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Two-scale convergence method and Homogenization in porous media

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Dedication

With deep love and sincere gratitude, I dedicate this humble work to those who have been my support and guiding light at every step of my journey.

To the dearest person in my life, my beloved **mother**, who instilled in me the love of knowledge and ambition, whose prayers have been the secret of my success, and whose patience and strength have been my greatest motivation to keep going. May God protect her and keep her as a crown above my head.

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To everyone who supported me with a kind word, a sincere prayer, or a noble gesture, near or far.

And to myself, who remained patient, worked hard, and believed that after exhaustion comes success, and after sleepless nights comes achievement worthy of ambition.

I dedicate this work with great pride and honor.

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Finally, I pray that Almighty Allah accepts this work, makes it beneficial, and grants success in future endeavors.

Contents

Notations

- Ω : an open set of R^n
- Y : The unit cell
- x : Macroscopic variable
- $y = \frac{x}{\varepsilon}$: Microscopic variable
- ∇_x, ∇_y : Gradient with respect to x or y
- div_y , div_x : Divergence with respect to x or y
- $\partial_n = n \cdot \nabla$: Normal derivative
- $\dfrac{\partial^k}{\partial x_i^k}$: k^{th} partial derivative
- $|\alpha| = \alpha_1 + \cdots + \alpha_n$
- $\rightarrow$: Strong convergence
- $\rightharpoonup$: Weak convergence
- $\rightharpoonup^2$: Two-scale convergence
- $|Y|$: Measure of Y

- $C^1_{\text{per}}(Y)$: Space of Y-periodic continuously differentiable functions.
- $H^1_{\text{per}}(Y)$: Subspace of $H^1(Y)$ of Y periodic functions
- $L^p(\Omega, C^1_{\text{per}}(Y))$: The space of measurable function $u : x \in \Omega \to u(x) \in C^\infty_{\text{per}}(Y)$ such that $\|u(x)\|_{C^\infty(Y)} \in L^p(\Omega)$.
- $\mathcal{D}(\Omega, C^\infty_{\text{per}}(Y))$: The space of infinitely differentiable functions with compact support in Ω with respect to the first argument and taking values in $C^\infty_{\text{per}}(Y)$ with respect to the second argument.
- $L^2(\Omega \times Y)$: space of square-integrable functions on $\Omega \times Y$.

Introduction

Homogenization theory ([5],[7]) is considered one of the most important modern fields in applied mathematics and partial differential equations. It aims to study mathematical models involving rapidly oscillating coefficients or microscopic heterogeneous structures. Such models arise in many physical and engineering applications, especially in fluid mechanics, porous media, composite materials, and heat transfer.

Among the most powerful methods developed in this field is the Two-Scale Convergence method introduced by G. Nguetseng [5] and G. Allaire [3]. This method provides a rigorous mathematical framework for describing both microscopic and macroscopic behaviors simultaneously, and for passing to the limit in heterogeneous problems in order to derive effective homogenized models.

In this work, we apply two-scale convergence method to study the homogenization of Stokes equations ([3],[5])

$$\begin{cases} \nabla p^\varepsilon - \varepsilon^2 \mu \Delta u^\varepsilon = f & \text{in } \Omega^\varepsilon \\ \operatorname{div}\, u^\varepsilon = 0 & \text{in } \Omega^\varepsilon \\ u^\varepsilon = 0 & \text{on } \partial\Omega^\varepsilon \end{cases}$$

in a periodic porous medium Ω^ϵ with a Dirichlet boundary condition where we denote by u^ε and p^ε be the velocity and pressure of the fluid, and f the density of forces acting on the fluid. The porous medium is contained in a domain Ω, and its fluid part is denoted by

Ω^ϵ. From a mathematical point of view, Ω^ϵ is a periodically perforated domain, i.e., it has many small holes of size ε which represent solid obstacles that the fluid cannot penetrate.

This work is compesd of three chapters:

The first chapter is devoted to the presentation of some fundamental notions required for the study, including different types of convergence, Sobolev spaces, and periodic functions. In the second chapter, we introduce the concept of two-scale convergence together with its main results and properties. Then, we apply two-scale convergence method to the homogenization of a model problem. The last chapter deals with a more involved application of this method, the Stokes equations in periodic porous media in order to derive the homogenized system describing the effective macroscopic behavior of fluid flow inside such structures.

Chapter 1

Generalities

In this chapter we recall some fundamental notions and basic results required for our study. For more details, see the following references [6],[7].

1.1 Strong, Weak and Weak* Convergence

Let $(E, \|\cdot\|_E)$ be a Banach space and E' its dual endowed with the norm

$$\|f\|_{E'} = \sup \frac{\langle f, u\rangle_{E',E}}{\|u\|_E}.$$

Definition 1.1.1 (Strong convergence [7]) *A sequence (u_n) in E is said to converge strongly to u if*

$$\lim_{n\to\infty} \|u_n - u\|_E = 0.$$

This strong convergence is denoted $u_n \to u$ in E .

Definition 1.1.2 (Weak convergence [7]) *A sequence (u_n) in E is said to converges weakly to u if*

$$\forall f \in E', \quad \langle f, u_n\rangle_{E',E} \to \langle f, u\rangle_{E',E}.$$

This weak convergence is denoted $u_n \rightharpoonup u$ *in* E .

Proposition 1.1.1 *[7]*

i) *Strong convergence implies weak convergence.*

ii) *If* $\dim E = N < +\infty$, *then strong and weak convergences are equivalent.*

Proposition 1.1.2 *[7] Let* $u_n \rightharpoonup u$ *be a sequence weakly convergent to* u *in* E. *Then*

i) (u_n) *is a bounded sequence in* E *i.e., there exists a constant* C *independent of* n *such that*

$$\forall n \in \mathbb{N}, |u_n\|_E \leq C$$

ii) *The norm on* E *is lower semi -continuous with respect to the weak convergence,i.e..* $\|u\|_E \leq \liminf\limits_{n\to\infty} \|u_n\|_E$.

Theorem 1.1.1 (Eberlein–Šmuljan [7]) *Let* E *be reflexive and* (u_n) *bounded in* E . *Then*

i) *There exists a subsequence* (u_{n_k}) *of* (u_n) *and* $u \in E$ *such that* $k \to \infty$

$$u_{n_k} \rightharpoonup u.$$

ii) *If all weakly convergent subsequences of* (u_{n_k}) *have the same limit* u . *Then the whole sequence* (u_n) *weakly converges to* u, *i.e..*

$$u_n \rightharpoonup u.$$

Proposition 1.1.3 *[7] Let* $(u_n) \subset E$ *and* $(v_n) \subset E'$ *such that*

$$\begin{cases} u_n \rightharpoonup u \text{ weakly in } E \\ v_n \to v \text{ strongly in } E' \end{cases}$$

Then

$$\lim_{n\to\infty} \langle v_n, u_n \rangle_{E',E} = \langle v, u \rangle_{E',E}$$

Definition 1.1.3 (Weak* convergence [7]) *A sequence (f_n) in E' is said to converge weakly* to f iff*

$$\forall u \in E,\ \langle f_n, u\rangle_{E',E} \to \langle f, u\rangle_{E',E}$$

This weak convergence is denoted*

$$f_n \rightharpoonup^* f \text{ in } E'$$

Proposition 1.1.4 *[7] Every weakly convergent sequence in E' is weakly* convergent. However, the opposite is false unless E is reflexive.*

Proposition 1.1.5 *[7] Let (f_n) be a sequence weakly* convergent to f in E' .Then*

1. *(f_n) is a bounded sequence in E ,i.e. there exists a constant C independent of n such that*

$$\forall n \in \mathbb{N}, \|f_n\|_{E'} \ \le C$$

2. *The norm is lower semi-continuous with respect to the weak* convergence, i.e.*

$$\|f\|_{E'} \le \liminf_{n\to\infty} \|f_n\|_{E'}.$$

Theorem 1.1.2 *[7] Let E be a separable Banach space and let (u_n) be a bounded sequence in E' . Then.*

1. *There exists a subsequence (f_{n_k}) of (f_n) and $f \in E'$ such that, as $k \to \infty$*

$$f_{n_k} \rightharpoonup^* f \quad \text{in } E'.$$

2. *If each weakly* convergent subsequence of (f_n) has the same limit f, then the whole sequence (f_n) weakly* converges to f, i.e.*

$$f_n \rightharpoonup f \quad \text{in } E'.$$

1.2 Weak convergence in L^p and Sobolev spaces

Let Ω be a bounded open set in $\mathbb{R}^n$

Definition 1.2.1 *[7] Let $p \in \mathbb{R}$ with $1 \leq p < \infty$. Define*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R},\ f \text{ measurable and } \int_\Omega |f|^p \, dx < \infty \right\}$$

$$L^\infty(\Omega) = \{ f : \Omega \to \mathbb{R},\ f \text{ measurable and } \exists C > 0,\ |f(x)| \leq C \text{ a.e on } \Omega \}$$

Proposition 1.2.1 *[7]*

- *The set $L^p(\Omega)$ is a Banach space for $1 \leq p \leq +\infty$ the norm*

$$\|f\|_{L^p(\Omega)} = \begin{cases} \left(\int_\Omega |f|^p \, dx \right)^{1/p} & \text{if } p < +\infty, \\ \inf\{ C \ ;\ |f(x)| \leq C \text{ a.e. on } \Omega \} & \text{if } p = +\infty. \end{cases}$$

- *The space $L^p(\Omega)$ is separable for $1 \leq p < +\infty$ and it is reflexive for $1 < p < +\infty$.*

- *If $p = 2$, the space $L^2(\Omega)$ is a Hilbert space for the scalar product*

$$(f, g)_{L^2(\Omega)} = \int_\Omega f(x) g(x) \, dx.$$

- *If $1 \leq p < \infty$, then $u_n \rightharpoonup u$ in $L^p(\Omega)$ if*

$$\int_\Omega u_n \phi \, dx \to \int_\Omega u \phi \, dx, \quad \forall \phi \in L^q(\Omega),$$

  with $1/p + 1/q = 1$.

- *If $p = +\infty$, then $f_n \overset{*}{\rightharpoonup} f$ in $L^\infty(\Omega)$ if*

$$\int_\Omega f_n \phi \, dx \to \int_\Omega f \phi \, dx, \quad \forall \phi \in L^1(\Omega).$$

Lemma 1.2.1 *[7] Let p, q and r be three reals in $[1, +\infty[$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$*
Let (u_n) be a sequence in $L^p(\Omega)$ and (v_n) be a sequence in $L^q(\Omega)$ such that:

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in } L^p(\Omega), \\ v_n \to v & \text{strongly in } L^q(\Omega). \end{cases}$$

Then

$$u_n v_n \rightharpoonup uv \quad \text{weakly in } L^r(\Omega).$$

Theorem 1.2.1 (Compact Injection [7]) *Let Ω be a bounded open in $\mathbb{R}^n$ such that $\partial\Omega$ is Lipschitzian.*

- *If $1 \leq p < n$, then $W^{1,p}(\Omega) \subset L^q(\Omega),\ \forall q \in \left[1, \frac{np}{n-p}\right]$, with compact injection*
- *If $p = n$, then $W^{1,p}(\Omega) \subset L^q(\Omega),\ \forall q \in [1, +\infty]$ and the injection is compact.*
- *If $p > n$, then $W^{1,p}(\Omega) \subset C(\overline{\Omega}),\ \forall q \in \left[1, \frac{np}{n-p}\right]$ with compact injection.*

Remark 1.2.1 *[7] Compact injection makes it possible to move from weak convergence to strong convergence as follows: let $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$.*

- *If $1 \leq p < n$, then $u_n \to u$ strongly in $L^q(\Omega)$ with $1 \leq q < \frac{np}{n-p}$.*
- *If $p = n$, then $u_n \to u$ strongly in $L^q(\Omega)$ with $1 \leq q < +\infty$.*
- *If $p > n$, then $u_n \to u$ strongly in $L^\infty(\Omega)$.*

1.3 Periodic Functions

We denote Y the interval in $\mathbb{R}^n$ defined by

$$Y =]0, l_1[\times \cdots \times]0, l_n[$$

where $l_1, \ldots, l_n$ are given positive numbers. We will refer to Y as the reference period.

Definition 1.3.1 (Periodic function [7]) *Let $Y =]0, l_1[\times \cdots \times]0, l_n[$ and f be a function defined a.e. on $\mathbb{R}^n$. The function f is called Y-periodic iff*

$$f(x + kl_ie_i) = f(x) \quad \text{a.e. on } \mathbb{R}^n,\ \forall k \in \mathbb{Z},\ \forall i \in \{1, \ldots, n\},$$

where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$. In the case $n = 1$, we simply say that f is l_1-periodic.

Definition 1.3.2 *[7] Let Ω be a bounded open set of $\mathbb{R}^n$ and f a function in $L^1(\Omega)$. The mean value of f over Ω is the real number $M_\Omega(f)$ given by*

$$M_\Omega(f) = \langle f \rangle = \frac{1}{|\Omega|} \int_\Omega f(y)\, dy.$$

Lemma 1.3.1 *[7] Let f be a Y-periodic function in $L^1(Y)$. Let y_0 be a fixed point in $\mathbb{R}^n$ and denote by Y_0 the translated set of Y, defined by $Y_0 = y_0 + Y$.*

Set

$$f^\varepsilon(x) = f\left(\frac{x}{\varepsilon}\right) \quad a.e. \ on \ \mathbb{R}^n.$$

Then

$$\text{i)} \int_{Y_0} f(y)\,dy = \int_Y f(y)\,dy,$$

$$\text{ii)} \int_{\varepsilon Y_0} f^\varepsilon(x)\,dx = \int_{\varepsilon Y} f^\varepsilon(x)\,dx = \varepsilon^n \int_Y f(y)\,dy.$$

Theorem 1.3.1 *[7] Let $1 \leq p \leq +\infty$ and f be a Y-periodic function in $L^p(Y)$. Set*

$$f_\varepsilon(x) = f\left(\frac{x}{\varepsilon}\right) \quad a.e. \ on \ \mathbb{R}^n.$$

Then, if $p < +\infty$, as $\varepsilon \to 0$

$$f_\varepsilon \rightharpoonup M_\Omega(f) \quad weakly \ in \ L^p(\Omega).$$

If $p = +\infty$, one has

$$f_\varepsilon \overset{*}{\rightharpoonup} M_\Omega(f) \quad weakly* \ in \ L^\infty(\Omega).$$

1.4 Some classes of Sobolev spaces

- We denote $C^\infty_{\text{per}}(Y)$ be the subset of $C^\infty(\mathbb{R}^n)$ of Y-periodic functions.
- We denote by $H^1_{\text{per}}(Y)$ the closure of $C^\infty_{\text{per}}(Y)$ for the H^1-norm.
- We denote by $D(\Omega, C^\infty_{\text{per}}(Y))$ the space of infinitely differentiable functions with compact support in Ω with respect to the first argument and taking values in $C^\infty_{\text{per}}(Y)$ with respect to the second argument.
- We denote by $L^p(\Omega, C^\infty_{\text{per}}(Y))$ the space of measurable functions $u : x \in \Omega \longmapsto u(x) \in C^\infty_{\text{per}}(Y)$ such that $\|u(x)\|_{C^\infty_{\text{per}}(Y)} \in L^p(\Omega)$.

- We denote by $L^p_{\text{per}}(\Omega, C(Y))$ the set of measurable functions $u : x \in \Omega \longmapsto u(x) \in C(Y)$ such that $\|u(x)\|_{C(Y)} \in L^p_{\text{per}}(\Omega)$, where $L^p_{\text{per}}(\Omega)$ is the subspace of $L^p(\Omega)$ consisting of Y-periodic functions.

Proposition 1.4.1 *[7] Let $u \in H^1_{\text{per}}(Y)$. Then, u has the same trace on the opposite faces of Y.*

Proposition 1.4.2 *[7] The space $L^p(\Omega, C^\infty_{\text{per}}(Y))$ is dense in $L^2(\Omega, L^2(Y)) = L^2(\Omega \times Y)$.*

Theorem 1.4.1 (Lax–Milgram Theorem [7]) *Let V be a real Hilbert space and let $a(\cdot, \cdot) : V \times V \to \mathbb{R}$ be a bilinear form satisfying:*

- ***Continuity:*** *There exists a constant $C > 0$ such that*

$$|a(u, v)| \leq C\|u\|_V \|v\|_V \quad \forall u, v \in V.$$

- ***Coercivity:*** *There exists a constant $\alpha > 0$ such that*

$$a(v, v) \geq \alpha \|v\|_V^2 \quad \forall v \in V.$$

Then, for every $f \in V'$, there exists a unique $u \in V$ such that

$$a(u, v) = f(v) \quad \forall v \in V.$$

Chapter 2

Two-scale convergence method

This chapter is devoted to introduce the concept of two-scale convergence its main and properties. We also point out how some results concerning two-scale convergence are related to classical results for weak and strong convergence. Furthermore, details can be found in [2],[3] and [5].

2.1 Two-scale convergence

Definition 2.1.1 (Two-scale convergence [5]) *Let $(u^\varepsilon)_{\varepsilon>0}$ be a sequence of functions from $L^2(\Omega)$. We say that u^ε two-scale converges to a limit $u_0 \in L^2(\Omega \times Y)$, and write*

$$u^\varepsilon \xrightarrow{2} u_0 \quad \text{as } \varepsilon \to 0$$

if for any function $\psi \in \mathcal{D}(\Omega, C^\infty_{\mathrm{per}}(Y))$ one has

$$\lim_{\varepsilon\to 0} \int_\Omega u^\varepsilon(x)\psi\left(x, \frac{x}{\varepsilon}\right) dx = \frac{1}{|Y|} \int_{\Omega\times Y} u_0(x,y)\psi(x,y)\, dy\, dx.$$

Remark 2.1.1
- *two-scale convergence holds also for any ψ of the form $\psi(x,y) = \psi_1(x)\psi_2(y)$, where $\psi_1 \in C_0^\infty(\Omega)$ and $\psi_2 \in C^\infty_{\mathrm{per}}(Y)$.*

- *If a sequence* (u^ε) *two-scale converges, then it is bounded in* $L^2(\Omega)$.

Definition 2.1.2 (Admissible functions [5]) *We say that a function* $\psi \in \mathcal{D}(\Omega, C^\infty_{\text{per}}(Y))$ *is admissible if*

$$\lim_{\varepsilon\to 0} \int_\Omega \left[\psi\left(x, \frac{x}{\varepsilon}\right)\right]^2 dx = \frac{1}{|Y|}\int_{\Omega\times Y} [\psi(x,y)]^2 \, dy \, dx.$$

The functions in $L^p(\Omega, C^\infty_{\text{per}}(Y))$ *as well as* $L^p_{\text{per}}(\Omega, C(Y))$ *are admissible.*

Theorem 2.1.1 (Compactness theorem [5]) *Let* $(u^\varepsilon)_{\varepsilon>0}$ *be a bounded sequence in* $L^2(\Omega)$. *There exists a subsequence, still denoted by* (u^ε), *and a function* $u_0 \in L^2(\Omega \times Y)$ *such that* (u^ε) *two-scale converges to* u_0.

2.2 Properties of two-scale convergence

The main property of two-scale convergence is the convergence of norms.

Proposition 2.2.1 *Let* $(u^\varepsilon)_{\varepsilon>0}$ *be a sequence in* $L^2(\Omega)$ *that two-scale converges to a limit* $u_0 \in L^2(\Omega \times Y)$. *Then*

$$u^\varepsilon \rightharpoonup \bar{u}_0 = \frac{1}{|Y|}\int_Y u_0(x,y)\, dy \quad \textit{weakly in } L^2(\Omega),$$

and we have

$$\liminf_{\varepsilon\to 0} \|u^\varepsilon\|_{L^2} \geq \|u_0\|_{L^2(\Omega\times Y)} \geq \|\bar{u}_0\|_{L^2(\Omega)}.$$

Proof. We take a test function $\psi(x,y) = \psi(x)$ that does not depend on the y variable in the definition of the two-scale convergence, we get

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_\Omega u^\varepsilon(x)\,\psi(x)\,dx &= \frac{1}{|Y|}\int_{\Omega\times Y} u_0(x,y)\,\psi(x)\,dy\,dx \\
&= \int_\Omega \left(\frac{1}{|Y|}\int_Y u_0(x,y)\,dy\right)\psi(x)\,dx \\
&= \int_\Omega \bar{u}_0\,\psi(x)\,dx
\end{aligned}$$

Then, $u^\varepsilon \rightharpoonup \bar{u}_0$ weakly in $L^2(\Omega)$. On the other hand, from the density, there exists a sequence $(\psi_n)_{n\in\mathbb{N}} \subset L^2(\Omega, C_{\text{per}}(Y))$ such that for $n \to \infty$,

$$\psi_n \to u_0 \quad \text{strongly in } L^2(\Omega \times Y).$$

Now, we consider

$$\int_\Omega \big[u^\varepsilon(x) - \psi_n(x,\frac{x}{\varepsilon})\big]^2\,dx = \int_\Omega \big[u^\varepsilon(x)\big]^2\,dx - 2\int_\Omega u^\varepsilon(x)\psi_n(x,\frac{x}{\varepsilon})\,dx + \int_\Omega \big[\psi_n(x,\frac{x}{\varepsilon})\big]^2\,dx \geq 0$$

which gives

$$\int_\Omega \big[u^\varepsilon(x)\big]^2 dx \geq 2\int_\Omega u^\varepsilon(x)\psi_n\left(x,\frac{x}{\varepsilon}\right)\,dx - \int_\Omega \left[\psi_n\left(x,\frac{x}{\varepsilon}\right)\right]^2 dx.$$

Passing to the lim inf as $\varepsilon \to 0$, we obtain

$$\liminf_{\varepsilon\to 0}\int_\Omega \big[u^\varepsilon(x)\big]^2 dx \geq \frac{2}{|Y|}\int_{\Omega\times Y} u_0(x,y)\psi_n(x,y)\,dy\,dx - \frac{1}{|Y|}\int_{\Omega\times Y}\big[\psi_n(x,y)\big]^2\,dy\,dx.$$

Passing to the limit as $n \to +\infty$, we obtain

$$\liminf_{\varepsilon\to 0}\int_\Omega \big[u^\varepsilon(x)\big]^2 dx \geq \frac{1}{|Y|}\int_{\Omega\times Y}\big[u_0(x,y)\big]^2\,dy\,dx.$$

On the other hand, we have

$$\|\bar{u}_0\|^2_{L^2(\Omega)} = \int_\Omega [\bar{u}_0(x)]^2 dx = \int_\Omega \left[\frac{1}{|Y|}\int_Y u_0(x,y)\,dy\right]^2 dx.$$

Using Cauchy-Schwarz inequality leads to

$$\begin{aligned}
\|\bar{u}_0\|^2_{L^2(\Omega)} &\leq \int_\Omega \left[\left(\frac{1}{|Y|}\int_Y (u_0(x,y))^2\,dy\right)^{\frac{1}{2}}\left(\int_Y dy\right)^{\frac{1}{2}}\right]^2 dx \\
&= \int_\Omega\int_Y (u_0(x,y))^2\;dy\,dx \\
&= \|u_0\|^2_{L^2(\Omega\times Y)} \\
&\leq \liminf_{\varepsilon\to 0}\|u_\varepsilon\|^2_{L^2(\Omega)}.
\end{aligned}$$

■

Remark 2.2.1 *The two-scale convergence implies the weak convergence.*

As for L^2 convergence, we also have the strong version of the preceding proposition.

Proposition 2.2.2 *Let $(u^\varepsilon)_{\varepsilon>0}$ be a sequence of functions in $L^2(\Omega)$ which two-scale converges to $u_0 \in L^2(\Omega\times Y)$. Suppose furthermore, that*

$$\lim_{\varepsilon\to 0}\|u^\varepsilon\|_{L^2(\Omega)} = \frac{1}{|Y|}\|u_0\|_{L^2(\Omega\times Y)}$$

Then, for any sequence $(v^\varepsilon)_{\varepsilon>0}$ that two-scale converges to a limit $v_0 \in L^2(\Omega \times Y)$ we have

$$u^\varepsilon v^\varepsilon \rightharpoonup \frac{1}{|Y|}\int_Y u_0(x,y)v_0(x,y)\,dy, \quad \text{in } \mathcal{D}'(\Omega)$$

Furthermore, if u_0 belongs to $L^2(\Omega, C_{\text{per}}(Y))$, we have

$$\lim_{\varepsilon\to 0}\left\| u^\varepsilon(x) - u_0\left(x, \frac{x}{\varepsilon}\right)\right\|_{L^2(\Omega)} = 0$$

Proof. We consider a sequence function $\psi_n \in L^2(\Omega, C_{\text{per}}(Y))$ that converges to $u_0 \in L^2(\Omega \times Y)$. We have

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_\Omega \left[u^\varepsilon(x) - \psi_n(x, \frac{x}{\varepsilon})\right]^2 dx &= \lim_{\varepsilon\to 0}\left[\int_\Omega \left[u^\varepsilon(x)\right]^2 dx - 2\int_\Omega u^\varepsilon(x)\psi_n(x,\frac{x}{\varepsilon})\,dx + \int_\Omega \left[\psi_n(x,\frac{x}{\varepsilon})\right]^2 dx\right] \\
&= \frac{1}{|Y|}\int_{\Omega\times Y}\left[u_0(x,y)\right]^2 dy\,dx - \frac{2}{|Y|}\int_{\Omega\times Y} u_0(x,y)\psi_n(x,y)\,dy\,dx \\
&\quad + \frac{1}{|Y|}\int_{\Omega\times Y}\left[\psi_n(x,y)\right]^2 dy\,dx \\
&= \frac{1}{|Y|}\int_{\Omega\times Y}\left[u_0(x,y) - \psi_n(x,y)\right]^2 dy\,dx
\end{aligned} \tag{2.1}$$

Thus

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0}\int_\Omega \left[u^\varepsilon(x) - \psi_n\left(x, \frac{x}{\varepsilon}\right)\right]^2 dx = 0$$

Now, for any $\phi \in \mathcal{D}(\Omega)$, one has

$$\int_\Omega u^\varepsilon(x)v^\varepsilon(x)\phi(x)\,dx = \int_\Omega \psi_n\left(x,\frac{x}{\varepsilon}\right)v^\varepsilon(x)\phi(x)\,dx + \int_\Omega \left[u^\varepsilon(x) - \psi_n\left(x,\frac{x}{\varepsilon}\right)\right]v^\varepsilon(x)\phi(x)\,dx$$

Using Holder inequality and the fact that $(v^\varepsilon)_{\varepsilon>0}$ is bounded, we obtain

$$\int_\Omega \left[u^\varepsilon(x) - \psi_n\left(x,\frac{x}{\varepsilon}\right)\right]v^\varepsilon(x)\phi(x)\,dx \le C\left(\int_\Omega \left[u^\varepsilon(x) - \psi_n\left(x,\frac{x}{\varepsilon}\right)\right]^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega (\phi(x))^2\,dx\right)^{\frac{1}{2}}$$

which gives at the limit

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0}\int_\Omega \left[u^\varepsilon(x) - \psi_n\left(x,\frac{x}{\varepsilon}\right)\right]v^\varepsilon(x)\phi(x)\,dx = 0 \tag{2.2}$$

we have also

$$\begin{aligned}
\lim_{n\to\infty}\lim_{\varepsilon\to 0}\int_\Omega \psi_n\left(x,\frac{x}{\varepsilon}\right)v^\varepsilon(x)\phi(x)\,dx &= \lim_{n\to\infty}\frac{1}{|Y|}\int_{\Omega\times Y}\psi_n(x,y)v_0(x,y)\phi(x)\,dy\,dx \\
&= \frac{1}{|Y|}\int_{\Omega\times Y}u_0(x,y)v_0(x,y)\phi(x)\,dy\,dx
\end{aligned} \tag{2.3}$$

Therefore, from (2.2) and (2.3) it yields

$$\lim_{\varepsilon\to 0}\int_\Omega u^\varepsilon(x)\,v^\varepsilon(x)\,\phi(x)\,dx = \frac{1}{|Y|}\int_{\Omega\times Y} u_0(x,y)\,v_0(x,y)\,\phi(x)\,dy\,dx$$
$$= \int_\Omega \left[\frac{1}{|Y|}\int_Y u_0(x,y)\,v_0(x,y)\,dy\right]\phi(x)\,dx$$

Thus,

$$u^\varepsilon v^\varepsilon \rightharpoonup \frac{1}{|Y|}\int_Y u_0(x,y)v_0(x,y)\,dy, \quad \text{in } \mathcal{D}'(\Omega)$$

Furthermore, if u_0 belongs to $L^2(\Omega, C_{\text{per}}(Y))$, we take $\psi_n = u_0$ in the beginning of the proof to obtain

$$\int_\Omega \left[u^\varepsilon(x) - \psi_n(x,\frac{x}{\varepsilon})\right]^2 dx = \left[\int_\Omega \left[u^\varepsilon(x)\right]^2 dx - 2\int_\Omega u^\varepsilon(x)\psi_n(x,\frac{x}{\varepsilon})\,dx + \int_\Omega \left[\psi_n(x,\frac{x}{\varepsilon})\right]^2 dx\right]$$

Passage to the limit as $\varepsilon \to 0$, we get:

First term:

$$\lim_{\varepsilon\to 0}\int_\Omega |u^\varepsilon(x)|^2 dx = \frac{1}{|Y|}\int_{\Omega\times Y} |u_0(x,y)|^2 dydx.$$

Second term:

$$\lim_{\varepsilon\to 0}\int_\Omega u^\varepsilon(x)\,\psi_n\Big(x,\frac{x}{\varepsilon}\Big)\,dx = \frac{1}{|Y|}\int_{\Omega\times Y} |u_0(x,y)|^2 dydx.$$

Third term:

$$\lim_{\varepsilon\to 0}\int_\Omega \left|\psi_n\Big(x,\frac{x}{\varepsilon}\Big)\right|^2 dx = \frac{1}{|Y|}\int_{\Omega\times Y} |u_0(x,y)|^2 dydx.$$

Combining the three limits, we obtain

$$\lim_{\varepsilon\to 0}\int_\Omega \left|u^\varepsilon(x) - u_0\Big(x,\frac{x}{\varepsilon}\Big)\right|^2 dx = \frac{1}{|Y|}\int_{\Omega\times Y}|u_0|^2 - 2\frac{1}{|Y|}\int_{\Omega\times Y}|u_0|^2 + \frac{1}{|Y|}\int_{\Omega\times Y}|u_0|^2 = 0.$$

Therefore

$$\lim_{\varepsilon\to 0}\left\|u^\varepsilon(x) - u_0\left(x,\frac{x}{\varepsilon}\right)\right\|_{L^2(\Omega)} = 0$$

∎

Theorem 2.2.1 *Let $(u^\varepsilon)_{\varepsilon>0}$ be a bounded sequence in $H^1(\Omega)$. Then, there exist $u_0 \in H^1(\Omega)$ and $u_1 \in L^2(\Omega, H^1_{\text{per}}(Y))$ such that, up to a subsequence, u^ε two-scale converges to u_0 and ∇u^ε two-scale converges to $\nabla_x u_0 + \nabla_y u_1$.*

Proof. Since u^ε (resp. ∇u^ε) is bounded in $L^2(\Omega)$ (resp. $[L^2(\Omega)]^n$), then there exists $u \in L^2(\Omega \times Y)$ (resp. $v \in [L^2(\Omega \times Y)]^n$) such that up to extracting a subsequence,

$$u^\varepsilon \xrightharpoonup{2} u$$

$$\nabla u^\varepsilon \xrightharpoonup{2} v$$

This means that for any test functions $\psi \in \mathcal{D}(\Omega, C^\infty_{\text{per}}(Y))$ and $\phi \in [\mathcal{D}(\Omega, C^\infty_{\text{per}}(Y))]^n$, we have

$$\lim_{\varepsilon\to 0} \int_\Omega u^\varepsilon(x)\psi\left(x, \frac{x}{\varepsilon}\right)\, dx = \frac{1}{|Y|}\int_{\Omega\times Y} u(x,y)\psi(x,y)\, dy\, dx \tag{2.4}$$

$$\lim_{\varepsilon\to 0} \int_\Omega \nabla u^\varepsilon(x)\phi\left(x, \frac{x}{\varepsilon}\right)\, dx = \frac{1}{|Y|}\int_{\Omega\times Y} v(x,y)\phi(x,y)\, dy\, dx \tag{2.5}$$

Using Green's formula to get

$$\begin{aligned}
\int_\Omega \nabla u^\varepsilon(x)\, \phi\left(x, \frac{x}{\varepsilon}\right)\, dx &= \sum_{i=1}^n \int_\Omega \frac{\partial u^\varepsilon}{\partial x_i}\, \phi_i\left(x, \frac{x}{\varepsilon}\right)\, dx \\
&= -\sum_{i=1}^n \int_\Omega u^\varepsilon\, \frac{\partial \phi_i}{\partial x_i}\left(x, \frac{x}{\varepsilon}\right)\, dx \\
&= -\sum_{i=1}^n \int_\Omega u^\varepsilon \left(\frac{\partial}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_i}\right)\phi_i\left(x, \frac{x}{\varepsilon}\right)\, dx \\
&= -\sum_{i=1}^n \int_\Omega u^\varepsilon \frac{\partial \phi_i}{\partial x_i}\left(x, \frac{x}{\varepsilon}\right)\, dx - \frac{1}{\varepsilon}\sum_{i=1}^n \int_\Omega u^\varepsilon \frac{\partial \phi_i}{\partial y_i}\left(x, \frac{x}{\varepsilon}\right)\, dx \\
&= -\int_\Omega u^\varepsilon\ \mathrm{div}_x\, \phi\left(x, \frac{x}{\varepsilon}\right)\, dx - \frac{1}{\varepsilon}\int_\Omega u^\varepsilon\ \mathrm{div}_y\, \phi\left(x, \frac{x}{\varepsilon}\right)\, dx
\end{aligned} \tag{2.6}$$

Then, multiplying by ε, one has

$$\int_\Omega u^\varepsilon\, \mathrm{div}_y\, \phi\left(x, \frac{x}{\varepsilon}\right) dx = -\varepsilon \int_\Omega u^\varepsilon\, \mathrm{div}_x\, \phi\left(x, \frac{x}{\varepsilon}\right) dx - \varepsilon \int_\Omega \nabla u^\varepsilon(x)\phi\left(x, \frac{x}{\varepsilon}\right) dx \tag{2.7}$$

Passing to the limit as $\varepsilon \to 0$ yields

$$\frac{1}{|Y|}\int_{\Omega\times Y} u(x,y)\, \mathrm{div}_y\, \phi(x,y)\, dy\, dx = 0 \tag{2.8}$$

Using Green's formula and taking $\phi(x,y) = \phi_1(x)\phi_2(y)$, we get

$$\begin{aligned}
\int_{\Omega\times Y} u(x,y)\, \mathrm{div}_y\, \phi(x,y)\, dy\, dx &= -\int_{\Omega\times Y} \nabla_y u(x,y)\phi(x,y)\, dy\, dx \\
&= -\int_{\Omega\times Y} \nabla_y u(x,y)\phi_1(x)\phi_2(y)\, dy\, dx \quad = 0
\end{aligned}$$

Rearranging this later with the help of Fubini's theorem yields

$$\int_\Omega \left(\int_Y \nabla_y u(x,y) \phi_2(y)\, dy \right) \phi_1(x)\, dx = 0, \quad \forall \phi_1 \in C_0^\infty(\Omega),$$

we deduce that

$$\int_Y \nabla_y u(x,y)\phi_2(y)\, dy = 0, \quad \forall \phi_2 \in C_{\text{per}}^\infty(Y),$$

Then,

$$\nabla_y u(x,y) = 0, \quad \text{a.e. on } \Omega \times Y$$

This implies that $u(x,y)$ does not depend on y, and therefore $u(x,y) = u_0(x)$.
Let now $\phi \in [\mathcal{D}(\Omega, C_{\text{per}}^\infty(Y))]^n$ such that $\operatorname{div}_y \phi = \displaystyle\sum_{i=1}^n \frac{\partial \phi_i}{\partial y_i} = 0$. From (2.6) we get

$$\int_\Omega \nabla u^\varepsilon(x)\, \phi\left(x, \frac{x}{\varepsilon}\right) dx = -\int_\Omega u^\varepsilon \operatorname{div}_x \phi\left(x, \frac{x}{\varepsilon}\right) dx \tag{2.9}$$

which, passing to the limit, leads to

$$\frac{1}{|Y|}\int_{\Omega\times Y} v(x,y)\, \phi(x,y)\, dydx = -\frac{1}{|Y|}\int_{\Omega\times Y} u_0(x) \operatorname{div}_x \phi(x,y)\, dydx \tag{2.10}$$

$$= \frac{1}{|Y|}\int_{\Omega\times Y} \nabla_x u_0(x)\, \phi(x,y)\, dydx \tag{2.11}$$

Thus, for any $\phi \in [\mathcal{D}(\Omega, C_{\text{per}}^\infty(Y))]^n$ such that $\operatorname{div}_y \phi = 0$, one has

$$\int_{\Omega\times Y} \big[v(x,y) - \nabla_x u_0(x)\big] \phi(x,y)\, dy\, dx = 0$$

We now make use of a classical result. It states that if $\langle v, \phi\rangle_{L^2} = 0$, for any ϕ such that $\operatorname{div} \phi = 0$, then v is a gradient. Let

$$F = \left\{ g \in L^2(\Omega \times Y),\ \exists p \in L^2(\Omega, H_{\text{per}}^1(Y)),\ g = \nabla_y p \right\}$$

be the set of gradient functions and

$$E = \left\{ \psi \in L^2(\Omega\times Y),\ \operatorname{div}_y \psi = 0 \right\}$$

be the set of functions with zero divergence. Then,

$$L^2(\Omega\times Y) = F \oplus E \quad \textit{and} \quad E = F^\perp$$

and since

$$\int_{\Omega\times Y} \big[v(x,y) - \nabla_x u_0(x)\big]\phi(x,y)\,dy\,dx = 0, \quad \forall\phi \in E$$

Therefore $v(x,y) - \nabla_x u_0(x) \in F$, which implies that there exists a unique function $u_1 \in L^2(\Omega, H^1_{\mathrm{per}}(Y))$ such that

$$v(x,y) - \nabla_x u_0(x) = \nabla_y u_1(x,y).$$

■

2.3 Homogenization of a second order elliptic equation

We consider the model problem

$$\begin{cases} -\operatorname{div}\big(A^\varepsilon(x)\nabla u^\varepsilon(x)\big) = f(x) & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{2.12}$$

where $A(y)$ is a Y-periodic matrix satisfying the coercivity hypothesis, whose variational formulation is

$$\begin{cases} \text{Find } u^\varepsilon \in H^1_0(\Omega) \text{ such that} \\ \displaystyle\int_\Omega A^\varepsilon(x)\nabla u^\varepsilon(x)\cdot\nabla v(x)\,dx = \int_\Omega f(x)v(x)\,dx, \quad \forall v \in H^1_0(\Omega) \end{cases} \tag{2.13}$$

We recall that equation (2.12) admits a unique solution u^ε in $H^1_0(\Omega)$ which satisfies the a priori estimate

$$\|u^\varepsilon\|_{H^1_0(\Omega)} \leq C\|f\|_{L^2(\Omega)}$$

where C is a positive constant which does not depend on ε. Therefore, Theorem 2.4.1 applies and, up to the extraction of a subsequence that we still denote by (u^ε) for simplicity, one can assume that

$$\begin{aligned} u^\varepsilon &\rightharpoonup u_0 \quad \text{weakly in } H^1(\Omega), \\ u^\varepsilon &\to u_0 \quad \text{strongly in } L^2(\Omega), \\ u^\varepsilon &\overset{2}{\rightharpoonup} u_0 \quad \text{two-scale}, \\ \nabla u^\varepsilon &\overset{2}{\rightharpoonup} \nabla_x u_0(x) + \nabla_y u_1(x,y) \quad \text{two-scale} \end{aligned}$$

where $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega, H^1_{\text{per}}(Y))$. The idea consists in taking a suitable test function in (2.13) and use the convergences above to pass to the limit. Namely, we consider $v_0 \in \mathcal{D}(\Omega)$ and $v_1 \in \mathcal{D}(\Omega, C^\infty_{\text{per}}(Y))$, clearly $v_0 + \varepsilon v_1 \in H_0^1(\Omega)$. We use the test function

$$v(x) = v_0(x) + \varepsilon v_1\left(x, \frac{x}{\varepsilon}\right)$$

in (2.13). Since

$$\nabla v(x) = \nabla_x v_0(x) + \varepsilon \nabla_x v_1\left(x, \frac{x}{\varepsilon}\right) + \nabla_y v_1\left(x, \frac{x}{\varepsilon}\right)$$

We obtain,

$$\int_\Omega A^\varepsilon(x)\nabla u^\varepsilon(x)\left[\nabla_x v_0(x) + \varepsilon\nabla_x v_1\left(x, \frac{x}{\varepsilon}\right) + \nabla_y v_1\left(x, \frac{x}{\varepsilon}\right)\right] dx = \int_\Omega f(x)\left[v_0(x) + \varepsilon v_1\left(x, \frac{x}{\varepsilon}\right)\right] dx,$$

Which can be rewritten as follows:

$$\begin{aligned}\int_\Omega A^\varepsilon(x)\nabla u^\varepsilon(x)\left[\nabla_x v_0(x) + \nabla_y v_1\left(x, \frac{x}{\varepsilon}\right)\right] dx + \varepsilon\int_\Omega A^\varepsilon(x)\nabla u^\varepsilon(x)\nabla_x v_1\left(x, \frac{x}{\varepsilon}\right) dx \\ = \int_\Omega f(x)\left[v_0(x) + \varepsilon v_1\left(x, \frac{x}{\varepsilon}\right)\right] dx\end{aligned} \tag{2.14}$$

We now pass to the limit in each term. For the first term, we write

$$\int_\Omega A^\varepsilon(x)\nabla u^\varepsilon(x)\left[\nabla_x v_0(x) + \nabla_y v_1\left(x, \frac{x}{\varepsilon}\right)\right] dx = \int_\Omega \nabla u^\varepsilon(x)\left({}^tA^\varepsilon\right)(x)\left[\nabla_x v_0(x) + \nabla_y v_1\left(x, \frac{x}{\varepsilon}\right)\right] dx$$

since ${}^tA^\varepsilon \in L^\infty(Y)$ and $v_0(x) + \varepsilon v_1(x,y) \in L^2_{\text{per}}(Y, C(\Omega))$ so that $\left({}^tA^\varepsilon\right)(x)\left[\nabla_x v_0(x) + \nabla_y v_1\left(x, \frac{x}{\varepsilon}\right)\right]$ can be used as test function in the two-scale convergence of ∇u^ε. Consequently,

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_\Omega \nabla u^\varepsilon(x)\left({}^tA^\varepsilon\right)(x)\left[\nabla_x v_0(x) + \nabla_y v_1\left(x, \frac{x}{\varepsilon}\right)\right] dx \\ &= \frac{1}{|Y|}\int_{\Omega\times Y}\left[\nabla_x u_0 + \nabla_y u_1(x,y)\right]\left({}^tA\right)(y)\left[\nabla_x v_0(x) + \nabla_y v_1(x,y)\right] dydx\end{aligned} \tag{2.15}$$

For the second term, the Holder inequality and the fact that $A^\varepsilon\nabla u^\varepsilon$ is bounded in $L^2(\Omega)$ one derives that

$$\lim_{\varepsilon\to 0} \varepsilon \int_\Omega A^\varepsilon(x)\nabla u^\varepsilon(x)\nabla_x v_1\left(x,\frac{x}{\varepsilon}\right)dx = 0 \tag{2.16}$$

For the last term, notice that, by the definition of v_0 and v_1 one has that

$$v_0 + \varepsilon v_1 \rightharpoonup v_0 \quad \text{weakly in } H_0^1(\Omega)$$

Then

$$\lim_{\varepsilon\to 0} \int_\Omega f(x)\left[v_0(x) + \varepsilon v_1\left(x,\frac{x}{\varepsilon}\right)\right]dx = \int_\Omega f(x)v_0(x)dx \tag{2.17}$$

Hence, passing to the limit in (2.14) as $\varepsilon \to 0$, we finally get

$$\int_{\Omega\times Y} \left[\nabla_x u_0 + \nabla_y u_1(x,y)\right]\left({}^tA\right)(y)\left[\nabla_x v_0(x) + \nabla_y v_1(x,y)\right]dydx = \int_\Omega f(x)v_0(x)dx$$

which can be rewritten as

$$\int_{\Omega\times Y} A(y)\left[\nabla_x u_0 + \nabla_y u_1(x,y)\right]\left[\nabla_x v_0(x) + \nabla_y v_1(x,y)\right]dy\,dx = \int_\Omega f(x)v_0(x)\,dx \tag{2.18}$$

Let us show that this equation is a variational formulation of the homogenized problem in the Hilbert space

$$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega, H_{per}^1(Y)).$$

and that the hypotheses of the Lax-Milgram theorem are fulfilled. Endowing this space with the norm

$$\|V\|_{\mathcal{H}} = \|v_0\|^2_{H_0^1(\Omega)} + \|v_1\|^2_{L^2(\Omega,H^1_{per}(Y))}, \quad \forall V = (v_0,v_1) \in H$$

By density, (2.17) holds true for any (v_0, v_1) in the space $\mathcal{H}$. We check the conditions of the Lax- Milgram theorem for the variational formulation

$$\begin{cases} \text{Find } U = (u_0,u_1) \in \mathcal{H} \text{ such that} \\ a(U,V) = L(V), \quad \forall V = (v_0,v_1) \in \mathcal{H} \end{cases} \tag{2.19}$$

where

$$a(U,V) = \int_{\Omega\times Y} A(y)\left[\nabla_x u_0 + \nabla_y u_1(x,y)\right]\left[\nabla_x v_0(x) + \nabla_y v_1(x,y)\right]dy\,dx$$

and

$$L(V) = \int_\Omega f(x) v_0(x)\, dx$$

The linear form satisfies

$$\begin{aligned} |L(V)| &\le \|f\|_{L^2} \, \|v_0\|_{L^2} \\ &\le C \|f\|_{L^2} \, \|V\|_H \end{aligned}$$

Next, the bilinear form satisfies

$$\begin{aligned} |a(U,V)| &\le \|A\|_{L^\infty} \Big[\|\nabla_x u_0\|_{L^2} + \|\nabla_y u_1\|_{L^2(\Omega\times Y)} \Big] \Big[\|\nabla_x v_0\|_{L^2} + \|\nabla_y v_1\|_{L^2(\Omega\times Y)} \Big] \\ &\le \|A\|_{L^\infty} \, \|U\|_{\mathcal{H}} \, \|V\|_{\mathcal{H}} \end{aligned}$$

and is therefore continuous on $\mathcal{H}$. Observe now that one has

$$\begin{aligned} a(V,V) &= \int_{\Omega\times Y} A(y) \big[\nabla_x v_0 + \nabla_y v_1(x,y)\big] \big[\nabla_x v_0(x) + \nabla_y v_1(x,y)\big] \, dy \, dx \\ &\ge \alpha \, \|\nabla_x v_0 + \nabla_y v_1\|^2_{L^2(\Omega\times Y)} \end{aligned}$$

On the other hand,

$$\|\nabla_x v_0 + \nabla_y v_1\|^2_{L^2(\Omega\times Y)} = \|v_0\|^2_{H^1_0(\Omega)} + \|\nabla_y u_1\|^2_{L^2(\Omega\times Y)} + 2 \int_{\Omega\times Y} \nabla_x v_0(x) \nabla_y v_1(x,y) \, dy \, dx$$

Using the Green formula and the periodicity of v_1, we have

$$\begin{aligned} \int_{\Omega\times Y} \nabla_x v_0(x) \nabla_y v_1(x,y) \, dy dx &= \int_\Omega \nabla_x v_0(x) \bigg(\int_Y \nabla_y v_1(x,y) \, dy \bigg) dx \\ &= \int_\Omega \nabla_x v_0(x) \bigg(\int_{\partial Y} v_1(x,y) n(y) \, ds_y \bigg) dx \\ &= 0 \end{aligned}$$

Then

$$\begin{aligned} a(V,V) &\ge \alpha \Big(\|v_0\|^2_{H^1_0(\Omega)} + \|\nabla_y u_1\|^2_{L^2(\Omega\times Y)} \Big) \\ &= \alpha \|V\|^2_{\mathcal{H}} \end{aligned}$$

Hence, we can apply the Lax-Milgram theorem to obtain the existence and uniqueness of $(u_0, u_1) \in \mathcal{H}$, the solution of (2.18), for any $(v_0, v_1) \in \mathcal{H}$. Choosing now first $v_0 = 0$ and after $v_1 = 0$, we see that (2.18) is equivalent to the problem

$$\begin{cases} -\mathrm{div}_y\Big(A(y)\nabla_y u_1(x,y)\Big) = \mathrm{div}_y\Big(A(y)\nabla_x u_0\Big) & \text{in } \Omega \times Y, \\ -\mathrm{div}_x\Big[\int_Y A(y)\big(\nabla_x u_0 + \nabla_y u_1(x,y)\big)dy\Big] = f & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega, \\ u_1 \text{ Y-periodic.} \end{cases} \tag{2.20}$$

It is easily seen that (2.18) is equivalent to the usual homogenized and cell problem through the relation

$$u_1(x,y) = \chi(y) \cdot \nabla_x u_0(x) + \bar{u}_1(x).$$

Consequently, the whole sequence in (2.18) converges to u_0. Let $\{e_j\}$ be the unit normal basis of $\mathbb{R}^N$. Treating x as a parameter in the second equation in (2.20) and writing

$$\nabla u(x) = \sum_{j=1}^{N} \frac{\partial u}{\partial x_j}(x)\, e_j,$$

we may introduce the following cell problem: define $\chi^j(x,y)$ as the solution of

$$\begin{cases} -\operatorname{div}_y\big(A(x,y)\,[\nabla_y \chi^j(x,y) - e_j]\big) = 0 & \text{in } Y, \\ \chi^j \text{ is } Y\text{-periodic in } y. \end{cases} \tag{2.21}$$

Then one can immediately see that

$$u_1(x,y) = -\sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(x)\, \chi^i(x,y),$$

where u is the only unknown to be determined.

Substituting u_1 in the first equation in (2.20), we get the single equation for u as

$$\begin{cases} -\operatorname{div}\big(A^*(x)\nabla u(x)\big) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.22}$$

where $A^* = (q_{ij}(x))$ with

$$q_{ij}(x) = \int_Y A(x,y)\,[\nabla_y \chi^j - e_j] \cdot [\nabla_y \chi^i - e_i]\, dy.$$

Chapter 3

Application to Stokes Equation in Porous Media

This chapter deals with the two-scale limit of the Stokes equations in a porous medium with a Dirichlet boundary condition in order to derive the homogenized system describing the effective macroscopic behavior of fluid flow inside such structures. For more details, see the following references [3], [4] and[5].

3.1 Setting of the problem

In this section, the two-scale convergence method is applied to the homogenization of the steady Stokes equations in a porous medium. The porous medium is contained in a domain Ω, and its fluid part is denoted by Ω^ε. From a mathematical point of view, Ω^ε is a periodically perforated domain, i.e.. it has many small holes of size ε which represent solid obstacles that fluid cannot penetrate. The motion of the fluid in Ω^ε is governed by the steady Stokes equations, complemented with a Dirichlet boundary condition. We denote by u^ε and p^ε the velocity and pressure of the fluid, and f the density of forces acting on the fluid. (u^ε and f are vector-valued functions, while p^ε is scalar). The fluid viscosity is a positive constant μ that we scale by a factor ε^2 (where ε is the period). The system of equations is:

$$\begin{cases} \nabla p^\varepsilon - \varepsilon^2 \mu \Delta u^\varepsilon = f & \text{in } \Omega^\varepsilon \\ \text{div } u^\varepsilon = 0 & \text{in } \Omega^\varepsilon \\ u^\varepsilon = 0 & \text{on } \partial\Omega^\varepsilon \end{cases} \tag{3.1}$$

From now on, we assume without loss of generality that $\mu = 1$.

Existence and uniqueness of a solution $(u^\varepsilon,p^\varepsilon)$ in $H_0^1(\Omega^\varepsilon) \times L^2(\Omega^\varepsilon)$ is a well-known result as long as $f \in L^2(\Omega^\varepsilon)$(see [8]).

We can therefore proceed to the homogenization, that is seeking the limits of u^ε and p^ε as $\varepsilon \to 0$. We consider a periodic structure in Ω. To this end , we assume that the unit cell Y can be decomposed as

$$Y = Y_s \cup Y_f, \qquad Y_s \cap Y_f = \varnothing,$$

where Y_s denotes the solid part of the unit cell and Y_f denotes the fluid part (see Fig. 3.1). Scaling down this structure by a factor of ε and repeating it throughout the domain leads

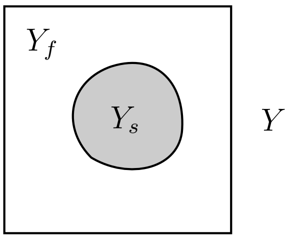

Figure 3.1: The unit cell of the porous medium composed of a solid part Y_s and a fluid part Y_f.

to a periodically structured medium, as illustrated in Fig. 3.2, consisting of alternating solid and fluid regions. More precisely, from the macroscopic point of view, the fluid domain is defined by

$$\Omega^\varepsilon = \Omega \setminus \bigcup_{i=1}^{N(\varepsilon)} Y_{s,i}^\varepsilon = \Omega \cap \bigcup_{i=1}^{N(\varepsilon)} Y_{f,i}^\varepsilon, \tag{3.2}$$

where each $Y_{f,i}^\varepsilon$ (respectively $Y_{s,i}^\varepsilon$) is a translated copy of εY_f (respectively εY_s). Furthermore, the total number of periodic cells satisfies $N(\varepsilon) = |\Omega|\, \varepsilon^{-n}\big(1 + o(1)\big)$.

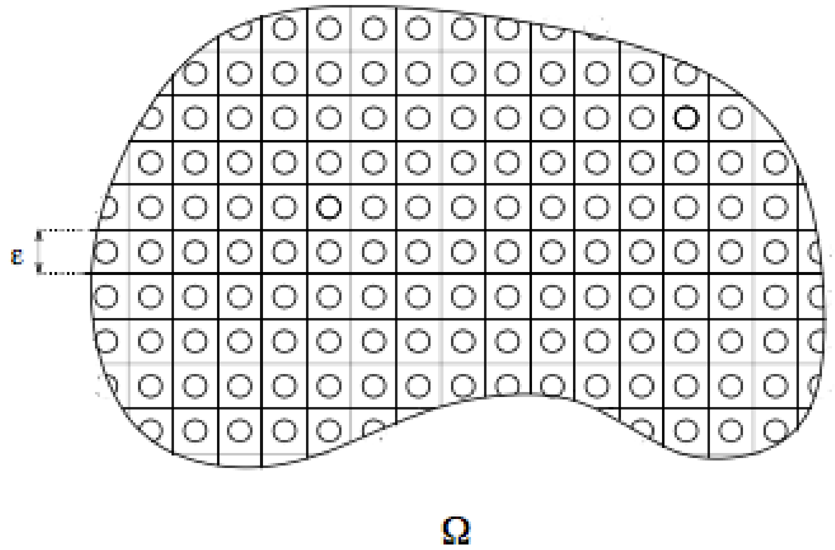

Figure 3.2: Periodic porous domain Ω^ε.

3.2 Extensions and a priori estimates

The next step is to obtain a priori estimates of the solution (u^ε,p^ε) which are independent of ε. These estimates will be used to extract weakly convergent subsequences. But to do so, the sequence (u^ε,p^ε) needs to be defined in a fixed Sobolev space, independent of ε. It is therefore necessary to extend the solution u^ε and the pressure p^ε suitably to the whole domain Ω. Since $u^\varepsilon = 0$ on $\partial\Omega^\varepsilon$, we may extend u^ε by zero inside the solid part, by setting

$$\tilde{u}^\varepsilon(x) = \begin{cases} u^\varepsilon(x) & \text{if } x \in \Omega^\varepsilon, \\ 0 & \text{if } x \in \Omega \setminus \Omega^\varepsilon. \end{cases} \tag{3.3}$$

For p^ε, the situation is slightly more involved. Classically, the pressure is defined up to an additive constant, and we must extend p^ε in such a way that when p^ε is locally constant, it remains constant (with the same value) inside the solid part. We therefore define, for $i \in \{1, \dots, N(\varepsilon)\}$,

$$\tilde{p}^\varepsilon(x) = \begin{cases} p^\varepsilon(x) & \text{if } x \in Y^\varepsilon_{f,i}, \\ \frac{1}{|Y^\varepsilon_{f,i}|} \int_{Y^\varepsilon_{f,i}} p^\varepsilon \, dx & \text{if } x \in Y^\varepsilon_{s,i}. \end{cases} \tag{3.4}$$

Proposition 3.2.1 (A Priori Estimates [3]) *The extensions $\tilde{u}^\varepsilon$ and $\tilde{p}^\varepsilon$ of the solution*

$(u^\varepsilon, p^\varepsilon)$, defined in (3.3), (3.4), satisfy the a priori estimates

$$\|\tilde{u}^\varepsilon\|_{L^2(\Omega)^n} + \varepsilon\|\nabla\tilde{u}^\varepsilon\|_{L^2(\Omega)^{n\times n}} \leq C \tag{3.5}$$

$$\|\tilde{p}^\varepsilon\|_{L^2(\Omega)/\mathbb{R}} \leq C, \tag{3.6}$$

where the constant C does not depend on ε.

Proof. We first notice that on any $Y^\varepsilon_{f,i}$ one has due to Poincaré inequality

$$\|u^\varepsilon\|_{L^2(Y^\varepsilon_{f,i})} \leq \varepsilon\|\nabla u^\varepsilon\|_{L^2(Y^\varepsilon_{f,i})}.$$

From which we deduce

$$\|\tilde{u}^\varepsilon\|_{L^2(Y^\varepsilon_i)} \leq \varepsilon\|\nabla\tilde{u}^\varepsilon\|_{L^2(Y^\varepsilon_i)},$$

which by summing over all cells gives

$$\|\tilde{u}^\varepsilon\|_{L^2(\Omega)} \leq \varepsilon\|\nabla\tilde{u}^\varepsilon\|_{L^2(\Omega)}. \tag{3.7}$$

Now, multiplying the original Stokes equation by u^ε and integrating over Ω^ε, we obtain

$$-\varepsilon^2\int_{\Omega^\varepsilon}\Delta u^\varepsilon\cdot u^\varepsilon\,dx + \int_{\Omega^\varepsilon}\nabla p^\varepsilon\cdot u^\varepsilon\,dx = \int_{\Omega^\varepsilon} f\cdot u^\varepsilon\,dx.$$

Integrating by parts, we obtain

$$\varepsilon^2\int_{\Omega^\varepsilon}|\nabla u^\varepsilon|^2\,dx - \int_{\Omega^\varepsilon}p^\varepsilon\operatorname{div}u^\varepsilon\,dx + \int_{\partial\Omega^\varepsilon}p^\varepsilon\,u^\varepsilon\cdot n\,d\sigma = \int_{\Omega^\varepsilon} f\cdot u^\varepsilon\,dx.$$

Since $\operatorname{div}u^\varepsilon = 0$ and $u^\varepsilon|_{\partial\Omega^\varepsilon} = 0$, and using the extension, we get

$$\begin{aligned}\|\varepsilon\nabla\tilde{u}^\varepsilon\|^2_{L^2(\Omega)} &= \|\varepsilon\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)}\\ &\leq \|f\|_{L^2(\Omega)}\,\|u^\varepsilon\|_{L^2(\Omega^\varepsilon)}\\ &= \|f\|_{L^2(\Omega)}\,\|\tilde{u}^\varepsilon\|_{L^2(\Omega)}\end{aligned}$$

Using (3.7), we obtian

$$\|\varepsilon\nabla\tilde{u}^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} \leq \varepsilon\|f\|_{L^2(\Omega)}\,\|\nabla\tilde{u}^\varepsilon\|_{L^2(\Omega)}$$

Then

$$\varepsilon\|\nabla\tilde{u}^\varepsilon\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)} \tag{3.8}$$

Due to (3.7) and (3.8), it yields :

$$\begin{aligned}\|\tilde{u}^\varepsilon\|_{L^2(\Omega)} + \varepsilon\|\nabla\tilde{u}^\varepsilon\|_{L^2(\Omega)} &\le C\|\nabla\tilde{u}^\varepsilon\|_{L^2(\Omega)} + C\|f\|_{L^2(\Omega)}\\ &\le C_1\varepsilon\|f\|_{L^2(\Omega)} + C_2\|f\|_{L^2(\Omega)}\\ &= (C_1+C_2)\|f\|_{L^2(\Omega)}\end{aligned}$$

Therefore

$$\|\tilde{u}^\varepsilon\|_{L^2(\Omega)} + \varepsilon\|\nabla\tilde{u}^\varepsilon\|_{L^2(\Omega)} \le C$$

The estimate for the pressure is complicated. We refer the interested reader to the references [3], [9] and [10].

■

3.3 Two-scale convergence limits

Throughout this section we assume for convenience that |Y|=1.

Lemma 3.3.1 *There exist functions $u_0(x,y) \in L^2\big(\Omega; H^1_{\mathrm{per}}(Y)^n\big)$ and $p(x) \in L^2(\Omega)/\mathbb{R}$ such that, up to a subsequence, the sequences $\tilde{u}^\varepsilon$, $\varepsilon\nabla\tilde{u}^\varepsilon$, and $\tilde{p}^\varepsilon$ two-scale converge to u_0, $\nabla_y u_0$, and $p(x)$, respectively. Furthermore, u_0 satisfies*

$$\begin{cases}\operatorname{div}_y u_0(x,y) = 0 \quad \textit{in } \Omega\times Y \quad \textit{and} \quad \operatorname{div}_x\Big(\displaystyle\int_Y u_0(x,y)\,dy\Big) = 0 \quad \textit{in } \Omega,\\[2ex] u_0(x,y) = 0 \quad \textit{in } \Omega\times Y_s \quad \textit{and} \quad \Big(\displaystyle\int_Y u_0(x,y)\,dy\Big)\cdot n = 0 \quad \textit{on } \partial\Omega.\end{cases} \tag{3.9}$$

Proof. Thanks to the bounds established in Proposition 3.2.1,

$$\|\tilde{u}^\varepsilon\|_{L^2(\Omega)} \le C, \qquad \varepsilon\|\nabla\tilde{u}^\varepsilon\|_{L^2(\Omega)} \le C, \qquad \|\tilde{p}^\varepsilon\|_{L^2(\Omega)} \le C,$$

Theorem 2.3.1 implies the existence of three functions $u_0(x,y), \xi_0(x,y)$ and $p_0(x,y)$ such that, up to a subsequence, The sequences $\tilde{u}^\epsilon$, $\varepsilon\nabla\tilde{u}^\varepsilon$ and $\tilde{p}^\epsilon$ two-scale converge to u_0,ξ_0 and p_0 respectively. Equivalently, for any test functions ψ, φ and $\phi \in \mathcal{D}\big(\Omega; C^\infty_{per}(Y)\big)^n$, one has

$$\lim_{\varepsilon\to 0}\int_\Omega \tilde{u}^\varepsilon(x)\,\psi\Big(x,\frac{x}{\varepsilon}\Big)\,dx = \int_{\Omega\times Y} u_0(x,y)\,\psi(x,y)\,dx\,dy, \tag{3.10}$$

$$\lim_{\varepsilon\to 0}\int_\Omega \varepsilon\nabla\tilde{u}^\varepsilon(x)\varphi\Big(x,\frac{x}{\varepsilon}\Big)\,dx = \int_{\Omega\times Y} \xi_0(x,y)\varphi(x,y)\,dx\,dy, \tag{3.11}$$

$$\lim_{\varepsilon\to 0}\int_\Omega \tilde{p}^\varepsilon(x)\,\phi\Big(x,\frac{x}{\varepsilon}\Big)\,dx = \int_{\Omega\times Y} p_0(x,y)\,\phi(x,y)\,dx\,dy. \tag{3.12}$$

We now identify the limit of the gradient. Integrating by parts, and using the chain rule with $y = x$, we obtain

$$\begin{aligned}\int_\Omega \varepsilon\nabla\tilde{u}^\varepsilon\varphi\left(x,\frac{x}{\varepsilon}\right)dx &= -\varepsilon\int_\Omega u^\varepsilon \mathrm{div}\left(\varphi\left(x,\frac{x}{\varepsilon}\right)\right)dx\\ &= -\varepsilon\int_\Omega u^\varepsilon\left[\mathrm{div}_x + \frac{1}{\varepsilon}\mathrm{div}_y\right]\left(\varphi\left(x,\frac{x}{\varepsilon}\right)\right)dx\\ &= -\varepsilon\int_\Omega u^\varepsilon \mathrm{div}_x\varphi\left(x,\frac{x}{\varepsilon}\right)dx - \int_\Omega u^\varepsilon \mathrm{div}_y\varphi\left(x,\frac{x}{\varepsilon}\right)dx\end{aligned}$$

Passing to the two-scale limit yields

$$\begin{aligned}\int_{\Omega\times Y}\xi_0(x,y)\varphi(x,y)dxdy &= -\lim_{\varepsilon\to 0}\int_\Omega u^\varepsilon \mathrm{div}_y\varphi\left(x,\frac{x}{\varepsilon}\right)dx\\ &= -\int_{\Omega\times Y} u_0(x,y)\mathrm{div}_y\varphi\left(x,\frac{x}{\varepsilon}\right)dx\end{aligned}$$

Integrating by parts with respect to the variable y, and using the periodicity of φ, gives

$$\int_{\Omega\times Y}\nabla_y u_0(x,y)\varphi(x,y)\,dx\,dy = \int_{\Omega\times Y}\xi_0(x,y)\varphi(x,y)\,dx\,dy.$$

Since this identity holds for every test function φ, it follows that

$$\xi_0 = \nabla_y u_0.$$

Next, we prove that the pressure is independent of the microscopic variable y. Multiplying the first equation of system (3.1) by $\varepsilon\psi\big(x,\frac{x}{\varepsilon}\big)$ integrating over Ω, and passing to the limit, we obtain

$$\int_{\Omega\times Y} p_0(x,y)\ \mathrm{div}_y\,\psi(x,y)\,dx\,dy = 0.$$

Since this relation is valid for all periodic test functions ψ, we deduce that

$$\nabla_y p_0 = 0.$$

Hence p_0 does not depend on y, and there exists $p(x) \in L^2(\Omega)/\mathbb{R}$ such that

$$p_0(x,y) = p(x).$$

Finally, we derive the incompressibility conditions. Starting from

$$\operatorname{div} u^\varepsilon = 0,$$

Multiplying by a test function $\varphi \in \mathcal{D}(\Omega, C^\infty_{\text{per}}(Y))$ and integrating by parts over Ω, we get

$$\begin{aligned}
0 = \int_\Omega \operatorname{div}\, \tilde{u}^\varepsilon \varphi\left(x, \frac{x}{\varepsilon}\right) dx &= -\int_\Omega u^\varepsilon \nabla\varphi\left(x, \frac{x}{\varepsilon}\right) dx \\
&= -\int_\Omega u^\varepsilon \left(\nabla_x + \frac{1}{\varepsilon}\nabla_y\right) \varphi\left(x, \frac{x}{\varepsilon}\right) dx \\
&= -\int_\Omega u^\varepsilon \nabla_x \varphi\left(x, \frac{x}{\varepsilon}\right) dx - \frac{1}{\varepsilon}\int_\Omega u^\varepsilon \nabla_y \varphi\left(x, \frac{x}{\varepsilon}\right) dx
\end{aligned}$$

Multiplying by ε and passing to the limit $\varepsilon \to 0$ leads to

$$\int_{\Omega\times Y} u_0(x,y) \nabla_y \varphi(x,y)\, dxdy = 0$$

which means

$$\operatorname{div}_y u_0 = 0 \quad \text{in } \Omega \times Y$$

Now, taking a test function $\varphi \in H^1(\Omega)$ independent of y and does not vanish on the boundary $\partial\Omega$. We have after integrating by parts and using $\tilde{u}^\varepsilon = 0$ on $\partial\Omega$

$$0 = -\int_\Omega \tilde{u}^\varepsilon \nabla_x \phi\left(x, \frac{x}{\varepsilon}\right) dx$$

Passing to the limit leads to

$$0 = \int_{\Omega\times Y} u_0(x,y) \nabla_x \phi(x)\, dydx$$

Integrating by parts, we get

$$\begin{aligned}
0 &= -\int_{\Omega\times Y} \operatorname{div}_x u_0(x,y)\phi(x)\, dydx - \int_{\partial\Omega\times Y} u_0(x,y)\cdot n\, \phi(x) ds \\
&= \int_\Omega \operatorname{div}_x \left(\int_Y u_0(x,y) dy\right) \phi(x)\, dx + \int_{\partial\Omega} \left[\int_Y u_0(x,y)dy\right] \cdot n\, \phi(x) ds
\end{aligned}$$

which gives

$$\operatorname{div}_x \left(\int_Y u_0(x,y)\,dy \right) = 0 \qquad \text{in } \Omega.$$

$$\left(\int_Y u_0(x,y)\,dy \right) \cdot n = 0 \quad \text{on } \partial\Omega.$$

■

3.4 Two-scale homogenized stokes system

Theorem 3.4.1 *The sequence $(\tilde{u}^\varepsilon, \tilde{p}^\varepsilon)$ two-scale converges to the unique solution $(u_0(x,y), p(x))$ such that*

$$\begin{cases} -\Delta_y u_0 + \nabla_y p_1 = f - \nabla_x p & \text{in } \Omega \times Y_f \\ \operatorname{div}_y u_0(x,y) = 0 & \text{in } \Omega \times Y_f \\ \operatorname{div}_x \left(\int_Y u_0(x,y)\,dy \right) = 0 & \text{in } \Omega \\ u_0(x,y) = 0 & \text{in } \Omega \times Y_s \\ \left(\int_Y u_0(x,y)\,dy \right) \cdot n = 0 & \text{on } \partial\Omega \\ y \mapsto u_0(x,y), p_1(x,y) & \text{are } Y\text{-periodic} \end{cases} \tag{3.13}$$

Remark 3.4.1 *In (3.13), p denotes the macroscopic pressure and p_1 denotes the microscopic pressure, which takes into account the microscopic structure.*

Proof. Applying lemma 3.3.1 there exists a two-scale limit $u_0(x,y) \in L^2\big(\Omega; H^1_{per}(Y)^n\big)$ such that, up to a subsequence, the sequences $\tilde{u}^\varepsilon$ and $\varepsilon\nabla\tilde{u}^\varepsilon$ two-scale converge to u_0 and $\nabla_y u_0$ respectively. Furthermore, u_0 satisfies

$$\begin{cases} \operatorname{div}_y u_0(x,y) = 0 & \text{in } \Omega \times Y \\ \operatorname{div}_x \left(\displaystyle\int_Y u_0(x,y)\,dy \right) = 0 & \text{in } \Omega \\ u_0(x,y) = 0 & \text{in } \Omega \times Y_f, \\ \left(\displaystyle\int_Y u_0(x,y)\,dy \right) \cdot n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.14}$$

and there exists a two-scale limit $p(x) \in L^2(\Omega \times Y)$ such that, up to a subsequence, The next step in the two-scale convergence method is to multiply the equation (3.1) by a test function which shares the same characteristics as u_0. Therefore we now take $\varphi(x,y)$ a vector valued test function that satisfies

$$\operatorname{div}_y \varphi(x,y) = 0 \text{ on } \Omega \times Y, \tag{3.15}$$

$$\operatorname{div}_x \left(\int_Y \varphi(x,y)\, dy \right) = 0 \text{ in } \Omega, \tag{3.16}$$

$$\left(\int_Y \varphi(x,y)\, dy \right) \cdot n = 0 \text{ on } \partial\Omega, \tag{3.17}$$

$$\varphi(x,y) = 0 \text{ in } \Omega \times Y_s. \tag{3.18}$$

Multiplying the original Stokes equation by $\varphi\big(x, \frac{x}{\varepsilon}\big)$ we get after integrating by parts:

$$\begin{aligned} &\varepsilon^2 \int_\Omega \nabla \tilde{u}^\varepsilon \cdot \nabla_x \varphi\Big(x, \frac{x}{\varepsilon}\Big)\, dx + \varepsilon \int_\Omega \nabla \tilde{u}^\varepsilon \cdot \nabla_y \varphi\Big(x, \frac{x}{\varepsilon}\Big)\, dx \\ &\quad - \int_\Omega \tilde{p}^\varepsilon \operatorname{div}_x \varphi\Big(x, \frac{x}{\varepsilon}\Big)\, dx - \frac{1}{\varepsilon} \int_\Omega \tilde{p}^\varepsilon \operatorname{div}_y \varphi\Big(x, \frac{x}{\varepsilon}\Big)\, dx = \int_\Omega f(x) \cdot \varphi\Big(x, \frac{x}{\varepsilon}\Big)\, dx. \end{aligned} \tag{3.19}$$

Using the assumptions above, we infer

$$\begin{aligned} &\varepsilon^2 \int_\Omega \nabla \tilde{u}^\varepsilon \cdot \nabla_x \varphi\Big(x, \frac{x}{\varepsilon}\Big)\, dx + \varepsilon \int_\Omega \nabla \tilde{u}^\varepsilon \cdot \nabla_y \varphi\Big(x, \frac{x}{\varepsilon}\Big)\, dx \\ &\qquad - \int_\Omega \tilde{p}^\varepsilon \operatorname{div}_x \varphi\Big(x, \frac{x}{\varepsilon}\Big)\, dx = \int_\Omega f(x) \cdot \varphi\Big(x, \frac{x}{\varepsilon}\Big)\, dx. \end{aligned} \tag{3.20}$$

and we may pass to the (two-scale) limit as $\varepsilon \to 0$ to get

$$\int_{\Omega\times Y} \nabla_y u_0(x,y) \cdot \nabla_y \varphi(x,y)\, dx\, dy - \int_{\Omega\times Y} p(x) \operatorname{div}_x \varphi(x,y)\, dx\, dy = \int_{\Omega\times Y} f(x) \cdot \varphi(x,y)\, dx\, dy \tag{3.21}$$

which further simplifies into

$$\int_{\Omega\times Y} \nabla_y u_0(x,y) \cdot \nabla_y \varphi(x,y)\, dx\, dy = \int_{\Omega\times Y} f(x) \cdot \varphi(x,y)\, dx\, dy. \tag{3.22}$$

By density, the preceding variational formulation (3.22) holds for any test function in the Hilbert space V defined by:

$$V = \left\{ \varphi(x,y) \in \big(L^2\left(\Omega; H^1_{per}(Y)\right)\big) \;\middle|\; \begin{array}{l} \varphi(x,y) = 0 \text{ on } \Omega \times Y_s, \quad \operatorname{div}_y \varphi(x,y) = 0, \\ \operatorname{div}_x \left(\int_Y \varphi(x,y)\, dy \right) = 0, \quad \left(\int_Y \varphi(x,y)\, dy \right) \cdot n = 0 \text{ on } \partial\Omega \end{array} \right\} \tag{3.23}$$

The Lax-Milgram Theorem applies directly to prove that this variational formulation has a unique solution $u_0 \in V$. This uniqueness characterizes the two-scale limit, and therefore the whole sequence $(\tilde{u}^\epsilon)$ converges to u_0.

Eventually, in order to recover the pressure terms , we need to characterize the orthogonal of V with respect to the $L^2(\Omega \times Y_f)$ scalar product. This was done in [1], where it is shown that

$$V^\perp = \left\{ \nabla_x \varphi(x) + \nabla_y \varphi_1(x,y) \quad \text{with} \quad \varphi \in H^1(\Omega) \quad \text{and} \quad \varphi_1 \in L^2\left(\Omega; L^2_{per}(Y_f)/\mathbb{R}\right) \right\} \tag{3.24}$$

We obtain as a consequence the existence of $q_0 \in L^2(\Omega)/\mathbb{R}$ and $q_1 \in L^2\left(\Omega; L^2_{per}(Y_f)/\mathbb{R}\right)$ that satisfy

$$\begin{aligned}\int_{\Omega\times Y} \nabla_y u_0 \cdot \nabla_y \varphi(x,y)\,dx\,dy - \int_{\Omega\times Y} q_0(x)\operatorname{div}_x \varphi(x,y)\,dx\,dy - \int_{\Omega\times Y} q_1(x,y)\operatorname{div}_y \varphi(x,y)\,dx\,dy \\ = \int_{\Omega\times Y} f(x)\cdot\varphi(x,y)\,dx\,dy\end{aligned}$$

This would give the existence of $p\,(= q_0)$ and $p_1\,(= q_1)$ in Theorem 3.4.1 except that we need to prove that q_0 is indeed the two-scale limit of p^ϵ. In order to do so, we take a test function $\varphi(x,y)$ which satisfies only $\operatorname{div}_y \varphi = 0$ into the equation (3.1). We deduce from the calculation before, that:

$$\int_{\Omega\times Y} \nabla_y u_0 \cdot \nabla_y \varphi(x,y)\,dx\,dy - \int_{\Omega\times Y} p(x)\operatorname{div}_x \varphi(x,y)\,dx\,dy = \int_{\Omega\times Y} f(x)\cdot\varphi(x,y)\,dx\,dy$$

This permits us to deduce that:

$$\int_{\Omega\times Y} q_0(x)\operatorname{div}_x \varphi(x,y)\,dx\,dy = \int_{\Omega\times Y} p(x)\operatorname{div}_x \varphi(x,y)\,dx\,dy$$

or, in other words, that $q_0 = p$. ■

Theorem 3.4.2 *The extension $(\tilde{u}^\varepsilon, \tilde{p}^\varepsilon)$ of the solution of (3.1) converges weakly in $L^2(\Omega)^n \times L^2(\Omega/\mathbb{R})$to is the unique solution (u,p) of the homogenized problem*

$$\begin{cases} u(x) = A\big(f(x) - \nabla p(x)\big), & \textit{in } \Omega, \\ \operatorname{div} u(x) = 0, & \textit{in } \Omega, \\ u(x)\cdot n = 0, & \textit{on } \partial\Omega. \end{cases} \tag{3.25}$$

where A is a symmetric, positive definite tensor defined by its entries

$$A_{ij} = \int_{Y_f} \nabla w_i(y) \cdot \nabla w_j(y)\, dy, \tag{3.26}$$

with $(e_i)_{1 \leq i \leq n}$ is the canonical basis of $\mathbb{R}^n$, and $(w_i(y), q_i(y))$ denotes the unique solution in $H^1_{per}(Y_f)^n \times L^2_{per}(Y_f)$ of the local, or unit cell, Stokes problem

$$\begin{cases} \nabla q_i - \Delta w_i = e_i & \text{in } Y_f, \\ \operatorname{div} w_i = 0 & \text{in } Y_f, \\ w_i = 0 & \text{in } Y_s, \\ q_i,\ w_i \text{ are } Y\text{-periodic.} \end{cases} \tag{3.27}$$

where $(e_i)_{1 \leq i \leq N}$ is the canonical basis of $\mathbb{R}^n$, Furthermore, the two-scale homogenized problem (3.13) is equivalent to (3.25)-(3.26) through the relation

$$u_0(x,y) = \sum_{i=1}^{n} \left[f_i(x) - \frac{\partial P}{\partial x_i}(x) \right] w_i(y) \tag{3.28}$$

Proof. We first remark that if we take x as a fixed parameter, (u_0, p_1) satisfies a Stokes equation on the unit cell

$$\begin{cases} -\Delta_y u_0 + \nabla_y p_1 = f - \nabla_x p \quad \text{in } Y_f \\ \operatorname{div}_y u_0 = 0 \quad \text{in } Y_f \\ u_0 = 0 \quad \text{on } \partial Y_f \\ u_0(x,y), p_1(x,y) \text{ are } Y\text{-periodic} \end{cases} \tag{3.29}$$

Since the problem is linear with respect to the driving force $f(x) - \nabla p(x)$, the solution can be expressed as a linear combination of the elementary cell solutions (w_i, q_i), $1 \leq i \leq n$, introduced in (3.27). Therefore,

$$u_0(x,y) = \sum_{i=1}^{N} \left(f_i(x) - \frac{\partial p}{\partial x_i}(x) \right) w_i(y).$$

This representation separates the macroscopic variable x from the microscopic variable y. The coefficients depend only on the macroscopic force and pressure gradient, whereas the functions w_i describe the geometry of the periodic porous structure.

To obtain the effective macroscopic velocity, we average u_0 over the fluid part Y_f of the reference cell. Define

$$u(x) = \int_{Y_f} u_0(x, y)\, dy.$$

Substituting the expression of u_0 into the above formula yields

$$u(x) = \sum_{i=1}^{N} \left(f_i(x) - \frac{\partial p}{\partial x_i}(x) \right) \int_{Y_f} w_i(y)\, dy.$$

Let $A = (A_{ij})$ denote the permeability tensor defined by

$$A_{ij} = \int_{Y_f} \nabla w_i(y) \cdot \nabla w_j(y)\, dy.$$

We now derive an equivalent expression for A_{ij}. Consider the local Stokes problem

$$\nabla q_i - \Delta w_i = e_i \qquad \text{in } Y_f.$$

Multiplying this equation by w_j and integrating over Y_f, we obtain

$$\int_{Y_f} \nabla q_i \cdot w_j\, dy - \int_{Y_f} \Delta w_i \cdot w_j\, dy = \int_{Y_f} e_i \cdot w_j\, dy.$$

Since w_j is divergence free, we have

$$\int_{Y_f} \nabla q_i \cdot w_j\, dy = -\int_{Y_f} q_i \operatorname{div}(w_j)\, dy = 0.$$

Moreover, using integration by parts and the periodic boundary conditions together with the homogeneous boundary condition on Y_s, all boundary terms vanish. Hence,

$$\int_{Y_f} \nabla w_i \cdot \nabla w_j\, dy = \int_{Y_f} e_i \cdot w_j\, dy.$$

Therefore,

$$A_{ij} = \int_{Y_f} w_j(y) \cdot e_i\, dy.$$

Substituting this identity into the expression of $u(x)$, we obtain

$$u(x) = A\big(f(x) - \nabla p(x)\big).$$

Furthermore, the incompressibility condition satisfied by the two-scale limit gives

$$\operatorname{div} u = 0 \quad \text{in } \Omega.$$

The impermeability condition on the exterior boundary is

$$u \cdot n = 0 \qquad \text{on } \partial\Omega.$$

Consequently, the pair (u, p) satisfies the homogenized problem

$$\begin{cases} \quad u(x) = A\big(f(x) - \nabla p(x)\big) & \text{in } \Omega, \\ \operatorname{div} u(x) = 0 & \text{in } \Omega, \\ u(x) \cdot n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.30}$$

Since the tensor A is symmetric and positive definite, the above system is well posed. Indeed, substituting into the divergence equation leads to a second-order elliptic equation for the pressure p with a Neumann boundary condition. Standard elliptic theory ensures the existence and uniqueness of p (up to an additive constant), and consequently of u. Therefore,

$$\tilde{u}^{\varepsilon} \rightharpoonup u \qquad \text{weakly in } L^2(\Omega)^N,$$

Hence, (u, p) is the unique solution of the homogenized problem (3.28). ■

Remark 3.4.2 *The homogenized problem* (3.30) *is called Darcy's law (i.e. the flow rate u is proportional to the balance of forces including the pressure).*

Conclusion and Perspectives

In this research, we have investigated the powerful mathematical framework of the **Two-Scale Convergence method** and its fundamental application to the theory of periodic homogenization. This approach has proven to be a rigorous and elegant tool for resolving complex partial differential equations with rapidly oscillating coefficients.

In the first part of this work, we established the essential mathematical foundations, reviewing different notions of convergence, the structures of Sobolev spaces, and the properties of periodic functions. These generalities provided the necessary groundwork to rigorously define the concept of two-scale convergence and its crucial compactness theorems.

We then demonstrated the efficiency of this method by applying it first to a model second-order elliptic equation. Through the choice of appropriate test functions and the application of the Lax-Milgram theorem, we successfully derived the homogenized system and the associated cell problems.

Finally, the core application of this research focused on the steady **Stokes equations inside a periodic porous medium**. Faced with the challenge of a domain varying with ε, we utilized suitable extensions for both velocity and pressure to establish uniform *a priori* estimates. Passing to the two-scale limit allowed us to eliminate the microscopic constraints and recover the macroscopic effective flow description.

Future Perspectives

While this work outlines the classical periodic framework, several interesting paths remain open for future research:

1. **Non-Periodic and Stochastic Homogenization:** Extending the two-scale convergence method to media with random or non-periodic heterogeneous structures, which are closer to real-world geological formations.

2. **Non-Linear Flows:** Applying the method to non-linear fluid models, such as the Navier-Stokes equations, where inertial effects cannot be neglected.

3. **Numerical Simulations:** Implementing the derived local cell problems in specialized finite element software (such as FreeFem++ or MATLAB) to numerically compute the permeability tensor A for different geometric configurations of the solid matrix Y_s.

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ملخص

تهدف هذه الدراسة إلى بحث النماذج الرياضية ذات المعاملات سريعة التذبذب أو البنى المجهرية غير المتجانسة باستخدام طريقة التقارب ثنائي المقياس. نبدأ بعرض بعض المفاهيم الأساسية اللازمة للدراسة، بما في ذلك أنواع التقارب المختلفة، فضاءات سوبوليف، والدوال الدورية. ثم نقدم مفهوم التقارب ثنائي المقياس إلى جانب أهم نتائج الاندماج وخصائصه. بعد ذلك، نطبق هذه الطريقة على تجانس معادلات ستوكس في الأوساط المسامية الدورية من أجل استنتاج النظام المتجانس الذي يصف السلوك العياني الفعال لتدفق المائع داخل هذه البنى.

الكلمات المفتاحية: التجانس، التقارب ثنائي المقياس، معادلات ستوكس، الأوساط المسامية.

Abstract

This research aims to study mathematical models involving rapidly oscillating coefficients or microscopic heterogeneous structures using the Two-Scale Convergence method. We begin by presenting some fundamental notions required for the study, including different types of convergence, Sobolev spaces, and periodic functions. Then, we introduce the concept of two-scale convergence together with its main compactness results and properties. After that, we apply this method to the homogenization of the Stokes equations in periodic porous media in order to derive the homogenized system describing the effective macroscopic behavior of fluid flow inside such structures.

Keywords: Homogenization, Two-scale convergence, Stokes equations, Porous media.

Résumé

Cette recherche vise à étudier des modèles mathématiques impliquant des coefficients à oscillations rapides ou des structures hétérogènes microscopiques en utilisant la méthode de convergence à deux échelles. Nous commençons par présenter quelques notions fondamentales requises pour l'étude, y compris différents types de convergence, les espaces de Sobolev et les fonctions périodiques. Ensuite, nous introduisons le concept de convergence à deux échelles ainsi que ses principaux résultats de compacité et ses propriétés. Après cela, nous appliquons cette méthode à l'homogénéisation des équations de Stokes dans des milieux poreux périodiques afin de dériver le système homogénéisé décrivant le comportement macroscopique effectif de l'écoulement des fluides à l'intérieur de ces structures.

Mots-clés : Homogénéisation, Convergence à deux échelles, Équations de Stokes, Milieux poreux.