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**By: Timizar Malak**

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# Solvability and Numerical Results for a Nonlinear Singular Pseudo-Hyperbolic Coupled System with Nonlocal Boundary Conditions via the Homotopy Analysis Transform Method

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# شكر وتقدير

ما سلكنا البدايات إلا بتيسيره وما بلغنا النهايات إلا بتوفيقه وما حققنا الغايات إلا بفضله فالحمد لله الذي وفقني لتثمين هذه الخطوة في مسيرتي الدراسية . الحمد لله الذي لا يترك قلباً مكسوراً دون جبر, ولا يضيع صبراً مهما طال به الطريق . اليوم أقف هنا, لا لألتفت للخلف, بل لأتذكر كم مرة إخترت التقدم حين كان التراجع أسهل . هذا التخرج لم يكن أمنية عابرة, كان رحلة صعبة, و صراعا هادئاً, تعلمت فيه أن الصبر ليس إنتظاراً, بل ثبات... وأن الإستمرار إنتصار.ها أنا اليوم أتوج اللحظات الأخيرة من هذا الطريق بعد تعب و مشقة دامت سنوات في سبيل الحلم والعلم , حملت في طياتها أمنيات الليالي , و أصبحت اليوم قرة عين ,ها أنا اليوم أقف على عتبة تخرجي بشهادة الماستر , أقطف ثمار تعبي, وأرفع قبعتي بكل فخر, فاللهم لك الحمد اذا رضيت ,ولك الحمد بعد الرضا, لأنك وفقتني لإتمام هذا النجاح وتحقيق حلمي. بكل حب وفخر أهدي ثمرة نجاحي وتخرجي : إلى نفسي الطموحة التي قالت أنا لها, سأنالها وأخيرا ها أنا نلتها ... إلى نفسي التي مشت بثبات في دروب لم تكن سهلة...إلى تلك الروح التي تعثرت و لكنها نهضت . إلى الذي زين اسمي بأجمل الألقاب و دعمني بلا حدود و أعطاني بلا مقابل ...إلى من علمني أن الدنيا كفاح... وسلاحها العلم والمعرفة الى الذي لم يبخل عني بأي شيء إلى من سعى لأجل راحتي و نجاحي إلى أعظم رجل في الكون...أبي العزيز. إلى من جعل الله الجنة تحت أقدامها و إحتضنتني بقلبها قبل يدها ...إلى تلك الحبيبة ذات القلب النقي إلى من أوصاني الرحمان بها برا و إحسانا إلى من سعت وعانت من أجلي إلى من كان دعائها سر نجاحي ...أمي الحبيبة . إلى السند الحقيقي حين يرحل الجميع ,في صمتك دعم , و في وجودك أمان توأم الروح و بهجة الأيام من شاركني التعب و الفرح طيلة المشوار ... أخي الغالي . إلى من شددت عضدي بهم فكانوا لي ينابيع ارتوي منها إلى خيرة أيامي وصفوتها إلى قرة عيني ...إلى إخواتي الغاليات إلى من كانوا عونا وسندا في هذا الطريق ... لأصحاب الشدائد و الأزمات إلى من أفاضني بمشاعره و نصائحه المخلصة... عائلتي الكريمة

♡

# إهداء

الحمد والشكر للحي القيوم أولا وأخيرا وإمتثالا لقوله صلى الله عليه و سلم:
"من لا يشكر الناس لا يشكر الله"
أتوجه بجزيل الشكر وجميل العرفان للأستاذ المشرف "د.قاسمي لطفي"الذي تكرم بقبول الإشراف على هذه المذكرة وعلى جميع التوجيهات و الملاحظات والنصائح .
كما لا يفوتني أن أتقدم بوافر التقدير و الإحترام لأعضاء اللجنة المحترمين على عناء قراءة المذكرة وقبولها و تصويبها.
وكذلك أتقدم بخالص الشكر إلى كل من درسني من أساتذة كلية الرياضيات بجامعة ورقلة والى كل موظفي الجامعة جزاهم الله كل خير .
وفي الأخير أشكر كل من قدم لي يد العون والمساعدة من قريب أو من بعيد وأسأل الله عز وجل أن يجعل ذلك في ميزان حسناتهم إنه قريب مجيب .

ملاك & تيميزار

# Contents

# الملخّص

تتناول هذه الرسالة دراسة وجود ووحدانية الحل لنظام شبه زائدي غير خطي مرفق بشروط حدودية من نوع نيومان وشروط حدودية تكاملية. في البداية، يتم وضع تقدير مسبق لجملة المعادلات الخطية المترابطة، ونثبت كثافة مدى التطبيق الناتج عن هذه الجملة في فضاء هيلبرت. بعد ذلك، ومن خلال تطبيق عملية تكرارية استنادًا إلى النتائج المستخلصة من المسألة الخطية، يتم إثبات وجود ووحدانية الحل الضعيف لجملة المعادلات الغير الخطية.

في الجزء العددي، تم استخدام طريقة التحليل الهموتوبي مع التحويل للحصول على حلول عددية تقريبية. وتم بناء مخطط تكراري يعتمد على طريقة التحليل الهموتوبي ، وتم توضيح فعاليته من خلال أمثلة عددية، بما في ذلك الجداول والرسوم البيانية، مما يؤكد كفاءة هذه الطريقة في حل أنظمة المعادلات التفاضلية المترابطة.



## الكلمات المفتاحيّة:

جمل معادلات غير خطية مترابطة، الحلول الضعيفة، طريقة التقديرات المسبقة، طريقة التحليل الهموتوبي مع التحويل، شروط حدودية غير محلية.

# Abstract

This thesis investigates the existence and uniqueness of solutions for a coupled nonlinear pseudo-hyperbolic system with Neumann and integral boundary conditions. We first establish a priori estimates for the associated linear system and prove that the range of the operator generated by the problem is dense. Then, by applying an iterative process based on the results obtained for the linear system, we establish the existence, uniqueness dependence of the weak solution for the nonlinear problem.

In the numerical part, the Homotopy Analysis Transform Method (HATM) is employed to obtain approximate solutions. An iterative scheme is constructed based on HATM, and its effectiveness is illustrated through numerical examples, including tables and graphs, which confirm the efficiency of the method in solving coupled systems of differential equations.



### Keywords:

Coupled nonlinear systems, Weak solutions, Energy Inequality Method, Homotopy Analysis Transform Method (HATM), Nonlocal boundary conditions.

# Introduction

Partial differential systems constitute one of the most significant areas of contemporary mathematical research, having witnessed remarkable advancements over the past decades. They have become indispensable tools for modeling complex phenomena in physics, engineering, and applied sciences [**1**]. The increasing interest in these systems is largely due to their capability to describe dynamics with long-term memory effects and nonlocal interactions-features . Notably, such equations have found prominent practical applications, including the modeling of heat transfer in heterogeneous materials and the analysis of the motion of charged particles in intricate electromagnetic fields. In these contexts, the solutions of partial differential systems provide insights that surpass those obtainable through traditional modeling approaches [**2**].

Amidst these developments, the study of differential equations subject to integral (non-local) conditions has attracted considerable attention. Such conditions provide a crucial extension to classical mathematical models, particularly in situations where direct boundary measurements are infeasible or only averaged or aggregate data are accessible. One of the pioneering contributions in this area was made by Cannon [**3**], who employed integral conditions to analyze heat transfer in metallic materials via heat-section equations. This foundational work paved the way for subsequent studies, including significant contributions by Lonkin and Yorshuk [**4**, **5**], who applied sophisticated techniques such as the Fourier method and energy estimates to rigorously investigate the existence and uniqueness of solutions in these systems. More recent research [**6**] has further extended these analyses to encompass nonlinear coupled systems, highlighting the growing complexity and applicability of differential equations with non-local constraints.

From a theoretical standpoint, considerable research efforts have been devoted to investigating the qualitative properties of such equations, with particular emphasis on establishing the existence and uniqueness of solutions. To this end, a variety of powerful analytical techniques have been employed, including fixed-point principles and the Lax--Milgram theorem [**7**]. Among these approaches, energy estimate methods originally introduced in the pioneering works of Petrovsky [**8**] have proven to be especially effective and have become a fundamental tool in the analysis of systems subject to integral boundary conditions. Over time, this framework has been significantly developed and extended, enabling the treatment of a broad class of differential equations, as demonstrated in [**9**].

To address the inherent difficulties in solving nonlocal coupled systems, a variety of advanced numerical techniques have been developed in recent years to produce highly accurate approximate solutions. These approaches can be broadly classified into two main categories: classical methods, such as the Finite Element Method (FEM) and the Finite Difference Method (FDM), and modern analytical–numerical methods, including the Adomian Decomposition Method (ADM) [**10**], the Homotopy Analysis Method (HAM) [**11**], and the Homotopy Analysis Transform Method (HATM).

Among these, the HATM represents a significant advancement in the numerical treatment of differential equations. It combines the strengths of the classical homotopy analysis framework with integral transform techniques, leading to a more efficient and flexible solution procedure. The core idea of this method consists in transforming the original problem into a more tractable form, which enables the construction of a convergent sequence of iterative approximations to the solution.

This approach is particularly distinguished by its ability to accelerate convergence, improve the accuracy of the obtained solutions, and provide a high degree of flexibility when dealing with complex and nonlinear systems. Consequently, the HATM has emerged as a powerful and reliable tool for the analysis of nonlocal coupled differential systems [**12**].

In this thesis, we are concerned with the study of a nonlinear coupled system of the form:

$$\begin{cases} u_{tt} - \frac{1}{x}(x u_x)_x - \frac{1}{x}(x u_x)_{xt} + z_1 v + u_t = f(x,t,u,v,u_x,v_x), \\ v_t - \frac{1}{x}(x v_x)_x - \frac{1}{x}(x v_x)_{xt} + z_2 u = g(x,t,u,v,u_x,v_x), \\ u(x,0) = \varphi_1(x), \quad u_t(x,0) = \varphi_2(x), \\ v(x,0) = \psi_1(x), \\ u_x(1,t) = 0, \quad v_x(1,t) = 0, \quad \int_0^1 x u dx = 0, \quad \int_0^1 x v dx = 0, \end{cases} \tag{1}$$

where: $Q = \Omega \times [0,T] = \{(x,t) : 0 < x < 1, 0 \le t \le T\}$ .

The functions $f$ and $g$ are $L^2\left(0,T; L^2_\rho(\Omega)\right)$ given Lipschitzian functions, for all $(x,t) \in Q$. The functions $\varphi_1, \varphi_2$ and $\psi_1$ are in $H^1_\rho(\Omega)$, and $z_1, z_2$ are positive constants. In some physical problems, $u$ is the displacement, $v$ is the difference absolute temperature, $f$ is an external force, $g$ is a heat supply. The present coupled system provides a suitable framework for applying the Homotopy Analysis Transform Method (HATM) to obtain approximate solutions, subsequent to proving the solvability of the system through the energy inequality method. This approach aims to emphasize the capability of the proposed techniques in effectively addressing complex nonlinear systems.

**Chapter One:** This chapter lays the theoretical foundation of the study by introducing the necessary function spaces together with their definitions and norms. It also presents the key lemmas and inequalities that will be employed in the subsequent analysis. Moreover, particular attention is given to the relationship between orthogonality and density in Hilbert spaces, along with an outline of the energy inequality method.

**Chapter Two:** We study the solvability of a nonlinear coupled system arising, for example, in heat conduction models involving both thermodynamic and conductive temperatures. The associated linear system is first reformulated within an operator framework. An a priori estimate, in the form of an energy inequality, is established, leading to the uniqueness of a strong solution.

Next, we demonstrate that the range of the corresponding operator is dense, which allows us to infer the existence of a strong solution. Building upon these results, an iterative procedure is employed to address the nonlinear problem, through which the existence, uniqueness of the weak solution are established.

**Chapter Three:** In this chapter, we construct an iterative scheme based on the Homotopy Analysis Transform Method (HATM) to approximate the solution of a singular linear pseudo-hyperbolic coupled system. The performance of the proposed approach is highlighted through two representative examples, demonstrating its accuracy and computational efficiency.

Finally, we highlight the principal findings and discuss the overarching conclusions derived from the study.

# Notation

| Symbol | Meaning |
|---|---|
| $\mathscr{L}, l$ | Linear operators. |
| $D(L)$ | Domain of definition of the operator $L$. |
| $\mathrm{R}(L)$ | Image of the operator $L$. |
| $L_p^2(\Omega)$ | Space of square integrable function $u$ with weight function $p$, defined on $\Omega$ |
| $HAM$ | Homotopy analysis method. |
| $HATM$ | Homotopy Analysis Transform Method. |
| $H_p^1(\Omega)$ | A weighted Sobolev space |

# Chapter one

## Preliminaries and functions spaces

### Contents

This chapter presents function spaces, their norms, and the fundamental inequalities and tools that will be used throughout this thesis.

## 1.1 Functions spaces

**Definition 1.1.1: See[15]**

e denote by $L^2(\Omega)$ the space of square-integrable functions on $\Omega$. A complex-valued function $u$ on $\Omega$ is square-integrable if it is measurable and satisfies $|u|^2 \in L^1(\Omega)$. The norm in $L^2(\Omega)$ is given by:

$$\|u\|_{L^2(\Omega)} = \left(\int_\Omega |u(x)|^2 dx\right)^{\frac{1}{2}}.$$

**Definition 1.1.2: See[19, 16]**

o investigate the problem under consideration, we introduce the necessary function spaces. We denote by $L^2_p(\Omega)$ the Hilbert space of weighted square-integrable functions, endowed with the inner product

$$(u,v)_{L^2_p(\Omega)} = (xu,v)_{L^2(\Omega)} = \int_\Omega xuv\,dx, \tag{1.1}$$

with associated norm

$$\|u\|^2_{L^2_p(\Omega)} = \|\sqrt{x}u\|^2_{L^2(\Omega)} = \left(\int_\Omega xu^2 dx\right)^{\frac{1}{2}}. \tag{1.2}$$

The space $H^1_p(\Omega)$ denotes the weighted Sobolev space with norm:

$$\|u\|^2_{H^1_p(\Omega)} = \|u\|^2_{L^2_p(\Omega)} + \|u_x\|^2_{L^2_p(\Omega)} < \infty. \tag{1.3}$$

Let $L^2(0,T;L^2_p(\Omega))$ be the space consisting of all mesurable functions:
$u:[0,T] \longrightarrow L^2_p(\Omega)$ with scalar product:

$$(u,v)_{L^2(0,T;L^2_p(\Omega))} = \int_0^T (u,v)_{L^2_p(\Omega)} dt, \tag{1.4}$$

associated with finite norm

$$\|u\|_{L^2(0,T;L^2_p(\Omega))} = \int_0^T \|u\|^2_{L^2_p(\Omega)}. \tag{1.5}$$

The space $L^2(0,T;H^1_p(\Omega))$ consists of all functions that are square-integrable in the Bochner sense, with inner product

$$(u,v)_{L^2(0,T;H^1_p(\Omega))} = \int_0^T (u(.,t),v(.,t))_{H^1_p(\Omega)} dt, \tag{1.6}$$

and the associated norm is given by

$$\|u\|_{L^2(0,T;H^1_p(\Omega))} = \int_0^T \|u(.,t)\|^2_{L^2_p(\Omega)} dt + \int_0^T \|u_x(.,t)\|^2_{L^2_p(\Omega)} dt. \tag{1.7}$$

## 1.2 Lemma and Some Inequalities

**Lemma 1.2.1: Gronwall's lemma (See [16])**

et $f_1$, $f_2$, and $f_3$ be nonnegative functions on the interval $[0,T]$, with $f_1$ and $f_2$ integrable on $[0,T]$, and $f_3$ bounded and nondecreasing on $[0,T]$. Let $C$ be a positive constant. Then, the inequality

$$\int_0^t f_1(\tau)\,d\tau + f_2(t) \leq e^{Ct} f_3(t), \tag{1.8}$$

is a direct consequence of

$$\int_0^t f_1(\tau)\,d\tau + f_2(t) \leq f_3(t) + C\int_0^t f_2(\tau)\,d\tau. \tag{1.9}$$

We need the following **Cauchy-$\varepsilon$ inequality** (see [**16**]):

$$ab \leq \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2, \quad \forall \varepsilon > 0, \tag{1.10}$$

where $a$ and $b$ are positive numbers.
For our purposes, we employ **Poincaré-type inequalities** (see [**16**])

$$\|\Im_x(\xi u)\|^2_{L^2(\Omega)} \leqslant \frac{b^3}{2}\|u(.,t)\|^2_{L^2_\rho(\Omega)}, \tag{1.11}$$

$$\left\|\Im_x^2(\xi u)\right\|^2_{L^2(\Omega)} \leqslant \frac{b^2}{2}\|\Im_x(\xi u)\|^2_{L^2(\Omega)}, \tag{1.12}$$

$$\|\Im_x(\xi u)\|^2_{L^2_\rho(\Omega)} \leqslant b\,\|\Im_x(\xi u)\|^2_{L^2(\Omega)}, \tag{1.13}$$

$$\|u(x,t)\|^2_{L^2_\rho(\Omega)} \leq \frac{b^2}{4}\|u_x(x,t)\|^2_{L^2_\rho(\Omega)}, \tag{1.14}$$

$$\|v(x,t)\|^2_{L^2_\rho(\Omega)} \leq \frac{b^2}{4}\|v_x(x,t)\|^2_{L^2_\rho(\Omega)}, \tag{1.15}$$

where: $\Im_x(\xi v) = \int_0^x \xi v(\xi,t)\,d\xi$, $\quad \Im_x^2(\xi v) = \int_0^x \int_0^\xi \eta v(\eta,t)\,d\eta$.

## 1.3 The Relationship between Orthogonality and Density in Hilbert Spaces

**Proposition 1**

Let $M$ be a vector subspace of a Hilbert space $H$. Then, $M$ is dense in $H$ **if and only if** $M^\perp = \{0\}$.

**Proof**

**Part 1:** Assume $M$ is dense in $H$. Let $f \in H$, and suppose $\{f_n\}$ is a sequence in $M$ converging to $f$. For every $n \in \mathbb{N}$, we have:

$$\langle f, f_n \rangle_H = 0.$$

Taking the limit as $n \to \infty$, we obtain $\langle f, f \rangle_H = 0$, which implies $f = 0$. Thus, $M^\perp = \{0\}$.
**Part 2:** Conversely, assume $M^\perp = \{0\}$. Then:

$$(M^\perp)^\perp = \{0\}^\perp = H.$$

Since $M \subseteq (M^\perp)^\perp$, it follows that $M \subseteq H$. Moreover, $(M^\perp)^\perp \subseteq M$ implies $(M^\perp)^\perp = M$. Combining these results, we have:

$$M^\perp \subseteq M \implies H \subseteq M.$$

Therefore, $M$ is dense in $H$.

## 1.4 The Energy Inequality Method

(See [**8**]) The Energy Inequality method, also known as a priori estimates, is an important mathematical framework for solving boundary value problems for differential equations, particularly Cauchy problems for hyperbolic equations. This method was developed by E. G. Petrovsky. The fundamental idea consists of transforming the problem into an operator form and analyzing the behavior of the solution through norm estimates.

First, the problem is formulated in the operator form:

$$Lu = \mathscr{F}, \quad \forall u \in D(L), \tag{1.16}$$

where the operator $L$ maps from a Banach space $B$ to a Hilbert space $H$. The uniqueness of the solution is then studied by establishing the following energy inequality:

$$\|u\|_B \leq c\|Lu\|_H, \tag{1.17}$$

This inequality bounds the norm of the solution $u$ in terms of the norm of $Lu$, and is typically derived by multiplying the equation under study by an operator $Mu$ (which involves $u$, its derivatives, and a suitable weight function) and performing the necessary computations. The choice of $Mu$ depends strongly on the specific equation and the boundary conditions.

It is then shown that $L$ admits a closure $\overline{L}$, so that the solution $u$ can be regarded as a strong solution of the equation:

$$\overline{L}u = \mathscr{F}, \quad u \in D(\overline{L}), \tag{1.18}$$

the inequality (1.17) is extended to the closure $\overline{L}$, yielding:

$$\|u\|_B \leq c\|\overline{L}u\|_H, \tag{1.19}$$

which confirms the uniqueness of the solution to equation (1.18).

To guarantee the existence of a solution, the image of the operator $\overline{L}$ must be both closed and dense in the Hilbert space $H$.

**Explanation of the Existence Proof:** To prove existence, we employ the energy inequality (1.19) together with the properties of the operator $\overline{L}$. We analyze the image of $\overline{L}$, denoted by $\text{Im}(\overline{L})$.

If this image is closed in $H$ (meaning that any convergent sequence of outputs in $H$ belongs to the image) and dense (meaning it is sufficiently rich throughout $H$), then the given data $\mathscr{F}$ for the equation $\overline{L}u = \mathscr{F}$ is compatible with the operator $\overline{L}$.

Under these conditions, the energy inequality ensures that the mapping $u \mapsto \overline{L}u$ admits a bounded inverse on its image. Consequently, the equation $\overline{L}u = \mathscr{F}$ can be transformed into a suitable framework to guarantee the existence of a solution $u \in D(\overline{L})$. In other words, if the data $\mathscr{F}$ is consistent with the nature of the process represented by $\overline{L}$ (expressed by closedness and density), then the energy inequality ensures the existence of a solution that is properly bounded and well defined.

## 1.5 Homotopy Analysis Transform Method

**Definition 1.5.1**

In the framework of the Homotopy Analysis Method (HAM), the homotopy function is defined as a continuous mapping that deforms an initial approximation of the solution into the exact solution of a given nonlinear problem.
Let us consider the nonlinear problem:

$$\mathscr{N}[u(x,t)] = 0, \tag{1.20}$$

where $\mathscr{N}$ is a nonlinear operator.
The homotopy function is constructed as follows:

$$H(u,p) = (1-p)\mathscr{L}[u-u_0] + p\,\hbar\,\mathscr{N}[u] = 0, \quad p \in [0,1], \tag{1.21}$$

where:

- $p \in [0,1]$ is an embedding parameter,
- $u_0(x,t)$ is an initial approximation of the solution,
- $\mathscr{L}$ is an auxiliary linear operator,
- $\mathscr{N}$ is the original nonlinear operator,
- $\hbar \neq 0$ is a convergence-control parameter.

The main idea of the homotopy is based on the following properties:

- When $p = 0$, we have

$$H(u,0) = \mathscr{L}[u-u_0] = 0 \quad \Rightarrow \quad u = u_0. \tag{1.22}$$

- When $p = 1$, we obtain

$$H(u,1) = \hbar\,\mathscr{N}[u] = 0 \quad \Rightarrow \quad \mathscr{N}[u] = 0. \tag{1.23}$$

Thus, the homotopy provides a continuous deformation from the initial approximation $u_0$ to the exact solution $u$ of the nonlinear problem, see[**14**].

**Definition 1.5.2**

The Laplace transform $\mathscr{L}$ of the ordinary derivative of integer order $n$ of a function $u(x,t)$ is defined as:

$$\mathscr{L}\left[\frac{\partial^n}{\partial t^n}u(x,t)\right]=s^n\mathscr{L}\{u(x,t)\}-\sum_{k=0}^{n-1}s^{n-k-1}u^{(k)}(x,0), \tag{1.24}$$

where $u^{(k)}(x,0)$ denotes the $k$-th derivative of $u(x,t)$ evaluated at $t=0$.

Consider the following ordinary differential equation of integer order:

$$\frac{\partial^n}{\partial t^n}u(x,t)+R(u(x,t))+N(u(x,t))=F(x,t), \quad n\in\mathbb{N}, \tag{1.25}$$

where $u(x,t)$ is a differentiable function, $\frac{\partial^n}{\partial t^n}$ is the ordinary derivative of order $n$, $R$ is a linear differential operator, $N$ denotes a nonlinear differential operator, and $F(x,t)$ is a known source function.

Applying the Laplace transform to both sides of equation (1.25) gives:

$$s^n\mathscr{L}\{u(x,t)\}-\sum_{k=0}^{n-1}s^{n-k-1}u^{(k)}(x,0)+\mathscr{L}\{R(u(x,t))+N(u(x,t))\}=\mathscr{L}\{F(x,t)\}. \tag{1.26}$$

# Chapter two

## Existence and Uniqueness of Solutions for Linear and Nonlinear Problems

**Contents**

## 2.1 Reformulation of the linear problem

We consider a linear singular pseudo-hyperbolic coupled system of the form:

$$\begin{aligned}\mathscr{L}_1(u,v) &= \frac{\partial^2 u}{\partial t^2} + -\frac{1}{x}(xu_x)_x - \frac{1}{x}(xu_x)_{xt} + z_1 v + u_t = f(x,t),\\ \mathscr{L}_2(u,v) &= \frac{\partial v}{\partial t} + -\frac{1}{x}(xv_x)_x - \frac{1}{x}(xv_x)_{xt} + z_2 u = g(x,t),\end{aligned} \tag{2.1}$$

associated with the initial conditions:

$$\begin{aligned}&\ell_1 u = u(x,0) = \varphi_1(x), \quad \ell_2 u = u_t(x,0) = \varphi_2(x),\\ &\ell_3 v = v(x,0) = \psi_1(x),\end{aligned} \tag{2.2}$$

and the Neumann and integral boundary conditions:

$$u_x(1,t) = 0, \quad v_x(1,t) = 0, \int_0^1 xu\,dx = 0, \quad \int_0^1 xv\,dx = 0. \tag{2.3}$$

We assume that there exists a solution $(u,v) \in (C^{2,2}(\overline{Q}))^2$ consisting of the set of functions together with their partial derivatives of order 2 in $x$ and $t$, which are continuous on $\overline{Q}$.
The solution of system 2.1-2.3 can be regarded as the solution of operator equation

$$LW = \mathscr{F}, \tag{2.4}$$

where $W, LW$ and $\mathscr{F}$ are respectively the pairs:

$$\begin{aligned}W &= (u,v),\\ LW &= (L_1 u, L_2 v),\\ \mathscr{F} &= (\mathscr{F}_1, \mathscr{F}_2),\end{aligned} \tag{2.5}$$

with

$$L_1 u = \{\mathscr{L}_1 u, \ell_1 u, \ell_2 u\}, \quad L_2 v = \{\mathscr{L}_2 v, \ell_3 v\}, \tag{2.6}$$

and

$$\mathscr{F}_1 = \{f, \varphi_1, \varphi_2\}, \quad \mathscr{F}_2 = \{g, \psi_1\}. \tag{2.7}$$

The operator $L$ is considered from a space $B$ into a space $H$, where $B$ is a Banach space consisting of all functions $(u,v) \in \left(L^2\left(0,T;L^2_\rho(Q)\right)\right)^2$ satisfying conditions 2.3 and having the finite norm:

$$\begin{aligned}\|(u,v)\|_B^2 = \sup_{0\le t\le T} &\Big\{\left(\|u_t\|^2_{L^2_\rho(\Omega)} + \|u_x\|^2_{L^2_\rho(\Omega)} + \|\Im_x(\xi u_t)\|^2_{L^2(\Omega)} + \|v_x\|^2_{L^2_\rho(\Omega)}\right)\\ &+ \|u_t\|^2_{L^2(0,T,L^2_\rho(\Omega)} + \|\Im_x(\xi u_t)\|^2_{L^2(0,T,L^2_\rho(\Omega)}\Big\},\end{aligned} \tag{2.8}$$

and $H = \left\{L^2(0,T,L^2_\rho(\Omega))\right\}^2 \times \left\{H^1_\rho(\Omega)\right\}^3$ is the Hilbert space consisting of vector-valued functions

$$\mathscr{F} = (\{f, \varphi_1, \varphi_2\}, \{g, \psi_1\}), \tag{2.9}$$

with norm

$$\|\mathscr{F}\|_H^2 = \|f\|^2_{L^2(0,T,L^2_\rho(\Omega))} + \|g\|^2_{L^2(0,T,L^2_\rho(\Omega))} + \left\|\varphi_1\right\|^2_{H^1_\rho(\Omega)} + \left\|\varphi_2\right\|^2_{H^1_\rho(\Omega)} + \left\|\psi_1\right\|^2_{H^1_\rho(\Omega)}. \tag{2.10}$$

Let $D(L)$ be the domain of definition of the operator $L$, defined by:

$$D(L) = \begin{cases}(u,v) \in \left(L^2\left(0,T;L^2_\rho(\Omega)\right)\right)^2 \text{ such that } \frac{\partial^2 u}{\partial t^2}, \frac{\partial v}{\partial t},\\ u_x, v_x, u_{xx}, v_{xx}, u_{tx}, v_{tx}, u_{txx}, v_{txx} \in L^2\left(0,T;L^2_\rho(\Omega)\right),\\ u_x(1,t) = 0, \quad v_x(1,t) = 0, \int_0^1 xu\,dx = 0, \quad \int_0^1 xv\,dx = 0.\end{cases} \tag{2.11}$$

## 2.2 Uniqueness of the Solution to the Linear System

In this section, we prove the uniqueness result for the linear coupled system (2.1)-(2.3), that is we establish an energy inequality for the operator $L$ and we give some of its consequences.

**Theorem 1**

For any $(u,v)\in D(L)$, $f,g\in L^2\left(0,T;L^2_\rho(Q)\right)$ and $\varphi_1,\varphi_2,\psi_1\in H^1_\rho(\Omega)$, the solution of the problem (2.1)-(2.3) verifies the a priori bound

$$\|(u,v)\|_B^2\le C_2\|\mathscr{F}\|_H^2, \tag{2.12}$$

where

$$C_1=\max\left\{z_1+\frac{3}{2},\frac{z_1}{2}+2,\frac{z_2^2}{2}+1\right\}, \tag{2.13}$$

$$C_2=C_1\exp(C_1T). \tag{2.14}$$

**Proof**

Taking the inner products in $L^2_\rho(\Omega)$, of the partial differential equations in (2.1) and the operators $\mathscr{M}_1u=u_t-\mathfrak{I}_x^2(\xi u_t)$, $\mathscr{M}_2v=v_t-\mathfrak{I}_x^2(\xi v_t)$ respectively, we have

$$\begin{aligned}
&(u_{tt}u_t)_{L^2_\rho(\Omega)}-\left(u_{tt},\mathfrak{I}_x^2(\xi u_t)\right)_{L^2_\rho(\Omega)}-\left(\frac{1}{x}(xu_x)_x,u_t\right)_{L^2_\rho(\Omega)}+\left(\frac{1}{x}(xu_x)_x,\mathfrak{I}_x^2(\xi u_t)\right)_{L^2_\rho(\Omega)}\\
&-\left(\frac{1}{x}(xu_x)_{xt},u_t\right)_{L^2_\rho(\Omega)}+\left(\frac{1}{x}(xu_x)_{xt},\mathfrak{I}_x^2(\xi u_t)\right)_{L^2_\rho(\Omega)}+(z_1v,u_t)_{L^2_\rho(\Omega)}-\left(z_1v,\mathfrak{I}_x^2(\xi u_t)\right)_{L^2_\rho(\Omega)}\\
&+(u_t,u_t)_{L^2_\rho(\Omega)}+\left(u_t,\mathfrak{I}_x^2(\xi u_t)\right)_{L^2_\rho(\Omega)}\\
&(v_t,v_t)_{L^2_\rho(\Omega)}-\left(v_t,\mathfrak{I}_x^2(\xi v_t)\right)_{L^2_\rho(\Omega)}-\left(\frac{1}{x}(xv_x)_x,v_t\right)_{L^2_\rho(\Omega)}+\left(\frac{1}{x}(xv_x)_x,\mathfrak{I}_x^2(\xi v_t)\right)_{L^2_\rho(\Omega)}\\
&-\left(\frac{1}{x}(xv_x)_{xt},v_t\right)_{L^2_\rho(\Omega)}+\left(\frac{1}{x}(xv_x)_{xt},\mathfrak{I}_x^2(\xi v_t)\right)_{L^2_\rho(\Omega)}+(z_2u,v_t)_{L^2_\rho(\Omega)}-\left(z_2u,\mathfrak{I}_x^2(\xi v_t)\right)_{L^2_\rho(\Omega)}\\
&=\left(f,u_t\right)_{L^2_\rho(\Omega)}-\left(f,\mathfrak{I}_x^2(\xi u_t)\right)_{L^2_\rho(\Omega)}+(g,v_t)_{L^2_\rho(\Omega)}-(g,\mathfrak{I}_x^2(\xi v_t))_{L^2_\rho(\Omega)}.
\end{aligned} \tag{2.15}$$

Using the conditions (2.3) and applying standard integration by parts to some terms on the left-hand side of (2.15), we obtain:

$$(u_{tt}, u_t)_{L^2_p(\Omega)} = \frac{1}{2}\frac{\partial}{\partial t}\|u_t\|^2_{L^2_p(\Omega)}, \tag{2.16}$$

$$\left(u_{tt}, \Im^2_x(\xi u_t)\right)_{L^2_p(\Omega)} = \frac{1}{2}\frac{\partial}{\partial t}\|\Im^1_x(\xi u_t)\|^2_{L^2(\Omega)}, \tag{2.17}$$

$$-\left(\frac{1}{x}(xu_x)_x, u_t\right)_{L^2_p(\Omega)} = \frac{1}{2}\frac{\partial}{\partial t}\|u_x\|^2_{L^2_p(\Omega)}, \tag{2.18}$$

$$-\left(\frac{1}{x}(xu_x)_x, \Im^2_x(\xi u_t)\right)_{L^2_p(\Omega)} = -(u_x, \Im_x(\xi u_t))_{L^2_p(\Omega)}, \tag{2.19}$$

$$-\left(\frac{1}{x}(xu_x)_{xt}, u_t\right)_{L^2_p(\Omega)} = \|u_{xt}\|^2_{L^2_p(\Omega)}, \tag{2.20}$$

$$-\left(\frac{1}{x}(xu_x)_{xt}, \Im^2_x(\xi u_t)\right)_{L^2_p(\Omega)} = -(u_{xt}, \Im_x(\xi u_t))_{L^2_p(\Omega)}, \tag{2.21}$$

$$(z_1 v, u_t)_{L^2_p(\Omega)} = z_1(v, u_t)_{L^2_p(\Omega)}, \tag{2.22}$$

$$\left(z_1 v, -\Im^2_x(\xi u_t)\right)_{L^2_p(\Omega)} = -z_1\left(\Im^2_x(\xi v), u_t\right)_{L^2_p(\Omega)}, \tag{2.23}$$

$$(u_t, u_t)_{L^2_p(\Omega)} = \|u_t\|^2_{L^2_p(\Omega)}, \tag{2.24}$$

$$\left(u_t, \Im^2_x(\xi u_t)\right)_{L^2_p(\Omega)} = \|\Im^2_x(\xi u_t))\|^2_{L^2(\Omega)}, \tag{2.25}$$

$$(v_t, v_t)_{L^2_p(\Omega)} = \|v_t\|^2_{L^2_p(\Omega)}, \tag{2.26}$$

$$-\left(\frac{1}{x}(xv_x)_x, v_t\right)_{L^2_p(\Omega)} = \frac{1}{2}\frac{\partial}{\partial t}\|v_x\|^2_{L^2_p(\Omega)}, \tag{2.27}$$

$$-\left(\frac{1}{x}(xv_x)_{xt}, v_t\right)_{L^2_p(\Omega)} = \|v_{xt}\|^2_{L^2_p(\Omega)}, \tag{2.28}$$

$$(z_2 u, v_t)_{L^2_p(\Omega)} = z_2(u, v_t)_{L^2_p(\Omega)}, \tag{2.29}$$

$$-\left(\frac{1}{x}(xv_x)_{xt}, \Im^2_x(\xi v_t)\right)_{L^2_p(\Omega)} = -(v_{xt}, \Im_x(\xi v_t))_{L^2_p(\Omega)}, \tag{2.30}$$

$$\left(\frac{1}{x}(xv_x)_x, \Im^2_x(\xi v_t)\right)_{L^2_p(\Omega)} = -(v_x, \Im_x(\xi v_t))_{L^2_p(\Omega)}, \tag{2.31}$$

$$\left(z_2 u, -\Im^2_x(\xi v_t)\right)_{L^2_p(\Omega)} = -z_2\left(\Im^2_x(\xi u), v_t\right)_{L^2_p(\Omega)}, \tag{2.32}$$

$$\left(v_t, -\Im^2_x(\xi v_t)\right)_{L^2_p(\Omega)} = \|\Im^2_x(\xi_t))\|^2_{L^2(\Omega)}, \tag{2.33}$$

$$\left(f, u_t\right)_{L^2_p(\Omega)} = \left(f, u_t\right)_{L^2_p(\Omega)}, \tag{2.34}$$

$$\left(f, -\Im^2_x(\xi v_t)\right)_{L^2_p(\Omega)} = \left(f, -\Im^2_x(\xi v_t)\right)_{L^2_p(\Omega)}, \tag{2.35}$$

$$\left(g, u_t\right)_{L^2_p(\Omega)} = \left(g, u_t\right)_{L^2_p(\Omega)}, \tag{2.36}$$

$$\left(g, -\Im^2_x(\xi v_t)\right)_{L^2_p(\Omega)} = \left(g, -\Im^2_x(\xi v_t)\right)_{L^2_p(\Omega)}. \tag{2.37}$$

Substituting (2.16)--(2.37) into (2.15) and using the symmetry of the problem, we obtain:

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\|u_x\|^2_{L^2_p(\Omega)}+\|v_x\|^2_{L^2_p(\Omega)}+\|u_t\|^2_{L^2_p(\Omega)}+\|\Im^1_x(\xi u_t)\|^2_{L^2(\Omega)}\right)+\|u_{xt}\|^2_{L^2_p(\Omega)}+\|u_t\|^2_{L^2_p(\Omega)} \tag{2.38}$$
$$+\|\Im^1_x(\xi u_t)\|^2_{L^2(\Omega)}+\|v_{xt}\|^2_{L^2_p(\Omega)}+\|v_t\|^2_{L^2_p(\Omega)}+\|\Im^1_x(\xi u_t)\|^2_{L^2(\Omega)}$$
$$\le (f,u_t)_{L^2_p(\Omega)}-\left(f,\Im^2_x(\xi u_t)\right)_{L^2_p(\Omega)}+(g,v_t)_{L^2_p(\Omega)}-(g,\Im^2_x(\xi v_t))_{L^2_p(\Omega)}$$
$$+(z_1 v,u_t)_{L^2_p(\Omega)}-(u_x,\Im_x(\xi u_t))_{L^2_p(\Omega)}-(u_{xt},\Im_x(\xi u_t))_{L^2_p(\Omega)}+\left(z_2 u,\Im^2_x(\xi v)\right)_{L^2_p(\Omega)}$$
$$+(z_2 u,v_t)_{L^2_p(\Omega)}-(v_x,\Im_x(\xi v_t))_{L^2_p(\Omega)}-(v_{xt},\Im_x(\xi v_t))_{L^2_p(\Omega)}-\left(z_1 v,\Im^2_x(\xi u)\right)_{L^2_p(\Omega)}.$$

By applying the Cauchy-$\varepsilon$ inequality (1.10) and the Poincaré-type inequalities (1.11)-(1.15) to the terms on the right-hand side of (2.38), we obtain:

$$\left(f,u_t\right)_{L^2_p(\Omega)}\le\frac{1}{2}\|f\|^2_{L^2_p(\Omega)}+\frac{1}{2}\|u_t\|^2_{L^2_p(\Omega)}, \tag{2.39}$$

$$-\left(f,\Im^2_x(\xi u_t)\right)_{L^2_p(\Omega)}\le\frac{1}{2}\|f\|^2_{L^2_p(\Omega)}+\frac{1}{2}\|\Im^2_x(\xi u_t)\|^2_{L^2_p(\Omega)}\le\frac{1}{2}\|f\|^2_{L^2_p(\Omega)}+\frac{1}{8}\|u_t\|^2_{L^2_p(\Omega)}, \tag{2.40}$$

$$-(z_1 v,u_t)_{L^2_p(\Omega)}\le\frac{z_1}{8}\|v_x\|^2_{L^2_p(\Omega)}+\frac{z_1}{2}\|u_t\|^2_{L^2_p(\Omega)}, \tag{2.41}$$

$$-(u_x,\Im_x(\xi u_t))_{L^2_p(\Omega)}\le\frac{1}{2}\|u_x\|^2_{L^2_p(\Omega)}+\frac{1}{2}\|\Im_x(\xi u_t)\|^2_{L^2_p(\Omega)}, \tag{2.42}$$

$$-(u_{xt},\Im_x(\xi u_t))_{L^2_p(\Omega)}\le\|u_{xt}\|^2_{L^2_p(\Omega)}+\frac{1}{4}\|\Im_x(\xi u_t)\|^2_{L^2_p(\Omega)}, \tag{2.43}$$

$$\left(z_2 u,\Im^2_x(\xi v_t)\right)_{L^2_p(\Omega)}\le\frac{z_2^2}{4}\|u_x\|^2_{L^2_p(\Omega)}+\frac{1}{4}\|\Im_x(\xi u_t)\|^2_{L^2_p(\Omega)}, \tag{2.44}$$

$$(z_2 u,v_t)_{L^2_p(\Omega)}\le\frac{z_2^2}{8}\|u_x\|^2_{L^2_p(\Omega)}+\frac{1}{2}\|v_t\|^2_{L^2_p(\Omega)}, \tag{2.45}$$

$$-(v_{xt},\Im_x(\xi v_t))_{L^2_p(\Omega)}\le\|v_{xt}\|^2_{L^2_p(\Omega)}+\frac{1}{4}\|\Im_x(\xi v_t)\|^2_{L^2_p(\Omega)}, \tag{2.46}$$

$$-(v_x,\Im_x(\xi v_t))_{L^2_p(\Omega)}\le\|v_x\|^2_{L^2_p(\Omega)}+\frac{1}{4}\|\Im_x(\xi v_t)\|^2_{L^2_p(\Omega)}, \tag{2.47}$$

$$\left(g,v_t\right)_{L^2_p(\Omega)}\le\|g\|^2_{L^2_p(\Omega)}+\frac{1}{2}\|v_t\|^2_{L^2_p(\Omega)}, \tag{2.48}$$

$$-\left(g,\Im^2_x(\xi v_t)\right)_{L^2_p(\Omega)}\le\frac{1}{2}\|g\|^2_{L^2_p(\Omega)}+\frac{1}{4}\|\Im^2_x(\xi v_t)\|^2_{L^2_p(\Omega)}. \tag{2.49}$$

Inserting (2.39)--(2.49) into (2.38) yields:

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\|u_t\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi u_t)\|^2_{L^2(\Omega)}+\|u_x\|^2_{L^2_p(\Omega)}+\|v_x\|^2_{L^2_p(\Omega)}\right)+\|u_t\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi v_t)\|^2_{L^2(\Omega)}$$
$$\le C_1\left(\|v_x\|^2_{L^2_p(\Omega)}+\|u_t\|^2_{L^2_p(\Omega)}+\|u_x\|^2_{L^2_p(\Omega)}+\|\Im_x(\xi u_t)\|^2_{L^2(\Omega)}+\|f\|^2_{L^2_p(\Omega)}+\|g\|^2_{L^2_p(\Omega)}\right), \tag{2.50}$$

where

$$C_1=\max\left\{z_1+\frac{3}{2},\frac{z_1}{2}+2,\frac{z_2^2}{2}+1\right\}. \tag{2.51}$$

Using the Gronwall's lemma (1.8), by taking: $f_2(t) = \|u\|^2_{H^2_p(\Omega)} + \|v\|^2_{H^2_p(\Omega)}$, it leads to

$$\left(\|u_t\|^2_{L^2_p(\Omega)} + \|u_x\|^2_{L^2_p(\Omega)} + \|\Im_x(\xi u_t)\|^2_{L^2(\Omega)} + \|v_x\|^2_{L^2_p(\Omega)}\right) + \int_0^t \|u_\tau\|^2_{L^2_p(\Omega)} d\tau + \int_0^t \|\Im_x(\xi u_\tau)\|^2_{L^2_p(\Omega)} d\tau$$
$$\leq C_1 \exp(C_1 T)\left[\int_0^t \|f\|^2_{L^2_p(\Omega)} d\tau + \int_0^t \|g\|^2_{L^2_p(\Omega)} d\tau + \|u_t(x,0)\|^2_{L^2_p(\Omega)} + \|u_x(x,0)\|^2_{L^2_p(\Omega)}\right.$$
$$\left. + \frac{1}{2}\|u_t(x,0)\|^2_{L^2_p(\Omega)} + \|v_x(x,0)\|^2_{L^2_p(\Omega)}\right], \tag{2.52}$$

where: $C_2 = C_1 \exp(C_1 T)$.
Then, by substituting the initial conditions (2.2) into (2.52), and after some calculations, we obtain:

$$\left(\|u_t\|^2_{L^2_p(\Omega)} + \|u_x\|^2_{L^2_p(\Omega)} + \|\Im_x(\xi u_t)\|^2_{L^2(\Omega)} + \|v_x\|^2_{L^2_p(\Omega)}\right) + \int_0^t \|u_\tau\|^2_{L^2_p(\Omega)} d\tau + \int_0^t \|\Im_x(\xi u_\tau)\|^2_{L^2_p(\Omega)} d\tau$$
$$\leq C_2\left[\int_0^t \|f\|^2_{L^2_p(\Omega)} d\tau + \int_0^t \|g\|^2_{L^2_p(\Omega)} d\tau + \|\varphi_1\|^2_{H^1_p(\Omega)} + \|\varphi_2\|^2_{H^1_p(\Omega)} + \|\psi_1\|^2_{H^1_p(\Omega)}\right], \tag{2.53}$$

and since its right-hand side does not depend on $t$, we take the upper bound on the left-hand side with respect to $t$ from 0 to $T$, and we have the a priori estimate

$$\sup_{0\leq t\leq T}\left[\|u_t\|^2_{L^2_p(\Omega)} + \|u_x\|^2_{L^2_p(\Omega)} + \|\Im_x(\xi u_t)\|^2_{L^2(\Omega)} + \|v_x\|^2_{L^2_p(\Omega)}\right] + \|u_t\|^2_{L^2(0,T,L^2_p(\Omega))} + \|\Im_x(\xi u_\tau)\|^2_{L^2(0,T,L^2_p(\Omega)}$$
$$\leq C_2\left[\|f\|^2_{L^2(0,T,L^2_p(\Omega))} + \|g\|^2_{L^2(0,T,L^2_p(\Omega))} + \|\varphi_1\|^2_{H^1_p(\Omega)} + \|\varphi_2\|^2_{H^1_p(\Omega)} + \|\psi_1\|^2_{H^1_p(\Omega)}\right], \tag{2.54}$$

this leads to:

$$\|(u,v)\|^2_B \leq C_2 \|\mathscr{F}\|^2_H. \tag{2.55}$$

By employing the final energy inequality (2.55), we can establish the uniqueness of the solution to the coupled system (2.1)--(2.3). Suppose $W_1 = (u_1, v_1)$ and $W_2 = (u_2, v_2)$ are two solutions of the system (2.1)--(2.3).

$$\begin{cases} LW_1 = F, \\ LW_2 = F, \end{cases} \implies LW_1 - LW_2 = 0. \tag{2.56}$$

Since $L$ is a linear operator, we obtain:

$$L(W_1 - W_2) = 0. \tag{2.57}$$

According to (2.55), we have:

$$\|W_1 - W_2\|_B \leq \sqrt{C_2}\|0\|_H = 0, \tag{2.58}$$

which implies the uniqueness of the solution.

$$W_1 = W_2. \tag{2.59}$$

.

It can be proved in a standard way that the operator $L : B \to H$ is closable. Let $\overline{L}$ be its closure.

**Proposition 2**

The operator $L : B \to H$ has a closure.

**Proof**

The proof can be established in a similar way as in [**20**] .

These are some consequences of Theorem (2.12).

**Corollary 1**

There exists a positive constant $C$ such that

$$\|(u,v)\|_B \leq C \|\overline{L}(u,v)\|_H, \quad \forall (u,v) \in D(\overline{L}), \tag{2.60}$$

where: $C = \sqrt{C_2}$.

The inequality (2.60) means that inequality (2.12) can be extended to strong solutions after passing to limit.

Inequality (2.60) leads to the following results:

**Corollary 2**

A strong solution of system (2.1)-(2.3) is unique and depends continuously on $\mathscr{F} = (\mathscr{F}_1, \mathscr{F}_2) \in H$, where $\mathscr{F}_1 = \{f, \varphi_1, \varphi_2\}$ and $\mathscr{F}_2 = \{g, \psi_1\}$.

**Corollary 3**

The set $R(\overline{L})$ of $\overline{L}$ is closed in $H$ and $R(\overline{L}) = \overline{R(L)}$.

This last corollary shows that in order to prove that system (2.1)-(2.3) has a strong solution for arbitrary $(\mathscr{F}_1, \mathscr{F}_2) \in H$, it is sufficient to prove that the range of $L$ is dense in $H$, that is $\overline{R(L)} = H$.

## 2.3 Existence of Solutions to the Linear System

**Proposition 3**

If for some function: $Y(x,t) = (y_1, y_2) \in L^2\left(0, T; L^2_\rho(\Omega)\right)^2$, and for all $W(x,t) = (u,v) \in D_0(L) = \{W/W \in D(L) : \ell_1 u = 0, \ell_2 u = 0, \ell_3 v = 0\}$, (satisfying homogeneous initial conditions), we have:

$$(\mathscr{L}W, Y)_{L^2(0,T;L^2_\rho(\Omega))} = \left(\mathscr{L}_1 u, y_1\right)_{L^2(0,T;L^2_\rho(\Omega))} + \left(\mathscr{L}_2 v, y_2\right)_{L^2(0,T;L^2_\rho(\Omega))} = 0, \tag{2.61}$$

then $Y$ is vanishes "a.e" in the domain $Q$.

**Proof**

We first set

$$W = (u, v) = (\Im_t^2(\tilde{u}), \Im_t^2(\tilde{v})), \tag{2.62}$$
$$Y = (y_1, y_2) = (\Im_t(\tilde{u}) - \Im_x^2(\xi \Im_t(\tilde{u})), \Im_t(\tilde{v}) - \Im_x^2(\xi \Im_t(\tilde{v}))), \tag{2.63}$$

where

$$\Im_t(\tilde{w}) = \int_0^t \tilde{w}(x,s)ds, \tag{2.64}$$
$$\Im_t^2(\tilde{w}) = \int_0^t \int_0^s \tilde{w}(x,z)dzds, \tag{2.65}$$
$$\Im_x^2(\xi \Im_t(\tilde{w}))) = \int_0^x \int_0^\xi \int_0^t \eta\, \tilde{w}(\eta,s)dsd\eta d\xi. \tag{2.66}$$

We suppose that the function $\tilde{u}$, $\tilde{v}$ satisfying conditions (2.3) and such that

$$\tilde{u}, \tilde{v}, \tilde{u}_x, \tilde{v}_x, \Im_t(\tilde{u}), \Im_t(\tilde{v}), \Im_t^2(\tilde{u}), \Im_t^2(\tilde{v}), x\Im_t^2(\tilde{u}_x), x\Im_t^2(\tilde{v}_x), \Im_x^2(\xi\Im_t(\tilde{u})), \Im_x^2(\xi\Im_t(\tilde{v})), \frac{\partial^2 \tilde{u}}{\partial t^2}, \frac{\partial \tilde{v}}{\partial t} \in L^2(Q). \tag{2.67}$$

Now replacing (2.62) and (2.63) in the relation (2.61), we obtain

$$\begin{aligned}
&\left(\Im_t^2(\tilde{u}_{tt})_{L^2_p(\Omega)}, \Im_t(\tilde{u})\right)_{L^2_p(\Omega)} - \left(\frac{1}{x}\left(x\Im_t^2(\tilde{u}_x)\right)_x, \Im_t(\tilde{u})\right)_{L^2_p(\Omega)} - \left(\frac{1}{x}\left(x\Im_t^2(\tilde{u}_x)\right)_{xt}, \Im_t(\tilde{u})\right)_{L^2_p(\Omega)} \\
&+ \left(z_1\Im_t^2(\tilde{v}), \Im_t(\tilde{u})\right)_{L^2_p(\Omega)} + \left(\Im_t^2(\tilde{u})_t, \Im_t(\tilde{u})\right)_{L^2_p(\Omega)} + \left(\Im_t^2(\tilde{u})_{tt}, -\Im_x^2(\xi\Im_t(\tilde{u}))\right)_{L^2_p(\Omega)} \\
&- \left(\frac{1}{x}\left(x\Im_t^2(\tilde{u}_x)\right)_x, -\Im_x^2(\xi\Im_t(\tilde{u}))\right)_{L^2_p(\Omega)} - \left(\frac{1}{x}(x\Im_t^2(\tilde{u}_x))_{xt}, -\Im_x^2(\xi\Im_t(\tilde{u}))\right)_{L^2_p(\Omega)} \\
&+ \left(z_1\Im_t^2(\tilde{v}), -\Im_x^2(\xi\Im_t(\tilde{u})_t)\right)_{L^2_p(\Omega)} + \left(\Im_t^2(\tilde{u}), -\Im_x^2(\xi\Im_t(\tilde{u}))\right)_{L^2_p(\Omega)} + \left(\Im_t^2(\tilde{v}), \Im_t(\tilde{v})\right)_{L^2_p(\Omega)} \\
&- \left(\frac{1}{x}\left(x\Im_t^2(\tilde{v}_x)\right)_x, \Im_t(\tilde{v})\right)_{L^2_p(\Omega)} - \left(\frac{1}{x}(x\Im_t^2(\tilde{v}_x))_{xt}, \Im_t(\tilde{v})\right) + \left(z_2\Im_t^2(\tilde{u}), \Im_t(\tilde{u})\right)_{L^2_p(\Omega)} \\
&+ \left(\Im_t^2(\tilde{v})_t, -\Im_x^2(\xi\Im_t(\tilde{v}))\right)_{L^2_p(\Omega)} - \left(\frac{1}{x}(x\Im_t^2(\tilde{v}_x)_x, -\Im_x^2(\xi\Im_t(\tilde{v}))\right)_{L^2_p(\Omega)} \\
&- \left(\frac{1}{x}(x\left(\Im_t^2(\tilde{v}_x)\right)_{xt}, -\Im_x^2(\xi\Im_t(\tilde{u}))\right)_{L^2_p(\Omega)} + \left(z_2\Im_t^2(\tilde{u}), -\Im_x^2(\xi\Im_t(\tilde{v}))\right)_{L^2_p(\Omega)} = 0.
\end{aligned} \tag{2.68}$$

Consider the following Poincaré inequality:

$$\|\Im_t^2(w)\|^2_{L^2_p(\Omega)} \leq \frac{T^2}{2}\|\Im_t(w)\|^2_{L^2_p(\Omega)}, \tag{2.69}$$

then, using conditions (2.3), and computation each term of (2.68), gives

$$\left(\Im_t^2(\tilde{u}),\Im_t(\tilde{u})\right)_{L_p^2(\Omega)}=\frac{1}{2}\frac{\partial}{\partial t}\|\Im_t(\tilde{u})\|^2_{L_p^2(\Omega)},\tag{2.70}$$

$$-\left(\frac{1}{x}\left(x\Im_t^2(\tilde{u}_x)\right)_x,\Im_t(\tilde{u})\right)_{L_p^2(\Omega)}=\frac{1}{2}\frac{\partial}{\partial t}\|\Im_t^2(\tilde{u}_x)\|^2_{L_p^2(\Omega)},\tag{2.71}$$

$$-\left(\frac{1}{x}\left(x\Im_t^2(\tilde{u}_x)\right)_{xt},\Im_t(\tilde{u})\right)_{L_p^2(\Omega)}=\|\Im_t(\tilde{u}_x)\|^2_{L_p^2(\Omega)},\tag{2.72}$$

$$\left(z_1\Im_t^2(\tilde{v}),\Im_t(\tilde{u})\right)_{L_p^2(\Omega)}=z_1\left(\Im_t^2(\tilde{v}),\Im_t(\tilde{u})\right)_{L_p^2(\Omega)},\tag{2.73}$$

$$\left(\Im_t^2(\tilde{u})_t,\Im_t(\tilde{u})\right)_{L_p^2(\Omega)}=\|\Im_t(\tilde{u})\|^2_{L_p^2(\Omega)},\tag{2.74}$$

$$\left(\Im_t^2(\tilde{u})_{tt},-\Im_x^2(\xi\Im_t(\tilde{u}))\right)_{Lp^2(\Omega)}=\frac{1}{2}\frac{\partial}{\partial t}\|\Im_x^1(\xi\Im_t(\tilde{u}))\|^2_{L^2(\Omega)},\tag{2.75}$$

$$\left(-\frac{1}{x}\left(x\Im_t^2(\tilde{u}_x)\right)_x,-\Im_x^2(\xi\Im_t(\tilde{u}))\right)_{L_p^2(\Omega)}=-\left(\Im_t^2(\tilde{u}_x)_x,\Im_x(\xi\Im_t(\tilde{u}))\right)_{L_p^2(\Omega)},\tag{2.76}$$

$$\left(-\frac{1}{x}(x\Im_t^2(\tilde{u}_x))_{xt},-\Im_x^2(\xi\Im_t(\tilde{u}))\right)_{L_p^2(\Omega)}=-(\Im_t(\tilde{u}_x)_{xt},\Im_x(\xi\Im_t(\tilde{u})))_{L_p^2(\Omega)},\tag{2.77}$$

$$\left(z_1\Im_t^2(\tilde{v}),-\Im_x^2(\xi\Im_t(\tilde{u}))\right)_{L_p^2(\Omega)}=\left(z_1\Im_t^2(\tilde{v}),-\Im_x^2(\xi\Im_t(\tilde{u}))\right)_{L_p^2(\Omega)},\tag{2.78}$$

$$\left(\Im_t^2(\tilde{u}),-\Im_x^2(\xi\Im_t(\tilde{u}))\right)_{L_p^2(\Omega)}=\|\Im_x\Im_t(\xi\tilde{u})\|^2_{L_p^2(\Omega)},\tag{2.79}$$

$$\left(\Im_t^2(\tilde{v}),\Im_t(\tilde{v})\right)_{L_p^2(\Omega)}=\|\Im_t(\tilde{v})\|^2_{L_p^2(\Omega)},\tag{2.80}$$

$$-\left(\frac{1}{x}\left(x\Im_t^2(\tilde{v}_x)\right)_x,\Im_t(\tilde{v})\right)_{L_p^2(\Omega)}=\frac{1}{2}\frac{\partial}{\partial t}\|\Im_t^2(\tilde{v}_x)\|^2_{L_p^2(\Omega)},\tag{2.81}$$

$$-\left(\frac{1}{x}(x\Im_t^2(\tilde{v}_x))_{xt},\Im_t(\tilde{v})\right)_{L_p^2(\Omega)}=\|\Im_t(\tilde{v}_x)\|^2_{L_p^2(\Omega)},\tag{2.82}$$

$$\left(z_2\Im_t^2(\tilde{u}),\Im_t(\tilde{u})\right)_{L_p^2(\Omega)}=\left(z_2\Im_t^2(\tilde{u}),\Im_t(\tilde{u})\right)_{L_p^2(\Omega)},\tag{2.83}$$

$$-\left(\Im_t^2(\tilde{v}),\Im_x^2(\xi\Im_t(\tilde{v}))\right)_{L_p^2(\Omega)}=\|\Im_x\Im_t(\xi\tilde{v})\|^2_{L^2(\Omega)},\tag{2.84}$$

$$\left(-\frac{1}{x}(x\Im_t^2(\tilde{v}_x))_x,-\Im_x^2(\xi\Im_t(\tilde{v}))\right)_{L_p^2(\Omega)}=-\left(\Im_t^2(\tilde{v}_x)_x,\Im_x(\xi\Im_t(\tilde{v}))\right)_{L_p^2(\Omega)},\tag{2.85}$$

$$-\left(\frac{1}{x}\left(x\Im_t^2(\tilde{v}_x)\right)_{xt},\Im_x^2(\xi\Im_t(\tilde{u}))\right)_{L_p^2(\Omega)}=-(\Im_t(\tilde{v}_x)_x,-\Im_x(\xi\Im_t(\tilde{v})))_{L_p^2(\Omega)},\tag{2.86}$$

$$\left(z_2\Im_t^2(\tilde{u}),-\Im_x^2(\xi\Im_t(\tilde{v}))\right)_{L_p^2(\Omega)}=\left(z_2\Im_t^2(\tilde{u}),\Im_x^2(\xi\Im_t(\tilde{v}))\right)_{L_p^2(\Omega)}.\tag{2.87}$$

Insertion of equations (2.70)--(2.87), into (2.68), yields

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial t}\left(\|\Im_t(\tilde{u})\|^2_{L_p^2(\Omega)}+\|\Im_t^2(\tilde{u}_x)\|^2_{L_p^2(\Omega)}+\|\Im_x(\xi\Im_t(\tilde{u}))\|^2_{L^2(\Omega)}+\|\Im_t^2(\tilde{v}_x)\|^2_{L_p^2(\Omega)}\right)\\&\leq M_1\left[\left(\|\Im_t(\tilde{u})\|^2_{L_p^2(\Omega)}+\|\Im_t^2(\tilde{u})\|^2_{L_p^2(\Omega)}+\|\Im_t(\tilde{v})\|^2_{L_p^2(\Omega)}+\|\Im_x(\xi\Im_t(\tilde{u}))\|^2_{L_p^2(\Omega)}\right)\right],\end{aligned}\tag{2.88}$$

where

$$M_1=\max\left[\frac{T^2z_1^2}{2}+\frac{T^2z_2^2}{4}+\frac{T^4z_2^2}{32},\frac{1}{2T^2z_1},\frac{T^2z_1}{2}+\frac{1}{4},\frac{T^2z_1^2}{4}\right].\tag{2.89}$$

Taking into account the homogeneous initial conditions in $D_0(L)$, replacing $t$ by $\tau$ in (2.88), and integrating with respect to $\tau$ from 0 to $t$, we obtain:

$$\begin{aligned}
&\left(\|\mathfrak{I}_t(\tilde{u})\|^2_{L^2_p(\Omega)}+\|\mathfrak{I}^2_t(\tilde{u}_x)\|^2_{L^2_p(\Omega)}+\|\mathfrak{I}_x(\xi\mathfrak{I}_t(\tilde{u}))\|^2_{L^2(\Omega)}+\|\mathfrak{I}^2_t(\tilde{v}_x)\|^2_{L^2_p(\Omega)}\right)\\
&+\left(2\int_0^t(\|\mathfrak{I}_\tau(\tilde{u})\|^2_{L^2_p(\Omega)}d\tau+2\int_0^t\|\mathfrak{I}_x(\xi\mathfrak{I}_\tau\tilde{u})\|^2_{L^2_p(\Omega)}d\tau)\right)\\
&\leq 2M_1\left[\int_0^t\left(\|\mathfrak{I}_\tau(\tilde{u})\|^2_{L^2_p(\Omega)}+\|\mathfrak{I}^2_\tau(\tilde{u}_x)\|^2_{L^2_p(\Omega)}+\|\mathfrak{I}_x(\xi\mathfrak{I}_\tau(\tilde{u}))\|^2_{L^2_p(\Omega)}+\|\mathfrak{I}_t(\tilde{v}_x)\|^2_{L^2_p(\Omega)}\right)d\tau\right].
\end{aligned}\tag{2.90}$$

Using the Gronwall's lemma (1.8), by taking:

$$f_2(t)=\|\mathfrak{I}_t(\tilde{u})\|^2_{L^2_p(\Omega)}+\|\mathfrak{I}^2_t(\tilde{u}_x)\|^2_{L^2_p(\Omega)}+\|\mathfrak{I}_x(\xi\mathfrak{I}_t(\tilde{u}))\|^2_{L^2(\Omega)}+\|\mathfrak{I}^2_t(\tilde{v}_x)\|^2_{L^2_p(\Omega)},\tag{2.91}$$

it follows that

$$\begin{aligned}
&\left(\|\mathfrak{I}_t(\tilde{u})\|^2_{L^2_p(\Omega)}+\|\mathfrak{I}^2_t(\tilde{u}_x)\|^2_{L^2_p(\Omega)}+\|\mathfrak{I}_x(\xi\mathfrak{I}_t(\tilde{u}))\|^2_{L^2(\Omega)}+\|\mathfrak{I}^2_t(\tilde{v}_x)\|^2_{L^2_p(\Omega)}\right)\\
&+\left(2\int_0^t(\|\mathfrak{I}_\tau(\tilde{u})\|^2_{L^2_p(\Omega)}d\tau+2\int_0^t\|\mathfrak{I}_x(\xi\mathfrak{I}_\tau\tilde{u})\|^2_{L^2_p(\Omega)}d\tau)\right)\leq 0.
\end{aligned}\tag{2.92}$$

Hence, we deduce from (2.92) that $Y=(y_1,y_2)=(0,0)$ almost everywhere in $Q$.

**Theorem 2**

For any $f,g\in L^2\left(0,T;L^2_\rho(Q)\right)$ and $\varphi_1,\varphi_2,\psi_1\in H^1_\rho(\Omega)$, there exists a unique strong solution $W=\overline{L}^{-1}\mathscr{F}=\overline{L^{-1}}\mathscr{F}$ of the system (2.1)-(2.3), where $\mathscr{F}=(\mathscr{F}_1,\mathscr{F}_2)\in H$, $\mathscr{F}_1=\{f,\varphi_1,\varphi_2\}$, $\mathscr{F}_2=\{g,\psi_1\}$, $W=(u,v)$ and

$$\|W\|_B\leq C\|LW\|_H,\tag{2.93}$$

for a positive constant $C$, independent of $W$.

**Proof**

It is sufficient to prove that the range $R(L)$ of $L$ is dense in $H$. Suppose that for some: $Y=(\mathscr{Y}_1,\mathscr{Y}_2)=(\{y_1,y_2,y_3\},\{y_4,y_5\})\in H$, the orthogonal of $R(L)$, so that

$$\begin{aligned}
(LW,Y)_H&=(\{L_1u,L_2v\},\{\mathscr{Y}_1,\mathscr{Y}_2\})_H\\
&=\left(\{(\mathscr{L}_1u,\ell_1u,\ell_2u),(\mathscr{L}_2v,\ell_3v)\},\{(y_1,y_2,y_3),(y_4,y_5)\}\right)_H\\
&=\left(\mathscr{L}_1u,y_1\right)_{L^2(0,T;L^2_\rho(\Omega))}+\left(\ell_1u,y_2\right)_{H^1_\rho(\Omega)}+\left(\ell_2u,y_3\right)_{H^1_\rho(\Omega)}\\
&+\left(\mathscr{L}_2v,y_4\right)_{L^2(0,T;L^2_\rho(\Omega))}+\left(\ell_3v,y_5\right)_{H^1_\rho(\Omega)}=0.
\end{aligned}\tag{2.94}$$

We must show that: $Y=0$. Putting $W\in D_0(L)$ in (2.94), we get

$$\left(\mathscr{L}_1u,y_1\right)_{L^2(0,T;L^2_\rho(\Omega))}+\left(\mathscr{L}_2v,y_4\right)_{L^2(0,T;L^2_\rho(\Omega))}=0,\quad\forall W\in D_0(L),\tag{2.95}$$

hence proposition (2.3) implies that: $y_1 = y_4 = 0$, the relation (2.95), implies that

$$\left(\ell_1 u, y_2\right)_{H^1_\rho(\Omega)} + \left(\ell_2 u, y_3\right)_{H^1_\rho(\Omega)} + \left(\ell_3 v, y_5\right)_{H^1_\rho(\Omega)} = 0, \quad \forall W \in D_0(L). \tag{2.96}$$

The three sets $\ell_1 u, \ell_3 u$ and $\ell_2 u$ are independent, and the images of the trace operator $\ell_1, \ell_3$ and $\ell_2$ are respectively everywhere dense in the Hilbert spaces $H^1_\rho(\Omega)$, then it follows from 2.96, that $y_2 = y_3 = y_5 = 0$ almost everywhere in $Q$. Hence $\overline{R(L)} = H$.

## 2.4 Existence and Uniqueness of Solutions for the Nonlinear System

We are now in a position to solve the nonlinear system (1). Relying on the results obtained previously, we apply an iterative process to establish the existence and uniqueness of the weak solution of the nonlinear system. If $(u, v)$ is a solution of system (1) and $(\psi, \phi)$ is a solution of the homogeneous system:

$$\begin{cases} \frac{\partial^2 \psi}{\partial t^2} + -\frac{1}{x}\left(x\psi_x\right)_x - \frac{1}{x}\left(x\psi_x\right)_{xt} + z_1\phi + \psi_t = 0, \\ \frac{\partial \phi}{\partial t} + -\frac{1}{x}\left(x\phi_x\right)_x - \frac{1}{x}\left(x\phi_x\right)_{xt} + z_2\psi = 0, \\ \psi(x,0) = \varphi_1(x), \quad \psi_1(x,0) = \varphi_2(x), \\ \phi(x,0) = \psi_1(x), \\ \psi_x(1,t) = 0, \quad \phi_x(1,t) = 0, \quad \int_0^1 x\psi\, dx = 0, \quad \int_0^1 x\phi\, dx = 0. \end{cases} \tag{2.97}$$

Then $(U, V) = (u - \psi, v - \phi)$ is a solution of

$$\begin{cases} \frac{\partial^2 U}{\partial t^2} + -\frac{1}{x}(xU_x)_x - \frac{1}{x}(xU_x)_{xt} + z_1 V + U_t = F(x,t,U,V,U_x,V_x), \\ \frac{\partial V}{\partial t} + -\frac{1}{x}(xV_x)_x - \frac{1}{x}(xV_x)_{xt} + z_2 U = G(x,t,U,V,U_x,V_x), \\ U(x,0) = 0, \quad U_t(x,0) = 0, \quad V(x,0) = 0, \\ \int_0^1 xU\, dx = 0, \quad \int_0^1 xV\, dx = 0, \quad U_x(1,t) = 0, \quad V_x(1,t) = 0, \end{cases} \tag{2.98}$$

where $F$ and $G$ are Lipschitzian functions

$$\begin{aligned} |F(x,t,u_1,v_1,w_1,z_1) - F(x,t,u_2,v_2,w_2,z_2)| &\le \delta_1(|u_1-u_2| + |v_1-v_2| + |w_1-w_2| + |z_1-z_2|), \\ |G(x,t,u_1,v_1,w_1,z_1) - G(x,t,u_2,v_2,w_2,z_2)| &\le \delta_2(|u_1-u_2| + |v_1-v_2| + |w_1-w_2| + |z_1-z_2|), \end{aligned} \tag{2.99}$$

for all $(x,t) \in Q$.
According to theorem (2.3), system (2.97) has a unique solution that depends continuously on $\varphi_1, \varphi_2, \psi_1 \in H^1_\rho(\Omega)$. We should prove that system (2.98) has a unique weak solution, for this end, we constitute the weak formulation of the system (2.98).

Suppose that $w, U, V \in C^2(Q)$, such that:

$$w(x,T) = 0, \quad w_t(x,T) = 0, \quad \int_0^1 x w(x,t)\, dx = 0. \tag{2.100}$$

Consider the identity

$$(\mathscr{L}_1 U, \Im_x(\xi w))_{L^2_\rho(Q)} + (\mathscr{L}_2 V, \Im_x(\xi w))_{L^2_\rho(Q)} = (F, \Im_x(\xi w))_{L^2_\rho(Q)} + (G, \Im_x(\xi w))_{L^2_\rho(Q)}, \tag{2.101}$$

In light of the above assumptions, we obtain

$$(U_{tt},\Im_x(\xi))_{L^2(0,T,L_p^2(\Omega))}=(U_{tt},(\Im_x(\xi w)))_{L^2(0,T,L_p^2(\Omega))}, \tag{2.102}$$
$$-\left(\frac{1}{x}(xU_x)_x,\Im_x(\xi w)\right)_{L^2(0,T,L_p^2(\Omega))}=(U_x,xw)_{L^2(0,T,L_p^2(\Omega))}, \tag{2.103}$$
$$-\left(\frac{1}{x}(xU_x)_{xt},\Im_x(\xi w)\right)_{L^2(0,T,L_p^2(\Omega))}=-(U_x,xw_t)_{L^2(0,T,L_p^2(\Omega))}, \tag{2.104}$$
$$(z_1V,\Im_x(\xi w))_{L^2(0,T,L_p^2(\Omega))}=-z_1(\Im_x(\xi V),w)_{L^2(0,T,L_p^2(\Omega))}, \tag{2.105}$$
$$(U_t,\Im_x(\xi\omega))_{L^2(0,T,L_p^2(\Omega))}=-(U,\Im_x(\xi\omega))_{L^2(0,T,L_p^2(\Omega))}, \tag{2.106}$$
$$(F,\Im_x(\xi w))_{L^2(0,T,L_p^2(\Omega))}=-(\Im_x(\xi F),w)_{L^2(0,T,L_p^2(\Omega))}, \tag{2.107}$$
$$(V_t,\Im_x(\xi\omega))_{L^2(0,T,L_p^2(\Omega))}=(V_t,\Im_x(\xi\omega))_{L^2(0,T,L_p^2(\Omega))}, \tag{2.108}$$
$$-\left(\frac{1}{x}(xV_x)_x,\Im_x(\xi\omega)\right)_{L^2(0,T,L_p^2(\Omega))}=(V_x,x\omega)_{L^2(0,T,L_p^2(\Omega))}, \tag{2.109}$$
$$-\left(\frac{1}{x}(xV_x)_x,\Im_x(\xi\omega)\right)_{L^2(0,T,L_p^2(\Omega))}=-(V_x,x\omega)_{L^2(0,T,L_p^2(\Omega))}, \tag{2.110}$$
$$(-z_2U,\Im_x(\xi\omega))_{L^2(0,T,L_p^2(\Omega))}=-z_2(\Im_x(\xi U,\omega))_{L^2(0,T,L_p^2(\Omega))}, \tag{2.111}$$
$$(G,\Im_x(\xi\omega))_{L^2(0,T,L_p^2(\Omega))}=-(\Im_x(\xi G,\omega))_{L^2(0,T,L_p^2(\Omega))}. \tag{2.112}$$

Using the symmetry in the system, and inserting equations (2.102)-(2.112) into (2.101), yields:

$$\begin{aligned}&(U_{tt},(\Im_x(\xi w)))_{L^2(0,T,L_p^2(\Omega))}+(U_x,(xw))_{L^2(0,T,L_p^2(\Omega))}-(U_x,xw_t)_{L^2(0,T,L_p^2(\Omega))}\\&-z_1(\Im_x(\xi V),w)_{L^2(0,T,L_p^2(\Omega))}-(U,\Im_x(\xi w_t))_{L^2(0,T,L_p^2(\Omega))}-(V_t,\Im_x(\xi w))_{L^2(0,T,L_p^2(\Omega))}\\&+(V_x,xw)_{L^2(0,T,L_p^2(\Omega))}-(V_x,xw_t),w)_{L^2(0,T,L_p^2(\Omega))}-z_2(\Im_x(\xi U,w))_{L^2(0,T,L_p^2(\Omega))}\\&=-(\Im_x(\xi F),w)_{L^2(0,T,L_p^2(\Omega))}+(G,\Im_x(\xi w))_{L^2(0,T,L_p^2(\Omega))}.\end{aligned} \tag{2.113}$$

We denote the LHS in (2.113) by $A(w,U,V)$, we have

$$A(w,U,V)=(w,\Im_x(\xi F))_{L^2(0,T,L_p^2(\Omega))}+(w,\Im_x(\xi G))_{L^2(0,T,L_p^2(\Omega))}. \tag{2.114}$$

**Definition 2.4.1**

A function $(U,V)\in(L^2(0,T,H_p^1(\Omega)))^2$ is called a weak solution of problem (2.98) if (2.114) hold.

We now consider the iterated system

$$\begin{cases}\frac{\partial^2U^{(n)}}{\partial t^2}+-\frac{1}{x}\left(xU_x^{(n)}\right)_x-\frac{1}{x}\left(xU_x^{(n)}\right)_{xt}+z_1V^{(n)}+U_t^{(n)}=F\left(x,t,U^{(n-1)},V^{(n-1)},U_x^{(n-1)},V_x^{(n-1)}\right),\\ \frac{\partial V^{(n)}}{\partial t}+-\frac{1}{x}\left(xV_x^{(n)}\right)_x-\frac{1}{x}\left(xV_x^{(n)}\right)_{xt}+z_2U^{(n)}=G\left(x,t,U^{(n-1)},V^{(n-1)},U_x^{(n-1)},V_x^{(n-1)}\right),\\ U^{(n)}(x,0)=0,\quad U_t^{(n)}(x,0)=0,\quad V^{(n)}(x,0)=0,\\ \int_0^1 xU^{(n)}dx=0,\quad \int_0^1 xV^{(n)}dx=0,\quad U_x^{(n)}(1,t)=0,\quad V_x^{(n)}(1,t)=0,\end{cases} \tag{2.115}$$

where the iterated sequence $\left\{U^{(n)},V^{(n)}\right\}_{n\geq 0}$ is constructed in the following way: Given: $\left(U^{(0)},V^{(0)}\right)=(0,0)$ and the element $\left(U^{(n-1)},V^{(n-1)}\right)$, then for $n=1,2,\ldots$, we solve the problem (2.115). According to theorem (2.3), for fixed $n$, each problem (2.115) has a unique solution $\left(U^{(n)},V^{(n)}\right)$.

If we set $\left(\mathscr{U}^{(n)}(x,t),\mathscr{V}^{(n)}(x,t)\right)=\left(U^{(n+1)}(x,t)-U^{(n)}(x,t),V^{(n+1)}(x,t)-V^{(n)}(x,t)\right)$, then we have the new problem

$$\begin{cases} \frac{\partial \mathscr{U}^{(n)}}{\partial t^2}+-\frac{1}{x}\left(x\mathscr{U}_x^{(n)}\right)_x-\frac{1}{x}\left(x\mathscr{U}_x^{(n)}\right)_{xt}+z_1\mathscr{V}^{(n)}+\mathscr{U}_t^{(n)}=H_1^{(n-1)}(x,t),\\ \frac{\partial \mathscr{V}^{(n)}}{\partial t}+-\frac{1}{x}\left(x\mathscr{V}_x^{(n)}\right)_x-\frac{1}{x}\left(x\mathscr{V}_x^{(n)}\right)_{xt}+z_2\mathscr{U}^{(n)}=H_2^{(n-1)}(x,t),\\ \mathscr{U}^{(n)}(x,0)=0,\quad \mathscr{U}_t^{(n)}(x,0)=0,\quad \mathscr{V}^{(n)}(x,0)=0,\quad \mathscr{V}_t^{(n)}(x,0)=0,\\ \int_0^1 x\mathscr{U}^{(n)}dx=0,\quad \int_0^1 x\mathscr{V}^{(n)}dx=0,\quad \mathscr{U}_x^{(n)}(1,t)=0,\quad \mathscr{V}_x^{(n)}(1,t)=0, \end{cases} \tag{2.116}$$

where

$$H_1^{(n-1)}(x,t)=F\left(x,t,U^{(n)},U_x^{(n)},V^{(n)},V_x^{(n)}\right)-F\left(x,t,U^{(n-1)},U_x^{(n-1)},V^{(n-1)},V_x^{(n-1)}\right), \tag{2.117}$$
$$H_2^{(n-1)}(x,t)=G\left(x,t,U^{(n)},U_x^{(n)},V^{(n)},V_x^{(n)}\right)-G\left(x,t,U^{(n-1)},U_x^{(n-1)},V^{(n-1)},V_x^{(n-1)}\right). \tag{2.118}$$

**Lemma 2.4.1**

Assume that conditions (2.99) hold, then for the coupled linearized system (2.116), we have a priori estimate:

$$\|\mathscr{U}^{(n)}\|_{L^2(0,T,H_p^1(\Omega))}+\|\mathscr{V}^{(n)}\|_{L^2(0,T,H_p^1(\Omega))}\le K^*\left(\|\mathscr{U}^{(n-1)}\|_{L^2(0,T,H_p^1(\Omega))}+\|\mathscr{V}^{(n-1)}\|_{L^2(0,T,H_p^1(\Omega))}\right), \tag{2.119}$$

where $K^*$ is a positive constant given by:

$$k^*=4\mathscr{C}^{**}\exp^{T\mathscr{C}^{**}}T(\sigma_1^2+\sigma_2^2). \tag{2.120}$$

**Proof**

The consideration of the inner products in $L_\rho^2(\Omega)$, of the PDEs in (2.116) and the integro-differential operators $\mathscr{M}_1\mathscr{U}^{(n)}=\mathscr{U}_t^{(n)}-\Im_x^2(\xi\mathscr{U}_t^{(n)})$, $\mathscr{M}_2\mathscr{V}^{(n)}=\mathscr{V}_t^{(n)}-\Im_x^2(\xi\mathscr{V}_t^{(n)})$ respectively, gives the equation

$$\begin{aligned}
&\left(\mathscr{U}_{tt}^{(n)},\mathscr{U}_t^{(n)}-\Im_x^2(\xi\mathscr{U}_t^{(n)})\right)_{L_p^2(\Omega)}-\left(\frac{1}{x}\left(x\mathscr{U}_x^{(n)}\right)_x,\mathscr{U}_t^{(n)}-\Im_x^2(\xi\mathscr{U}_t^{(n)})\right)_{L_p^2(\Omega)}\\
&-\left(\frac{1}{x}\left(x\mathscr{U}_x^{(n)}\right)_{xt},\mathscr{U}_t^{(n)}-\Im_x^2(\xi\mathscr{U}_t^{(n)})\right)_{L_p^2(\Omega)}+\left(z_1\mathscr{V}^{(n)},\mathscr{U}_t^{(n)}-\Im_x^2(\xi\mathscr{U}_t^{(n)})\right)_{L_p^2(\Omega)}\\
&\left(\mathscr{U}_t^{(n)}+\mathscr{U}_t^{(n)}-\Im_x^2(\xi\mathscr{U}_t^{(n)})\right)_{L_p^2(\Omega)}+\left(\mathscr{V}_t^{(n)},\mathscr{V}_t^{(n)}-\Im_x^2(\xi\mathscr{V}_t^{(n)})\right)_{L_p^2(\Omega)}\\
&-\left(\frac{1}{x}\left(x\mathscr{V}_x^{(n)}\right)_x,\mathscr{V}_t^{(n)}-\Im_x^2(\xi\mathscr{V}_t^{(n)})\right)_{L_p^2(\Omega)}\\
&-\left(\frac{1}{x}\left(x\mathscr{V}_x^{(n)}\right)_{xt},\mathscr{V}_t^{(n)}-\Im_x^2(\xi\mathscr{V}_t^{(n)})\right)_{L_p^2(\Omega)}+\left(z_2\mathscr{U}^{(n)},\mathscr{V}_t^{(n)}-\Im_x^2(\xi\mathscr{V}_t^{(n)})\right)_{L_p^2(\Omega)}\\
&=\left(H_1^{(n-1)},\mathscr{U}_t^{(n)}-\Im_x^2(\xi\mathscr{U}_t^{(n)})\right)_{L_p^2(\Omega)}+\left(H_2^{(n-1)},\mathscr{V}_t^{(n)}-\Im_x^2(\xi\mathscr{V}_t^{(n)})\right)_{L_p^2(\Omega)}.
\end{aligned} \tag{2.121}$$

As in the proof of Theorem (2.2), we obtain

$$\|\mathscr{U}^{(n)}\|_{H_p^1(\Omega)}^2+\|\mathscr{V}^{(n)}\|_{H_p^1(\Omega)}^2\le K^*\left(\int_0^T\|H_1^{(n-1)}\|_{L_p^2(\Omega)}^2d\tau+\int_0^T\|H_2^{(n-1)}\|_{L_p^2(\Omega)}^2d\tau\right). \tag{2.122}$$

We now turn to the right-hand side of the inequality (2.122). Using the Lipschitz conditions

(2.98), we obtain:

$$\int_0^T \|(H_i^{(n-1)})\|^2_{L^2_p(\Omega)} d\tau \leq 4\delta_i^2 \left(\|\mathscr{U}^{(n-1)}\|^2_{L^2(0,T,H^1_p(\Omega))} + \|\mathscr{V}^{(n-1)}\|^2_{L^2(0,T,H^1_p(\Omega))}\right), \quad i = 1,2. \tag{2.123}$$

Hence, inequality (2.122) becomes

$$\|\mathscr{U}^{(n)}\|^2_{H^1_p(\Omega)} + \|\mathscr{V}^{(n)}\|^2_{H^1_p(\Omega)} \leq 4K^*\left(\delta_1^2 + \delta_2^2\right)\left(\|\mathscr{U}^{(n-1)}\|^2_{L^2(0,T,H^1_p(\Omega))} + \|\mathscr{V}^{(n-1)}\|^2_{L^2(0,T,H^1_p(\Omega))}\right). \tag{2.124}$$

By integrating both sides of (2.124) with respect to $t$ over the interval $[0, T]$, we obtain

$$\|\mathscr{U}^{(n)}\|^2_{L^2(0,T,H^1_p(\Omega))} + \|\mathscr{V}^{(n)}\|^2_{L^2(0,T,H^1_p(\Omega))} \leq K^*\left(\|\mathscr{U}^{(n-1)}\|^2_{L^2(0,T,H^1_p(\Omega))} + \|\mathscr{V}^{(n-1)}\|^2_{L^2(0,T,H^1_p(\Omega))}\right), \tag{2.125}$$

where $K^*$ is given by (2.120). This achieves the proof of Lemma (2.4).

**Theorem 3**

Suppose that conditions (2.99) hold, and $K^* < \frac{1}{4}$ , then the nonlinear coupled system (2.98) admits a weak solution in $L^2(0, T, H^1_p(\Omega))$.

**Proof**

From Lemma (2.4), we conclude that the series $\sum_{n=1}^{\infty} \mathscr{U}^{(n)}$ and $\sum_{n=1}^{\infty} \mathscr{V}^{(n)}$ converge, if $K^* < \frac{1}{4}$. Indeed, inequality (2.125) implies

$$\begin{aligned} \|U^{(n)}\|_{L^2(0,T,H^1_p(\Omega))} &\leq \sqrt{K^*}\left(\|\mathscr{U}^{(n-1)}\|^2_{L^2(0,T,H^1_p(\Omega))} + \|\mathscr{V}^{(n-1)}\|^2_{L^2(0,T,H^1_p(\Omega))}\right)^{\frac{1}{2}}, \\ \|V^{(n)}\|_{L^2(0,T,H^1_p(\Omega))} &\leq \sqrt{K^*}\left(\|\mathscr{U}^{(n-1)}\|^2_{L^2(0,T,H^1_p(\Omega))} + \|\mathscr{V}^{(n-1)}\|^2_{L^2(0,T,H^1_p(\Omega))}\right)^{\frac{1}{2}}, \end{aligned} \tag{2.126}$$

it follows from (2.126) that

$$\|U^{(n)}\|_{L^2(0,T,H^1_p(\Omega))} + \|V^{(n)}\|_{L^2(0,T,H^1_p(\Omega))} \leq \sqrt{K^*}\left(\|\mathscr{U}^{(n-1)}\|^2_{L^2(0,T,H^1_p(\Omega))} + \|\mathscr{V}^{(n-1)}\|^2_{L^2(0,T,H^1_p(\Omega))}\right)^{\frac{1}{2}}. \tag{2.127}$$

Now since

$$\|U^{(n)} + V^{(n)}\|_{L^2(0,T,H^1_p(\Omega))} + \leq \|\mathscr{U}^{(n-1)}\|^2_{L^2(0,T,H^1_p(\Omega))} + \|\mathscr{V}^{(n-1)}\|^2_{L^2(0,T,H^1_p(\Omega))}, \tag{2.128}$$

then, we infer from (2.127) and (2.128) that

$$\begin{aligned} \left\|\mathscr{U}^{(n)} + \mathscr{V}^{(n)}\right\|_{L^2(0,T;H^1_\rho(\Omega))} &\leq 2\sqrt{K^*}\left(\left\|\mathscr{U}^{(n-1)}\right\|^2_{L^2(0,T;H^1_\rho(\Omega))} + \left\|\mathscr{V}^{(n-1)}\right\|^2_{L^2(0,T;H^1_\rho(\Omega))}\right)^{1/2} \\ &\leq 2\sqrt{K^*}\left(\left\|\mathscr{U}^{(n-1)} + \mathscr{V}^{(n-1)}\right\|^2_{L^2(0,T;H^1_\rho(\Omega))}\right)^{1/2} \\ &= 2\sqrt{K^*}\left\|\mathscr{U}^{(n-1)} + \mathscr{V}^{(n-1)}\right\|_{L^2(0,T;H^1_\rho(\Omega))}. \end{aligned} \tag{2.129}$$

Inequality (2.129) shows that the series $\sum_{n=1}^{\infty}\left(\mathscr{U}^{(n)}+\mathscr{V}^{(n)}\right)=\sum_{n=1}^{\infty}\mathscr{U}^{(n)}+\sum_{n=1}^{\infty}\mathscr{V}^{(n)}$ converges if $K^*<1/4$. Since $\left(\mathscr{U}^{(n)},\mathscr{V}^{(n)}\right)=\left(U^{(n+1)}-U^{(n)},V^{(n+1)}-V^{(n)}\right)$, then it follows that the sequence $\left(U^{(n)},V^{(n)}\right)_{n\in N}$ with $U^{(n)}$ and $V^{(n)}$ defined by

$$U^{(n)}(x,t)=\sum_{k=0}^{n-1}\mathscr{U}^{(k)}(x,t)+U^{(0)}(x,t)=\sum_{k=0}^{n-1}\left(U^{(k+1)}-U^{(k)}\right)+U^{(0)}(x,t),\quad n=1,2,\ldots, \tag{2.130}$$

and

$$V^{(n)}(x,t)=\sum_{k=0}^{n-1}\mathscr{V}^{(k)}(x,t)+V^{(0)}(x,t)=\sum_{k=0}^{n-1}\left(V^{(k+1)}-V^{(k)}\right)+V^{(0)}(x,t),\quad n=1,2,\ldots, \tag{2.131}$$

converge to an element $(U,V)\in\left(L^2\left(0,T;H_\rho^1(\Omega)\right)\right)^2$, which must be proved that it is a solution of problem (2.98). In other words, $(U,V)$ must satisfy (2.114).
From the iterated system (2.115), we have:

$$\begin{aligned}A\left(w,U^{(n)},V^{(n)}\right)=&\left(w,\Im_x\left(\xi F\left(\xi,t,U^{(n-1)},U_\xi^{(n-1)},V^{(n-1)},V_\xi^{(n-1)}\right)\right)\right)_{L^2(0,T,L_p^2(\Omega))}\\&+\left(w,\Im_x\left(\xi G\left(\xi,t,U^{(n-1)},U_\xi^{(n-1)},V^{(n-1)},V_\xi^{(n-1)}\right)\right)\right)_{L^2(0,T,L_p^2(\Omega))}.\end{aligned} \tag{2.132}$$

We infer from (2.132) that

$$\begin{aligned}&A\left(w,U^{(n)}-U,V^{(n)}-V\right)+A(w,U,V)\\&=\left(w,\Im_x\left(\xi F\left(\xi,t,U^{(n-1)},U_\xi^{(n-1)},V^{(n-1)},V_\xi^{(n-1)}\right)\right)-\Im_x\left(\xi F\left(\xi,t,U,U_\xi,V,V_\xi\right)\right)\right)_{L^2(0,T,L_p^2(\Omega))}\\&+\left(w,\Im_x\left(\xi G\left(\xi,t,U^{(n-1)},U_\xi^{(n-1)},V^{(n-1)},V_\xi^{(n-1)}\right)\right)-\Im_x\left(\xi G\left(\xi,t,U,U_\xi,V,V_\xi\right)\right)\right)_{L^2(0,T,L_p^2(\Omega))}\\&+\left(w,\Im_x\left(\xi F\left(\xi,t,U,U_\xi,V,V_\xi\right)\right)\right)_{L^2(0,T,L_p^2(\Omega))}+\left(w,\Im_x\left(\xi G\left(\xi,t,U,U_\xi,V,V_\xi\right)\right)\right)_{L^2(0,T,L_p^2(\Omega))}.\end{aligned} \tag{2.133}$$

Now from the PDEs in (2.115), we obtain

$$\begin{aligned}&A\left(w,U^{(n)}-U,V^{(n)}-V\right)\\&=\left(w,\frac{\partial^2}{\partial t^2}\Im_x\left(\xi\left(U^{(n)}-U\right)\right)\right)_{L^2(0,T,L_p^2(\Omega))}-\left(w,\Im_x\left(\frac{\partial}{\partial\xi}\left(\xi\frac{\partial}{\partial\xi}\left(U^{(n)}-U\right)\right)\right)\right)_{L^2(0,T,L_p^2(\Omega))}\\&-\left(w,\frac{\partial}{\partial t}\Im_x\left(\frac{\partial}{\partial\xi}\left(\xi\frac{\partial}{\partial\xi}\left(U^{(n)}-U\right)\right)\right)\right)_{L^2(0,T,L_p^2(\Omega))}+z_1\left(w,\Im_x\left(\xi\left(V^{(n)}-V\right)\right)\right)_{L^2(0,T,L_p^2(\Omega))}\\&+\left(w,\frac{\partial}{\partial t}\Im_x\left(\xi\left(U^{(n)}-U\right)\right)\right)_{L^2(0,T,L_p^2(\Omega))}\\&+\left(w,\frac{\partial}{\partial t}\Im_x\left(\xi\left(V^{(n)}-V\right)\right)\right)_{L^2(0,T,L_p^2(\Omega))}-\left(w,\Im_x\left(\frac{\partial}{\partial\xi}\left(\xi\frac{\partial}{\partial\xi}\left(V^{(n)}-V\right)\right)\right)\right)_{L^2(0,T,L_p^2(\Omega))}\\&-\left(w,\frac{\partial}{\partial t}\Im_x\left(\frac{\partial}{\partial\xi}\left(\xi\frac{\partial}{\partial\xi}\left(V^{(n)}-V\right)\right)\right)\right)_{L^2(0,T,L_p^2(\Omega))}+z_2\left(w,\Im_x\left(\xi\left(U^{(n)}-U\right)\right)\right)_{L^2(0,T,L_p^2(\Omega))}\\&=\left(w,\Im_x\left(\xi F\left(\xi,t,\left(U^{(n-1)}-U\right),\left(U_\xi^{(n-1)}-U_\xi\right),\left(V^{(n-1)}-V\right),\left(V_\xi^{(n-1)}-V_\xi\right)\right)\right)\right)_{L^2(0,T,L_p^2(\Omega))}\\&+\left(w,\Im_x\left(\xi G\left(\xi,t,\left(U^{(n-1)}-U\right),\left(U_\xi^{(n-1)}-U_\xi\right),\left(V^{(n-1)}-V\right),\left(V_\xi^{(n-1)}-V_\xi\right)\right)\right)\right)_{L^2(0,T,L_p^2(\Omega))}.\end{aligned} \tag{2.134}$$

Conditions on functions $w, U, V,$ and integration of each term on the right-hand side of (2.134), yield

$$
\begin{aligned}
&A\left(w, U^{(n)}-U, V^{(n)}-V\right)\\
&=-\left(U^{(n)}-U, \mathfrak{I}_x(\xi w)_{tt}\right)_{L^2(0,T,L_p^2(\Omega))}-\left(\frac{\partial}{\partial x}\left(U^{(n)}-U\right), xw\right)_{L^2(0,T,L_p^2(\Omega))}\\
&+\left(\frac{\partial}{\partial x}\left(U^{(n)}-U\right), xw_t\right)_{L^2\left(0,T;L_p^2(\Omega)\right)}-z_1\left(V^{(n)}-V, \mathscr{I}_x(\xi w)\right)_{L^2\left(0,T;L_p^2(\Omega)\right)}\\
&+\left(U^{(n)}-U, \mathscr{I}_x(\xi w_t)\right)_{L^2\left(0,T;L_p^2(\Omega)\right)}+\left(V^{(n)}-V, \mathscr{I}_x(\xi w_t)\right)_{L^2\left(0,T;L_p^2(\Omega)\right)}\\
&-\left(V^{(n)}-V, \mathscr{I}_x(\xi w)_t\right)_{L^2\left(0,T;L_p^2(\Omega)\right)}-\left(\frac{\partial}{\partial x}\left(V^{(n)}-V\right), xw\right)_{L^2\left(0,T;L_p^2(\Omega)\right)}\\
&+\left(\frac{\partial}{\partial x}\left(V^{(n)}-V\right), xw_t\right)_{L^2\left(0,T;L_p^2(\Omega)\right)}-z_2\left(U^{(n)}-U, \mathscr{I}_x(\xi w)\right)_{L^2\left(0,T;L_p^2(\Omega)\right)}. \qquad (2.135)
\end{aligned}
$$

Applying of the Cauchy–Schwarz inequality, to the terms on the RHS of (2.135) gives

$$-\left(U^{(n)}-U, \mathfrak{I}_x(\xi w)_{tt}\right)_{L^2(0,T,L_p^2(\Omega))} \le \left\|U^{(n)}-U\right\|_{L^2(0,T,L_p^2(\Omega))}\|\mathfrak{I}_x(\xi w)\|_{L^2(0,T,L_p^2(\Omega))}, \qquad (2.136)$$

$$-\left(\frac{\partial}{\partial x}\left(U^{(n)}-U\right), xw\right)_{L^2(0,T,L_p^2(\Omega))} \le \left\|\frac{\partial}{\partial x}\left(U^{(n)}-U\right)\right\|_{L^2(0,T,L_p^2(\Omega))}\|w\|_{L^2(0,T,L_p^2(\Omega))}, \qquad (2.137)$$

$$+\left(\frac{\partial}{\partial x}\left(U^{(n)}-U\right), xw_t\right)_{L^2(0,T,L_p^2(\Omega))} \le \left\|\frac{\partial}{\partial x}\left(U^{(n)}-U\right)\right\|_{L^2(0,T,H_p^1(\Omega))}\|w_t\|_{L^2(0,T,L_p^2(\Omega))}, \qquad (2.138)$$

$$-z_1\left(V^{(n)}-V, \mathfrak{I}_x(\xi w)\right)_{L^2(0,T,L_p^2(\Omega))} \le z_1\left\|V^{(n)}-V\right\|_{L^2(0,T,L_p^2(\Omega))}\|\mathfrak{I}_x(\xi w)\|_{L^2(0,T,L_p^2(\Omega))}, \qquad (2.139)$$

$$-\left(V^{(n)}-V, \mathfrak{I}_x(\xi w)_t\right)_{L^2(0,T,L_p^2(\Omega))} \le \left\|V^{(n)}-V\right\|_{L^2(0,T,L_p^2(\Omega))}\left\|\mathfrak{I}_x(\xi w)_t\right\|_{L^2(0,T,L_p^2(\Omega))}, \qquad (2.140)$$

$$-\left(\frac{\partial}{\partial x}\left(V^{(n)}-V\right), xw\right)_{L^2(0,T,L_p^2(\Omega))} \le \left\|\frac{\partial}{\partial x}\left(V^{(n)}-V\right)\right\|_{L^2(0,T,L_p^2(\Omega))}\|w\|_{L^2(0,T,L_p^2(\Omega))}, \qquad (2.141)$$

$$+\left(\frac{\partial}{\partial x}\left(V^{(n)}-V\right), xw_t\right)_{L^2(0,T,L_p^2(\Omega))} \le \left\|\frac{\partial}{\partial x}\left(V^{(n)}-V\right)\right\|_{L^2(0,T,L_p^2(\Omega))}\|w_t\|_{L^2(0,T,L_p^2(\Omega))}, \qquad (2.142)$$

$$-z_2\left(U^{(n)}-U, \mathfrak{I}_x(\xi w)\right)_{L^2(0,T,L_p^2(\Omega))} \le z_2\left\|U^{(n)}-U\right\|_{L^2(0,T,L_p^2(\Omega))}\|\mathfrak{I}_x(\xi w)\|_{L^2(0,T,L_p^2(\Omega))}, \qquad (2.143)$$

$$\left(U^{(n)}-U, \mathfrak{I}_x(\xi w_t)\right)_{L^2(0,T,L_p^2(\Omega))} \le \left\|U^{(n)}-U\right\|_{L^2(0,T,L_p^2(\Omega))}\|\mathfrak{I}_x(\xi w_t)\|_{L^2(0,T,L_p^2(\Omega))}. \qquad (2.144)$$

Combination of equality (2.135) and inequalities (2.136)-(2.144), leads to

$$
\begin{aligned}
&A\left(w, U^{(n)}-U, V^{(n)}-V\right)\\
&\le l_1\left(\left\|U^{(n)}-U\right\|_{L^2(0,T,H_p^1(\Omega))}\right)\Bigg(\|\mathfrak{I}_x(\xi w)_{tt}\|+\|\mathfrak{I}_x(\xi w)\|_{L^2(0,T,L_p^2(\Omega))}+\|w\|_{L^2(0,T,L_p^2(\Omega))}\\
&+\|w_t\|_{L^2(0,T,L_p^2(\Omega))}\Bigg)\\
&+l_2\left(\left\|V^{(n)}-V\right\|_{L^2(0,T,H_p^1(\Omega))}\right)\Bigg(\|\mathfrak{I}_x(\xi w)_t\|+\|\mathfrak{I}_x(\xi w)\|_{L^2(0,T,L_p^2(\Omega))}+\|w\|_{L^2(0,T,L_p^2(\Omega))}\\
&+\|w_t\|_{L^2(0,T,L_p^2(\Omega))}\Bigg),
\end{aligned}
$$

where:

$$l_1 = l_2 = \sqrt{2}\max(1, z_1, z_2).$$

On the other side of (2.134), we have

$$\begin{aligned}
&\left(w, \Im_x\left(\xi F\left(\xi, t, \left(U^{(n-1)} - U\right), \left(U_\xi^{(n-1)} - U_\xi\right), \left(V^{(n-1)} - V\right), \left(V_\xi^{(n-1)} - V_\xi\right)\right)\right)\right)_{L^2(0,T,L_p^1(\Omega))} \\
&+\left(w, \Im_x\left(\xi G\left(\xi, t, \left(U^{(n-1)} - U\right), \left(U_\xi^{(n-1)} - U_\xi\right), \left(V^{(n-1)} - V\right), \left(V_\xi^{(n-1)} - V_\xi\right)\right)\right)\right)_{L^2(0,T,L_p^1(\Omega))} \\
&\leq \sqrt{2}\max\{\delta_1, \delta_2\}\left(\left\|U^{(n-1)} - U\right\|_{L^2(0,T,H_p^1(\Omega))} + \left\|V^{(n-1)} - V\right\|_{L^2(0,T,H_p^1(\Omega))}\right)\|w\|_{L^2(0,T,L_p^2(\Omega))}.
\end{aligned} \tag{2.145}$$

As $n \longrightarrow \infty$, it follows from (2.145)-- (2.145) and (2.133) that

$$A(w, U, V) = (w, \Im_x(\xi F))_{L^2(0,T,L_p^2(\Omega))} + (w, \Im_x(\xi G))_{L^2(0,T,L_p^2(\Omega))} \tag{2.146}$$

To conclude that problem (2.98) admits a weak solution,

It remains now to prove the uniqueness of solution of system (2.98).

**Theorem 4**

If hypotheses (2.99) are satisfied, then the system (2.98) has only one solution.

**Proof**

Suppose that $(U_1, V_1), (U_2, V_2) \in (L^2(0, T, H_p^1(\Omega)))^2$ are two different solutions of the system (2.98), then: $(\mathcal{U}, \mathcal{V}) = (U_1 - U_2, V_1 - V_2) \in (L^2(0, T, H_p^1(\Omega)))^2$ verifies

$$\begin{cases}
\frac{\partial^2}{\partial t^2}\mathcal{U} - \frac{1}{x}(x\mathcal{U}_x)_x - \frac{1}{x}(x\mathcal{U}_x)_{xt} + z_1\mathcal{V} + \mathcal{U}_t = H_1(x,t), \\
\frac{\partial}{\partial t}\mathcal{V} - \frac{1}{x}(x\mathcal{V}_x)_x - \frac{1}{x}(x\mathcal{V}_x)_{xt} + z_2\mathcal{U} = H_2(x,t), \\
\mathcal{U}(x,0) = 0, \quad \mathcal{U}_t(x,0) = 0, \quad \mathcal{V}(x,0) = 0, \\
\int_0^1 x\mathcal{U}\,dx = 0, \quad \int_0^1 x\mathcal{V}\,dx = 0, \quad \mathcal{U}_x(1,t) = 0, \quad \mathcal{V}_x(1,t) = 0,
\end{cases} \tag{2.147}$$

where

$$H_1(x,t) = F(x, t, U_1, U_{x1}, V_1, V_{x1}) - F(x, t, U_2, U_{x2}, V_2, V_{x2}), \tag{2.148}$$
$$H_2(x,t) = G(x, t, U_1, U_{x1}, V_1, V_{x1}) - G(x, t, U_2, U_{x2}, V_2, V_{x2}). \tag{2.149}$$

By applying the same procedure used in the proof of Lemma (2.4), we obtain:

$$\|\mathcal{U}\|^2_{L^2(0,T,H_p^1(\Omega))} + \|\mathcal{V}\|^2_{L^2(0,T,H_p^1(\Omega))} \leq K^*\left(\|\mathcal{U}\|^2_{L^2(0,T,H_p^1(\Omega))} + \|\mathcal{V}\|^2_{L^2(0,T,H_p^1(\Omega))}\right), \tag{2.150}$$

where $K^*$ is the same constant as in Lemma (2.4). Since $K^* < 1$, it follows from (2.150) that

$$(1 - K^*)\left(\|\mathcal{U}\|^2_{L^2(0,T,H_p^1(\Omega))} + \|\mathcal{V}\|^2_{L^2(0,T,H_p^1(\Omega))}\right) = 0, \tag{2.151}$$

which implies that : $(\mathcal{U}, \mathcal{V}) = (U_1 - U_2, V_1 - V_2) = (0,0)$, and hence

$$(U_1, V_1) = (U_2, V_2) \in (L^2(0, T, H_p^1(\Omega)))^2. \tag{2.152}$$

This achieves the proof of Theorem (2.4).

# Chapter three

## Numerical Results via the Homotopy Analysis Transform Method

**Contents**

## 3.1 Homotopy Analysis Method (HAM)

In this section, we present the Homotopy Analysis Method (HAM), originally introduced by Liao in 1992 as a versatile tool for solving both linear and nonlinear equations [**11**]. Liao later provided a systematic formulation in 2003, which significantly enhanced its theoretical understanding and practical application [**13**]. Here, we review the fundamental principles of HAM and extend its framework to systems of partial differential equations. Over the years, HAM has proven to be highly effective for addressing complex nonlinear equations and coupled systems [**14**].

We consider the following coupled system (see[**17**]):

$$\begin{aligned} \mathcal{N}_1[u(x,t)] &= 0, \\ \mathcal{N}_2[v(x,t)] &= 0, \end{aligned} \tag{3.1}$$

where $\mathcal{N}_1$ and $\mathcal{N}_2$ are linear or nonlinear operators, $x$ and $t$ denote the independent variables, and $u(x,t)$ and $v(x,t)$ are unknown functions. For simplicity, all boundary or initial conditions are ignored, as they can be treated in a similar manner.

Based on the zero-order deformation equation constructed by Liao (see [**18**]), we present the following **zero-order deformation equation** in a similar manner:

$$\begin{aligned} (1-q)\mathcal{L}[\phi(x,t;q)-u_0(x,t)] &= qh\mathcal{N}_1(\phi(x,t;q)), \\ (1-q)\mathcal{L}[\psi(x,t;q)-v_0(x,t)] &= qh\mathcal{N}_2(\psi(x,t;q)), \end{aligned} \tag{3.2}$$

where $q \in [0,1]$ is the embedding parameter, $h$ is a nonzero auxiliary parameter, and $\mathcal{L}$ is an auxiliary linear operator of non-integer order satisfying the property $\mathcal{L}(C) = 0$, where $C$ is a constant. The functions $u_0(x,t)$ and $v_0(x,t)$ are initial guesses of $u(x,t)$ and $v(x,t)$, respectively. Moreover, $\phi(x,t;q)$ and $\psi(x,t;q)$ are unknown functions depending on the independent variables $x$, $t$, and $q$.

It is important to note that there is considerable freedom in choosing the auxiliary parameter $h$ in HAM.

If $q = 0$ and $q = 1$, then we have:

$$\begin{aligned} \phi(x,t;0) &= u_0(x,t), \qquad and \qquad \phi(x,t;1) = u(x,t), \\ \psi(x,t;0) &= v_0(x,t), \qquad and \qquad \psi(x,t;1) = v(x,t). \end{aligned} \tag{3.3}$$

Thus, as $q$ increases from 0 to 1, the solutions $\phi(x,t;q)$ and $\psi(x,t;q)$ vary from the initial guesses $u_0(x,t)$ and $v_0(x,t)$ to the exact solutions $u(x,t)$ and $v(x,t)$.

Expanding $\phi(x,t;q)$ and $\psi(x,t;q)$ into Taylor series with respect to $q$, we obtain:

$$\begin{aligned} \phi(x,t;q) &= u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) q^m, \\ \psi(x,t;q) &= v_0(x,t) + \sum_{m=1}^{\infty} v_m(x,t) q^m, \end{aligned} \tag{3.4}$$

where:

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;q)}{\partial q^m}|_{q=0}, \qquad and \qquad v_m(x,t) = \frac{1}{m!}\frac{\partial^m \psi(x,t;q)}{\partial q^m}|_{q=0}. \tag{3.5}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $h$, and the auxiliary function are properly chosen, then the series (3.4) converges at $q = 1$. Consequently, we have:

$$\begin{aligned} u(x,t) &= u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t), \\ v(x,t) &= v_0(x,t) + \sum_{m=1}^{\infty} v_m(x,t), \end{aligned} \tag{3.6}$$

which must be one of the solutions of the original linear or nonlinear equation, as proved in [18].

When $h=-1$, equation (3.2) becomes:

$$\begin{aligned}(1-q)\mathscr{L}\left[\phi(x,t;q)-u_0(x,t)\right]+q\mathscr{N}_1(\phi(x,t;q))=0,\\ (1-q)\mathscr{L}\left[\psi(x,t;q)-v_0(x,t)\right]+q\mathscr{N}_2(\psi(x,t;q))=0.\end{aligned} \tag{3.7}$$

According to (3.4), the governing equations can be derived from the zero-order deformation equation (3.2).

Define the vectors:

$$\begin{aligned}\vec{u_m}(x,t)=\{u_0(x,t),u_1(x,t),u_2(x,t),u_3(x,t),...,u_m(x,t)\},\\ \vec{v_m}(x,t)=\{v_0(x,t),v_1(x,t),v_2(x,t),v_3(x,t),...,v_m(x,t)\}.\end{aligned} \tag{3.8}$$

Differentiating (3.2) $m$ times with respect to the embedding parameter $q$, then setting $q=0$, and finally dividing by $m!$, we obtain the so-called **$m$th-order deformation equations**:

$$\begin{aligned}\mathscr{L}\left[u_m(x,t)-\chi_m u_{m-1}(x,t)\right]=h\mathscr{R}_1(u_{m-1}(x,t)),\\ \mathscr{L}\left[v_m(x,t)-\chi_m v_{m-1}(x,t)\right]=h\mathscr{R}_2(v_{m-1}(x,t)),\end{aligned} \tag{3.9}$$

where :

$$\begin{aligned}\mathscr{R}_1(u_{m-1}(x,t))=\frac{1}{(m-1)!}\frac{\partial^{m-1}\mathscr{N}_1(\phi(x,t;q))}{\partial q^{m-1}}\bigg|_{q=0},\\ \mathscr{R}_2(v_{m-1}(x,t))=\frac{1}{(m-1)!}\frac{\partial^{m-1}\mathscr{N}_2(\psi(x,t;q))}{\partial q^{m-1}}\bigg|_{q=0},\end{aligned} \tag{3.10}$$

and

$$\mathscr{X}_m=\begin{cases}0 & m\leq 1,\\ 1 & m>1.\end{cases} \tag{3.11}$$

Applying $\mathscr{L}^{-1}$ to both sides of (3.9), we obtain:

$$\begin{aligned}u_m(x,t)=\chi_m u_{m-1}(x,t)+h\mathscr{L}^{-1}[\mathscr{R}_1(u_{m-1}(x,t))],\\ v_m(x,t)=\chi_m v_{m-1}(x,t)+h\mathscr{L}^{-1}[\mathscr{R}_2(v_{m-1}(x,t))].\end{aligned} \tag{3.12}$$

In this way, it is easy to obtain $u_m$ and $v_m$ for $m\geq 1$. At the $m$th order, we have:

$$u(x,t)=\sum_{m=1}^{M}u_m(x,t),\qquad v(x,t)=\sum_{m=1}^{M}v_m(x,t). \tag{3.13}$$

When $M\to\infty$, we obtain an accurate approximation of the original equation (3.1). For the convergence of the above method, we refer the reader to Liao's work.

If equation (3.1) admits a unique solution, then this method will produce that unique solution. If equation (3.1) does not possess a unique solution, the HAM will provide one solution among many possible solutions.

## 3.2 Iterative Scheme Based on the Homotopy Analysis Transform Method

The HATM method, also known as the Modified Homotopy Analysis Method, is considered one of the most efficient and accurate techniques for solving mathematical models of integer and fractional orders. It is distinguished by its ability to handle a wide range of problems across various scientific fields, as demonstrated in [**17**, **18**].

To solve the problem (2.1)–(2.2), we begin by applying the Laplace transform to both sides of each equation in the system, this yields:

$$\begin{aligned} &s^2\mathscr{L}[u(x;t)]-s\,u(x;0)-u_t(x;0)+\mathscr{L}[\mathscr{R}_1 u(x;t)+\mathscr{N}_1 u(x;t)]=\mathscr{L}[f(x,t)],\\ &s\mathscr{L}[v(x,t)]-v(x;0)+\mathscr{L}[\mathscr{R}_2 v(x;t)+\mathscr{N}_2 v(x;t)]=\mathscr{L}[g(x,t)], \end{aligned} \tag{3.14}$$

thus, we obtain:

$$\begin{aligned} &\mathscr{L}[u(x,t)]-\frac{1}{s}\phi_1(x)-\frac{1}{s^2}\phi_2(x)-\frac{1}{s^2}\mathscr{L}[\frac{1}{x}(x u_x)_x+\frac{1}{x}(x u_x)_{xt}-z_1 v-u_t+f]=0,\\ &\mathscr{L}[v(x,t)]-\frac{1}{s}\psi_1(x)-\frac{1}{s}\mathscr{L}[\frac{1}{x}(x v_x)_x+\frac{1}{x}(x v_x)_{xt}-z_2 u+G]=0. \end{aligned} \tag{3.15}$$

Next, we introduce two nonlinear operators, denoted by $\mathscr{N}_1$ and $\mathscr{N}_2$, defined as follows:

$$\begin{aligned} \mathscr{N}_1\left[\tilde{\phi}_1(x,t;q),\tilde{\phi}_2(x,t;q)\right]=&\mathscr{L}\left\{\tilde{\phi}_1(x,t;q)\right\}-\frac{1}{s}\tilde{\phi}_1(x)-\frac{1}{s^2}\tilde{\phi}_2(x)-\frac{1}{s^2}\mathscr{L}\left\{\frac{1}{x}(x\tilde{\phi}_{1x})_x\right.\\ &\left.+\frac{1}{x}(x\tilde{\phi}_{1x})_{xt}-z_1\tilde{\phi}_2(x,t;q)-\tilde{\phi}_{1t}+f(x,t)\right\},\\ \mathscr{N}_1\left[\tilde{\phi}_1(x,t;q),\tilde{\phi}_2(x,t;q)\right]=&\mathscr{L}\left\{\tilde{\phi}_{2t}(x,t;q)\right\}-\frac{1}{s}\psi_1(x)-\frac{1}{s}\tilde{\phi}_1(x)-\frac{1}{s}\mathscr{L}\left\{\frac{1}{x}(x\tilde{\phi}_{2x})_x\right.\\ &\left.+\frac{1}{x}(x\tilde{\phi}_{2x})_{xt}-z_2\tilde{\phi}_1(x,t;q)+G(x,t)\right\}. \end{aligned} \tag{3.16}$$

Hence, we take the zeroth deformation equations as:

$$\begin{aligned} (1-q)\mathscr{L}\left[\tilde{\phi}_1(x,t;q)-u_0(x,t)\right]=q\hbar_1\mathscr{N}_1\left[\tilde{\phi}_1,\tilde{\phi}_2\right],\\ (1-q)\mathscr{L}\left[\tilde{\phi}_2(x,t;q)-v_0(x,t)\right]=q\hbar_2\mathscr{N}_2\left[\tilde{\phi}_1,\tilde{\phi}_2\right], \end{aligned} \tag{3.17}$$

which implies that the $m$th-order deformation equations are given by:

$$\begin{aligned} \mathscr{L}\left[u_m(x,t)-\chi_m u_{m-1}(x,t)\right]=\hbar_1\Re_1(\vec{u}_{m-1},\vec{v}_{m-1}),\\ \mathscr{L}\left[v_m(x,t)-\chi_m v_{m-1}(x,t)\right]=\hbar_2\Re_2(\vec{u}_{m-1},\vec{v}_{m-1}), \end{aligned} \tag{3.18}$$

where:

$$\begin{aligned} &\Re_1(\tilde{u}_{m-1},\tilde{v}_{m-1})=\mathscr{L}[u_{m-1}]-(1-\chi_m)(\frac{1}{s}\phi_1(x)+\frac{1}{s^2}\phi_2(x))\\ &-\frac{1}{s^2}\mathscr{L}[\frac{1}{x}(x u_{m-1})_x+\frac{1}{x}(x u_{m-1})_{xt}-z_1 v_{m-1}-u_{m-1}+(1-\chi_m)F(x,t)],\\ &\Re_2(\tilde{u}_{m-1},\tilde{v}_{m-1})=\mathscr{L}[v_{m-1}]-(1-\chi_m)(\frac{1}{s}\psi_1(x))\\ &-\frac{1}{s}\mathscr{L}[\frac{1}{x}(x v_{m-1})_x+\frac{1}{x}(x v_{m-1})_{xt}-z_2 u_{m-1}+(1-\chi_m)G(x,t)]. \end{aligned} \tag{3.19}$$

Therefore, successive terms of the approximate series solution can be computed recursively from the iterative schemes:

$$\begin{aligned} u_m(x,t)=\chi_m u_{m-1}(x,t)+\hbar_1\mathscr{L}^{-1}[\Re_1(\vec{u}_{m-1},\vec{v}_{m-1})], \qquad m\geq 1\\ v_m(x,t)=\chi_m v_{m-1}(x,t)+\hbar_2\mathscr{L}^{-1}[\Re_2(\vec{u}_{m-1},\vec{v}_{m-1})], \qquad m\geq 1 \end{aligned} \tag{3.20}$$

and the solution will be given as:

$$\begin{aligned} u(x,t)=u_0(x,t)+\sum_{i=1}^{\infty}u_i(x,t),\\ v(x,t)=v_0(x,t)+\sum_{i=1}^{\infty}v_i(x,t). \end{aligned} \tag{3.21}$$

## 3.3 Numerical Results

In this section, we employ the iterative scheme (3.20), derived using the HATM, to numerically solve a set of examples. These examples are used to assess the efficiency of the proposed scheme in handling problems of the form (2.1)–(2.2).

**Example 3.3.1**

Consider a coupled system (2.1) with $z_1 = z_2 = 1$:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{1}{x}(x u_x)_x - \frac{1}{x}(x u_x)_{xt} + z_1 v + u_t = \ln x - e^{-t}, \\ \frac{\partial v}{\partial t} - \frac{1}{x}(x v_x)_x - \frac{1}{x}(x v_x)_{xt} + z_2 u = \ln x + 2e^{-t}, \end{cases} \tag{3.22}$$

subject to the following initial conditions:

$$\begin{aligned} \phi_1 &= u(x,0) = \ln x + 1, \\ \psi_1 &= v(x,0) = \ln x - 1, \\ \phi_2 &= u_t(x;0) = -1. \end{aligned} \tag{3.23}$$

**Solution 1**

Let $u_0(x,t) = u(x,0) = \ln x + 1$ and $v_0(x,t) = v(x,0) = \ln x - 1$. In view of (3.20), by setting $n = 1$ and $\hbar_1 = \hbar_2 = \hbar$, the first term of the series solution is given by:

$$\begin{aligned} u_1(x,t) &= \chi_1 u_0(x,t) + \hbar_1 \mathscr{L}^{-1}\left[\Re_1(\vec{u}_0, \vec{v}_0)\right], \\ &= \hbar_1 \mathscr{L}^{-1}\left[\mathscr{L}\left\{u_0(x,t)\right\} - (1-\chi_1)\left(\frac{1}{s}\phi_1(x) + \frac{1}{s^2}\phi_2(x)\right)\right. \\ &\quad \left. - \frac{1}{s^2}\mathscr{L}\left\{\frac{1}{x}(x u_{0x})_x + \frac{1}{x}(x u_{0x})_{xt} - v_0 - (u_0)_t + (1-\chi_1)F\right\}\right], \\ &= \hbar_1 \mathscr{L}^{-1}\left[\mathscr{L}\left\{\ln(x)+1\right\} - 1\left(\frac{\ln(x)+1}{s} - \frac{1}{s^2}\right) - \frac{1}{s^2}\mathscr{L}\left\{-\ln(x) + 1 - 0 + \ln(x) - e^{-t}\right\}\right], \\ &= \hbar_1 \mathscr{L}^{-1}\left[\frac{\ln(x)+1}{s} - \frac{\ln(x)+1}{s} + \frac{1}{s^2} - \frac{1}{s^2}\mathscr{L}\left\{1 - e^{-t}\right\}\right], \\ &= \hbar_1\left(2t - \frac{t^2}{2!} - 1 + e^{-t}\right). \end{aligned} \tag{3.24}$$

We now proceed to compute $v_1(x,t)$:

$$
\begin{aligned}
v_1(x,t) &= \chi_1 v_0(x,t) + \hbar_2 \mathscr{L}^{-1}\Big[\Re_1(\vec{u}_0,\vec{v}_0)\Big],\\
&= \chi_1 v_0(x,t) + \hbar_2 \mathscr{L}^{-1}\Bigg[\mathscr{L} v_0 - (1-\chi_1)(\frac{1}{s}\psi_1)\\
&- \frac{1}{s}\mathscr{L}\left\{\frac{1}{x}(x v_{0x})_x + \frac{1}{x}(x v_{0x})_{xt} - z_2 u_0 + (1-\chi_1)G\right\}\Bigg],\\
&= \hbar_2 \mathscr{L}^{-1}\Bigg[\mathscr{L}(\ln(x)-1) - \frac{1}{s}(\ln(x)-1) - \frac{1}{s}\mathscr{L}\{-\ln(x)-1+\ln(x)+2e^{-t}\}\Bigg],\\
&= \hbar_2 \mathscr{L}^{-1}\Bigg[\left(\frac{1}{s^2} - \frac{2}{s}\times\frac{1}{s+1}\right)\Bigg] = \hbar_2 \mathscr{L}^{-1}\Bigg[\frac{1}{s^2} - \frac{2}{s} + \frac{2}{s+1}\Bigg] = \hbar_2\Big[t-2+2e^{-t}\Big].
\end{aligned}
\tag{3.25}
$$

The next step is to compute $u_2(x,t)$

$$
\begin{aligned}
u_2(x,t) &= \chi_2 u_1(x,t) + \hbar \mathscr{L}^{-1}\Big[\mathscr{L} u_1 - (1-\chi_2)(\frac{1}{s}\phi_1 + \frac{1}{s^2}\phi_2)\Big]\\
&- \frac{1}{s^2}\mathscr{L}\left\{\frac{1}{x}(x u_{1x})_x + \frac{1}{x}(x u_{1x})_x - v_1 - (u_1)_t + (1-\chi_2)F\right\},\\
&= u_1(1+\hbar) + \hbar \mathscr{L}^{-1}\Bigg[-\frac{1}{s^2}\mathscr{L}\left\{-\hbar t + 2\hbar - 2\hbar e^{-t} - \hbar(2-t-e^{-t})\right\}\Bigg],\\
&= u_1(1+\hbar) + \hbar^2 \mathscr{L}^{-1}\Bigg[\frac{1}{s^2}\mathscr{L}\left\{e^{-t}\right\}\Bigg],\\
&= u_1(1+\hbar) + \hbar^2 \mathscr{L}^{-1}\Bigg[\frac{1}{s^2} * \frac{1}{s+1}\Bigg],\\
&= u_1(1+\hbar) + \hbar^2 \mathscr{L}^{-1}\Bigg[\frac{1-s}{s^2} + \frac{1}{s+1}\Bigg],\\
&= u_1(1+\hbar) + \hbar^2 \mathscr{L}^{-1}\Bigg[\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\Bigg],\\
&= u_1(1+\hbar) + \hbar^2\Big(t-1+e^{-t}\Big).
\end{aligned}
\tag{3.26}
$$

Next, we compute $v_2(x,t)$:

$$\begin{aligned}
v_2(x,t) &= \chi_2 v_1(x,t) + \hbar \mathscr{L}^{-1}\Bigg[\mathscr{L}\, v_1 - (1-\chi_2)(\frac{1}{s}\psi_1) \\
&\quad - \frac{1}{s}\mathscr{L}\left\{\frac{1}{x}(x v_{1x})_x + \frac{1}{x}(x v_{1x})_{xt} - z_2 u_1 + (1-\chi_1)G\right\}\Bigg], \\
&= v_1(1+\hbar) - \hbar \mathscr{L}^{-1}\left[\frac{1}{s}\mathscr{L}\{-u_1\}\right], \\
&= v_1(1+\hbar) + \hbar^2 \mathscr{L}^{-1}\left[\frac{1}{s}(\frac{2}{s^2} - \frac{1}{2!}\frac{2!}{s^3} - \frac{1}{s} + \frac{1}{s+1})\right], \\
&= v_1(1+\hbar) + \hbar^2 \mathscr{L}^{-1}\left[\frac{2}{s^3} - \frac{1}{s^4} - \frac{1}{s^2} + \frac{1}{s+1} * \frac{1}{s}\right], \\
&= v_1(1+\hbar) + \hbar^2\left[t^2 - \frac{t^3}{3!} - t - e^{-t} + 1\right].
\end{aligned} \tag{3.27}$$

We now determine the value of $u_3(x,t)$:

$$\begin{aligned}
u_3(x,t) &= \chi_3 u_2 + \hbar \mathscr{L}^{-1}\Bigg[\mathscr{L}(u_2) - (1-\chi_3)(\frac{1}{s}\phi_1 + \frac{1}{s^2}\phi_2) \\
&\quad - \frac{1}{s^2}\mathscr{L}\left\{\frac{1}{x}(x u_{2x})_x + \frac{1}{x}(x u_{2x})_{xt} - v_2 - (u_2)_t + (1-\chi_3)F\right\}\Bigg], \\
&= u_2(1+\hbar) - \hbar \mathscr{L}^{-1}\left[\frac{1}{s^2}\mathscr{L}\left\{-v_2 - (u_2)_t\right\}\right], \\
&= u_2(1+\hbar) + \hbar^3 \mathscr{L}^{-1}\left[\frac{1}{s^2}\mathscr{L}\left\{t^2 - \frac{t^3}{3!} - t + 2 - 2e^{-t}\right\}\right], \\
&= u_2(1+\hbar) + \hbar^3 \mathscr{L}^{-1}\left[\frac{1}{s^2}\Big(\frac{2!}{s^2} - \frac{1}{s^4} + \frac{1}{s^2} + \frac{2}{s} - \frac{2}{s+1}\Big)\right], \\
&= u_2(1+\hbar) + \hbar^3 \mathscr{L}^{-1}\left[\frac{2!}{s^5} - \frac{1}{S^6} - \frac{1}{s^4} + \frac{2}{s^3} - \frac{2}{s^2} * \frac{1}{s+1}\right], \\
&= u_2(1+\hbar) + \hbar^3\left[2\frac{t^4}{4!} - \frac{t^5}{5!} - \frac{t^3}{3!} + 2\frac{t^2}{2!} - 2t + 2 - 2e^{-t}\right].
\end{aligned} \tag{3.28}$$

Next, we determine $v_3(x,t)$:

$$\begin{aligned} v_3(x,t) &= \chi_3 v_2 + \hbar \mathscr{L}^{-1}\Big[\mathscr{L}(v_2) - (1-\chi_3)(\frac{1}{s}\psi_1) \\ &\quad - \frac{1}{s}\mathscr{L}\Big\{\frac{1}{x}(x v_{2x})_x + \frac{1}{x}(x v_{2x})_{xt} - z_2 u_2 + (1-\chi_3)G\Big\}\Big], \\ &= v_2(1+\hbar) - \hbar \mathscr{L}^{-1}\Big[\frac{1}{s}\mathscr{L}\Big(-u_2\Big)\Big], \\ &= v_2(1+\hbar) + \hbar^3 \mathscr{L}^{-1}\Big[\frac{1}{s}\mathscr{L}\Big\{t-1+e^{-t}\Big\}\Big], \\ &= v_2(1+\hbar) + \hbar^3 \mathscr{L}^{-1}\Big[\frac{1}{s^3} - \frac{1}{s^2} + \Big(\frac{1}{s} * \frac{1}{s+1}\Big)\Big], \\ &= v_2(1+\hbar) + \hbar^3\Big[\frac{t^2}{2!} - \frac{t}{1!} - e^{-t} + 1\Big]. \end{aligned} \tag{3.29}$$

If we set $\hbar = -1$ in the previously calculated terms, we obtain:

$$\begin{aligned} u_0(x;t) &= u(x;t) = \ln x + 1, \\ v_0(x;t) &= v(x;t) = \ln x - 1, \\ u_1(x;t) &= -2t + \frac{t^2}{2!} + 1 - e^{-t}, \\ v_1(x;t) &= -t + 2 - 2e^{-t}, \\ u_2(x;t) &= t - 1 + e^{-t}, \\ v_2(x;t) &= t^2 - \frac{t^3}{3!} - t - e^{-t} + 1, \\ u_3(x;t) &= -2\frac{t^4}{4!} + \frac{t^5}{5!} + \frac{t^3}{3!} - 2\frac{t^2}{2!} + 2t - 2 + 2e^{-t}, \\ v_3(x;t) &= -\frac{t^2}{2!} + t + e^{-t} - 1. \end{aligned} \tag{3.30}$$

Hence, the solution of the coupled system is given by:

$$\begin{cases} u(x,t) = & u_0(x;t) + \sum_{i=1}^{\infty} u_i(x;t), \\ v(x,t) = & v_0(x;t) + \sum_{i=1}^{\infty} v_i(x;t), \end{cases} \tag{3.31}$$

$$\begin{cases} u(x;t) = \ln x + 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \ldots, \\ v(x;t) = \ln x - 1 + t - \frac{t^2}{2!} + \frac{t^3}{3!} - \ldots \end{cases} \tag{3.32}$$

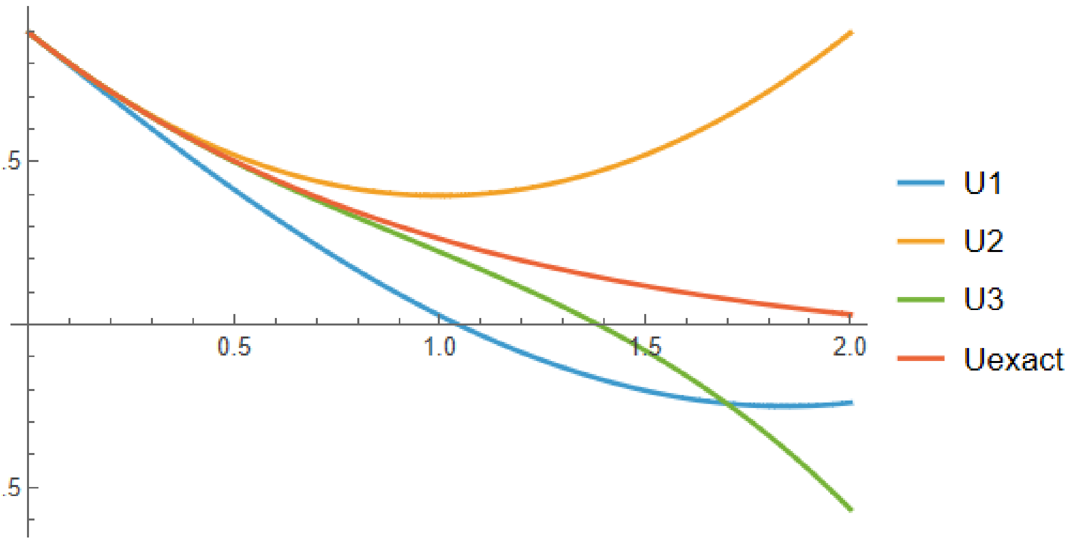
The series (3.32) converges to the following analytical function (exact solution):

$$\begin{cases} u(x;t) = \ln x + e^{-t}, \\ v(x;t) = \ln x - e^{-t}. \end{cases} \tag{3.33}$$

Figure 3.1 shows the graph of the truncated series solution for different numbers of terms in Example 1 at $x = 0.9$ and $\bar{h} = -1$. The figure illustrates the rapid convergence of these approximate

Out[21]= 0.9

Out[30]=




Out[1]= 0.9

Out[10]=

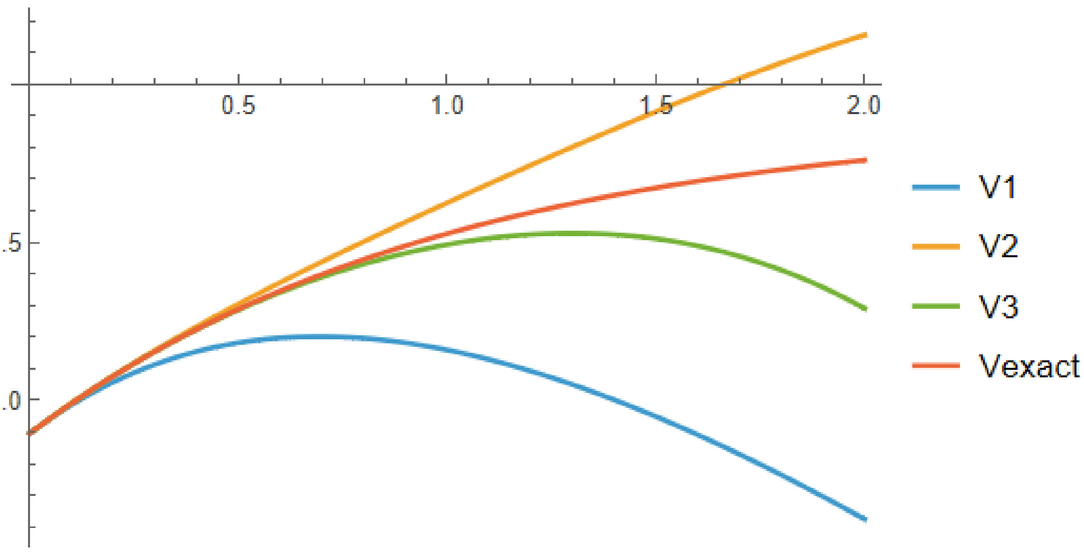



**Figure 3.1:**
shows the convergence of numerical approximations to the analytical solutions of $u(x,t)$ and $v(x,t)$ under varying $m$ at $x = 0.9$, with $m = 1, 2, 3$, $h = -1$.

solutions.

In Table 3.1, we present the numerical solutions of Example 1 obtained from the $k$th-order truncated series solutions $U_k = u_0 + u_1 + ... + u_k$ and $V_k = v_0 + v_1 + ... + v_k$ generated by the HATM for several values of $k$, $x$, and $t$, rounded to six significant digits. These results illustrate the rapid convergence of the truncated solutions after only a few terms. As shown in the table, both methods demonstrate good performance for this example.

| $t$ | $x$ | $m$ | $(U_{(m)}, V_{(m)})$ | $(U_{exact}, V_{exact})$ | $Error$ |
|---|---|---|---|---|---|
| 0.5 | 0.9 | 1 | 0.413109 | 0.50117 | 0.0880613 |
| | | | -0.818422 | -0.711891 | 0.106531 |
| | | 2 | 0.519639 | 0.50117 | 0.0184693 |
| | | | -0.695786 | -0.711891 | 0.0161053 |
| | | 3 | 0.498586 | 0.50117 | 0.00258392 |
| | | | -0.695786 | -0.711891 | 0.00236399 |
| 0.1 | 0.3 | 1 | -0.30381 | -0.299135 | 0.00467484 |
| | | | -2.11365 | -2.10881 | 0.00483742 |
| | | 2 | -0.298973 | -0.299135 | 0.000162582 |
| | | | -2.10865 | -2.10881 | 0.000158497 |
| | | 3 | -0.299135 | -0.299135 | $4.1653 \times 10^{-6}$ |
| | | | -2.10881 | -2.10881 | $4.0847 \times 10^{-6}$ |

**Table 3.1:** Comparison between approximate and exact solutions for parameters $h = -1$, $m = 1, 2, 3$.

The numerical results show that the proposed algorithm is highly efficient and exhibits stable convergence as the approximation order $m$ increases. High accuracy is achieved at low orders, with errors falling below $10^{-6}$ at $m = 3$. These results confirm the robustness of the method and its effective balance between computational cost and numerical accuracy, making it suitable for practical engineering applications.

### Example 3.3.2

Consider a coupled system (2.1) with $z_1 = z_2 = 1$:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{1}{x}(x u_x)_x - \frac{1}{x}(x u_x)_{xt} + z_1 v + u_t = f(x,t), \\ \frac{\partial v}{\partial t} - \frac{1}{x}(x v_x)_x - \frac{1}{x}(x v_x)_{xt} + z_2 u = g(x,t), \end{cases} \tag{3.34}$$

where $f$ and $g$ are defined by:

$$\begin{aligned} f(x,t) &= \ln x\left[t^3 + 3t^2 + 6t\right], \\ g(x,t) &= \ln x\left[t^3 + 2t\right], \end{aligned} \tag{3.35}$$

associated with the following initial conditions:

$$\begin{aligned} \phi_1 &= u(x,0) = 0, \\ \phi_2 &= u_t(x,0) = 0, \\ \psi_1 &= v(x,0) = 0. \end{aligned} \tag{3.36}$$

**Solution 2**

Let $u_0(x,t)=u(x,0)=0$ and $v_0(x,t)=v(x,0)=0$. In view of (3.20), by setting $n=1$ and $\hbar_1=\hbar_2=\hbar$, the first term of the series solution is given by:

$$\begin{aligned}
u_1(x,t) &= \chi_1 u_0(x,t)+\hbar\mathscr{L}^{-1}\Big[\Re_1(\vec{u}_0,\vec{v}_0)\Big],\\
&= \chi_1 u_0+\hbar\mathscr{L}^{-1}\Bigg[\mathscr{L}u_0-(1-\chi_1)\Big(\frac{1}{s}\phi_1+\frac{1}{s^2}\phi_2\Big)\\
&\quad -\frac{1}{s^2}\mathscr{L}\Big\{\frac{1}{x}(xu_{0x})_x+\frac{1}{x}(xu_{0x})_{xt}\quad -z_1v_0-u_{0t}+(1-\chi_1)F\Big\}\Bigg],\\
&= \hbar\mathscr{L}^{-1}\Big[-\frac{1}{s^2}\mathscr{L}(F)\Big]=\hbar\ln x\mathscr{L}^{-1}\Big[-\frac{3!}{s^6}-3\frac{2!}{s^5}-\frac{6}{s^4}\Big],\\
&= -\hbar\ln x\Big[3!\frac{t^5}{5!}+3\frac{2!}{4!}t^4+\frac{6t^3}{3!}\Big],\\
&= -\hbar\ln x\Big(\frac{t^5}{20}+\frac{t^4}{4}+t^3\Big).
\end{aligned} \tag{3.37}$$

We now proceed to compute $v_1(x,t)$:

$$\begin{aligned}
v_1(x,t) &= \chi_1 v_0(x,t)+\hbar\mathscr{L}^{-1}\Bigg[\mathscr{L}(v_0)-(1-\chi_1)(\frac{1}{s}\psi_1)\\
&\quad -\frac{1}{s}\mathscr{L}\Big\{\frac{1}{x}(xv_{0x})_x+\frac{1}{x}(xv_{0x})_{xt}-z_2u_0+(1-\chi_1)G\Big\}\Bigg],\\
&= -\hbar\mathscr{L}^{-1}\Big[\frac{1}{s}\mathscr{L}(G)\Big]=-\hbar\ln x\mathscr{L}^{-1}\Big[\frac{3!}{s^5}+\frac{2}{s^3}\Big],\\
&= -\hbar\ln x\Big(\frac{t^4}{4}+t^2\Big).
\end{aligned} \tag{3.38}$$

The next step is to compute $u_2(x,t)$

$$\begin{aligned}
u_2(x,t) &= \chi_2 u_1(x,t)+\hbar\mathscr{L}^{-1}\Bigg[\mathscr{L}(u_1)-(1-\chi_2)(\frac{1}{s}\phi_1+\frac{1}{s^2}\phi_2)\\
&\quad -\frac{1}{s^2}\mathscr{L}\Big\{\frac{1}{x}(xu_{1x})_x+\frac{1}{x}(xu_{1x})_{xt}-z_1v_1-(u_1)_t+(1-\chi_2)F\Big\}\Bigg],\\
&= u_1+\hbar u_1-\hbar\mathscr{L}^{-1}\Big[\frac{1}{s^2}\mathscr{L}\Big\{-v_1-u_{1t}\Big\}\Big],\\
&= u_1(1+\hbar)+\hbar\mathscr{L}^{-1}\Big[\frac{1}{s^2}\mathscr{L}\Big\{-\hbar\ln x(\frac{t^4}{4}+t^2)-\hbar\ln x(\frac{t^4}{4}+t^3+3t^2)\Big\}\Big],\\
&= u_1(1+\hbar)-\hbar^2\ln x\mathscr{L}^{-1}\Big[\frac{1}{s^2}\mathscr{L}\Big\{\frac{t^4}{2}+4t^2+t^3\Big\}\Big],\\
&= u_1(1+\hbar)-\hbar^2\ln x\mathscr{L}^{-1}\Big[\frac{1}{2}\frac{4!}{s^7}+4\frac{2!}{s^5}+\frac{3!}{s^6}\Big],\\
&= u_1(1+\hbar)-\hbar^2\ln x\Big(\frac{t^6}{60}+\frac{t^4}{3}+\frac{t^5}{20}\Big).
\end{aligned} \tag{3.39}$$

Now, we calculate $v_2(x,t)$:

$$\begin{aligned}
v_2(x,t) &= \chi_2 v_1(x,t)+\hbar\mathscr{L}^{-1}\Big[\mathscr{L}(v_1)-(1-\chi_2)(\frac{1}{s}\psi_1)\\
&\quad -\frac{1}{s}\mathscr{L}\Big\{\frac{1}{x}(x v_{1x})_x+\frac{1}{x}(x v_{1x})_{xt}-u_1+(1-\chi_2)G\Big\}\Big],\\
&= v_1+\hbar v_1-\hbar\mathscr{L}^{-1}\Big[\frac{1}{s}\mathscr{L}\big\{-u_1(x,t)\big\}\Big],\\
&= v_1(1+\hbar)-\hbar^2\ln x\,\mathscr{L}^{-1}\Big[\frac{1}{s}\mathscr{L}\Big\{\frac{t^5}{20}+\frac{t^4}{4}+t^3\Big\}\Big],\\
&= v_1(1+\hbar)-\hbar^2\ln x\,\mathscr{L}^{-1}\Big[\frac{1}{20}.5!\frac{t^6}{6!}+\frac{1}{4}.4!\frac{t^5}{5!}+3!\frac{t^4}{4!}\Big],\\
&= v_1(1+\hbar)-\hbar^2\ln x\Big(\frac{t^6}{120}+\frac{t^5}{20}+\frac{t^4}{4}\Big).
\end{aligned}\tag{3.40}$$

We now determine the value of $u_3(x,t)$:

$$\begin{aligned}
u_3(x,t) &= \chi_3 u_2(x,t)+\hbar_1\mathscr{L}^{-1}\Big[\mathscr{L}u_2-(1-\chi_3)(\frac{1}{s}\phi_1(x)+\frac{1}{s^2}\phi_2(x))\\
&\quad -\frac{1}{s^2}\mathscr{L}\Big\{\frac{1}{x}(x u_{2x})_x+\frac{1}{x}(x u_{2x})_{xt}-z_1 v_2-(u_2)_t+(1-\chi_3)F\Big\}\Big],\\
&= u_2(1+\hbar)+\hbar\mathscr{L}^{-1}\Big[\frac{1}{s^2}\mathscr{L}\Big\{-v_2-(u_2)_t\Big\}\Big],\\
&= u_2(1+\hbar)-\hbar^3\ln x\,\mathscr{L}^{-1}\Big[\frac{1}{s^2}\mathscr{L}\Big\{\frac{t^6}{120}+\frac{3t^5}{20}+\frac{t^4}{2}+\frac{4t^3}{3}\Big\}\Big],\\
&= u_2(1+\hbar)-\hbar^3\ln x\,\mathscr{L}^{-1}\Big[\frac{6!}{120}.\frac{1}{s^9}+\frac{3.5!}{20}.\frac{1}{s^8}+\frac{4!}{2}.\frac{1}{s^7}+\frac{4.3!}{3}.\frac{1}{s^6}\Big],\\
&= u_2(1+\hbar)-\hbar^3\ln x\Big[\frac{6!}{120}.\frac{t^8}{8!}+\frac{3.5!}{20}.\frac{t^7}{7!}+\frac{4!}{2}.\frac{t^6}{6!}+\frac{4.3!}{3}.\frac{t^5}{5!}\Big],\\
&= u_2(1+\hbar)-\hbar^3\ln x\Big[\frac{t^8}{6720}+\frac{t^7}{280}+\frac{t^6}{60}+\frac{t^5}{15}\Big].
\end{aligned}\tag{3.41}$$

Next, we determine $v_3(x,t)$:

$$
\begin{aligned}
v_3(x,t) &= \chi_3 v_2 + \hbar \mathscr{L}^{-1}\Bigg[\mathscr{L}(v_2) - (1-\chi_3)(\frac{1}{s}\psi_1(x)) \\
&\quad - \frac{1}{s}\mathscr{L}\left\{\frac{1}{x}(x v_{2x})_x + \frac{1}{x}(x v_{2x})_{xt} - u_2 + (1-\chi_3)G\right\}\Bigg], \\
&= v_2(1+\hbar) - \hbar\mathscr{L}^{-1}\Big[\frac{1}{s}\mathscr{L}\Big(-u_2\Big)\Big], \\
&= v_2(1+\hbar) - \hbar^3 \ln x \mathscr{L}^{-1}\left[\frac{1}{s}\mathscr{L}\Big(\frac{t^6}{60} + \frac{t^5}{20} + \frac{t^4}{3}\Big)\right], \\
&= v_2(1+\hbar) - \hbar^3 \ln x \mathscr{L}^{-1}\left[\frac{6!}{60}.\frac{1}{s^8} + \frac{5!}{20}.\frac{1}{s^7} + \frac{4!}{3}.\frac{1}{s^6}\right], \\
&= v_2(1+\hbar) - \hbar^3 \ln x\Big(\frac{6!}{60}\frac{t^7}{7!} + \frac{5!}{20}\frac{t^6}{6!} + \frac{4!}{3}\frac{t^5}{5!}\Big), \\
&= v_2(1+\hbar) - \hbar^3 \ln x\Big(\frac{t^7}{420} + \frac{t^6}{120} + \frac{t^5}{15}\Big).
\end{aligned}
\tag{3.42}
$$

We now determine the value of $u_4(x,t)$:

$$
\begin{aligned}
u_4 &= \chi_4 u_3 + \hbar\mathscr{L}^{-1}\Big\{\mathscr{L} u_3 - (1-\chi_4)(\frac{1}{s}\phi_1(x) + \frac{1}{s^2}\phi_2(x))\Big\} \\
&\quad - \frac{1}{s^2}\mathscr{L}\left\{\frac{1}{x}(x u_{3x})_x + \frac{1}{x}(x u_{3x})_{xt} - z_1 v_3 - (u_3)_t + (1-\chi_3)F\right\}, \\
&= u_3(1+\hbar) - \hbar\mathscr{L}^{-1}\left[\frac{1}{s^2}\mathscr{L}\Big(-v_3 - (u_3)_t\Big)\right], \\
&= u_3(1+\hbar) - \hbar^4 \ln x \mathscr{L}^{-1}\left\{\frac{1}{s^2}\mathscr{L}\left[\frac{t^7}{280} + \frac{t^6}{30} + \frac{t^5}{6} + \frac{t^4}{3}\right]\right\}, \\
&= u_3(1+\hbar) - \hbar^4 \ln x \mathscr{L}^{-1}\left[\frac{7!}{280}\frac{1}{s^{10}} + \frac{6!}{30}\frac{1}{s^9} + \frac{5!}{6}\frac{1}{s^8} + \frac{4!}{3}\frac{1}{s^7}\right], \\
&= u_3(1+\hbar) - \hbar^4 \ln x\left[\frac{7!}{280}\frac{t^9}{9!} + \frac{6!}{30}\frac{t^8}{8!} + \frac{5!}{6}\frac{t^7}{7!} + \frac{4!}{3}\frac{t^6}{6!}\right], \\
&= u_3(1+\hbar) - \hbar^4 \ln x\Big(\frac{t^9}{20160} + \frac{t^8}{1680} + \frac{t^7}{252} + \frac{t^6}{90}\Big).
\end{aligned}
\tag{3.43}
$$

Next, we determine $v_4(x,t)$:

$$
\begin{aligned}
v_4 &= \chi_4 v_3 + \hbar \mathscr{L}^{-1}\Big[\mathscr{L} v_3 - (1-\chi_4)(\psi_1(x)) \\
&\quad - \frac{1}{s}\mathscr{L}\Big\{\frac{1}{x}(x v_{3x})_x + \frac{1}{x}(x v_{3x})_{xt} - u_3 + (1-\chi_4)G\Big\}\Big], \\
&= v_3(1+\hbar) + \hbar \mathscr{L}^{-1}\Big[\frac{1}{s}\mathscr{L}\Big(u_3\Big)\Big], \\
&= v_3(1+\hbar) - \hbar^4 \ln x \mathscr{L}^{-1}\Big[\frac{1}{s}\mathscr{L}\Big\{\frac{t^8}{6720} + \frac{t^7}{280} + \frac{t^6}{60} + \frac{t^5}{15}\Big\}\Big], \\
&= v_3(1+\hbar) - \hbar^4 \ln x \mathscr{L}^{-1}\Big[\frac{8!}{6720}\frac{1}{s^{10}} + \frac{7!}{280}\frac{1}{s^9} + \frac{6!}{60}\frac{1}{s^8} + \frac{5!}{15}\frac{1}{s^7}\Big], \\
&= v_3(1+\hbar) - \hbar^4 \ln x \Big[\frac{8!}{6720}\frac{t^9}{9!} + \frac{7!}{280}\frac{t^8}{8!} + \frac{6!}{60}\frac{t^7}{7!} + \frac{5!}{15}\frac{t^6}{6!}\Big], \\
&= v_3(1+\hbar) - \hbar^4 \ln x \Big[\frac{t^9}{60480} + \frac{t^8}{2240} + \frac{t^7}{420} + \frac{t^6}{90}\Big].
\end{aligned}
\tag{3.44}
$$

If we set $\hbar = -1$ in the previously calculated terms, we obtain:

$$
\begin{aligned}
u_0(x;t) &= u(x;t) = 0, \\
v_0(x;t) &= v(x;t) = 0, \\
u_1(x;t) &= \ln x(\frac{t^5}{20} + \frac{t^4}{4} + t^3), \\
v_1(x;t) &= \ln x(\frac{t^4}{4} + t^2), \\
u_2(x;t) &= \ln x(-\frac{t^6}{60} - \frac{t^4}{3} - \frac{t^5}{20}), \\
v_2(x;t) &= \ln x(-\frac{t^6}{120} - \frac{t^5}{20} - \frac{t^4}{4}), \\
u_3(x;t) &= \ln x(\frac{t^8}{6720} + \frac{t^7}{280} + \frac{t^6}{60} + \frac{t^5}{15}), \\
v_3(x;t) &= \ln x(\frac{t^7}{420} + \frac{t^6}{120} + \frac{t^5}{15}), \\
u_4(x;t) &= \ln x(-\frac{t^9}{20160} - \frac{t^8}{1680} - \frac{t^7}{252} - \frac{t^6}{90}), \\
v_4(x;t) &= \ln x(\frac{t^9}{60480} - \frac{t^8}{2240} - \frac{t^7}{420} - \frac{t^6}{90}).
\end{aligned}
\tag{3.45}
$$

Hence, the solution of the coupled system is given by:

$$
\begin{cases}
u(x,t) = & u_0(x;t) + \sum_{i=1}^{\infty} u_i(x;t), \\
v(x,t) = & v_0(x;t) + \sum_{i=1}^{\infty} v_i(x;t),
\end{cases}
\tag{3.46}
$$

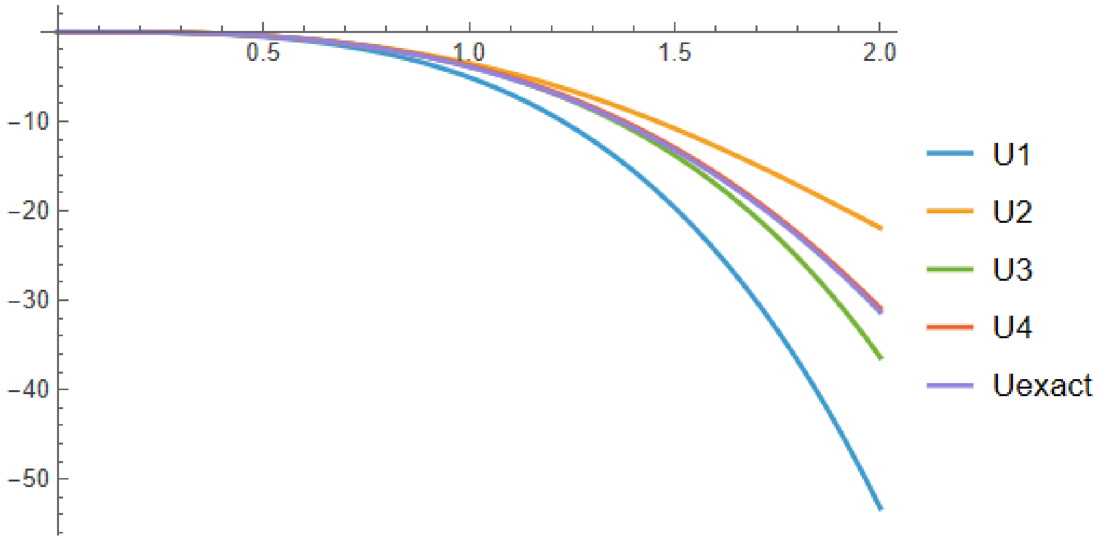
$$
\begin{cases}
u(x;t) = u_0 + u_1 + u_2 + u_3 + u_4 + .., \\
v(x;t) = v_0 + v_1 + v_2 + v_3 + v_4 + ...
\end{cases}
\tag{3.47}
$$

The series (3.46) converges to the following analytical function (exact solution):

$$\begin{cases} u(x;t) = t^3 . \ln x. \\ v(x;t) = t^2 . \ln x, \end{cases} \tag{3.48}$$

Out[424]= 0.02

Out[435]=




Out[571]= 0.02

Out[582]=

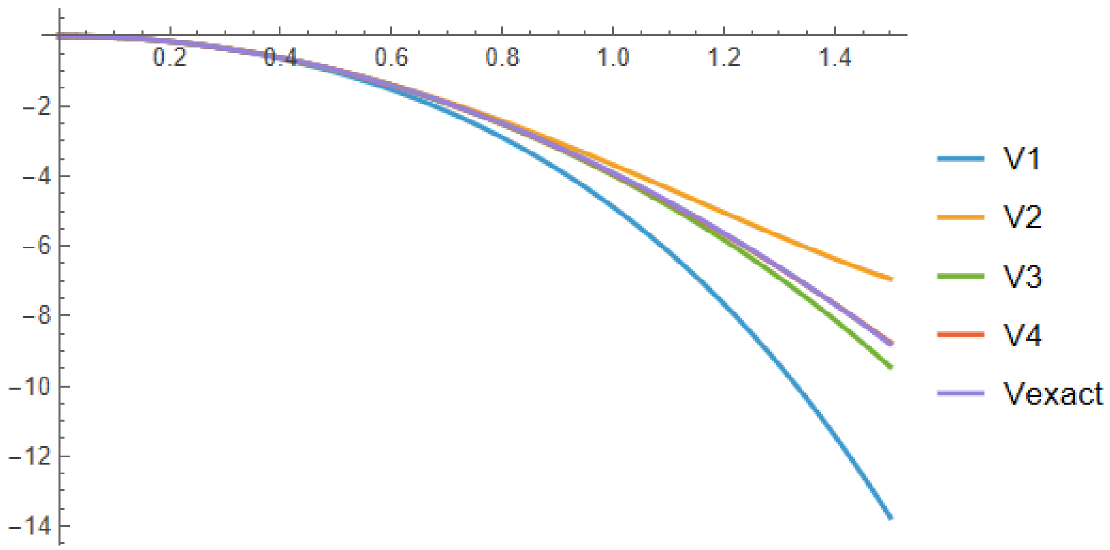



**Figure 3.2:**
shows the convergence of numerical approximations to the analytical solutions of $Uexact$ and $Vexact$ under varying $m$ at $x = 0.02$, with $m = 1,2,3,\ h = -1$.

Figure 3.2 shows the graph of the truncated series solution for different numbers of terms in Example 1 at $x = 0.02$ and $\bar{h} = -1$. The figure illustrates the rapid convergence of these approximate solutions.

In Table 3.2, we present the numerical solutions of Example 1 obtained from the $k$th-order truncated series solutions $U_k = u_0 + u_1 + u_2 + ... + u_k$ and $V_k = v_0 + v_1 + v_2 + ... + v_k$ generated by the HATM for several values of $k$, $x$, and $t$, rounded to six significant digits. These results illustrate the rapid convergence of the truncated solutions after only a few terms. As shown in the table, both methods demonstrate good performance for this example.

| $t$ | $x$ | $m$ | $(U_{(m)}, V_{(m)})$ | $(U_{exact}, V_{exact})$ | $Error$ |
|---|---|---|---|---|---|
| 0.4 | 0.09 | 1 | -0.170752 | -0.154109 | 0.0166437 |
| | | | -0.400682 | -0.385271 | 0.0154109 |
| | | 2 | -0.148807 | -0.154109 | 0.00530133 |
| | | | -0.383956 | -0.385271 | 0.00131506 |
| | | 3 | -0.15063 | -0.154109 | 0.0034788 |
| | | | -0.385692 | -0.385271 | 0.000420349 |
| 0.05 | 0.02 | 1 | -0.000495177 | -0.000489003 | $6.17366 \times 10^{-6}$ |
| | | | -0.00978617 | -0.00978006 | $6.11254 \times 10^{-6}$ |
| | | 2 | -0.000486964 | -0.000489003 | $2.03853 \times 10^{-6}$ |
| | | | -0.00978 | -0.00978006 | $6.16347 \times 10^{-8}$ |
| | | 3 | -0.000487047 | -0.000489003 | $1.956 \times 10^{-6}$ |
| | | | -0.00978008 | -0.00978006 | $2.03824 \times 10^{-8}$ |

**Table 3.2:** Comparison between approximate and exact solutions for parameters $h = -1$, $m = 1, 2, 3$.

The numerical results confirm that the proposed algorithm is both efficient and strongly convergent toward the exact solution as the approximation order $m$ increases. High accuracy is achieved at low orders, with errors dropping below $10^{-6}$ already at $m = 3$. This demonstrates the robustness of the method and its excellent balance between computational cost and numerical precision, making it suitable for efficient simulation of applied problems.

**Remark 1**

The detailed tables and graphical illustrations are presented in the thesis. All numerical results and visualizations were generated using **Mathematica (Wolfram, Version 14.3)**.

# Conclusion

This thesis has focused on the analysis of a nonlinear pseudo-hyperbolic system subject to both classical and nonlocal integral boundary conditions. The associated linear problem was first reformulated, and the existence and uniqueness of strong solutions were rigorously established in an appropriate Sobolev space. A priori bounds for the solution were derived, from which uniqueness directly follows. By employing density arguments, the solvability of the linear problem was fully justified. Building on these foundational results, the well-posedness of the nonlinear problem was investigated through an iterative process. This iterative approach proved particularly effective in managing the complexities introduced by nonlinearity, where small variations in the input can cause significant differences in the output.

The theoretical analysis relied heavily on the energy inequality method from functional analysis. It is worth noting that the application of this method becomes challenging in the presence of nonlinear source terms, singularities, and nonlocal integral conditions. Nevertheless, careful handling of these difficulties allowed us to obtain rigorous existence, uniqueness, and continuous dependence results for the weak solution of the nonlinear system.

In the numerical part of this work, an iterative scheme based on the Homotopy Analysis Transform Method (HATM) was developed to approximate the solution of the singular nonlinear pseudo-hyperbolic coupled system. Two representative examples were provided to illustrate the accuracy and efficiency of the scheme. Error analysis was conducted for the computed solutions, with results displayed both graphically and numerically. The plots of the truncated series solutions demonstrate convergence to the exact solution as the number of terms increases, indicating stability of the scheme and a systematic reduction of error. Numerical results reported in Tables further confirm that the absolute error decreases as the number of terms increases, demonstrating the reliability and precision of the HATM. Notably, the method achieves optimal performance for $h = -1$.

In summary, this research confirms that the HATM is an efficient, reliable, and easily implementable method for solving coupled nonlinear pseudo-hyperbolic systems with singularities and nonlocal boundary conditions, without requiring restrictive assumptions. The combination of rigorous theoretical analysis and robust numerical validation underscores the applicability and effectiveness of the approaches developed in this dissertation, providing a solid foundation for future research in the study of complex nonlinear differential systems.

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