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# The Dynamics of Entanglement and Steerability in the Double Jaynes–Cummings Model under Noiseless and Noisy Environments

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# General Introduction

# General Introduction

Atomic and Optical Physics is a science that seeks to understand and control the quantum behavior of matter at the smallest scale. it asks questions such as: what are the fundamental properties of single atoms, ions, molecules, and photons? these are the building blocks of this science, and those building blocks historically were available in the gas phase, gas of atom, gas of molecule or studying them in the field of collisions, collision between atom-atom, atom-ion or molecules and photons, However, in these days the field has moved away from being gas phase and has split into two fields: many-body system, which study ultra-cold atoms, Quantum gases and Bose-Einstein Condensation; and few-body system which leads us to Quantum Entanglement and Quantum Information Processing.

If we focus on the second field and want to redefine the Atomic Physics, we can say that the precision, preparation and manipulation which Atomic Physics has reached with Quantum System puts this Physics in a leading position at the front of exploring new aspects of Hilbert space, because we have gone from a big ensemble of many quantum states to a single quantum state, and now we can create a complexity by assembling a few photons or a few atoms into new entangled states so that we can take our system into very different regions of Hilbert space.

Speaking of Entanglement, this concept take us back to 1935, when Einstein,Podolsky and Rosen (EPR) published their paper entitled "can Quantum Mechanical Description of Physical Reality Be Considered Complete?" This work provoked an interesting response from Erwin Schrodinger who introduced the terms entangled state and formulated the concept of Quantum Steering which has gradually gained considerable attention, especially in the field of Quantum Information Science.

In this thesis we aim to establish a coherent connection between these concepts, beginning with the dynamics of a single quantum state and extending to Quantum Steering and Entanglement, as a physical platform, we consider one of the most precise and experimentally active single quantum state systems currently available: a single Atom interacting with a quantized electromagnetic field inside a cavity, commonly described by Jaynes–Cummings Model. We first investigates the atom-photon interaction within cavity quantum electrodynamics (QED) and then go deeper into the story of EPR paradox and Shrodinger Steering. Finally, in the last two chapters, we examine the dynamics of Entanglement and steerability in the double Jynes–Cummings Model, both as closed system and under the influence environmental effects.

# Chapter I

## Photon-Atom Interaction:

## The Jaynes–Cummings Model (JCM)

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# Photon-Atom Interaction: The Jaynes–Cummings Model (JCM)

## I.1 Introduction

We begin this chapter with a quotation by Nobel Prize Winner *Wolfgang ketterle*:

> "*We really have to understand the harmonic oscillator, the two-level system and the hydrogen atom, if we understand those systems very-well in all their glorious detail, then we have understood a lot of Physics.*"

The Jaynes–Cummings model is an important and actual topic in quantum physics, since it touches different branches of the most recent and outstanding research activities. To realize the Jaynes–Cummings model, it is necessary to understand how tho solve the two state-system. What makes this model particularly significant is that it describes a single two-level atom interacting with a single mode of electromagnetic field. More precisely, The Jaynes–Cummings model aims to find how quantization of the radiation field affects the predictions for the evolution of the states of a two-level atom, modeled as a two-state system with quantum harmonic oscillator. This description is identical to that of a spin state in rotating magnetic field which is governed by the fundamental spin 1/2 Hamiltonian.

Therefore, the Hamiltonian of Jaynes–Cummings model can be derived starting from understanding and realizing the Hamiltonian of spin 1/2 particle in a static magnetic field coupled to a rotating magnetic field, with an understanding of the Rabi Oscillation that frequently appears in the Jaynes–Cummings model. Finally a details solution of Jaynes–Cummings Hamiltonian is presented, along with a discussion of its experimental aspects in the field of Quantum Information Science.

## I.2 Dynamics of the Two-State System

### I.2.1 Spin Precession in a Magnetic Field

A two-state system does not just have two states. It has two basis states; the state space is the two dimensional complex vector space. For such a state space, the Hamiltonian can be viewed as the most general Hermitian $2 \times 2$ matrix.

Our two-state system will be a spin 1/2 particle, This is to prepare for the more complex example of spin 1/2 in a rotating magnetic field to be considered in the following section. The mathematics of two-state systems is always the same, making it possible to visualize any two-state system as a spin system.[21]

We begin this section by examine so called *Spin Precession.* The idea of Spin Precession arises from magnetic moment $\mu$ for a spinning charged particle with charge $q$ and mass $m$:

$$\vec{\mu} = \frac{q}{2m} \cdot \vec{L} \tag{I.1}$$

Here $\mu$ is the magnetic dipole moment, and $L$ is its angular momentum. In the quantum world, particles have spin angular momentum operators $\hat{S}$ and magnetic moment operators $\hat{\mu}$, and the above relation gets modified by the inclusion of a unit-free constant factor $g$ which differs for each particle; for the electron $g$=2

$$\vec{\mu} = g\frac{q}{2m} \cdot \hat{S} \tag{I.2}$$

$\mu$ can be written as:

$$\vec{\mu} = \gamma \hat{S} \tag{I.3}$$

$\gamma$ will summarize all the constant $g$, $q$ and $m$, it called gyromagnetic ratios, for the electron:

$$\gamma = 1.76 \times 10^{11} s^{-1} t^{-1}$$

If we insert the particle in a magnetic field $B$, the Hamiltonian $\hat{H}$ for the spin system is:

$$\hat{H} = -\hat{\mu} \cdot \vec{B} = -\gamma \hat{S} \cdot \vec{B} \tag{I.4}$$

For a magnetic field pointing along $z$- axis $\vec{B} = B\hat{z}$, we obtain:

$$\hat{H} = -\gamma \hat{S}_z B \tag{I.5}$$

and the associated time-evolution unitary operator is given by:

$$U(t) = \exp\left(-\frac{i}{\hbar}\hat{H}t\right) = \exp\left(-\frac{i}{\hbar}(-\gamma Bt)\hat{S}_z\right) \tag{I.6}$$

We have the unitary rotation operator $R_n(\alpha)$ defined by a unit vector $n$ and an angle $\alpha$:

$$R_n(\alpha) = \exp\left(-\frac{i\alpha \hat{S}_n}{\hbar}\right) \tag{I.7}$$

where $\hat{S}_n$ is defined as:

$$\hat{S}_n = n \cdot \hat{S}$$

Acting on a spin state, the operator rotates it by an angle $\alpha$ about the direction defined by the vector $n$. Comparing (I.6) and (I.7), we conclude that $U(t)$ generates a rotation by the angle $(-\gamma Bt)$ around the z-axis.

Consider a spin state that at $t = 0$ points along the direction specified by the angles $(\theta_0, \phi_0)$: (Note that $|+\rangle$ is spin state up and $|-\rangle$ is spin state down)

$$|\psi(0)\rangle = \cos\frac{\theta_0}{2}|+\rangle + \sin\frac{\theta_0}{2}e^{i\phi_0}|-\rangle \tag{I.8}$$

Therefore, the time-evolved state is given by:

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle$$

so that we find:

$$|\psi(t)\rangle = \cos\frac{\theta_0}{2}\exp\Big(\frac{1}{2}i\gamma Bt\Big)|+\rangle + \sin\frac{\theta_0}{2}e^{i\phi_0}\exp\Big(-\frac{1}{2}i\gamma Bt\Big)|-\rangle \tag{I.9}$$

where the eigenvalues of the spin states is:

$$\hat{H}|\pm\rangle = \mp\frac{1}{2}\gamma B\hbar|\pm\rangle$$

To identify the direction of the resulting state, we factor out the phase that multiplies the $|+\rangle$ state:

$$|\psi(t)\rangle = \exp\Big(\frac{1}{2}i\gamma Bt\Big)\Big[\cos\frac{\theta_0}{2}|+\rangle + \sin\frac{\theta_0}{2}\exp(i(\phi_0 - \gamma Bt))|-\rangle\Big] \tag{I.10}$$

recognize that the spin state points along the direction defined by angles:

$$\begin{aligned}\theta(t) &= \theta_0\\ \phi(t) &= \phi_0 - \gamma Bt\end{aligned} \tag{I.11}$$

Keeping $\theta$ constant while changing $\phi$ indeed corresponds to a rotation about the $z$-axis and, after time t, the spin has under goes precession around the magnetic field $B$ with angular frequency of magnitude $\omega = \gamma B$ which is called *Larmor frequency* $\omega_L$ (Fig I.1).[14]
The result in general is that in a time-independent magnetic field, spin states precess with Larmor frequency $\omega_L$ given by:

$$\vec{\omega_L} = -\gamma\vec{B} \tag{I.12}$$

Therefore, we can rewrite the Hamiltonian of a spin in a magnetic field (I.6):

$$\hat{H} = -\gamma\vec{B}\cdot\hat{S} = \vec{\omega_L}\cdot\hat{S} \tag{I.13}$$

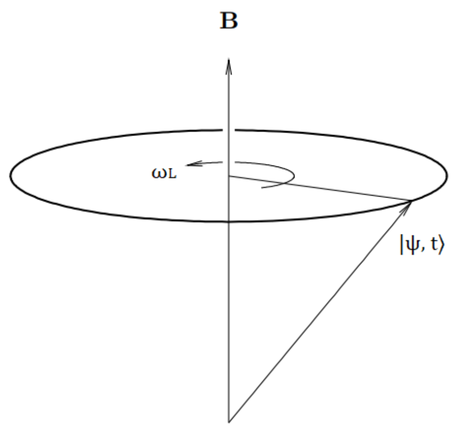


**Figure I.1:** In a constant magnetic field, spin state precesses about the field direction at frequency $\omega_L$

the time evolution operator becomes:

$$U(t) = \exp\Big(-i\frac{\vec{\omega_L}\hat{S}_n}{\hbar}\Big) \tag{I.14}$$

The precession of a spin state means the precession of the unit vector $n(t)$ that fixes the direction of the spin state, and we can say that the Hamiltonian rotates the spin state with $\omega_L$ around $n^{th}$ direction.

### I.2.2 Spin 1/2 in a Rotating Magnetic Field and Rabi Oscillation

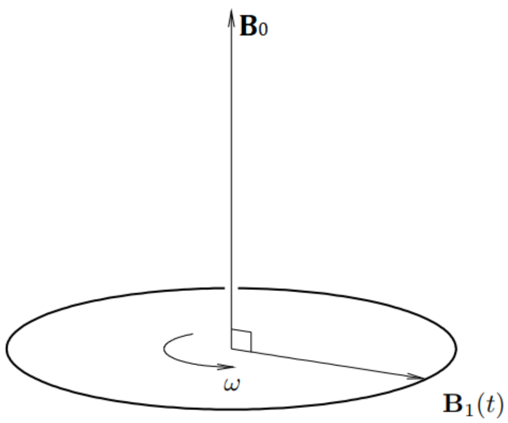


**Figure I.2:** Total magnetic field $\vec{B}(t)$ consists of $B_0\hat{z} \perp \vec{B_1}(t)$ rotates about $B_0$ at frequency $\omega$.

We consider in this case that our spin 1/2 system are subject to time-dependent magnetic field, this magnetic field has a time-independent $z$-component $B_0$ (Previous case) and a circularly polarized component representing a magnetic field $\vec{B_1}(t)$ that is perpendicular to $B_0$ and points along the $x$-axis at $t = 0$ with magnitude $B_1$ and rotating with frequency $\omega$ in the clockwise direction of the $(x, y)$ plane.[21][14]

$$\vec{B_1}(t) = B_1(\hat{x}\cos\omega t + \hat{y}\sin\omega t) \tag{I.15}$$

the total time-dependent magnetic field $\vec{B(t)}$ is:

$$\vec{B}(t) = B_0\hat{z} + B_1(\hat{x}\cos\omega t + \hat{y}\sin\omega t) \tag{I.16}$$

Associated to the longitudinal magnetic field $B_0$, the Larmor frequency $\omega_L$ is defined by:

$$\omega_L = -\gamma B_0 \tag{I.17}$$

the spin Hamiltonian is:

$$\hat{H}_s = -\gamma\vec{B}(t)\cdot\hat{S} = -\gamma[B_0\hat{S}_z + B_1(\hat{S}_x\cos\omega t + \hat{S}_y\sin\omega t] \tag{I.18}$$

we define the *Rabi frequency* $\omega_R$:

$$\omega_R = \gamma B_1 \tag{I.19}$$

then the spin Hamiltonian becomes:

$$\hat{H}_s = \frac{1}{2}\hbar\omega_0\sigma_z + \frac{1}{2}\hbar\omega_R(\sigma_x\cos\omega t + \sigma_y\sin\omega t) \tag{I.20}$$

This Hamiltonian is not only time dependent; the Hamiltonian at different times do not commute, this is therefore a nontrivial time-evolution problem.
this problem is solved by transforming to a rotating frame to obtain Hamiltonian that no longer depending on time, The Hamiltonian transformed by defining a unitary operator that generates rotations around the $z$-axis:

$$U(t) = e^{-i\frac{\omega t}{2}\sigma_z} \tag{I.21}$$

this corresponds to moving into a frame rotating at frequency $\omega$ around $z$ -axis.
we define the spin state of the rotating frame:

$$|\psi_r(t)\rangle = U(t)\,|\psi(t)\rangle \tag{I.22}$$

we have the original Schrodinger equation for $|\psi(t)\rangle$:

$$i\hbar\frac{d}{dt}|\psi\rangle = \hat{H}_s|\psi\rangle \tag{I.23}$$

then the Schrodinger equation for $|\psi_r(t)\rangle$ becomes:

$$i\hbar\frac{d}{dt}|\psi_r(t)\rangle = i\hbar\frac{d}{dt}(U(t)|\psi\rangle) = i\hbar\dot{U}(t)|\psi\rangle + i\hbar U(t)\frac{d}{dt}|\psi\rangle \tag{I.24}$$

the transformation is unitary, that implies $U^\dagger(t)U(t) = 1$, by multiplying Eq(I.24) from the left by $U^\dagger(t)U(t)$ and using Eq(I.23) we find the Schrodinger equation for the rotating-frame spin Hamiltonian $\hat{H}_r$:

$$i\hbar\frac{d}{dt}|\psi_r(t)\rangle = \hat{H}_r|\psi_r\rangle \tag{I.25}$$

where

$$\hat{H}_r = U(t)\hat{H}_s U^\dagger(t) + i\hbar\dot{U}(t)U^\dagger(t) \tag{I.26}$$

Using the identities of the Pauli matrices, we simplify the rotating-frame spin Hamiltonian and at the end we find:

$$\hat{H}_r = \frac{1}{2}\hbar(\delta\sigma_z + \omega_R\sigma_x) \tag{I.27}$$

where $\delta$ is the detuning frequency:

$$\delta = \omega - \omega_0$$

By performing some simple rearrangements and writing this Hamiltonian in terms of the magnetic fields $B_0$ and $B_1$, we can predict how the spin state evolve in the rotating frame. Thus, the rotating-frame spin Hamiltonian can be written as:

$$\hat{H}_r = -\gamma\vec{B}_{eff}\cdot\hat{S} \tag{I.28}$$

where the effective magnetic field $\vec{B}_{eff}$ is:

$$\vec{B}_{eff} = B_0(1 - \frac{\omega}{\omega_0})\hat{z} + B_1\hat{x} \tag{I.29}$$

the corresponding time-evolution operator $U_r$ is:

$$U_r = \exp\left(\frac{i}{\hbar}\gamma\vec{B}_{eff}\hat{S}t\right) \tag{I.30}$$

then, the spin state in the rotating-frame will evolve as:

$$\begin{aligned} |\psi_r(t)\rangle &= U_r|\psi_r(0)\rangle \\ &= \exp\left(\frac{i}{\hbar}\gamma\vec{B_{eff}}\hat{S}t\right)|\psi_r(0)\rangle \end{aligned} \tag{I.31}$$

Therefore, we have the same situation that we had in the previous case where the magnetic field is static, here the spin state is just precessing around the effective magnetic field with angular frequency of magnitude $\Omega_R = \gamma B_{eff}$ which is called *General Rabi frequency* $\Omega_R$.

The magnitude of effective magnetic field is:

$$B_{eff} = \sqrt{B_1^2 + B_0^2(1 - \frac{\omega}{\omega_0})^2} \tag{I.32}$$

Then, the General Rabi frequency is:

$$\Omega_R = \gamma B_{eff} = \sqrt{\delta^2 + \omega_R^2} \tag{I.33}$$

- In case on-resonance $\delta = 0$, the $z$-component is canceled. Therefore, the spin precesses around the static field $B_1$ at the Rabi frequency $\omega_R$ performing what is called Rabi flopping or Rabi oscillation. Assuming that the spin at $t = 0$ is up and after half cycle $t = \frac{\pi}{\omega_0}$ the spin points down.
- In case off-resonance $\delta \neq 0$ where $\omega > \omega_0$ or $\omega < \omega_0$, the term of $z$-component is non-zero, there would be $B_{eff}$, then the spin will precess at higher frequency which is General Rabi frequency and will never fully invert.

We have the rotating-frame spin Hamiltonian that giving in Eq(I.27), [14] written in the vector form:

$$\hat{H_{rot}} = \frac{\hbar\Omega_R}{2}\hat{n}\cdot\vec{\sigma} \tag{I.34}$$

where the unit vector $\hat{n}$ is:

$$\hat{n} = \frac{1}{\Omega_R}(\omega_R, 0, \delta)$$

Then the time evolution operator becomes:

$$U_r(t) = I\cos\frac{\Omega_R t}{2} - \frac{\delta\sigma_z + \omega_R\sigma_x}{\Omega_R} i\sin\frac{\Omega_R t}{2} \tag{I.35}$$

Suppose that the initial state is a spin up along $z$-direction:

$$|\psi(0)\rangle = |+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{I.36}$$

The time evolution of this state would be:

$$|\psi(t)\rangle = U(t)\,|+\rangle \tag{I.37}$$

Then we find:

$$|\psi(t)\rangle = \begin{pmatrix} \cos(\Omega_R t/2) - i\frac{\delta}{\Omega_R}\sin(\Omega_R t/2) \\ -i\frac{\omega_R}{\Omega_R}\sin(\Omega_R t/2) \end{pmatrix} \tag{I.38}$$

Therefore, the probability of spin flip $P_{+\to-}(t)$, known as the Rabi oscillation probability is the projection into $|-\rangle$, so that:

$$P_{+\to-}(t) = \left|\,\langle -|\psi(t)\rangle\,\right|^2 \tag{I.39}$$

The final result is:

$$P_{+\to-}(t) = \left(\frac{\omega_R}{\Omega_R}\right)^2 \sin^2\left(\frac{\Omega_R t}{2}\right) \tag{I.40}$$

In the resonant case $\delta = 0$, the probability becomes:

$$P_{+\to-}(t) = \sin^2(\frac{\omega_R t}{2}) \tag{I.41}$$

This behavior is known as *Rabi oscillation*, it is well-known in any tow-state system and will also be seen it in the Jaynes-Cummings model.

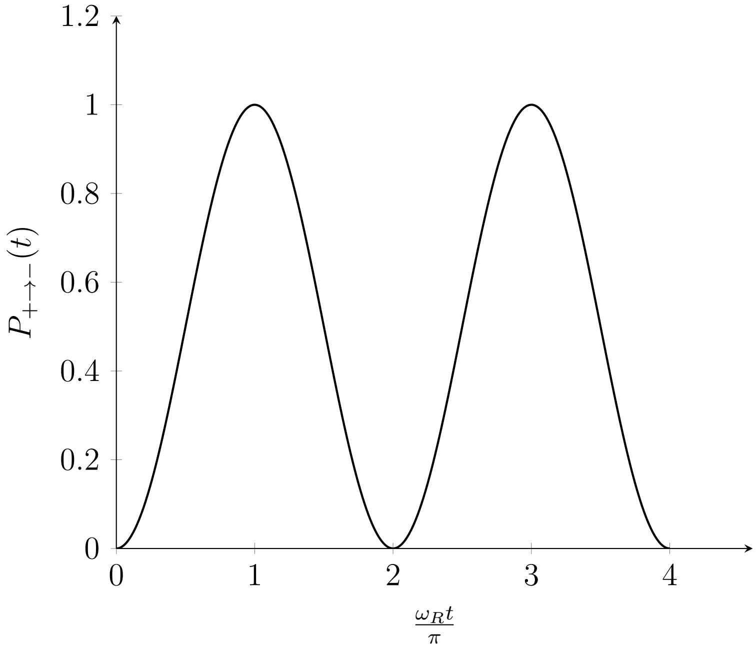


**Figure I.3:** The probability of spin flip $P_{+\to-}(t) = \sin^2\left(\frac{\omega_R t}{2}\right)$

Returning now to the lab-frame, we examine how the spin state evolves in time. This can be done using the same transformation that performed to move to the rotating frame. Thus, from Eq(I.22) we have:

$$|\psi(t)\rangle = U^{\dagger}(t)\,|\psi_r(t)\rangle \tag{I.42}$$

and we have the spin state evolution in the rotating-frame Eq(I.31), so that we plug for $|\psi_r(t)\rangle$ into the last equation, than we find:

$$|\psi(t)\rangle = U^{\dagger}(t)U_r(t)\,|\psi_r(0)\rangle \tag{I.43}$$

we have the same unitary operator so that $|\psi_r(0)\rangle = |\psi(0)\rangle$, therefore the time evolution of the spin states is giving by

$$|\psi(t)\rangle = \exp\left(\frac{i}{\hbar}\omega t\hat{S}_z\right)\exp\left(\frac{i}{\hbar}\gamma\vec{B}_{eff}\hat{S}t\right)|\psi_r(0)\rangle \tag{I.44}$$

Assuming that the spin at $t = 0$ is in the state up $|+\rangle$ along $z$-direction, and from the last equation we can say that:[21]

- In case off-resonance $\delta \neq 0$, the second exponent rotates the spin state around the axis of $\vec{B_{eff}}$ with frequency $\Omega_R$ and the first exponent produces a rotation around $z$-axis but much smaller than the second one because of General Rabi frequency $\Omega_R$. Therefore the spin state generate a cone and rotating very fast but the hole thing rotates around $z$-axis.

- In case on-resonance $\delta = 0$, the second exponent rotates the spin state around the axis of $\vec{B_1}$ with $\omega_R$. Therefore the spin goes down in the $y$-direction, but the first exponent produces a rotation around $z$-axis, so when the spin state goes down the other one rotates it, since $B_1$ is much smaller than $B_0$ and $\omega = \omega_0$ then the rotating around $z$-axis with $\omega_0$ is faster than the rotating around $B_1$. Fig(I.4).

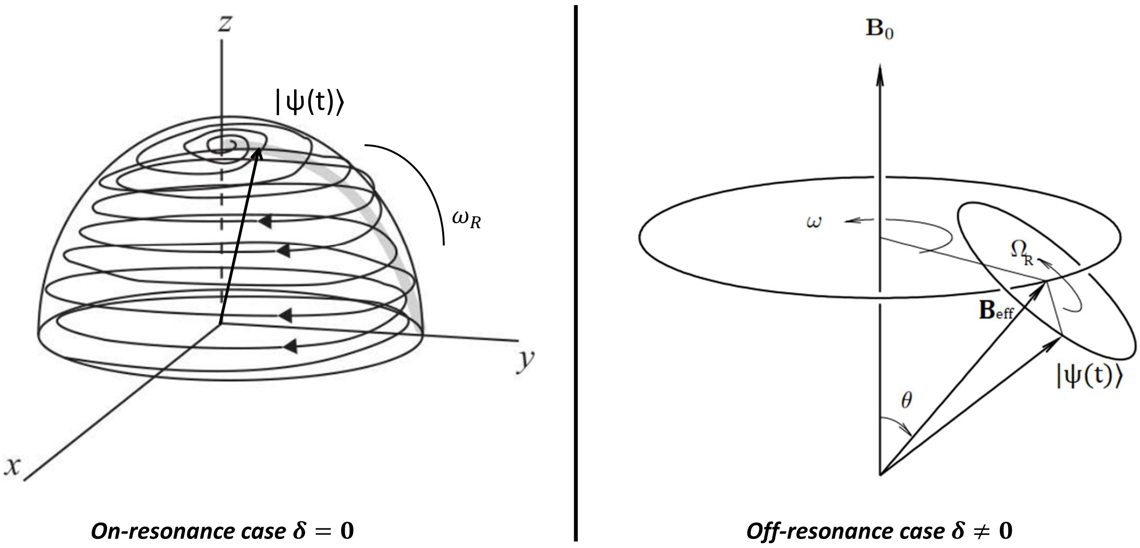


**Figure I.4:** Spin state evolves with Rabi frequency on resonance and off-resonance cases

## I.3 The Jaynes-Cummings Hamiltonian

As mentioned in the introduction, The Jaynes–Cummings model is a model describes a single two-level atom interacting with a single mode of electromagnetic field, more specifically, The natural framework for the implementation and experimental testing of the Jaynes–Cummings model is cavity quantum electrodynamics (QED), where fundamental experiments has been performed using microwave and optical cavities. Therefore this model describes a single atom inside a cavity. This is the typical situation in cavity quantum electrodynamics, when a single atom is coupled to a cavity mode (atom-cavity system). Historically, this model was originally proposed in 1963 by Edwin Jaynes and Fred Cummings in order to study the relationship between the quantum theory of radiation and the semi-classical theory in describing the phenomenon of spontaneous emission.[9][12][13]

The complete Hamiltonian of the Jaynes–Cummings Model is the quantum version of semi-classical description of atom-light interaction and can be expressed as:

$$\hat{H} = \hat{H}_a + \hat{H}_c + \hat{H_{int}} \tag{I.45}$$

where $\hat{H}_a$ is the unperturbed Hamiltonian of the atom, $\hat{H}_c$ is the cavity field Hamiltonian and $\hat{H}_{int}$ represents the atom-cavity field interacting Hamiltonian.[13][2]

We begin with the unperturbed Hamiltonian of the atom $\hat{H}_a$, to express it in a form two-level system analogous to a spin 1/2 Hamiltonian, we are shifting the ground state by $-\frac{1}{2}\hbar\omega_{eg}$. In the other word, usually to say that for an electronic transition we start at zero and go up, here we shifts things that the zero of energy is in the middle between the ground state $|g\rangle$ and the excited state $|e\rangle$ and then it looks like excited state is spin up and the ground state is spin down. so the unperturbed Hamiltonian of the atom $\hat{H}_a$ can be written as:

$$\hat{H}_a = \frac{1}{2}\hbar\omega_{eg}(|e\rangle\langle e| - |g\rangle\langle g|) \tag{I.46}$$

Shifting the ground state energy corresponds to Pauli matrix $\sigma_z$:

$$|e\rangle\langle e| - |g\rangle\langle g| = \sigma_z$$

Therefore:

$$\hat{H}_a = \frac{1}{2}\hbar\omega_{eg}\sigma_z \tag{I.47}$$

(The energy of excited state $|e\rangle$ is $E_e = \frac{1}{2}\hbar\omega_{eg}$ and that of ground state $|g\rangle$ is $E_g = -\frac{1}{2}\hbar\omega_{eg}$)
The Hamiltonian for the cavity field is nothing else than the Hamiltonian of the photon:

$$\hat{H}_c = \hbar\omega\hat{a}^\dagger\hat{a} \tag{I.48}$$

with Fock states $|n\rangle$, where $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators for the photons in the cavity mode obeying the bosonic commutation relation $\left[\hat{a}, \hat{a}^\dagger\right] = 1$.

Finally, in the dipole approximation, the atom-cavity interaction has the standard form Hamiltonian:

$$\hat{H_{int}} = -\hat{d} \cdot \vec{E} \tag{I.49}$$

$\vec{E}$ is the fully quantized electric field at the atomic position $z$ with time $t$

$$E(z,t) = \varepsilon_0\hat{e}(i\hat{a}^\dagger e^{i\omega t} - i\hat{a}e^{i\omega t}) \tag{I.50}$$

for convenience we chose $z = 0$ and $t = \frac{\pi}{2\omega}$, then we got:

$$E(0,t) = \varepsilon_0\hat{e}(\hat{a}^\dagger + \hat{a}) \tag{I.51}$$

with $\varepsilon_0$ being the field per photon within the cavity volume V:

$$\varepsilon_0 = \sqrt{\frac{\hbar\omega}{\epsilon V}}$$

$\hat{d}$ is the atomic dipole operator and can be expressed in terms of the basis $\left\{|e\rangle, |g\rangle\right\}$ as:

$$\hat{d} = |e\rangle\langle e| d |g\rangle\langle g| + |g\rangle\langle g| d |e\rangle\langle e| = d_{eg}|e\rangle\langle g| + d_{ge}|g\rangle\langle e| \tag{I.52}$$

it is useful to define the Pauli raising and lowering operators $\sigma_+$ and $\sigma_-$:

$$\sigma_\pm = \frac{1}{2}(\sigma_x \pm \sigma_y)$$

- $\sigma_+$ is the atomic raising operator: $\sigma_+|g\rangle = |e\rangle \Leftrightarrow \sigma_+ = |e\rangle\langle g|$
- $\sigma_-$ is the atomic lowering operator: $\sigma_-|e\rangle = |g\rangle \Leftrightarrow \sigma_- = |g\rangle\langle e|$

where they satisfy the commutation relation:

$$\left[\sigma_+, \sigma_-\right] = \sigma_z$$

then the atomic dipole operator becomes:

$$\hat{d} = d_{eg}\sigma_+ + d_{ge}\sigma_- \tag{I.53}$$

for the sake of simplicity, we take $d_{eg}$ and $d_{ge}$ to be real and positive. For $d_{eg} = d_{ge}$, we define the atom–cavity field coupling constant:

$$-d_{eg}\varepsilon_0 = \hbar g \tag{I.54}$$

$g$ is the atom–cavity field coupling constant. In some references, it is referred to as the single-photon Rabi frequency or the vacuum Rabi frequency, since it characterizes the interaction strength responsible for spontaneous emission.

With this approximation, the atom–cavity interacting Hamiltonian reduces to:

$$\hat{H_{int}} = \hbar g(\hat{a} + \hat{a}^\dagger)(\sigma_+ + \sigma_-) \tag{I.55}$$

Therefore, on substituting for $\hat{H}_a$, $\hat{H}_c$ and $\hat{H_{int}}$ from Eqs. (I.47), (I,48) and (I.55) into Eq. (I.45), that:

$$\hat{H} = \frac{1}{2}\hbar\omega_{eg}\sigma_z + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar g(\hat{a} + \hat{a}^\dagger)(\sigma_+ + \sigma_-) \tag{I.56}$$

This is the fully Quantum electrodynamics Hamiltonian for two-level system, well-known as *the Jaynes–Cummings Hamiltonian*

The Hilbert space on which this Hamiltonian acts is: $|atom\rangle \otimes |photon\rangle$
where the basis state of the atom is $\Big\{|e\rangle\,, |g\rangle\Big\}$ and the basis state of the photon is Fock states $\Big\{\,|n\rangle\,\Big\}$

Focusing on the interacting term: $(\hat{a} + \hat{a}^\dagger)(\sigma_+ + \sigma_-)$, there are four types of process:

- $\hat{a}\sigma_+$: corresponds to the process of atomic transition from the ground state $|g\rangle$ to the excited state $|e\rangle$ and annihilate of one photon $\Rightarrow$ absorption process
- $\hat{a}^\dagger\sigma_-$: corresponds to the process of atomic transition from the excited state $|e\rangle$ to the ground state $|g\rangle$ and create of one photon $\Rightarrow$ emission process

this two terms are called intuitive terms (co-rotating terms), which conserve energy and drive transitions

- $\hat{a}^\dagger\sigma_+$ corresponds to the process of atomic transition from the ground state $|g\rangle$ to the excited state $|e\rangle$ and create of one photon $\Rightarrow$ the gain of approximately $2\hbar\omega$
- $\hat{a}\sigma_-$ corresponds to the process of atomic transition from the excited state $|e\rangle$ to the ground state $|g\rangle$ and annihilate of one photon $\Rightarrow$ the loss of approximately $2\hbar\omega$

this two terms are called off-shell terms (counter-rotating terms), which do not conserve energy, they give only rise to shifts of energy.

Drooping the off-shell terms (counter-rotating terms) corresponds to what is called the Rotating Wave Approximation (RWA)

Therefore, the final result of the Jaynes–Cummings Hamiltonian is:

$$\hat{H} = \frac{1}{2}\hbar\omega_{eg}\sigma_z + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar g(\hat{a}\sigma_+ + \hat{a}^\dagger\sigma_-) \tag{I.57}$$

**Remarks**

1. The Rotating Wave Approximation means that we neglect rapidly oscillating interaction terms (the counter-rotating terms) because their time average is nearly zero, keeping only the near-resonant terms that slowly varying (co-rotating terms) which are actually drive transitions. In other word, if we take the interacting Hamiltonian $\hat{H_{int}}$ and transform it into the interacting picture $\hat{H_{ip}}$, then we have

$$\hat{H_{ip}} = \exp\left(-\frac{i\hat{H_{ac}}t}{\hbar}\right)\hat{H_{int}}\exp\left(\frac{i\hat{H_{ac}}t}{\hbar}\right) \tag{I.58}$$

where $\hat{H_{ac}}$ is the sum of the atomic and the cavity field Hamiltonians:

$$\hat{H_{ac}} = \hat{H_a} + \hat{H_c}$$

therefore the interacting Hamiltonian in the interacting picture is:

$$\hat{H_{ip}} = \hbar g(\hat{a}^\dagger\sigma_- e^{i(\omega-\omega_{eg})t} + \hat{a}\sigma_+ e^{-i(\omega-\omega_{eg})t} + \hat{a}^\dagger\sigma_+ e^{i(\omega+\omega_{eg})t} + \hat{a}\sigma_- e^{-i(\omega+\omega_{eg})t}) \tag{I.59}$$

$\hat{a}^\dagger\sigma_-$ and $\hat{a}\sigma_+$ correspond to slow oscillation (co-rotating terms) $e^{\pm i(\omega-\omega_{eg})t}$, both of which are near-resonant ($\omega \approx \omega_{eg} >> \Delta = \omega - \omega_{eg}$), while $\hat{a}^\dagger\sigma_+$ and $\hat{a}\sigma_-$ correspond to very fast oscillation (the counter-rotating terms) $e^{-i(\omega+\omega_{eg})t}$ which are non-resonant process ($\omega \approx \omega_{eg} \Rightarrow \omega + \omega_{eg} \approx 2\omega$), this fast oscillation average to zero over times longer than $1/\omega_{eg}$, so we neglect those terms and that is the Rotating Wave Approximation.

2. As we mentioned, the off-shell term that we neglected ($\hat{a}^\dagger\sigma_+$ and $\hat{a}\sigma_-$) do not conserve energy, they give only rise to shifts of energy, but those shifts are actually *Lamb shifts*. If we take the vacuum state $|n\rangle = |0\rangle$, no photon tn the cavity, and the atom in the ground state, then the only non-vanishing term is $\hat{a}^\dagger\sigma_+$ which is the operator for the Lamb shift.

   Therefore, if we would apply this operator to the bound state of en electron and atom, then we have done the first principle QED calculation of the Lamb shift.

3. The form of the interaction Hamiltonian remains the same, it does not matter which type of transition we consider (electric dipole, magnetic dipole or electric quadrupole) as long as the transition leads to creation or annihilation of photons. the only difference lies in the definition of the coupling constant -$g$- which is the single-photon Rabi frequency.

### I.4 Solution of the Jaynes-Cummings Hamiltonian

***Eigenstates and Eigenvalues***

We have the Jaynes–Cummings Hamiltonian .Eq(I.57),[2][9] this Hamiltonian can be written in another convenient form by noting that:

$$N = \hat{a}^{\dagger}\hat{a} + \frac{1}{2}\sigma_z \tag{I.60}$$

$N$ is the number of excitation, which is called the constant of motion, that is $\left[\hat{H}, \hat{N}\right] = 0$, so that we find:

$$\hat{H} = \hbar\omega N + \frac{1}{2}\hbar\Delta\sigma_z + \hbar g(\hat{a}\sigma_+ + \hat{a}^{\dagger}\sigma_-) \tag{I.61}$$

where $\Delta = \omega_{eg} - \omega$ is known as the *detuning* – the frequency difference between the field and the atomic resonance.

This Hamiltonian couples only pair of states:

$$|e\rangle \otimes |n\rangle \underset{\hat{a}\sigma_+}{\overset{\hat{a}^{\dagger}\sigma_-}{\rightleftharpoons}} |g\rangle \otimes |n+1\rangle$$

( label these atom-photon states as $|e\rangle \otimes |n\rangle \equiv |e, n\rangle$ and $|g\rangle \otimes |n+1\rangle \equiv |g, n+1\rangle$ )

then the coupling energy that corresponds to the absorption and emission process respectively is giving by:

$$\langle e, n| \hat{H} |g, n+1\rangle = \langle g, n+1| \hat{H} |e, n\rangle = \hbar g\sqrt{n+1} \tag{I.62}$$

this is the off-diagonal matrix element of this Hamiltonian, the diagonal elements are represented by the eigenvalues of the uncoupled system Hamiltonian. Considering that there is no interaction between the atom and the cavity field ($g = 0$) then the Jaynes-Cummings Hamiltonian becomes Hamiltonian of the uncoupled system which is:

$$\hat{H} = \hat{H_{ac}} = \hbar\omega N + \frac{1}{2}\hbar\Delta\sigma_z \tag{I.63}$$

the eigenstates of this uncoupled Hamiltonian are $|e, n\rangle$ and $|g, n+1\rangle$ with different eigenvalues given by:

$$\begin{aligned} E_{e,n} &= \langle e, n| \hat{H} |e, n\rangle = \hbar\omega(n + \frac{1}{2}) + \frac{1}{2}\hbar\Delta \\ E_{g,n+1} &= \langle g, n+1| \hat{H} |g, n+1\rangle = \hbar\omega(n + \frac{1}{2}) - \frac{1}{2}\hbar\Delta \end{aligned} \tag{I.64}$$

so we write the Jaynes–Cummings Hamiltonian in the form:

$$\hat{H} = \hbar\omega(n + \frac{1}{2}) + \frac{1}{2}\hbar(\Delta\sigma_z + 2g\sqrt{n+1}\sigma_x) \tag{I.65}$$

Indeed, this is exactly the form of a two-state Hamiltonian, and it is well-known that a two-state system undergoes oscillations between the two states known as Rabi oscillation. The new feature here is that we have two states each of which is a combined state of the atom and the quantized electromagnetic field. In this way the quantum state of the electromagnetic field is included in the two-state description. Nevertheless, this Hamiltonian is formally identical to spin 1/2 in a rotating magnetic field. Eq(I.25), where $B_x$ equivalent to $g\sqrt{n+1}$ which is called *n-photon rabi frequency* and defined as $g_n \equiv g\sqrt{n+1}$.

The matrix form of the Jaynes–Cummings Hamiltonian is:

$$\hat{H} = \hbar\omega(n+\frac{1}{2})I + \frac{1}{2}\hbar\begin{pmatrix} \Delta & 2g_n \\ 2g_n & -\Delta \end{pmatrix} \tag{I.66}$$

Finally, we obtain the eigenvalues of the Jaynes–Cummings Hamiltonian, known as *the dressed energies* which are given by:

$$\begin{aligned} E_{n+} &= \hbar\omega(n+\frac{1}{2}) + \frac{1}{2}\hbar\Omega_n \\ E_{n-} &= \hbar\omega(n+\frac{1}{2}) - \frac{1}{2}\hbar\Omega_n \end{aligned} \tag{I.67}$$

Again, $\Omega_n$ is the General Rabi frequency:

$$\Omega_n = \sqrt{\Delta^2 + 4g_n^2} \tag{I.68}$$

In the resonance case ($\Delta = 0$), one gets n-photon Rabi frequency $\Omega_n = 2g_n$. In this case the degeneracy of the two energy states of the system $E_{e,n}$ and $E_{g,n+1}$ is lifted and the energy separation is given by $\Delta E_n = 2g_n$. For non-resonance case, the energy separation is the General Rabi frequency $\Omega_n$. Fig(I.5)

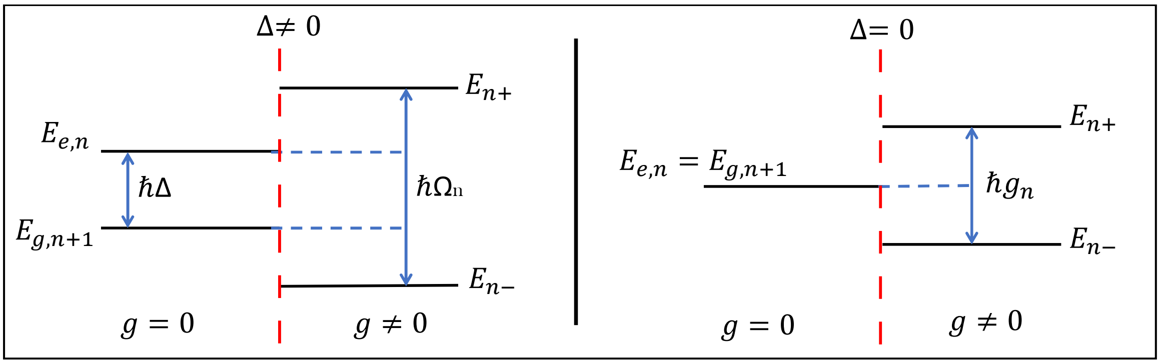


**Figure I.5:** Dressed energies level of JCM

Except for the ground state $|g,0\rangle$, which remains unchanged, the eigenstates of the Jaynes–Cummings Hamiltonian are arranged in doublets called the dressed states $|n\pm\rangle$ which expressed in terms of atom-photon states:

$$\begin{aligned} |n+\rangle &= \cos\frac{\theta}{2}|e,n\rangle + \sin\frac{\theta}{2}|g,n+1\rangle \\ |n-\rangle &= \cos\frac{\theta}{2}|g,n+1\rangle - \sin\frac{\theta}{2}|e,n\rangle \end{aligned} \tag{I.69}$$

where $\cos\theta$ and $\sin\theta$ are giving by:

$$\begin{aligned} \cos\theta &= \frac{\Delta}{\Omega_n} \\ \sin\theta &= \frac{2g_n}{\Omega_n} \end{aligned} \tag{I.70}$$

The dressed states are general entangled states of the global system atom-photon. The interaction Hamiltonian produces a unitary transformation on the basis states $|e, n\rangle$ and $|g, n+1\rangle$, which is simply a rotation of an angle $\theta$, that is varying between 0 and $\pi$. In the resonant case where $\Delta = 0$, it equals to $\frac{\pi}{2}$ for all n-values, then the dressed states becomes:

$$|n\pm\rangle = \frac{1}{\sqrt{2}}(|e, n\rangle \pm |g, n+1\rangle) \tag{I.71}$$

The eigenvalues of the uncoupled Hamiltonian $\hat{H_{ac}}$ (Eq I.64) are linear in the detuning frequency $\Delta$, that means that they cross at $\Delta = 0$ showing a degeneracy at resonance

$$E_{e,n} = E_{g,n+1} = \hbar\omega(n+1)$$

In this case, the dressed energies level do not cross, but they are separated by the effect of the interaction by n-phton Rabi frequency $2g_n = 2g\sqrt{n+1}$. On the other hand, quite off resonance $|\Delta| >> 2g_n$, the generalized Rabi frequency is $\Omega_n \approx |\Delta|$ then the dressed states are asymptotically equal to the uncoupled states. This physically means that the eigenstates of the total Hamiltonian $\hat{H}$ coincide with those of the uncoupled Hamiltonian $\hat{H_{ac}}$ and that equivalently means that $g \to 0$. Fig(I.6)

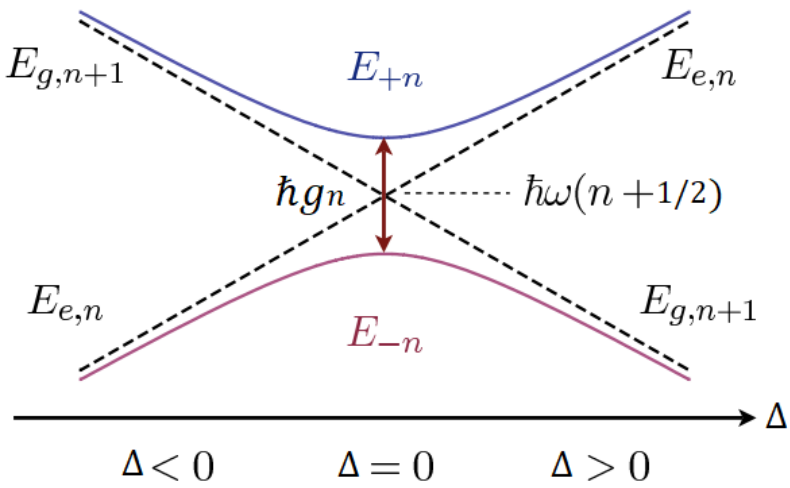


**Figure I.6:** Dressed energies in function of $\Delta$

## I.5 Dynamics of the Jaynes-Cummings Model

In this suction, we discuss how the Jaynes–Cummings states $|e,n\rangle$ and $|g,n+1\rangle$ evolves in time, this very important section for our discussion in the dynamics of entanglement and steerabilty in the doublet Jaynes-Cummings Atom.[9]

In general, the tame evolution operator defined as:

$$|\psi(t)\rangle = U(t)\,|\psi(0)\rangle$$

$$U(t) = e^{\frac{-i}{\hbar}\hat{H}t}$$

In this case, we have the dressed states are the eigenstates for the Jaynes–Cummings Hamiltonian:

$$\hat{H}\,|n\pm\rangle = \hbar\omega_n \pm \frac{1}{2}\hbar\Omega_n\,|n\pm\rangle \tag{I.72}$$

Therefore, we can write:

$$e^{\frac{-i}{\hbar}\hat{H}t}\,|n\pm\rangle = e^{-i\lambda^{\pm}t}\,|n\pm\rangle \tag{I.73}$$

where $\lambda$ begin the eigenvalues:

$$\begin{aligned} \lambda^{\pm} &= \omega_n \pm \frac{1}{2}\Omega_n \\ \omega_n &= \omega(n+\frac{1}{2}) \end{aligned} \tag{I.74}$$

Suppose that we have initially (t=0) an atom in the excited state $|e\rangle$ inside a cavity containing n-photons. The initial state $|\psi_e(0)\rangle = |e,n\rangle$, expanded on the dressed states basis is:

$$|\psi_e(0)\rangle = \cos\frac{\theta}{2}\,|n+\rangle - \sin\frac{\theta}{2}\,|n-\rangle \tag{I.75}$$

According to Eq (I.73), the time evolution of this state is:

$$|\psi_e(t)\rangle = e^{-i\lambda^{+}t}\cos\frac{\theta}{2}\,|n+\rangle - e^{-i\lambda^{-}t}\sin\frac{\theta}{2}\,|n-\rangle \tag{I.76}$$

by reverting to the uncoupled basis using Eq (I.69), we find:

$$|\psi_e(t)\rangle = e^{-i\lambda^{+}t}\cos^2\frac{\theta}{2}\,|e,n\rangle + e^{-i\lambda^{-}t}\sin^2\frac{\theta}{2}\,|e,n\rangle + \frac{\sin\theta}{2}(e^{-i\lambda^{+}t} - e^{-i\lambda^{-}})\,|g,n+1\rangle \tag{I.77}$$

for convenient and to simplify the equation, we introduce the constant coefficients L,M and N given by:

$$\begin{aligned} L &= \cos^2\frac{\theta}{2} = \frac{1}{2}(1+\frac{\Delta}{\Omega_n}) \\ M &= \sin^2\frac{\theta}{2} = \frac{1}{2}(1-\frac{\Delta}{\Omega_n}) \\ N &= \frac{\sin\theta}{2} = \frac{2g\sqrt{n+1}}{2\Omega_n} \end{aligned} \tag{I.78}$$

then the state simply becomes:

$$|\psi_e(t)\rangle = Le^{-i\lambda^{+}t}\,|e,n\rangle + Me^{-i\lambda^{-}t}\,|e,n\rangle + N(e^{-i\lambda^{+}t} - e^{-i\lambda^{-}t})\,|g,n+1\rangle \tag{I.79}$$

Similarly, if the atom is initially in the ground state $|g\rangle$ in a cavity containing n+1 photons, the time evolution of the initially state $|\psi_g(0)\rangle = |g, n+1\rangle$ in terms of the dressed state is:

$$|\psi_g(t)\rangle = e^{-i\lambda^+ t} \sin\frac{\theta}{2} |n+\rangle + e^{-i\lambda^- t} \cos\frac{\theta}{2} |n-\rangle \tag{I.80}$$

moving to the uncoupled basis, we find:

$$|\psi_g(t)\rangle = Le^{-i\lambda^+ t} |g, n+1\rangle + Me^{-i\lambda^- t} |g, n+1\rangle + N(e^{-i\lambda^+ t} - e^{-i\lambda^- t}) |e, n\rangle \tag{I.81}$$

From Eqs. (I.66) and (I.67), we observe that the atom–photon interaction leads to a reversible exchange of energy between the states $|e, n\rangle$ and $|g, n+1\rangle$. This exchange occurs at the General Rabi frequency $\Omega_n$, which characterizes the strength of the coupling between the atom and the cavity field. Consequently, the system undergoes coherent oscillations between these two states, known as Rabi oscillations, representing the quantum counterpart of the classical Rabi oscillation phenomenon.

## I.6 Spontaneous Emission and Vacuum Rabi Oscillation

Behind the main point of the Jaynes–Cummings Model which is characterizing the interaction between two-level atom with a single mode of electromagnetic field, this model provides an explanation for spontaneous emission. In the semi-classical description of electromagnetic field, photons can only be emitted through stimulation, since the Hamiltonian contains a classical electric field, Therefore, if the system is not driven by an external electric field, the emission process can not occur within this frame work.[9]

However, in reality photons can also be emitted into the vacuum, this phenomenon can not be explained within the semi-classical description and requires a fully quantized treatment of the electromagnetic field. For this reason we discuss the derivation of the Jaynes–Cummings Hamiltonian, its solution and the corresponding time evolution of the system. In particular we focus on this distinguish process and the observation of the Vacuum Rabi Oscillation.

The atom initially (t=0) in the excited state $|e\rangle$, inside a cavity with vacuum state ($n = 0$) at the resonant frequency ($\Delta = 0$), the general Rabi frequency now is:

$$\Omega_0 = 2g \tag{I.82}$$

the state evolves at the same way of n-photon Eq.(I.79) with $n = 0$ and $\Delta = 0$. In the interaction representation with respect to the pure phase $\omega$, the state $|e, 0\rangle$ at time t becomes:

$$|\psi_e(t)\rangle = \cos(gt) |e, 0\rangle + \sin(gt) |g, 1\rangle \tag{I.83}$$

In this case, the system undergoes coherent oscillations between $|e, 0\rangle$ and $|g, 1\rangle$ at the frequency $g$ which is the single-photon Rabi frequency.

The oscillation between $|e, 0\rangle$ and $|g, 1\rangle$ can also be viewed as an *'oscillatory spontaneous emission'*. An atom initially in the excited state $|e\rangle$ emits a photon while undergoing a transition to the ground state. In free space, the photon escapes and the atom remains in state $|\text{g}\rangle$. This is the usual irreversible spontaneous emission process. In the cavity, the photon emitted by the atom remains trapped. It can be absorbed, and then emitted again and so on. In a completely reversible coherent way and time unitary evolution.

The probability of finding the atom in the ground state with one photon in the cavity, $|g, 1\rangle$, is the same spin-flip probability obtained in the section on a spin in a rotating magnetic field .Eq(I.33).since both systems ultimately exhibit the same behave, namely *Rabi oscillation.* where $\omega_R$ corresponds to $2g$

Without doing calculations, the probability to find the system in state $|g, 1\rangle$ is:

$$P_g(t) = \sin^2(gt) \tag{I.84}$$

From the probability conservation, we find the probability of the system to be in the other state $|e, 0\rangle$ is:

$$P_e(t) = \cos^2(gt) \tag{I.85}$$

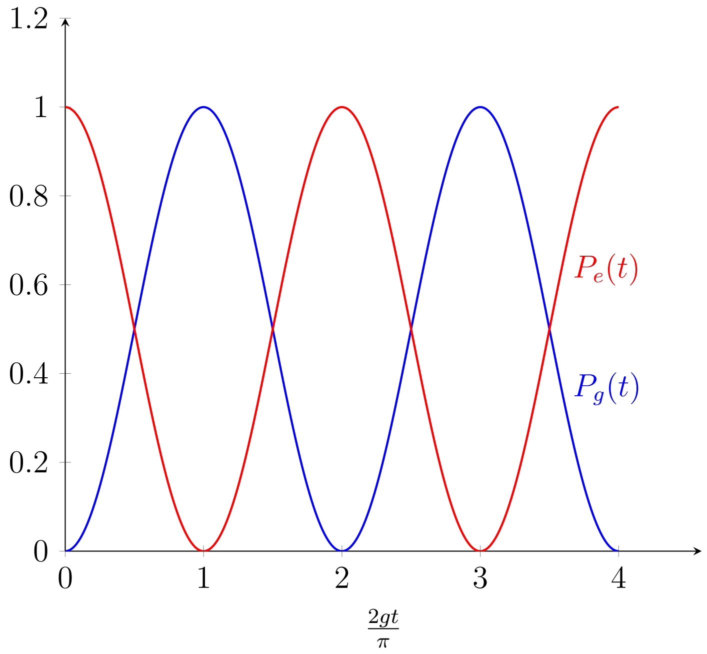


**Figure I.7:** The probability of finding a system in state $|g, 1\rangle$ and $|e, 0\rangle$ with $P_g(t) = \sin^2\left(\frac{2gt}{2}\right)$ and $P_e(t) = \cos^2\left(\frac{2gt}{2}\right)$

At $t = 0$, the atom-photon system is in the state $|e, 0\rangle$ where the atom is in the excited state with no photon in the cavity. As time evolves, the system oscillates between $|e, 0\rangle$ and $|g, 1\rangle$ until $gt = \frac{\pi}{2}$, at which point the system is entirely in the state $|g, 1\rangle$. The system then oscillates back to the state $|e, 0\rangle$; this is the phenomenon of *Rabi oscillation.*

## I.7 Experimental Aspects and the Application in Quantum Computation

The Jaynes–Cummings model has been theoretically and experimentally realized in several quantum platforms and constitutes a fundamental building block for quantum information processing including trapped ions and QED Circuit.[3] These experimental platforms have made the Jaynes–Cummings interaction a central mechanism for quantum state transfer, entanglement generation, and quantum gate implementation in modern quantum information technologies.

Experimental realizations of the Jaynes–Cummings interaction were achieved in cavity quantum electrodynamics experiments with Rydberg atoms.[4] A landmark demonstration was performed by the group of *Serge Haroche* in the early 1990s, culminating in the observation of vacuum Rabi oscillations reported in 1996.

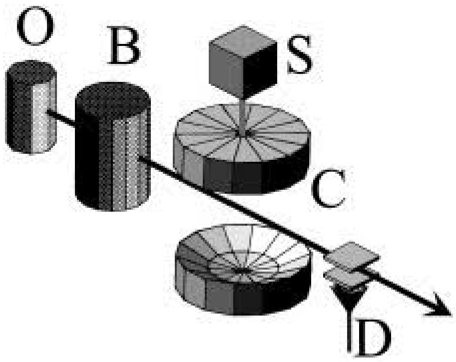


**Figure I.8:** Experimental setup of the cavity QED experiment

The experimental setup (sketched in figure I.8 ) that the group of *Serge Haroche* used to observe vacuum Rabi oscillations consists of a beam of rubidium atoms interacting with a single mode of a high-quality microwave cavity.

The apparatus is cooled to about 0.8 K in order to suppress thermal photons in the cavity, Rubidium atoms emerging from an oven **O** are prepared in a circular Rydberg state $|e\rangle$ with principal quantum number n=51 in a preparation zone **B**, These atoms are then launched in short pulses with an average velocity of approximately $v_0 = 350m/s$ toward the cavity **C**.

The cavity **C** is formed by two superconducting niobium mirrors and supports well-defined microwave modes. One of these modes is tuned into resonance with the atomic transition between the Rydberg states n=51 and n=50, corresponding to the transition $|e\rangle \to |g\rangle$. A weak static electric field is applied across the mirrors to stabilize the circular Rydberg orbit and to finely tune the atomic transition frequency via the Stark effect. The high quality factor of the cavity ensures a long photon lifetime, allowing coherent interaction between the atom and the cavity field during the atom's transit.

A stable microwave source **S** can inject a small coherent field into the cavity, with an energy ranging from zero to a few photons. After passing through the cavity, the atomic state is analyzed by state-selective field ionization, enabling the measurement of the transition probability between the excited and ground states and revealing the coherent atom–photon dynamics.

One of the most application of Jaynes–Cummings model in Quantum Information Processing is circuit Quantum electrodynamics (QED)[3], which studies the interaction of nonlinear superconducting circuits, acting as artificial atoms or as qubits, with quantized electromagnetic fields in the microwave frequency domain. So it can be viewed as a solid-state implementation of the concepts originally developed in cavity QED. Instead of natural atoms interacting with photons confined in an optical cavity, circuit QED employs superconducting circuits that behave as artificial atoms (qubits) coupled to microwave photons in a superconducting resonator. The interaction between the qubit and the resonator mode is well described by the Jaynes–Cummings Hamiltonian, making circuit QED a direct extension of cavity QED concepts to engineered quantum systems. This approach was strongly inspired by atomic cavity QED and allows the study and control of light–matter interaction in a highly tunable solid-state platform.

we do not go further into the details and equations here; we only wish to inecate that the Jaynes–Cummings Model is really fundamental building blocks for Quantum Information Processing, and has led to an independent and thriving field of research known tody as circuit QED.

# Chapter II

## EPR Paradox, Quantum Steering and Entanglement

# Chapter II

# EPR Paradox, Quantum Steering and Entanglement

## II.1 Introduction

Again, We begin this chapter with a quotation by *Erwin Schrodinger*:

> "*Entanglement is not one but rather the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought.*"

*Erwin Schrodinger* wrote this quotation in his famous paper "Discussion of Probability Relations between Separated Systems" [16]criticizing the paper of Einstein, Podolsky and Rosen (EPR paradox), and giving for the first time the definition of the entanglement.

In this chapter we go further into the EPR paradox and show that their argument was incorrect by discussing the so-called Bell inequalities by taking the two-level system as model, then we return to Schrodinger's idea by analyzing the Quantum Steering and Steerability inequality. Finally we present entanglement in a more rigorous way, including its definition, properties and measurement by introducing some quantities that will be useful for the application.

**properties of the singlet state of two particles of spin 1/2.**

In order to realize the EPR paradox in a simple way, we will take the spin 1/2 system as an example. Before starting with EPR argument, we introduce some properties of the single state of the two spin 1/2 particles that will be useful latter on.[21]

Suppose we have two entangled spin 1/2 particle (1 and 2), we take one of the bell states that:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle) \tag{II.1}$$

This state is the same for whatever direction $a$ we use to define **a** basis of spin states:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|a;+\rangle \otimes |a;-\rangle - |a;-\rangle \otimes |a;+\rangle) \tag{II.2}$$

In this singlet, we ask what is the probability $P(a+, b+)$ to find first particle in spin up along $a$ and the second particle with spin up along $b$, with $a$ and $b$ are two arbitrarily chosen unit vectors?

$$P(a+, b+) = \left|(\langle a+| \otimes \langle b+|)\, |\psi\rangle\right|^2 \tag{II.3}$$

by using the properties of the tensor product, the probability that we want is:

$$P(a+, b+) = \frac{1}{2}\left|\ \langle b+|a-\rangle\ \right|^2 \tag{II.4}$$

we recall that the overlap squared between two spin states is given by:

$$\left|\ \langle b+|a+\rangle\ \right|^2 = \cos^2\frac{\theta_{ab}}{2}$$

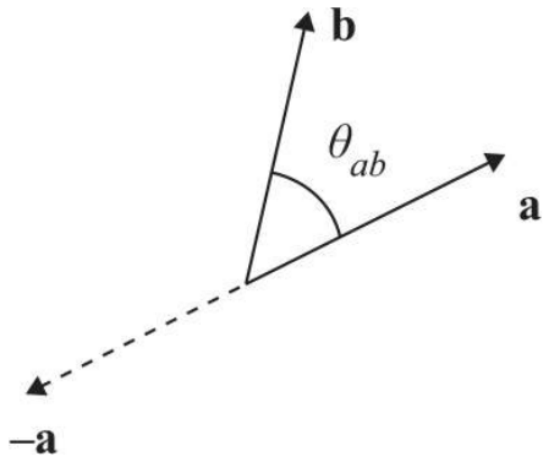


**Figure II.1:** Directions associated with the vectors a and b

from figure (II.1) we see that the angle between $-a$ and $b$ is $\pi - \theta_{ab}$, therefore:

$$P(a+, b+) = \frac{1}{2}\cos^2\frac{(\pi - \theta_{ab})}{2} \tag{II.5}$$

Our final result is therefore

$$P(a+, b+) = \frac{1}{2}\sin^2\frac{\theta_{ab}}{2} \tag{II.6}$$

if $b = -a$ then $P(a+, a-) = \frac{1}{2}$, which is what we expect. If we measure using orthogonal vectors, like the unit vectors $\hat{x}$ and $\hat{z}$, we get $P(z+, x+) = \frac{1}{4}$

### II.2 The EPR Paradox

In 1935, Einstein, Podolsky, and Rosen published their paper entitled "can Quantum Mechanical Description of Physical Reality Be Considered Complete?".[8] They started the paper by emphasizing the distinction between objective reality and the theoretical concepts used to describe it. They argued that a satisfactory physical theory must not only be correct in its predictions but also provide a complete description of physical reality.[5]

"Any serious consideration of a physical theory must take into account the distinction between the objective reality, which is independent of any theory, and the physical concepts with which the theory operates. These concepts are intended to correspond with the objective reality, and by means of these concepts we picture this reality to ourselves. In attempting to judge the success of a physical theory, we may ask ourselves two questions: the first one:'Is the theory correct?' and the second one: 'Is the description given by the theory complete?' It is only in the case in which positive answers may be given to both of these questions, that the concepts of the theory may be said to be satisfactory".[8]

The essence of the 1935 argument of Einstein, Podolsy and Rosen is a demonstration of the incompatibility between the notion of *local realism* and the *completeness* of quantum mechanics. This is understood as two assumptions of measurement:[21]

1. The result of a measurement on a particle in an entangled pair corresponds to some aspect of reality. If the measurement of an observable gives a value, that value was a definite property of the state before measurement.

2. The result of a measurement is independent of actions performed at a distant location at the same time.

Both assumptions seem eminently reasonable at first thought, but both are violated in quantum mechanics. the way physicists did in order to corresponds this aspects of reality and locality of measurements by introducing what is called *local hidden variable*.

A local hidden variable is a hypothetical parameter whose value is not accessible to the experimentalist and is introduced to provide a more complete description of physical systems. Let us denote by $\lambda$ a set of such hidden variables, assumed to be distributed according to a probability distribution $\rho(\lambda)$. In this framework, measurement outcomes are predetermined functions of both the measurement setting and the hidden variable. For instance, if Alice measures the spin along the $z$ or $x$ direction, the outcomes are described by two functions $A(z, \lambda)$ and $A(x, \lambda)$, each taking values $\pm 1$. These functions specify the result of any measurement once the value of $\lambda$ is fixed. Similarly, Bob's results are described by analogous functions, sharing the same hidden variable $\lambda$ for each entangled pair, reflecting the assumption of locality. The statistical predictions of the theory are then obtained by averaging over $\lambda$, weighted by the probability distribution $\rho(\lambda)$, in analogy with quantum expectation values.

However, such local hidden variable models fail to reproduce all the predictions of quantum mechanics, as demonstrated by Bell's inequalities, whose violation in experiments rules out any theory based on local hidden variables.

### II.2.1 Bell's inequality

Bell's inequality are provide a way to translate the EPR assumptions into mathematical equalities. as an example we consider a spin 1/2 system in order to derive these inequalities.[21]

Assuming that Alice and Bob share a pair of entangled spin 1/2 particles. Both perform measurements along $-z-$ direction. Therefore. According to state $|\psi\rangle$ (Eq II.1), if Alice finds spin up, then Bob finds spin down, and if Alice finds spin down, Bob finds spin up; this reflects their correlations which is interesting thing about Quantum Mechanics.

However, EPR say that there is nothing interesting about this correlation, the so-called entangled singlet pairs are just pairs of particles that have definite spins directions. Moreover, in this logic, the results of quantum mechanical measurements are reproduced if our large ensemble of pairs has the following distribution of states:

- 50% of pairs, particle 1 has spin up, and particle 2 has spin down.
- 50% of pairs, particle 1 has spin down, and particle 2 has spin up.

It is therefore not surprising that the results are correlated, since there is no quantum superposition in this description and the measurements yield definite spin values. This is where local hidden variables come into play: the spin of particle 1 is described as a function of a hidden variable $A(\lambda)$. Although the value of $\lambda$ is unknown, knowing it would determine the spin uniquely, since the system is assumed to have definite spin values.

The challenge for the EPR proposal is to keep reproducing the results of more complicated measurements. Suppose each of Alice and Bob can measure spin along two possible axes: $x-$ and $z-$ axes. They measure in any of these two directions. EPR logic would posit that in any entangled pair each particle has a definite state of spin in these two directions.

Following the logic of EPR, In this setup the observed quantum mechanical results are matched if our ensemble of pairs has the following properties:

- 25% of pairs have particle 1 in $(x+; z+)$ and particle 2 in $(x-; z-)$
- 25% of pairs have particle 1 in $(x+; z-)$ and particle 2 in $(x-; z+)$
- 25% of pairs have particle 1 in $(x-; z+)$ and particle 2 in $(x+; z-)$
- 25% of pairs have particle 1 in $(x-; z-)$ and particle 2 in $(x+; z-)$.

Note some perfect correlations in this EPR-like setup: particles 1 and 2 have opposite spins in each possible direction, so knowing the spin of one in a particular direction tells us the spin of the second. this is exactly corresponds to quantum mechanical calculations in Eq(II.6), for example the probability to find the first particle in spin up and the second particle in spin down along $-z-$ direction $P(z+, z-)$ then the first and the third cases apply, and thus this probability is 1/2. For the probability to find the first particle in spin up along $-z-$ direction and the second particle in spin up along $-x-$ direction $P(z+, x+)$ is 1/4 which corresponds only the third case. Therefore the quantum mechanical answers indeed arise for all the possibilities.

The core idea of Bell was to realize that when we can measure in three directions, the quantum mechanical answers cannot be reproduced by suitable ensembles with pairs of particles having their own definite spin states. Suppose each observer can measure along any one of the three vectors a, b and c. Each particle is just measured once, along one of these directions. Considering $N$ a total Number of a possibilities $N = \sum_{i=1} N_i$, following the EPR logic, contain particles with well-defined spins on these three directions. Labels for particles $(a+; b-; c+)$ that referring for measuring spin $(S_a = \frac{\hbar}{2}; S_b = -\frac{\hbar}{2}; S_c = \frac{\hbar}{2})$ respectively. EPR logic would try to give a distribution of pairs of different types that would match quantum mechanical results. The result of EPR distribution giving by the following table:

| Populations | Particle 01 | Particle 02 |
|---|---|---|
| $N_1$ | $(a+; b+; c+)$ | $(a-; b-; c-)$ |
| $N_2$ | $(a+; b+; c-)$ | $(a-; b-; c+)$ |
| $N_3$ | $(a+; b-; c+)$ | $(a-; b+; c-)$ |
| $N_4$ | $(a+; b-; c-)$ | $(a-; b+; c+)$ |
| $N_5$ | $(a-; b+; c+)$ | $(a+; b-; c-)$ |
| $N_6$ | $(a-; b+; c-)$ | $(a+; b-; c+)$ |
| $N_7$ | $(a-; b-; c+)$ | $(a+; b+; c-)$ |
| $N_8$ | $(a-; b-; c-)$ | $(a+; b+; c+)$ |

To find the probability $P(a, b)$, for example, we look for the lines in the table that have particle 1 in $a+$ as well as particle 2 in $b+$. The populations that satisfy this are $N_3$ and $N_4$, and their sum divided by the total number $N$ of pairs is the desired probability. In this way we record the following probabilities:

$$P(a+; b+) = \frac{N_3 + N_4}{N}; P(a+; c+) = \frac{N_2 + N_4}{N}; P(b+; c+) = \frac{N_7 + N_3}{N} \tag{II.7}$$

Consider now the trivially correct inequality:

$$N_3 + N_4 \leq N_3 + N_7 + N_4 + N_2 \tag{II.8}$$

Dividing by $N$ and using Eq(II.7) we obtain an inequality that is called *Bell's inequality*:

$$P(a+; b+) \leq P(a+; c+) + P(b+; c+) \tag{II.9}$$

It is in general, whatever the populations we chose, This inequality must hold. If quantum mechanics is true, given Eq(II.6) we would have:

$$\frac{1}{2}\sin^2\frac{\theta_{ab}}{2} \leq \frac{1}{2}\sin^2\frac{\theta_{ac}}{2} + \frac{1}{2}\sin^2\frac{\theta_{bc}}{2} \tag{II.10}$$

This inequality must hold, because we have Bell's inequality which is derived from the local realism, then if quantum mechanics is true, it must hold Bell's inequality. But this is violated for many choices of angles, considering the planar configuration in Fig(II.2):

$$\theta_{ab} = 2\theta, \theta_{ac} = \theta_{cb} = \theta$$

. For this situation, the inequality (II.10) becomes:

$$\frac{1}{2}\sin^2\theta \leq \sin^2\frac{\theta}{2} \tag{II.11}$$

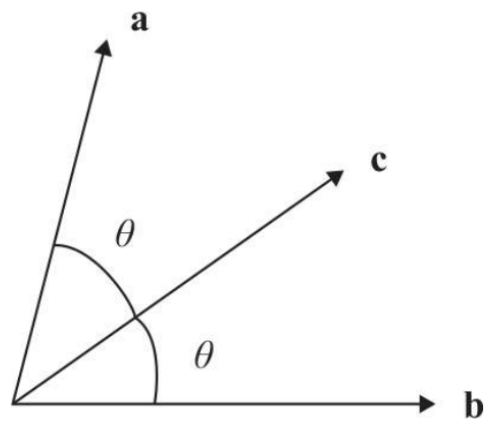


**Figure II.2:** A planar configuration for the unit vectors $a$, $b$ and $c$.

it is not true for all $\theta$, giving $\theta << 1$ then $\sin\theta = \theta$ for that:

$$\frac{1}{2}\theta \leq \frac{\theta^2}{4} \tag{II.12}$$

This inequality is violated for any $\theta \leq \frac{\pi}{2}$. Experimental results have confirmed that Bell inequalities are violated, and thus the original claim of local realism by EPR is incorrect. it was a great result of Bell to show that quantum mechanics is in contradiction with local realism. There is no way to keep the idea of quantum mechanics and put a hidden variable assuming that there is a real values that specify the particle properties or there is no distance effect in measurements.

### II.3 Quantum Steering

As mentioned in the introduction, EPR's argument prompted an interesting response from Schrödinger, who was not prepared to accept EPR's assumptions about the measurements and more that quantum mechanics was incomplete. But at the same time, he was not see any mistake with their argument. For this reason he described this argument by a Paradox, and that is why it is called EPR paradox.[5]

Clearly Schrödinger was also interested in implications arising from composite quantum systems described by nonfactorizable pure states. He described this situation By a famous terms:" If two separated bodies, each by itself known maximally, enter a situation in which they influence each other and separate again, then there occurs regularly... entanglement of our knowledge of the two bodies."[5][16]

By this definition of entanglement, Schrödinger then defined the process of disentanglement which occurs when a nondegenerate observable is measured on one body: "After establishing one representative by observation, the other one can be inferred simultaneously... this procedure will be called the disentanglement." This leads us directly to the EPR paradox, as Schrödinger describes it: "EPR called attention to the obvious but very disconcerting fact that even though we restrict the disentangling measurements to one system, the representative obtained for the other system is by no means independent of the particular choice of observations which we select for that purpose and which by the way are entirely arbitrary".[16][5]

Schrödinger describes this ability to affect the state of the remote subsystem as *steering*. In order to describes the quantum steering more accurate, let us consider the following scenario, Alice and Bob shared bipartite state, Alice aims to convince another Bob that a shared quantum state is entangled, despite Bob not trusting Alice. While Bob accepts the validity of quantum mechanics, he assumes that his subsystem can be described by a local hidden state (LHS) model. Returning to the physicists who explained the EPR assumption by local hidden theorem. In this model, Bob's system is assumed to be in a pre existing quantum state$|\lambda\rangle$, determined by a hidden variable $\lambda$, which is distributed according to some probability distribution $\rho(\lambda)$. Alice's reported results may depend on , but Bob's measurement outcomes are entirely determined by his local quantum state and are independent of Alice's choice of measurement, reflecting the assumption of locality.[11]

If the correlations between Alice's and Bob's results can be explained within such an LHS model, then Bob has no reason to conclude that the state is entangled, since Alice could simulate the correlations using classical knowledge of $\rho(\lambda)$. However, if no such model can reproduce the observed correlations, then the shared state must be entangled. In this case, Alice is said to "steer" Bob's state, meaning that her choice of measurements nonlocally affects the set of states describing Bob's system.[11]

Therefore, the definition formula of Steering is the ability to demonstrate that a quantum state is entangled by the choice of measurements, in a way that cannot be explained by a local hidden state (LHS) model.

### II.3.1 Steerability and Steering Inequality

Considering the density operator $\rho_{AB}$ that describes a shared bipartite quantum state between Alice and Bob. Let $D_1$ and $D_2$ denote the sets of observables on the Hilbert spaces corresponding to Alice's and Bob's systems $A$ and $B$, respectively. An element of $D_1$ is denoted by $x$, with its set of outcomes labeled as $i \in L(x)$. Similarly, this applies to Bob with an element of $D_2$ is $y$ and outcomes labeled as $j \in L(y)$.[1]

The bipartite quantum state $\rho_{AB}$ is steerable so that Alice can steer Bob's state if the joint probability cannot be expressed as the following decomposition:

$$P(i,j|x,y) = \sum_{\lambda} p(\lambda)p(i|x,\lambda)p(j|y,\rho_\lambda) \tag{II.13}$$

where $P(i,j|x,y)$ denotes the joint probability distribution that Alice performs measurement $x$ on subsystem $A$ with outcome $i$, and Bob performs measurement $y$ on subsystem $B$ with outcome $j$. The quantity $p(i|x,\lambda)$ represents the conditional probability of obtaining outcome $i$ when Alice performs the measurement $x$ on her subsystem, given that the hidden variable has value $\lambda$. Similarly, denotes the conditional probability of obtaining outcome $j$ for measurement setting $y$, given that Bob's subsystem is described by the quantum state $\rho_\lambda$, as determined by the Born rule which is:

$$p(j|y,\rho_\lambda) = Tr(M_j|\rho_\lambda^B)$$

where $M_j$ is the projection of the measurement.
Additionally, $p(\lambda)$ is the probability distribution over the hidden variables $\lambda$, satisfying $\rho_\lambda \geq 0$ and $\sum_\lambda \rho(\lambda) = 1$

For two-level systems, such as the Jaynes–Cummings model considered here, quantum steering can be tested using inequalities based on spin correlations. In particular, for a bipartite arbitrary two-level state $\rho$, one can employ the CJWR inequality.

$$F(\rho,\mu) = \frac{1}{\sqrt{3}}\left|\sum_{i=1}^{3}\Big\langle A_i \otimes B_i\Big\rangle\right| \leq 1 \tag{II.14}$$

violation of this inequality implies that the observed correlations cannot be explained by a local hidden state (LHS) model. Consequently, Alice is able to steer Bob's state through her choice of measurements, leading Bob to conclude that the shared state is entangled.

Here $A_i = \hat{u} \cdot \vec{\sigma}$ and $B_i = \hat{v} \cdot \vec{\sigma}$ where $\vec{\sigma}$ is is a vector composed of the Pauli matrices $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, $\hat{u}$ and $\hat{v}$ are real orthonormal vectors. $\mu = (\hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{v}_1, \hat{v}_2, \hat{v}_3)$ represents the set of measurement directions. Thus:

$$\Big\langle A_i \otimes B_i \Big\rangle = Tr(\rho A_i \otimes B_i) \tag{II.15}$$

representing the operator $M$ which has the form

$$M = \frac{1}{\sqrt{3}}(A_1 \otimes B_1 + A_2 \otimes B_2 + A_3 \otimes B_3)$$

then the CJWR inequality can be reformulated as:

$$\Big|\Big\langle M_\rho \Big\rangle\Big| \leq 1 \tag{II.16}$$

the maximal expectation value of the operator $M$ for arbitrary two-level state $\rho$ is given by:

$$F(\rho) = \max_{\mu} F(\rho, \mu) = \max \Big|\Big\langle M_\rho \Big\rangle\Big| = \sqrt{c_1^2 + c_2^2 + c_3^2} \tag{II.17}$$

where $c_k$ is are the singular values of the matrix $C$ which is called the correlation matrix of $\rho$ with entries $C_{ij}$ given by:

$$C_{ij} = Tr\Big[\rho(\sigma_i \otimes \sigma_j)\Big]$$

then the singular values $c_k$ of the matrix $C$ is the squar root of the eigenvalues $E_k$ of the matrix $C^\dagger C$.
Obviously, $F(\rho) \geq 1$ indicates that the quantum state $\rho$ is steerable; otherwise, $\rho$ is nonsteerable. The maximal violation value of the three-setting linear steering inequality is $F^{max} = \sqrt{3}$ for all two-level states.

A quantity $S(\rho)$ is defined as:

$$S(\rho) = \max\Big\{0, \frac{F(\rho) - 1}{\sqrt{3} - 1}\Big\} \tag{II.18}$$

Clearly, $S(\rho) > 0$ indicates that the quantum state $\rho$ violates the CJWR inequality, implying that it is steerable. Conversely, $S(\rho) = 0$ indicates that $\rho$ is non-steerable. Therefore, $S(\rho)$ can serve as a measure of steerability for a two-level state $\rho$

## II.4 Entanglement

In this section, we explore the concept of entanglement in greater detail. Although it was previously discussed in the context of the EPR paradox and quantum steering, no precise definition, properties, or quantitative measures were provided. These aspects will be addressed in this section.

Many physicists regard entanglement as one of the most fundamental features of quantum mechanics, often viewed as its purest manifestation. A great deal of research has been devoted to revealing the nonclassical nature of quantum mechanics and the remarkable phenomena that arise from entanglement. These nonclassical correlations constitute a valuable resource for various fields of development, enabling applications such as quantum teleportation, exponential speedup in quantum algorithms, and enhanced precision in atomic and optical experiments.

### Definition of the entangled state

Consider a composite system consisting two particles.[21] The first particle is described by vectors $v_i$ in a complex vector space $V$, while the second one is described by vectors $w_i$ in a complex vector space $W$. The general state $|\psi\rangle$ of the composite system belongs to the tensor product space $V \otimes W$ and can be written as:

$$|\psi\rangle = \sum_{ij} \alpha_{ij} v_i \otimes w_j \tag{II.19}$$

Entangled states are those states that cannot be described as separate states of individual subsystems. In such states, it is not possible to assign independent properties to each particle. Moreover, a measurement performed on one subsystem affects the state of the other, regardless of the distance between them. A quantum state is said to be entangled if it cannot be written as a product of states of its subsystems. More mathematically, we can ask the following: is there vectors $v^*$ and $w^*$ where $v^* \in V$ and $w^* \in W$ in such a way $|\psi\rangle$ (Eq II.19) can be written as a single term $v^* \otimes w^*$

$$|\psi\rangle = v^* \otimes w^* \tag{II.20}$$

If no such $v^*$ and $w^*$ exist, we say that $|\psi\rangle$ is an entangled state of the two particles. If, on the other hand, $v^*$ and $w^*$ exist, then you are able to describe the state of the particles in $|\psi\rangle$ independently: particle 1 is in state $v^*$ and particle 2 in state $w^*$, and we say that $|\psi\rangle$ is not an entangled state.

Entanglement is a basis-independent property. Indeed, if the state can be factorized into $v^* \otimes w^*$ for some basis choice in $V$ and $W$, it can be factorized for any other basis choice by simply rewriting $v^*$ and $w^*$ in the new basis. If the state cannot be factorized into $v^* \otimes w^*$ for some basis choice in $V$ and $U$, it cannot be factorized for any other basis choice because factorization with another basis choice would then imply factorization in the original basis choice.

the simplest example to illustrate this is the two state system which is tow dimensional vector space. Let $V$ have a basis $e_1$, $e_2$ and $W$ have a basis $f_1$, $f_2$. Then the most general state in $V \otimes W$ is:

$$|\psi\rangle = A_{11} e_1 \otimes f_1 + A_{12} e_1 \otimes f_2 + A_{21} e_2 \otimes f_1 + A_{22} e_2 \otimes f_2 \tag{II.21}$$

with coefficients $A_{ij}$ that can be encoded by a matrix A:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{II.22}$$

The state is not entangled and manages to factor it out if there exist constants $a_1$, $a_2$, $b_1$ and $b_2$ such that:

$$A_{11}e_1 \otimes f_1 + A_{12}e_1 \otimes f_2 + A_{21}e_2 \otimes f_1 + A_{22}e_2 \otimes f_2 = (a_1e_1 + a_2e_2) \otimes (b_1u_1 + b_2u_2) \tag{II.23}$$

To have a solution, we must have

$$\begin{cases} A_{11} = a_1b_1 \\ A_{12} = a_1b_2 \\ A_{21} = a_2b_1 \\ A_{22} = a_2b_2 \end{cases} \tag{II.24}$$

Combining these four expressions gives us a consistency condition:

$$A_{11}A_{22} - A_{21}A_{12} = a_1b_1a_2b_2 - a_2b_1a_1b_2 = 0 \tag{II.25}$$

That means that:

$$det(A) = 0 \tag{II.26}$$

Therefore, the constants $a_1$, $a_2$, $b_1$ and $b_2$ are exist if and only if $det(A) = 0$. That implies that the state $|\psi\rangle$ is not entangled. In other world, the state $|\psi\rangle$ is entangled and can not factorized if and only if $det(A) \neq 0$

$$|\psi\rangle = \sum_{ij} A_{ij}e_i \otimes u_j \neq v^* \otimes w^* \Leftrightarrow det(A) \neq 0 \tag{II.27}$$

we can take an example of one the Bell state that will be discussed of two spin 1/2 particle. Consider the state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle) \tag{II.28}$$

comparing with the notation that we use for the basis vector that: $e_1 = |+\rangle$, $e_2 = |-\rangle$ and $u_1 = |+\rangle$, $u_2 = |-\rangle$ then the state is:

$$|\psi\rangle = \frac{1}{\sqrt{2}}e_1 \otimes u_2 - \frac{1}{\sqrt{2}}e_2 \otimes u_1 \tag{II.29}$$

the associated matrix $A$ would be:

$$A = \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

since $det(A) = \frac{1}{2} \neq 0$ then the stat is entangled.

One of the most important properties of entanglement is nonlocality, as discussed in the context of the EPR paradox and quantum steering. The entanglement between two particles does not depend on the distance separating them; regardless of how far apart they are, the entanglement can persist. Another important property is its fragility under decoherence: entanglement is affected by interactions with the environment, which can lead to its degradation. This effect will be discussed in the next two chapters.

### II.4.1 Bell Basis State

Bell states are a set of four entangled orthonormal basis vectors in the state space of two spin one-half particles considered as Qubit. To describe this basis consider the tensor product $V \otimes W$, with $V$ and $W$ both the two dimensional complex vector space appropriate to two Qubit. Consider the following state:

$$|\phi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{II.30}$$

obviously, this state is entangled. its associated matrix is diagonal with equal entries of $\frac{1}{\sqrt{2}}$ and thus a nonzero determinant. Moreover, this state is unit normalized:

$$\langle\phi_0|\phi_0\rangle = 1 \tag{II.31}$$

We use this state as the first of our basis vectors for $V \otimes W$. Since this tensor product is four-dimensional, we need three more entangled basis states. They derived by the following:

$$|\phi_i\rangle = (I \otimes \sigma_i)\,|\phi_0\rangle \tag{II.32}$$

with $\sigma_i$ represent the Pauli matrices, $i = 0, 1, 2, 3$. Since the operator $I \otimes \sigma_i$ is unitary, it follows from the definition that all $|\phi_i\rangle$ are unit normalized.

Therefore, the full list of **Bell states** are giving by:

$$\begin{cases} |\phi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\ |\phi_1\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\ |\phi_2\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \\ |\phi_3\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \end{cases} \tag{II.33}$$

we can check the orthogonality by calculating $\langle\phi_i|\phi_j\rangle$, then we obtained:

$$\langle\phi_i|\phi_j\rangle = \delta_{ij} \tag{II.34}$$

Indeed, we have an orthonormal basis of entangled states.

The universal notation for Bell states that most references used which is also suitable for the next chapters are:

$$\begin{aligned} \left|\psi^{\pm}\right\rangle &= \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle) \\ \left|\phi^{\pm}\right\rangle &= \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle) \end{aligned} \tag{II.35}$$

### II.4.2 Measures of Entanglement

Measures of entanglement constitute one of the most important topics in quantum information theory and are, in fact, central to the study of entanglement, particularly in relation to quantum computation. In some approaches, entanglement is even characterized in terms of how it can be quantified. One of the most important information-theoretic quantities used to measure entanglement is **entropy.**[15]

Entropy is a key concept of quantum information theory. It measures how much uncertainty there is in the state of a physical system. In a classical information theory, the entropy is well-known as *Shannon entropy* that measures the uncertainty associated with a classical probability distribution of a random values $X$ is defined by:[15]

$$H(X) = \sum_x px \log p_x \tag{II.36}$$

In the Quantum world, Quantum states are described in a similar fashion, with density operators replacing probability distributions. Von Neumann defined the entropy of a quantum state $\rho$ by the formula:

$$S(\rho) = -Tr(\rho \log \rho) \tag{II.37}$$

In this formula logarithms are taken to base two, as usual. If $\lambda_x$ are the eigenvalues of $\rho$ then von Neumann's definition can be re-expressed:

$$S(\rho) = -\sum_x \lambda_x \log \lambda_x \tag{II.38}$$

If we want to redefine the entanglement, we would say that the entanglement $E$ of a pure state of a pair of quantum systems $|\psi_{AB}\rangle$ is defined as the entropy of either member of the pair.

$$E(|\psi_{AB}\rangle) = S(\rho_A) = S(\rho_B) \tag{II.39}$$

Here $\rho_A$ is the partial trace of $\rho_{AB}$ over subsystem B, and $\rho_B$ is defined similarly.

$$\rho_A = Tr_B(\rho_{AB}) = \sum_{e_B} \langle e_B| \rho_{AB} |e_B\rangle \tag{II.40}$$

with $e_B$ are the basis state of the system $B$. For instance of pure entangled state, we take one of the bell state (Eq II.35) with labeled as two Qubits as $Qubit_A \otimes Qubit_B$:

$$\left|\psi^+\right\rangle = \frac{1}{\sqrt{2}}(|0_A 1_B\rangle + |1_A 0_B\rangle)$$

the density matrix of the total state $\rho_{AB}$ is:

$$\rho_{AB} = \left|\psi^+\right\rangle \left\langle\psi^+\right| \tag{II.41}$$

then we find $\rho_A$ and $\rho_B$ by partial trace of $\rho_{AB}$:

$$\rho_A = \rho_B = \frac{1}{2} I \tag{II.42}$$

Therefore, Von Neumann entropy of the system $A$ or $B$ is:

$$S(\rho_A) = S(\rho_B) = 1 \tag{II.43}$$

this implies that the subsystems $A$ and $B$ are completely random states, which refer to that the state $|\psi_{AB}\rangle$ is entangled state. If we have another state that the subsystem has $S(\rho) = 0$, that implies that the subsystems are pure, which refer to that the total state is not entangled.

The entanglement of formation of the mixed state $\rho$ is then defined as the average entanglement of the pure states of the decomposition, minimized over all decompositions of $\rho$[17]:

$$E(\rho) = \min \sum_i p_i E(\psi_i) \tag{II.44}$$

The quantification of entanglement can be mathematically involved and sometimes difficult to interpret. To address this, William K. Wootters [17] introduced a powerful quantity known as the concurrence, which provides a convenient measure of entanglement.

He showed that the entanglement of formation can be expressed as a function of the concurrence:

$$E(|\psi\rangle) = f\Big(C(|\psi\rangle)\Big) \tag{II.45}$$

where $C$ is the concurrence, and the function $f(C)$ is monotonically increasing, ranging from 0 to 1 as $C$ varies from 0 to 1. Therefore, concurrence itself can be used as a measure of entanglement.

The concept is also convenient for expressing the entanglement of a pure bipartite state $\rho_{AB}$[6]. the concurrence is defined as:

$$C(\rho_{AB}) = \sqrt{2(1 - Tr(\rho_A^2)} \tag{II.46}$$

the concurrence of the mixed bipartite state[17] $\rho$ is defined as:

$$C(\rho) = \max \Big\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\Big\} \tag{II.47}$$

the $\lambda_i$'s are the eigenvalues, in decreasing order, of the Hermitian matrix $R$:

$$R = \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}$$

Alternatively, one can say that the$\lambda_i$'s are the square roots of the eigenvalues of the non-Hermitian matrix $\rho\tilde{\rho}$. where $\tilde{\rho}$ is giving by:

$$\tilde{\rho} = (\sigma_2 \otimes \sigma_2)\rho(\sigma_2 \otimes \sigma_2)$$

The application of these formulas motivates the search for a simpler expression of concurrence. In this context, Joseph H. Eberly[20], in his studies of decoherence and the Jaynes–Cummings model, introduced a convenient formula for a particular class of mixed bipartite states known as X-states, which are defined by the density matrix:

$$\rho = \begin{pmatrix} a & 0 & 0 & w \\ 0 & b & z & 0 \\ 0 & z^* & c & 0 \\ w^* & 0 & 0 & d \end{pmatrix} \tag{II.48}$$

For such states, the concurrence takes the simple form:

$$C(\rho) = 2\max \Big\{0, |z| - \sqrt{ad}, |w| - \sqrt{bc}\Big\} \tag{II.49}$$

These states frequently arise in open quantum systems and in the study of decoherence dynamics. We will use this form in the analysis of entanglement in the Jaynes–Cummings model.

# Chapter III

## The Dynamics of Entanglement and Steerability in the Double Jaynes–Cummings Model under Noiseless Environment

# Chapter III

# The Dynamics of Entanglement and Steerability in the Double Jaynes–Cummings Model under Noiseless Environment

## III.1 Introduction

Understanding how quantum correlations emerge, evolve, and degrade in physical systems is one of the central challenges of quantum information science. Among these correlations, entanglement and quantum steering play a fundamental role, not only in revealing the nonclassical nature of quantum mechanics but also in enabling advanced quantum technologies.

The Jaynes–Cummings model is one of the most useful physical platforms for quantum computation, as entanglement and steerability are fundamental properties in quantum information theory, distinguishing quantum computation from classical computation. Therefore, the investigation of the dynamics of entanglement and steerability within the Jaynes–Cummings model constitutes an active and important area of research. In particular, it provides a clear framework for studying light–matter interaction at the quantum level and its role in quantum technologies.

In this chapter, we first discuss the entanglement between two atoms described by the Jaynes–Cummings model and, conversely, the entanglement between two cavity field modes.[9] These correlations are generated and controlled through the manipulation of atomic states and cavity modes via Rabi oscillations.

Based on the works of Joseph H. Eberly[19] and Ya- Ben and others[1], we will investigate the dynamics of entanglement and steerability in the double Jaynes–Cummings model, including the so-called sudden death effect, without considering environmental decoherence. This allows us to focus on the intrinsic quantum dynamics of the system. The analysis will rely on the tools and concepts that have been rigorously developed in the previous chapters.

### III.2 Entanglement of the Atoms and Photons in the Jaynes-Cummings Model

The dynamics of the Rabi oscillation in vacuum corresponds to a unitary evolution in the Hilbert space spanned by the states $\{|e,0\rangle, |g,0\rangle\}$. This evolution generally produces entanglement between the atom and field partners (section II.3). By combining such rotations on composite systems involving the cavity field and successive atoms crossing the cavity, it is possible to engineer complex entangled states. These experiments provide an interesting testing ground for quantum information physics.

#### III.2.1 Entanglement Between Two Jaynes–Cummings Atoms

In the section (II.3), we discussed the dynamics of the Jaynes –Cummings states and showed how $|e,n\rangle$ and $|g,n+1\rangle$ evolve in time. In this section, we focus on the vacuum Rabi oscillations where the Atom-cavity field system oscillates between $|e,0\rangle$ and $|g,1\rangle$. This evolution can be viewed as arising from the coupling of two qubits: Atomic qubit (spanned by $|e\rangle$ and $|g\rangle$) and cavity field qubit (spanned by $|0\rangle$ and $|1\rangle$ corresponding to the vacuum the one-photon states). Preparing the system initially in one of the three states $|g,0\rangle$, $|g,1\rangle$ and $|e,0\rangle$, we note that the state $|g,0\rangle$ remains unchanged. For the other two state, respectively, the time evolution based on Eq(I.79) and Eq(I.81) in the resonance case $\Delta = 0$ is given by:

$$\begin{aligned}|\psi_g(t)\rangle &= \cos(gt)\,|g,1\rangle - \sin(gt)\,|e,0\rangle \\ |\psi_e(t)\rangle &= \cos(gt)\,|e,0\rangle + \sin(gt)\,|g,1\rangle\end{aligned} \tag{III.1}$$

This vacuum Rabi evolution is equivalent to the following Quantum circuit:

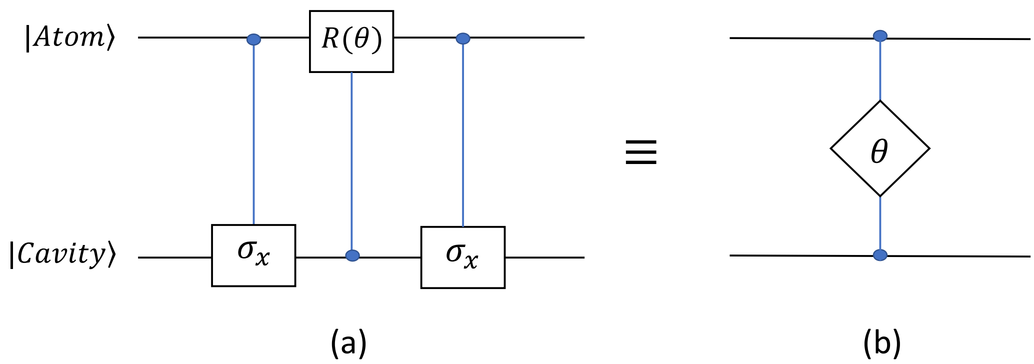


**Figure III.1:** (a): Quantum circuit of vacuum Rabi oscillation (b): simplified circuit

The quantum circuit representing two qubits where the top line represents the atom qubit and the bottom one the field qubit. Two control-not gates (conditional-$\sigma_x$ operations) in which the atom is the control and the field is the target, where they are sandwiching a controlled-$R(\theta)$ gate of the atom-qubit, conditioned to the state of the field. The angle of the $R(\theta)$ rotation is $\theta = gt$ and its axis in the Bloch sphere representation is $OY$, so that:

$$R(\theta) = e^{-i\theta\sigma_y/2} = \cos(gt) - i\sin(gt)\sigma_y \tag{III.2}$$

We denote by $U_{VR}(t)$ the unitary operator acting on the two-qubit Hilbert space, which represents the effective evolution associated with the vacuum Rabi oscillation and corresponds to the sequence of quantum gates in the equivalent quantum circuit.

$$U_{VR}(t) = U_{\sigma_x} R(\theta) U_{\sigma_x} \tag{III.3}$$

Expressed explicitly in the computational basis of the two qubits, this transformation is:

$$\begin{aligned}
U_{VR}(t)\,|e,0\rangle &= \cos(gt)\,|e,0\rangle + \sin(gt)\,|g,1\rangle \\
U_{VR}(t)\,|g,1\rangle &= \cos(gt)\,|g,1\rangle - \sin(gt)\,|e,0\rangle \\
U_{VR}(t)\,|g,0\rangle &= |g,0\rangle \\
U_{VR}(t)\,|e,1\rangle &= |e,1\rangle
\end{aligned} \tag{III.4}$$

In this circuit, the atomic qubit is successively the control, the target and then the control again. This representation stresses that the two qubits play a symmetrical role in the vacuum Rabi oscillation, both being in turn control and target. we analyzed the properties of the three specific Rabi pulses that we discussed in section(I.4):

- $\pi/4$- Vacuum Rabi pulse: consider $\theta = \pi/4$ corresponding to an effective interaction time $t = \pi/4g$. Starting from $|e.0\rangle$, the atom–cavity state becomes:

  $$U(\frac{\pi}{4g})\,|e,0\rangle = \frac{1}{\sqrt{2}}(|e,0\rangle + |g,1\rangle) \tag{III.5}$$

  This is a maximally entangled atom–field state which remains frozen after the atom has left the cavity, as long as field relaxation, the dominant cause of decoherence, can be neglected.

- $\pi/2$-Vacuum Rabi pulse: The effective interaction time $t = \pi/4g$ defines a $\pi/2$-quantum Rabi pulse corresponding to the transformations:

  $$|e,0\rangle \rightarrow |g,1\rangle \quad \text{and} \quad |g,1\rangle \rightarrow |e,0\rangle \tag{III.6}$$

  The excitation is swapped between the atom and the cavity.

- $\pi$-Vacuum Rabi pulse: consider the $\pi$-Rabi pulse corresponding to $t = \pi/g$ After it, the atom–field system returns to the initial state, In fact, it produces a global phase shift of the atom–field state, which turns out to be very useful for quantum information processing in Cavity QED experiments. This pulse realizes the transformations:

  $$|e,0\rangle \rightarrow -\,|e,0\rangle \quad \text{and} \quad |g,1\rangle \rightarrow -\,|g,1\rangle$$

Using these Rabi pulses, we are now ready to generate an entangled state between two atoms in the form of a Bell state. The experimental setup used for the Jaynes–Cummings model described in Section (I.6), with slight modifications, can be employed to create entanglement between two atoms sequentially injected into the same cavity. As illustrated in Fig(III.2), two atoms cross the cavity one after the other, where the first atom $A_1$ is initially prepared in the excited state $|e\rangle$, while the second atom $A_2$ is prepared in the ground state $|g\rangle$.

The first atom $A_1$ interacts with the empty cavity for a $\pi/4$-Vacuum Rabi pulse Eq (III.5). This interaction prepares the entangled state between atom $A_1$ and the cavity giving by:

$$|g\rangle_2 \otimes |e\rangle_1 \otimes |0\rangle \rightarrow |g\rangle_2 \otimes \frac{1}{\sqrt{2}}(|e,0\rangle + |g,1\rangle) \tag{III.7}$$

then copy the cavity state on to the second atom $A_2$. which means that a coherent transfer of quantum information from the cavity field to the second atom. Through a $\pi/2$-Rabi pulse Eq(III.6) the excitation stored in the cavity is completely transferred to the second

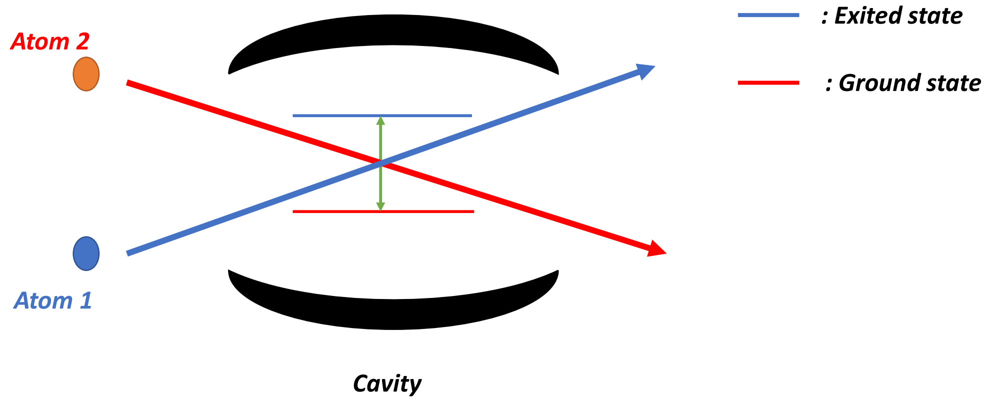


**Figure III.2:** set up experiments to generate two entangled JC atoms

atom leaving the cavity in the vacuum state and resulting in an entangled state between the two atoms, from Eq(III.7) we find:

$$|g\rangle_2 \otimes \frac{1}{\sqrt{2}}(|e,0\rangle + |g,1\rangle) \rightarrow \frac{1}{\sqrt{2}}(|e_1,g_2\rangle + |g_1,e_2\rangle) \otimes |0\rangle \tag{III.8}$$

therefore, we obtain two entangled atoms in the form of Bell state:

$$\left|\psi^+\right\rangle = \frac{1}{\sqrt{2}}(|e_1 g_2\rangle + |g_1 e_2\rangle) \tag{III.9}$$

Note that the atoms are entangled without having directly interacted. The entanglement is *catalysed* by the cavity, which ends up unchanged and unentangled.

### III.2.2 Entanglement Between Two Jaynes–Cummings cavity modes

The Bell atom-pair experiment uses a cavity field as a catalyst to entangle two atoms. Inversely, an atom can be employed in the same set-up to entangle two field modes. A photon in one of the two polarization modes $M_a$ and $M_b$. The resulting state is an entangled superposition of $|1_a, 0_b\rangle$ and $|0_a, 1_b\rangle$, two field states in which a single photon is present in either one of two modes with slightly different frequencies.

In this scheme, a single atom is used as a mediator to generate entanglement between two cavity modes. Initially, the atom $A_1$, prepared in the excited state $|e\rangle$, interacts resonantly with mode $M_a$ via a $\pi/4$-Rabi pulse, creating an entangled atom–field state:

$$|e\rangle \otimes |0\rangle_1 \otimes |0\rangle_b \rightarrow \frac{1}{\sqrt{2}}(|e,0_1\rangle + |g,1_1\rangle) \otimes |0_b\rangle \tag{III.10}$$

The atom is then rapidly tuned into resonance with the second mode $M_b$ ($\Delta = \delta$), where it undergoes a $\pi/2$-Rabi pulse. During this interaction, the atomic excitation is coherently transferred to mode $M_b$, effectively mapping the atomic component of the superposition onto the second field mode. Owing to the large frequency detuning between the two modes ($\delta >> 2g$), the evolution of mode $M_a$ remains frozen during this process.

$$\frac{1}{\sqrt{2}}(|e,0_1\rangle + |g,1_1\rangle) \otimes |0_b\rangle \rightarrow \frac{1}{\sqrt{2}}(|0_a,1_b\rangle + |0_a,1_b\rangle) \otimes |g\rangle \tag{III.11}$$

As a result, when the atom leaves the cavity in the ground state $|g\rangle$, the two modes are left in a maximally entangled state of the form:

$$\left|\psi^+\right\rangle = \frac{1}{\sqrt{2}}(|0_a 1_b\rangle + |0_a 1_b\rangle) \tag{III.12}$$

### III.3 Dynamics of Entanglement in Tow Jaynes-Cummings Atoms

We have now reached our intended purpose, which is investigated the entanglement dynamics of two isolated atoms, each in its own Jaynes-Cummings cavity. Based on the works of Joseph H. Eberly (2006)[19] we want to show the details calculations that he was derived and understanding more the results that he show on his article, the same thing we will do on the work of Y.Ben (2025) [1] with the steerability dynamics. without consideration of the effect of environment which we will work on it in the next chapter.

Consider the “double Jaynes-Cummings” model consisting of two two-level atoms. Each one is in a perfect one mode near-resonant cavity and interacts with its initially unexcited cavity mode, but each is completely isolated from the other atom and cavity. Fig (III.3)

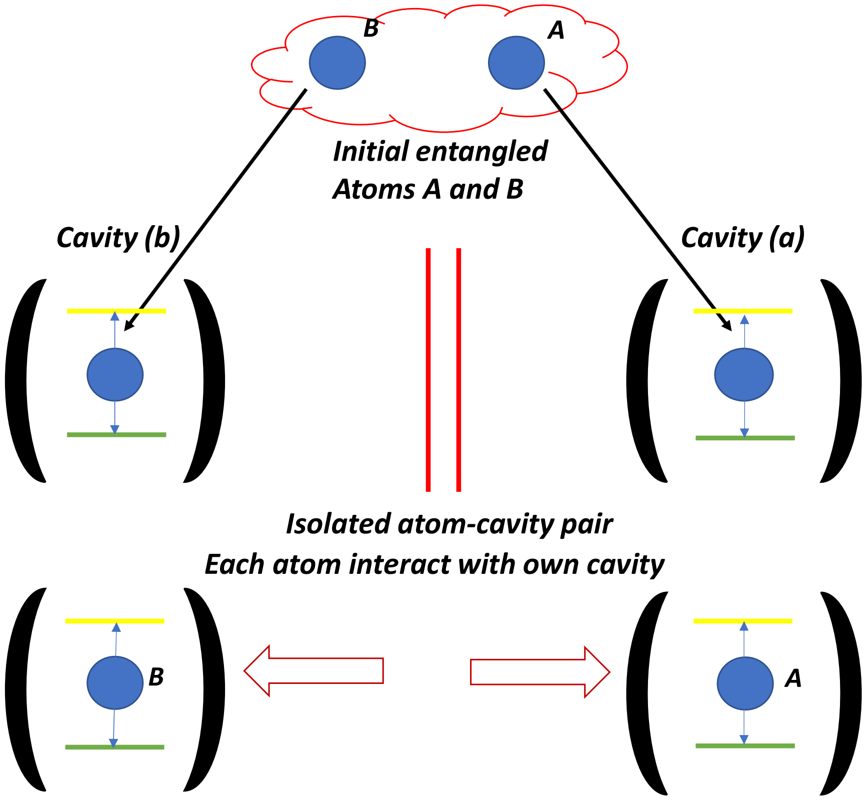


**Figure III.3:** set up for study dynamics of entanglement in double JC atoms

By tracing over the cavity modes at time $t$ we are left with a mixed state of atoms $A$ and $B$. this led us to forcibly creating a two qubit scenario, and various measures of entanglement are available. we use the Wootters' concurrence $C(\rho)$ that we discussed in details in the section of measure of entanglement.

The double Jaynes-Cummings Hamiltonian for our system may be written as, $\hbar = 1$:

$$\hat{H}_{tot} = \omega\sigma_z^A + \omega\sigma_z^B + \nu a^\dagger a + \nu b^\dagger b + g(a\sigma_+ + a^\dagger\sigma_-) + g(b\sigma_+ + b^\dagger\sigma_-) \tag{III.13}$$

Clearly there will be no interaction between atom $A$ and atom $B$ or between cavity $a$ and cavity $b$. The eigenstates of this Hamiltonian are products of the dressed states of the separate Jaynes–Cummings systems, which we discussed in the first chapter.

Assuming that both cavities are prepared initially in the vacuum state $|0\rangle_a \otimes |0\rangle_b$ and the two atoms are in a pure entangled state, Under these assumptions, there is never more than one photon in each cavity, so the cavity mode is essentially equivalent to a two-level system. This allows a uniform measure of quantum entanglement – concurrence – for both atoms and the cavity modes. J-H Eberly focus on the two types of Bell-like states as an atomic initial state:

$$|\psi(0)\rangle = \cos\alpha\,|eg\rangle + \sin\alpha\,|ge\rangle \quad \text{and} \quad |\phi(0)\rangle = \cos\alpha\,|ee\rangle + \sin\alpha\,|gg\rangle \tag{III.14}$$

with the first index denoting the state of atom $A$ and the second denoting the state of atom $B$. It is evident that when $\alpha = \pi/4$, the standard Bell states are reproduced.

### III.3.1 Entanglement Dynamics of the $\psi$-type Bell-like States

We begin with Bell-like State $|\psi(0)\rangle$, the initial state for the total system is given by:

$$\begin{aligned}|\Psi(0)\rangle &= |\psi(0)\rangle \otimes |00\rangle \\ &= \cos\alpha\,|eg00\rangle + \sin\alpha\,|ge00\rangle\end{aligned} \tag{III.15}$$

In order to find the evolution of the system $|\Psi(t)\rangle$, we use Eq(I.79) and Eq(I.81) from the section of the dynamics of the Jaynes–Cummings model.

Except for the ground state $|g,0\rangle$, which remains unchanged, the total state at time $t$ is giving by:

$$\begin{aligned}|\Psi(t)\rangle =& \cos\alpha\Big[(Le^{-i\lambda^+t}\,|e,0\rangle + Me^{-i\lambda^-t}\,|e,0\rangle) \otimes |g,0\rangle \\ &+N(e^{-i\lambda^+t} - e^{-i\lambda^-t})\,|g,1\rangle \otimes |g,0\rangle\Big] \\ &+\sin\alpha\Big[(Le^{-i\lambda^+t}\,|e,0\rangle + Me^{-i\lambda^-t}\,|e,0\rangle \otimes |g,0\rangle \\ &+N(e^{-i\lambda^+t} - e^{-i\lambda^-t})\,|g,1\rangle) \otimes |g,0\rangle\Big]\end{aligned} \tag{III.16}$$

In the way of J-H Eberly did we find:

$$|\Psi(t)\rangle = x_1\,|eg00\rangle + x_2\,|ge00\rangle + x_3\,|gg10\rangle + x_4\,|gg01\rangle \tag{III.17}$$

where the coefficients stand for the following time-dependent formulas:

$$\begin{aligned}x_1 =&(Le^{-i\lambda^+t} + Me^{-i\lambda^-t})\cos\alpha \\ x_2 =&(Le^{-i\lambda^+t} + Me^{-i\lambda^-t})\sin\alpha \\ x_3 =&N(e^{-i\lambda^+t} - e^{-i\lambda^-t})\cos\alpha \\ x_4 =&N(e^{-i\lambda^+t} - e^{-i\lambda^-t})\sin\alpha\end{aligned} \tag{III.18}$$

Note here that $\lambda^{\pm}$ and the constant coefficients $L$, $M$, and $N$ are given by Eq(I.74) and Eq(I.78) respectively. The information about the entanglement of two atoms is contained in the reduced density matrix $\rho^{AB}$ for the two atoms which can be obtained from Eq(III.16) by tracing out the photonic parts $a$ and $b$ of the total pure state $\rho^{\Psi} = |\Psi(t)\rangle\langle\Psi(t)|$:

$$\rho_{\Psi}^{AB} = \sum \langle ab|\,\rho^{\Psi}\,|ab\rangle \tag{III.19}$$

where the basis $|ab\rangle$ are the photonic states $\Big\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\Big\}$. then we obtained:

$$\begin{aligned}\rho_{\Psi}^{AB} =&|x_1|^2\,|eg\rangle\langle eg| + x_1^*x_2\,|ge\rangle\langle eg| + x_1x_2^*\,|eg\rangle\langle ge| \\ &+|x_2|^2\,|ge\rangle\langle ge| + (|x_3|^2 + |x_4|^2)\,|gg\rangle\langle gg|\end{aligned} \tag{III.20}$$

The explicit $4 \times 4$ matrix written in the basis $\Big\{ |ee\rangle , |eg\rangle , |ge\rangle , |gg\rangle \Big\}$ is given by:

$$\rho_{\Psi}^{AB} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & |x_1|^2 & x_1 x_2^* & 0 \\ 0 & x_1^* x_2 & |x_2|^2 & 0 \\ 0 & 0 & 0 & |x_3|^2 + |x_4|^2 \end{pmatrix} \tag{III.21}$$

Now we want to measure the entanglement between this two atoms after the system evolves. The measurement of this correlation is done by calculating the concurrence. we discussed in details the concurrence in section (II.3.3), and we can see in this case that $\rho^{AB}$ is exactly of the form of X-state Eq(II.48), and the concurrence of this like state is given by Eq(II.49). by corresponding we find the coefficient $a = \omega = 0$, $b = |x_1|^2$, $c = |x_2|^2$ and $|z| = |x_1||x_2|$ then the concurrence would be:

$$C_{\Psi}^{AB}(t) = 2 \max \Big\{ 0, |x_1||x_2| \Big\} \tag{III.22}$$

It is helpful to calculate $|x_i|^2$, so that we obtained:

$$\begin{aligned} |x_1|^2 =& \Big( 1 - 4N^2 \sin^2 (\delta t/2) \Big) \cos^2 \alpha \\ |x_2|^2 =& \Big( 1 - 4N^2 \sin^2 (\delta t/2) \Big) \sin^2 \alpha \\ |x_3|^2 =& \Big( 4N^2 \sin^2 (\delta t/2) \Big) \cos^2 \alpha \\ |x_4|^2 =& \Big( 4N^2 \sin^2 (\delta t/2) \Big) \sin^2 \alpha \end{aligned} \tag{III.23}$$

Then, it can be shown that:

$$|x_1||x_2| = \frac{1}{2} \sin (2\alpha) \Big[ 1 - 4N^2 \sin^2 (\delta t/2) \Big]$$

Therefore, we obtained the result that J-H Eberly obtained which is giving by:

$$C_{\Psi}^{AB}(t) = \sin (2\alpha) \Big[ 1 - 4N^2 \sin^2 (\delta t/2) \Big] \tag{III.24}$$

where $\delta = \lambda^{+} - \lambda^{-}$. A particularly interesting example is the case of zero detuning ($\Delta = 0$) in which the concurrence becomes:

$$C_{\Psi}^{AB}(t) = \sin (2\alpha) \cos^2 (gt) \tag{III.25}$$

As shown in Fig(III.4) with three different values of $\alpha$, $\alpha = \pi/4, \alpha = \pi/6$ and $\alpha = \pi/8$.

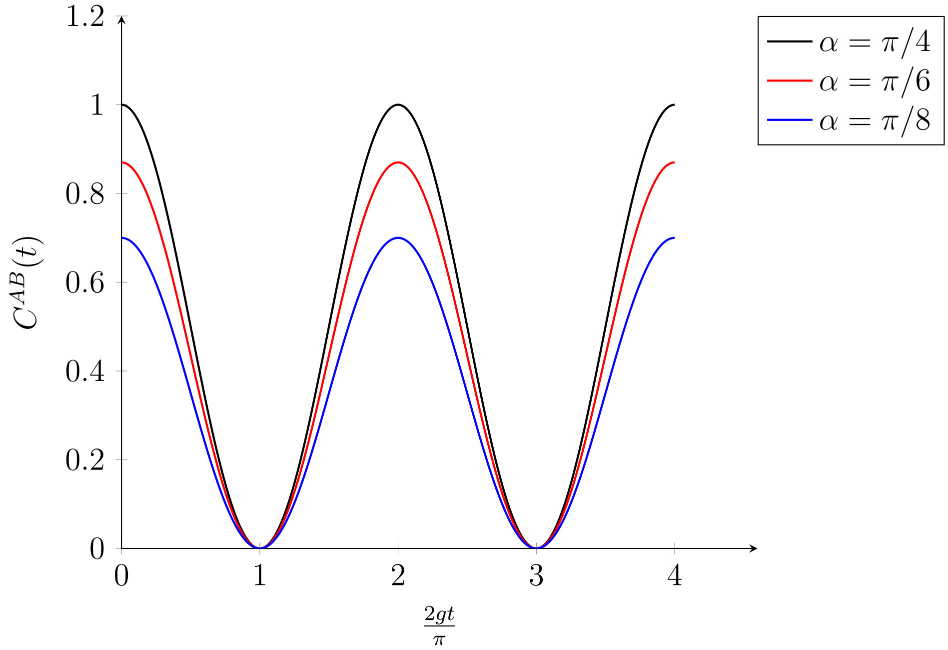


**Figure III.4:** The concurrence for atom-atom entanglement with the initial atomic state $|\psi(0)\rangle$ for zero detuning $\Delta = 0$ with parameter $\alpha$

### III.3.2 Entanglement Dynamics of the $\phi$-type Bell-like States

We choose now the other Bell-like state $|\phi(0)\rangle$ as an atomic initial state, the initial state for the total system is giving by:

$$\begin{aligned}|\Phi(0)\rangle &= |\phi(0)\rangle \otimes |00\rangle \\ &= \cos\alpha\,|ee00\rangle + \sin\alpha\,|gg00\rangle\end{aligned} \tag{III.26}$$

In the same way of the first case, the state of the total system at time $t$ can be expressed:

$$\begin{aligned}|\Phi(t)\rangle = \cos\alpha\Big[&Le^{-i\lambda^+t}|e,0\rangle + Me^{-i\lambda^-t}|e,0\rangle + N(e^{-i\lambda^+t} - e^{-i\lambda^-t})|g,1\rangle\Big] \\ &\otimes \cos\alpha\Big[Le^{-i\lambda^+t}|e,0\rangle + Me^{-i\lambda^-t}|e,0\rangle + N(e^{-i\lambda^+t} - e^{-i\lambda^-t})|g,1\rangle\Big] \\ &+ \sin\alpha\,|gg00\rangle\end{aligned} \tag{III.27}$$

simplified in trims of the coefficient:

$$|\Phi(t)\rangle = y_1|ee00\rangle + y_2|eg01\rangle + y_3|ge10\rangle + y_4|gg11\rangle + y_5|gg00\rangle \tag{III.28}$$

where the coefficients are now given by:

$$\begin{aligned}y_1 &= \cos\alpha\Big[(Le^{-i\lambda^+t} + Me^{-i\lambda^-t})^2\Big] \\ y_2 &= \cos\alpha\Big[(Le^{-i\lambda^+t} + Me^{-i\lambda^-t})N(e^{-i\lambda^+t} - e^{-i\lambda^-t})\Big] \\ y_3 &= \cos\alpha\Big[(Le^{-i\lambda^+t} + Me^{-i\lambda^-t})N(e^{-i\lambda^+t} - e^{-i\lambda^-t})\Big] \\ y_4 &= \cos\alpha\Big[N^2(e^{-i\lambda^+t} - e^{-i\lambda^-t})^2\Big] \\ y_5 &= \sin\alpha\end{aligned} \tag{III.29}$$

by tracing the cavity's state. The density matrix of the two atoms $\rho^{AB}$ In the basis $\left\{ |ee\rangle, |eg\rangle, |ge\rangle, |gg\rangle \right\}$ is giving by:

$$\rho_{\Phi}^{AB} = \begin{pmatrix} |y_1|^2 & 0 & 0 & y_1 y_5 \\ 0 & |y_2|^2 & 0 & 0 \\ 0 & 0 & |y_3|^2 & 0 \\ y_1^* y_5 & 0 & 0 & |y_4|^2 + |y_5|^2 \end{pmatrix} \tag{III.30}$$

Again, we find the density matrix in the form of X-state. Therefore the concurrence between the two atoms $A$ and $B$ is giving by:

$$C_{\Phi}^{AB}(t) = 2 \max \left\{ 0, |y_1||y_5| - |y_2||y_3| \right\} \tag{III.31}$$

then we obtained:

$$C_{\Phi}^{AB}(t) = [1 - 4N^2 \sin^2(\delta t/2)] \Big[ \sin(2\alpha) - 8N^2 \sin^2(\delta t/2) \cos^2 \alpha \Big] \tag{III.32}$$

For $\Delta = 0$, it becomes:

$$C_{\Phi}^{AB}(t) = \cos^2(gt) \Big[ \sin(2\alpha) - \sin^2(gt) \cos^2 \alpha \Big] \tag{III.33}$$

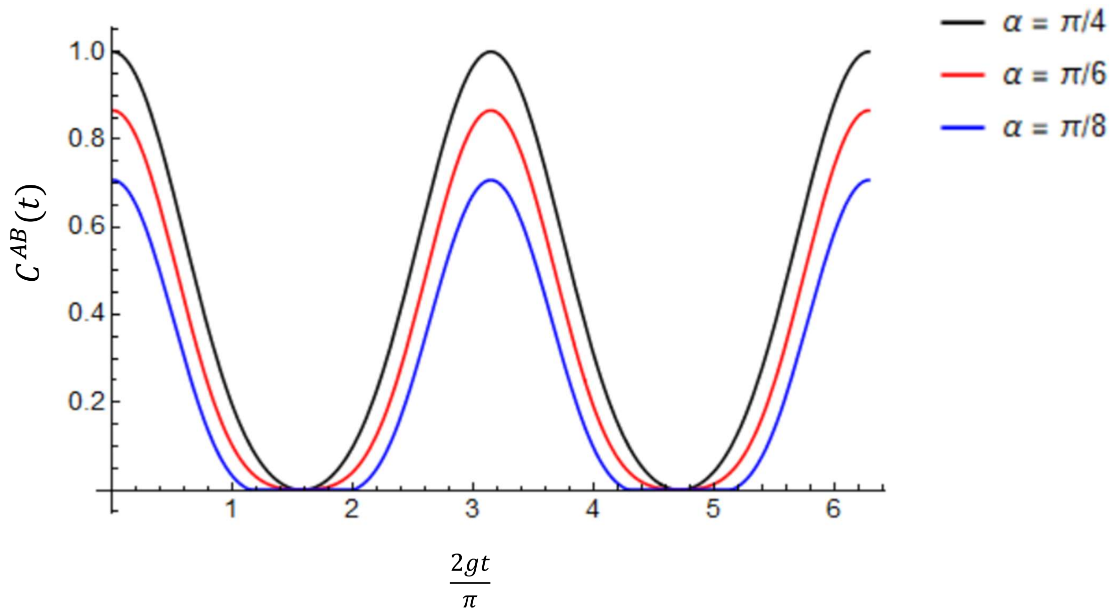


**Figure III.5:** The concurrence for atom-atom entanglement with the initial atomic state $|\phi(0)\rangle$ for zero detuning $\Delta = 0$ with parameter $\alpha$

### III.3.3 Analysis of results

- In the both cases of Bells-like states, the entanglement between two atoms $A$ and $B$ disappear suddenly, showing what is called sudden death of entanglement. The main result is the appearance of this phenomenon in a new environment, the sudden death happened without decoherence in the traditional sort. That is because the cavities in our double Jaynes–Cummings model are lossless, they are as far from being standard decoherence reservoirs as possible. Another thing we can notice in $\phi$-type Bell like state is that the duration of zero entanglement intervals depends on the degree of entanglement of the initial state (the magnitude of $\alpha$). The smaller the initial degree of entanglement is, the longer the state will stay in the disentangled separable state.[19]

- The entanglement was not destroyed in this case, but rather to an internal redistribution of quantum correlations among the subsystems (atom-photon and photon-photon), we do not focus on the other correlations, but if we calculate the concurrence between the photons or between the atoms and the photons we find that the entanglement between any two subsystems get arise.[19]

- We observe that the entanglement revives periodically after each sudden-death event, a behavior known as the revival phenomenon. In other words, the lost entanglement reappears in the atomic subsystem at finite times. This can be understood from the coherent Rabi oscillations of the atom–cavity system: as the system evolves, it periodically returns close to its initial state, allowing the entanglement initially shared between the atoms to re-emerge. This behavior reflects a memory effect, often associated with the coherent dynamics of Jaynes–Cummings cavities. Clearly, in the absence of interactions with the local cavities, the entanglement between the two atoms would remain constant.[19]

- For the comparison between the $\psi$-type and $\phi$-type Bell states, we observe that both exhibit periodic death–revival dynamics of entanglement. However, the $\phi$-type Bell state shows wider time intervals during which the concurrence vanishes completely, compared to the $\psi$-type state.

  For clarity, the mere touching of zero by the concurrence is not considered as entanglement sudden death; rather, sudden death requires the concurrence to remain zero over a finite time interval. In this sense, the $\psi$-type state does not exhibit true sudden death, but only oscillatory minima where the concurrence momentarily reaches zero. In contrast, the $\phi$-type Bell state displays genuine entanglement sudden death, characterized by finite time intervals of vanishing concurrence. This difference can be understood from the structure of the two states within the Jaynes–Cummings dynamics. The $\psi$-type Bell state involves a single excitation distributed between the two atoms, which restricts the evolution to the single-excitation subspace and leads to more regular oscillatory behavior. On the other hand, the $\phi$-type Bell state is a superposition of zero- and two-excitation components. As a result, its evolution involves a larger set of coupled atom–field states, leading to a more complex redistribution of quantum correlations. This enhanced spreading of correlations over multiple subsystems explains the appearance of extended time intervals in which the atom–atom entanglement completely vanishes.[18]

### III.4 Dynamics of Steerability in Two Jaynes-Cummings Atoms

In this section we want to investigate the steerability dynamics of double Jayens–Cummings model under noiseless environment. Following the section of quantum steering and steerabilty inequality known as CJWR inequality and Based on work of Ya-Xi ben and his team (2025)[1]. we have showed how they calculated the steerability measurement quantity $F(\rho)$ and $S(\rho)$ and explain the results in more details.

For the same scenario mentioned above, we have two tow-level atom $A$ and $B$ each in its own Jaynes– Cummings cavity $a$ and $b$ respectively. Again There will be no interaction between the subsystems $Aa$ and $Bb$, and initially, the cavities are in the vacuum state while the atoms are in an entangled state. The quantum steering task for the Jaynes–Cummings model is as we discussed in the section of the steering inequality where Alice has the system $Aa$ characterized measurements $x \in D_1$ and Bob has the system $Bb$ characterized measurements $y \in D_2$ as shown in Fig(III.6)

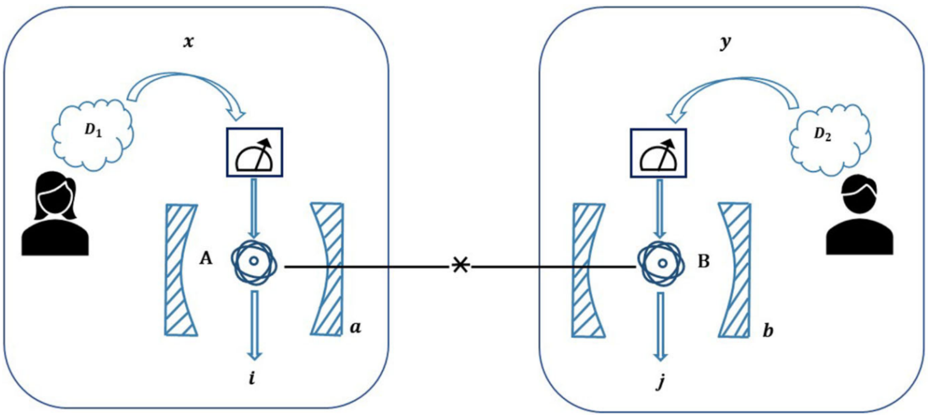


**Figure III.6:** Steering task in the double JCM

#### III.4.1 Steerability Dynamics of the $\psi$-type Bell-like States

The same study that have done in the previous section, we started with the Bell-like State $|\psi(0)\rangle$. The initial state of the system is shown in Eq(III.15) .The information about the steerability between the two atoms is contained in the reduced density matrix $\rho^{AB}$ of the two atoms, which we already have from Eq(III.21).

The information about steerability means how much the CJWR inequality got violated, the quantity that shows this violation is $F(\rho^{AB})$. Therefore we have to calculate the quantity $F(\rho^{AB})$ that is shown in Eq(II.17). which is depend on the the singular values of the correlation matrix $C$. Using Eq(II.18) to find the elements of the correlation matrix, we obtained:

$$C = \begin{pmatrix} x_1x_2^* + x_2x_1^* & ix_1x_2^* - ix_2x_1^* & 0 \\ ix_2x_1^* - ix_1x_2^* & x_1x_2^* + x_2x_1^* & 0 \\ 0 & 0 & |x_3|^2 + |x_4|^2 - |x_1|^2 - |x_2|^2 \end{pmatrix} \tag{III.34}$$

Then the singular values of $C$ are:

$$\begin{aligned} c_1 &= 2x_1x_2^* \\ c_2 &= 2x_2x_1^* \\ c_3 &= |x_3|^2 + |x_4|^2 - |x_1|^2 - |x_2|^2 \end{aligned} \tag{III.35}$$

the maximal violation of the CJWR inequality for $\Delta = 0$ is:

$$F(\rho_{\Psi}^{AB}) = \sqrt{c_1^2 + c_2^2 + c_3^2} = \sqrt{2\sin^2(2\alpha)\cos^2(gt) + \cos^2(2gt)} \tag{III.36}$$

To provide a more intuitively variation of the steerability, using $S(\rho_{\Psi}^{AB})$ from Eq(II.19). we find:

$$S(\rho_{\Psi}^{AB}) = \max\left\{0, \frac{1}{\sqrt{3}-1}\left[\sqrt{2\sin^2(2\alpha)\cos^4(gt) + \cos^2(2gt)} - 1\right]\right\} \tag{III.37}$$

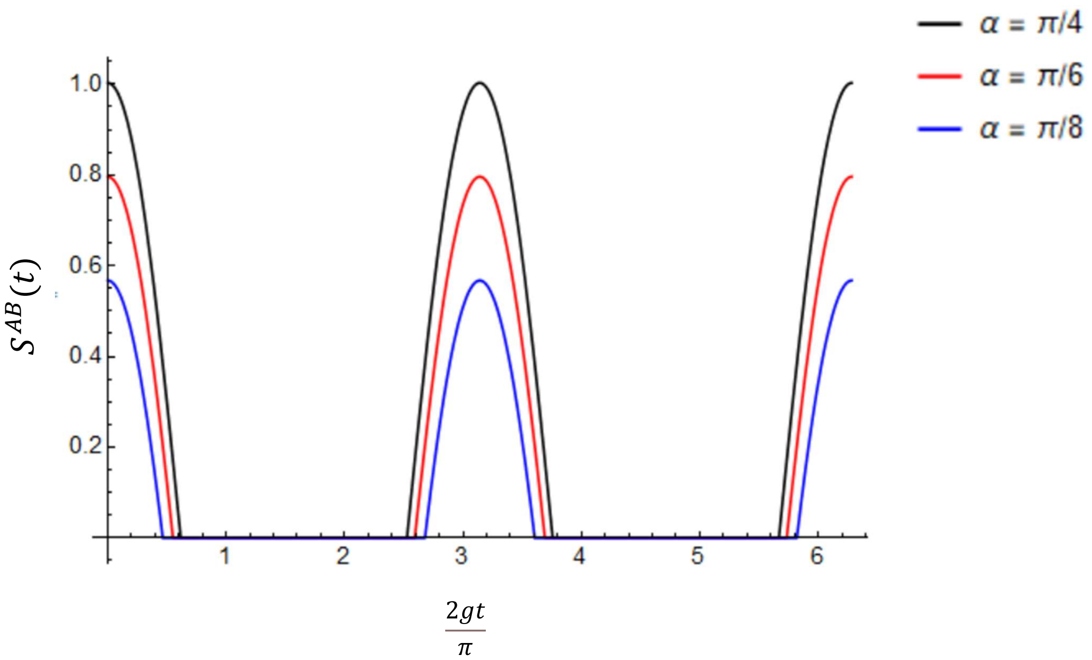


**Figure III.7:** The Steerability for atom-atom entanglement with the initial atomic state $|\psi(0)\rangle$ for zero detuning $\Delta = 0$ with parameter $\alpha$

### III.4.2 Steerability Dynamics of the $\phi$-type Bell-like States

We study now the other Bell state $|\phi(0)\rangle$ as an atomic initial state. The evolution of the system is shown in Eq(III.26). The reduced density matrix $\rho^{AB}$ of the two atoms is obtained in Eq(III.29).

Using the same method as the previous case we calculated the correlation matrix $C$ which is:

$$C = \begin{pmatrix} (y_1 + y_1^*)y_5 & iy_1^* y_5 - iy_1 y_5 & 0 \\ iy_1^* y_5 - iy_1 y_5 & -(y_1 + y_1^*)y_5 & 0 \\ 0 & 0 & |y_1|^2 + |y_4|^2 + |y_5|^2 - |y_2|^2 - |y_3|^2 \end{pmatrix} \tag{III.38}$$

The singular values of this matrix are giving by:

$$\begin{aligned} c_1 &= -2y_5\sqrt{y_1 y_1^*} \\ c_2 &= 2y_5\sqrt{y_1 y_1^*} \\ c_3 &= |y_1|^2 + |y_4|^2 + |y_5|^2 - |y_2|^2 - |y_3|^2 \end{aligned} \tag{III.39}$$

Then the maximal violation of the CJWR inequality for zero detuning $\Delta = 0$ is:

$$\begin{aligned} F(\rho_{\Phi}^{AB}) &= \sqrt{c_1^2 + c_2^2 + c_3^2} \\ &= \sqrt{2\sin^2(2\alpha)\cos^4(gt) + (\cos^2\alpha\cos^2(2gt) + \sin^2\alpha)^2} \end{aligned} \tag{III.40}$$

Therefore, for steerability measurement $S(\rho^{AB})$ we obtained:

$$S(\rho_{\Phi}^{AB}) = \max\left\{0, \frac{1}{\sqrt{3}-1}\left[\sqrt{2\sin^2(2\alpha)\cos^4(gt) + (\cos^2\alpha\cos^2(2gt) + \sin^2\alpha)^2} - 1\right]\right\} \tag{III.41}$$

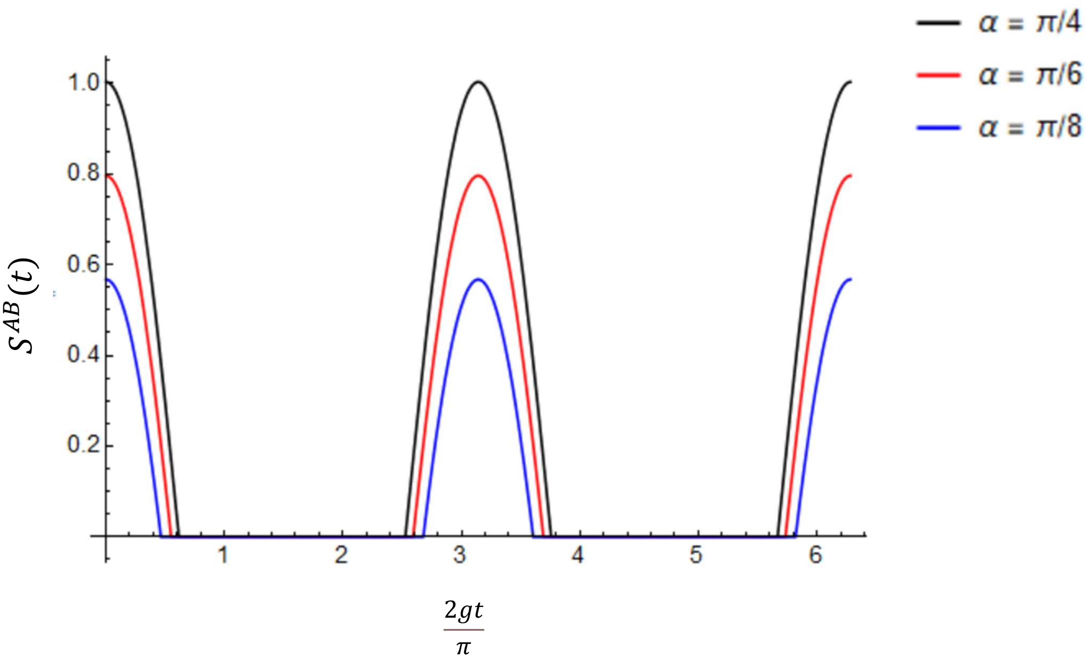


**Figure III.8:** The Steerability for atom-atom entanglement with the initial atomic state $|\phi(0)\rangle$ for zero detuning $\Delta = 0$ with parameter $\alpha$

### III.4.3 Analysis of results

- Like the entanglement case, the steerability between the system $A$ and system $B$ fall abruptly to zero and remains zero for a finite time interval across a continuous range of $\alpha$ values. This phenomenon, known as steering sudden death following the same statement of entanglement. Again, the duration of zero steerability intervals depends on the degree of entanglement of the initial state (the magnitude of $\alpha$). The smaller the initial degree of entanglement, the longer the state will remain in a state that satisfies the CJWR inequality.[1]

- In comparison between the concurrence $C^{AB}$ and steerability measurement $S(\rho^{AB})$, we can see that both of them exhibit qualitatively identical dynamical behavior, including sudden death and revival. This indicates a strong correspondence between entanglement and steering dynamics in this model. The coincidence between the sudden death and revival of concurrence and steering suggests that, in the noiseless double Jaynes–Cummings model, steering is governed by the same coherent excitation exchange mechanism responsible for entanglement sudden death and resurrection.

- It is worth noting that steering represents a stronger form of quantum correlation than entanglement; indeed, every steerable state is entangled, but not every entangled state is steerable. Consequently, we can see that there exist time intervals in which the concurrence remains nonzero $C^{AB} > 0$ while the steerability vanishes $S(\rho^{AB}) = 0$, indicating that steering can undergo sudden death before entanglement itself disappears.

- Unlike the entanglement dynamics in comparison between the $\psi$-type and $\phi$-type Bell states, the steering exhibits sudden death for both Bell-like initial states, whereas entanglement sudden death appears only for the symmetric $\phi$-type Bell states, reflects the stronger and more fragile nature of steering compared with entanglement. [1]

# Chapter IV

## The Dynamics of Entanglement and Steerability in the Double Jaynes–Cummings Model under Noisy Environment

# Chapter IV

# The Dynamics of Entanglement and Steerability in the Double Jaynes–Cummings Model under Noisy Environment

## IV.1 Introduction

We discussed in the previous chapter the Dynamics of Entanglement and Steerability in the double lossless Jaynes–Cummings model. In this chapter we will consider that the quantum system is interacting with the environment which is well-known as reservoir in the Atomic Physics context. The information about the system's state is always leaking into the correlations building up between the quantum system and the reservoir. In other words, the environment is continuously 'watching' the system and collecting information about it. The mere possibility that this information could be retrieved or virtually, is enough to destroy the coherences of the system's wave function. The concept of a wave function then becomes inadequate and the quantum system must in general be described not by a state vector, rather by a density matrix.[9]

We begin this chapter by understanding how to deal with the quantum state-reservoir system by discussing the master equation, which is standard technique in quantum optics that describe the dynamical evolution of the quantum system coupled to reservoir. Referring to the construction of the Lindblad form and the jump operator.

Again, Based on the works of Ya- Ben and others (2025) we will check the solution of the mater equation of a Single Jaynes-Cummings model interacting with Phase dumping reservoir. Moving to our purpose which is investigating the dynamics of entanglement and steerability by applying this solution into double Jaynes–Cummings model each of one interacting with the same reservoir with initially entangled atoms.

### IV.2 Lindblad Form and the Master Equation

Consider a quantum system $S$ with Hamiltonian $\hat{H}_S$ interacting with reservoir $\hat{H}_R$, the Hamiltonian of the total system then becomes[7]:

$$H = H_S + H_R + V \tag{IV.1}$$

where $V$ is the interaction between the quantum state $S$ and the reservoir $R$. As mentioned in the introduction, the wave function in this model described by the density matrix $\rho$. For the global system $S + R$, the evolution of the density operator $\rho$ obeys the equations:

$$\frac{d}{dt}\rho(t) = \frac{1}{i\hbar}\Big[H, \rho(t)\Big] \tag{IV.2}$$

The solution of this equation will be obtained by finding the so called the Lindblad generator $\mathcal{L}$ acting on $\rho$:

$$\frac{d}{dt}\rho(t) = \mathcal{L}(\rho) \tag{IV.3}$$

We are interested in the dynamics of the quantum system only. Its density matrix $\rho_S$ is obtained by performing a partial trace over the eigenstates of the reservoir.

$$\rho_S(t) = Tr_R(\rho(t)) \tag{IV.4}$$

The initial density matrix of the quantum system $\rho_S$ is transformed into a new one, $L(\rho_S)$.

$$\rho_S \to L(\rho_S) = L\rho_S L^\dagger$$

The process $L(\rho_S)$ known as Quantum linear maps or what is called Quantum Channel which can always be cast in a simple finite form of the 'Kraus sum representation' (Kraus 1983).[10] Fixing the bises $\Big\{|e\rangle\Big\}$ for the reservoir, and expand the quantum operation as:

$$L = \sum_e K_e \otimes |e\rangle$$

where $K_e$ is Kreaus operators which is a linear map of the quantum system $\rho_S$. By tracing out the reservoir $|e\rangle$ we obtained:

$$\begin{aligned} L(\rho_S) =& Tr_R\Big[L\rho_S L^\dagger\Big] \\ =& Tr_R\Big[\sum_e K_e\rho_S K_e^\dagger \otimes |e\rangle\langle e|\Big] = \sum_e K_e\rho_S K_e^\dagger \end{aligned} \tag{IV.5}$$

This is called the Kraus sum representation, then the quantum system $\rho_S$ is transformed as:

$$\rho_S \to \sum_e K_e\rho_S K_e^\dagger$$

#### Properties of Kraus operators

1. Kraus operators are unitary operators: A linear map $L$ is unitary transformation, so that $LL^\dagger = 1$. For Kreaus sum representation we find:'

$$\Big(\sum_e K_e^\dagger \otimes \langle e|\Big)\Big(\sum_e K_e \otimes |e\rangle\Big) = \sum_e K_e^\dagger K_e = 1 \Rightarrow Kreaus - operator - condition$$

2. Linearity and Hermiticity preserving:

$$L(\rho) = \sum_e K_e \rho K_e^\dagger$$

$L(\rho)$ is clearly linear, then we check the Hermiticity preserving, if we have $\rho^\dagger = \rho$ then:

$$L(\rho)^\dagger = \sum_e K_e \rho^\dagger K_e^\dagger = L(\rho)$$

3. The quantum channel of Kreaus operator should be Trace-Preserving

$$Tr(\rho) = Tr(L(\rho))$$

$$Tr(L(\rho)) = Tr \sum_e K_e \rho K_e^\dagger = Tr\Big(\sum_e K_e^\dagger K_e\Big)\rho = Tr(\rho)$$

4. Completely Positivity:

$$\rho \geq 0 \Rightarrow L(\rho) \otimes I \geq 0$$

If any quantum operation follow this properties then it is called CPTP maps ( Completely Positivity Trace-Preserving). Therefore, the Kreaus quantum channel are CPTP maps.

### Assumptions about the environment

In order to derive the master equation that is describe the open system dynamics, there are two important assumptions about the environment:[7]

- The environment is assumed to remain in its vacuum state and is not significantly affected by its interaction with the system, owing to its large number of degrees of freedom. This corresponds to the **Born approximation.**
- Furthermore, the dynamics are assumed to be memoryless, meaning that the system does not retain correlations with its past evolution. This corresponds to the **Markovian approximation.**

### The Lindblad Master Equation

Under the Born and Markovian approximations, and using perturbation theory applied to Eq. (IV.1), the reduced dynamics of the system can be described by a quantum dynamical semigroup. By exploiting the properties of the Kraus operator-sum representation, one arrives at the Lindblad form of the generator of the evolution:

$$\mathcal{L}(\rho) = \sum_e \Big( L_e \rho L_e^\dagger - \frac{1}{2} L_e^\dagger L_e \rho - \frac{1}{2} \rho L_e^\dagger L_e \Big) \tag{IV.6}$$

Consequently, the time evolution of the system density operator $\rho_S$ is governed by the Lindblad master equation (1976):

$$\frac{d}{dt}\rho_S = -\frac{i}{\hbar}[H_S, \rho_S] + \sum_e \Big( L_e \rho L_e^\dagger - \frac{1}{2} L_e^\dagger L_e \rho - \frac{1}{2} \rho L_e^\dagger L_e \Big). \tag{IV.7}$$

$L_e$ are the jump operators that characterize the interaction of the system with its environment, by specifying the different decoherence mechanisms acting on it.

## IV.3 State Evolution of a Single Jaynes-Cummings Model under Phase Dumping Reservoir

Consider now our quantum system is a single Jaynes–Cummings model with well-known Hamiltonian begin in Eq(I.57). This system evolves under effect of reservoir, and based on work of Xi-Ya Ben, the environment that was considered is Phase Dumping. The time evolution of the Jaynes–Cummings model is governed by the Lindblad master equation under the standard Born-Markovian and the weak system-reservoir coupling approximations, Eq(IV.7)

The jump operator $L_e$ that represents the effects the of reservoir on the system in case of phase dumping is $\sigma_z$. Here we only consider the phase damping reservoir acting on the atom, the cavity decoherence is typically neglected due to its lower dephasing rate compared to other dissipative channels, so the jump operator is $L_e = (\sigma_z \otimes I)$. Then the master equation is giving by:

$$\frac{d}{dt}\rho(t) = \frac{1}{i\hbar}\Big[H, \rho(t)\Big] + \gamma\Big[(\sigma_z \otimes I)\rho(\sigma_z \otimes I) - \frac{1}{2}(\sigma_z \otimes I)(\sigma_z \otimes I)\rho - \frac{1}{2}\rho(\sigma_z \otimes I)(\sigma_z \otimes I)\Big] \quad \text{(IV.8)}$$

Therefore, the master equation simplified into:

$$\frac{d}{dt}\rho(t) = \frac{1}{i\hbar}\Big[H, \rho(t)\Big] + \gamma\Big[(\sigma_z \otimes I)\rho(t)(\sigma_z \otimes I) - \rho(t)\Big] \quad \text{(IV.9)}$$

Consider the scenario that the cavity has at most one excitation $n = 1$, so the states of the cavity is $|0\rangle$ and $|1\rangle$, then the system can be modeled as a two-qubit system. For that, the Hamiltonian of a single Jaynes–Cummings model in case of zero dituning $\Delta = 0$ can be expressed as:

$$\begin{aligned} \hat{H} =& E_+ |n+\rangle \langle n+| + E_- |n-\rangle \langle n-| \\ =& \frac{\omega}{2} |e0\rangle \langle e0| + g |e0\rangle \langle g1| + g |g1\rangle \langle e0| + \frac{\omega}{2} |g1\rangle \langle g1| \end{aligned} \quad \text{(IV.10)}$$

In the $\Big\{|e0\rangle, |e1\rangle, |g0\rangle, |g1\rangle\Big\}$ basis, the matrix form of this Hamiltonian is:

$$H = \begin{pmatrix} \frac{\omega}{2} & 0 & 0 & g \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ g & 0 & 0 & \frac{\omega}{2} \end{pmatrix} \quad \text{(IV.11)}$$

By substitution in the master equation Eq(IV.7). and expressed $\rho(t)$ as $\rho_{ij}$ we find eight linear differential equations simplified into three groups. Note that we do not consider the evolution of the state $|e1\rangle$ due to the assumption of at most one excitation $n = 1$ in the atom–cavity system. Under this restriction, the doubly excited state $|e1\rangle$ is excluded from the dynamics. Moreover, the ground state $|g0\rangle \langle g0|$ remains unchanged during the evolution. Therefore, we obtained:

$$\begin{cases} \dot{\rho}_{11} = ig\rho_{14} - ig\rho_{41} \\ \dot{\rho}_{14} = ig\rho_{11} - 2\gamma\rho_{14} - ig\rho_{44} \\ \dot{\rho}_{41} = -ig\rho_{11} - 2\gamma\rho_{41} + ig\rho_{44} \\ \dot{\rho}_{11} = -ig\rho_{14} + ig\rho_{41} \end{cases} \quad \text{(IV.12)}$$

$$\begin{cases} \dot{\rho}_{13} = -\Big(2\gamma + \dfrac{i\omega}{2}\Big)\rho_{13} - ig\rho_{43} \\ \dot{\rho}_{43} = -ig\rho_{13} - \dfrac{i\omega}{2}\rho_{43} \end{cases} \tag{IV.13}$$

$$\begin{cases} \dot{\rho}_{31} = \Big(-2\gamma + \dfrac{i\omega}{2}\Big)\rho_{31} + ig\rho_{34} \\ \dot{\rho}_{34} = +ig\rho_{31} + \dfrac{i\omega}{2}\rho_{34} \end{cases} \tag{IV.14}$$

The solutions of these equations are presented in Ref. [X]. However, after carefully verifying these solutions, we found some minor inconsistencies, which seem to originate from typographical oversights. For consistency, we have corrected these solutions and adopted the corrected forms in the present work.

For the first group Eq(IV.12), the initial conditions are giving as $\rho_{11}(0) = 1$, $\rho_{14}(0) = \rho_{41}(0) = 0$ and $\rho_{44}(0) = 0$, then the solutions are giving by:

$$\begin{cases} \rho_{11}(t) = \dfrac{1}{2} + \dfrac{1}{2}e^{-\gamma t}\Big[\cosh\Big(\sqrt{\gamma^2 - 4g^2 t}\Big) + \dfrac{\gamma}{\sqrt{\gamma^2 - 4g^2}}\sinh\Big(\sqrt{\gamma^2 - 4g^2 t}\Big)\Big] \\ \rho_{14}(t) = \dfrac{ig}{\sqrt{4g^2 - \gamma^2}}e^{-\gamma t}\sin\Big(\sqrt{4g^2 - \gamma^2 t}\Big) \\ \rho_{41}(t) = \dfrac{-ig}{\sqrt{4g^2 - \gamma^2}}e^{-\gamma t}\sin\Big(\sqrt{4g^2 - \gamma^2 t}\Big) \\ \rho_{44}(t) = \dfrac{1}{2} - \dfrac{1}{2}e^{-\gamma t}\Big[\cosh\Big(\sqrt{\gamma^2 - 4g^2 t}\Big) + \dfrac{\gamma}{\sqrt{\gamma^2 - 4g^2}}\sinh\Big(\sqrt{\gamma^2 - 4g^2 t}\Big)\Big] \end{cases} \tag{IV.15}$$

For the system of Eq(II.13) , considering the initial conditions $\rho_{13}(0) = 1$ and $\rho_{43}(0) = 0$, the solutions are:

$$\begin{cases} \rho_{13}(t) = \exp\Big(-(\gamma + \dfrac{i\omega}{2})t\Big)\Big[\cosh\Big(\sqrt{\gamma^2 - g^2 t}\Big) - \dfrac{\gamma}{\sqrt{\gamma^2 - g^2}}\sinh\Big(\sqrt{\gamma^2 - g^2 t}\Big)\Big] \\ \rho_{43}(t) = \dfrac{-ig}{\sqrt{g^2 - \gamma^2}}\exp\Big\{\Big(-(\gamma + \dfrac{i\omega}{2})t\Big)\Big\}\sin\Big(\sqrt{g^2 - \gamma^2 t}\Big) \end{cases} \tag{IV.16}$$

Additionally, when the initial conditions for the system of Eq. are given as $\rho_{31}(0) = 1$ and $\rho_{34}(0) = 0$, its solutions can also be obtained as:

$$\begin{cases} \rho_{13}(t) = \exp\Big(-(\gamma - \dfrac{i\omega}{2})t\Big)\Big[\cosh\Big(\sqrt{\gamma^2 - g^2 t}\Big) - \dfrac{\gamma}{\sqrt{\gamma^2 - g^2}}\sinh\Big(\sqrt{\gamma^2 - g^2 t}\Big)\Big] \\ \rho_{43}(t) = \dfrac{ig}{\sqrt{g^2 - \gamma^2}}\exp\Big\{\Big(-(\gamma + \dfrac{i\omega}{2})t\Big)\Big\}\sin\Big(\sqrt{g^2 - \gamma^2 t}\Big) \end{cases} \tag{IV.17}$$

## IV.4 Dynamics of Entanglement in Tow Jaynes-Cummings Atoms under Phase Dumping Reservoir

After figure out how the single Jaynes–Cummings system evolves in time under phase dumping reservoir, we are ready now to investigate the dynamics of entanglement for double Jaynes–Cummings atoms, each ones is affected by the phase damping noise. we continue to consider the scenario where two Janes–Cummings models are completely identical and no communication between them. Like the noiseless case, we begin the study by considering the Bell-like state $|\psi(0)\rangle$ as atomic initial states. The density operator for the initial state of the entire four-qubit system is giving by:

$$\rho_\Psi(0) = |\Psi(0)\rangle \langle\Psi(0)| \tag{IV.18}$$

where

$$|\Psi(0)\rangle = |\psi(0)\rangle \otimes |00\rangle \tag{IV.19}$$

then the initial state can be expressed as:

$$\begin{aligned}\rho_\Psi(0) = \cos^2\alpha\, |e0\rangle\langle e0| \otimes |g0\rangle\langle g0| + \sin\alpha\cos\alpha\, |e0\rangle\langle g0| \otimes |g0\rangle\langle e0| \\ + \sin\alpha\cos\alpha\, |g0\rangle\langle e0| \otimes |e0\rangle\langle g0| + \sin^2\alpha\, |g0\rangle\langle g0| \otimes |e0\rangle\langle e0|\end{aligned} \tag{IV.20}$$

the initial state $\rho_\Psi(0)$ evolves into $\rho_\Psi(t)$ at time $t$ is giving by:

$$\begin{aligned}\rho_\Psi(t) = \cos^2\alpha\Big[&\rho_{11}|e0\rangle\langle e0| + \rho_{14}|e0\rangle\langle g1| + \rho_{41}|g1\rangle\langle e0| \\ &+ \rho_{44}|g1\rangle\langle g1| \otimes |g0\rangle\langle g0|\Big] \\ &+ \sin\alpha\cos\alpha\Big[\rho_{13}|e0\rangle\langle g0| + \rho_{43}|g1\rangle\langle g0| \otimes \rho_{31}|g0\rangle\langle e0| + \rho_{34}|g0\rangle\langle g1|\Big] \\ &+ \sin\alpha\cos\alpha\Big[\rho_{31}|g0\rangle\langle e0| + \rho_{34}|g0\rangle\langle g1| \otimes \rho_{13}|e0\rangle\langle g0| + \rho_{43}|g1\rangle\langle g0|\Big] \\ &+ \sin^2\alpha\Big[\,|g0\rangle\langle g0| \otimes \alpha\rho_{11}|e0\rangle\langle e0| + \rho_{14}|e0\rangle\langle g1| \\ &+ \rho_{41}|g1\rangle\langle e0| + \rho_{44}|g1\rangle\langle g1|\Big]\end{aligned} \tag{IV.21}$$

By tracing Eq. over the degrees of freedom of cavity field $a$ and $b$, we get the reduced state of atoms $A$ and $B$ at any time $t$:

$$\rho_\Psi^{AB}(t) = Tr_{ab}(\rho_\Psi(t)) \tag{IV.22}$$

it seems messy a little bit and get confuse to calculate, so we used the following notation:

$$\sum_{ab} \langle ab|\, Aa \otimes Bb\, |ab\rangle = \sum_{ab} \langle a|\, Aa\, |a\rangle \otimes \langle b|\, Bb\, |b\rangle$$

then we obtained:

$$\begin{aligned}\rho_\Psi^{AB}(t) = \cos^2\alpha\Big(\rho_{11}|eg\rangle\langle eg| + \rho_{44}|gg\rangle\langle gg|\Big) + \sin\alpha\cos\alpha\rho_{13}\rho_{31}|ge\rangle\langle eg| \\ + \sin\alpha\cos\alpha\rho_{13}\rho_{31}|eg\rangle\langle ge| + \sin^2\alpha\Big(\rho_{11}|ge\rangle\langle ge| + \rho_{44}|gg\rangle\langle gg|\Big)\end{aligned} \tag{IV.23}$$

In the basis $\left\{|ee\rangle, |eg\rangle, |ge\rangle, |gg\rangle\right\}$, the reduced density matrix $\rho_{\Psi}^{AB}(t)$ is now given by:

$$\rho_{\Psi}^{AB}(t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos^2\alpha\rho_{11} & \sin\alpha\cos\alpha\rho_{13}\rho_{31} & 0 \\ 0 & \sin\alpha\cos\alpha\rho_{13}\rho_{31} & \sin^2\alpha\rho_{11} & 0 \\ 0 & 0 & 0 & \rho_{44} \end{pmatrix} \tag{IV.24}$$

this state is exactly of the form X-state. Following the same methods of the previous case, then we obtained the concurrence between atoms $A$ and $B$ whchi is giving by:

$$C_{\Psi}^{AB}(t) = 2\max\left\{0, |\sin\alpha\cos\alpha\rho_{13}\rho_{31}|\right\} \tag{IV.25}$$

we have $\rho_{31} = \rho_{13}^*$, then we obtained:

$$C_{\Psi}^{AB}(t) = |\sin 2\alpha||\rho_{13}|^2 \tag{IV.26}$$

Expressing $\rho_{13}$ in terms of damping ratio $r = \frac{\gamma}{g}$, defined as the ratio between the phase damping constant $\gamma$ and the single photon Rabi frequency $g$, by setting $r = 0.1$ we obtained damped Rabi oscillations.

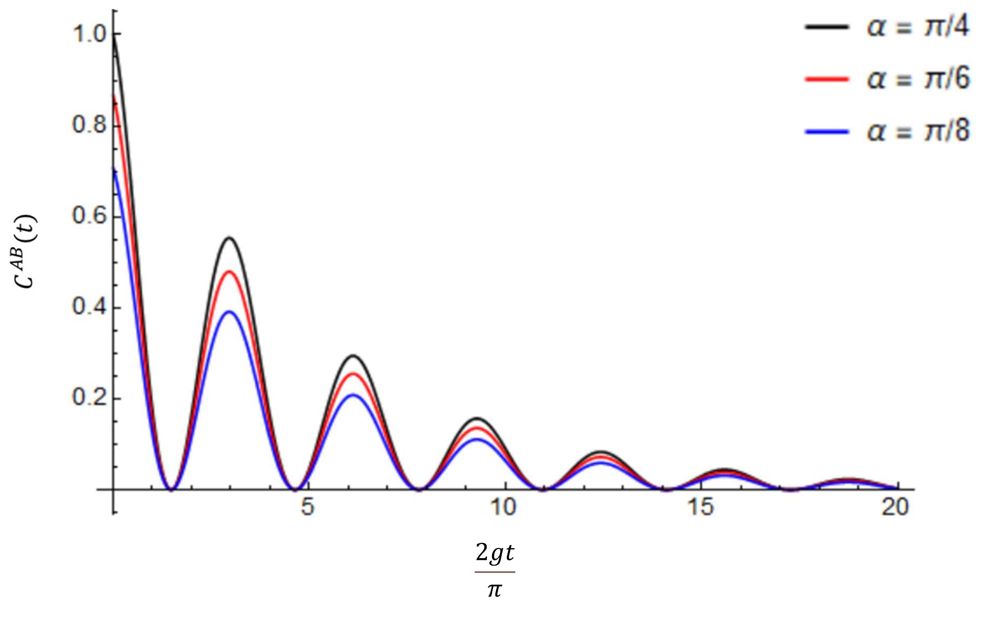


**Figure IV.1:** The concurrence for atom-atom entanglement with the initial atomic state $|\psi(0)\rangle$ for zero detuning $\Delta = 0$ with parameter $\alpha$ under phase dumping reservoir

Next, we examine entanglement dynamics for the other Bell-like state $|\phi(0)\rangle$ as atomic initial states. The density matrix for the total system is giving by:

$$\rho_\Phi(0) = |\Phi(0)\rangle \langle\Phi(0)| \tag{IV.27}$$

where

$$|\Phi(0)\rangle = |\phi(0)\rangle \otimes |00\rangle \tag{IV.28}$$

then, the initial state would be:

$$\begin{aligned}\rho_\Phi(0) = {} & \cos^2\alpha\, |e0\rangle \langle e0| \otimes |e0\rangle \langle e0| + \sin\alpha\cos\alpha\, |e0\rangle \langle g0| \otimes |e0\rangle \langle g0| \\ & + \sin\alpha\cos\alpha\, |g0\rangle \langle e0| \otimes |g0\rangle \langle e0| + \sin^2\alpha\, |g0\rangle \langle g0| \otimes |g0\rangle \langle g0|\end{aligned} \tag{IV.29}$$

Using the same approach, the time evolution of this state can be expressed as:

$$\begin{aligned}\rho_\Phi(t) = {} & \cos^2\alpha\Big[(\rho_{11}|e0\rangle\langle e0| + \rho_{14}|e0\rangle\langle g1| + \rho_{41}|g1\rangle\langle e0| + \rho_{44}|g1\rangle\langle g1|) \\ & \otimes (\rho_{11}|e0\rangle\langle e0| + \rho_{14}|e0\rangle\langle g1| + \rho_{41}|g1\rangle\langle e0| + \rho_{44}|g1\rangle\langle g1|)\Big] \\ & + \sin\alpha\cos\alpha\Big[(\rho_{13}|e0\rangle\langle g0| + \rho_{43}|g1\rangle\langle g0|) \otimes (\rho_{13}|e0\rangle\langle g0| + \rho_{43}|g1\rangle\langle g0|)\Big] \\ & + \sin\alpha\cos\alpha\Big[(\rho_{31}|g0\rangle\langle e0| + \rho_{34}|g0\rangle\langle g1|) \otimes (\rho_{31}|g0\rangle\langle e0| + \rho_{34}|g0\rangle\langle g1|)\Big] \\ & + \sin^2\alpha\, |g0\rangle\langle g0| \otimes |g0\rangle\langle g0|\end{aligned} \tag{IV.30}$$

By tracing out the cavity states from the last equation we find the density matrix $\rho^{AB}$ of the two atoms:

$$\begin{aligned}\rho_\Phi^{AB}(t) = {} & \cos^2\alpha\rho_{11}^2 |ee\rangle\langle ee| + \sin\alpha\cos\alpha\rho_{13}^2 |ee\rangle\langle gg| + \sin\alpha\cos\alpha\rho_{13}^2 |gg\rangle\langle ee| \\ & + \sin^2\alpha\, |gg\rangle\langle gg| + \cos^2\alpha\rho_{11}\rho_{44} |eg\rangle\langle eg| + \cos^2\alpha\rho_{11}\rho_{44} |ge\rangle\langle ge| \\ & + \cos^2\alpha\rho_{44}^2 |gg\rangle\langle gg|\end{aligned} \tag{IV.31}$$

In the bases $\Big\{|ee\rangle, |eg\rangle, |ge\rangle, |gg\rangle\Big\}$, the matrix form of $\rho_\Phi^{AB}$ is giving by:

$$\rho_\Phi^{AB}(t) = \begin{pmatrix} \cos^2\alpha\rho_{11}^2 & 0 & 0 & \sin\alpha\cos\alpha\rho_{13}^2 \\ 0 & \cos^2\alpha\rho_{11}\rho_{44} & 0 & 0 \\ 0 & 0 & \cos^2\alpha\rho_{11}\rho_{44} & 0 \\ \sin\alpha\cos\alpha\rho_{31}^2 & 0 & 0 & \rho_{44} + \cos^2\alpha\rho_{44}^2 \end{pmatrix} \tag{IV.32}$$

Again, this state is in the form of X-state, therefore the concurrence of the atoms $A$ and $B$ is giving by:

$$C_\Phi^{AB}(t) = \max\Big\{0, f(t)\Big\} \tag{IV.33}$$

where

$$f(t) = \sin 2\alpha|\rho_{13}|^2 - 2\cos^2\alpha\rho_{11}\rho_{44} \tag{IV.34}$$

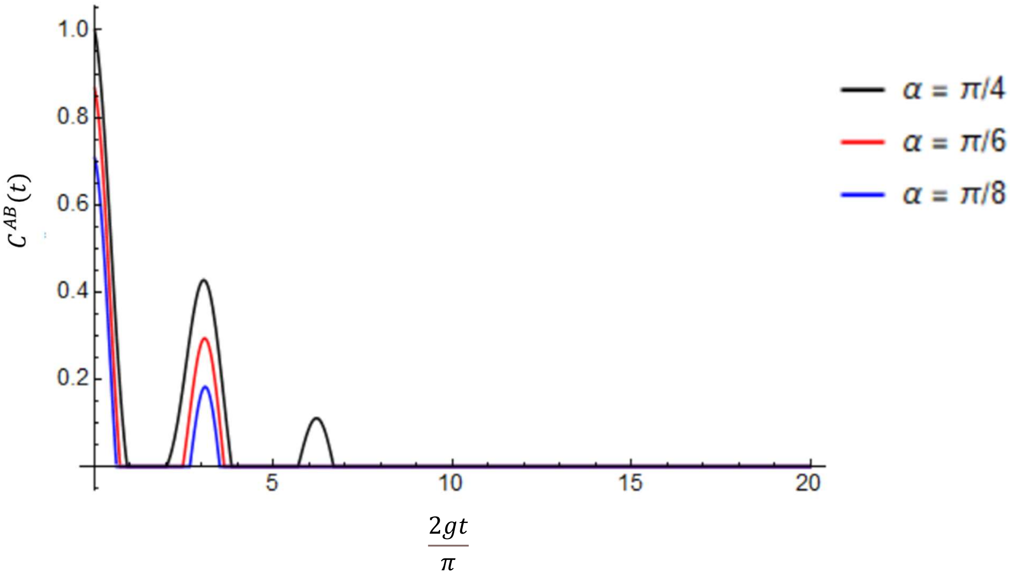


**Figure IV.2:** The concurrence for atom-atom entanglement with the initial atomic state $|\phi(0)\rangle$ for zero detuning $\Delta = 0$ with parameter $\alpha$ under phase dumping reservoir

### IV.4.1 Analysis of results

- Again, the entanglement dynamics between the atoms exhibits a periodic behavior, death and revival compared to the noiseless environment, the periodicity remains unchanged. The difference here, by the existence of phase damping effect on the atoms, the amplitude of the revival got decreased until the concurrence vanishes defining what is called the damped Rabi oscillation.

- The appearance of damped Rabi oscillations is attributed to dephasing decoherence acting on the atomic state, which induces a random phase between the excited state $|e\rangle$ and the ground state $|g\rangle$. This phase randomization progressively degrades the coherent interaction between the atom and the cavity field. Consequently, the off-diagonal elements of the density matrix, which are responsible for quantum coherence and entanglement, are gradually suppressed. As a result, the amplitude of the Rabi oscillations decreases over time, and each revival appears with a reduced amplitude until the concurrence eventually vanishes asymptotically.

- As discussed in the noiseless case, the $\psi$-type Bell state is more robust against entanglement sudden death compared to the $\phi$-type Bell state, and the presence of a phase-damping reservoir confirms this behavior. Although the concurrence eventually vanishes for both states, it disappears significantly earlier for the $\phi$-type state than for the $\psi$-type state. This result highlights the fundamental difference between symmetric and antisymmetric Bell states, as discussed by J. H. Eberly.

### IV.5 Dynamics of Steerability in Two Jaynes-Cummings Atoms under Phase Dumping Reservoir

Subsequently, we will dive into the evolution characteristics of the steerability between two atoms in the double Jaynes–Cummings model under the dephasing damping. The same scenario that we have in the dynamics of entanglement, and the same method that we used in the case of noiseless environment. Calculating the singular values of the correlation matrix that is driven from the reduced density matrix of the two atoms, then calculate the maximum violation of CJWR inequality that applies for the steerability measurement $S(\rho)$.

Beginning with Bell-like state $|\psi(0)\rangle$ as initial atomic state. We derived the reduced density matrix $\rho_{\Psi}^{AB}$ of atoms $A$ and $B$ in the previous section Eq. For that, the correlation matrix of this state is giving by:

$$C = \begin{pmatrix} \sin 2\alpha\rho_{13}\rho_{31} & 0 & 0 \\ 0 & \sin 2\alpha\rho_{13}\rho_{31} & 0 \\ 0 & 0 & \rho_{11}-\rho_{44} \end{pmatrix} \tag{IV.35}$$

then the singular values of this matrix are:

$$\begin{aligned} c_1 =& \sin 2\alpha\rho_{13}\rho_{31} \\ c_2 =& \sin 2\alpha\rho_{13}\rho_{31} \\ c_3 =& \rho_{11}-\rho_{44} \end{aligned} \tag{IV.36}$$

Finally, we obtain the maximal violation of CJWR inequality by calculating $F(\rho^{AB})$:

$$F(\rho_{\Psi}^{AB}) = \sqrt{2\sin^2(2\alpha)\rho_{13}^2\rho_{31}^2 + (\rho_{11}-\rho_{44})^2} \tag{IV.37}$$

Therefore the steerability measurement $S(\rho_{\Psi}^{AB})$ is giving by:

$$S(\rho_{\Psi}^{AB}) = \max\left\{0, \frac{1}{\sqrt{3}-1}\left[\sqrt{2\sin^2(2\alpha)\rho_{13}^2\rho_{31}^2 + (\rho_{11}-\rho_{44})^2} - 1\right]\right\} \tag{IV.38}$$

For $\phi$-type Bell-like state as initial atomic state, we have the reduced density matrix $\rho_{\Phi}^{AB}$ of atoms $A$ and $B$ Eq . Then the correlation matrix of this state is:

$$C = \begin{pmatrix} \sin\alpha\cos\alpha(\rho_{13}^2+\rho_{31}^2) & i\sin\alpha\cos\alpha(\rho_{31}^2-\rho_{13}^2) & 0 \\ i\sin\alpha\cos\alpha(\rho_{31}^2-\rho_{13}^2) & -\sin\alpha\cos\alpha(\rho_{13}^2+\rho_{31}^2) & 0 \\ 0 & 0 & \cos^2\alpha(\rho_{11}-\rho_{44})^2+\sin^2\alpha \end{pmatrix} \tag{IV.39}$$

The corresponding singular values are giving by:

$$\begin{aligned} c_1 =& \sin(2\alpha)\rho_{13}\rho_{31} \\ c_2 =& -\sin(2\alpha)\rho_{13}\rho_{31} \\ c_3 =& \cos^2\alpha(\rho_{11}-\rho_{44})^2+\sin^2\alpha \end{aligned} \tag{IV.40}$$

Therefore, the maximal violation of the CJWR inequality is:

$$F(\rho_{\Phi}^{AB}) = \sqrt{2\sin^2(2\alpha)\rho_{13}^2\rho_{31}^2 + (\cos^2\alpha(\rho_{11}-\rho_{44})^2+\sin^2\alpha)^2} \tag{IV.41}$$

Applying for steerability measurement we obtained:

$$S(\rho_{\Phi}^{AB}) = \max\left\{0, \frac{1}{\sqrt{3}-1}\left[\sqrt{2\sin^2(2\alpha)\rho_{13}^2\rho_{31}^2 + (\cos^2\alpha(\rho_{11}-\rho_{44})^2+\sin^2\alpha)^2} - 1\right]\right\} \tag{IV.42}$$

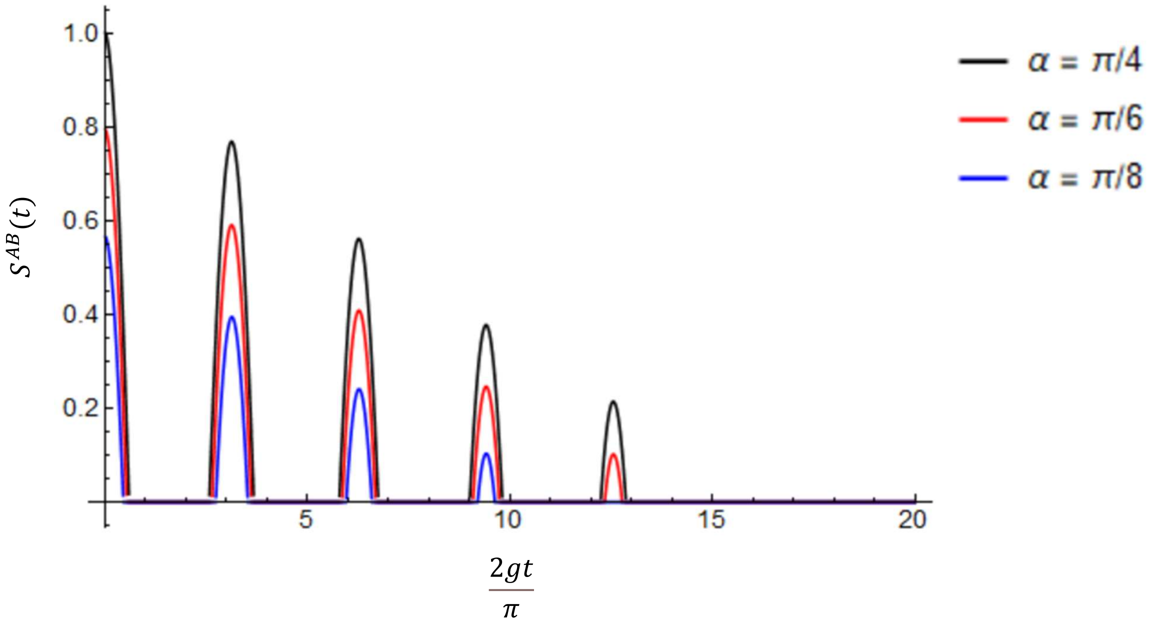


**Figure IV.3:** The steerability for atom-atom entanglement with the initial atomic state $|\psi(0)\rangle$ for zero detuning $\Delta = 0$ with parameter $\alpha$ under phase dumping reservoir

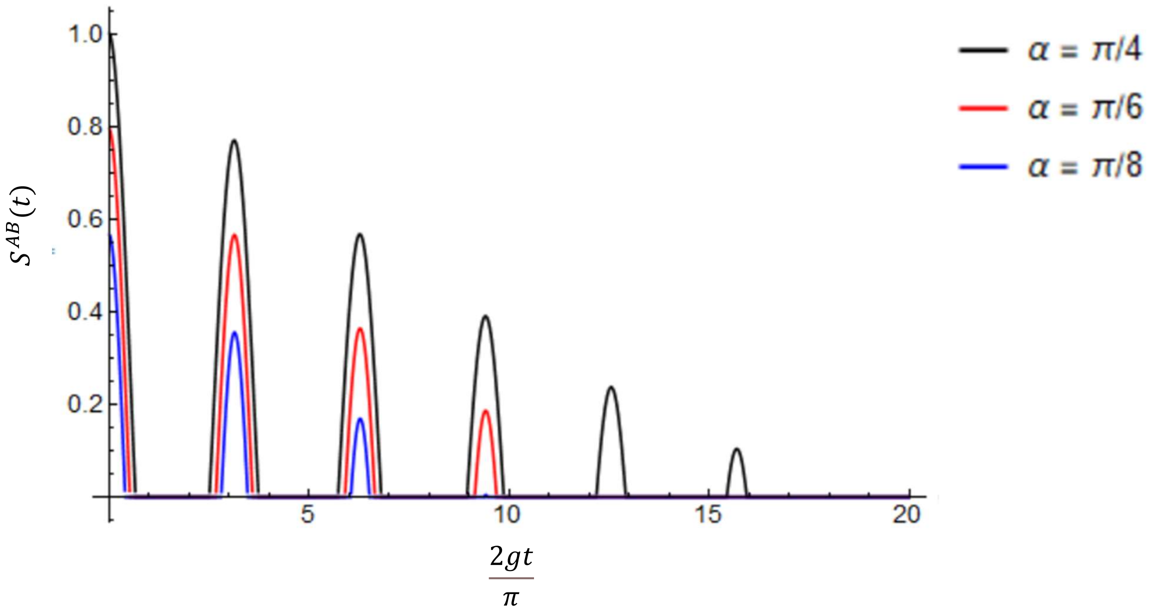


**Figure IV.4:** The steerability for atom-atom entanglement with the initial atomic state $|\phi(0)\rangle$ for zero detuning $\Delta = 0$ with parameter $\alpha$ under phase dumping reservoir

### IV.5.1 Analysis results

- From Fig(IV.3) and Fig(IV.4), it can be observed that under the influence of dephasing noise, the evolution of quantum steerability between the atoms exhibits a periodic behavior, characterized by gradual disappearance, death, and revival. However, due to the disturbance of the noise, the survival time of steerability after each revival decreases successively, and the maximum attainable value diminishes with each cycle, eventually leading to its complete demise.[1]

- A comparison between the steerability dynamics of the $\psi$-type and $\phi$-type Bell states reveals a behavior that contrasts with that of entanglement dynamics. In particular, when the initial state is of $\phi$-type, the steerability persists for a longer duration than for the $\psi$-type state under phase-damping noise. This indicates that, in the presence of dephasing, the state $|\phi\rangle_{AB}$ can retain steerability for relatively longer times.

  In contrast, as discussed previously for entanglement dynamics, the $\psi$-type Bell state is more robust against entanglement sudden death than the $\phi$-type state. This difference highlights that entanglement and steering respond differently to decoherence and are quantified by distinct criteria, especially in mixed states under phase-damped environments.

- We observe an interesting feature when comparing the concurrence (illustrated in Figs. (IV.1) and (IV.2)) with the steerability. There exist time intervals during which the concurrence vanishes, while the steerability appears to persist.

  It is important to note that, while concurrence provides a complete and exact measure of entanglement for two-qubit systems, the steering criterion employed in Ref. [X] (based on CJWR inequalities) constitutes only a sufficient condition for steerability. Therefore, the apparent persistence of steerability beyond the vanishing of concurrence does not imply the existence of steerable but separable states, but rather reflects the limitations of the adopted steering criterion.

  This result opens an interesting avenue for future research on the interplay between entanglement and steering under decoherence.

# Conclusion

# Conclusion

In order to advance quantum computation and deepen our understanding of quantum information theory, it is essential to study the dynamics of its fundamental building blocks. These include the physical platforms of atomic physics, such as atoms, ions, molecules, and photons. In other words, one must analyze the dynamics of their interactions from the perspective of quantum information concepts. In this thesis we considered the Jaynes–Cummings Model as atomic interactions platform with entanglement and steerability as concepts of quantum computation. It is investigating the dynamics of entanglement and steerability of two types of Bell-like states in the double Janes–Cummings Model under both noiseless and phase-damping noise environments.

The main results that we conclude this thesis that in the absence of noise, both initial states exhibit periodic sudden death-revival phenomena in the atom-atom system's steerability, while entanglement between the two Jaynes–Cummings atoms only experiences sudden death in the case of the symmetric $\phi$-type Bell initial state, where $\psi$-type Bell state experiences only the mere touching of zero by the concurrence which is not considered as entanglement sudden death. This led us to confirm what J-H Eberly obtained that the $\psi$-type Bell state is robust against entanglement sudden death than $\phi$-type Bell state.

Remarkably, under phase-damping noise, the $\psi$-type Bell state preserves its robustness against entanglement sudden death compared to the $\phi$-type Bell state. In particular, the entanglement of the $\phi$-type state vanishes significantly earlier than that of the $\psi$-type state. In contrast, the steerability dynamics exhibit a different behavior, where the $\phi$-type Bell state maintains steerability for relatively longer times than the $\psi$-type state. This difference highlights both the fundamental distinction between symmetric and antisymmetric Bell states and the fact that entanglement and steering respond differently to decoherence. Additionally, we comparing between steerability and entanglement under dephasing noise that the concurrence vanish earlier than steering. With consider that steerability measurement under CJWR inequalities constitutes only a sufficient condition while concurrence provides a complete and exact measure of entanglement for two-qubit systems, this the apparent persistence of steerability beyond the vanishing of concurrence reflects the limitations of the adopted steering criterion which opens an interesting research on the interplay between entanglement and steering under decoherence.

Quantum computation based on the Jaynes–Cummings model constitutes a rich and active field of research, particularly in the presence of decoherence, which remains one of the main challenges for quantum information processing. From an experimental perspective, this framework provides a crucial link between theoretical models and practical implementations in quantum technologies. This dual aspect makes the field especially appealing, as it allows one to explore the fundamental nature of quantum systems from an information-theoretic point of view, while simultaneously contributing to the development of quantum computation and emerging quantum technologies.

# Abstract

In this thesis, we investigated two of the most important topics in quantum information science, namely entanglement and quantum steerability, within one of the most widely studied and active platforms in atomic physics: the Jaynes–Cummings model, which describes the dynamics of atom–photon interactions. In particular, we studied the dynamics of entanglement and steerability in the double Jaynes–Cummings model, both in the absence of environmental effects and in the presence of decoherence, where the environment was treated as a reservoir coupled to the system.

The first two chapters of this thesis were devoted to the fundamental concepts on which our study is based. In the first chapter, we discussed atom–photon interactions through the cavity QED model known as the Jaynes–Cummings model. We began by introducing the concept of Rabi oscillations, which represent the central feature of the atom–cavity field dynamics. As an intuitive classical analogy, we considered a two-state spin- system interacting with a magnetic field in order to understand the physical meaning of Rabi oscillations.

We then introduced the quantum electrodynamics Hamiltonian, namely the Jaynes–Cummings Hamiltonian, which describes the interaction between the atom and the quantized electromagnetic field. The Hamiltonian was solved by deriving its eigenvalues and eigenstates, known as dressed states, followed by a discussion of the dynamics of the Jaynes–Cummings states. Finally, we studied the phenomenon of vacuum Rabi oscillations, corresponding to the interaction between an atom and an initially empty cavity.

In the second chapter, we introduced the concepts of entanglement and quantum steering by discussing the EPR paradox and the limitations of local hidden-variable theories through Bell's inequality, together with Schrödinger's response to the EPR argument. We then moved to the characterization and quantification of entanglement and steerability. In particular, we discussed the CJWR inequality as a criterion for quantum steering, as well as entropy and concurrence as measures of entanglement, especially for pure states and mixed states such as X-states.

The main objectives of our study were presented in Chapters Three and Four. In Chapter Three, we investigated the time evolution of entanglement and steerability without considering environmental effects, whereas in Chapter Four the influence of a dephasing reservoir was taken into account. We first discussed how to generate two entangled Jaynes–Cummings atoms and, conversely, how to generate two entangled Jaynes–Cummings cavity modes. We then presented the detailed calculations describing a system of two entangled Jaynes–Cummings atoms, each interacting with its own vacuum cavity. By performing the partial trace over the photonic degrees of freedom, we obtained the reduced density matrix of the two atoms, which enabled us to quantify both entanglement and steerability.Based on the works of J.-H. Eberly, Ya-Xi Ban, and our own analysis, we presented and discussed several important results, including the phenomena of sudden death of entanglement and steerability, the periodic death–revival behavior, the compar-

ison between different types of Bell states, and the differences between concurrence and steerability measurements.

In the final chapter, we briefly introduced the Lindblad formalism and the master equation by discussing the Kraus representation, the assumptions made about the environment, and the role of the jump operators. We then solved the master equation for a single Jaynes–Cummings model under a phase-damping reservoir. After analyzing the time evolution of the single Jaynes–Cummings system in the presence of dephasing, we considered a system of two initially entangled atoms, as in the noiseless case, with each atom independently subjected to phase damping. We presented the detailed calculations describing the evolution of the system and evaluated both the concurrence and the steerability measures. The obtained results revealed important differences between entanglement and steerability quantification under decoherence, as well as their distinct responses for the two types of Bell states.

This work opens the door to further investigations of entanglement and quantum steering within the Jaynes–Cummings model, particularly concerning the comparison between concurrence and steerability under decoherence. Such a comparison is important for improving our understanding of the quantification of quantum correlations and may contribute to the development of more accurate methods for quantum computation and quantum information processing.

**Keywords:**

Jaynes–Cummings model, Entanglement, Steerability, Phase Damping reservoir.

# الملخص

قمنا في هذه الاطروحة بدراسة أهم مفهومين في مجال ميكانيك الكم عامة وفي مجال الحوسبة الكمومية خاصة وهما التشابك الكمومي وقابلية التوجيه، وذلك في نظام يعتبر من بين أكثر الأنظمة دراسة في فيزياء الذرة وهو نموذج Jaynes—Cummings الذي يدرس ديناميكا التفاعل بين الذرة والفوتون داخل تجويف كهروديناميكي كمومي. ركزت الدراسة في ديناميكا التطور لكل من التشابك وقابلية التوجيه لذرتين متشابكتين كل منهما في تجويف فارغ وذلك بتجاهل ضوضاء الوسط الخارجي تارة، وأخذه بعين الاعتبار تارة أخرى.

تناولنا في كل من الفصلين الأول والثاني مراجعة أهم المفاهيم الأساسية التي قمنا بتطبيقها في الدراسة الفعلية والتي تمحورت في الفصلين الثالث والرابع، ففي الفصل الأول تم تعريف مفهوم ظاهرة اهتزاز رابي الذي يمثل نقطة الارتكاز في ديناميكا التفاعل بين الذرة والفوتون وذلك بأخذ نظام المستويين للنصف اللف الذاتي (Tow-State Spin ½ System) كمثال كلاسيكي يفسر هذه ظاهر.

شرعنا بعد ذلك باستخلاص الهاميلتونيان التفاعل بين الذرة والفوتون الذي يعرف بـ J-C Hamiltonian وقدمنا بذلك حلول لهذا الهميلتونيان بحساب القيم الذاتية والحالات الذاتية التي تعرف بالحالات المغطاة (Dressed State)، وتابعنا بذلك دراسة كيفية تطور حالات النظام خلال الزمن والتي سمحت لنا بدراسة اهتزاز رابي في الفارغ الذي يفسر ظاهرة الانبعاث التلقائي والتي توافق وجود الذرة في حالة مثارة داخل تجويف في حالة فراغ.

أما الفصل الثاني فتمركز على إعطاء مفهوم للتشابك والتوجيه الكمومي وذلك بمناقشة EPR Paradox وقيود نظريات القيم المخفية المحلية من خلال متراجحة Bell مشيرين بذلك الى ردود العالم Schrödinger حول حجج اينشتاين، انتقلنا بعد ذلك الى الوصف الرياضي لكل من التشابك والتوجيه الكمومي من خلال مناقشة متراجحة CJWR كمعيار قياس للقابلية التوجيه وكذا الانتروبي والترافقية (concurrence) كمعايير قياس للتشابك الكمومي، مركزين بذلك لحالة الأكثر انتشار في نظام JC والمعروفة بالحالة المختلطة X.

في الفصل الثالث بدأنا بدراسة كيفية توليد تشابك بين ذرتين وبين فوتونين في التجويف الكهروديناميكي معتمدين بذلك على أعمال Serge Haroche و Jean-Michel Raimond ، ثم بدأنا بعد ذلك بدراسة تطور الزمني لذرتين متشابكتين مبدئيا كل منهما في تجويف كهروديناميكي كمومي فارغ بحيث أن التجويفين منعزلين تماما، بعد حسابات دقيقة توصلنا الى مصفوفة الكثافة التي تصف حالة الذرتين في أي لحظة زمنية t من خلال طريقة التتبع الجزئي (Partial Trace) التي تسمح بالتخلص من درجات الحرية للفوتونات، سمحت لنا مصفوفة الكثافة للذرتين بقياس كل من التشابك وقابلية التوجيه من خلال حساب الترافقية المتعلقة بالتشابك والانتهاك الاعظمي لمتراجحة CJWR لقابلية التوجيه، وبالاعتماد على أعمال J-H Eberly وXi-Ya Beng وعلى تحليلاتنا المحتملة تمكنا من تقديم العديد من النتائج المهمة والتي تشمل ظاهرة الموت المفاجئ للتشابك (ESD)، السلوك الدوري للموت والانبعاث، المقارنة بين نتائج حالتي Bell والاختلافات في معايير القياس بين الترافقية وقياس قابلية التوجيه.

قدمنا في الفصل الأخير، بشكل مختصر صيغ Lindblad ومعادلات ماستر البصرية من خلال مناقشة مؤثرات Kreaus، إضافة الى الشروط الموضوعة على ضوضاء الوسط الخارجي والتي تشمل تقريب Born وتقريب Marckove وبذلك انتقلنا الى حل معادلات الماستر البصرية لنظام Jaynes—Cummings بافتراض أن ضوضاء الوسط الخارجي عبارة عن طور مخمد يؤثر على الذرة وذلك بتوليد تخامد طوري بين الحالة المثارة والحالة الأرضية للذرة. وبعد معرفة كيفية تطور النظام في وجود هذا النوع من الضوضاء، اعتبرنا نظام ذرتين متشابكتين مبدئيا تتعرض كليهما لضوضاء الطور المخمد كل منهما في تجويف كهروديناميكي في حالة فراغ كما في الحالة الأولى، قدمنا حسابات دقيقة حول كيفية تطور هذا النظام عبر الزمن ومن خلال طريقة التتبع الجزئي تحصلنا على مصفوفة الكثافة التي تصف حالة الذرتين التي سمحت لنا بحساب الترافقية وقياس قابلية التوجيه، النتائج المتحصل عليها كشفت اختلافات مهمة بين التشابك والتوجيه الكمومي وكذا التصرف المتميز لكل منهما لحالات Bell.

يفتح هذا العمل مواضيع بحث معمقة حول دراسة التشابك والتوجيه الكمومي لنظام Jaynes—Cummings خاصة بمناقشة قيود متراجحة CJWR مقارنة بصيغ الترافقية تحت وجود الضوضاء، هذه المقارنة ستكون جد مهمة في فهمنا للتحديد الكمي للترابطات الكمومية والتي قد تساهم في تطوير طرق أكثر دقة للمعالجة الحوسبة الكمومية.



***الكلمات المفتاحية:***

نموذج Jaynes—Cummings، التشابك الكمومي، قابلية التوجيه، ضوضاء التخامد الطوري.

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