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Existence and Asymptotic Behaviour of Solutions to a Viscoelastic Marine Riser System

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Abstract

This work is devoted to the study of the stabilization of a flexible marine riser coupled with vessel dynamics. More precisely, we establish an exponential decay result for the total energy of the considered system. The stabilization is achieved through the application of an appropriate boundary control acting at the upper end of the riser, with the objective of reducing and attenuating the vibrations efficiently over time.

The mathematical analysis is based on the multiplier method combined with suitable energy estimates and Lyapunov functionals. The marine riser is modeled by a viscoelastic beam equation incorporating memory effects through a relaxation kernel of standard type with a slight perturbation.

Under appropriate assumptions on the relaxation function and system parameters, we prove the well-posedness of the problem and derive sufficient conditions ensuring the exponential stability of the system. The obtained results show that the combined effect of the viscoelastic damping and the boundary feedback control guarantees a rapid dissipation of the energy associated with the marine riser dynamics.

ملخص:

يهدف هذا البحث إلى دراسة الاستقرار الأسي لأنبوب الرفع البحري المرن المقترن بديناميكيات السفينة. وبشكل أكثر تحديدًا، نبرهن على أن الطاقة الكلية للنظام تتناقص تناقصًا أسيًا مع مرور الزمن. ويتحقق هذا الاستقرار من خلال تطبيق نظام تحكم حدي مناسب عند الطرف العلوي الأنبوب الرفع بهدف الحد من الاهتزازات وتخميدها بكفاءة، مما يسهم في تحسين الأداء الديناميكي للنظام.

يعتمد التحليل الرياضي على طريقة المضاعفات، مدعومة بتقديرات مناسبة للطاقة وبناء دوال ليابونوف، في حين ينمذج أنبوب الرفع البحري بواسطة معادلة عتبة لزجة - مرنة تأخذ في الاعتبار تأثيرات الذاكرة من خلال نواة استرخاء من النوع القياسي مع اضطراب طفيف.

وبالاعتماد على فرضيات مناسبة تتعلق بدالة الاسترخاء ومعلمات النظام، نثبت حسن الوضع الرياضي للمسألة، ونستخلص شروطا كافية لضمان الاستقرار الأسي للنظام. وتبين النتائج المتوصل إليها أن التأثير المشترك للتخميد اللزج - المرن والتحكم الارتجاعي الحدي يؤدي إلى تبديد سريع للطاقة المرتبطة بديناميكيات أنبوب الرفع البحري مما يؤكد فعالية استراتيجية الاستقرار المقترحة.

CONTENTS

INTRODUCTION

Oil Platform

The continuous growth of the global population, combined with the rapid industrial and economic development of many countries, has led to a substantial increase in the worldwide demand for energy. Among the various energy sources, oil and natural gas remain dominant due to their high energy density and their essential role in transportation, industry, and electricity generation. Consequently, the demand for these resources has risen sharply over the past decades.

In response to this growing demand and the progressive depletion of easily accessible onshore reserves, energy companies have increasingly turned their attention to offshore exploration and production. The relatively high market prices of oil and gas further encourage significant investments in this sector, as offshore fields often contain large untapped reserves. As a result, a considerable proportion of the global energy supply is now derived from offshore production activities.

Today, there are more than ten thousand offshore installations operating worldwide, ranging from small units to large and highly complex structures. These installations function continuously, 24 hours a day, in some of the most active offshore regions, including the Gulf of Mexico, Southeast Asia, the North Sea, offshore Brazil, and various parts of the Far East. Offshore structures encompass not only fixed and floating platforms but also a variety of specialized vessels designed for drilling, production, storage, and transportation.

Offshore platforms are generally classified into two principal categories: fixed platforms and floating platforms. Fixed platforms are anchored directly to the seabed and are typically used in relatively shallow waters where such installation is technically and economically feasible. In contrast, floating platforms are designed to operate in deep and ultra-deep waters, where fixed structures become impractical or impossible to install. These floating systems, such as semi-submersibles, drillships, and floating production storage and offloading units (FPSOs), offer greater flexibility and can be relocated when necessary.

Due to the increasing shift toward deep-water exploration, floating platforms now represent the majority of offshore installations. Their design must account for complex environmental conditions, including waves, currents, and wind forces, which can significantly affect their stability and performance. As a result, understanding the dynamic behavior and ensuring the structural integrity of these systems are critical challenges in offshore engineering.

Stabilization

Control theory for partial differential equations arises in various contexts and provides a mathematical framework for analyzing and influencing the behavior of dynamical systems. It addresses three fundamental problems: controllability, observability, and stabilization. Controllability concerns the possibility of steering a system from any initial state to any desired final state using an appropriate control input. Observability deals with the ability to determine the state of a system from its outputs. Stabilization, which is the central topic of this thesis, focuses on designing control mechanisms that ensure the decay of the system's energy toward zero over time.

In this thesis, we study the stabilization of a viscoelastic problem within the framework of control theory. We consider an evolution system described by partial differential equations over a time interval $[0, T]$. The main question is whether it is possible to drive the solutions from a given initial state (at time $t = 0$) to a desired final state (at time $t = T$) by means of an appropriate control applied either on the boundary or on a subregion of the domain. This controllability aspect is closely related to stabilization, as exact controllability implies exponential stabilizability for linear systems through appropriate feedback controls. Significant research efforts have been devoted to this topic (see J.-L. Lions [28, 29], I. Lasiecka and R. Triggiani [26, 31], H. O. Fattorini [25], D. L. Russell [39], and E. Zuazua [43, 44]).

From a control theory perspective, stabilization aims to attenuate vibrations by guaranteeing the decay of the energy of solutions toward zero through a dissipation mechanism, typically achieved by designing feedback controls that introduce damping. The stabilization problem consists of three main objectives: determining the asymptotic behavior of the energy $E(t)$, studying its limit, and, if this limit is zero, providing an estimate of the decay rate. The decay rate is particularly important in control applications, as it determines how quickly vibrations are attenuated.

There exist several types of stabilization, classified according to the decay rate of the energy:

- **Strong stabilization** consists of studying whether the energy decays to zero, i.e.,
$$E(t) \to 0, \quad \text{as } t \to +\infty.$$
This guarantees that the energy eventually vanishes, but does not provide information about the speed of decay.

- **Exponential stabilization** concerns the fastest decay, namely when the energy converges to zero at an exponential rate:
$$E(t) \leq Ce^{-\delta t}, \quad \forall t > 0,$$
for some positive constants C and δ. This is particularly desirable because it guarantees rapid decay and provides robustness with respect to perturbations.

- **Polynomial or logarithmic stabilization** concerns intermediate situations where the decay is not exponential, but of polynomial type,

$$E(t) \leq \frac{C}{t^{\alpha}}, \quad \forall t > 0,$$

or logarithmic type,

$$E(t) \leq \frac{C'}{\log(1+t)^{k}}, \quad \forall t > 0,$$

for some positive constants C, C', α, and k. These slower decay rates typically arise when the dissipation mechanism is weak or localized.

The design of stabilizing feedback laws for infinite-dimensional systems relies on powerful control-theoretic methods. Lyapunov methods involve constructing a suitable functional, often related to the energy, whose derivative along the solutions is negative definite, guaranteeing energy decay and allowing estimation of the decay rate. The Riccati equation arises in linear quadratic regulator (LQR) problems, providing optimal feedback controls that minimize a quadratic cost functional. For partial differential equations, however, solving the Riccati equation is computationally intensive and often requires approximations or numerical methods.

The interplay between controllability and stabilization is of particular interest. It is well established that for linear systems, exact controllability implies exponential stabilizability through appropriate feedback controls. This relationship, established through the work of J.-L. Lions, forms the foundation for many stabilization results. Conversely, stabilizability is a weaker notion than controllability, as it only requires the system to be driven to zero asymptotically. The penalization method, introduced by Lions, connects these concepts by transforming a controllability problem into a stabilization problem through the addition of a penalization term.

Significant contributions to control theory for stabilization have been made over the past decades. In 1986, J.-L. Lions developed a general method for exponential stabilization of exactly controllable reversible linear systems. His approach relies on control theory and the penalization method, but does not provide an explicit procedure for constructing the feedback operator, nor an estimate of the energy decay rate. Despite these limitations, Lions' work laid the theoretical foundations for subsequent developments. In 1987, V. Komornik and E. Zuazua weakened the boundary condition introduced by G. Chen, generalizing results to domains with regular and connected boundaries at the cost of modifying the boundary conditions. Their work demonstrated the importance of boundary conditions in the design of stabilizing feedback controls and highlighted the interplay between domain geometry and control effectiveness.

In this thesis, we aim to contribute to the understanding of how control theory techniques can be applied to the stabilization of viscoelastic systems. We investigate the asymptotic behavior of the energy and seek to determine the type of stabilization achievable under various dissipation mechanisms and boundary conditions. By combining control-theoretic methods with the properties of viscoelastic materials, we hope to provide insights into the design of effective control strategies for this important class of systems.

Viscoelastic Material

Elasticity refers to the ability of a material to return to its original shape after the applied stress is removed.

Viscosity describes the resistance of a material to deformation or flow under applied stress.

A viscoelastic material is a material whose behavior lies between that of an ideal elastic solid, capable of storing and releasing energy after deformation, and a viscous fluid, capable of dissipating energy. Viscoelastic materials, such as polymers, combine elasticity and viscosity, allowing them to both store energy and dissipate it over time.

The viscoelastic component provides a natural damping mechanism, causing the energy associated with deformations to decay and driving the system toward equilibrium. From a mathematical point of view, this effect is often modeled using integro-differential operators, which account for the material's memory of past deformations.

It is well established that viscoelastic materials inherently provide damping, due to their unique ability to retain memory of past deformations. Mathematically, this damping effect is often modeled using integro-differential operators, such as:

$$\int_0^t h\,(t-s)\,\Delta\,(s)\,ds,$$

where h is the memory kernel. This integral term reflects the fact that the material's stress at any given time depends not only on its current state, but also on the entire history of past stresses the material has undergone.

Problem Statement

The system is represented by a beam structure [40] as follows:

$$\begin{cases} \rho u_{tt} + EI\left(u_{xxxx} - \displaystyle\int_0^t h(t-s)u_{xxxx}(s)\,ds\right) - Tu_{xx} = 0, \\ \qquad \forall (x,t) \in (0,L)\times[0,\infty), \\ u(0,t) = u_x(0,t) = u_{xx}(L,t) = 0, \\ u_{xxx}(0,t) = u_{xx}(0,t), \\ EI\left(-u_{xxx}(L,t) + \displaystyle\int_0^t h(t-s)u_{xxx}(L,s)\,ds\right) + Tu_x(L,t) \\ \qquad = A(t) - d_S u_t(L,t) - M_S u_{tt}(L,t), \quad \forall t \in [0,\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in (0,L). \end{cases} \tag{1}$$

The physical configuration of the system is shown in Figure 1.

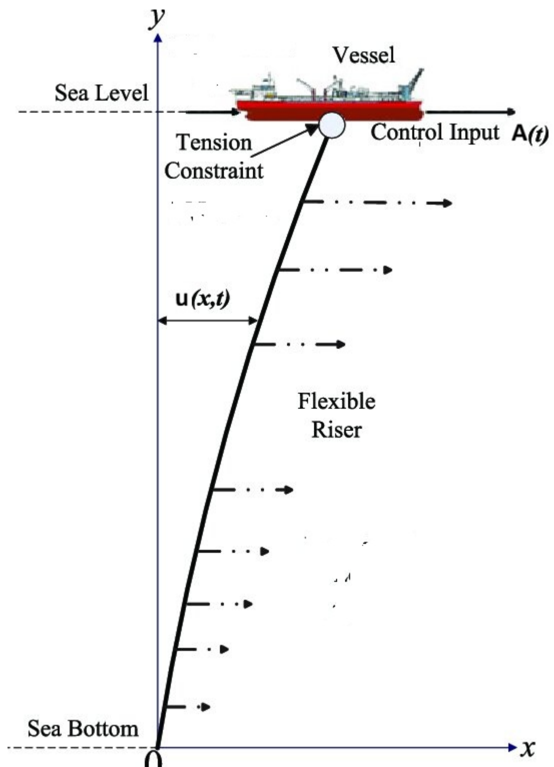

Fig 1. Flexible Marine Riser

where

- $u(x,t)$ represents the displacement of the riser at position x at time t.
- $\rho > 0$ is the uniform mass per unit length of the riser.
- EI is the bending stiffness of the riser.
- T is the tension in the riser.
- M_S represents the mass of the vessel.
- $u\left(L,t\right)$ is the position of the vessel.
- $u_t\left(L,t\right)$ is the velocity of the vessel.
- L denotes the length of the riser.
- d_S is the damping coefficient of the vessel.

Objective of the Work

In this work, we focus on the analysis of the dynamics of the riser. This is motivated by their high cost and limited lifespan. As mentioned above, a riser is a special pipeline that transports oil or gas (in our case) from a subsea well to the floating platform at the sea surface. Therefore, the riser must operate vertically from the seabed to the surface. Unfortunately, it is constantly subjected to different internal and external hydrostatic pressures, which result in undesired vibrations. Among these, vortex-induced vibrations are the most severe, as they can lead to a premature reduction in lifespan. Other disturbances arise from seismic ground movements, irregular fluid flow, current turbulence, wave transmission, and so on.All these phenomena have direct negative effects on the reliability and efficiency of the riser. They can cause fatigue fractures, poor quality, low productivity, and even leaks that may result in ecological disasters. The fact that the riser is slender and spans a long distance amplifies small disturbances into potentially destructive effects. Therefore, special attention must be given to improving riser performance. They must be continuously monitored and carefully maintained. Significant efforts are being made to reduce riser vibrations, including the use of new material technologies, innovative design techniques, and the application of appropriate controllers. Risers are generally made of composite steel tubes (such as carbon fiber and aluminum). In the present work, we propose a method to dissipate vibrational energy by using a material with high damping properties.

For the study of the stabilization of oil platforms, it is important to mention the work of L. Seghour and al [6], [40], [41].

Organization of the Thesis:

This thesis is organized as follows. In Chapter 1, we present some preliminary results and fundamental tools that will be used throughout this work. Chapter 2 is devoted to establishing the existence and uniqueness of the solution for the viscoelastic problem (1). Finally, Chapter 3 addresses the stabilization of the problem (1) by using the multiplier method.

CHAPTER 1

PRELIMINARY RESULTS AND NOTATION

To facilitate the reading of this work, we recall some basic notions from functional analysis that will be used throughout the analysis, together with their main properties. These preliminaries are essential for the rigorous study of the well-posedness and stability of the considered system, and they provide the mathematical framework in which the problem will be formulated. In particular, we briefly review fundamental results concerning normed and Banach spaces, Hilbert spaces, weak and strong convergence, as well as compactness arguments. We also recall several classical inequalities and embedding results that play a crucial role in the derivation of energy estimates and in the application of Lyapunov techniques for proving exponential decay of solutions. Proofs and illustrations of most theorems and results of this chapter can be checked in [7].

1.1 Reflexive space

Definition 1.1.1 (The dual space)**.** *We define the dual space X' of a normed linear space X to be the space of bounded linear functionals (linear forms) on X. That is*

$$X' = \{\ell : X \to \mathbb{R} \quad \textit{linear and bounded}\}.$$

Remark 1.1.1. *The space X' endowed with the norm:*

$$\|\ell\|_{X'} = \sup_{\|x\|\leq 1} |\ell(x)|, \ell \in X',$$

is a Banach space.

The dual space of the space X' is called the bidual of X, and it is denoted by X''.

Definition 1.1.2. *Let X be a Banach space. The mapping*

$$\begin{array}{rcl} J : X & \longmapsto & X'' \\ x & \longmapsto & J_x \end{array} \quad \textit{where} \quad \begin{array}{rcl} J_x : X' & \longmapsto & \mathbb{R} \\ f & \longmapsto & J_x(f) = \langle f, x\rangle \end{array},$$

is called the canonical embedding of X into X''.

Proposition 1.1.1. *Let X be a Banach space and J be the canonical embedding. Then*

1. *J is injective,*
2. *J is linear and bounded,*
3. *J is isometric. That is $\|J_x\| = \|x\|$, for all $x \in X$.*

Remark 1.1.2. *In general*

$$\{J_x\}_{x\in X} \subset X''.$$

Definition 1.1.3 (Reflexive space)**.** *The Banach space X is said to be reflexive if :*

$$\{J_x\}_{x\in X} = X''.$$

Theorem 1.1.1. *The Hilbert spaces are reflexive.*

1.2 Continuous functions spaces

Let Ω be an open set of $\mathbb{R}^n$, $(n \geq 1)$, and $k \geq 1$ is integer.

Definition 1.2.1. *We note by $C(\Omega)$ (respectively $C^k(\Omega)$) to the space of continuous functions on Ω (respectively the space of functions which are k times differentiable whose differentials of order k are continuous).*

Definition 1.2.2. *We We denote by $C^k(\overline{\Omega})$ the space of restrictions to $\overline{\Omega}$ of elements of $C^k(\mathbb{R}^n)$.*

Definition 1.2.3. *We define the space $C^\infty(\Omega)$ to be the space of infinitely differentiable functions.*

Definition 1.2.4. *The space $C_0^\infty(\Omega)$ is defined to be the space of elements of $C^\infty(\Omega)$ whose supports are compact.*

Remark 1.2.1. *$C_0^\infty(\Omega)$ is called the space of test functions and it is denoted by $D(\Omega)$.*

1.3 Lebesgue spaces

Let Ω be open subset of $\mathbb{R}^n$ where $n \in \mathbb{N}^*$. we endow Ω with the Lebesgue measure dx. Let p be any real number in $[1, +\infty]$.

1.3.1 Lebesgue space $L^p(\Omega)$

Definition 1.3.1. *Let $p \in [1, +\infty)$, $L^p(\Omega)$ is the linear-space of (classes of) measurable functions $u : \Omega \to \mathbb{R}$ such that the function $|u|^p : \Omega \to \mathbb{R}$ is Lebesgue integrable:*

$$L^p(\Omega) = \left\{ u \in \Omega \to \mathbb{R} \quad \textit{measurable such that} \quad \int_\Omega |u(x)|^p dx < \infty \right\}.$$

Definition 1.3.2. *$L^\infty(\Omega)$ is the space of (classes) of measurable functions $u : \Omega \to \mathbb{R}$ which a essentially bounded on Ω.*

$$L^\infty(\Omega) = \{ u \in \Omega \to \mathbb{R} \quad \textit{measurable such that} \quad \exists C > 0 \quad |u(x)| \leq C \to a.e\ in\ \Omega \}.$$

Theorem 1.3.1. *Let $p \in (1, +\infty)$ and define on $L^p(\Omega)$ the norm:*

$$\|u\|_{L^p(\Omega)} = \int_\Omega |u(x)|^p dx.$$

Then $L^p(\Omega)$ is a Banach space.

Theorem 1.3.2. *The space $L^\infty(\Omega)$ endowed with the norm:*

$$\|u\|_{L^\infty(\Omega)} = ess_\Omega \sup |u(x)|,$$

is a Banach space.

Remark 1.3.1.

$$ess_\Omega \sup |u(x)| = \inf \{ C > 0 \quad |u| \leq C\ a.e\ on\ \Omega \}.$$

Proposition 1.3.1. *$L^2(\Omega)$ endowed with the inner product:*

$$\langle u, v \rangle_{L^2(\Omega)} = \int_\Omega u(x)v(x)dx \quad u, v \in L^2(\Omega),$$

is a Hilbert space.

Properties 1.3.1. *The space $L^p(\Omega)$ has the following properties.*

1. *For $p \in (1, \infty)$, $L^p(\Omega)$ is reflexive and separable and the dual space of $L^p(\Omega)$ is $L^q(\Omega)$ where q is the conjugate of p.*

2. *The space $L^1(\Omega)$ is separable and not reflexive. The dual space of $L^1(\Omega)$ is $L^\infty(\Omega)$.*

3. *The space $L^\infty(\Omega)$ is neither reflexive non separable. The space $L^1(\Omega)$ is strictly contained in the dual of $L^\infty(\Omega)$.*

Definition 1.3.3. *The space $L^1_{loc}(\Omega)$ is defined by*

$$L^1_{loc}(\Omega) = \left\{ f : \Omega \to \mathbb{R} \;\middle|\; f\mathbf{1}_K \in L^1(\Omega) \textit{ for every compact } K \subset \Omega \right\},$$

where $\mathbf{1}_K$ denotes the characteristic function of K.

Corollary 1.3.1. *Let $f \in L^1_{loc}(\Omega)$ such that*

$$\int_\Omega f\psi\, dx = 0 \quad \forall \psi \in C_0^\infty(\Omega),$$

then $f = 0$ a.e. on Ω.

1.4 Sobolev spaces

The spaces so-called Sobolev spaces are particularly useful when it comes to partial differential equations. In fact, weak solutions to some important PDEs are guaranteed to exist in suitable Sobolev spaces, even when there are no strong solutions in spaces of continuous functions with derivatives understood in the classical sense.

Throughout this section, Ω denotes an open subset of $\mathbb{R}^n$ and p is any real number in the interval $[1, \infty)$.

1.4.1 Weak derivative

Let $u, v \in L^1_{\text{loc}}(\Omega)$ and let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n$ be a multi-index.

Definition 1.4.1. *We say that v is the weak derivative of order α of u, and we write*

$$D^\alpha u = v, \tag{1.1}$$

if for every $\psi \in \mathcal{D}(\Omega)$ we have

$$\int_\Omega u\, D^\alpha \psi\, dx = (-1)^{|\alpha|} \int_\Omega v\, \psi\, dx. \tag{1.2}$$

Remark 1.4.1. *A weak α-th partial derivative of u, if it exists, is uniquely defined up to a set of measure zero.*

1.4.2 Sobolev space $W^{1,p}(\Omega)$

Definition 1.4.2. *The Sobolev space $W^{1,p}(\Omega)$ is defined by*

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) \;\middle|\; \frac{\partial u}{\partial x_i} \in L^p(\Omega),\ 1 \le i \le n \right\}, \tag{1.3}$$

where the partial derivative $\dfrac{\partial u}{\partial x_i}$ is understood in the weak sense.

Proposition 1.4.1. *The space $W^{1,p}(\Omega)$ endowed with the norm*

$$\|u\|_{W^{1,p}(\Omega)} = \left[\|u\|^p_{L^p(\Omega)} + \sum_{i=1}^n \left\| \frac{\partial u}{\partial x_i} \right\|^p_{L^p(\Omega)} \right]^{\frac{1}{p}}, \quad p \in [1, \infty),$$

is a Banach space.

Remark 1.4.2. *In the particular case when $p = 2$, the space $W^{1,2}(\Omega)$ endowed with the inner product*

$$\langle u, v \rangle_{W^{1,2}(\Omega)} = \langle u, v \rangle_{L^2(\Omega)} + \langle \nabla u, \nabla v \rangle_{L^2(\Omega)}$$

is a Hilbert space.

Notation 1.4.1. *The space $W^{1,2}(\Omega)$ is usually denoted by $H^1(\Omega)$.*

1.4.3 Sobolev space $W^{m,p}(\Omega)$

Let m be a nonnegative integer.

Definition 1.4.3. *The Sobolev space $W^{m,p}(\Omega)$ is defined by*

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) \mid D^\alpha u \in L^p(\Omega),\ \alpha \in \mathbb{N}^n,\ |\alpha| \le m\}. \tag{1.4}$$

Proposition 1.4.2. *If we endow the space $W^{m,p}(\Omega)$ with the norm*

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha|\le m} \|D^\alpha u\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}}, \quad p \in [1,\infty),$$
$$\|u\|_{W^{m,\infty}(\Omega)} = \max_{|\alpha|\le m} \|D^\alpha u\|_{L^\infty(\Omega)},$$

then it is a Banach space.

Remark 1.4.3. *In the particular case when $p = 2$, the space $W^{m,2}(\Omega)$ endowed with the inner product*

$$\langle u, v\rangle_{W^{m,2}(\Omega)} = \sum_{|\alpha|\le m} \langle D^\alpha u, D^\alpha v\rangle_{L^2(\Omega)}, \quad u, v \in W^{m,2}(\Omega),$$

is a Hilbert space.

Notation 1.4.2. *The space $W^{m,2}(\Omega)$ is usually denoted by $H^m(\Omega)$.*

Proposition 1.4.3. *Let m be a nonnegative integer. Then the space $W^{m,p}(\Omega)$ is separable for $1 \le p < \infty$.*

1.4.4 The space $W_0^{m,p}(\Omega)$

Definition 1.4.4. *The space $W_0^{m,p}(\Omega)$ is defined to be the closure of $\mathcal{D}(\Omega)$ in $W^{m,p}(\Omega)$.*

Proposition 1.4.4. *If Ω is bounded, then*

$$W_0^{m,p}(\Omega) = \{u \in W^{m,p}(\Omega) \mid D^\alpha u = 0 \text{ on } \partial\Omega \text{ for all } |\alpha| \le m-1\}.$$

Notation 1.4.3. *The space $W_0^{m,2}(\Omega)$ is denoted by $H_0^m(\Omega)$.*

Remark 1.4.4. *From the previous proposition, we have*

$$H_0^1(\Omega) = \left\{u \in H^1(\Omega) \;\middle|\; u = 0 \text{ on } \partial\Omega\right\}.$$

Remark 1.4.5. *Since $\mathcal{D}(\mathbb{R}^n)$ is dense in $H^1(\mathbb{R}^n)$, we have*

$$H_0^1(\mathbb{R}^n) = H^1(\mathbb{R}^n).$$

Definition 1.4.5. *The space $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$.*

Remark 1.4.6. *If $f \in H^{-1}(\Omega)$, we define the norm*

$$\|f\|_{H^{-1}(\Omega)} = \sup\left\{\langle f, u\rangle \;\middle|\; u \in H_0^1(\Omega),\ \|u\|_{H_0^1(\Omega)} \le 1\right\},$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

Theorem 1.4.4 (Trace Theorem)**.** *Let Ω be an open and bounded subset of $\mathbb{R}^n$ with boundary $\partial\Omega$ of class C^1. Then there exists a continuous linear operator*

$$\begin{aligned} \gamma_0 : H^1(\Omega) &\to L^2(\partial\Omega), \\ u &\mapsto \gamma_0 u, \end{aligned}$$

that coincides with the usual restriction operator on $\mathcal{D}(\overline{\Omega})$.

Proposition 1.4.5. *Let $\gamma_0(H^1(\Omega))$ be the image of $H^1(\Omega)$ by the trace operator γ_0. Then*

$$\begin{aligned} \gamma_0(H^1(\Omega)) &= H^{1/2}(\partial\Omega), \\ \ker\gamma_0 &= H_0^1(\Omega). \end{aligned}$$

Definition 1.4.6. *Let Ω be bounded with boundary of class C^m, $m \in \mathbb{N}$, and let γ_0 be the trace operator defined by*

$$\begin{aligned} \gamma_0 : H^m(\Omega) &\to L^2(\partial\Omega), \\ u &\mapsto \gamma_0 u. \end{aligned}$$

We define the space $H^{m-\frac{1}{2}}(\partial\Omega)$ to be the image of $H^m(\Omega)$ by the trace operator γ_0. That is,

$$H^{m-\frac{1}{2}}(\partial\Omega) = \gamma_0(H^m(\Omega)). \tag{1.5}$$

In particular, we have

$$H^{\frac{1}{2}}(\partial\Omega) = \gamma_0(H^1(\Omega)). \tag{1.6}$$

Remark 1.4.7. *The space $H^{m-\frac{1}{2}}(\partial\Omega)$ is endowed with the norm*

$$\|u\|_{H^{m-\frac{1}{2}}(\partial\Omega)} = \inf_{\substack{v\in H^m(\Omega)\\ \gamma_0 v = u}} \|v\|_{H^m(\Omega)}. \tag{1.7}$$

Definition 1.4.7. *The space $H^{-1/2}(\partial\Omega)$ is defined to be the dual space of $H^{1/2}(\partial\Omega)$.*

Theorem 1.4.5. *Suppose $u \in H^1(\Omega)$ such that $\Delta u \in L^2(\Omega)$. Then the normal derivative $\dfrac{\partial u}{\partial \nu}$ defines an element of $H^{-1/2}(\partial\Omega)$.*

1.4.5 Sobolev inequalities

Definition 1.4.8. *For $1 \le p < n$, we define the Sobolev conjugate p^* of p by:*

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}. \tag{1.8}$$

Theorem 1.4.6 (Gagliardo-Nirenberg-Sobolev inequality)**.** *Assume* $1 \leq p < n$*, then there exists a positive constant* C *depending on* n *and* p *such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|\nabla u\|_{L^p(\mathbb{R}^n)}, \quad \forall u \in C_c^1(\mathbb{R}^n). \tag{1.9}$$

Theorem 1.4.7. *Let* Ω *be bounded and* $\partial\Omega$ *is* C^1*, assume* $1 \leq p < n$ *and* $u \in W^{1,p}(\Omega)$ *then*

$$\begin{aligned} &u \in L^{p^*}(\Omega), \\ &\|u\|_{L^{p^*}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}. \end{aligned}$$

where C *is a constant depending on* n, p *and* Ω*.*

Theorem 1.4.8. *Assume* Ω *is bounded and* $u \in W_0^{1,p}(\Omega)$ *where* $1 \leq p < n$*, then*

$$\|u\|_{L^q(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}, \tag{1.10}$$

where $q \in [1, p^*]$ *and* C *is a constant depending on* p, q, n *and* Ω*.*

Theorem 1.4.9. *Let* Ω *be bounded and* $\partial\Omega$ *is* C^1*, then for* $u \in W_0^{1,n}(\Omega)$*, we have the estimate*

$$\|u\|_{L^q(\Omega)} \leq C\|u\|_{W_0^{1,n}(\Omega)},$$

for all $q \in [1, \infty)$*, where* C *is a constant depending on* n *and* Ω*.*

Remark 1.4.8. *This estimate is sometimes called Poincaré's inequality.*

The norm $\|\nabla u\|_{L^p(\Omega)}$ *on* $W_0^{1,p}(\Omega)$ *is equivalent to* $\|u\|_{W^{1,p}(\Omega)}$ *if* Ω *is bounded.*

Theorem 1.4.10 (Morrey's inequality)**.** *Let suppose* $n < p \leq \infty$ *then there exists a constant* C *depending on* n *and* p *such that*

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}, \tag{1.11}$$

for all $u \in C^1(\mathbb{R}^n)$ *where*

$$\gamma = 1 - \frac{n}{p}. \tag{1.12}$$

1.4.6 Compactness

Definition 1.4.9. *Let* X, Y *be Banach spaces with* $X \subset Y$*. We say that* X *is compactly embedded in* X *and we write:*

$$X \subset\subset Y. \tag{1.13}$$

provided

1. *There is some constant* $C > 0$*, such that*

$$\|x\|_Y \leq C\|x\|_X \quad (x \in X). \tag{1.14}$$

2. *Each bounded sequence in* X *is precompact in* Y*.*

Theorem 1.4.11 (Rellich Kondrachov compact theorem). *Let Ω be bounded and $\partial\Omega$ is C^1. Then we have the following compact embeddings*

$$W^{1.p}(\Omega) \subset\subset L^q(\Omega); \qquad \forall q \in [1, p^*) \text{ if } \quad p < n, \tag{1.15}$$
$$W^{1.p}(\Omega) \subset\subset L^q(\Omega); \qquad \forall q \in [p, \infty) \text{ if } \quad p = n, \tag{1.16}$$
$$W^{1.p}(\Omega) \subset\subset C(\overline{\Omega}) \qquad \text{if } \quad p > n. \tag{1.17}$$

Remark 1.4.9. *Since $p^* > p$ and $p^* \to +\infty$ as $p \to n$ then we have in particular*

$$W^{1.p}(\Omega) \subset\subset L^q(\Omega), \quad \forall q \in [1, \infty]. \tag{1.18}$$

1.4.7 Spaces involving time

We study some Sobolev spaces, in particular functions mapping time into Banach spaces. Let X be a Banach space with norm $\|.\|$.

Definition 1.4.10 (The space $L^p(0, T, X)$). *The space $L^p(0, T, X)$ consists of all measurable functions:*

$$u : [0, T] \to X, \tag{1.19}$$

with

1.
$$\|u\|_{L^p(0,T,X)} = \left(\int_0^T \|u(t)\|^p \right)^{\frac{1}{p}} < \infty, \quad if 1 \leq p < \infty. \tag{1.20}$$

2.
$$\|u\|_{L^p(0,T,X)} = ess \sup_{0 \leq t \leq T} \|u(t)\| < \infty \quad if p = +\infty. \tag{1.21}$$

Definition 1.4.11. *The space $C([0, T], X)$ consists of all functions:*

$$u : [0, T] \to X, \tag{1.22}$$

with

$$\|u\|_{C([0,T],X)} = \max_{0 \leq t \leq T} \|u(t)\|. \tag{1.23}$$

Definition 1.4.12. *Let $u \in L^1(0, T, X)$, then $v \in L^1(0, T, X)$ is said to be the weak derivative of u and we write:*

$$u' = v,$$

provided

$$\int_0^T u(t)\psi'(t)dt = -\int_0^T v'(t)\psi(t)dt, \tag{1.24}$$

for every ψ in $D(0, T)$.

Definition 1.4.13.

$$\|u\|_{W^{1,p}(0,T,X)} = \left(\int_0^T (\|u(t)\|^p + \|u'(t)\|^p)\, dt \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty,$$
$$\|u\|_{W^{1,\infty}(0,T,X)} = \operatorname{ess\,sup}_{0 \leq t \leq T} (\|u(t)\| + \|u'(t)\|), \quad p = \infty.$$

Notation 1.4.12. *We write $H^1(0,T,X)$ instead of $W^{1,2}(0,T,X)$.*

Theorem 1.4.13. *Let $u \in W^{1.p}(0,T,X)$ with $1 \leq p < \infty$ then*

1. $u \in C([0,T],X)$.
2. $u(t) = u(s) + \int_s^t u'(\tau)d\tau \quad (0 \leq s \leq \tau \leq T)$.
3. $\max\limits_{0\leq t\leq T} \|u(t)\| \leq c\|u'(t)\|_{W^{1.p}(0,T,X)}$.

1.5 Weak and weak* topology

Let X be a normed linear space.

1.5.1 Weak topology

Definition 1.5.1. *We define the weak topology on X to be the smallest topology on X to keep all the linear forms on X (elements of X') continuous. This topology is denoted by $\sigma(X,X')$.*

Definition 1.5.2. *A sequence $\{x_n\}$ in X is said to be convergent weakly in X if it converges in the weak topology $\sigma(X,X')$ and we write:*

$$x_n \rightharpoonup x \, in \, X,$$

instead of:

$$x_n \to x \, in \, \sigma(X,X').$$

Remark 1.5.1. *The sequence $\{x_n\}$ converges strongly in X. That is*

$$x_n \to x \, in \, X,$$

if:

$$\lim_{n\to+\infty} \|x_n - x\| = 0.$$

Proposition 1.5.1.

$$x_n \rightharpoonup x \quad in \, X \; iff \, \forall f \in X' \quad \langle f, x_n\rangle \to \langle f, x\rangle. \tag{1.25}$$

Theorem 1.5.1. *Let $\{x_n\}$ and $\{f_n\}$ be two sequences in X and X' respectively, then*

$$\begin{aligned} &x_n \to x \quad in \, X \implies x_n \rightharpoonup x \quad in \, X, \\ &x_n \rightharpoonup x \quad in \, X \implies \{x_n\} is \, bounded \, in \, X and \quad \|x\| \leq \liminf \|x_n\|, \\ &\begin{cases} x_n \rightharpoonup x \quad in \, X \\ \\ f_n \to f \quad in \, X' \end{cases} \implies \langle f, x_n\rangle \to \langle f, x\rangle. \end{aligned}$$

1.5.2 Weak* topology

Definition 1.5.3 (Weak* topology)**.** *We define the weak* topology on X' to be the topology to keep all canonical embeddings $\{J_x\}_{x \in X}$ continuous.*

Remark 1.5.2. *Let refer to this topology by $\sigma(X', X)$:*

1. *In general, we have*
$$\sigma(X', X) \subset \sigma(X', X''),$$
where $\sigma(X', X'')$ is the weak topology on X'.

2. *If X is reflexive then we have*
$$\sigma(X', X) = \sigma(X', X'').$$

1.5.3 Extraction

Theorem 1.5.2 (Banach-Alaougle-Bourbaki)**.** *The closed unit ball:*

$$B_{X'} = \{f \in X', \quad \|f\| \leq 1\},$$

*is compact in the weak*topology $\sigma(X', X)$.*

Corollary 1.5.1. *If X is reflexive then closed unit ball of X is compact in the weak topology $\sigma(X, X')$.*

Theorem 1.5.3 (Kakutani)**.** *A Banach space X is reflexive if and only if B_X is compact in the weak topology $\sigma(X, X')$.*

Corollary 1.5.2. *Every closed subspace of a* ***reflexive*** *Banach space is reflexive.*

Corollary 1.5.3. *A Banach space X is reflexive if and only X' is reflexive.*

Corollary 1.5.4. *Let X be a reflexive Banach space, then any closed convex bounded set is compact in the weak topology $\sigma(X', X)$.*

Theorem 1.5.4. *Let X be a separable Banach space. Then X' is sequentially compact in the weak* topology $\sigma(X', X)$.*

Theorem 1.5.5 (Eberlem-Smulian)**.** *Let X be a reflexive Banach space. Then X is sequentially compact in the weak topology $\sigma(X, X')$.*

1.5.4 Application(case L^p spaces)

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with smooth boundary. We recall the notions of weak and weak-$*$ convergences in the Lebesgue space $L^p(\Omega)$.

Case(1) $p \in (1, \infty)$

- In this case, the dual space of $L^p(\Omega)$ is given by
$$(L^p(\Omega))' = L^q(\Omega),$$
where
$$\frac{1}{p} + \frac{1}{q} = 1.$$
- If $\{f_n\}$ is a sequence in $L^p(\Omega)$ and $f \in L^p(\Omega)$ then
$$f_n \rightharpoonup f \quad in\ L^p(\Omega) \quad \text{iff } \ \forall g \in L^q(\Omega) : \int_\Omega f_n g dx \to \int_\Omega f g dx.$$

Case(2) $p = 1$

In this case we have
$$(L^1(\Omega))' = L^\infty(\Omega).$$
If $\{f_n\}$ is a sequence in $L^1(\Omega)$ and $f \in L^1(\Omega)$ then
$$f_n \rightharpoonup f \quad in\ L^1(\Omega) \quad \text{iff } \ \forall g \in L^\infty(\Omega) : \int_\Omega f_n g dx \to \int_\Omega f g dx.$$

Case(3) $p = +\infty$

- In this case, the dual space of $L^\infty(\Omega)$ is strictly contained in $L^1(\Omega)$. That is
$$(L^\infty(\Omega))' \supsetneq L^1(\Omega),$$
and so the weak convergence in $L^\infty(\Omega)$ does not make sense. However, since $L^\infty(\Omega)$ is the dual space of $L^1(\Omega)$ we can define the weak* convergence in $L^\infty(\Omega)$.
- If $\{f_n\}$ is a sequence in $L^\infty(\Omega)$ and $f \in L^\infty(\Omega)$ then
$$f_n \overset{*}{\rightharpoonup} f \quad in\ L^\infty(\Omega) \quad \text{iff } \ \forall g \in L^1(\Omega) : \int_\Omega f_n g dx \to \int_\Omega f g dx.$$

Theorem 1.5.6. *Let $p \in (1, \infty)$ and $\{f_n\}$ be a bounded sequence in $L^p(\Omega)$. Then we can find a subsequence $\{f_{n_k}\}$ which converges weakly to some $f \in L^p(\Omega)$.*

Theorem 1.5.7. *If $\{f_n\}$ is a bounded subsequence in $L^\infty(\Omega)$ then we can find a subsequence $\{f_{n_k}\}$ which converges weakly* to some $f \in L^\infty(\Omega)$.*

Proposition 1.5.2. *Let $p \in (1, \infty)$ and q be the conjugate of p, let $\{f_n\}$ be a sequence in $L^p(\Omega)$ which converges weakly to some $f \in L^p(\Omega)$. then*

i)
$$\|f\|_{L^p(\Omega)} \leq \liminf \|f_n\|_{L^p(\Omega)}.$$

ii) If:
$$g_n \to g \quad in\ L^q(\Omega),$$
then
$$\int_\Omega f_n g_n \to \int_\Omega f g.$$

iii)
$$\|f_n\|_{L^p(\Omega)} \to \|f\|_{L^p(\Omega)} \implies f_n \to f \quad in\ L^p(\Omega).$$

1.6 Some Inequalities

Lemma 1.6.1 (Hölder inequality)**.** *Let $1 < p \leq \infty$ and let q be the conjugate exponent of p. If $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, then*

$$\int_\Omega |u(x)v(x)|\, dx \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}. \tag{1.26}$$

Lemma 1.6.2 (Young inequality)**.** *Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. For all $a, b \geq 0$ and $\eta > 0$,*

$$ab \leq \eta a^p + C_\eta b^q, \quad C_\eta = \frac{1}{q(\eta p)^{p/q}}. \tag{1.27}$$

Lemma 1.6.3 (Poincaré inequality)**.** *Let $w \in H_0^1(0, L)$. Then there exists a constant $C > 0$ such that*

$$\|w\|_{L^2(0,L)} \leq C \|w_x\|_{L^2(0,L)}.$$

Theorem 1.6.1 (Gronwall inequality)**.** *Let $f \in C([0, T])$, $g \in L^1(0, T)$ with $f(t) \geq 0$ and $g(t) \geq 0$. If*

$$f(t) \leq c + \int_0^t g(s) f(s)\, ds,$$

then

$$f(t) \leq c \exp\left(\int_0^t g(s)\, ds \right), \quad \forall t \in [0, T].$$

CHAPTER 2

STUDY OF THE EXISTENCE AND UNIQUENESS OF THE SOLUTION

2.1 Introduction

In this chapter we study the existence and uniqueness of solutions for the following problem:

$$
\begin{cases}
\rho u_{tt} + EI\left(u_{xxxx} - \displaystyle\int_0^t h(t-s)u_{xxxx}(s)\,ds\right) - Tu_{xx} = 0, \\
\qquad \forall (x,t) \in (0,L) \times [0,\infty), \\
u(0,t) = u_x(0,t) = u_{xx}(L,t) = 0, \\
u_{xxx}(0,t) = u_{xx}(0,t), \\
EI\left(-u_{xxx}(L,t) + \displaystyle\int_0^t h(t-s)u_{xxx}(L,s)\,ds\right) + Tu_x(L,t) \\
\qquad = A(t) - d_S u_t(L,t) - M_S u_{tt}(L,t), \quad \forall t \in [0,\infty), \\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in (0,L).
\end{cases}
\tag{2.1}
$$

The proof is based on the Galerkin method, which consists of three steps:

- Construction of approximate solutions.
- Establishing a priori estimates on these approximate solutions to show existence and uniqueness for the approximated problem.
- Passing to the limit using compactness property.

We define the following spaces:

$$V = \{u \in H^2(0,L) : u(0) = u_x(0) = 0\},$$

endowed with the norm

$$\|w\|_V^2 = \|w_{xx}\|_2^2,$$

$$W = \{u \in V \cap H^4(0,L) : u_{xx}(0) = u_{xx}(L) = 0\},$$

with the norm

$$\|w\|_W^2 = \|w_{xx}\|_2^2 + \|w_{xxxx}\|_2^2,$$

and the scalar product

$$(u,v) = \int_0^L uv dx.$$

In this work, we aim to demonstrate the well-posedness by taking

$$A(t) = d_S u_t(L,t). \tag{2.2}$$

To establish the existence of solutions to problem (2.1), we assume the following hypotheses:

(A1) Hypotheses on the kernel $h(t)$:
Suppose the kernel satisfies

$$\begin{gathered} h : [0,\infty) \to [0,\infty) \text{ is a } C^2 \text{ non-increasing function such that} \\ h \in W^{1,2}(0,\infty) \cap W^{1,1}(0,\infty), \quad h(t) \geq 0, \quad \forall t \geq 0, \\ 1 - \int_0^\infty h(s)\,ds = 1 - \overline{h} > 0. \end{gathered} \tag{2.3}$$

(A2) Hypotheses on the initial conditions:
Suppose $u^0 \in W$, $u^1 \in L^2(0,L)$ satisfy the compatibility condition

$$\begin{cases} EIu_{xxxx}(x,0) - Tu_{xx}(x,0) = 0, \\ u(0,0) = u_x(0,0) = u_{xx}(L,0) = 0, \\ u_{xxx}(0,0) = u_{xx}(0,0), \\ EI(-u_{xxx}(L,0)) + Tu_x(L,0) = -M_S u_{tt}(L,0), \end{cases}$$

equivalently,

$$\begin{cases} EIu^0_{xxxx} - Tu^0_{xx} = 0, \\ u^0(0) = u^0_x(0) = u^0_{xx}(L) = 0, \\ u^0_{xxx}(0) = u^0_{xx}(0), \\ EI(-u^0_{xxx}(L)) + Tu^0_x(L) = -M_S u_{tt}(L,0). \end{cases} \tag{2.4}$$

We define the classical energy associated with system (2.1) by

$$E(t) = E_k(t) + E_p(t), \ t \geq 0,$$

where

$$E_k(t) = \tfrac{M_s}{2}\left[u_t(L,t)\right]^2 + \tfrac{\rho}{2}\textstyle\int_0^L |u_t|^2\,dx, \ t \geq 0$$

represents the kinetic energy and

$$E_p(t) = \tfrac{EI}{2}\textstyle\int_0^L |u_{xx}|^2\,dx + \tfrac{T}{2}\int_0^L |u_x|^2\,dx, \ t \geq 0$$

represents the potential energy.

Our main theorem can be stated as follows.

2.2 Main Result Theorem (Well-Posedness)

The main result concerning existence and uniqueness reads as follows:

Theorem 2.2.1. *Suppose $u_0 \in W$, $u_1 \in L^2(0, L)$ and the hypotheses (2.2)-(2.4) hold. Then the problem (2.1) admits a unique solution such that*

$$u \in L^\infty(0, \infty, V), \quad u' \in L^\infty(0, \infty, V), \quad u'' \in L^\infty(0, \infty, L^2(0, L)).$$

2.3 Proof of the Theorem

2.3.1 Existence of the Solution

The proof is based on three a priori estimates, which are crucial for establishing existence, uniqueness, and the global continuation of the solution. These estimates control the norms of the solution and its derivatives, allowing us to apply compactness arguments and Gronwall's lemma.

Proof. The variational formulation associated with (2.1) is: find $u(t) \in W$ such that

$$\begin{gathered} \rho \int_0^L u_{tt} w dx + EI \int_0^L u_{xx} w_{xx} dx - EI \int_0^L w_{xx} \int_0^t h(t-s) u_{xx}(s) ds dx \\ +T \int_0^L u_x w_x dx + M_S u_{tt} w(L, t) = 0, \quad \forall w \in V. \end{gathered} \tag{2.5}$$

Let $\{w^j\}_{j=1}^\infty$ be an orthogonal system of W such that $\{u^0, u^1\} \in \operatorname{Span}\{w^1, w^2\}$. For $m \in \mathbb{N}$, let

$$W_m = \operatorname{span}\{w^1, w^2, \ldots, w^m\}.$$

We look for the approximate solution

$$u^m(t) = \sum_{j=1}^m k^j(t) w^j,$$

such that for all $w \in W_m$ it satisfies the approximated problem

$$\begin{cases} \rho \int_0^L u_{tt}^m w dx + EI \int_0^L u_{xx}^m w_{xx} dx - EI \int_0^L w_{xx} \int_0^t h(t-s) u_{xx}^m(s) ds dx \\ +T \int_0^L u_x^m w_x dx = -M_S u_{tt}^m w(L, t), \quad \forall w \in W, \\ u^m(0) = u^{0m} \to u^0 \text{ in } W, \\ u_t^m(0) = u^{1m} \to u^1 \text{ in } L^2(0, L). \end{cases} \tag{2.6}$$

By standard ODE methods (such as the Picard-Lindelöf theorem), we can prove the existence of a unique local solution to the approximate problem (2.6) on the interval $[0, t^m)$, where $t^m > 0$ is the maximal time of existence. Then, using the a priori estimates established below, this solution can be extended to the closed interval $[0, T]$ for any $T > 0$ (and in particular to $[0, \infty)$), provided these estimates ensure that the solution does not blow up in finite time. **First a priori estimate:** Replace w by u_t^m in (2.6) to obtain

$$\tfrac{d}{dt}\left[\tfrac{1}{2}\rho\|u_t^m\|^2+\tfrac{1}{2}EI\|u_{xx}^m\|^2+\tfrac{1}{2}T\|u_x^m\|^2+\tfrac{1}{2}M_S u_t^m(L,t)^2\right]$$

$$=\tfrac{d}{dt}EI\textstyle\int_0^L u_{xx}^m\int_0^t h(t-s)u_{xx}^m(s)dsdx \tag{2.7}$$

$$-EI\textstyle\int_0^L u_{xx}^m\int_0^t h'(t-s)u_{xx}^m(s)dsdx-EIh\left(0\right)\|u_{xx}^m\|^2,$$

we have used the fact that

$$\frac{d}{dt}EI\left(\int_0^L u_{xx}^m\int_0^t h(t-s)u_{xx}^m(s)dsdx\right)$$

$$=EI\int_0^L u_{xx}^m\int_0^t h'(t-s)u_{xx}^m(s)dsdx$$

$$+EI\int_0^L u_{xxt}^m\int_0^t h(t-s)u_{xx}^m(s)dsdx+EIh\left(0\right)\|u_{xx}^m\|^2.$$

Now, using Cauchy-Schwarz, Young inequalities, the properties of $h(t)$, we get

$$-EI\left(\int_0^L u_{xx}^m\int_0^t h'(t-s)u_{xx}^m(s)dsdx\right)$$

$$\le\tfrac{EI}{2}\left\|u_{xx}^m\right\|^2-EI\tfrac{1}{2}\left(\textstyle\int_0^L\int_0^t h'\left(t-s\right)\left|u_{xx}^m\left(s\right)\right|^2dsdx\right)^2 \tag{2.8}$$

$$\le\tfrac{EI}{2}\left\|u_{xx}^m\right\|^2-EI\tfrac{1}{2}\left\|h'\right\|_{L^1(0,\infty)}\textstyle\int_0^L\int_0^t h'\left(t-s\right)\left|u_{xx}^m\left(s\right)\right|^2dsdx$$

$$=\tfrac{EI}{2}\left\|u_{xx}^m\right\|^2-EI\tfrac{1}{2}\left\|h'\right\|_{L^1(0,\infty)}\textstyle\int_0^t h'\left(t-s\right)\left\|u_{xx}^m\left(s\right)\right\|^2ds.$$

We replace (2.8) in (2.7), we get

$$\tfrac{d}{dt}\left[\tfrac{1}{2}\rho\|u_t^m\|^2+\tfrac{1}{2}EI\|u_{xx}^m\|^2+\tfrac{1}{2}T\|u_x^m\|^2+\tfrac{1}{2}M_S u_t^m(L,t)^2\right]$$

$$\le\tfrac{d}{dt}EI\textstyle\int_0^L u_{xx}^m\int_0^t h(t-s)u_{xx}^m(s)dsdx$$

$$+\tfrac{EI}{2}\left\|u_{xx}^m\right\|^2+EI\tfrac{1}{2}\left\|h'\right\|_{L^1(0,\infty)}\textstyle\int_0^t h'\left(t-s\right)\left\|u_{xx}^m\left(s\right)\right\|^2ds \tag{2.9}$$

$$-EIh\left(0\right)\|u_{xx}^m\|^2.$$

we have

$$\int_0^\tau\int_0^t h'\left(t-s\right)\left\|u_{xx}^m\left(s\right)\right\|^2dsdt=\int_0^\tau\left(\left\|u_{xx}^m\left(s\right)\right\|^2\int_s^\tau h'\left(t-s\right)dt\right)ds$$

if we set $r=t-s$, then

$$\int_s^\tau h'\left(t-s\right)dt=\int_0^{\tau-s}h'\left(r\right)dr\le\int_0^\infty\left|h'\left(r\right)\right|dr=\left\|h'\right\|_{L^1(0,\infty)},$$

we get

$$\int_0^\tau \int_0^t h'(t-s)\left\|u_{xx}^m(s)\right\|^2 dsdt \le ||h'||_{L^1(0,\infty)} \int_0^t \left\|u_{xx}^m(s)\right\|^2 ds. \tag{2.10}$$

Integrating (2.9) over $(0,t)$ and using (2.10), we obtain

$$\begin{array}{c} \left[\frac{1}{2}\rho\left\|u_t^m\right\|^2+\frac{1}{2}EI\left\|u_{xx}^m\right\|^2+\frac{1}{2}T\left\|u_x^m\right\|^2+\frac{1}{2}Msu_t^m(L,t)^2\right] \\ \le \frac{1}{2}E(0)+EI\int_0^L u_{xx}^m\int_0^t h(t-s)u_{xx}^m(s)dsdx \\ +\frac{EI}{2}\int_0^t ||u_{xx}^m||^2 ds+EI\frac{1}{2}||h'||_{L^1(0,\infty)}\left(||h'||_{L^1(0,\infty)}\int_0^t\left\|u_{xx}^m(s)\right\|^2 ds\right) \\ -EIh(0)\int_0^t\|u_{xx}^m\|^2 ds, \end{array} \tag{2.11}$$

as we have

$$\int_0^L u_{xx}^m \int_0^t h(t-s)u_{xx}^m(s)dsdx$$

$$\le \epsilon\,||u_{xx}^m||^2+\frac{1}{4\epsilon}||h||_{L^1(0,\infty)}||h||_{L^\infty(0,\infty)}\int_0^t\left\|u_{xx}^m(s)\right\|^2 ds,$$

then (2.9) becomes

$$\left[\frac{1}{2}\rho\left\|u_t^m\right\|^2+\frac{1}{2}EI\left\|u_{xx}^m\right\|^2+\frac{1}{2}T\left\|u_x^m\right\|^2+\frac{1}{2}Msu_t^m(L,t)^2\right]$$

$$\le E(0)+EI\epsilon\,||u_{xx}^m||^2+EI\frac{1}{4\epsilon}||h||_{L^1(0,\infty)}||h||_{L^\infty(0,\infty)}\int_0^t\left\|u_{xx}^m(s)\right\|^2 ds$$

$$+\frac{EI}{2}\int_0^t ||u_{xx}^m||^2 ds+EI\frac{1}{2}||h'||_{L^1(0,\infty)}\left(||h'||_{L^1(0,\infty)}\int_0^t\left\|u_{xx}^m(s)\right\|^2 ds\right)$$

$$-EIh(0)\int_0^t\|u_{xx}^m\|^2 ds,$$

choosing ϵ sufficiently small, we get

$$\left[\frac{1}{2}\rho\left\|u_t^m\right\|^2+\frac{1}{2}EI\left\|u_{xx}^m\right\|^2+\frac{1}{2}T\left\|u_x^m\right\|^2+\frac{1}{2}Msu_t^m(L,t)^2\right]$$

$$\le E(0)+EI\frac{1}{4\epsilon}||h||_{L^1(0,\infty)}||h||_{L^\infty(0,\infty)}\int_0^t\left\|u_{xx}^m(s)\right\|^2 ds$$

$$\begin{array}{c} +\frac{EI}{2}\int_0^t ||u_{xx}^m||^2 ds+EI\frac{1}{2}||h'||_{L^1(0,\infty)}\left(||h'||_{L^1(0,\infty)}\int_0^t\left\|u_{xx}^m(s)\right\|^2 ds\right) \\ -EIh(0)\int_0^t\|u_{xx}^m\|^2 ds. \end{array}$$

That is

$$\left\|u_t^m\right\|^2+\left\|u_{xx}^m\right\|^2+\left\|u_x^m\right\|^2+u_t^m(L,t)^2$$

$$\le C_1+C_2\int_0^t\left(\left\|u_{xx}^m(s)\right\|^2\right).$$

Applying Gronwall's lemma yields

$$\|u_t^m\|^2+\|u_{xx}^m\|^2+\|u_x^m\|^2+u_t^m(L,t)^2 \le M_1, \tag{2.12}$$

where M_1 is a positive constant.

We thus obtain a local solution $u^m(t)$ on an interval $[0,T]$. The a priori estimates derived above allow us to extend this solution globally, that is, for $T=\infty$.

Second a priori estimate:

$$\rho\int_0^L u_{tt}^m w dx + EI\int_0^L u_{xx}^m w_{xx} dx - EI\int_0^L w\int_0^t h(t-s)u_{xx}^m(s)dsdx + T\int_0^L u_x^m w_x dx$$
$$= -M_s u_{tt}^m w\left(L,t\right), \quad \forall w \in V. \tag{2.13}$$

First, we estimate $u_{tt}^m(0)$ in L^2. To this end, setting $t=0$ and $w = u_{tt}^m(0)$ in (2.13), we get

$$\rho\int_0^L u_{tt}^m u_{tt}^m(0)dx + EI\int_0^L u_{xx}^m\left(0\right)u_{ttxx}^m(0) + T\int_0^L u_x^m\left(0\right)u_{xtt}^m(0)dx$$
$$= -M_s u_{tt}^m u_{tt}^m\left(L,0\right),$$

then

$$\rho\left\|u_{tt}^m\left(0\right)\right\|^2 + EI\left(u_{xx}^0, u_{ttxx}^m(0)\right) + T\left(u_x^0, u_{xtt}^m(0)\right)$$
$$= -M_s u_{tt}^m u_{tt}^m\left(L,0\right). \tag{2.14}$$

We have

$$T\left(u_x^0, u_{xtt}^m(0)\right) = -T\left(u_{xx}^0, u_{tt}^m(0)\right) + Tu_x\left(L,0\right)u_{tt}^m\left(L,0\right),$$

and

$$EI\left(u_{xx}^0, u_{xxtt}^m(0)\right) = -EI\left(u_{xxx}^0\left(L\right), u_{tt}^m(L,0)\right) + EI\left(u_{xxxx}^0\left(L\right), u_{tt}^m(0)\right),$$

then (2.14) becomes

$$\rho\left\|u_{tt}^m\left(0\right)\right\|^2 - EI\left(u_{xxx}^0\left(L\right), u_{tt}^m(L,0)\right) + EI\left(u_{xxxx}^0\left(L\right), u_{tt}^m(0)\right)$$
$$-T\left(u_{xx}^0, u_{tt}^m(0)\right) + Tu_x\left(L,0\right)u_{tt}^m\left(L,0\right)$$
$$= -M_s u_{tt}^m u_{tt}^m\left(L,0\right), \ \forall\ w \in V.$$

According to the compatibility condition (2.4)

$$EI(-u_{xxx}^0(L)) + Tu_x^0(L) = -M_S u_{tt}(L,0)$$

i. e

$$EI\left(-u_{xxx}^0\left(L\right)\right) + Tu_x^0\left(L\right) = -M_S u_{tt}^0\left(L\right).$$

Then

$$\rho\left\|u_{tt}^m\left(0\right)\right\|^2 + EI\left(u_{xxxx}^0\left(L\right), u_{tt}^m(0)\right) - T\left(u_{xx}^0, u_{tt}^m(0)\right) = 0.$$

We deduce that

$$\rho\left\|u_{tt}^m\left(0\right)\right\|^2 \leq EI\left\|u_{xxxx}^0\left(L\right)\right\|\left\|u_{tt}^m\left(0\right)\right\| + T\left\|u_{xx}^0\right\|\left\|u_{tt}^m(0)\right\|,$$

$$\rho \left\| u_{tt}^{m}(0) \right\| \leq EI \left\| u_{xxxx}^{0}(L) \right\| + T \left\| u_{xx}^{0} \right\|,$$

from (2.13), we get

$$\left\| u_{tt}^{m}(0) \right\|^{2} \leq M_{2}, \forall m \in \mathbb{N}.$$

where M_2 denoting a constant independent of m.

We now estimate the terms u_{tt} and u_{xxt} in the L^2. Differentiating (2.6) with respect to time and multiplying by $u_{tt}(t)$, we obtain

$$\begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt}\left[\rho \left\| u_{tt}^{m} \right\|^{2} + \left\| u_{xxt}^{m} \right\|^{2} + T \left\| u_{xt}^{m} \right\|^{2} + Msu_{tt}^{m}(L,t)^{2}\right] + EIh(0)\left\| u_{xxt}^{m} \right\|^{2} \\ &= \tfrac{d}{dt} EI \int_{0}^{L} u_{xxt}^{m} \int_{0}^{t} h'(t-s) u_{xx}^{m}(s) ds dx - EI \int_{0}^{L} u_{xxt}^{m} \int_{0}^{t} h''(t-s) u_{xx}^{m}(s) ds dx \\ &\quad - EIh'(0) \int_{0}^{L} u_{xx}^{m} u_{xxt}^{m} dx - EIh(0) \tfrac{d}{dt} \int_{0}^{L} u_{xx}^{m} u_{xxt}^{m} dx, \end{aligned} \tag{2.15}$$

using (2.3), we get

$$\begin{aligned} &EI \int_{0}^{L} u_{xxt}^{m} \int_{0}^{t} h''(t-s) u_{xx}^{m}(s) ds dx \\ &\leq \tfrac{1}{4\varepsilon} \left\| u_{xxt}^{m} \right\|^{2} + \varepsilon \left\| h'' \right\|_{L^{1}(0,\infty)}^{2} \int_{0}^{t} h''(t-s) \left\| u_{xx}^{m}(s) \right\|^{2} ds. \end{aligned} \tag{2.16}$$

By using (2.16), and integrating (2.15) over $(0,t)$, we get

$$\begin{aligned} &\tfrac{1}{2}\left[\rho \left\| u_{tt}^{m} \right\|^{2} + \left\| u_{xxt}^{m} \right\|^{2} + T \left\| u_{xt}^{m} \right\|^{2} + Msw_{tt}^{m}(L,t)^{2}\right] + EIh(0)\left\| u_{xxt}^{m} \right\|^{2} \\ &\leq EI \int_{0}^{L} u_{xxt}^{m} \int_{0}^{t} h'(t-s) u_{xx}^{m}(s) ds dx - EIh'(0) \int_{0}^{t} \int_{0}^{L} u_{xx}^{m} u_{xxt}^{m} dx ds \\ &\quad + EI\varepsilon \left\| h'' \right\|_{L^{1}(0,\infty)}^{2} \int_{0}^{t} \left\| u_{xx}^{m}(s) \right\|^{2} ds + EI \tfrac{1}{4\varepsilon} \int_{0}^{t} \left\| u_{xxt}^{m}(s) \right\|^{2} ds \\ &\quad - EIh(0) \int_{0}^{L} u_{xx}^{m} u_{xxt}^{m} dx. \end{aligned} \tag{2.17}$$

Since we have

$$\begin{aligned} EIh(0) \int_{0}^{L} u_{xx}^{m} u_{xxt}^{m} dx &\leq \tfrac{(EIh(0))^{2}}{4\varepsilon} \left\| u_{xx}^{m} \right\|^{2} + \varepsilon \left\| u_{xxt}^{m} \right\|^{2}, \\ \left| EIh'(0) \int_{0}^{t} \int_{0}^{L} u_{xx}^{m} u_{xxt}^{m} dx ds \right| &\leq \varepsilon (EIh'(0))^{2} \int_{0}^{t} \left\| u_{xx}^{m}(s) \right\|^{2} ds + \tfrac{1}{4\varepsilon} \int_{0}^{t} \left\| u_{xxt}^{m}(s) \right\|^{2} ds, \\ EI \int_{0}^{L} u_{xxt}^{m} \int_{0}^{t} h'(t-s) u_{xx}^{m}(s) ds dx &\leq \tfrac{(EI)^{2}}{4\varepsilon} \left\| h' \right\|_{L^{1}(0,\infty)} \left\| h' \right\|_{L^{\infty}(0,\infty)} \int_{0}^{t} \left\| u_{xx}^{m}(s) \right\|^{2} ds \\ &\quad + \varepsilon \left\| u_{xxt}^{m}(s) \right\|^{2}, \end{aligned} \tag{2.18}$$

We replace (2.18) in (2.17), yields

$$\tfrac{1}{2}\left[\rho\left\|u_{tt}^{m}\right\|^{2}+\left\|u_{xxt}^{m}\right\|^{2}+T\left\|u_{xt}^{m}\right\|^{2}+Msw_{tt}^{m}\left(L,t\right)^{2}\right]+EIh(0)\left\|u_{xxt}^{m}\right\|^{2}$$

$$\leq\tfrac{(EI)^{2}}{4\varepsilon}\left\|h'\right\|_{L^{1}(0,\infty)}\left\|h'\right\|_{L^{\infty}(0,\infty)}\int_{0}^{t}\left\|u_{xx}^{m}\left(s\right)\right\|^{2}ds+\varepsilon\left\|u_{xxt}^{m}\left(s\right)\right\|^{2}$$

$$+\varepsilon\left(EIh'(0)\right)^{2}\int_{0}^{t}\left\|u_{xx}^{m}\left(s\right)\right\|^{2}ds+\tfrac{1}{4\varepsilon}\int_{0}^{t}\left\|u_{xxt}^{m}\left(s\right)\right\|^{2}ds$$

$$+EI\varepsilon\left\|h''\right\|_{L^{1}(0,\infty)}^{2}\int_{0}^{t}\left\|u_{xx}^{m}(s)\right\|^{2}ds+EI\tfrac{1}{4\varepsilon}\int_{0}^{t}\left\|u_{xxt}^{m}\left(s\right)\right\|^{2}ds$$

$$+\tfrac{(EIh(0))^{2}}{4\varepsilon}\left\|u_{xx}^{m}\right\|^{2}+\varepsilon\left\|u_{xxt}^{m}\right\|^{2},$$

for ε sufficiently small and taking into account (2.12), we obtain

$$\tfrac{1}{2}\left[\rho\left\|u_{tt}^{m}\right\|^{2}+\left\|u_{xxt}^{m}\right\|^{2}+T\left\|u_{xt}^{m}\right\|^{2}+Msw_{tt}^{m}\left(L,t\right)^{2}\right]+EIh(0)\int_{0}^{t}\left\|u_{xxt}^{m}\left(s\right)\right\|^{2}ds$$

$$\leq c+c'\int_{0}^{t}\left(\left\|u_{tt}^{m}\right\|^{2}+\left\|u_{xxt}^{m}\left(s\right)\right\|^{2}+\left\|u_{xt}^{m}\right\|^{2}+u_{tt}^{m}(L,s)^{2}\right)ds.$$

where c, c' are positive constants. Applying Gronwall's lemma yields:

$$\left\|u_{tt}^{m}\right\|^{2}+\left\|u_{xxt}^{m}\right\|^{2}+\left\|u_{xt}^{m}\right\|^{2}+u_{tt}^{m}\left(L,t\right)^{2}\leq M_{3}\tag{2.19}$$

where M_3 is a positive constant. From the estimates (2.12) et (2.19), we get

$$\begin{cases} u^{m} \text{ is bounded in } L^{\infty}(0,\infty,V)\\ u_{t}^{m} \text{ is bounded in } L^{\infty}(0,\infty,V)\\ u_{tt}^{m} \text{ is bounded in } L^{\infty}(0,\infty,L^{2}\left(\Omega\right))\end{cases}$$

Passing to the limit: Consequently, there exists a subsequence u^m, that satisfies

$$\begin{cases} u^{m} \rightarrow^{*} u \text{ in } L^{\infty}(0,\infty,V)\\ u_{t}^{m} \rightarrow^{*} u_{t} \text{ in } L^{\infty}(0,\infty,V)\\ u_{tt}^{m} \rightarrow^{*} u_{tt} \text{ in } L^{\infty}(0,\infty,L^{2}\left(\Omega\right))\end{cases}$$

"Using the previous results, we can pass to the limit in (2.9) and conclude the existence of global solutions on $[0, T]$. To prove uniqueness, we employ standard arguments together with Gronwall's inequality.

solution follows from standard arguments combined with Gronwall's inequality.
□

2.3.2 Uniqueness of the solution

Proof. We now demonstrate that the solution is unique. To this end, taking the relation (2.2), let u_1 and u_2 be two solutions of problem (2.1). Denote $w = u_1 - u_2$. Then w satisfies, for any test function $v \in V$, the identity:

$$\begin{aligned} \rho \int_0^L w_{tt} v \, dx + EI \int_0^L w_{xx} v_{xx} \, dx - EI \int_0^L v_{xx} \int_0^t h(t-s) w_{xx}(s) \, ds \, dx \\ + T \int_0^L w_x v_x \, dx + M_S w_{tt}(L,t) v(L) = 0, \quad \forall v \in V, \end{aligned} \tag{2.20}$$

with initial conditions

$$w(0) = 0, \qquad w_t(0) = 0.$$

Choose $v = w_t(t)$ in (2.20) and use the facts that

$$\begin{aligned} \rho \int_0^L w_{tt} w_t \, dx &= \frac{\rho}{2} \frac{d}{dt} \|w_t\|^2_{L^2(0,L)}, \\ EI \int_0^L w_{xx} w_{xxt} \, dx &= \frac{EI}{2} \frac{d}{dt} \|w_{xx}\|^2_{L^2(0,L)}, \\ T \int_0^L w_x w_{xt} \, dx &= \frac{T}{2} \frac{d}{dt} \|w_x\|^2_{L^2(0,L)}, \\ M_S \, w_{tt}(L,t) \, w_t(L,t) &= \frac{M_S}{2} \frac{d}{dt} \left(w_t(L,t)^2 \right). \end{aligned}$$

Then we obtain

$$\begin{aligned} \frac{d}{dt} \left(\frac{\rho}{2} \|w_t\|^2 + \frac{EI}{2} \|w_{xx}\|^2 + \frac{T}{2} \|w_x\|^2 + \frac{M_S}{2} w_t(L,t)^2 \right) \\ = EI \int_0^L w_{xxt}(x,t) \int_0^t h(t-s) w_{xx}(x,s) \, ds \, dx. \end{aligned} \tag{2.21}$$

Noticing that

$$\begin{aligned} \int_0^L w_{xxt}(x,t) \int_0^t h(t-s) w_{xx}(x,s) \, ds \, dx \\ &= \frac{d}{dt} \int_0^L w_{xx}(x,t) \int_0^t h(t-s) w_{xx}(x,s) \, ds \, dx \\ &\quad - h(0) \|w_{xx}(\cdot,t)\|^2_{L^2(0,L)} \\ &\quad - \int_0^L w_{xx}(x,t) \int_0^t h'(t-s) w_{xx}(x,s) \, ds \, dx, \end{aligned}$$

then (2.21) yields

$$\begin{aligned} \frac{d}{dt} \Bigg(\frac{\rho}{2} \|w_t\|^2 + \frac{EI}{2} \|w_{xx}\|^2 + \frac{T}{2} \|w_x\|^2 + \frac{M_S}{2} w_t(L,t)^2 \\ - EI \int_0^L w_{xx} \int_0^t h(t-s) w_{xx}(s) \, ds \, dx \Bigg) \end{aligned}$$

$$= -EIh(0)\|w_{xx}\|^2 - EI\int_0^L w_{xx}\int_0^t h'(t-s)w_{xx}(s)\,ds\,dx. \quad (2.22)$$

Integrating (2.22) from 0 to t yields

$$\begin{aligned}
&\frac{\rho}{2}\|w_t\|^2 + \frac{EI}{2}\|w_{xx}\|^2 + \frac{T}{2}\|w_x\|^2 + \frac{M_S}{2}w_t(L,t)^2 \\
&= EI\int_0^L w_{xx}\int_0^t h(t-s)w_{xx}(s)\,ds\,dx \\
&\quad - EIh(0)\int_0^t \|w_{xx}\|^2\,ds \\
&\quad - EI\int_0^t\int_0^L w_{xx}(s)\int_0^s h'(s-\tau)w_{xx}(\tau)\,d\tau\,dx\,ds.
\end{aligned} \quad (2.23)$$

We now estimate the right-hand side of (2.23). For the first term, applying the Cauchy-Schwarz inequality in x and then in s gives, for any $\varepsilon > 0$,

$$\begin{aligned}
\left|\int_0^L w_{xx}\int_0^t h(t-s)w_{xx}(s)\,ds\,dx\right| &\le \frac{\varepsilon}{2}\|w_{xx}\|^2 \\
&\quad + \frac{1}{2\varepsilon}\|h\|_{L^1(0,\infty)}\int_0^t |h(t-s)|\|w_{xx}(s)\|^2\,ds.
\end{aligned} \quad (2.24)$$

For the third term, we have

$$\left|\int_0^t\int_0^L w_{xx}(s)\int_0^s h'(s-\tau)w_{xx}(\tau)\,d\tau\,dx\,ds\right| \le \|h'\|_{L^1(0,\infty)}\int_0^t \|w_{xx}(s)\|^2\,ds. \quad (2.25)$$

Insert estimates (2.24) and (2.25) into (2.23) and choose $\varepsilon = EI\,h(0)$ (which is positive by assumption (H1)). Then we obtain

$$\begin{aligned}
&\frac{\rho}{2}\|w_t\|^2 + \frac{EI}{2}\|w_{xx}\|^2 + \frac{T}{2}\|w_x\|^2 + \frac{M_S}{2}w_t(L,t)^2 \\
&\qquad \le C\int_0^t \Big(\|w_{xx}(s)\|^2 + \|w_t(s)\|^2 + w_t(L,s)^2\Big)\,ds,
\end{aligned} \quad (2.26)$$

where C is a positive constant independent of t. Set

$$\Phi(t) = \frac{\rho}{2}\|w_t\|^2 + \frac{EI}{2}\|w_{xx}\|^2 + \frac{T}{2}\|w_x\|^2 + \frac{M_S}{2}w_t(L,t)^2.$$

then, (2.26) becomes

$$\Phi(t) \le C\int_0^t \Phi(s)\,ds,$$

Since $\Phi(0) = 0$, Gronwall's lemma implies $\Phi(t) = 0$ for all $t \ge 0$. Consequently,

$$\|w_t\| = 0, \quad \|w_{xx}\| = 0, \quad w_t(L,t) = 0 \quad \forall t \ge 0.$$

From $\|w_{xx}\| = 0$ and the boundary conditions $w(0) = 0$, $w_x(0) = 0$, we deduce $w_x = 0$ and then $w = 0$ almost everywhere. Hence $u_1 = u_2$, proving uniqueness. □ □

CHAPTER 3

EXPONENTIAL STABILIZATION OF A SYSTEM WITH A RELAXATION KERNEL

3.1 Introduction

In this section, we investigate the exponential stabilization of an Euler–Bernoulli beam model involving a relaxation kernel that accounts for memory effects in the material. Our main objective is to analyze the asymptotic behavior of the corresponding solutions as time tends to infinity. In particular, we aim to establish sufficient conditions ensuring that the total energy of the system decays at an exponential rate.

To achieve this goal, we rely on suitable energy functionals combined with Lyapunov techniques. These tools allow us to capture the dissipative mechanisms induced by the relaxation kernel and to derive explicit decay estimates.

In this section, we study the stabilization of the following Euler–Bernoulli beam problem:

$$\begin{cases} \rho u_{tt} + EI\left(u_{xxxx} - \displaystyle\int_0^t h(t-s)u_{xxxx}(s)\,ds\right) - Tu_{xx} = 0, \\ \qquad \forall (x,t) \in (0,L)\times[0,\infty), \\ u(0,t) = u_x(0,t) = u_{xx}(L,t) = 0, \\ u_{xxx}(0,t) = u_{xx}(0,t), \\ EI\left(-u_{xxx}(L,t) + \displaystyle\int_0^t h(t-s)u_{xxx}(L,s)\,ds\right) + Tu_x(L,t) \\ \qquad = A(t) - d_S u_t(L,t) - M_S u_{tt}(L,t), \quad \forall t \in [0,\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in (0,L). \end{cases} \tag{3.1}$$

Our goal in this section is to state and prove the exponential decay of solutions under a suitable control. We start by our assumptions on the kernel $h(t)$:
(H1) $h\,(t) \geq 0$ for all $t \geq 0$ and $0 < \overline{h} = \int_0^\infty h(s)ds < 1$.
(H2) $0 < h'\,(t) + \gamma h\,(t) \leq \xi\,(t)$ for almost all $t > 0$ where γ is a positive constant and $\xi(t)$ is a nonnegative function.

We denote by $\overline{\xi}$ the expression $\overline{\xi} = \int_0^\infty \xi(s)ds$. To stabilize the system 3.1, we propose the following boundary control law, for $t \geq 0$

$$A(t) = \frac{-K}{u_t(L,t)}\left(u_{xxx}^2(L,t) + u^2(L,t) + \left(\int_0^t h(t-s)u_{xxx}(L,s)\,ds\right)^2\right), \tag{3.2}$$

where K is a positive constant. The velocity of the displacement is assumed nonzero or else there is no need for control.

Theorem 2: *Under hypotheses (H1), (H2), the control law (3.2) and the condition*

$$\int_0^\infty e^{\beta s}\xi(s)\,ds < \infty$$

for some $\beta > 0$, the energy of the system (3.1) decays to zero exponentially; that is, there exist positive constants C and b such that

$$E(t) \leq Ce^{-bt},\ t \geq 0$$

provided that $\overline{\xi}_\sigma = \int_0^\infty e^{\beta s}\xi(s)\,ds$ is small enough.

Remark 3.1.1. *The existence and uniqueness have been established in the previous chapter by taking the condition (3.2).*

3.2 Modified energy

The energy associated with the problem (3.1) is defined by

$$E(t) = E_k(t) + E_p(t),\ t \geq 0,$$

where

$$E_k(t) = \frac{M_s}{2}\left[u_t(L,t)\right]^2 + \frac{\rho}{2}\int_0^L |u_t|^2\,dx,\ t \geq 0 \tag{3.3}$$

represents the kinetic energy

$$E_p(t) = \frac{EI}{2}\int_0^L |u_{xx}|^2\,dx + \frac{T}{2}\int_0^L |u_x|^2\,dx,\ t \geq 0 \tag{3.4}$$

represents the potential energy.

Lemma 3.2.1. *We have the following equality*

$$\begin{array}{c} \frac{1}{2}\frac{d}{dt}\left(h\Box u_{xx}\right)(t) \\ = -\int_0^L u_{xxt}\int_0^t h(t-s)u_{xx}(s)\,dsdx + \frac{1}{2}\left(h'\Box u_{xx}\right)(t) \\ +\frac{1}{2}\frac{d}{dt}\left\{\int_0^t h(s)ds\int_0^L |u_{xx}|^2\,dx\right\} - \frac{1}{2}h(t)\int_0^L |u_{xx}|^2\,dx \end{array}$$

where

$$\left(h\Box u_{xx}\right)(t) = \int_0^L\int_0^t h(t-s)\left|u_{xx}(s) - u_{xx}(t)\right|^2\,dsdx,\ t \geq 0. \tag{3.5}$$

Proof. See [6] □

Lemma 3.2.2. *The modified energy associated to problem (3.1) is given by*

$$\mathcal{E}(t) = \frac{EI}{2}\left(1 - \int_0^t h(s)\, ds\right) \int_0^L |u_{xx}|^2 \, dx + \frac{T}{2} \int_0^L |u_x|^2 \, dx + \frac{EI}{2} (h \square u_{xx})(t)$$

$$+ \frac{M_s}{2} [u_t(L,t)]^2 + \frac{\rho}{2} \int_0^L |u_t|^2 \, dx, \ \ t \geq 0.$$

such that

$$\begin{aligned} \mathcal{E}'(t) = -\frac{EI}{2} h(t) \int_0^L |u_{xx}|^2 \, dx + \frac{EI}{2} (h' \square u_{xx})(t) - d_S [u_t(L,t)]^2 \\ + A(t)\, u_t(L,t), \ \ t \geq 0. \end{aligned} \tag{3.6}$$

Proof. By multiplying the first equation of the system (3.1) by $u_t(t)$,

$$\begin{aligned} \rho \int_0^L u_{tt} u_t dx + EI \int_0^L u_{xxxx} u_t dx - EI \int_0^L u_t \int_0^t h(t-s) u_{xxxx}(s) ds dx \\ -T \int_0^L u_{xx} u_t dx = 0, \end{aligned} \tag{3.7}$$

Integrating over $[0, L]$ leads to the following relation:

$$\begin{aligned} EI \int_0^L u_{xxxx} u_t dx &= EI \left(u_{xxx} u_t \, |_0^L - \int_0^L u_{xxx} u_{tx} dx \right) \\ &= EI \left(u_{xxx}(L,t)\, u_t(L,t) - u_{xxx}(0,t)\, u_t(0,t) - \int_0^L u_{xxx} u_{tx} dx \right), \end{aligned}$$

using the boundary conditions

$$u(0,t) = u_x(0,t) = u_{xx}(L,t) = 0,$$

we get

$$EI \int_0^L u_{xxxx} u_t dx = EI \left(u_{xxx}(L,t)\, u_t(L,t) - \int_0^L u_{xxx} u_{tx} dx \right). \tag{3.8}$$

On the other hand, we have

$$EI \int_0^L u_t \int_0^t h(t-s) u_{xxxx}(s) ds dx$$

$$= EI \left(u_t \int_0^t h(t-s) u_{xxx}(s) ds \, |_0^L - \int_0^L u_{tx} \int_0^t h(t-s) u_{xxxx}(s) ds dx \right) \tag{3.9}$$

$$= EI \left(\left(u_t(L,t) \int_0^t h(t-s) u_{xxx}(L,s) ds \right) - \int_0^L u_{tx} \int_0^t h(t-s) u_{xxxx}(s) ds dx \right).$$

We have

$$\begin{aligned} T \int_0^L u_{xx} u_t dx &= T \left(u_x u_t \, |_0^L - \int_0^L u_x u_{tx} dx \right) \\ &= T \left(u_x(L,t)\, u_t(L,t) - \int_0^L u_x u_{tx} dx \right). \end{aligned} \tag{3.10}$$

By substituting the equations into (3.8)-(3.10) into (3.7), we get

$$\begin{aligned} \rho \frac{1}{2} \frac{d}{dt} \|u_t\|^2 + EI \left(u_{xxx}(L,t)\, u_t(L,t) - \int_0^L u_{xxx} u_{tx} dx \right) \\ -EI \left(u_t(L,t) \int_0^t h(t-s) u_{xxx}(L,s) ds - \int_0^L u_{tx} \int_0^t h(t-s) u_{xxx}(s) ds dx \right) \\ -T \left(u_x(L,t)\, u_t(L,t) - \int_0^L u_x u_{tx} dx \right) = 0, \end{aligned} \tag{3.11}$$

we have

$$\int_0^L u_{xxx}u_{tx}dx = u_{xx}u_{tx}\mid_0^L - \int_0^L u_{xx}u_{txx}dx$$

$$= u_{xx}u_{tx}\mid_0^L - \int_0^L u_{xx}u_{txx}dx$$

$$= u_{xx}(L,t)\,u_{tx}(L,t) - u_{xx}(0,t)\,u_{tx}(0,t) - \int_0^L u_{xx}u_{txx}dx,$$

then

$$-\int_0^L u_{xxx}u_{tx}dx = -u_{xx}(L,t)\,u_{tx}(L,t) + u_{xx}(0,t)\,u_{tx}(0,t) + \int_0^L u_{xx}u_{txx}dx, \quad (3.12)$$

and

$$\int_0^L u_{tx}\int_0^t h(t-s)u_{xxx}(s)dsdx$$

$$= u_{tx}\int_0^t h(t-s)u_{xx}(s)ds\mid_0^L - \int_0^L u_{txx}\int_0^t h(t-s)u_{xx}(s)dsdx \quad (3.13)$$

$$= u_{tx}(L,t)\int_0^t h(t-s)u_{xx}(L,s)\,ds - \int_0^L u_{txx}\int_0^t h(t-s)u_{xx}(s)dsdx;$$

We have

$$\frac{d}{dt}\|u_x\|^2 = \frac{d}{dt}\int_0^L |u_x|^2\,dx = 2\int_0^L u_xu_{tx}dx, \quad (3.14)$$

$$T\int_0^L u_xu_{tx}dx \;= \frac{1}{2}T\frac{d}{dt}\|u_x\|^2, \quad (3.15)$$

$$\frac{d}{dt}\|u_{xx}\|^2 = \frac{d}{dt}\int_0^L |u_{xx}|^2\,dx = 2\int_0^L u_{xx}u_{txx}dx, \quad (3.16)$$

$$EI\int_0^L u_{xx}u_{txx}dx \;= \frac{1}{2}EI\frac{d}{dt}\|u_{xx}\|^2. \quad (3.17)$$

Replacing (3.12), (3.13), and (3.14)-(3.17) in (3.11), we get

$$\tfrac{1}{2}\rho\tfrac{d}{dt}\|u_t\|^2 + EIu_{xxx}(L,t)\,u_t(L,t) - EIu_{xx}(L,t)\,u_{tx}(L,t) + \tfrac{1}{2}EI\tfrac{d}{dt}\|u_{xx}\|^2$$

$$-EIu_t(L,t)\int_0^t h(t-s)u_{xxx}(L,s)ds - EIu_{tx}(L,t)\int_0^t h(t-s)u_{xx}(L,s)\,ds$$

$$-EI\int_0^L u_{txx}\int_0^t h(t-s)u_{xx}(s)dsdx - Tu_x(L,t)\,u_t(L,t) - \tfrac{1}{2}T\tfrac{d}{dt}\|u_x\|^2 = 0, \quad (3.18)$$

from the third third equation of problem (3.1) we can write

$$EIu_t(L,t)\int_0^t h(t-s)\,u_{xxx}(L,s)\,ds$$

$$+Tu_t(L,t)\,u_x(L,t) - u(t)\,u_t(L,t) + u_t(L,t)\,d_S u_t(L,t) \quad (3.19)$$

$$+u_t(L,t)\,M_S u_{tt}(L,t) = EIu_t(L,t)\,u_{xxx}(L,t) \quad \forall t\in[0,\infty).$$

from the lemma 3.2.1, we have

$$-\int_0^L u_{xxt}\int_0^t h(t-s)u_{xx}(s)dsdx = -\tfrac{1}{2}(h'\Box u_{xx})(t)$$

$$-\tfrac{1}{2}\tfrac{d}{dt}\left\{\left(\int_0^t h(s)ds\right)\int_0^L |u_{xx}|^2\,dx\right\} + \tfrac{1}{2}h(t)\int_0^L |u_{xx}|^2\,dx + \tfrac{1}{2}\tfrac{d}{dt}(h\Box u_{xx})(t). \quad (3.20)$$

If we replace (3.19) and (3.20) in (3.18), we get

$$\begin{aligned}
&\tfrac{1}{2}\rho\tfrac{d}{dt}\left\|u_t\right\|^2+EIu_t\left(L,t\right)\textstyle\int_0^t h\left(t-s\right)u_{xxx}\left(L,s\right)ds\\
&+Tu_t\left(L,t\right)u_x\left(L,t\right)-u\left(t\right)u_t\left(L,t\right)+u_t\left(L,t\right)d_Su_t\left(L,t\right)\\
&+u_t\left(L,t\right)M_Su_{tt}\left(L,t\right)-EIu_{xx}\left(L,t\right)u_{tx}\left(L,t\right)+\tfrac{1}{2}EI\tfrac{d}{dt}\left\|u_{xx}\right\|^2\\
&-EIu_t\left(L,t\right)\textstyle\int_0^t h(t-s)u_{xxx}(L,s)ds-EIu_{tx}\left(L,t\right)\int_0^t h(t-s)u_{xx}\left(L,s\right)ds\\
&+EI\tfrac{1}{2}\tfrac{d}{dt}\left(h\Box u_{xx}\right)(t)-EI\tfrac{1}{2}\left(h\Box u_{xx}\right)(t)+EI\tfrac{1}{2}h(t)\textstyle\int_0^L\left|u_{xx}\right|^2dx\\
&-EI\tfrac{1}{2}\tfrac{d}{dt}\left\{\textstyle\int_0^t h(s)ds\int_0^L\left|u_{xx}\right|^2dx\right\}-Tu_x\left(L,t\right)u_t\left(L,t\right)-\tfrac{1}{2}T\tfrac{d}{dt}\left\|u_x\right\|^2=0.
\end{aligned}\tag{3.21}$$

Since we have

$$u\left(0,t\right)=u_x\left(0,t\right)=u_{xx}\left(L,t\right)=0,$$

then (3.21) gives us

$$\begin{aligned}
&\tfrac{d}{dt}\left[\tfrac{1}{2}\rho\left\|u_t\right\|^2+\tfrac{1}{2}EI\left\|u_{xx}\right\|^2+EI\tfrac{1}{2}\left(h\Box u_{xx}\right)(t)+\tfrac{1}{2}Msu_t^2\left(L,t\right)\right]\\
&+\tfrac{d}{dt}\left[-EI\tfrac{1}{2}\textstyle\int_0^t h(s)ds\int_0^L\left|u_{xx}\right|^2dx-\tfrac{1}{2}T\left\|u_x\right\|^2\right]\\
&=EI\tfrac{1}{2}\left(h\Box u_{xx}\right)(t)-EI\tfrac{1}{2}h(t)\textstyle\int_0^L\left|u_{xx}\right|^2dx\\
&+A\left(t\right)u_t\left(L,t\right)-d_Su_t\left(L,t\right)^2,
\end{aligned}$$

we put

$$\begin{aligned}
\xi(x)=\tfrac{1}{2}\rho\left\|u_t\right\|^2+\tfrac{1}{2}EI\left(1-\textstyle\int_0^t h(s)ds\right)\left\|u_{xx}\right\|^2+\tfrac{1}{2}EI\left(h\Box u_{xx}\right)(t)\\
-\tfrac{1}{2}T\left\|u_x\right\|^2+\tfrac{1}{2}Msu_t^2\left(L,t\right),
\end{aligned}$$

then

$$\frac{d}{dt}\xi(x)=\frac{1}{2}EI\left(h\Box u_{xx}\right)(t)-\frac{1}{2}EIh(t)\left|\left|u_{xx}\right|\right|^2+u_t\left(L,t\right)A\left(t\right)-ds\ u_t^2\left(L,t\right).$$

$\Box$

3.3 Lyapunov Functionals

Neither $E\left(t\right)$ nor $\mathcal{E}\left(t\right)$ are ready yet to provide the exponential decay. For this reason we shall ‘modify’ further the functional $\mathcal{E}\left(t\right)$ with the following ones

$$L\left(t\right):=\lambda_0\mathcal{E}\left(t\right)+\textstyle\sum_{i=1}^5\lambda_i\Phi_i\left(t\right),\ t\geq0 \tag{3.22}$$

où

$$\Phi_1(t) = \rho \int_0^L u_t u dx,$$

$$\Phi_2(t) = -\rho \int_0^L u_t \int_0^t h(t-s)(u(t) - u(s))\, dsdx,$$

$$\Phi_3(t) = \int_0^L \int_0^t H_\sigma(t-s) |u_{xx}(s)|^2 \, dsdx,$$

$$\Phi_4(t) = \int_0^t H_\sigma(t-s) |u(L,s) - u(L,t)|^2 \, dsdx,$$

$$\Phi_5(t) = \int_0^L \int_0^t \psi_\sigma(t-s) |u_{xx}(s)|^2 \, dsdx,$$

where

$$\psi_\sigma(t) = e^{-\sigma t} \int_t^{+\infty} \xi(s) e^{\sigma s} ds,$$

$$H_\sigma(t) = e^{-\sigma t} \int_t^{+\infty} h(s) e^{\sigma s} ds, \ t \geq 0$$

and $\sigma > 0$ is a constant to be determined later.

We consider the functional

$$L(t) := \mathcal{E}(t) + \sum_{i=1}^{5} \lambda_i \Phi_i(t), \ t \geq 0$$

for some $\lambda_i > 0$, $i = 1, ..., 5$ to be determined.
It is our intention to prove that this new functional $L(t)$ satisfies a differential inequality of the form $L'(t) \leq -CL(t)$ for some positive constant C to derive the exponential decay of $L(t)$. To pass to $E(t)$ we will need some 'equivalence' between the two functionals.

Proposition 3.3.1. *There exist two positive constants $\rho_i > 0$, $i = 1, 2$ such that*

$$\rho_1 [\mathcal{E}(t) + \Phi_3(t) + \Phi_4(t) + \Phi_5(t)] \leq L(t) \leq \rho_2 [\mathcal{E}(t) + \Phi_3(t) + \Phi_4(t) + \Phi_5(t)], \tag{3.23}$$

for all $t \geq 0$.

Proof. **Proof**. Applying (1.26) and (1.27), we find

$$\Phi_1(t) \leq \frac{\rho}{2} \int_0^L |u_t|^2 dx + \frac{\rho L^4}{2} \int_0^L |u_{xx}|^2 dx, \ t \geq 0,$$

and

$$\Phi_2(t) = \rho \textstyle\int_0^L u_t \int_0^t h(t-s)(u(t) - u(s))\, dsdx$$

$$\leq \tfrac{\rho}{2} \textstyle\int_0^L |u_t|^2 dx + \tfrac{\rho}{2} \int_0^L \left(\int_0^t h(t-s)(u(t) - u(s)) \right)^2 dsdx$$

$$\leq \tfrac{\rho}{2} \textstyle\int_0^L |u_t|^2 dx + \tfrac{\rho}{2} \overline{h} L^4 (h \Box u_{xx})(t),$$

then

$$\Phi_2(t) \leq \frac{\rho}{2} \int_0^L |u_t|^2 dx + \frac{\rho L^4 \overline{h}}{2} (h \Box u_{xx})(t), \ t \geq 0.$$

These two relations imply that

$$L(t) \le \tfrac{\rho}{2}(1+\lambda_1+\lambda_2)\int_0^L |u_t|^2\,dx + \tfrac{1}{2}\left(EI+\lambda_2\rho L^4\overline{h}\right)(h\square u_{xx})(t)$$

$$+\tfrac{1}{2}\left(EI\left(1-\int_0^t h(s)\,ds\right)+\lambda_1\rho L^4\right)\int_0^L |u_{xx}|^2\,dx + \tfrac{T}{2}\int_0^L |u_x|^2\,dx$$

$$+\tfrac{M_s}{2}[u_t(L,t)]^2 + \lambda_3\Phi_3(t)+\lambda_4\Phi_4(t)+\lambda_5\Phi_5(t),\ t\ge 0.$$

On the other hand, we have

$$2L(t) \ge \rho(1-\lambda_1-\lambda_2)\int_0^L |u_t|^2\,dx + \left(EI-\lambda_2\rho L^4\overline{h}\right)(h\square u_{xx})(t)$$

$$+\left(EI\left(1-\overline{h}\right)-\lambda_1\rho L^4\right)\int_0^L |u_{xx}|^2\,dx + T\int_0^L |u_x|^2\,dx + M_s[u_t(L,t)]^2$$

$$+2\lambda_3\Phi_3(t)+2\lambda_4\Phi_4(t)+2\lambda_5\Phi_5(t),\ t\ge 0.$$

Therefore

$$\rho_1(\mathcal{E}(t)+\Phi_3(t)+\Phi_4(t)+\Phi_5(t)) \le L(t) \le \rho_2\mathcal{E}(t)+\Phi_3(t)+\Phi_4(t)\ +\Phi_5(t)$$

for some constants $\rho_i>0$, $i=1,2$ provided that

$$\lambda_1 < \min\left(1, \frac{EI\left(1-\overline{h}\right)}{\rho L^4}\right),\ \lambda_2 < \min\left(\frac{EI}{\rho L^4\overline{h}}, 1-\lambda_1\right).$$

□

3.4 Technical Lemmas

For the stabilization analysis, we will make use of the following technical lemmas. These results play a fundamental role in the proof, as they provide the essential estimates and inequalities required to control the energy and to construct an appropriate Lyapunov functional. They will be repeatedly used throughout the stabilization argument.

Lemma 3.4.1. *The derivative of $\Phi_1(t)$ fulfills the following estimate*

$$\Phi_1'(t) \le 2EI\delta_1 u^2(L,t) + EI\left(\delta_1+\overline{h}-1\right)\int_0^L |u_{xx}|^2\,dx + \rho\int_0^L u_t^2 dx$$

$$+\tfrac{EI}{4\delta_1}\left(\int_0^t h(t-s)\,u_{xxx}(L,s)\,ds\right)^2 + \tfrac{EI}{4\delta_1}u_{xxx}^2(L,t) + \tfrac{EI}{4\delta_1}\overline{h}(h\square u_{xx})(t) \qquad (3.24)$$

$$+Tu(L,t)\,u_x(L,t) - T\int_0^L |u_x|^2\,dx,\ \delta_1>0, t\ge 0.$$

Proof. A differentiation of $\Phi_1(t)$ with respect to t along solutions of (3.1) gives

$$\begin{aligned}\Phi_1'(t) &= \rho\int_0^L (u_{tt}u+u_t^2)\,dx\\ &= \int_0^L u\left(-EIu_{xxxx}+EI\int_0^t h(t-s)\,u_{xxxx}(s)\,ds + Tu_{xx}\right)dx\\ &\quad +\rho\int_0^L u_t^2 dx,\ t\ge 0.\end{aligned}$$

We set

$$\Phi_1'(t) = B_1 + B_2 + \rho \int_0^L u_t^2 dx, \tag{3.25}$$

where

$$B_1 = \int_0^L (-EIuu_{xxxx} + Tuu_{xx})\, dx, \tag{3.26}$$

and

$$B_2 = EI \int_0^L u(t) \int_0^t h(t-s)\, u_{xxxx}(s)\, dsdx. \tag{3.27}$$

Integrating the expression in (3.26) by parts and using the boundary conditions in (3.1), we obtain

$$B_1 = -EIu(L,t)\, u_{xxx}(L,t) - EI \textstyle\int_0^L u_{xx}^2 dx + Tu(L,t)\, u_x(L,t) - T \textstyle\int_0^L u_x^2 dx. \tag{3.28}$$

Applying **Young's Inequality** (1.27) in (3.28) yields

$$\begin{array}{c} B_1 \leq \delta_1 EIu^2(L,t) + \frac{EI}{4\delta_1} u_{xxx}^2(L,t) \\ -EI \int_0^L u_{xx}^2 dx + Tu(L,t)\, u_x(L,t) - T \int_0^L u_x^2 dx, \ t \geq 0. \end{array} \tag{3.29}$$

After integrating the expression of B_2 in (3.27) by parts and using the boundary conditions, we get

$$B_2 = EI \left[u(L,t) \int_0^t h(t-s)\, u_{xxx}(L,s)\, ds + \int_0^L u_{xx} \int_0^t h(t-s)\, u_{xx}(s)\, dsdx \right]. \tag{3.30}$$

Clearly, by adding and subtracting $u_{xx}(t)$ inside the inner integral, we obtain

$$\textstyle\int_0^t h(t-s)\, u_{xx}(s)\, ds = h(t)\, u_{xx}(s) + \int_0^t h(t-s)\, (u_{xx}(s) - u_{xx}(t))\, ds,$$

hence

$$\begin{array}{c} EI \int_0^L u_{xx}(t) \int_0^t h(t-s)\, u_{xx}(s)\, dsdx = EIh(t) \int_0^L |u_{xx}(s)|^2 dx \\ \\ +EI \int_0^L u_{xx}(t) \int_0^t h(t-s)\, (u_{xx}(s) - u_{xx}(t))\, dsdx. \end{array} \tag{3.31}$$

Using Young's inequality

$$\int ab \leq \delta_1 \int a^2 + \frac{1}{4\delta_1} \int b^2,$$

we get

$$EI \int_0^L u_{xx}(t) \int_0^t h(t-s)\, (u_{xx}(s) - u_{xx}(t))\, dsdx \tag{3.32}$$

$$\tag{3.33}$$

$$\leq EI\delta_1 \int_0^L |u_{xx}|^2 dx + \frac{EI}{4\delta_1} \int_0^L \left(\int_0^t h(t-s)\, (u_{xx}(s) - u_{xx}(t))\, ds \right)^2 dx$$

$$\tag{3.34}$$

$$\leq EI\delta_1 \int_0^L |u_{xx}|^2 dx + \frac{EI}{4\delta_1} \overline{h} \int_0^L \int_0^t h(t-s)\, |(u_{xx}(s) - u_{xx}(t))|^2 dsdx$$

$$\le EI\delta_1 \int_0^L |u_{xx}|^2 dx + \frac{EI}{4\delta_1}\overline{h}\,(h\Box u_{xx})(t), \tag{3.35}$$

if we replace (3.32) in (3.31) yields

$$\begin{gathered} EI \int_0^L u_{xx} \int_0^t h(t-s)\, u_{xx}(s)\, dsdx \\ \le EI\left(\delta_1 + \overline{h}\right) \int_0^L |u_{xx}|^2 dx + \frac{EI}{4\delta_1}\overline{h}\,(h\Box u_{xx})(t), \;\; \delta_1 > 0, \;\; t \ge 0. \end{gathered} \tag{3.36}$$

We have

$$EIu(L,t) \int_0^t h(t-s)\, u_{xxx}(L,s)\, ds \le EI\delta_1 u^2(L,t) + \frac{EI}{4\delta_1}\left(\int_0^t h(t-s)\, u_{xxx}(L,s)\, ds\right)^2. \tag{3.37}$$

Replacing the relations (3.36) and (3.37) in (3.30), we get

$$\begin{gathered} B_2 \le EI\left(\delta_1 u^2(L,t) + \frac{1}{4\delta_1}\left(\int_0^t h(t-s)\, u_{xxx}(L,s)\, ds\right)^2\right) \\ +EI\left(\delta_1 + \overline{h}\right) \int_0^L |u_{xx}|^2 dx + \frac{EI\overline{h}}{4\delta_1}(h\Box u_{xx})(t), \;\; t \ge 0. \end{gathered} \tag{3.38}$$

Now back to (3.25), by(3.29) and (3.38) we see that

$$\begin{gathered} \Phi_1'(t) \le 2EI\delta_1 u^2(L,t) + EI\left(\delta_1 + \overline{h} - 1\right) \int_0^L |u_{xx}|^2 dx + \rho \int_0^L u_t^2 dx \\ +\frac{EI}{4\delta_1}\left(\int_0^t h(t-s)\, u_{xxx}(L,s)\, ds\right)^2 + \frac{EI}{4\delta_1} u_{xxx}^2(L,t) + \frac{EI}{4\delta_1}\overline{h}\,(h\Box u_{xx})(t) \\ +Tu(L,t)\, u_x(L,t) - T\int_0^L |u_x|^2 dx, \;\; t \ge 0. \end{gathered}$$

□

Lemma 3.4.2. *The derivative of $\Phi_2(t)$ satisfies the following estimate*

$$\begin{gathered} \Phi_2'(t) \le \left(\frac{EI\overline{h}}{2\delta_1} + \frac{TL^4\overline{h}}{4\delta_1} + \frac{\rho\gamma\overline{h}L^4}{4\delta_1}\right)(h\Box u_{xx})(t) + \left(EI\delta_1 + T\delta_1\right)\int_0^L u_{xx}^2 dx \\ +\left(\rho\gamma\delta_1 + \rho\delta_2 - \rho h_0\right)\int_0^L u_t^2 dx + \frac{EI}{4\delta_1} u_{xxx}^2(L,t) \\ +2EI\delta_1\overline{h}\int_0^t h(t-s)\,|u(L,s) - u(L,t)|^2 ds + \frac{EI}{4\delta_1}\left(\int_0^t h(t-s)\, u_{xxx}(L,s)\, ds\right)^2 \\ +EI\delta_1\overline{h}\int_0^L\int_0^t h(t-s)\,|u_{xx}(s)|^2 dsdx + \frac{\rho L^4}{4\delta_2}\int_0^t \xi(s)\, ds\, (\xi\Box u_{xx})(t), \;\; t \ge t_0. \end{gathered} \tag{3.39}$$

Proof. For $\Phi_2(t)$ we have

$$\Phi_2(t) = -\rho\int_0^L u_t \int_0^t h(t-s)\,(u(t) - u(s))\, dsdx$$

$$\begin{gathered} \Phi_2'(t) = \rho\int_0^L u_{tt}\int_0^t h(t-s)\,(u(s) - u(t))\, dsdx - \rho\int_0^L u_t^2 dx \int_0^t h(s)\, ds \\ +\rho\int_0^L u_t \int_0^t h'(t-s)\,(u(s) - u(t))\, dsdx, \;\; t \ge 0. \end{gathered} \tag{3.40}$$

Using the first equation of (3.1), we find

$$\Phi_2'(t) = -EI \int_0^L u_{xxxx} \int_0^t h(t-s)(u(s)-u(t))\,dsdx$$

$$+EI \int_0^L \left[\int_0^t h(t-s)\,u_{xxxx}(s)\,ds \int_0^t h(t-s)(u(s)-u(t))\,ds\right] dx$$

$$+T \int_0^L u_{xx} \int_0^t h(t-s)(u(s)-u(t))\,dsdx - \rho \int_0^L u_t^2 dx \int_0^t h(s)\,ds$$

$$+\rho \int_0^L u_t(t) \int_0^t h'(t-s)(u(s)-u(t))\,dsdx,\ \ t \geq 0.$$

We set

$$\Phi_2'(t) = C_1 + C_2 + C_3 + C_4 + C_5, \tag{3.41}$$

where

$$C_1 = -EI \int_0^L u_{xxxx} \int_0^t h(t-s)(u(s)-u(t))\,dsdx, \tag{3.42}$$

$$C_2 = EI \int_0^L \left[\int_0^t h(t-s)\,u_{xxxx}(s)\,ds \int_0^t h(t-s)(u(s)-u(t))\,ds\right] dx, \tag{3.43}$$

$$C_3 = \rho \int_0^L u_t \int_0^t h'(t-s)(u(s)-u(t))\,dsdx, \tag{3.44}$$

$$C_4 = -\rho \int_0^L u_t^2 dx \int_0^t h(s)\,ds, \tag{3.45}$$

and

$$C_5 = T \int_0^L u_{xx} \int_0^t h(t-s)(u(s)-u(t))\,dsdx. \tag{3.46}$$

Let us estimate these terms separately. First an integrating by parts in (3.42) gives

$$\begin{aligned} C_1 = -EIu_{xxx}(L,t) \int_0^t h(t-s)(u(L,s)-u(L,t))\,ds \\ +EI \int_0^L u_{xxx} \int_0^t h(t-s)(u_x(s)-u_x(t))\,dsdx, \end{aligned} \tag{3.47}$$

and a second integration by parts in (3.47) leads to

$$C_1 = -EIu_{xxx}(L,t) \int_0^t h(t-s)(u(L,s)-u(L,t))\,ds$$

$$-EI \int_0^L u_{xx} \int_0^t h(t-s)(u_{xx}(s)-u_{xx}(t))\,dsdx.$$

Then

$$\begin{aligned} C_1 \leq \tfrac{EI}{4\delta_1} u_{xxx}^2(L,t) + \delta_1 EI \left(\int_0^t h(t-s)(u(L,s)-u(L,t))\,ds\right)^2 \\ +EI \left(\delta_1 \int_0^L u_{xx}^2 dx\right) + \tfrac{EI\overline{h}}{4\delta_1}(h\Box u_{xx})(t),\ \ \delta_1 > 0,\ \ t \geq 0. \end{aligned} \tag{3.48}$$

Similarly, an integration by parts in (3.43) yields

$$\begin{aligned} C_2 = EI \left(\int_0^t h(t-s)\,u_{xxx}(L,s)\,ds\right) \left(\int_0^t h(t-s)(u(L,s)-u(L,t))\,ds\right) \\ -EI \int_0^L \left(\int_0^t h(t-s)\,u_{xxx}(s)\,ds\right) \left(\int_0^t h(t-s)(u_x(s)-u_x(t))\,ds\right) dx, \end{aligned} \tag{3.49}$$

and another one in (3.49) gives

$$C_2 = EI\left(\int_0^t h\left(t-s\right)u_{xxx}\left(L,s\right)ds\right)\left(\int_0^t h\left(t-s\right)\left(u\left(L,s\right)-u\left(L,t\right)\right)ds\right)$$

$$+EI\int_0^L\left(\int_0^t h\left(t-s\right)u_{xx}\left(s\right)ds\right)\left(\int_0^t h\left(t-s\right)\left(u_{xx}\left(s\right)-u_{xx}\left(t\right)\right)ds\right)dx.$$

Therefore

$$\begin{aligned} C_2 &\leq \tfrac{EI}{4\delta_1}\left(\int_0^t h\left(t-s\right)u_{xxx}\left(L,s\right)ds\right)^2 \\ &+EI\overline{h}\delta_1\int_0^t h\left(t-s\right)\left|u\left(L,s\right)-u\left(L,t\right)\right|^2 ds \\ &+EI\delta_1\overline{h}\int_0^L\int_0^t h\left(t-s\right)\left|u_{xx}\left(s\right)\right|^2 dsdx+\tfrac{EI\overline{h}}{4\delta_1}\left(h\Box u_{xx}\right)\left(t\right),\ t\geq 0. \end{aligned} \tag{3.50}$$

We have

$$\begin{aligned} C_3 &= \rho\int_0^L u_t\int_0^t h'\left(t-s\right)\left(u\left(s\right)-u\left(t\right)\right)dsdx \\ &= -\rho\int_0^L u_t\int_0^t \gamma h\left(t-s\right)\left(u\left(s\right)-u\left(t\right)\right)dsdx \\ &+\rho\int_0^L u_t\int_0^t\left(h'\left(t-s\right)+\gamma h\left(t-s\right)\right)\left(u\left(s\right)-u\left(t\right)\right)dsdx. \end{aligned}$$

We have

$$\begin{aligned} C_3 &= \rho\int_0^L u_t\int_0^t\left(h'\left(t-s\right)+\gamma h\left(t-s\right)\right)\left(u\left(s\right)-u\left(t\right)\right)dsdx \\ &= \rho\gamma\int_0^L u_t\int_0^t h\left(t-s\right)\left(u\left(s\right)-u\left(t\right)\right)dsdx \\ &+\rho\int_0^L u_t\int_0^t h'\left(t-s\right)\left(u\left(s\right)-u\left(t\right)\right)dsdx \\ &\leq \rho\gamma\delta_1\int_0^L u_t^2dx+\tfrac{\rho\gamma\overline{h}L^4}{4\delta_1}\left(h\Box u_{xx}\right)\left(t\right) \\ &+\rho\delta_2\int_0^L u_t^2dx+\tfrac{\rho L^4}{4\delta_2}\left(\int_0^t\xi\left(s\right)ds\right)\left(\xi\Box u_{xx}\right)\left(t\right),\ t\geq 0, \end{aligned} \tag{3.51}$$

where $\delta_2 > 0$.
For $t \geq t_0 > 0$, the fact

$$\int_0^t h\left(s\right)ds \geq \int_0^{t_0} h\left(s\right)ds = h_0 > 0,$$

implies that

$$-\rho\int_0^t h\left(s\right)ds\int_0^L u_t^2dx \leq -\rho h_0\int_0^L u_t^2dx,\ t\geq t_0>0.$$

Further, we have

$$C_5 \leq T\delta_1\int_0^L u_{xx}^2dx+\frac{TL^4\overline{h}}{4\delta_1}\left(h\Box u_{xx}\right)\left(t\right),\ t\geq t_0. \tag{3.52}$$

Substituting (3.48), (3.50)-(3.52) and (3.45) into (3.41), we arrive at

$$\Phi_2'(t) \leq \left(\frac{EI\overline{h}}{2\delta_1} + \frac{TL^4\overline{h}}{4\delta_1} + \frac{\rho\gamma\overline{h}L^4}{4\delta_1}\right)(h\Box u_{xx})(t) + (EI\delta_1 + T\delta_1)\int_0^L u_{xx}^2 dx$$

$$+ (\rho\gamma\delta_1 + \rho\delta_2 - \rho h_0)\int_0^L u_t^2 dx + \frac{EI}{4\delta_1}u_{xxx}^2(L,t)$$

$$+2EI\delta_1\overline{h}\int_0^t h(t-s)\left|u(L,s) - u(L,t)\right|^2 ds + \frac{EI}{4\delta_1}\left(\int_0^t h(t-s)\,u_{xxx}(L,s)\,ds\right)^2$$

$$+EI\delta_1\overline{h}\int_0^L\int_0^t h(t-s)\left|u_{xx}(s)\right|^2 ds dx + \frac{\rho L^4}{4\delta_2}\int_0^t \xi(s)\,ds\,(\xi\Box u_{xx})(t),\ \ t \geq t_0.$$

□

Lemma 3.4.3. *The derivative of* $\Phi_3(t)$ *satisfies*

$$\Phi_3'(t) = -\sigma\Phi_3(t) - \int_0^L\int_0^t h(t-s)\left|u_{xx}(s)\right|^2 ds dx + \overline{h_\sigma}\int_0^L \left|u_{xx}\right|^2 dx,\ \ t \geq 0, \quad (3.53)$$

where

$$\overline{h_\sigma} = \int_0^\infty e^{\sigma s} h(s)\,ds.$$

Lemma 3.4.4. *The derivative of* $\Phi_4(t)$ *satisfies the following estimate*

$$\Phi_4'(t) = -\sigma\Phi_4(t) - \int_0^t h(t-s)\left|u(L,s) - u(L,t)\right|^2 ds$$
$$-2u_t(L,t)\int_0^t H_\sigma(t-s)\,(u(L,s) - u(L,t))\,ds,\ \ t \geq 0. \quad (3.54)$$

Lemma 3.4.5. *The derivative of* $\Phi_5(t)$ *obeys the following estimate*

$$\Phi_5'(t) = -\sigma\Phi_5(t) - \int_0^L\int_0^t \xi(t-s)\left|u_{xx}(s)\right|^2 ds dx + \overline{\xi}_\sigma\int_0^L \left|u_{xx}\right|^2 dx,\ \ t \geq 0, \quad (3.55)$$

where

$$\overline{\xi}_\sigma = \int_0^\infty e^{\sigma s}\xi(s)\,ds.$$

3.5 Proof of our main result

Proof. of Theoreme

Gathering all the previous estimates (3.2), (3.6), (3.24), (3.39)-(3.55) and using the facts that

$$u(L,t)^2 \leq L\int_0^L u_x(x,t)^2 dx, \quad (3.56)$$

$$\left(\int_0^t H_\sigma(t-s)\left[u(L,s) - u(L,t)\right]ds\right)^2 \leq \overline{H}_\sigma\Phi_4(t),$$

where

$$\overline{H}_\sigma = \int_0^\infty H_\sigma(s)\,ds,$$

we get, for $t \geq t_0 > 0$

$$\begin{array}{c} L'(t) \leq -\left(\rho\lambda_2 h_0 - \rho\lambda_1 - \rho\gamma\lambda_2\delta_1 - \rho\lambda_2\delta_2\right)\int_0^L u_t^2 dx - \lambda_1 T \int_0^L u_x^2(x,t)\,dx \\ -\sigma\lambda_3\Phi_3(t) - \sigma\lambda_5\Phi_5(t) + \left(\frac{2\lambda_4}{4\delta_3} - d_S\right)u_t(L,t)^2 + \lambda_4\left(2\delta_3\overline{H}_\sigma - \sigma\right)\Phi_4(t) \\ +\left(2\lambda_2 EI\delta_1\overline{h} - \lambda_4\right)\int_0^t h(t-s)\left|u(L,s) - u(L,t)\right|^2 ds \\ -\left(\lambda_3 - \lambda_2 EI\delta_1\overline{h}\right)\int_0^L\int_0^t h(t-s)\left|u_{xx}(s)\right|^2 dsdx \\ -\left(\lambda_1 EI - \lambda_3\overline{h_\sigma} - \lambda_2 EI\delta_1 - \lambda_1 EI\left(\delta_1 + \overline{h}\right)\right)\int_0^L u_{xx}^2 dx \\ +\left(\lambda_2 T\delta_1 + \lambda_1 L\delta_4 + \lambda_5\overline{\xi}_\sigma\right)\int_0^L u_{xx}^2 dx + \frac{\overline{h}\left[\lambda_1 EI + \lambda_2\left(TL^4 + 2\lambda_2 EI + \rho\gamma L^4\right)\right]}{4\delta_1} \\ \times\left(h\Box u_{xx}\right)(t) + \frac{\rho\lambda_2 L^4}{4\delta_2}\int_0^t \xi(s)\,ds\left(\xi\Box u_{xx}\right)(t) + \frac{EI}{2}\left(h'\Box u_{xx}\right)(t) \\ -\lambda_5\int_0^L\int_0^t \xi(t-s)\left|u_{xx}(s)\right|^2 dsdx + \left(\frac{EI(\lambda_1+\lambda_2)}{4\delta_1} - K\right)u_{xxx}^2(L,t) \\ +\left(\frac{EI(\lambda_1+\lambda_2)}{4\delta_1} - K\right)\left(\int_0^t h(t-s)\,u_{xxx}(L,s)\,ds\right)^2 \\ +\left(2\lambda_1 EI\delta_1 + \frac{\lambda_1 T^2}{4\delta_4} - K\right)u^2(L,t), \end{array} \tag{3.57}$$

where we have used the estimation

$$Tu(L,t)\,u_x(L,t) \leq \delta_4 L\int_0^L u_{xx}^2 dx + \frac{T^2}{4\delta_4}u^2(L,t).$$

Let

$$P = \frac{\overline{h}}{4\delta_1}\left\{\lambda_1 EI + \lambda_2\left(TL^4 + 2EI + \rho\gamma L^4\right)\right\},$$

then under assumption (**H2)**, we can write

$$0 < h'(t) \leq \xi(t) - \gamma h(t),$$

and

$$\begin{array}{c} \frac{1}{2}\left(\frac{\rho\lambda_2 L^4}{2\delta_2}\overline{\xi} + EI\right)\int_0^L\int_0^t \xi(t-s)\left|u_{xx}(s) - u_{xx}(t)\right|^2 dsdx \\ = \frac{1}{2}\left(\frac{\rho\lambda_2 L^4}{2\delta_2}\overline{\xi} + EI\right)\left(\xi\Box u_{xx}\right)(t) \\ \leq 2\left(\frac{\rho\lambda_2 L^4}{4\delta_2}\overline{\xi} + \frac{EI}{2}\right)\int_0^t\int_0^L \xi(t-s)\left|u_{xx}(s)\right|^2 dxds \\ +2\overline{\xi}\left(\frac{\rho\lambda_2 L^4}{4\delta_2}\overline{\xi} + \frac{EI}{2}\right)\int_0^L u_{xx}^2 dx, \end{array}$$

then

$$\begin{array}{c} P\left(h\Box u_{xx}\right)(t) + \frac{\rho\lambda_2 L^4}{4\delta_2}\left(\int_0^t \xi(s)\,ds\right)\left(\xi\Box u_{xx}\right)(t) + \frac{EI}{2}\left(h'\Box u_{xx}\right)(t) \\ \leq -\left(\frac{\gamma EI}{2} - P\right)\left(h\Box u_{xx}\right)(t) + \frac{1}{2}\left(\frac{\rho\lambda_2 L^4}{2\delta_2}\overline{\xi} + EI\right)\left(\xi\Box u_{xx}\right)(t) \\ \leq -\left(\frac{\gamma EI}{2} - P\right)\left(h\Box u_{xx}\right)(t) + 2\overline{\xi}\left(\frac{\rho\lambda_2 L^4}{4\delta_2}\overline{\xi} + \frac{EI}{2}\right)\int_0^L u_{xx}^2 dx \\ +2\left(\frac{\rho\lambda_2 L^4}{4\delta_2}\overline{\xi} + \frac{EI}{2}\right)\int_0^t\int_0^L \xi(t-s)\left|u_{xx}(s)\right|^2 dxds,\ t \geq 0. \end{array} \tag{3.58}$$

Next, substituting (3.58) into (3.57), we obtain for $t \geq t_0 > 0$

$$
\begin{aligned}
L'(t) \leq & -\left[\rho\lambda_2\left(h_0 - \gamma\delta_1 - \delta_2\right) - \lambda_1\rho\right]\int_0^L u_t^2 dx - \lambda_1 T \int_0^L u_x^2(x,t)\,dx \\
& -\sigma\lambda_3\Phi_3(t) - \sigma\lambda_5\Phi_5(t) + \left(\frac{2\lambda_4}{4\delta_3} - d_S\right)u_t(L,t)^2 + \lambda_4\left(2\delta_3\overline{H}_\sigma - \sigma\right)\Phi_4(t) \\
& + \left(2\lambda_2 EI\delta_1\overline{h} - \lambda_4\right)\int_0^t h(t-s)\left[u(L,s) - u(L,t)\right]^2 ds - \left(\lambda_3 - \lambda_2 EI\delta_1\overline{h}\right) \\
& \times \int_0^L\int_0^t h(t-s)\,u_{xx}^2(x,s)\,ds dx - \left[\lambda_1 EI - \lambda_3\overline{h_\sigma} - \lambda_2 EI\delta_1 - \lambda_1 EI\left(\delta_1 + \overline{h}\right)\right] \\
& \times \int_0^L u_{xx}^2 dx - \left[-\lambda_2 T\delta_1 - \lambda_1 L\delta_4 - \lambda_5\overline{\xi_\sigma} - \overline{\xi}\left(\frac{\rho\lambda_2 L^4}{2\delta_2}\overline{\xi} + EI\right)\right]\int_0^L u_{xx}^2 dx \\
& + \left(\frac{\rho\lambda_2 L^4}{2\delta_2}\overline{\xi} + EI - \lambda_5\right)\int_0^t\int_0^L \xi(t-s)\left|u_{xx}(s)\right|^2 dx ds - \left(\frac{\gamma EI}{2} - P\right)\left(h\Box u_{xx}\right)(t) \\
& + \left(\frac{EI(\lambda_1+\lambda_2)}{4\delta_1} - K\right)\left[u_{xxx}^2(L,t) + \left(\int_0^t h(t-s)\,u_{xxx}(L,s)\,ds\right)^2\right] \\
& + \left(2\lambda_1 EI\delta_1 + \frac{\lambda_1 T^2}{4\delta_4} - K\right)u^2(L,t),
\end{aligned}
\tag{3.59}
$$

Now we start selecting the different parameters in such a way that all the coefficients in the right hand side of (36) are negative. That is

$$
\begin{cases}
\rho\lambda_2 h_0 - \rho\lambda_1 - \rho\gamma\lambda_2\delta_1 - \rho\lambda_2\delta_2 > 0, \\
\frac{\lambda_4}{2\delta_3} - d_S < 0, \\
2\delta_3\overline{H}_\sigma - \sigma < 0, \\
2\lambda_2 EI\delta_1\overline{h} - \lambda_4 < 0, \\
\lambda_3 - \lambda_2 EI\delta_1\overline{h} > 0, \\
\lambda_1 EI - \lambda_3\overline{h_\sigma} - \lambda_2 EI\delta_1 - \lambda_1 EI\left(\delta_1 + \overline{h}\right) - \lambda_2 T\delta_1 \\
-\lambda_1 L\delta_4 - \lambda_5\overline{\xi}_\sigma - \overline{\xi}\left(\frac{\rho\lambda_2 L^4}{2\delta_2}\overline{\xi} + EI\right) > 0, \\
\frac{\rho\lambda_2 L^4}{2\delta_2}\overline{\xi} + EI - \lambda_5 < 0, \\
\frac{\overline{h}}{4\delta_1}\left\{\lambda_1 EI + \lambda_2\left(TL^4 + 2EI + \rho\gamma L^4\right)\right\} < \gamma\frac{EI}{2}, \\
\frac{EI(\lambda_1+\lambda_2)}{4\delta_1} - K < 0, \\
2\lambda_1 EI\delta_1 + \frac{\lambda_1 T^2}{4\delta_4} - K < 0.
\end{cases}
\tag{3.60}
$$

For the last three conditions in (3.60), λ_1 and λ_2 will be selected small enough. Let us ignore the parameters δ_1 and δ_4 for a moment, we need

$$
\begin{cases}
\lambda_1 < \lambda_2\left(h_0 - \delta_2\right), \\
\frac{\lambda_4}{2\delta_3} < d_S, \\
2\delta_3\overline{H}_\sigma < \sigma, \\
\lambda_5\overline{\xi_\sigma} + \overline{\xi}\left(\frac{\rho\lambda_2 L^4}{2\delta_2}\overline{\xi} + EI\right) < \lambda_1 EI\left(1 - \overline{h}\right), \\
\frac{\rho\lambda_2 L^4}{2\delta_2}\overline{\xi} + EI < \lambda_5.
\end{cases}
$$

Notice that we have only a smallness condition on λ_3 in the sixth condition in (3.60). So we forget about it for a while. Let $\delta_2 = \frac{h_0}{2}$, $\delta_3 = \frac{\sigma}{4\overline{H}_\sigma}$ and $\lambda_4 = \delta_3 d_S$. The system reduces to

$$
\begin{cases}
\lambda_1 < \lambda_2\frac{h_0}{2}, \\
\lambda_5\overline{\xi}_\sigma + \overline{\xi}\left(\frac{\rho\lambda_2 L^4}{2\delta_2}\overline{\xi} + EI\right) < \lambda_1 EI\left(1 - \overline{h}\right), \\
\frac{\rho\lambda_2 L^4}{h_0}\overline{\xi} + EI < \lambda_5.
\end{cases}
$$

Take, for instance, $\lambda_1 = \frac{h_0}{4}\lambda_2$, then we must have

$$\left(\frac{\rho\lambda_2 L^4}{h_0}\overline{\xi}+EI\right)\overline{\xi}_\sigma+\overline{\xi}\left(\frac{\rho\lambda_2 L^4}{2\delta_2}\overline{\xi}+EI\right)$$
$$<\lambda_5\overline{\xi}_\sigma+\overline{\xi}\left(\frac{\rho\lambda_2 L^4}{2\delta_2}\overline{\xi}+EI\right)<\frac{h_0}{2}\lambda_2 EI\left(1-\overline{h}\right).$$

We will be able to find λ_5 if

$$\left(\frac{\rho\lambda_2 L^4}{h_0}\overline{\xi}+EI\right)\overline{\xi}_\sigma+\overline{\xi}\left(\frac{\rho\lambda_2 L^4}{2\delta_2}\overline{\xi}+EI\right)<\frac{h_0}{2}\lambda_2 EI\left(1-\overline{h}\right),$$

that is

$$\left(\overline{\xi}_\sigma+\overline{\xi}\right)EI<\left(\frac{h_0}{2}EI\left(1-\overline{h}\right)-\frac{\rho L^4}{h_0}\overline{\xi}\,\overline{\xi}_\sigma-\frac{\rho L^4}{h_0}\overline{\xi}^2\right)\lambda_2.$$

This condition holds for small $\overline{\xi}_\sigma$. Roughly, as $\overline{\xi}\le\overline{\xi}_\sigma$ it suffices to assume that

$$\frac{\rho L^4}{h_0}\overline{\xi}_\sigma^2<\frac{h_0}{8}EI\left(1-\overline{h}\right),$$

that is

$$\overline{\xi}_\sigma<\frac{\sqrt{2}h_0}{4L^2}\sqrt{\frac{EI\left(1-\overline{h}\right)}{\rho}}$$

and

$$\overline{\xi}_\sigma<\frac{h_0}{8}\lambda_2\left(1-\overline{h}\right),$$

i.e

$$\overline{\xi}_\sigma<\min\left\{\frac{\sqrt{2}h_0}{4L^2}\sqrt{\frac{EI\left(1-\overline{h}\right)}{\rho}},\frac{h_0}{8}\lambda_2\left(1-\overline{h}\right)\right\}. \tag{3.61}$$

Having imposed the above condition on $\overline{\xi}_\sigma$ we can determine λ_5 (in terms of λ_2 and $\overline{\xi}_\sigma$). Then we select δ_1 and δ_4. In turn, λ_2 will be fixed through K and γ according to (3.60).
Finally, we get

$$L'(t)\le -C'\left(\mathcal{E}(t)+\Phi_3(t)+\Phi_4(t)+\Phi_5(t)\right),$$

for some positive constant C'. In virtue of Proposition 1 we entail that

$$L'(t)\le\frac{-C'}{\rho_2}L(t)=-bL(t),\ t\ge t_0>0.$$

We deduce that $L(t)\le L(0)e^{-bt}$, $t\ge t_0>0$ and thereafter $\mathcal{E}(t)\le\frac{L(0)}{\rho_1}e^{-bt}$, $t\ge t_0>0$.
The existence of an $M>0$ such that $\mathcal{E}(t)\le Me^{-bt}$ on $[0,t_0]$ is trivial. The proof is complete. $\square$

GENERAL CONCLUSION

In conclusion, this work provides a rigorous mathematical analysis of the stabilization problem for a flexible marine riser interacting with vessel dynamics. By introducing an appropriate boundary feedback control, we have shown that the vibrations of the system can be effectively damped and that the associated energy decreases exponentially with time.

The analysis relies on the construction of suitable Lyapunov functionals together with the multiplier method, allowing us to overcome the difficulties caused by the viscoelastic memory effects and the distributed delay term. The obtained stability result highlights the important role played by the boundary control and the relaxation mechanism in enhancing the dynamic performance of the system.

These results contribute to the theoretical understanding of marine riser stabilization problems and may serve as a basis for further investigations on more general coupled systems, nonlinear models, or different types of damping and delay effects arising in engineering applications.

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