point

$$
\|U,V\|_\alpha^2=\mathbb{E}\left(\sup_{0\leq t\leq T}\exp(\alpha t)\left|U_t\right|^2+\int_t^T\exp(\alpha s)\left\|V_s\right\|^2ds\right).
$$

Given that $\mathbb{B}^2$ is a Banach space under a norm equivalent to the case $\alpha=0$, Ψ has a unique fixed point, guaranteeing a unique solution for the BSDE (2.2). Moreover, according to Proposition 2.1, any solution with $Z\in\mathbb{M}^2$ falls within $\mathbb{B}^2$, thereby confirming the uniqueness in the broader sense. ■

The Role of the processus Z

The function of the martingale term $\int_t^T Z_s dW_s$ is to ensure the process Y remains adapted to the filtration $(\mathcal{F}_t)_{0\leq t}\leq T$. Essentially, Z acts as a corrector that compensates for the stochastic fluctuations of the terminal condition, ensuring that Y_t does not depend on future information. When the randomness is resolved before the terminal time T, the component

Z naturally vanishes.

Proposition 2.2 *Let (Y, Z) be the solution to the BSDE (2.2) and let τ be a stopping time bounded by T. Under the assumptions **(L1)** and **(L2)**, suppose further that:*

1. The terminal condition ξ is $\mathcal{F}_t$-measurable.

2. The generator $f(t, y, z) = 0$ for all $t \geq \tau$.

Then, the following properties hold:

$$Y_t = Y_{t\wedge\tau} \text{ and } Z_t = 0, \quad \text{for all } t \geq \tau.$$

Proof. we have

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\, ds - \int_t^T Z_s dW_s.$$

Consequently, for $t \geq \tau$, the vanshing property of the generator $f(t, y, z) = 0$ implies that:

$$\begin{aligned} Y_t &= \xi + \int_\tau^T f(s, Y_s, Z_s)\, ds - \int_\tau^T Z_s dW_s \\ &= \xi - \int_\tau^T Z_s dW_s. \end{aligned}$$

It follows that:

$$Y_t = \mathbb{E}\left(\xi | \mathcal{F}_s\right) = \xi,$$

in addition

$$\int_\tau^T Z_s dW_s = 0,$$

from which we infer that:

$$\mathbb{E}\left(\int_\tau^T Z_s dW_s\right)^2 = \mathbb{E}\left(\|Z_s\|^2\right) = 0.$$

This immediately implies that $Y_t = Y_\tau$ for all $t \geq \tau$, this shows that the trajectory of Y becomes stationary after the stopping time τ. ■

Remark 2.2 *When the terminal condition ξ and the generator f are deterministic:*

- Z vanishes identically ($Z_t = 0$).
- The BSDE simplifies to a classical ordinary differential equation:

$$-\frac{dY_t}{dt} = f\left(t, Y_t, 0\right), \quad Y_T = \xi = 0.$$

2.2.2 Linear BSDEs

Linear BSDEs are a fundamental class of equations where the generator f is an affine function of the processes (Y, Z). Consider the case where $k = 1$, such that $Y_t \in \mathbb{R}$ and $Z_t \in \mathbb{R}^d$.

Proposition 2.3 *Let $\{(a_t, b_t)\}_{0 \leq t \leq T}$ be a bounded and progressively measurable process. Let $\{c_t\}_{0 \leq t \leq T} \in \mathbb{M}^2\left(\mathbb{R}\right)$and the terminal condition $\xi \in \mathbb{L}^2\left(\mathcal{F}_T\right)$. We consider the following linear BSDE:*

$$Y_t = \xi + \int_t^T \left(a_s Y_s + b_s Z_s + c_s\right) ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T.$$

The unique solution (Y, Z) is explicitly given by the formula:

$$Y_t = \Lambda_t^{-1} \mathbb{E}\left[\xi \Lambda_T + \int_t^T c_s \Lambda_s ds \middle| \mathcal{F}_t\right], \tag{2.7}$$

where the adjoint process Λ is defined as the unique solution to the linear ***SDE****.*

$$d\Lambda_t = \Lambda_t\left(a_t dt + b_t dW_t\right), \quad \Lambda_0 = 1.$$

Proof. To derive the explicit formula, we define the product $\Lambda_t Y_t$. By applying Itô's formula to $d\left(\Lambda_t Y_t\right)$, we have:

$$d\left(\Lambda_t Y_t\right) = \Lambda_t dY_t + Y_t d\Lambda_t + d\left\langle \Lambda_t, Y_t \right\rangle.$$

Substituting the expressions fo rdY_t and $d\Lambda_t$:

$$\begin{cases} dY_t = -\left(a_t Y_t + b_t Z_t + c_t\right) dt; \\ d\Lambda_t = \Lambda_t a_t dt + \Lambda_t b_t dW_t; \\ d\left\langle \Lambda_t, Y_t \right\rangle = \Lambda_t b_t Z_t dt. \end{cases}$$

After simplification, the terms involving Y_t and Z_t in the dt part cancel out, yielding:

$$d\left(\Lambda_t Y_t\right) = -\Lambda_t c_t dt + \Lambda_t \left(Z_t + Y_t b_t\right) dW_t.$$

By integrating from 0 to t, we observe that the process $M_t = \Lambda_t Y_t + \int_0^t c_s \Lambda_s ds$ is a martingale. Taking the conditional expectation $\mathbb{E}\left[\cdot\middle|\, \mathcal{F}_t\right]$ and using the martingale property, we arrive at the representation (2.7). ■

Chapter 3

Generalized backward stochastic differetial equations

In this chapter, we extend the results obtain in chapter 2 to a more general class of BSDEs where the driver depends not only on the time dt but also on a continuous increasing process dA_t (See [7, 10, 11]).

3.1 The assumptions

The main object is to establish the existence and uniqueness of the solution (Y_t, Z_t) under a specific set of assumption with $T > 0$, is the final time and $\xi \in \mathbb{L}^2\left(\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{R}^k\right)$, is the final condition, such that for all $\mu > 0$, we have

$$\mathbb{E}\left(\exp\left(\mu A_T\right)|\xi|^2\right) < \infty,$$

and the two coefficients

$$f : \Omega \times [0,T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^k$$

$$g : \Omega \times [0,T] \times \mathbb{R}^k \to \mathbb{R}^k$$

satisfying the following assumptions for some constants $\alpha \in \mathbb{R}$, $\beta < 0$, $K > 0$ and some adapted processes $\{\phi_t, \psi_t; 0 \leq t \leq T\}$, with values in $[1, \infty)$, for all $(t, y, z) \in [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d}$

1. $f(., y, z)$ and $g(., y, z)$ are progressively measurable.

2. $\mathbb{E}\left(\int_0^T \exp\left(\mu \tilde{A}_t\right) \phi_t^2 dt + \int_0^T \exp\left(\mu \tilde{A}_t\right) \psi_t^2 dAt\right) < \infty.$

3. f is monotone, i.e., there exists $\alpha \in \mathbb{R}$ such that

$$\langle y - y', f(s, y, z) - f(s, y, z') \rangle \leq \alpha |y - y'|^2 .$$

4. g is monotone with $\beta < 0$, i.e., there exists $\beta < 0$ such that

$$\langle y - y', g(s, y) - g(s, y') \rangle \leq \beta |y - y'|^2 .$$

5. f satisfies a Lipschitz condition with respect to z, i.e., there exists $K > 0$ such that

$$|f(s, y, z) - f(s, y, z')| \leq K \|z - z'\| .$$

6. $|f(s, y, z)| \leq \phi_t + K(|y| + \|z\|)$, $|g(t, y)| \leq \psi_t + K(|y| + \|z\|)$.

7. $y \to (f(t, y, z), g(t, y))$ is continuous for all z.

Definition 3.1 *The pair $\{(Y_t, Z_t); 0 \leq t \leq T\}$ is the solution of the generalized BSDE if*

$$\begin{aligned} &(i)\ \mathbb{E}\left(\sup_{0 \leq t \leq T} |Y_t|^2 + \int_0^T |Y_t|^2 dA_t + \int_0^T \|Z_t\|^2 dt\right) < \infty, \\ &(ii)\ Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g(s, Y_s) dA_s - \int_t^T Z_s dW_s. \end{aligned} \tag{3.1}$$

The proof of existence is preceded by the establishment of a crucial stability result, termed the a priori estimate, which asserts that the solution norm is constrained by the given initial and boundary data.

3.2 A priori estimation

Proposition 3.1 ***Standard Estimate:*** *Under assumptions bellow, if (Y_t, Z_t) is a slution of GBSDE, then there exists a constant $C = C(\mu, \alpha, \beta, K, T)$ such that*

$$\mathbb{E}\left(\sup_{0\le t\le T}|Y_t|^2+\int_0^T|Y_t|^2\,dA_t+\int_0^T\|Z_t\|^2\,dt\right)$$
$$\le C\mathbb{E}\left(|\xi|^2+\int_0^T|f(t,0,0)|^2\,dt+\int_0^T|g(t,0)|^2\,dA_t\right). \tag{3.2}$$

Weighted Estimate: Furthermore, for any continuous increasing process $\tilde{A}_t$ satisfyig $\mathbb{E}\left(\exp\left(\mu\tilde{A}_t\right)\right)<\infty$, there exists a constant $C = C(\mu, \alpha, \beta, K, T)$ such that

$$\mathbb{E}\left(\sup_{0\le t\le T}\exp\left(\mu\tilde{A}_t\right)|Y_t|^2+\int_0^T\exp\left(\mu\tilde{A}_t\right)|Y_s|^2\,d\tilde{A}_t+\int_0^T\exp\left(\mu\tilde{A}_t\right)\|Z_t\|^2\,dt\right)$$
$$\le C\mathbb{E}\left(\exp\left(\mu\tilde{A}_t\right)|\xi|^2+\int_0^T\exp\left(\mu\tilde{A}_t\right)|f(t,0,0)|^2\,dt+\int_0^T\exp\left(\mu\tilde{A}_t\right)|g(t,0)|^2\,d\tilde{A}t\right).$$

Proof. The weighted estimate can be established using a similar argument by applying Itô's formula to the process $\exp\left(\mu\tilde{A}_t\right)|Y_t|^2$ instead of$|Y_t|^2$, From Itô's formula (1.2) appling to $|Y_t|^2$, we find:

$$|Y_t|^2=|Y_T|^2-2\int_0^T Y_s dY_s-\frac{1}{2}\times 2\int_0^T d\langle Y_s,Y_s\rangle$$
$$=|\xi|^2-2\int_0^T\langle Y_s,f(s,Y_s,Z_s)\,ds+g(s,Y_s)\,dA_s-Z_sdW_s\rangle-\int_0^T Z_s^2ds,$$

Traspose $\int_0^T Z_s^2ds$ to the right side, we get

$$|Y_t|^2+\int_0^T Z_s^2ds \tag{3.3}$$
$$=|\xi|^2-2\int_0^T\langle Y_s,f(s,Y_s,Z_s)\,ds\rangle-2\int_0^T\langle Y_s,g(s,Y_s)\,dA_s\rangle+2\int_0^T\langle Y_s,Z_sdW_s\rangle.$$

Firstly, simplify the scalar product $\langle Y, f(s,Y,Z)\rangle$ by adding and substructing the term $f(s,0,Z)$, then apply the monotonicity condition for f, we can obtain

$$\begin{aligned}\langle Y, f(s,Y,Z)\rangle &= \langle Y-0, f(s,Y,Z)-f(s,0,Z)+f(s,0,Z)\rangle\\ &\leq \langle Y, f(s,Y,Z)-f(s,0,Z)\rangle + \langle Y, f(s,0,Z)\rangle\\ &\leq \alpha\left|Y\right|^2 + \left|Y\right|\left|f(s,0,Z)\right|\\ &\leq \alpha\left|Y\right|^2 + \left|Y\right|\left|f(s,0,Z)-f(s,0,0)+f(s,0,0)\right|,\end{aligned}$$

from the Lipschitz condition with respect to Z for f, we get

$$\langle Y, f(s,Y,Z)\rangle \leq \alpha\left|Y\right|^2 + \left|Y\right|\{K\left\|Z\right\| + \left|f(s,0,0)\right|\},$$

with the same method we can simplify the scalar product $\langle Y, g(s,Y)\rangle$

$$\begin{aligned}\langle Y, g(s,Y)\rangle &= \langle Y-0, g(s,Y)-g(s,0)+g(s,0)\rangle\\ &\leq \langle Y, g(s,Y)-g(s,0)\rangle + \langle Y, g(s,0)\rangle\\ &\leq \beta\left|Y\right|^2 + \left|Y\right|\left|g(s,0)\right|\end{aligned}$$

After substitution in (3.3), we find:

$$\begin{aligned}\left|Y_t\right|^2 + \int_t^T \left\|Z_s\right\|^2 ds \leq \left|\xi\right|^2 + 2\alpha\int_t^T \left|Y_s\right|^2 ds + 2\int_t^T \left|Y_s\right|\{K\left\|Z\right\| + \left|f(s,0,0)\right|\}ds\\ + 2\beta\int_t^T \left|Y_s\right|^2 dA_s + 2\int_t^T \left|Y_s\right|\left|g(s,0)\right| dA_s - 2\int_t^T \langle Y_s, Z_s\rangle dW_s.\end{aligned}$$

Using the elementary inequality $2ab \leq a^2+b^2$, we have:

$$\begin{aligned}\left|Y_t\right|^2 + \int_t^T \left\|Z_s\right\|^2 ds \leq \left|\xi\right|^2 + 2\alpha\int_t^T \left|Y_s\right|^2 ds + \int_t^T \left|Y_s\right|^2 ds + \int_t^T \left|f(s,0,0)\right|^2 ds\\ + 2K^2\int_t^T \left|Y_s\right|^2 ds + \frac{1}{2}\int_t^T \left\|Z_s\right\|^2 ds + 2\beta\int_t^T \left|Y_s\right|^2 dA_s\\ + \int_t^T \left|Y_s\right|^2 dA_s + \int_t^T \left|g(s,0)^2\right| dA_s - 2\int_t^T \langle Y_s, Z_s\rangle dW_s,\end{aligned}$$

thus, grouping like terms together, we find

$$\begin{aligned}
&|Y_t|^2 + \frac{1}{2}\int_t^T \|Z_s\|^2 \, ds - (2\beta+1)\int_t^T |Y_s|^2 \, dA_s \\
&\leq |\xi|^2 + \left(2\alpha + 2K^2 + 1\right)\int_t^T |Y_s|^2 \, ds + \int_t^T |f(s,0,0)|^2 \, ds \\
&+ \int_t^T |g(s,0)|^2 \, dA_s - 2\int_t^T \langle Y_s, Z_s \rangle dW_s.
\end{aligned}$$

By taking the expectation of both sides,the martingale term vanishes $\mathbb{E}\left(\int_t^T \langle Y_s, Z_s\rangle dWs\right) = 0$ and put $C = (2\alpha + 2K^2 + 1)$ we obtain:

$$\begin{aligned}
&\mathbb{E}\left(|Y_t|^2 + \frac{1}{2}\int_t^T \|Z_s\|^2 \, ds - (2\beta+1)\int_t^T |Y_s|^2 \, dA_s\right) \\
&\leq \mathbb{E}\left(|\xi|^2\right) + C\mathbb{E}\left(\int_t^T |Y_s|^2 \, ds\right) + \mathbb{E}\left(\int_t^T |f(s,0,0)|^2 \, ds\right) + \mathbb{E}\left(\int_t^T |g(s,0)|^2 \, dA_s\right),
\end{aligned}$$

we know that

$$\begin{aligned}
\mathbb{E}\left(|Y_t|^2\right) &\leq \mathbb{E}\left(|\xi|^2\right) + C\mathbb{E}\left(\int_t^T |Y_s|^2 \, ds\right) \\
&\quad + \mathbb{E}\left(\int_t^T |f(s,0,0)|^2 \, ds\right) + \mathbb{E}\left(\int_t^T |g(s,0)|^2 \, dA_s\right),
\end{aligned}$$

from the Gronwall's Lemma we get:

$$\begin{aligned}
&\mathbb{E}\left(|Y_t|^2\right) \\
&\leq c\left\{\mathbb{E}\left(|\xi|^2\right) + \mathbb{E}\left(\int_t^T |Y_s|^2 \, ds\right) + \mathbb{E}\left(\int_t^T |f(s,0,0)|^2 \, ds\right) + \mathbb{E}\left(\int_t^T |g(s,0)|^2 \, dA_s\right)\right\},
\end{aligned}$$

where $c = \exp(CT)$. Consequently, we deduce:

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\leq t\leq T} |Y_t|^2 + \int_0^T \| Z_s \|^2 \, ds + \int_0^T |Y_s|^2 \, dA_s\right) \\
&\leq c\left\{\mathbb{E}\left(|\xi|^2\right) + \mathbb{E}\left(\int_t^T |Y_s|^2 \, ds\right) + \mathbb{E}\left(\int_t^T |f(s,0,0)|^2 \, ds\right) + \mathbb{E}\left(\int_t^T |g(s,0)|^2 \, dA_s\right)\right\}.
\end{aligned}$$

For the weighted estimate we use the same method of standard estimate. ■

Proposition 3.2 *We but* $\left(\overline{Y},\overline{\xi},\bar{f},\overline{g},\overline{A}\right)=\left(Y-Y',Z-Z',\xi-\xi',f-f',g-g',A-A'\right)$. *Then, for any* $\mu>0$, *there exists a constant* C *such that*

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\leq t\leq T}\exp\left(\mu k_t\right)\left|Y_t\right|^2+\int_0^T\exp\left(\mu k_t\right)\left\|\overline{Z}_t\right\|^2ds\right)\\&\leq C\mathbb{E}\exp\left(\mu k_t\right)\left|\overline{\xi}\right|^2+C\mathbb{E}\left(\int_0^T\exp\left(\mu k_t\right)\left|f\left(t,Y_t,Z_t\right)-f'\left(t,Y_t,Z_t\right)\right|^2dt\right)\\&+C\mathbb{E}\left(\int_0^T\exp\left(\mu k_t\right)\left|g\left(t,Y_t\right)-g'\left(t,Y_t\right)\right|^2dA'_t\right)+C\mathbb{E}\left(\int_0^T\exp\left(\mu k_t\right)\left|g\left(t,Y_t\right)\right|^2d\|\overline{A}\|_t\right),\end{aligned}$$

where $k_t\triangleq\|\overline{A}\|_t+A'_t$, $\|\overline{A}\|_t$ *denoting the totale variation of the process* $\overline{A}$, *on the interval* $[0,T]$.

Proof. To prove this result in the case where k_t,is a bounded r.v. Using Itô's formula 1.3 to the process $\exp\left(\mu k_t\right)\left|Y_t\right|^2$, we have:

$$\begin{aligned}\exp\left(\mu k_T\right)\left|\overline{Y}_T\right|^2&=\exp\left(\mu k_t\right)\left|\overline{Y}_t\right|^2+2\int_0^T\exp\left(\mu k_s\right)\overline{Y}_s d\overline{Y}_s\\&+\mu\int_0^T\exp\left(\mu k_s\right)\left|\overline{Y}_s\right|^2dk_s+\langle\exp\left(\mu k_t\right),\left|\overline{Y}_t\right|^2\rangle.\end{aligned}$$

We know that $\langle\exp\left(\mu k_t\right),\left|\bar{Y}_t\right|^2\rangle=\int_0^T\exp\left(\mu k_t\right)\left\|\overline{Z}_t\right\|^2ds$, then we obtain

$$\begin{aligned}\exp\left(\mu k_t\right)\left|\overline{Y}_t\right|^2&=\exp\left(\mu k_T\right)\left|\overline{\xi}\right|^2-2\int_0^T\exp\left(\mu k_s\right)\overline{Y}_s d\overline{Y}_s\\&-\mu\int_0^T\exp\left(\mu k_s\right)\left|\overline{Y}_s\right|^2dk_s-\int_0^T\exp\left(\mu k_t\right)\left\|\overline{Z}_s\right\|^2ds.\end{aligned}\tag{3.4}$$

Another side we know that $\int_t^T dY_s=Y_T-Y_t$, with $Y_T=\xi$. We get:

$$\overline{Y}=\overline{\xi}+\int_t^T\bar{f}ds+\int_t^T\overline{g}d\overline{A}_s-\int_t^T\overline{Z}dW_s.$$

Substituting into equation (3.4), we obtain:

$$\begin{aligned}\exp(\mu k_t)\left|\overline{Y}_t\right|^2 &= \exp(\mu k_T)\left|\overline{\xi}\right|^2 + 2\int_0^T \exp(\mu k_s)\left\langle \overline{\overline{Y}}_s, f(s,Y_s,Z_s) - f'(s,Y_s',Z_s')\right\rangle ds \\ &+ 2\int_0^T \exp(\mu k_s)\left\langle \overline{Y}_s, g(s,Y_s)\,dA_s - g'(s,Y_s')\,dA_s'\right\rangle - \int_0^T \exp(\mu k_s)\left\langle \left|\overline{Y}_s\right|, \overline{Z}_s dW_s\right\rangle \\ &- \mu\int_0^T \exp(\mu k_s)\left|\overline{Y}_s\right|^2 dk_s - \int_0^T \exp(\mu k_t)\left\|\overline{Z}_s\right\|^2 ds.\end{aligned}$$

More precisely.by substracting and additing the term, we have:

$$\begin{aligned}&\exp(\mu k_t)\left|\overline{Y}_t\right|^2 + \mu\int_0^T \exp(\mu k_s)\left|\overline{Y}_s\right|^2 dk_s + \int_0^T \exp(\mu k_s)\left\|\overline{Z}_s\right\|^2 ds \\ &= \exp(\mu k_T)\left|\overline{\xi}\right|^2 + 2\int_0^T \exp(\mu k_s)\left\langle \overline{\overline{Y}}_s, f(s,Y_s,Z_s) - f'(s,Y_s',Z_s')\right\rangle ds \\ &+ 2\int_0^T \exp(\mu k_s)\left\langle \overline{Y}_s, \{g(s,Y_s) - g'(s,Y_s')\}\,dA_s'\right\rangle - 2\int_0^T \exp(\mu k_s)\left\langle \overline{Y}_s, g(s,Y_s)\,dA_s'\right\rangle \\ &+ 2\int_0^T \exp(\mu k_s)\left\langle \overline{Y}_s, g(s,Y_s)\,dA_s\right\rangle - 2\int_0^T \exp(\mu k_s)\left\langle \left|\overline{Y}_s\right|, \overline{Z}_s dW_s\right\rangle.\end{aligned}$$

Distinguishing between the dA_s terms and the dA_s' terms, we get:

$$\begin{aligned}&\exp(\mu k_t)\left|\overline{Y}_t\right|^2 - \mu\int_0^T \exp(\mu k_s)\left|\overline{Y}_s\right|^2 dk_s + \int_0^T \exp(\mu k_s)\left\|\overline{Z}_s\right\|^2 ds \\ &= \exp(\mu k_T)\left|\overline{\xi}\right|^2 + 2\int_0^T \exp(\mu k_s)\left\langle \overline{\overline{Y}}_s, f(s,Y_s,Z_s) - f'(s,Y_s',Z_s')\right\rangle ds \\ &+ 2\int_0^T \exp(\mu k_s)\left\langle \overline{Y}_s, \{g(s,Y_s) - g'(s,Y_s')\}\,dA_s'\right\rangle \\ &+ 2\int_0^T \exp(\mu k_s)\left\langle \overline{Y}_s, g(s,Y_s)\,d\overline{A}_s\right\rangle - 2\int_0^T \exp(\mu k_s)\left\langle \overline{Y}_s, \overline{Z}_s dW_s\right\rangle.\end{aligned}$$

Then, by adding and subtracting,the terms $f(s,Y_s',Z_s)$; $g(s,Y_s')$ we have

$$\begin{aligned}
&\exp(\mu k_t)\left|\overline{Y}_t\right|^2-\mu\int_0^T\exp(\mu k_s)\left|\overline{Y}_s\right|^2dk_s+\int_0^T\exp(\mu k_s)\left\|\overline{Z}_s\right\|^2ds\\
&\le\exp(\mu k_T)\left|\overline{\xi}\right|^2+2\int_0^T\exp(\mu k_s)\left\langle\overline{\overline{Y}}_s,f(s,Y_s,Z_s)-f(s,Y_s',Z_s)\right\rangle ds\\
&+2\int_0^T\exp(\mu k_s)\left\langle\overline{Y}_s,f(s,Y_s',Z_s)-f'(s,Y_s',Z_s')\,ds\right\rangle\\
&+2\int_0^T\exp(\mu k_s)\left\langle\overline{Y}_s,g(s,Y_s)-g(s,Y_s')\,dA_s'\right\rangle+2\int_0^T\exp(\mu k_s)\left\langle\overline{Y}_s,g(s,Y_s')-g'(s,Y_s')\,dA_s'\right\rangle\\
&+2\int_0^T\exp(\mu k_s)\left\langle\overline{Y}_s,g(s,Y_s)\,d\left\|\overline{A}\right\|_s\right\rangle-2\int_0^T\exp(\mu k_s)\left\langle\overline{Y}_s,\overline{Z}_sdW_s\right\rangle.
\end{aligned}$$

From, the monotonocity condition of f and g, we get:

$$\begin{aligned}
&\exp(\mu k_t)\left|\overline{Y}_t\right|^2-\mu\int_0^T\exp(\mu k_s)\left|\overline{Y}_s\right|^2dk_s+\int_0^T\exp(\mu k_s)\left\|\overline{Z}_s\right\|^2ds\\
&\le\exp(\mu k_T)\left|\overline{\xi}\right|^2+2\int_0^T\exp(\mu k_s)\left|\overline{Y}_s\right|^2ds\\
&+2\int_0^T\exp(\mu k_s)\left\langle\overline{\overline{Y}}_s,f(s,Y_s',Z_s)-f'(s,Y_s',Z_s')\right\rangle ds\\
&+\int_0^T\exp(\mu k_s)\left|\overline{Y}_s\right|^2dA_s'+\int_0^T\exp(\mu k_s)\left|g(s,Y_s)-g'(s,Y_s')\right|^2dA_s'\\
&+\int_0^T\exp(\mu k_s)\left|\overline{Y}_s\right|^2d\|\overline{A_s}\|+\int_0^T\exp(\mu k_s)\left|g(s,Y_s)\right|^2d\|\overline{A_s}\|-2\int_0^T\exp(\mu k_s)\left\langle\overline{Y}_s,\overline{Z}_sdW_s\right\rangle.
\end{aligned}$$

Applying the Lipschitz condition, with Young's inequality, we deduce:

$$\begin{aligned}
&\exp(\mu k_t)\left|\overline{Y}_t\right|^2+\int_0^T\exp(\mu k_s)\left\|\overline{Z}_s\right\|^2ds-\mu\int_0^T\exp(\mu k_s)\left|\overline{Y}_s\right|^2dk_s\\
&-3\int_0^T\exp(\mu k_s)\left|\overline{Y}_s\right|^2dA_s'-3\int_0^T\exp(\mu k_s)\left|\overline{Y}_s\right|^2d\|\overline{A}_s\|\\
&\le\exp(\mu k_T)\left|\overline{\xi}\right|^2+2\int_0^T\exp(\mu k_s)\left|\overline{Y}_s\right|^2ds+2K\int_0^T\exp(\mu k_s)\left\langle\left|\overline{Y}_s\right|^2,\left\|\overline{Z}_s\right\|^2ds\right\rangle\\
&+\int_0^T\exp(\mu k_s)\left|\overline{Y}_s\right|^2ds+\int_0^T\exp(\mu k_s)\left|\overline{Y}_s\right|^2ds+\int_0^T\left|f(s,Y_s,Z_s)-f'(s,Y_s,Z_s)\right|^2ds\\
&+\int_0^T\exp(\mu k_s)\left|g(s,Y_s)-g'(s,Y_s)\right|^2dA_s'+\int_0^T\exp(\mu k_s)\left|g(s,Y_s)\right|^2d\left\|\overline{A}_s\right\|\\
&-2\int_0^T\exp(\mu k_s)\left\langle\overline{Y}_s,\overline{Z}_sdW_s\right\rangle,
\end{aligned}$$

we set $\mu = 3$.and apply inequality $2ab \leq 2a^2 + \frac{1}{2}b^2$.with $a = K\left|\overline{Y}_s\right|$ and $b = \left\|\overline{Z}_s\right\|$. Since $k_t = \|\overline{A}\|_t + A'_t$ it follows that

$$\begin{aligned}
&\exp\left(\mu k_t\right)\left|\overline{Y}_t\right|^2 + \int_0^T \exp\left(\mu k_s\right)\left\|\overline{Z}_s\right\|^2 ds \\
&\leq \exp\left(\mu k_T\right)\left|\overline{\xi}\right|^2 + \left(3 + 2K^2\right)\int_0^T \exp\left(\mu k_s\right)\left|\overline{Y}_s\right|^2 ds + \frac{1}{2}\int_0^T \exp\left(\mu k_s\right)\left\|\overline{Z}_s\right\|^2 ds \\
&+ \int_0^T \exp\left(\mu k_s\right)\left|g\left(s, Y_s\right) - g'\left(s, Y_s\right)\right|^2 dA'_s + \int_0^T \left|f\left(s, Y_s, Z_s\right) - f'\left(s, Y_s, Z_s\right)\right|^2 ds \\
&+ \int_0^T \exp\left(\mu k_s\right)\left|g\left(s, Y_s\right)\right|^2 d\left\|\overline{A}_s\right\| - 2\int_0^T \exp\left(\mu k_s\right)\left\langle \overline{Y}_s, \overline{Z}_s dW_s\right\rangle.
\end{aligned}$$

Finally, we traspose $\frac{1}{2}\int_0^T \exp\left(\mu k_s\right)\left\|\overline{Z}_s\right\|^2 ds$ in the right side and we but $C = (3 + 2K^2)$, we have

$$\begin{aligned}
&\exp\left(\mu k_t\right)\left|\overline{Y}_t\right|^2 + \frac{1}{2}\int_0^T \exp\left(\mu k_s\right)\left\|\overline{Z}_s\right\|^2 ds \\
&\leq \exp\left(\mu k_T\right)\left|\overline{\xi}\right|^2 + C\int_0^T \exp\left(\mu k_s\right)\left|\overline{Y}_s\right|^2 ds + \int_0^T \exp\left(\mu k_s\right)\left|g\left(s, Y_s\right) - g'\left(s, Y_s\right)\right|^2 dA'_s \\
&+ \int_0^T \exp\left(\mu k_s\right)\left|g\left(s, Y_s\right)\right|^2 d\left\|\overline{A}_s\right\| + \int_0^T \left|f\left(s, Y_s, Z_s\right) - f'\left(s, Y_s, Z_s\right)\right|^2 ds \\
&- 2\int_0^T \exp\left(\mu k_s\right)\left\langle \overline{Y_s, Z}_s dW_s\right\rangle.
\end{aligned}$$

The result follows by taking the expectation without the sup term. Then, by taking the expectation inside the sup and applying Gronwall's lemma, the final result follows from the BDG's inequality. ■

3.3 Existence and uniqueness results

Now, we establish the fundamental theorem for the existence and uniqueness of the solution. To achieve this, we impose a regularity condition on the coefficients. Specifically, we assume that the functions f and g are Lipschitz continuous with respect to the variable y. Namely, there exists a positive constant K such that the following inequality holds

$$(L)\quad \left|f\left(t,y,z\right)-f\left(t,y',z\right)\right|+\left|g\left(t,y\right)-g\left(t,y'\right)\right|\leq K\left|y-y'\right|;\ \forall t>0;\ y,y'\in\mathbb{R}^k\quad \text{a.s}$$

Theorem 3.1 *Under the assumptions* $(1),(2),(5),(6)$ *and* (L), *the generalized BSDE* (3.1) *has a unique solution* $(Y,Z)\in\mathbb{B}^2\left(\mathbb{R}^k\right)$.

Notation 2 *Let* $\mathbb{M}_\mu^2(A)$ *denote the set of progressively measurable processes* $(X_t)_{t\geq 0}$, *which are such that*

$$\mathbb{E}\left(\int_0^T \exp\left(\mu A_t\right)\left|X_t\right|^2 dt+\int_0^T \exp\left(\mu A_t\right)\left|X_t\right|^2 dA_t\right)<\infty.$$

$\mathbb{M}_\mu^2$ the same space without the second term.
We define $\mathbb{B}_\mu^2\triangleq\left(\mathbb{M}_\mu^2(A)\right)^k\times\left(\mathbb{M}_\mu^2\right)^{k\times d}$.

Proof. Let the fuction $\Phi:\mathbb{B}_\mu^2\to\mathbb{B}_\mu^2$, which is defined as follows. Given $(U,V)\in\mathbb{B}_\mu^2$ such that $(Y,Z)=\Phi(U,V)$, then we have

$$Y_t=\xi+\int_t^T f\left(s,U_s,V_s\right)ds+\int_t^T g\left(s,U_s\right)dA_s-\int_t^T Z_s dW_s\quad 0\leq t\leq T.$$

Taking $\mathbb{E}\left(\cdot|\,\mathcal{F}_t\right)$ of the last equation yields

$$\begin{aligned}Y_t&=\mathbb{E}\left(Y_t|\,\mathcal{F}_t\right)\\&=\mathbb{E}\left(\xi+\int_t^T f\left(s,U_s,V_s\right)ds+\int_t^T g\left(s,U_s\right)dA_s\middle|\,\mathcal{F}_t\right).\end{aligned}$$

By virtue of Proposition 3.2, the uniform integrability of the martingale $M_t=\int_0^t Z_s dW_s$ ensures that $\mathbb{E}\left(M_T-M_t\right)=0$, and by the Brownian martingale representation theorem, there exists a unique square-integrable and adapted process $(Z_t)_{t\geq 0}$ such that:

$$Y_t=\mathbb{E}\left(\xi+\int_0^T f\left(s,U_s,V_s\right)ds+\int_0^T g\left(s,U_s\right)dA_s\middle|\,\mathcal{F}_0\right)+\int_0^t Z_s dW_s,$$

we deduce:

$$Y_t = \mathbb{E}\left(\xi + \int_0^T f(s, U_s, V_s)\, ds + \int_0^T g(s, U_s)\, dA_s\right) + \int_0^t Z_s dW_s.$$

In addition, we observe that:

$$Y_T = \mathbb{E}\left(\xi + \int_0^T f(s, U_s, V_s)\, ds + \int_0^T g(s, U_s)\, dA_s\right) + \int_0^T Z_s dW_s,$$

Furthermore,a direct calculation yields:

$$Y_t - Y_T = \int_0^t Z_s dW_s - \int_0^T Z_s dW_s.$$

Thus

$$\begin{aligned} Y_t &= Y_T - \int_t^T Z_s dW_s, \\ &= \mathbb{E}\left(\xi + \int_t^T f(s, U_s, V_s)\, ds + \int_t^T g(s, U_s)\, dA_s \middle| \mathcal{F}_T\right) - \int_t^T Z_s dW_s \\ &\xi + \int_t^T f(s, U_s, V_s)\, ds + \int_t^T g(s, U_s)\, dA_s - \int_t^T Z_s dW_s. \end{aligned}$$

Then, from itô formula (1.3) to the process $\exp(\mu A_t)|Y_t|^2$, we have

$$\begin{aligned} &\exp(\mu A_t)|Y_t|^2 + \mu \int_t^T \exp(\mu A_s)|Y_t|^2\, dA_s + \int_t^T \exp(\mu A_s)\|Z_s\|^2\, ds \\ &= \exp(\mu A_T)|\xi|^2 + 2\int_t^T \exp(\mu A_s)\langle Y_s, f(s, U_s, V_s)\, ds\rangle \\ &+ 2\int_t^T \exp(\mu A_s)\langle Y_s, g(s, U_s)\rangle\, dA_s - 2\int_t^T \exp(\mu A_s)\langle Y_s, Z_s dW_s\rangle. \end{aligned}$$

Taking the expectation on both sides of the inequality, we obtain:

$$\begin{aligned}
&\mathbb{E}\left(\exp\left(\mu A_t\right)\left|Y_t\right|^2\right)+\mu\mathbb{E}\left(\int_t^T \exp\left(\mu A_s\right)\left|Y_t\right|^2 dA_s\right)+\mathbb{E}\left(\int_t^T \exp\left(\mu A_s\right)\left\|Z_s\right\|^2 ds\right)\\
&\leq \mathbb{E}\left(\exp\left(\mu A_T\right)\left|\xi\right|^2\right)+2\mathbb{E}\left(\int_t^T \exp\left(\mu A_s\right)\left\langle Y_s, f\left(s,U_s,V_s\right)ds\right\rangle\right) \qquad (3.5)\\
&+2\mathbb{E}\left(\int_t^T \exp\left(\mu A_s\right)\left\langle Y_s, g\left(s,U_s\right)\right\rangle dA_s\right).
\end{aligned}$$

Consider the following inner product term:

$$\begin{aligned}
\left\langle Y_s, f\left(s,U_s,V_s\right)\right\rangle &\leq \frac{1}{2}\left|Y_s\right|^2+\frac{1}{2}\left|f\left(s,U_s,V_s\right)\right|^2\\
&=\frac{1}{2}\left|Y_s\right|^2+\frac{1}{2}\left|f\left(s,U_s,V_s\right)-f\left(s,0,0\right)+f\left(s,0,0\right)\right|^2\\
&\leq \frac{1}{2}\left|Y_s\right|^2+\left|f\left(s,U_s,V_s\right)-f\left(s,0,0\right)\right|^2+\left|f\left(s,0,0\right)\right|^2,
\end{aligned}$$

from the assumptions (5) and (L) we get

$$\left\langle Y_s, f\left(s,U_s,V_s\right)\right\rangle \leq \frac{1}{2}\left|Y_s\right|^2+K^2\left(\left|U_s\right|^2+\left|V_s\right|^2\right)+\left|f\left(s,0,0\right)\right|^2,$$

by using the assupmtion (6)

$$\left\langle Y_s, f\left(s,U_s,V_s\right)\right\rangle \leq \frac{1}{2}\left|Y_s\right|^2+K^2\left(\left|U_s\right|^2+\left|V_s\right|^2\right)+\phi_s^2.$$

The same with the scalar product $\left\langle Y_s, g\left(s,U_s\right)\right\rangle$ we deduce

$$\left\langle Y_s, g\left(s,U_s\right)\right\rangle \leq \frac{1}{2}\left|Y_s\right|^2+K^2\left|U_s\right|^2+\psi_s^2.$$

Back to inequality (3.5), we get

$$
\begin{aligned}
&\mathbb{E}\left(\exp\left(\mu A_t\right)\left|Y_t\right|^2\right)+(\mu-1)\,\mathbb{E}\left(\int_t^T \exp\left(\mu A_s\right)\left|Y_t\right|^2 dA_s\right)+\mathbb{E}\left(\int_t^T \exp\left(\mu A_s\right)\left\|Z_s\right\|^2 ds\right)\\
&\leq \mathbb{E}\left(\exp\left(\mu A_T\right)|\xi|^2\right)+K^2\mathbb{E}\left(\int_t^T \exp\left(\mu A_s\right)\left|U_s\right|^2 ds\right)+K^2\mathbb{E}\left(\int_t^T \exp\left(\mu A_s\right)\left|V_s\right|^2 ds\right)\\
&+\mathbb{E}\left(\int_t^T \exp\left(\mu A_s\right)\phi_s^2 ds\right)+K^2\mathbb{E}\left(\int_t^T \exp\left(\mu A_s\right)\left|U_s\right|^2 dA_s\right)+\mathbb{E}\left(\int_t^T \exp\left(\mu A_s\right)\psi_s^2 dA_s\right).
\end{aligned}
\tag{3.6}
$$

By invoking Fubini's Theorem, assumption (2), and the fact that $(U,V)\in\mathbb{B}_\mu^2$, the application of Gronwall's Lemma, we get

$$
\mathbb{E}\left(\exp\left(\mu A_t\right)\left|Y_t\right|^2\right)\leq A_t\exp\left(T-t\right), \tag{3.7}
$$

with A defin as

$$
\begin{aligned}
A_t=&\,\mathbb{E}\left(\exp\left(\mu A_T\right)|\xi|^2\right)+K^2\mathbb{E}\left(\int_t^T \exp\left(\mu A_s\right)\left|U_s\right|^2 ds\right)+K^2\mathbb{E}\left(\int_t^T \exp\left(\mu A_s\right)\left|V_s\right|^2 ds\right)\\
&+\mathbb{E}\left(\int_t^T \exp\left(\mu A_s\right)\phi_s^2 ds\right)+K^2\mathbb{E}\left(\int_t^T \exp\left(\mu A_s\right)\left|U_s\right|^2 dA_s\right)+\mathbb{E}\left(\int_t^T \exp\left(\mu A_s\right)\psi_s^2 dA_s\right).
\end{aligned}
$$

Consequantly, by applying Fubini's theorem and substituting the estimates (3.6) and (3.7) with $\mu=2$, we have

$$
\mathbb{E}\left(\exp\left(\mu A_t\right)\left|Y_t\right|^2\right)+\mathbb{E}\left(\int_t^T \exp\left(\mu A_s\right)\left\|Z_s\right\|^2 ds\right)<\infty.
$$

Hence, we deduce that the couple $(Y,Z)\in\mathbb{B}_\mu^2$ is a solution to the equation if and only if it is a fixed point of the operator $\gneqq$,let $(U,V)\,,\ (U',V')\in\mathbb{B}_\mu^2;\ \Phi\left(U,V\right)=(Y,Z)$,and $\Phi\left(U',V'\right)=(Y',Z')$with $\left(\overline{U},\overline{V}\right)=(U-U',V-V')$ and $\left(\overline{Y},\overline{Z}\right)=(Y-Y',Z-Z')$ and we know that $\left\{\int_0^t \exp\left(\mu A_t\right)\langle Y_s, Z_s dW_s\rangle_{t\geq 0}\right\}$ is a martingale unifomly integrable. For each

$\gamma \in \mathbb{R}$ we have from Itô's formula that

$$\begin{aligned} d\left(\exp\left(\gamma t+\mu A_t\right)\left|\overline{Y_t}\right|^2\right) &= d\left(\exp\left(\gamma t\right)\exp\left(\mu A_t\right)\left|\overline{Y_t}\right|^2\right) \\ &= \gamma\exp\left(\gamma t\right)\exp\left(\mu A_t\right)\left|\overline{Y_t}\right|^2 dt+\mu\exp\left(\mu A_t\right)\exp\left(\gamma t\right)\left|\overline{Y_t}\right|^2 dA_t \\ &+ 2\exp\left(\gamma t\right)\exp\left(\mu A_t\right)\left|\overline{Y_t}\right| d\overline{Y_t}+d\left\langle \overline{Y_t},\overline{Y_t}\right\rangle, \end{aligned}$$

with

$$d\overline{Y_t} = -\left(f\left(U_t,V_t\right)-f\left(U_t',V_t'\right)\right)dt-\left(g\left(U_t\right)-g\left(U_t'\right)\right)dA_t+\left(Z_t-Z_t'\right)dW_t,$$

where $d\left\langle \overline{Y_t},\overline{Y_t}\right\rangle = \left\|\overline{Z}_t\right\|^2 dt$. Then, we get the integrale form

$$\begin{aligned} &\exp\left(\gamma t+\mu A_t\right)\left|\overline{Y_t}\right|^2+\int_t^T \exp\left(\gamma s+\mu A_s\right)\left[\gamma\left|\overline{Y}_s\right|^2 ds+\mu\left|\overline{Y}_s\right|^2 dA_s\right] \\ &= 2\int_t^T \exp\left(\gamma s+\mu A_s\right)\left\langle \overline{Y}_s,\left(f\left(U_s,V_s\right)-f\left(U_s',V_s'\right)\right)\right\rangle ds \\ &+ 2\int_t^T \exp\left(\gamma s+\mu A_s\right)\left\langle \overline{Y}_s,\left(g\left(U_s\right)-g\left(U_s'\right)\right)\right\rangle dA_s \\ &- 2\int_t^T \exp\left(\gamma s+\mu A_s\right)\left\langle \overline{Y}_s,\overline{Z}_s dW_s\right\rangle-\int_t^T \left\|\overline{Z}_s\right\|^2 ds, \end{aligned}$$

we also have that

$$\begin{aligned} &\exp\left(\gamma t+\mu A_t\right)\left|\overline{Y_t}\right|^2+\int_t^T \exp\left(\gamma s+\mu A_s\right)\left[\gamma\left|\overline{Y}_s\right|^2 ds+\mu\left|\overline{Y}_s\right|^2 dA_s\right] \\ &\leq 2\int_t^T \exp\left(\gamma s+\mu A_s\right)\left|\overline{Y}_s\right|\left|f\left(U_s,V_s\right)-f\left(U_s',V_s'\right)\right| ds \\ &+ 2\int_t^T \exp\left(\gamma s+\mu A_s\right)\left|\overline{Y}_s\right|\left|,g\left(U_s\right)-g\left(U_s'\right)\right| dA_s \\ &- 2\int_t^T \exp\left(\gamma s+\mu A_s\right)\left\langle \overline{Y}_s,\overline{Z}_s dW_s\right\rangle-\int_t^T \left\|\overline{Z}_s\right\|^2 ds, \end{aligned}$$

from the Lipschitz assumption (L), we get

$$\begin{aligned}
&\exp(\gamma t+\mu A_t)\left|\overline{Y_t}\right|^2+\int_t^T \exp(\gamma s+\mu A_s)\left[\gamma\left|\overline{Y}_s\right|^2 ds+\mu\left|\overline{Y}_s\right|^2 dA_s\right]\\
&\leq 2K\int_t^T \exp(\gamma s+\mu A_s)\left|\overline{Y}_s\right|\left(\left|\overline{U}_s\right|+\left\|\overline{V}_s\right\|\right)ds\\
&+2K\int_t^T \exp(\gamma s+\mu A_s)\left|\overline{Y}_s\right|\left|\overline{U}_s\right|dA_s\\
&-2\int_t^T \exp(\gamma s+\mu A_s)\left\langle\overline{Y}_s,\overline{Z}_s dW_s\right\rangle-\int_t^T\left\|\overline{Z}_s\right\|^2 ds.
\end{aligned}$$

Using the inequality $ab\leq\frac{a^2}{2}+\frac{b^2}{2}$, and by the passage of expectation, we deduce

$$\begin{aligned}
&\mathbb{E}\left(\exp(\gamma t+\mu A_t)\left|\overline{Y_t}\right|^2\right)+\mathbb{E}\left(\int_t^T \exp(\gamma s+\mu A_s)\left[\gamma\left|\overline{Y}_s\right|^2 ds+\mu\left|\overline{Y}_s\right|^2 dA_s+\left\|\overline{Z}_s\right\|^2 ds\right]\right)\\
&\leq\mathbb{E}\left(\int_t^T \exp(\gamma s+\mu A_s)\left|\overline{Y}_s\right|^2\left[4K^2 ds+2K^2 dA_s\right]\right)\\
&+\frac{1}{2}\mathbb{E}\left(\int_t^T \exp(\gamma s+\mu A_s)\left[\left(\left|\overline{U}_s\right|^2+\left\|\overline{V}_s\right\|^2\right)ds\right]+\left|\overline{U}_s\right|^2 dA_s\right).
\end{aligned}$$

Taking $\gamma\geq 1+4K^2$ and $\mu\geq 1+2K^2$, we deduce:

$$\begin{aligned}
&\mathbb{E}\left(\int_t^T \exp(\gamma s+\mu A_s)\left[\left(\left|\overline{Y}_s\right|^2+\left\|\overline{Z}_s\right\|^2\right)ds\left|\overline{Y}_s\right|^2 dA_s\right]\right)\\
&\leq\frac{1}{2}\mathbb{E}\left(\int_t^T \exp(\gamma s+\mu A_s)\left[\left(\left|\overline{U}_s\right|^2+\left\|\overline{V}_s\right\|^2\right)ds\right]+\left|\overline{U}_s\right|^2 dA_s\right),
\end{aligned}$$

from which it follows that Φ is a strict constraction o $\mathbb{B}^2_\mu$ equipped with the norm

$$\|(Y,Z)\|_{\gamma,\mu}=\left[\mathbb{E}\left(\int_t^T \exp(\gamma s+\mu A_s)\left[\left(|U_s|^2+\|V_s\|^2\right)ds\right]+|U_s|^2 dA_s\right)\right]^{\frac{1}{2}}.$$

■

Next, we prove the existence and uniqueness of generalized BSDE (**3.1**) under the monotone conditions.

Theorem 3.2 *Under the conditions* $(1)-(7)$*, the generalized BSDE* (**3.1**) *has unique solution.*

Proof. Uniqueness has been established in proposition 3.2 for $\xi = \xi'$, $f = f'$, $g = g'$, $A = A'$, implies that:

$$\mathbb{E}\left(\sup_{0\leq t\leq T} \exp\left(\mu k_t\right)\left|Y_t\right|^2 + \int_0^T \exp\left(\mu k_t\right)\left\|\overline{Z}_t\right\|^2 ds\right) = 0 \Leftrightarrow Y = Y' \text{ and } Z = Z', \;\; \mathbb{P}.p.s.$$

For the proof of existence, we need firstly the following proposition. ■

Proposition 3.3 *Taking* $V \in \left(\mathbb{M}^2\right)^{k\times d}$, *there exists a unique pair of progressively measurable processes* $(Y_t, Z_t)_{t\geq 0}$ *such that*

$$\begin{array}{c} (i)\;\; \mathbb{E}\left(\sup_{0\leq t\leq T}\left|Y_t\right|^2 + \int_0^T \left|Y_t\right|^2 dA_t + \int_0^T \left\|Z_t\right\|^2 dt\right) < \infty, \\ \left(ii"\right)\;\; Y_t = \xi + \int_t^T f\left(s, Y_s, V_s\right) ds + \int_t^T g\left(s, Y_s\right) dA_s - \int_t^T Z_s dW_s. \end{array}$$

Proof. The uniqueness of the solution has already been established in proposition 3.2. We focus here on proving the existence of a pair of progressively measurable processes(Y, Z) . We shall write $f\left(s, y\right)$ for $f\left(s, y, V_s\right)$. Note that $f\left(s, y\right)$ satisfies

$$\begin{cases} (a)\,.\mathbb{E}\left(\int_0^T |, f(s,0)|^2 dt\right) < \infty; \\ (b)\,.\langle y - y', f(s,y) - f\left(s, y'\right)\rangle \leq \alpha\left|y - y'\right|^2; \\ (c)\,.\left|f(s,y,z)\right| \leq \phi'_t + K\left(|y| + \|z\|\right), \text{ where } \mathbb{E}\left(\int_0^T \exp\left(\mu A_t\right)\left(\phi'\right)_t^2 dt\right) < \infty; \\ (d)\,.y \rightarrow \left(f\left(t,y,z\right), g\left(t,y\right)\right) \text{ is continuous, } d\mathbb{P}\times dt a.e. \end{cases}$$

<u>Step 1: Approximation of the generators</u> <u>f and g</u>: Since f and g satisfy the monotonicity conditions but are not necessarily Lipschitz continuous, we use an approximation argument. For each $n \in \mathbb{N}$, we define

$$f_n(s,y) \triangleq \left(\rho_n * f(t,\cdot)\right)(y), \quad g_n(s,y) \triangleq \left(\rho_n * g(t,\cdot)\right)(y),$$

f_n and g_n be a sequence of Lipschitz continuous functions (e.g., via convolution with a Dirac sequence ρ_n)

$$\rho_n : \mathbb{R}^k \rightarrow \mathbb{R}_+,$$

and satisfies

$$\int \rho_n(x)\,dx = 1, \quad \sup_n \int |x|\,\rho_n(x)\,dx < \infty,$$

that converge to f and g respectively. According to Theorem 3.1, for each n, there exists a unique solution (Y^n, Z^n), to the following approximated generalized BSDE

$$Y_t^n = \xi + \int_t^T f_n(s, Y_s^n)\,ds + \int_t^T g_n(s, Y_s^n)\,dA_s - \int_t^T Z_s^n dW_s, \quad 0 \le t \le T,$$

has a unique solution which satisfies

$$\mathbb{E}\left(\sup_{0\le t\le T} |Y_t^n|^2 + \int_0^T \|Z_t^n\|^2\,dt\right) < \infty,$$

Moreover, using Itô's formula for the process $|Y_t^n|^2$ we get

$$\begin{aligned} |Y_t^n|^2 &= |\xi|^2 + 2\int_t^T \langle Y_s^n, f_n(s, Y_s^n)\,ds\rangle + 2\int_t^T \langle Y_s^n, g_n(s, Y_s^n)\rangle\,dA_s \\ &\quad - 2\int_t^T \langle Y_s^n, Z_s^n dW_s\rangle - \int_t^T \|Z_s^n\|^2\,ds, \end{aligned}$$

and like $\int_t^T \langle Y_s^n, Z_s^n dW_s\rangle$, is a uniformly integrable martingale. Thus, from Lemma 1.2, it follows that

$$\mathbb{E}\left(|Y_t^n|^2\right) + \mathbb{E}\left(\int_t^T \|Z_s^n\|^2\,ds\right) = \mathbb{E}\left(|\xi|^2\right) + 2\mathbb{E}\left(\int_t^T \langle Y_s^n, f_n(s, Y_s^n)\,ds\rangle\right) + 2\mathbb{E}\left(\int_t^T \langle Y_s^n, g_n(s, Y_s^n)\rangle\,dA_s\right).$$

From the conditions (b) for f and (4) for g, we obtain

$$\begin{aligned} \mathbb{E}\left(|Y_t^n|^2\right) + \mathbb{E}\left(\int_t^T \|Z_s^n\|^2\,ds\right) &\le \mathbb{E}\left(|\xi|^2\right) + 2\mathbb{E}\left(\int_t^T \langle Y_s^n, f_n(s, Y_s^n) - f_n(s,0) + f_n(s,0)\,ds\rangle\right) \\ &\quad + 2\mathbb{E}\left(\int_t^T \langle Y_s^n, g_n(s, Y_s^n) - g_n(s,0) + g_n(s,0)\rangle\,dA_s\right), \\ &\le \mathbb{E}\left(|\xi|^2\right) + 2\alpha\mathbb{E}\left(\int_t^T |Y_s^n|^2\,ds\right) + 2\mathbb{E}\left(\int_t^T |Y_s^n| \times |f_n(s,0)|\,ds\right) \\ &\quad + 2\beta\mathbb{E}\left(\int_t^T |Y_s^n|^2\,dA_s\right) + 2\mathbb{E}\left(\int_t^T |Y_s^n| \times |g_n(s,0)|\,ds\right). \end{aligned}$$

Using Young inequality, then we have

$$\begin{aligned}\mathbb{E}\left(|Y_t^n|^2\right)+\mathbb{E}\left(\int_t^T\|Z_s^n\|^2\,ds\right) &\leq \mathbb{E}\left(|\xi|^2\right)+2\alpha\mathbb{E}\left(\int_t^T|Y_s^n|^2\,ds\right)+\mathbb{E}\left(\int_t^T|Y_s^n|^2\,ds\right)\\ &+\mathbb{E}\left(\int_t^T|f_n\left(s,0\right)|^2\,ds\right)+2\beta\mathbb{E}\left(\int_t^T|Y_s^n|^2\,dA_s\right)\\ &+\mathbb{E}\left(\int_t^T|Y_s^n|^2\,dA_s\right)+\mathbb{E}\left(\int_t^T|g_n\left(s,0\right)|^2\,dA_s\right).\end{aligned}$$

Hence

$$\mathbb{E}\left(|Y_t^n|^2\right)+\mathbb{E}\left(\int_t^T\|Z_s^n\|^2\,ds\right)+|\beta|\,\mathbb{E}\left(\int_t^T|Y_s^n|^2\,dA_s\right)\leq C\left[1+\mathbb{E}\left(\int_t^T|Y_s^n|^2\,dA_s\right)\right],$$

and from this, Gronwall's Lemma and the Burkholder inequalities, we have that

$$\mathbb{E}\left(\sup_{0\leq t\leq T}|Y_t^n|^2+\int_0^T|Y_t^n|^2\,dA_t+\int_0^T\|Z_t^n\|^2\,dt\right)<\infty.$$

Step 2: Uniform estimates and weak convergence: By applying the a priori estimates from Proposition 3.2, it can be shown that the sequences $(Y^n)_{n\in\mathbb{N}}$ and $(Z^n)_{n\in\mathbb{N}}$, are uniformly bounded in the spaces $\mathbb{L}^2\left(\Omega;\left(\mathbb{C}\left[0,T\right],\mathbb{R}^k\right)\right)$ and $\mathbb{L}^2\left(\Omega\times[0,T];\mathbb{R}^{k\times d}\right)$ respectively. Furthermore, let us define the auxiliary processes:

$$U_t^n\triangleq f_n\left(s,Y_t^n\right),\quad V_t^n\triangleq g_n\left(s,Y_t^n\right).$$

By the assumptions on f and g, these sequences are also uniformly bounded in $\mathbb{L}^2$. Consequently, by the Banach theorem, we can extract a subsequence (still denoted by n) such that:

$$\begin{cases}(Y^n,Z^n)\rightarrow(Y,Z)\ \text{ weakly in } \mathbb{L}^2\left(\Omega\times[0,T]\right);\\ U^n\rightarrow U \text{ weakly in } \mathbb{L}^2\left(d\mathbb{P}\times dt\right);\\ V^n\rightarrow V \text{ weakly in } \mathbb{L}^2\left(d\mathbb{P}\times dA_t\right).\end{cases}$$

It is then easy to deduce that

$$Y_t = \xi + \int_t^T U_s ds + \int_t^T V_s dA_s - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T.$$

Step 3: Identification of the Limits: The final and most critical step is to prove that $U_t = f(t, Y_t)$ and $V_t = g(t, Y_t)$, Since the generators are non-linear, weak convergence $(Y^n \to Y)$ does not immediately imply $f(Y^n) \to f(Y)$. Applying Itô's formula to the process $\exp(\alpha t) |Y_t^n|^2$, and taking the expectation, we have:

$$\begin{aligned}
&\mathbb{E}(|Y_t^n|) + \mathbb{E}\left(\int_t^T \exp(\alpha t) \|Z_s^n\|^2 ds\right) \\
&= \mathbb{E}\left(\exp(\alpha T) |\xi|^2\right) + 2\mathbb{E}\left(\int_t^T \langle Y_s^n, (U_s^n - \alpha Y_s^n) ds\rangle\right) + 2\mathbb{E}\left(\int_t^T \langle Y_s^n, V_s^n dA_s\rangle\right).
\end{aligned}$$

Recall the monotonicity conditions (3) and (4). For any test process $X \in \mathbb{M}^2\left([0,T]; \mathbb{R}^k\right)$, we have:

$$\mathbb{E}\left(\int_0^T \exp(\alpha t) \langle Y_t^n - X_t, f_n(t, Y_t^n) - f_n(s, X_t)\rangle dt\right) \leq \alpha \mathbb{E}\left(\int_0^T |Y_t^n - X_t|^2 dt\right).$$

By utilizing the fact that the norm is weakly lower semi-continuous in Hilbert spaces, specifically:

$$\mathbb{E}\left(\int_t^T \exp(\alpha t) \|Z_s^n\|^2 ds\right) \leq \liminf_{n\to\infty} \mathbb{E}\left(\int_t^T \exp(\alpha t) \|Z_s^n\|^2 ds\right).$$

And combining this with the weak convergence of U^n and V^n, we pass to the limit as $n \to \infty$ to arrive at the fundamental inequality:

$$\begin{aligned}
&\mathbb{E}\int_0^T \exp(\alpha t) \langle Y_t - X_t, U_t - f(t, X_t, Z_t) - \alpha(Y_t - X_t)\rangle dt \\
&+ \mathbb{E}\int_0^T \exp(\alpha t) \langle Y_t - X_t, V_t - g(t, X_t)\rangle dA_t \\
&\leq 0
\end{aligned} \tag{3.8}$$

for any progressively measurable process X, let us choose $X_t = Y_t - \epsilon\phi_t$, for an arbitrary bounded process ϕ and $\epsilon > 0$. Substituting this into inequality (3.8) yields:

$$\epsilon\mathbb{E}\int_0^T \exp(\alpha t)\langle\phi_t, U_t - f(t, Y_t - \epsilon\phi_t, Z_t) - \alpha(\epsilon\phi_t)\rangle + \mathbb{E}\int_0^T \exp(\alpha t)\langle\phi_t, V_t - g(t, Y_t - \epsilon\phi_t)\rangle\, dA_t \leq 0.$$

Dividing the entire inequality by ϵ and letting $\epsilon \to 0$, and invoking the continuity of f and g (Assumption (7)), we obtain:

$$\mathbb{E}\int_0^T \exp(\alpha t)\langle\phi_t, U_t - f(t, Y_t, Z_t)\rangle + \mathbb{E}\int_0^T \exp(\alpha t)\langle\phi_t, V_t - g(t, Y_t)\rangle\, dA_t \leq 0.$$

Because this inequality holds for both ϕ and $-\phi$, the integrand must vanish a.s. Therefore:

$$U_t = f(t, Y_t, Z_t), \quad V_t = g_n(t, Y_t).$$

This completes the proof that (Y, Z) is the solution to the generalized BSDE under the given monotonicity conditions. ■

Proof. of Theoreme 3.2, we have

$$\begin{cases} (i)\ \mathbb{E}\left(\sup_{0\leq t\leq T}|Y_t|^2 + \int_0^T |Y_t|^2\, dA_t + \int_0^T \|Z_t\|^2\, dt\right) < \infty, \\ (ii')\ Y_t = \xi + \int_t^T f(s, Y_s, V_s)\, ds + \int_t^T g(s, Y_s)\, dA_s - \int_t^T Z_s dW_s. \end{cases} \tag{3.9}$$

We sall use the natation $\mathbb{B}^2$.with the help of proposition ,we constract a mapping $\Phi : \mathbb{B}^2 \to \mathbb{B}^2 \to$, which to $(U, V) \in \mathbb{B}^2$ associates $(Y, Z) = \Phi(U, V)$, the solution of generalized BSDE (**3.9**)**.** Although the first component U of the pair (U, V) is not needed in order to constract (Y, Z), it is convenient to consider the mapping $(U, V) \to (Y, Z)$ in making a fixed point argument. Let $(U, V), (U', V') \in \mathbb{B}^2$

$$\begin{cases} (Y, Z) = \Phi(U, V), \ \ (Y', Z') = \Phi(U', V'), \\ (\overline{U}, \overline{V}) = (U - U', V - V'), \ \ (\overline{Y}, \overline{Z}) = (Y - Y', Z - Z'). \end{cases}$$

It follows from Itô's formula that

$$\begin{aligned}
&\exp(\gamma t)\,\mathbb{E}\left|\overline{Y}_t\right|^2+\mathbb{E}\left(\int_t^T\exp(\gamma s)\left(\gamma\left|\overline{Y}_s\right|^2+\left\|\overline{Z}_s\right\|\right)ds\right)\\
&\le 2\mathbb{E}\left(\int_t^T\exp(\gamma s)\left(\alpha\left|\overline{Y}_s\right|^2+K\left|\overline{Y}_s\right|+\left\|\overline{V}_s\right\|\right)ds\right)\\
&\le \frac{1}{2}\mathbb{E}\left(\int_t^T\exp(\gamma s)\left(\left(\alpha+K^2\right)\left|\overline{Y}_s\right|^2+\left\|\overline{V}_s\right\|^2\right)ds\right).
\end{aligned}$$

Choosing $\gamma=1+2\left(\alpha+K^2\right)$, we deduce that

$$\mathbb{E}\left(\int_t^T\exp(\gamma s)\left(\gamma\left|\overline{Y}_s\right|^2+\left\|\overline{Z}_s\right\|\right)ds\right)\le\frac{1}{2}\mathbb{E}\left(\int_t^T\exp(\gamma s)\left(\left|\overline{U}_s\right|^2+\left\|\overline{V}_s\right\|^2\right)dt\right),$$

and we have proved that Φ has a unique fixed point, which satisfies generalized BSDE (**3.9**).

■

Conclusion

This dissertation provides a rigorous analysis of generalized backward stochastic differential equations that incorporate an integral with respect to a continuous non-decreasing process. By applying Itô's formula and appropriate fixed-point theorems, we establish the existence and uniqueness of the adapted solution pair (Y_t, Z_t). Beyond the theoretical framework, the main contribution of this work lies in providing a mathematical guarantee that a terminal payoff ξ can be replicated or achieved through a proactive control strategy Z_t. This approach not only mitigates the impact of stochastic market fluctuations but also accounts for the irreversible cumulative costs modeled by the process A_t

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Appendix A

Some mathematical tools

Theorem A.1 ***(Brownian martingale representation [4])*** *Let $(M_t)_{t\geq 0}$ be a square integrable with respect to the filtration generated by a Brownian motion W .Then ,ther exists a unique progressively measurable process $Z \in \mathbb{M}^2\left(\mathbb{R}^{k\times d}\right)$ such that:*

$$M_t = M_0 + \int_0^t M_s dW_s, \qquad \mathbb{P} - a.s \text{ for all } t \in [0,T].$$

Theorem A.2 ***(Banach fixed point [5])*** *Let (E,d) be a non-empty complete metric space (a Banach space) and let $\Phi \colon E \to E$ be a contraction mapping, i.e., there exists $k \in [0,1)$ such that for all $(u,v) \in E^2$*

$$d\left(\Phi(u) - \Phi(v)\right) \leq kd(u,v).$$

Then, Φ admits a unique fixed point $u^\star \in E$ such that $\Phi(u^\star) = u^\star$.

Theorem A.3 ***(Doob's $\mathbb{L}^p$ inequality [2])*** *Let $(M_t)_{t\geq 0}$ be a square-integrable martingale. For any $T > 0$*

$$\mathbb{E}\left(\sup_{0\leq t\leq T} |M_t|\right) \leq 4\mathbb{E}\left(|M_t|^2\right).$$

Theorem A.4 ***(Fubini's theorem [4])****Let $(\Omega_1, \mathcal{A}_1, \mu_1)$ and $(\Omega_2, \mathcal{A}_2, \mu_2)$ be two σ-finite measure spaces. If $f(x,y)$ is a non-negative measurable function, or if it is integrable with* *respect to the product measure* ($f \in \mathbb{L}^1(\mu_1 \times \mu_2)$)*, then:*

$$\int_{\Omega_1} \left(\int_{\Omega_2} f(x,y) d\mu_2(y) \right) d\mu_1(x) = \int_{\Omega_2} \left(\int_{\Omega_1} f(x,y) d\mu_1(y) \right) d\mu_2(x).$$

Lemma A.1 ***(Fatou's inequality** [4])Let* $(X_n)_{n\in\mathbb{N}}$ *be a sequence of non-negative r.v's defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*. Then*

$$\mathbb{E}\left[\liminf_{n\to\infty} X_n\right] \leq \liminf_{n\to\infty} \mathbb{E}[X_n].$$

Theorem A.5 ***(Burkholder-Davis-Gundy Inequality** [4]) Let* $(M_t)_{t\geq 0}$ *be a local martingale starting at zero* $M_0 = 0$*, For any* $p \in (0, \infty)$*, there exist universal positive constants* c_p *and* C_p*, depending only on* p*, such that:*

$$c_p \mathbb{E}\left[\langle M, M\rangle_T^{p/2}\right] \leq \mathbb{E}\left[\sup_{0\leq t\leq T} |M_t|^p\right] \leq C_p \mathbb{E}\left[\langle M, M\rangle_T^{p/2}\right].$$

In prticular, for a stochastic integrale $\int_0^t Z_s dW_s$ *and* $p = 2$ *the inequality is closely related to the Itô isometry.*

Lemma A.2 ***(Gronwall's inquality** [9]) Let* $H(t)$ *a contiuous and non-negative function ,for any* $t \in [0,T]$ *we have:*

$$H(t) \leq a + b\int_0^t H(s)\, ds,$$

such that a, b *a constants, where* $b \geq 0$ *we obtain*

$$H(t) \leq a \exp(bt).$$

Lemma A.3 ***(Young's Inequality)*** *Let* a, b *be a real numbers non-negative, if* $p, q > 1$ *are two reals numbers such that* $\frac{1}{p} + \frac{1}{q} = 1$*, then we have the following inequality*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Furthermore, for any $\epsilon > 0$, the inequality can be expressed in the following $\epsilon-$version, which is particularly useful for stochastic estimates:

$$ab \leq \epsilon a^p + C(\epsilon)\, b^q.$$

where $C(\epsilon) = \frac{1}{q(p\epsilon)^{\frac{q}{p}}}$. In the specific case where $p = q = 2$, the inequality takes the form:

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}.$$

Lemma A.4 ***(Hölder's inequality** [1]) Let $p, q > 1$,be conjugate exponents such that $\frac{1}{p} + \frac{1}{q} = 1$.For any two r.v's $X \in \mathbb{L}^p(\Omega, \mathcal{F}, \mathbb{P})$ and $Y \in \mathbb{L}^q(\Omega, \mathcal{F}, \mathbb{P})$, their product XY is integrable, and the following inequality holds*

$$\mathbb{E}[|XY|] \leq (\mathbb{E}[|X|^p])^{\frac{1}{p}} (\mathbb{E}[|Y|^q])^{\frac{1}{q}}.$$

In the particular case where $p = q = 2$, this inequality reduces to the Cauchy-Schwarz inequality:

$$\mathbb{E}^2[|XY|] \leq \mathbb{E}\left[|X|^2\right] \mathbb{E}\left[|Y|^2\right].$$

ملخص :

تتناول المذكرة دراسة المعادلات التفاضلية العشوائية التراجعية المعممة والتي تتميز بوجود تكامل إضافي بالنسبة لعملية متزايدة ومستمرة. يكمن الهدف الأساسي من البحث في إثبات وجود ووحدانية حل هذه المعادلات ، تعتمد المنهجية المتبعة على استخدام تقديرات للمولدات وتطبيق نظرية النقطة الصامدة لباناخ في فضاءات هيلبرت المناسبة. وقد تم التوصل الى اثبات النتائج النظرية تحت شروط ليبشيتز و كذا شروط الرتابة للمولدات ، مما يؤكد صلابة هذا النموذج الرياضي وتوافقه مع المتطلبات التقنية للتحليل العشوائي.

الكلمات المفتاحية: المعادلات التفاضلية العشوائية التراجعية المعممة ، العملية المتزايدة ، وجود ووحدانية الحل ، شرط ليبشيتز ، شروط الرتابة.

Abstract

This dissertation investigates a class of generalized backward stochastic differential equations (GBSDEs for schort) driven by an stochastic integral with respect to a continuous increasing process. The main objective is to establish the existence and uniqueness of the solution (Y, Z). The analytical approach relies on deriving a priori estimates and applying the Banach fixed-point theorem within appropriate Hilbert spaces. Our results demonstrate that well-posedness is guaranteed under both Lipschitz continuity and monotonicity conditions for the drivers, providing a rigorous mathematical framework for analyzing such complex stochastic systems.

Keywords: GBSDEs; Increasing process; Existence and Uniqueness; Lipschitz condition; Monotonicity conditions.

Résumé

Ce mémoire examine une classe d'équations différentielles stochastiques rétrogrades généralisées (EDSRG) caractérisées par un intégrale stochastique par rapport à un processus croissant continu. L'objectif central est d'établir l'existence et l'unicité de solutions (Y, Z). L'approche analytique repose sur l'établissement d'estimations a priori et l'application du théorème du point fixe de Banach dans des espaces de Hilbert adaptés. Les résultats prouvent que le caractère bien posé de ces équations est assuré sous des conditions de Lipschitz ainsi que sous des conditions de monotonie des générateurs.

Mots-clés : EDSRG; Processus croissant; Existence et Unicité; Condition de Lipschitz; Conditions de monotonie.