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Dedication

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*To myself, because I did not give up.
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To the one who was judged, underestimated, or sometimes betrayed, but who chose to transform every wound into ambition.*

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Metchat Djihad, 2026

Abbreviations and Notations

The different abbreviations and notations used throughout this dissertation are explained below.

Probabilistic Notation

$\mathbb{E}[X]$	Expectation of X .
$\mathbb{E}[X \mid \mathcal{F}]$	Conditional expectation of X given the σ -field $\mathcal{F}$.
$\text{Var}(X)$	Variance of the random variable X .
$(\Omega, \mathcal{F}, P)$	Probability space.
Ω	Fundamental space of a random experiment.
P	Probability measure.
$\mathcal{F}_t$	Filtration at time t .

Abbreviations

BSDEs	Backward stochastic differential equations.
a.s.	Almost surely.
a.e.	Almost everywhere.
i.e.	That is .
r.v.	Random variable.

Stochastic Processes and Mathematical Symbols

B	Brownian motion (also denoted W).
$\mathbf{1}_B$	Indicator function of the set B .
$\lim$	Limit.
$\lim^+$	Limit from the right.
$\inf$	Infimum.
$\sup$	Supremum.
$\min$	Minimum.
$\max$	Maximum.

Contents

Acknowledgements	i
Dedication	ii
Abbreviations and Notations	iii
Introduction	1
1 Stochastic Analysis	3
1.1 Processes and Martingales	3
1.1.1 Processes	3
1.1.2 L^p Spaces	3
1.1.3 Filtration	4
1.1.4 Martingales	5
1.2 Gaussian Processes	7
1.3 Brownian Motion	7
1.3.1 Properties of Brownian Motion	9
1.4 Total Variation and Quadratic Variation	10
1.5 Stochastic Integration	14
1.6 Itô's Formula	17
1.7 Itô Processes	17
2 Backward Doubly Stochastic Differential Equations	19
2.1 Setting and Notation	19
2.2 The BDSDE under Lipschitz Conditions	20
2.2.1 Extension of Itô's Formula	21
2.3 Uniqueness and Existence of Solutions: Proof of Theorem 2.1	22
3 BDSDEs with Locally Monotone Coefficients	24
3.1 BDSDEs with Locally Monotone Coefficients	24
3.2 Proof of Theorem 3.1	28

Conclusion	31
A Some Mathematical Tools	32
Abstract	36

Introduction

Backward doubly stochastic differential equations (BDSDEs) currently occupy a prominent place in the field of stochastic analysis. The origin of these equations traces back to the pioneering work of Bismut [5], who introduced linear BSDEs within the framework of the stochastic maximum principle. However, it was not until the seminal work of Pardoux and Peng [9] that nonlinear BSDEs theory truly began to flourish under a Lipschitz condition. Subsequently, Pardoux and Peng [10] extended this framework to backward doubly stochastic differential equations (BDSDEs), which incorporate both a forward integral with respect to a Brownian motion and a backward integral with respect to an independent backward martingale measure. This doubly nature makes BDSDEs particularly suitable for applications in stochastic control and partial differential equations.

Interest in these equations has grown considerably due to their concrete applications, particularly in stochastic partial differential equations (SPDEs) and mathematical finance. As demonstrated by Bally and Matoussi [4] and others, BDSDEs provide a natural probabilistic representation for solutions of semi-linear SPDEs, thereby establishing a powerful connection between probability theory and partial differential equations.

A nonlinear BDSDE is defined as follows: let ξ be a square-integrable $\mathcal{F}_T$ -measurable random variable, and let $f(t, \omega, Y, Z)$ be a progressively measurable function. Then, the associated BDSDE is written as:

$$\begin{cases} dY_t = -f(t, Y_t, Z_t) dt + Z_t dB_t + Z_t d\overleftarrow{B}_t, & t \in [0, T], \\ Y_T = \xi, \end{cases} \quad (1)$$

where $(Y_t, Z_t)_{t \in [0, T]}$ represents the adapted solution, f is called the generator (or driver), ξ is the terminal condition, B_t is a standard Brownian motion, and $\overleftarrow{B}_t$ denotes an independent backward Brownian motion. The presence of the backward integral $Z_t d\overleftarrow{B}_t$ introduces additional analytical challenges compared to classical BSDEs.

In the classical framework established by Pardoux and Peng [10], the well-posedness of BDSDEs relies on the assumption that the generator f is uniformly Lipschitz continuous in (y, z) . Consequently, research has focused on relaxing these hypotheses,

moving from the global Lipschitz condition to conditions such as local Lipschitz, linear growth, and later to monotonicity conditions. In this direction, Mao [8] proposed weaker conditions based on a non-Lipschitz approach. More recently, efforts have been directed toward introducing **local monotonicity** conditions, which allow the generator to behave poorly for large values of y while remaining well-behaved locally. This generalization encompasses both classical monotonicity and Mao's condition, thereby allowing for the treatment of a broader class of equations with coefficients that are monotone only in a local sense.

The main objective of this thesis is to study the solvability of BDSDEs when the generator satisfies this **local monotonicity** condition. We structure our work around three main axes:

- **Chapter 1:** This introductory part lays the necessary theoretical foundations. We recall key concepts of stochastic analysis, ranging from martingales and Brownian motion to stochastic differential equations, without omitting the essential tools of Itô calculus.
- **Chapter 2:** We dedicate this chapter to the study of the standard framework. We present the classical existence and uniqueness results obtained by Pardoux and Peng [10] for BDSDEs when the coefficients of the equation are globally Lipschitz, thus providing a benchmark for subsequent extensions.
- **Chapter 3:** This final chapter constitutes the core of our contribution. We analyze BDSDEs under **local monotonicity**. Building on techniques such as a priori estimates, approximation procedures, and Gronwall's inequality, we demonstrate that the existence and uniqueness of solutions remain valid under this less restrictive condition, thereby extending the classical results to a significantly broader class of equations with locally monotone coefficients.

Chapter 1

Stochastic Analysis

In all that follows, we work on a complete probability space $(\Omega, \mathcal{A}, P)$.

1.1 Processes and Martingales

1.1.1 Processes

Definition 1.1. A (random) process on the probability space $(\Omega, \mathcal{A}, P)$ is a family of random variables $(X_t)_{t \in [0, T]}$. It is therefore a function of two variables:

$$X: [0, T] \times \Omega \rightarrow \mathbb{R}.$$

Remark 1.1. The variable $t \in [0, T]$ represents time, but one could also consider $t \in \mathbb{R}$ or $t \in \mathbb{R}^2$. Similarly, the arrival space could be much more complex than $\mathbb{R}$.

Remark 1.2. We can view a process as a function that associates to each $\omega \in \Omega$ a function from $[0, T]$ to $\mathbb{R}$, $t \mapsto X_t(\omega)$, called the trajectory of the process.

Definition 1.2. We say that X is a continuous process (a.s.) if it has continuous trajectories, i.e. $t \mapsto X_t(\omega)$ is in C^0 for almost all $\omega \in \Omega$.

1.1.2 L^p Spaces

Definition 1.3. For all $p \in [1, \infty)$, we denote:

$$L^p := \{X \text{ r.v. such that } \|X\|_{L^p} := (\mathbb{E}[|X|^p])^{1/p} < \infty\},$$

$$\mathcal{P}_{[0, T]}^p := \{(\phi_s)_{0 \leq s \leq T} \text{ process, } [0, T] \times \Omega\text{-measurable, such that } \|\phi\|_{\mathcal{P}^p} := \left(\mathbb{E} \left[\int_0^T |\phi_s|^p ds \right] \right)^{1/p} < \infty\}.$$

1.1.3 Filtration

Definition 1.4. A filtration $(\mathcal{F}_t)_{t \in [0, T]}$ is an increasing collection of sub- σ -algebras of $\mathcal{F}$; i.e.

$$\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F},$$

for all $s, t \in [0, T]$ such that $s \leq t$.

Remark 1.3. $\mathcal{F}_t$ represents the amount of information available at time t ; it is therefore logical that this amount increases with time.

Definition 1.5. A process $(X_t)_{t \in [0, T]}$ is said to be $\mathcal{F}$ -adapted if the random variable X_t is $\mathcal{F}_t$ -measurable for all $t \in [0, T]$.

Proposition 1.1. For $p \geq 1$, the spaces L^p equipped with $\|\cdot\|_{L^p}$ and $\mathcal{P}_{[0, T]}^p$ equipped with $\|\cdot\|_{\mathcal{P}^p}$ are Banach spaces (i.e. complete).

Proof. We will skip the proof; the space L^p is complete, and $\mathcal{P}_{[0, T]}^p$ is likewise complete for the same reasons: instead of considering functions of a single variable, it suffices to consider functions of two variables $(t; \omega)$. $\square$

Proposition 1.2. If X is $\mathcal{F}$ -adapted, then the random variable X_s is $\mathcal{F}_t$ -measurable for all $s \leq t$ and all $t \in [0, T]$.

Proof. This result is natural when reasoning in terms of information: the σ -algebra $\mathcal{F}_t$ is larger than the σ -algebra $\mathcal{F}_s$, so if X_s is known given the information $\mathcal{F}_s$, then it is known given the information $\mathcal{F}_t$. Analytically, let B be a Borel set of $\mathbb{R}$. Then the preimage of B under X_s , namely $X_s^{-1}(B)$, is an element of $\mathcal{F}_s$ and is therefore an element of $\mathcal{F}_t \supseteq \mathcal{F}_s$. $\square$

Definition 1.6. The filtration generated by a process X , denoted by $\mathcal{F}^X$, is the increasing sequence of σ -algebras $(\mathcal{F}_t^X)_{t \in [0, T]}$ generated by $(X_t)_{t \in [0, T]}$, i.e.

$$\mathcal{F}_t^X = \sigma(X_s, s \leq t),$$

for all $t \in [0, T]$.

Remark 1.4. $\mathcal{F}^X$ is the smallest filtration that makes X adapted. A σ -algebra is said to be complete when it contains the collection of negligible sets of $(\Omega, \mathcal{F}, P)$, which is defined by:

$$\mathcal{N} := \{N \subseteq \Omega : \exists A \in \mathcal{F}, N \subseteq A, P(A) = 0\}.$$

Definition 1.7. A filtration $(\mathcal{F}_t)_{t \in [0, T]}$ is complete when $\mathcal{F}_t$ is complete for every $t \in [0, T]$ (which is equivalent to $\mathcal{F}_0$ being complete).

Remark 1.5. To complete a filtration it suffices to replace $\mathcal{F}_t$ by $\sigma(\mathcal{F}_t \cup \mathcal{N})$ for every $t \in [0, T]$. The completed filtration generated by a process X is generally called the natural filtration of the process. If the filtration is complete and X and Y are two processes, then for fixed t we have

$$X_t = Y_t \text{ a.s.} \implies X_t \text{ is } \mathcal{F}_t\text{-measurable} \Leftrightarrow Y_t \text{ is } \mathcal{F}_t\text{-measurable.}$$

It then follows that, if the filtration is complete and $(X^n)_{n \geq 0}$ is a sequence of processes, then for fixed t :

$$X_t^n \xrightarrow{\text{a.s.}} X_t \quad \text{and} \quad X_t^n \text{ is } \mathcal{F}_t\text{-measurable for all } n \geq 0, \implies X_t \text{ is } \mathcal{F}_t\text{-measurable.}$$

This result remains valid for convergence in L^p with $p \geq 1$. In this case, one can extract a subsequence that converges almost surely to the same limit; thus, the limit remains measurable:

$$X_t^n \xrightarrow{P} X_t \quad \text{and} \quad \forall n, X_t^n \text{ is } \mathcal{F}_t\text{-measurable} \implies X_t \text{ is } \mathcal{F}_t\text{-measurable.}$$

1.1.4 Martingales

Definition 1.8. A stochastic process $(M_t)_{t \in [0, T]}$ is an $\mathcal{F}$ -martingale (resp. $\mathcal{F}$ -supermartingale and $\mathcal{F}$ -submartingale) if:

- (i) M is $\mathcal{F}$ -adapted;
- (ii) $M_t \in L^1$, i.e. $\mathbb{E}[|M_t|] < \infty$, for all $t \in [0, T]$;
- (iii) $\mathbb{E}[M_t | \mathcal{F}_s] = M_s$ for all $s, t \in [0, T]$ such that $s \leq t$.

A process is a supermartingale (resp. a submartingale) if it satisfies properties (i) and (ii), and $\mathbb{E}[M_t | \mathcal{F}_s] \leq M_s$ (resp. $\mathbb{E}[M_t | \mathcal{F}_s] \geq M_s$) for all $s, t \in [0, T]$ such that $s \leq t$.

Remark 1.6. A martingale represents a fair game, a supermartingale a losing game, and a submartingale a winning game.

Proposition 1.3. Every martingale satisfies:

$$\mathbb{E}[M_t] = \mathbb{E}[M_0],$$

for all $t \in [0, T]$.

Proof. We have $\mathbb{E}[M_t] = \mathbb{E}[\mathbb{E}[M_t | \mathcal{F}_0]] = \mathbb{E}[M_0]$, for all $t \in [0, T]$. □

Proposition 1.4. Let M be an $\mathcal{F}$ -martingale and $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be a measurable convex function. If $\varphi(M_t) \in L^1$ for all $t \in [0, T]$, then $(\varphi(M_t))_{t \in [0, T]}$ is a submartingale.

Proof. $\varphi(M)$ is integrable and adapted, and the conditional Jensen's inequality yields:

$$\mathbb{E}[\varphi(M_t) \mid \mathcal{F}_s] \geq \varphi(\mathbb{E}[M_t \mid \mathcal{F}_s]) = \varphi(M_s),$$

for all $s, t \in [0, T]$ such that $s \leq t$. $\square$

Proposition 1.5. *Let M be a square-integrable $\mathcal{F}$ -martingale (i.e. $\mathbb{E}[|M_t|^2] < \infty$ for all $t \in [0, T]$), then*

$$\mathbb{E}[|M_t - M_s|^2 \mid \mathcal{F}_s] = \mathbb{E}[M_t^2 \mid \mathcal{F}_s] - M_s^2.$$

Proof. For every $t \in [0, T]$:

$$\begin{aligned} \mathbb{E}[|M_t - M_s|^2 \mid \mathcal{F}_s] &= \mathbb{E}[M_t^2 \mid \mathcal{F}_s] - 2M_s\mathbb{E}[M_t \mid \mathcal{F}_s] + \mathbb{E}[M_s^2 \mid \mathcal{F}_s] \\ &= \mathbb{E}[M_t^2 \mid \mathcal{F}_s] - 2M_s^2 + M_s^2 \\ &= \mathbb{E}[M_t^2 \mid \mathcal{F}_s] - M_s^2. \end{aligned}$$

Hence, it follows that $|M|^2$ is a submartingale. $\square$

Proposition 1.6 (L^p Stability of Martingales). *Let $p \geq 1$ and $(M^n)_{n \geq 0}$ be a sequence of $\mathcal{F}$ -martingales such that $M_t^n \in L^p$ for all $n \geq 0$ and $t \in [0, T]$. If the random variable M_t^n converges to M_t in L^p for every $t \in [0, T]$, then the limiting process M is a martingale and $M_t \in L^p$ for all $t \in [0, T]$.*

Proof. It is necessary to demonstrate three properties:

1. M is adapted;
2. For every $t \in [0, T]$, $M_t \in L^1$;
3. For $0 \leq s \leq t \leq T$, the martingale property holds: $\mathbb{E}[M_t \mid \mathcal{F}_s] = M_s$.

Step 1. For every n , M_t^n is $\mathcal{F}_t$ -measurable. Consequently, M_t is also $\mathcal{F}_t$ -measurable (assuming the filtration is complete).

Step 2. For any fixed t , since M_t^n converges to M_t in L^p , it follows by the properties of L^p spaces that the limit M_t belongs to L^p .

Step 3. For $0 \leq s \leq t \leq T$, we observe:

$$\|M_s^n - \mathbb{E}[M_t \mid \mathcal{F}_s]\|_{L^p} \leq \|M_t^n - M_t\|_{L^p} \xrightarrow{n \rightarrow \infty} 0.$$

Furthermore, since M_s^n converges to M_s in L^p , the term $M_s^n - \mathbb{E}[M_t \mid \mathcal{F}_s]$ converges to $M_s - \mathbb{E}[M_t \mid \mathcal{F}_s]$ in L^p . Given the limit is 0 in L^p , we conclude that $\mathbb{E}[M_t \mid \mathcal{F}_s] = M_s$ almost surely. $\square$

1.2 Gaussian Processes

Definition 1.9. A vector of random variables $(X_1, \dots, X_n)$ is a Gaussian vector if and only if every linear combination of the X_i is Gaussian; i.e. the random variable $\langle a, X \rangle$ defined by:

$$\langle a, X \rangle := \sum_{k=1}^n a_k X_k$$

is a Gaussian random variable for every $a = (a_1, \dots, a_n) \in \mathbb{R}^n$.

Proposition 1.7. If the vector (X_1, X_2) is Gaussian, the random variables X_1 and X_2 are independent if and only if $\text{cov}(X_1, X_2) = 0$.

Proof. If X_1 and X_2 are independent, then $\text{cov}(X_1, X_2) = \mathbb{E}[X_1 X_2] - \mathbb{E}[X_1]\mathbb{E}[X_2] = 0$. Conversely, if (X_1, X_2) is a Gaussian vector with $\text{cov}(X_1, X_2) = 0$, then the joint characteristic function factors: $\mathbb{E}[e^{i(u_1 X_1 + u_2 X_2)}] = \mathbb{E}[e^{iu_1 X_1}] \mathbb{E}[e^{iu_2 X_2}]$, which implies independence. $\square$

Proposition 1.8. Any vector consisting of independent Gaussian random variables is a Gaussian vector.

Proof. Let $X_1, \dots, X_n$ be independent Gaussian random variables. For any $a = (a_1, \dots, a_n) \in \mathbb{R}^n$, the linear combination $\sum_{k=1}^n a_k X_k$ is a sum of independent Gaussian random variables, hence Gaussian. Therefore $(X_1, \dots, X_n)$ is a Gaussian vector by definition. $\square$

Remark 1.7. Note that it is possible to find non-independent Gaussian random variables with zero covariance. The continuous-time version of Gaussian vectors are Gaussian processes.

Definition 1.10. A process $(X_t)_{t \in [0, T]}$ is called a Gaussian process if, for any n and any $t_1 < t_2 < \dots < t_n$ in $[0, T]$, the vector $(X_{t_1}, \dots, X_{t_n})$ is Gaussian.

1.3 Brownian Motion

Brownian motion is generally denoted by B (for Brown) or W (for Wiener).

Definition 1.11. Let $\mathcal{F}$ be a filtration. An $\mathcal{F}$ -Brownian motion (standard) is a process B satisfying:

1. B is $\mathcal{F}$ -adapted;
2. $B_0 = 0$ P -almost surely;
3. B is continuous, i.e. $t \mapsto B_t(\omega)$ is continuous for P -almost all $\omega \in \Omega$;

4. *B has independent increments: $B_t - B_s$ is independent of $\mathcal{F}_s$ for all $t, s \in [0, T]$ such that $s \leq t$;*
5. *B has stationary Gaussian increments: $B_t - B_s \sim \mathcal{N}(0, t - s)$ for all $t, s \in [0, T]$ such that $s \leq t$.*

Remark 1.8. Sometimes, when there is no ambiguity regarding the filtration to be considered, or when $\mathcal{F}$ is the natural filtration of the process B , we simply refer to it as *Brownian motion*.

Theorem 1.1 (Existence of Brownian Motion). *Brownian motion exists.*

Proof. This non-trivial result is assumed here without proof. The primary technical challenge lies in establishing the sample path continuity of Brownian motion. Brownian motion can be formally constructed as the limit of random walks as the time steps approach zero. $\square$

Proposition 1.9. *Let B be a Brownian motion and $\mathcal{F}$ its natural filtration. The following processes:*

$$B_t, \quad B_t^2 - t, \quad e^{\lambda B_t - \frac{\lambda^2}{2}t} \quad (\text{Exponential Brownian Motion})$$

are martingales.

Proof. These three processes are adapted. They are integrable because $B_t \sim \mathcal{N}(0, t)$, ensuring that its expectation, variance, and all finite moments exist. It remains to verify the martingale property (conditional expectation condition).

For the first process:

$$\mathbb{E}[B_t \mid \mathcal{F}_s] = \mathbb{E}[B_s \mid \mathcal{F}_s] + \mathbb{E}[B_t - B_s \mid \mathcal{F}_s] = B_s + (B_t - B_s) = B_s + 0 = B_s.$$

For the second process: Since B is a martingale, we utilise the independence of increments:

$$\mathbb{E}[B_t^2 - B_s^2 \mid \mathcal{F}_s] = \mathbb{E}[(B_t - B_s)^2 \mid \mathcal{F}_s] + \mathbb{E}[B_t - B_s \mid \mathcal{F}_s] = \text{Var}(B_t - B_s) = t - s.$$

Thus, $\mathbb{E}[B_t^2 - t \mid \mathcal{F}_s] = B_s^2 - s$.

For the third process: The condition is derived using the Laplace transform of a Gaussian distribution:

$$\mathbb{E}\left[e^{\lambda B_t - \frac{\lambda^2}{2}t} \mid \mathcal{F}_s\right] = e^{\lambda B_s - \frac{\lambda^2}{2}s} \cdot \mathbb{E}\left[e^{\lambda(B_t - B_s)}\right] \cdot e^{-\frac{\lambda^2}{2}(t-s)} = e^{\lambda B_s - \frac{\lambda^2}{2}s}.$$

$\square$

Theorem 1.2 (Characterization of Brownian Motion). *A process X is a Brownian motion if and only if it is a centered, continuous Gaussian process with covariance function $\text{cov}(X_s, X_t) = s \wedge t$, for $s, t \in [0, T]$.*

Proof. Forward direction: For $t_1 < t_2 < \dots < t_n$, the vector $(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}})$ consists of independent Gaussian random variables; therefore, it is Gaussian. Any linear combination of $B_{t_1}, \dots, B_{t_n}$ is a linear combination of these independent Gaussian variables and is thus also Gaussian. Consequently, the process B is Gaussian. It is continuous by hypothesis and centered since $\mathbb{E}[B_t] = \mathbb{E}[B_t - B_0] = 0$. For $s \leq t$, the covariance function is:

$$\text{cov}(B_s, B_t) = \mathbb{E}[B_s B_t] = \mathbb{E}[B_s(B_t - B_s)] + \mathbb{E}[B_s^2] = \mathbb{E}[B_s] \mathbb{E}[B_t - B_s] + \text{Var}(B_s) = 0 + s = s.$$

Converse direction: We verify the properties of the definition of BM sequentially:

- $\mathbb{E}[B_0^2] = \text{Var}(B_0) = \text{cov}(B_0, B_0) = 0$, hence $B_0 = 0$ a.s.
- B is continuous by hypothesis.
- Let $r_1 < \dots < r_n \leq s < t$. The vector $(B_{r_1}, \dots, B_{r_n}, B_t - B_s)$ is Gaussian. Since

$$\text{cov}(B_t - B_s, B_{r_i}) = r_i \wedge t - r_i \wedge s = r_i - r_i = 0,$$

it follows that $B_t - B_s$ is independent of any vector $(B_{r_1}, \dots, B_{r_n})$ and consequently, independent of the filtration $\mathcal{F}_s^B = \sigma(B_r, r \leq s)$.

- For $s \leq t$, $B_t - B_s$ is Gaussian and is therefore determined by its expectation $\mathbb{E}[B_t - B_s] = 0$ and its variance:

$$\text{Var}(B_t - B_s) = \text{Var}(B_t) + \text{Var}(B_s) - 2\text{cov}(B_t, B_s) = t + s - 2s = t - s.$$

Thus, $B_t - B_s \sim \mathcal{N}(0, t - s)$. Since the distribution of $B_t - B_s$ depends only on the difference $t - s$, the increments are stationary.

□

1.3.1 Properties of Brownian Motion

Remark 1.9. *The fact that the process $\frac{1}{a}B_{a^2t}$ remains a Brownian motion represents the fractal property of Brownian motion. Its behaviour is invariant under scaling (self-similarity). This also reflects the fact that the trajectories of Brownian motion have infinite length.*

We cite several properties of Brownian motion without proof, intended to provide a better understanding of its behaviour:

1. If B is a Brownian motion, then:

$$\frac{B_t}{t} \xrightarrow{t \rightarrow \infty} 0 \quad \text{a.s.}$$

This follows from the Law of Large Numbers, as the increments $B_i - B_{i-1}$ are independent and identically distributed (i.i.d.) with law $\mathcal{N}(0, 1)$.

2. Brownian motion has continuous trajectories, yet they are *nowhere differentiable*.
3. If B is a Brownian motion, then:

$$\limsup_{t \rightarrow \infty} B_t = +\infty \quad \text{and} \quad \liminf_{t \rightarrow \infty} B_t = -\infty.$$

Thus, Brownian motion visits every point in the space almost surely.

4. Brownian motion returns to any given point infinitely many times almost surely.
5. Brownian motion is a Markov process:

$$s \leq t, \quad B_t / \mathcal{F}_s \stackrel{d}{=} B_t / B_s.$$

In conclusion, Brownian motion exhibits extreme oscillation. It is both continuous and inherently unpredictable.

1.4 Total Variation and Quadratic Variation

Definition 1.12. Let X be a process on the interval $[0, T]$. We define the infinitesimal variation of order p associated with a partition $\pi_n = (t_1^n, \dots, t_n^n)$ of $[0, T]$ as:

$$V_T^{(p)}(\pi_n) := \sum_{i=1}^n |X_{t_i^n} - X_{t_{i-1}^n}|^p.$$

If $V_T^{(p)}(\pi_n)$ converges in a certain sense (e.g. L^2 convergence or almost sure convergence) as the mesh size

$$|\pi_n| := \max_{i \leq n} |t_{i+1}^n - t_i^n| \rightarrow 0,$$

then the limit is independent of the chosen partition. We refer to this limit as the *variation of order p of X over $[0, T]$* . Specifically:

- For $p = 1$, the limit is called the *total variation* of X over $[0, T]$.
- For $p = 2$, the limit is called the *quadratic variation* of X over $[0, T]$, denoted by $[X]_T$.

Remark 1.10. *If the total variation of a process exists almost surely, it is given by:*

$$V_T^{(1)} := \sup_{\pi} \sum_{i=1}^n |X_{t_i} - X_{t_{i-1}}| \quad a.s.,$$

where π denotes the set of all possible partitions of $[0, T]$. Conversely, if this supremum is finite, the process admits a total variation. This follows from the monotonicity of the infinitesimal variation with respect to the partition: if $\pi' \supseteq \pi$, then $V_T^{(1)}(\pi') \geq V_T^{(1)}(\pi)$, a property derived naturally from the triangle inequality

$$|x - y| \leq |x - z| + |z - y|.$$

The total variation of a process is interpreted as the path length (arc length) of its trajectories.

Proposition 1.10. *The quadratic variation of Brownian motion on $[0, T]$ exists in L^2 (i.e. the infinitesimal variation converges in the L^2 -norm) and is equal to T . Furthermore, if the sequence of partitions (π_n) satisfies the condition $\sum_{n=1}^{\infty} |\pi_n| < \infty$, the convergence holds in the almost sure sense. Thus:*

$$[B]_T = T.$$

Proof. The quadratic variation of Brownian motion is given by:

$$V_T^{(2)}(\pi_n) := \sum_{i=1}^n |B_{t_i^n} - B_{t_{i-1}^n}|^2.$$

Recall that if $X \sim \mathcal{N}(0, \sigma^2)$, then $\mathbb{E}[X^2] = \sigma^2$ and $\text{Var}(X^2) = 2\sigma^4$. Consequently:

$$\mathbb{E}[V_T^{(2)}(\pi_n)] = \sum_{i=1}^n \mathbb{E}[(B_{t_i^n} - B_{t_{i-1}^n})^2] = \sum_{i=1}^n (t_i^n - t_{i-1}^n) = T.$$

By denoting the mesh size as $|\pi_n| := \max |t_i^n - t_{i-1}^n|$, we obtain:

$$\text{Var}(V_T^{(2)}(\pi_n)) = \sum_{i=1}^n \text{Var}[(B_{t_i^n} - B_{t_{i-1}^n})^2] = 2 \sum_{i=1}^n (t_i^n - t_{i-1}^n)^2 \leq 2T|\pi_n| \xrightarrow{n \rightarrow \infty} 0.$$

Thus, $V_T^{(2)}(\pi_n) \xrightarrow{L^2} T$. To establish almost sure convergence, we employ Markov's inequality: for any $\varepsilon > 0$:

$$P(|V_T^{(2)}(\pi_n) - T| > \varepsilon) \leq \frac{\text{Var}(V_T^{(2)}(\pi_n))}{\varepsilon^2} \leq \frac{2T|\pi_n|}{\varepsilon^2}.$$

Hence, if $\sum_{i=1}^{\infty} |\pi_n| < \infty$, then for any $\varepsilon > 0$, $\sum_{n=1}^{\infty} P(|V_T^{(2)}(\pi_n) - T| > \varepsilon) < \infty$. $\square$

Proposition 1.11. *By the Borel–Cantelli Lemma, since $\sum_{n=1}^{\infty} P(|V_T^{(2)}(\pi_n) - T| > \varepsilon) < \infty$ for any $\varepsilon > 0$, it follows that $V_T^{(2)}(\pi_n) \rightarrow T$ almost surely.*

Proof. This is a direct application of the Borel–Cantelli Lemma: if $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(\limsup_n A_n) = 0$, i.e. only finitely many events A_n occur almost surely. Taking $A_n = \{|V_T^{(2)}(\pi_n) - T| > \varepsilon\}$ and using the bound $P(A_n) \leq \frac{2T|\pi_n|}{\varepsilon^2}$ with $\sum |\pi_n| < \infty$ gives the result. $\square$

Proposition 1.12. *For any sequence of partitions (π_n) satisfying $\sum_{n=1}^{\infty} |\pi_n| < \infty$, the total variation of Brownian motion on $[0, T]$ associated with these partitions converges almost surely to $+\infty$. Consequently, the total variation of Brownian motion is almost surely infinite:*

$$V_T^{(1)} := \sup_n V_T^{(1)}(\pi_n) = +\infty \quad a.s.$$

Proof. Let (π_n) be a sequence of partitions of $[0, T]$ such that $\sum_{n=1}^{\infty} |\pi_n| < \infty$. Then, for almost every path ω , we have the following inequality:

$$V_T^{(2)}(\pi_n) \leq \sup_{|u-v| \leq |\pi_n|} |B_u - B_v| \cdot V_T^{(1)}(\pi_n).$$

The left-hand side converges to T due to the almost sure convergence of the quadratic variation. The first term on the right-hand side tends to 0 because Brownian motion paths are uniformly continuous on $[0, T]$ (as they are continuous functions on a compact interval). Therefore, the second term on the right must necessarily tend toward infinity. In conclusion, the quadratic variation of Brownian motion is non-zero (equal to T in L^2), whereas its total variation is almost surely infinite. $\square$

Definition 1.13. *A process X is said to be a process of bounded variation on $[0, T]$ if its trajectories exhibit bounded variation almost surely:*

$$\sup_{\pi} \sum_{i=1}^n |X_{t_i} - X_{t_{i-1}}| < \infty \quad a.s.$$

Proposition 1.13. *A process is of bounded variation if and only if it can be represented as the difference of two non-decreasing processes.*

Proof. Assuming X is of bounded variation, for any $t \leq T$, let:

$$X_t^+ := \sup_{\pi \subset [0, t]} \sum_{i=1}^n |X_{t_i} - X_{t_{i-1}}| < \infty \quad a.s.,$$

where π denotes the partitions of $[0, t]$. Then, X_t can be decomposed as follows:

$$X_t = \frac{X_t + X_t^+}{2} - \frac{X_t^+ - X_t}{2} := X_t^{(1)} - X_t^{(2)}.$$

The processes $X_t^{(1)}$ and $X_t^{(2)}$ are non-decreasing since, for $s \leq t$, the inequality $|X_t - X_s| \leq X_t^+ - X_s^+$ holds. Conversely, if $X = X^{(1)} - X^{(2)}$, where $X^{(1)}$ and $X^{(2)}$ are non-decreasing, then for any partition of $[0, T]$:

$$\sum_{i=1}^n |X_{t_i} - X_{t_{i-1}}| \leq \sum_{i=1}^n (X_{t_i}^{(1)} - X_{t_{i-1}}^{(1)}) + \sum_{i=1}^n (X_{t_{i-1}}^{(2)} - X_{t_i}^{(2)}) = (X_T^{(1)} - X_0^{(1)}) + (X_0^{(2)} - X_T^{(2)}) < \infty.$$

Thus X is of bounded variation. $\square$

Proposition 1.14. *If X is a continuous process of bounded variation, its quadratic variation vanishes almost surely: $[X]_T = 0$.*

Proof. As previously established, for almost every ω :

$$V_T^{(2)}(\pi_n) \leq \sup_{|u-v| \leq |\pi_n|} |X_u - X_v| \cdot V_T^{(1)}(\pi_n).$$

Since X is continuous on the compact interval $[0, T]$, the supremum tends to zero as $|\pi_n| \rightarrow 0$. Given that the total variation of X is finite, the quadratic variation is necessarily zero almost surely. $\square$

The following summary table highlights the fundamental distinctions between Brownian motion and continuous processes of bounded variation:

Comparison of total and quadratic variation

Property	Total Variation	Quadratic Variation
Brownian Motion	$+\infty$	T
Continuous Process with Bounded Variation	$< \infty$	0

By construction of the quadratic variation, $[X]_T = [X, X]_T$; hence, it is natural to extend the mapping $X \mapsto [X] = [X, X]$ into a bilinear form.

Definition 1.14. *Let X and Y be two processes such that $[X]$, $[Y]$, and $[X + Y]$ have finite quadratic variation in L^2 . The quadratic covariation between the processes X and Y is defined as:*

$$[X, Y]_T := \frac{1}{2}([X + Y]_T - [X - Y]_T).$$

By construction, this definition ensures the mapping $(X, Y) \mapsto [X, Y]$ is bilinear:

$$[X + Y, X + Y]_T = [X, X]_T + 2[X, Y]_T + [Y, Y]_T.$$

Remark 1.11. $[X, Y]_T$ is identified as the limit in L^2 of $\sum_{i=1}^n |X_{t_i^n} - X_{t_{i-1}^n}| |Y_{t_i^n} - Y_{t_{i-1}^n}|$.

Proposition 1.15. *Let X be a continuous process of bounded variation with a quadratic variation in L^2 . Then:*

$$[X + Y]_T = [Y]_T,$$

which is equivalent to saying that $[X, Y]_T = 0$.

Proof. We have the following inequality:

$$\sum_{i=1}^n |X_{t_i} - X_{t_{i-1}}| |Y_{t_i} - Y_{t_{i-1}}| \leq \sqrt{\sum_{i=1}^n |X_{t_i} - X_{t_{i-1}}|^2} \sqrt{\sum_{i=1}^n |Y_{t_i} - Y_{t_{i-1}}|^2}.$$

The first factor is bounded by $\sup_{|u-v| \leq |\pi_n|} |X_u - X_v| \cdot V_T^{(1)}(X) \rightarrow 0$ as $|\pi_n| \rightarrow 0$, since X is continuous on the compact $[0, T]$ and has finite total variation. Therefore $[X, Y]_T = 0$ at the limit. $\square$

Remark 1.12. *We have just observed that the quadratic variation of Brownian motion over the interval $[0, T]$ is T . Recall that we previously established that the process $B_t^2 - t$ is a martingale. This is not coincidental; there exists a more general result, which we shall state without proof, that characterizes square-integrable martingales. This is known as the Doob–Meyer decomposition.*

Theorem 1.3 (Doob–Meyer Decomposition). *If M is a continuous square-integrable martingale (such that $\mathbb{E}[M_t^2] < \infty$ for all t), then $[M]$ is the unique, continuous, increasing process starting at 0 such that $M^2 - [M]$ is a martingale.*

1.5 Stochastic Integration

Our objective is to provide a formal definition for the random variable:

$$\int_0^T \phi_s dB_s.$$

When integrating a function g with respect to a differentiable function f , provided g is sufficiently smooth, the integral is defined as:

$$\int_0^T g(s) df(s) = \int_0^T g(s) f'(s) ds.$$

In cases where f is not differentiable but is simply of bounded variation, the integral can still be defined as the *Riemann–Stieltjes integral*:

$$\int_0^T g(s) df(s) = \lim_{|\pi_n| \rightarrow 0} \sum_{i=0}^{n-1} g(t_i) (f(t_{i+1}) - f(t_i)).$$

In our context, however, Brownian motion is not of bounded variation; consequently, this limit cannot be defined on a path-by-path basis. Nevertheless, since it possesses finite quadratic variation, it is natural to define the integral with respect to Brownian motion as the limit in L^2 of:

$$\int_0^T \phi_s dB_s := \lim_{|\pi_n| \rightarrow 0} \sum_{i=0}^{n-1} \phi_{t_i} (B_{t_{i+1}} - B_{t_i}).$$

We further require the integrand ϕ to be $\mathcal{F}$ -adapted, ensuring that ϕ_{t_i} remains independent of $B_{t_{i+1}} - B_{t_i}$.

Definition 1.15. A process $(\phi_t)_{t \in [0, T]}$ is called an elementary process if there exists a partition $0 = t_0 < t_1 < \dots < t_n = T$ and a discrete process $(\phi_i)_{0 \leq i \leq n-1}$, where each ϕ_i is $\mathcal{F}_{t_i}$ -measurable and belongs to L^2 , such that:

$$\phi_t = \sum_{i=0}^{n-1} \phi_i \mathbf{1}_{(t_i, t_{i+1}]}(t).$$

We denote by $\mathcal{E}$ the set of elementary processes, which is a subspace of $M^2(\mathcal{F}, [0, T])$.

Definition 1.16. Using the same notation, the stochastic integral between 0 and $t \leq T$ of an elementary process $\phi \in \mathcal{E}$ is the random variable defined by:

$$\int_0^t \phi_s dB_s := \sum_{i=0}^{k-1} \phi_i (B_{t_{i+1}} - B_{t_i}) + \phi_k (B_t - B_{t_k}), \quad \text{on } (t_k, t_{k+1}],$$

which can be written as:

$$\int_0^t \phi_s dB_s = \sum_{i=0}^n \phi_i (B_{t_{i+1} \wedge t} - B_{t_i \wedge t}).$$

Remark 1.13. We naturally define:

$$\int_s^t \phi_u dB_u := \int_0^t \phi_u dB_u - \int_0^s \phi_u dB_u.$$

Proposition 1.16 (Properties of the Stochastic Integral on $\mathcal{E}$). *On the set of elementary processes $\mathcal{E}$, the stochastic integral satisfies the following properties:*

1. **Linearity:** The mapping $\phi \mapsto \int_0^t \phi_s dB_s$ is linear.
2. **Path Continuity:** The mapping $t \mapsto \int_0^t \phi_s dB_s$ is a.s. continuous.
3. **Adaptivity:** The process $\left(\int_0^t \phi_s dB_s \right)_{t \in [0, T]}$ is $\mathcal{F}$ -adapted.
4. **Mean and Variance:** $\mathbb{E} \left[\int_0^t \phi_s dB_s \right] = 0$ and $\text{Var} \left(\int_0^t \phi_s dB_s \right) = \mathbb{E} \left[\int_0^t \phi_s^2 ds \right]$.

5. **Isometry Property:** $\left\| \int_0^t \phi_s dB_s \right\|_{L^2}^2 = \mathbb{E} \left[\int_0^t \phi_s^2 ds \right].$

6. **Conditional Properties:**

$$\mathbb{E} \left[\int_s^t \phi_u dB_u \mid \mathcal{F}_s \right] = 0, \quad \mathbb{E} \left[\left(\int_s^t \phi_v dB_v \right)^2 \mid \mathcal{F}_s \right] = \mathbb{E} \left[\int_s^t \phi_v^2 dv \mid \mathcal{F}_s \right].$$

7. **Generalized Result:**

$$\mathbb{E} \left[\int_s^t \phi_v dB_v \cdot \int_s^u \psi_v dB_v \mid \mathcal{F}_s \right] = \mathbb{E} \left[\int_s^{t \wedge u} \phi_v \psi_v dv \mid \mathcal{F}_s \right].$$

8. **Martingale Property:** $\left(\int_0^t \phi_s dB_s \right)_{t \in [0, T]}$ is an $\mathcal{F}$ -martingale.

9. **Martingale Characterization:** The process $\left(\int_0^t \phi_s dB_s \right)^2 - \int_0^t \phi_s^2 ds$ is an $\mathcal{F}$ -martingale.

10. **Quadratic Variation:** The quadratic variation of the stochastic integral is given by:

$$\left[\int_0^\cdot \phi_s dB_s \right]_t = \int_0^t \phi_s^2 ds.$$

11. **Quadratic Covariation:** The quadratic covariation between two stochastic integrals is given by:

$$\left[\int_0^\cdot \phi_s dB_s, \int_0^\cdot \psi_s dB_s \right]_{t \wedge u} = \int_0^{t \wedge u} \phi_s \psi_s ds.$$

Example 1.1. $\int_0^t dB_s = B_t - B_0 = B_t$ for all $t \in [0, T]$.

Theorem 1.4 (Extension of the Stochastic Integral). *There exists a unique linear mapping I from $M^2(\mathcal{F}, [0, T])$ into $\mathcal{M}^2([0, T])$ that coincides with the stochastic integral on the set of elementary processes $\mathcal{E}$ and satisfies the isometry property:*

$$\|I_t(\phi)\|_{L^2}^2 = \mathbb{E} \left[\int_0^t \phi_s^2 ds \right],$$

for all $t \in [0, T]$.

Proposition 1.17. *On $M^2(\mathcal{F}, [0, T])$, the stochastic integral satisfies the same properties as those stated above (Properties 1–11).*

Proposition 1.18 (Special case: non-random integrand). *If the process ϕ is not random but simply a function f of time, then in addition to the preceding properties, the*

stochastic integral—then referred to as a Wiener integral—is *Gaussian*:

$$\int_0^t f(s) dB_s \sim \mathcal{N}\left(0, \int_0^t f(s)^2 ds\right).$$

Proof. Indeed, $\int_0^t f(s) dB_s$ can be written as the limit in L^2 of random variables of the form:

$$\sum_{i=0}^{n-1} f(t_i)(B_{t_{i+1}} - B_{t_i}) \xrightarrow{L^2} \int_0^t f(s) dB_s.$$

Now, $\sum_{i=0}^{n-1} f(t_i)(B_{t_{i+1}} - B_{t_i})$ is Gaussian, being a linear combination of elements of the Gaussian vector $(B_{t_i}, \dots, B_{t_n})$. Therefore, $\int_0^t f(s) dB_s$ is Gaussian. Indeed, convergence in L^2 implies the convergence of the expectation, the variance, and consequently the characteristic function of the Gaussian random variables. Furthermore, any linear combination of $\int_0^{t_i} f(s) dB_s$ is also a Wiener integral and is thus Gaussian. Hence, the Wiener integral $\int_0^t f(s) dB_s$ is a Gaussian process characterized by:

$$\text{cov}\left(\int_0^t f(s) dB_s, \int_0^u g(s) dB_s\right) = \int_0^{t \wedge u} f(s)g(s) ds.$$

□

1.6 Itô's Formula

Theorem 1.5 (Itô's Formula for Brownian Motion). *Any function $f \in \mathcal{C}^2$ with a bounded second derivative satisfies almost surely:*

$$f(B_t) = f(B_0) + \int_0^t f'(B_s) dB_s + \frac{1}{2} \int_0^t f''(B_s) ds, \quad t \leq T.$$

The infinitesimal notation of this relation is:

$$df(B_t) = f'(B_t) dB_t + \frac{1}{2} f''(B_t) dt.$$

1.7 Itô Processes

Definition 1.17. *An Itô process is a process of the form:*

$$X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dB_s, \quad (1.1)$$

where X_0 is $\mathcal{F}_0$ -measurable, and μ and σ are two $\mathcal{F}$ -adapted processes satisfying the following integrability conditions:

$$\int_0^T |\sigma_s|^2 ds < \infty \quad a.s. \quad \text{and} \quad \int_0^T |\mu_s| ds < \infty.$$

In infinitesimal notation, this is written as:

$$dX_t = \mu_t dt + \sigma_t dB_t.$$

Proposition 1.19. *The quadratic variation over $[0, t]$ of an Itô process X given by (1.1) is:*

$$[X]_t = \left[\int_0^t \sigma_s dB_s \right]_t = \int_0^t \sigma_s^2 ds.$$

Proposition 1.20. *The quadratic covariation between two Itô processes X^1 and X^2 given by $dX_t^i = \mu_t^i dt + \sigma_t^i dB_t$ is:*

$$[X^1, X^2]_t = \int_0^t \sigma_s^1 \sigma_s^2 ds.$$

Definition 1.18. *For an element $\phi \in M^2(\mathcal{F}, [0, T])$ satisfying appropriate integrability conditions, the stochastic integral with respect to an Itô process X is defined as:*

$$\int_0^t \phi_s dX_s := \int_0^t \phi_s \sigma_s dB_s + \int_0^t \phi_s \mu_s ds.$$

Theorem 1.6 (Itô's Formula for Itô Processes). *Let f be a function of class $\mathcal{C}^2$. Then:*

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2} \int_0^t f''(X_s) d[X]_s \\ &= f(X_0) + \int_0^t f'(X_s) \mu_s ds + \int_0^t f'(X_s) \sigma_s dB_s + \frac{1}{2} \int_0^t f''(X_s) \sigma_s^2 ds. \end{aligned}$$

The infinitesimal notation of this relation is:

$$df(X_t) = f'(X_t) dX_t + \frac{1}{2} f''(X_t) d[X]_t.$$

Proposition 1.21 (Integration by Parts Formula). *Let X and Y be two Itô processes. Then:*

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + [X, Y]_t.$$

Alternatively, in infinitesimal notation:

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + d[X, Y]_t.$$

Chapter 2

Backward Doubly Stochastic Differential Equations

2.1 Setting and Notation

The Euclidean norm of a vector $x \in \mathbb{R}^k$ is denoted by $|x|$. For a $d \times d$ matrix A , we define the Frobenius norm as: $\|A\| = \sqrt{\text{Tr}(A^\top A)}$.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $T > 0$ be fixed throughout this paper. Let $(W_t)_{0 \leq t \leq T}$ and $(B_t)_{0 \leq t \leq T}$ be two mutually independent standard Brownian motion processes with values in $\mathbb{R}^d$ and in ℓ^2 , respectively, defined on $(\Omega, \mathcal{F}, P)$. Let $\mathcal{N}$ denote the class of P -null sets in $\mathcal{F}$. For each $t \in [0, T]$, we define the following filtration:

$$\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B,$$

where for any process η , we define $\mathcal{F}_{s,t}^\eta := \sigma(\eta_r - \eta_s; s \leq r \leq t) \vee \mathcal{N}$, and we set $\mathcal{F}_t^\eta := \mathcal{F}_{0,t}^\eta$.

Note that the collection $(\mathcal{F}_t)_{t \in [0, T]}$ is neither increasing nor decreasing, and thus it does not constitute a standard filtration.

For any $n \in \mathbb{N}$, let $M^2(0, T; \mathbb{R}^n)$ denote the set of (classes of $P \otimes dt$ a.e. equal) n -dimensional jointly measurable random stochastic processes $(\phi_t; t \in [0, T])$ which satisfy:

- (i) $\mathbb{E} \left[\int_0^T |\phi_t|^2 dt \right] < \infty$;
- (ii) ϕ_t is $\mathcal{F}_t$ -measurable, for almost every $t \in [0, T]$.

We similarly denote by $S^2(0, T; \mathbb{R}^n)$ the set of continuous n -dimensional random processes satisfying:

- (i) $\mathbb{E} [\sup_{0 \leq t \leq T} |Q_t|^2] < \infty$;
- (ii) Q_t is $\mathcal{F}_t$ -measurable for any $t \in [0, T]$.

2.2 The BDSDE under Lipschitz Conditions

Let the coefficients be defined as:

$$f : \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \rightarrow \mathbb{R}^k, \quad g : \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \rightarrow \mathbb{R}^{k \times \ell},$$

which are jointly measurable such that, for any $(y, z) \in \mathbb{R}^k \times \mathbb{R}^{k \times d}$,

$$f(\cdot, y, z) \in M^2(0, T; \mathbb{R}^k), \quad g(\cdot, y, z) \in M^2(0, T; \mathbb{R}^{k \times \ell}).$$

Furthermore, we assume there exist constants $c > 0$ and $0 < \alpha < 1$ such that for any $(\omega, t) \in \Omega \times [0, T]$, and any $(y_1, z_1), (y_2, z_2) \in \mathbb{R}^k \times \mathbb{R}^{k \times \ell}$:

$$\begin{aligned} |f(t, y_1, z_1) - f(t, y_2, z_2)|^2 &\leq c(|y_1 - y_2|^2 + \|z_1 - z_2\|^2), \\ \|g(t, y_1, z_1) - g(t, y_2, z_2)\|^2 &\leq c(|y_1 - y_2|^2 + \alpha \|z_1 - z_2\|^2). \end{aligned} \quad (\text{H.1})$$

Given a terminal condition $\xi \in L^2(\mathcal{F}_T; \mathbb{R}^K)$, we consider the following *backward doubly stochastic differential equation* (BDSDE):

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g(s, Y_s, Z_s) d\overleftarrow{B}_s - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T. \quad (2.1)$$

It is observed that the integral with respect to B_t is defined as a “backward Itô integral”, whereas the integral with respect to W_t is a standard forward Itô integral. These two formulations are specific instances of the Itô–Skorohod integral (cf. Nualart and Pardoux).

Theorem 2.1. *Under the aforementioned conditions, specifically hypothesis (H.1), Equation (2.1) admits a unique solution:*

$$(Y, Z) \in S^2(0, T; \mathbb{R}^k) \times M^2(0, T; \mathbb{R}^{k \times d}).$$

Proposition 2.1. *There exists a unique pair $(Y, Z) \in S^2(0, T; \mathbb{R}^k) \times M^2(0, T; \mathbb{R}^{k \times d})$ satisfying the simplified equation*

$$Y_t = \xi + \int_t^T f_s ds + \int_t^T g_s d\overleftarrow{B}_s - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T, \quad (2.2)$$

where $f \in M^2(0, T; \mathbb{R}^K)$, $g \in M^2(0, T; \mathbb{R}^{K \times \ell})$, and ξ are as previously defined.

Proof. Uniqueness is immediate. If (Y, Z) represents the difference between two solutions, then:

$$\bar{Y}_t = \int_t^T \bar{Z}_s dW_s, \quad 0 \leq t \leq T.$$

By orthogonality:

$$\mathbb{E} \left[\left| \int_t^T \bar{Z}_s dW_s \right|^2 \right] = 0 \implies |\bar{Y}_t|^2 + \mathbb{E} \left[\int_t^T \text{Tr}(\bar{Z}_s^\top \bar{Z}_s) ds \right] = 0,$$

which implies $\bar{Y}_t = 0$ for all $t \in [0, T]$ a.s. and $\bar{Z}_t = 0$ for almost all t under the product measure $dt \otimes dP$.

Existence. We define the filtration $(\mathcal{G}_t)_{0 \leq t \leq T}$ by $\mathcal{G}_t := \mathcal{F}_t^W$ and the square-integrable martingale:

$$M_t := \mathbb{E} \left[\xi + \int_0^T f_s ds + \int_0^T g_s d\overleftarrow{B}_s \mid \mathcal{G}_t \right], \quad 0 \leq t \leq T.$$

By applying a direct extension of Itô's martingale representation theorem, there exists a $\mathcal{G}_t$ -progressively measurable process (Z_t) taking values in $\mathbb{R}^{k \times d}$ such that $\mathbb{E}[\int_t^T |Z_t|^2 dt] < \infty$, and M_t can be represented as:

$$M_t = M_0 + \int_0^t Z_s dW_s.$$

Setting $Y_t = M_t - \int_0^t f_s ds - \int_0^t g_s d\overleftarrow{B}_s$, one can verify that (Y, Z) satisfies (2.2). The $\mathcal{F}_t$ -adaptedness of (Y_t, Z_t) follows by observing that Y_t involves only $\mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B$ -measurable quantities, and similarly for Z_t . $\square$

2.2.1 Extension of Itô's Formula

We shall need the following extension of the well-known Itô formula.

Lemma 2.1. *Let $\phi \in S^2(0, T; \mathbb{R}^k)$, $\mu \in M^2(0, T; \mathbb{R})$, $\nu \in M^2(0, T; \mathbb{R}^{k \times \ell})$, $\sigma \in M^2(0, T; \mathbb{R}^{k \times d})$ be such that:*

$$\phi_t = \phi_0 + \int_0^t \mu_s ds + \int_0^t \nu_s d\overleftarrow{B}_s + \int_0^t \sigma_s dW_s, \quad 0 \leq t \leq T.$$

Then, the squared norm $|\phi_t|^2$ satisfies:

$$\begin{aligned} |\phi_t|^2 &= |\phi_0|^2 + 2 \int_0^t \langle \phi_s, \mu_s \rangle ds + 2 \int_0^t \langle \phi_s, \nu_s d\overleftarrow{B}_s \rangle + 2 \int_0^t \langle \phi_s, \sigma_s dW_s \rangle \\ &\quad + \int_0^t \|\nu_s\|^2 ds + \int_0^t \|\sigma_s\|^2 ds. \end{aligned}$$

Furthermore, the expectation is given by:

$$\mathbb{E}[|\phi_t|^2] = \mathbb{E}[|\phi_0|^2] + 2\mathbb{E} \left[\int_0^t \langle \phi_s, \mu_s \rangle ds \right] + \mathbb{E} \left[\int_0^t \|\nu_s\|^2 ds \right] + \mathbb{E} \left[\int_0^t \|\sigma_s\|^2 ds \right].$$

Proof. The first identity is a combination of Itô's forward and backward formulae, applied to the process ϕ_t and the quadratic function $f(x) = |x|^2$. Let $0 = t_0 < t_1 < \dots < t_n = t$ be a partition of $[0, t]$. Consider the increment of the squared norm:

$$\begin{aligned} |\phi_{t_{i+1}}|^2 - |\phi_{t_i}|^2 &= 2\langle \phi_{t_i}, \phi_{t_{i+1}} - \phi_{t_i} \rangle + |\phi_{t_{i+1}} - \phi_{t_i}|^2 \\ &= 2\left\langle \phi_{t_i}, \int_{t_i}^{t_{i+1}} \mu_s ds \right\rangle + 2\left\langle \phi_{t_i}, \int_{t_i}^{t_{i+1}} \nu_s d\overleftarrow{B}_s \right\rangle + 2\left\langle \phi_{t_i}, \int_{t_i}^{t_{i+1}} \sigma_s dW_s \right\rangle \\ &\quad + \left| \int_{t_i}^{t_{i+1}} \nu_s d\overleftarrow{B}_s \right|^2 + \left| \int_{t_i}^{t_{i+1}} \sigma_s dW_s \right|^2 + r_i, \end{aligned}$$

where the remainder $r_i \rightarrow 0$ in probability as $\sup |t_{i+1} - t_i| \rightarrow 0$. Summing over i and passing to the limit yields the first identity. The second identity (expectation formula) follows from the first, since the stochastic integrals with respect to W and $\overleftarrow{B}$ have zero expectation, which is guaranteed by the Burkholder–Davis–Gundy inequality and the integrability assumptions on ν and σ . $\square$

2.3 Uniqueness and Existence of Solutions: Proof of Theorem 2.1

Proof. Uniqueness. Let (Y_t^1, Z_t^1) and (Y_t^2, Z_t^2) be two solutions to the system. We define the processes $\bar{Y}_t$ and $\bar{Z}_t$ for $0 \leq t \leq T$ as follows:

$$\bar{Y}_t = Y_t^1 - Y_t^2, \quad \bar{Z}_t = Z_t^1 - Z_t^2.$$

It follows that:

$$\bar{Y}_t = \int_t^T [f(s, Y_s^1, Z_s^1) - f(s, Y_s^2, Z_s^2)] ds + \int_t^T [g(s, Y_s^1, Z_s^1) - g(s, Y_s^2, Z_s^2)] d\overleftarrow{B}_s - \int_t^T \bar{Z}_s dW_s.$$

By applying Lemma 2.1 to the process $\bar{Y}_t$, we obtain:

$$\begin{aligned} |\bar{Y}_t|^2 + \int_t^T \|\bar{Z}_s\|^2 ds &= 2 \int_t^T \langle f(s, Y_s^1, Z_s^1) - f(s, Y_s^2, Z_s^2), \bar{Y}_s \rangle ds \\ &\quad + \int_t^T \|g(s, Y_s^1, Z_s^1) - g(s, Y_s^2, Z_s^2)\|^2 ds. \end{aligned}$$

Utilising assumption (H.1) and Young's inequality $ab \leq \frac{1}{2(1-\varepsilon)}a^2 + \frac{1-\varepsilon}{2}b^2$, we get:

$$|\bar{Y}_t|^2 + \frac{1-\alpha}{2} \int_t^T \|\bar{Z}_s\|^2 ds \leq c \int_t^T |\bar{Y}_s|^2 ds,$$

where $0 < \alpha < 1$ is the constant in (H.1). Invoking Gronwall's Lemma, it follows that $|\bar{Y}_t|^2 = 0$ for all $0 \leq t \leq T$, which further implies that $\int_0^T \|\bar{Z}_s\|^2 ds = 0$. This establishes uniqueness.

Existence. We construct the solution by defining a sequence of processes $\{(Y_t^i, Z_t^i)\}_{i=0,1,\dots}$ as follows: let $Y_t^0 = 0$ and $Z_t^0 = 0$. Given (Y_t^i, Z_t^i) , (Y_t^{i+1}, Z_t^{i+1}) is defined as the unique solution—constructed in accordance with Proposition 2.1—to the following equation:

$$Y_t^{i+1} = \xi + \int_t^T f(s, Y_s^i, Z_s^i) ds + \int_t^T g(s, Y_s^i, Z_s^i) d\overleftarrow{B}_s - \int_t^T Z_s^{i+1} dW_s.$$

Let $\bar{Y}_t^{i+1} = Y_t^{i+1} - Y_t^i$ and $\bar{Z}_t^{i+1} = Z_t^{i+1} - Z_t^i$. By applying the integration by parts formula (specifically the Itô product rule with an exponential weight) with $\lambda > 0$, we deduce that there exist constants $c, \lambda > 0$ such that:

$$\mathbb{E} \left[\int_t^T (c|\bar{Y}_s^{i+1}|^2 + \|\bar{Z}_s^{i+1}\|^2) e^{\lambda s} ds \right] \leq \frac{1+\alpha}{2} \mathbb{E} \left[\int_t^T (c|\bar{Y}_s^i|^2 + \|\bar{Z}_s^i\|^2) e^{\lambda s} ds \right].$$

By selecting $\lambda = 2c/(1-\alpha)$, since $\frac{1+\alpha}{2} < 1$, the sequence $\{(Y_t^i, Z_t^i)\}_{i=0,1,2,\dots}$ constitutes a Cauchy sequence in the space $M^2(0, T; \mathbb{R}^k) \times M^2(0, T; \mathbb{R}^{k \times d})$. It is then straightforward to conclude that $(Y_t^i)_{i=0,1,2,\dots}$ is also Cauchy in $S^2(0, T; \mathbb{R}^k)$, and that the limit $(Y_t, Z_t) = \lim_{i \rightarrow \infty} (Y_t^i, Z_t^i)$ solves (2.1). $\square$

Chapter 3

BDSDEs with Locally Monotone Coefficients

3.1 BDSDEs with Locally Monotone Coefficients

In this chapter, we investigate Backward Doubly Stochastic Differential Equations (BDSDEs) with locally monotone coefficients, extending the results established in [15] for a finite time horizon. Some of the proof techniques are inspired by the work of Bahlali et al. [1, 2, 3].

We assume the following hypotheses:

(H3) There exist constants $K > 0$ and $0 \leq \beta < 1$ such that:

$$|f(t, y, z)| \leq K(1 + |y|^\beta + \|z\|^\beta).$$

(H4) For every $N \in \mathbb{N}$, there exists a constant $\lambda_N \in \mathbb{R}$ such that, for any y_1, y_2, z satisfying $|y_1|, |y_2|, \|z\| \leq N$, the following monotonicity condition holds:

$$\langle y_1 - y_2, f(t, y_1, z) - f(t, y_2, z) \rangle \leq \lambda_N |y_1 - y_2|^2.$$

(H5) For every $N \in \mathbb{N}$, there exists a Lipschitz constant $L_N > 0$ such that, for any y, z_1, z_2 satisfying $|y|, \|z_1\|, \|z_2\| \leq N$:

$$|f(t, y, z_1) - f(t, y, z_2)| \leq L_N \|z_1 - z_2\|.$$

Remark 3.1. Since $|x|^\beta \leq 1 + |x|$ for $\beta \in [0, 1)$, hypothesis (H3) implies that:

$$|f(t, y, z)| \leq K(3 + |y| + \|z\|).$$

The primary result of this chapter is:

Theorem 3.1. *Let hypotheses (H1), (H2), and (H3)–(H5) hold. Furthermore, assume that:*

$$1 - L_N^\alpha \cdot N^2 \cdot \exp(2\lambda_N T + L_N^2 \cdot \frac{1}{\alpha} T) \cdot N^2 > 0, \quad \forall N \in \mathbb{N}, \quad (3.1)$$

where α is an arbitrarily fixed constant such that $0 < \alpha < 1 - \alpha$. Then the BDSDE (2.1) admits a unique solution $(Y, Z) \in S^2(0, T; \mathbb{R}^m) \times M^2(0, T; \mathbb{R}^{m \times d})$.

Prior to the proof of Theorem 3.1, we introduce the following lemma:

Lemma 3.1. *Under hypotheses (H2) and (H3)–(H5), there exists a sequence $(f_n)_{n \geq 1}$ such that:*

1. *For fixed n, ω, t , the function $f_n(t, \cdot, \cdot)$ is continuous;*
2. *For all n , $|f_n(t, y, z)| \leq |f(t, y, z)| \leq K(1 + |y|^\beta + \|z\|^\beta)$;*
3. *For every N , $\|f_n - f\|_N \rightarrow 0$ as $n \rightarrow \infty$, where the norm $\|\cdot\|_N$ is defined as:*

$$\|f\|_N^2 := \int_0^T \sup_{|y|, \|z\| \leq N} |f(t, y, z)|^2 dt;$$

4. *For each n , f_n is globally monotone in y . Moreover, for any n, N :*

$$\langle y_1 - y_2, f_n(t, y_1, z) - f_n(t, y_2, z) \rangle \leq \lambda_N |y_1 - y_2|^2,$$

for all t, y_1, y_2, z satisfying $|y_1|, |y_2|, \|z\| \leq N$;

5. *For each n , f_n is globally Lipschitz in z . Furthermore, for any n, N with $n \geq N$:*

$$|f_n(t, y, z_1) - f_n(t, y, z_2)| \leq L_N \|z_1 - z_2\|,$$

for all t, y, z_1, z_2 satisfying $|y|, \|z_1\|, \|z_2\| \leq N$.

Proof. Let $(\chi_n^y)_{n \geq 1} : \mathbb{R}^m \rightarrow [0, 1]$ be a sequence of smooth “cut-off” functions such that $\chi_n^y(u) = 1$ for $|u| \leq n$ and $\chi_n^y(u) = 0$ for $|u| \geq n + 1$. We define the sequence $(\chi_n^z)_{n \geq 1} : \mathbb{R}^{m \times d} \rightarrow [0, 1]$ in an analogous manner. Notably, χ_n^y and χ_n^z are continuously differentiable with bounded derivatives for each n . We define the approximated coefficient f_n as follows:

$$f_n(t, y, z) := f(t, y, z) \chi_n^y(y) \chi_n^z(z).$$

It is evident that f_n satisfies conditions (1) and (2). Assertion (3) follows directly from the Dominated Convergence Theorem. We now prove (4). Let $\Delta := \langle y_1 - y_2, f_n(t, y_1, z) - f_n(t, y_2, z) \rangle$.

We analyse the following cases:

(A1) If $\|z\| \geq n+1$, then $\chi_n^z(z) = 0$, which immediately implies $\Delta = 0$.

(A2) If $\|z\| \leq n+1$, $|y_1| \geq n+1$, and $|y_2| \geq n+1$, then $\chi_n^y(y_1) = \chi_n^y(y_2) = 0$, yielding $\Delta = 0$.

(A3) If $\|z\| \leq n+1$, $|y_1| \leq n+1$, and $|y_2| \leq n+1$, then by hypotheses (H3) and (H4):

$$\begin{aligned} \Delta &= \langle y_1 - y_2, f(t, y_1, z) - f(t, y_2, z) \rangle \chi_n^y(y_1) \chi_n^z(z) + \langle y_1 - y_2, f(t, y_2, z) \rangle (\chi_n^y(y_1) - \chi_n^y(y_2)) \chi_n^z(z) \\ &\leq \lambda_{n+1} |y_1 - y_2|^2 + K(3 + 2(n+1)) \sup_u |\nabla \chi_n^y(u)| \cdot |y_1 - y_2|^2 \\ &\leq (\lambda_{n+1} + K(3 + 2n + 5) \sup_u |\nabla \chi_n^y(u)|) |y_1 - y_2|^2. \end{aligned}$$

(A4) If $\|z\| \leq n+1$, $|y_1| \leq n+1$, and $|y_2| \geq n+1$, then $\chi_n^y(y_2) = 0$ and:

$$\Delta = \langle y_1 - y_2, f(t, y_1, z) \rangle \chi_n^y(y_1) \chi_n^z(z) \leq K(3 + 2(n+1)) \sup_u |\nabla \chi_n^y(u)| \cdot |y_1 - y_2|^2.$$

(A5) If $\|z\| \leq n+1$, $|y_1| \geq n+1$, and $|y_2| \leq n+1$: analogous to (A4).

Based on the derivations (A1)–(A5), we conclude that:

$$\Delta \leq (\lambda_{n+1} + K(3 + 2N + 1) \sup_u |\nabla \chi_n^y(u)|) |y_1 - y_2|^2,$$

which implies that f_n is globally monotone in y . Furthermore, if $n \geq N$ and $|y_1|, |y_2|, \|z\| \leq N$, then $f_n(t, y_i, z) = f(t, y_i, z)$. Consequently, assertion (4) follows directly from hypothesis (H4). Similar arguments yield assertion (5), with:

$$|f_n(t, y, z_1) - f_n(t, y, z_2)| \leq (L_{n+1} + K(3 + 2(n+1)) \sup_u |\nabla \chi_n^z(u)|) \|z_1 - z_2\|.$$

□

Proposition 3.1. Assume $\xi^1, \xi^2 \in L^2(\mathcal{F}_T; \mathbb{R}^m)$ and let f^1, f^2 and g satisfy conditions (H1), (H2), and (H3)–(H5). Furthermore, assume that condition (3.1) holds. Let $(Y^i, Z^i) \in S^2(0, T; \mathbb{R}^m) \times M^2(0, T; \mathbb{R}^{m \times d})$, for $i = 1, 2$, be the unique solution to the BDSDE:

$$Y_t^i = \xi^i + \int_t^T f^i(s, Y_s^i, Z_s^i) ds + \int_t^T g(s, Y_s^i, Z_s^i) d\overleftarrow{B}_s - \int_t^T Z_s^i dW_s.$$

Then, the following estimate holds:

$$\mathbb{E} \left[\sup_{0 \leq u \leq T} |Y_u^1 - Y_u^2|^2 \right] + \mathbb{E} \left[\int_t^T \|Z_s^1 - Z_s^2\|^2 dt \right] \leq c \Pi_1 * \Pi_2, \quad (3.2)$$

where:

$$\begin{aligned}\Pi_1 &= \mathbb{E}[|\xi^1 - \xi^2|^2] + \rho_N^2(f^1 - f^2) + N^{-2(1-\gamma)}, \\ \Pi_2 &= 1 + (1 + \mu_N^+ + \mathcal{L}_N^2) \exp(2\mu_N^+ T + \mathcal{L}_N^2 \theta^{-1} T),\end{aligned}$$

and θ is an arbitrarily fixed constant such that $0 < \theta < 1 - \bar{\alpha}$.

Proof. Since $|f^i(t, y, z)| \leq K(3 + |y| + \|z\|)$, applying Itô's formula to $|Y_t^i|^2$ yields:

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |Y_t^i|^2\right] + \mathbb{E}\left[\int_0^T \|Z_t^i\|^2 dt\right] \leq c\left(1 + \mathbb{E}[|\xi^i|^2] + \mathbb{E}\left[\int_0^T |g(t, 0, 0)|^2 dt\right]\right),$$

where $c > 0$ is a constant depending on T, K , and $\bar{\alpha}$. Let us define the following sets:

$$A := \{(\omega, s) : |Y_s^1| + |Y_s^2| + \|Z_s^1\| + \|Z_s^2\| \leq N\}, \quad \bar{A} := \Omega \times [0, T] \setminus A.$$

By applying Itô's formula once more, we obtain:

$$\begin{aligned}\mathbb{E}[|Y_t^1 - Y_t^2|^2] + \mathbb{E}\left[\int_t^T \|Z_s^1 - Z_s^2\|^2 ds\right] \\ &= \mathbb{E}[|\xi^1 - \xi^2|^2] + \mathbb{E}\left[\int_t^T \|g(s, Y_s^1, Z_s^1) - g(s, Y_s^2, Z_s^2)\|^2 ds\right] \\ &\quad + 2\mathbb{E}\left[\int_t^T \langle Y_s^1 - Y_s^2, f^1(s, Y_s^1, Z_s^1) - f^1(s, Y_s^2, Z_s^2) \rangle ds\right] \\ &\leq \mathbb{E}[|\xi^1 - \xi^2|^2] + K\mathbb{E}\left[\int_t^T |Y_s^1 - Y_s^2|^2 ds\right] + \bar{\alpha}\mathbb{E}\left[\int_t^T \|Z_s^1 - Z_s^2\|^2 ds\right] \\ &\quad + I_1 + I_2 + I_3 + I_4,\end{aligned}$$

where the terms I_1, I_2, I_3 , and I_4 are defined as follows:

$$\begin{aligned}I_1 &= 2\mathbb{E}\left[\int_t^T (Y_s^1 - Y_s^2) \cdot [f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2)] \mathbf{1}_A ds\right], \\ I_2 &= 2\mathbb{E}\left[\int_t^T (Y_s^1 - Y_s^2) \cdot [f^1(s, Y_s^1, Z_s^1) - f^1(s, Y_s^2, Z_s^1)] \mathbf{1}_{\bar{A}} ds\right], \\ I_3 &= 2\mathbb{E}\left[\int_t^T (Y_s^1 - Y_s^2) \cdot [f^1(s, Y_s^2, Z_s^1) - f^1(s, Y_s^2, Z_s^2)] \mathbf{1}_{\bar{A}} ds\right], \\ I_4 &= 2\mathbb{E}\left[\int_t^T (Y_s^1 - Y_s^2) \cdot [f^1(s, Y_s^2, Z_s^2) - f^2(s, Y_s^2, Z_s^2)] \mathbf{1}_{\bar{A}} ds\right].\end{aligned}$$

Estimation of Terms. We now proceed to estimate I_1, I_2, I_3 , and I_4 . Invoking

Hölder's inequality and Chebyshev's inequality, we can bound I_1 as:

$$I_1 \leq \mathbb{E} \left[\int_t^T |Y_s^1 - Y_s^2|^2 ds \right] + CN^{-2(1-\gamma)},$$

where the constant $C > 0$ depends on $T, K, \bar{\alpha}, \gamma, \mathbb{E}[|\xi^1|^2], \mathbb{E}[|\xi^2|^2]$, and $\mathbb{E} \left[\int_0^T |g(t, 0, 0)|^2 dt \right]$.

The local monotonicity of f^1 with respect to y yields:

$$I_2 \leq 2\mu_N^+ \mathbb{E} \left[\int_t^T |Y_s^1 - Y_s^2|^2 ds \right].$$

In light of the local Lipschitz condition of f^1 in z , it is straightforward to verify that for any $\beta > 0$:

$$I_3 \leq \beta^{-1} \mathbb{E} \left[\int_t^T |Y_s^1 - Y_s^2|^2 ds \right] + \beta \mathcal{L}_N^2 \mathbb{E} \left[\int_t^T \|Z_s^1 - Z_s^2\|^2 ds \right].$$

Finally, we obtain:

$$I_4 = \mathbb{E} \left[\int_t^T |Y_s^1 - Y_s^2|^2 ds \right] + \rho_N^2 (f^1 - f^2).$$

By choosing $\beta > 0$ such that $\theta := \beta \mathcal{L}_N^2 < 1 - \bar{\alpha}$, it follows that:

$$\begin{aligned} & \mathbb{E} |Y_t^1 - Y_t^2|^2 + (1 - \bar{\alpha} - \theta) \mathbb{E} \left[\int_t^T \|Z_s^1 - Z_s^2\|^2 ds \right] \\ & \leq \mathbb{E} [|\xi^1 - \xi^2|^2] + CN^{-2(1-\gamma)} + \rho_N^2 (f^1 - f^2) \\ & \quad + (2 + K + 2\mu_N^+ + \mathcal{L}_N^2 \theta^{-1}) \mathbb{E} \left[\int_t^T |Y_s^1 - Y_s^2|^2 ds \right]. \end{aligned}$$

The conclusion is then established by applying Gronwall's inequality and the Burkholder–Davis–Gundy (B-D-G) inequality. $\square$

3.2 Proof of Theorem 3.1

Proof. By setting $f^1 = f^2 = f$ and $\xi^1 = \xi^2 = \xi$, and passing to the limit as $N \rightarrow \infty$ in equation (3.2), we establish uniqueness. We now proceed to prove existence.

Let (f_n) be associated with f according to Lemma 3.1. By construction, f_n is globally monotone in y and globally Lipschitz in z for each n . Consequently, by Theorem 2.2 in [14], the following Backward Doubly Stochastic Differential Equation (BDSDE):

$$Y_t^n = \xi + \int_t^T f_n(s, Y_s^n, Z_s^n) ds + \int_t^T g(s, Y_s^n, Z_s^n) d\overleftarrow{B}_s - \int_t^T Z_s^n dW_s, \quad 0 \leq t \leq T,$$

admits a unique solution $(Y^n, Z^n) \in S^2(0, T, \mathbb{R}^m) * \mathcal{M}^2(0, T, \mathbb{R}^{m \times d})$ for each $n \in \mathbb{N}$.

Applying Itô's formula to $|Y_t^n|^2$, we have:

$$\mathbb{E}|Y_t^n|^2 + \mathbb{E} \left[\int_t^T |Z_s^n|^2 ds \right] = \mathbb{E}|\xi|^2 + 2\mathbb{E} \left[\int_t^T Y_s^n \cdot f_n(s, Y_s^n, Z_s^n) ds \right] + \mathbb{E} \left[\int_t^T \left| \int_s^T g(s, Y_s^n, Z_s^n) \right|^2 ds \right],$$

where

$$\begin{aligned} 2\mathbb{E} \left[\int_t^T Y_s^n \cdot f_n(s, Y_s^n, Z_s^n) ds \right] &\leq 2KE \left[\int_t^T |Y_s^n| (3 + |Y_s^n| + |Z_s^n|) ds \right] \\ &\leq 9T + \left(2K + K^2 + \frac{4K^2}{1-\varepsilon} \right) \mathbb{E} \left[\int_t^T |Y_s^n|^2 ds \right] + \frac{1-\varepsilon}{4} \mathbb{E} \left[\int_t^T |Z_s^n|^2 ds \right]. \end{aligned}$$

Furthermore, we have:

$$\begin{aligned} \mathbb{E} \left[\int_t^T |g(s, Y_s^n, Z_s^n)|^2 ds \right] &\leq \frac{1+\varepsilon}{2\varepsilon} \mathbb{E} \left[\int_t^T |g(s, Y_s^n, Z_s^n) - g(s, 0, 0)|^2 ds \right] + \frac{1+\varepsilon}{1-\varepsilon} \mathbb{E} \left[\int_t^T |g(s, 0, 0)|^2 ds \right] \\ &\leq \frac{(1+\varepsilon)K}{2\varepsilon} \mathbb{E} \left[\int_t^T |Y_s^n|^2 ds \right] + \frac{1+\varepsilon}{2} \mathbb{E} \left[\int_t^T |Z_s^n|^2 ds \right] \\ &\quad + \frac{1+\varepsilon}{1-\varepsilon} \mathbb{E} \left[\int_0^T |g(s, 0, 0)|^2 ds \right]. \end{aligned}$$

Hence:

$$\begin{aligned} \mathbb{E}|Y_t^n|^2 + \frac{1-\varepsilon}{4} \mathbb{E} \left[\int_t^T |Z_s^n|^2 ds \right] &\leq 9T + \mathbb{E}|\xi|^2 + \frac{1+\varepsilon}{1-\varepsilon} \mathbb{E} \left[\int_0^T |g(s, 0, 0)|^2 ds \right] \\ &\quad + \left(2K + \frac{(1+\varepsilon)K}{2\varepsilon} + K^2 + \frac{4K^2}{1-\varepsilon} \right) \mathbb{E} \left[\int_t^T |Y_s^n|^2 ds \right]. \end{aligned}$$

It then follows from Gronwall's inequality and the Burkholder–Davis–Gundy (B-D-G) inequality that:

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t^n|^2 \right] + \mathbb{E} \left[\int_0^T |Z_s^n|^2 ds \right] \leq c \left(1 + \mathbb{E}|\xi|^2 + \mathbb{E} \left[\int_0^T |g(s, 0, 0)|^2 ds \right] \right),$$

where $c > 0$ is a constant independent of n .

An analysis similar to the proof of Proposition 3.1 demonstrates that, for sufficiently large m and n :

$$\begin{aligned} &\mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t^n - Y_t^m|^2 \right] + \mathbb{E} \left[\int_0^T \|Z_t^n - Z_t^m\|^2 dt \right] \\ &\leq c \left(\|f_n - f_m\|_N^2 + \mathcal{L}_N^2 \Pi_1 + \left(1 + (1 + \mu_N^+ + \mathcal{L}_N^2) e^{2\mu_N^+ T + \mathcal{L}_N^2 \theta^{-1} T} \right) \right), \end{aligned}$$

where $c > 0$ is independent of m and n . By passing to the limit successively as

$n, m, N \rightarrow \infty$, we conclude that (Y^n, Z^n) is a Cauchy sequence in the space $S^2(0, T; \mathbb{R}^m) \times M^2(0, T; \mathbb{R}^{m \times d})$. Consequently, there exists a pair of processes (Y, Z) such that:

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |Y_t^n - Y_t|^2 \right] + \mathbb{E} \left[\int_0^T \|Z_t^n - Z_t\|^2 dt \right] \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

To assert that (Y, Z) constitutes a solution to the BDSDE (2.1), it remains only to verify that, along a subsequence:

$$\int_t^T f_n(s, Y_s^n, Z_s^n) ds \rightarrow \int_t^T f(s, Y_s, Z_s) ds \quad \text{in } L^2 \text{ as } n \rightarrow \infty. \quad (3.3)$$

Define the set $A_n := \{(\omega, s) : |Y_s^n| + |Y_s| + \|Z_s^n\| + \|Z_s\| \leq N\}$ and let $\bar{A}_n := \Omega \times [0, T] \setminus A_n$. One can readily verify that:

$$\begin{aligned} & \mathbb{E} \left[\int_t^T |f_n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)|^2 ds \right] \\ & \leq CN^{-2(1-\gamma)} \Pi_1 + 4\|f_n - f\|_N^2 + 4\mathcal{L}_N^2 \mathbb{E} \left[\int_t^T \|Z_s^n - Z_s\|^2 ds \right] \\ & + 2\mathbb{E} \left[\int_t^T |f(s, Y_s^n, Z_s) - f(s, Y_s, Z_s)|^2 ds \right], \end{aligned}$$

where $C > 0$ is a constant independent of n . Given that $\sup_{0 \leq t \leq T} |Y_t^n - Y_t|^2 \rightarrow 0$ and $\sup_n \sup_{0 \leq t \leq T} |Y_t^n|^2 < \infty$, there exists a subsequence (still denoted by Y^n) such that $Y_t^n \rightarrow Y_t$ almost everywhere and almost surely. Consequently, by applying the Dominated Convergence Theorem and invoking the continuity of f with respect to y , we obtain:

$$\mathbb{E} \left[\int_t^T |f(s, Y_s^n, Z_s) - f(s, Y_s, Z_s)|^2 ds \right] \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Then, by passing to the limit successively with respect to n and N , we demonstrate that (3.3) holds. Thus, the proof is complete. $\square$

Conclusion

In this work, we studied the existence and uniqueness of solutions for backward doubly stochastic differential equations (BDSDEs) within a framework of relaxed assumptions. Unlike classical approaches that rely solely on the global Lipschitz condition, our objective was to extend the validity of these results to a broader class of equations by considering the local monotonicity condition rather than a global one.

We first laid the necessary theoretical foundations, covering essential concepts from stochastic analysis, including conditional expectation, martingales, and Itô's calculus, all adapted to the specific structure of BDSDEs involving both forward and backward integrals. We then reviewed the classical existence and uniqueness results for BDSDEs under the standard Lipschitz hypothesis, which served as a reference point for our extensions.

Appendix A

Some Mathematical Tools

Theorem A.1 (Hölder's Inequality). *If $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then:*

$$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

Theorem A.2 (Cauchy–Schwarz Inequality). *Let $f, g \in L^2([0, T]; \mathbb{R})$. Then:*

$$\int_0^T |f(\omega)g(\omega)| d\omega \leq \left(\int_0^T |f(\omega)|^2 d\omega \right)^{1/2} \left(\int_0^T |g(\omega)|^2 d\omega \right)^{1/2}.$$

Lemma A.1 (Young's Inequality). *Let a, b be two real numbers and $p, q \geq 1$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Then:*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Lemma A.2 (Gronwall's Lemma). *Let g be a positive, measurable, and bounded function on $[0, T]$. Suppose there exist constants $a \geq 0$ and $b \geq 0$ such that for all $t \in [0, T]$:*

$$g(t) \leq a + b \int_0^T g(s) ds.$$

Then, for all $t \in [0, T]$:

$$g(t) \leq a e^{bt}.$$

Theorem A.3 (Jensen's Inequality). *Let $I \subseteq \mathbb{R}$ be an interval and $\varphi : I \rightarrow \mathbb{R}$ be a convex function. For any $n \geq 2$, for all $x_1, \dots, x_n \in I$, and for all $\lambda_1, \dots, \lambda_n \in [0, 1]$ satisfying $\sum_{i=1}^n \lambda_i = 1$, we have:*

$$\varphi\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \sum_{i=1}^n \lambda_i \varphi(x_i).$$

Theorem A.4 (Dominated Convergence Theorem). *Let $(f_n)_n$ be a sequence of measurable functions on a measurable space $(\Omega, \mathcal{F}, \mu)$ with real or complex values. Suppose*

that the sequence $(f_n)_n$ converges pointwise to f and there exists an integrable function g such that $|f_n(x)| \leq |g(x)|$ for all $n \in \mathbb{N}$ and all $x \in \Omega$. Then f is integrable and:

$$\lim_{n \rightarrow \infty} \int |f_n - f| d\mu = 0.$$

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الملخص

في هذه المذكرة، قننا بدراسة مسألة وجود ووحدانية الحلول للمعادلات التفاضلية العشوائية التراجعية ثنائية الاتجاه في إطار شروط محلية على المولد. ركزنا بشكل أساسي على الحالة التي يحقق فيها المولد شرط الرتبة المحلية بالنسبة للمتغير Y ، مع كونه يحقق شرط ليبشيتز بالنسبة للمتغير Z . إن الطبيعة ثنائية الاتجاه لهذه المعادلات، والتي تتضمن تكاملين (نسبة إلى حركة براونية وقياس مارتينغال معكوس)، تضيف تحدياً إضافياً مقارنةً بالمعادلات التفاضلية العشوائية التراجعية الكلاسيكية. من خلال الاعتماد على تقنيات التقريب، أثبتنا أن هذه المعادلات تمتلك حلاً وحيداً تحت هذه الافتراضات المحلية.

الكلمات المفتاحية: المعادلات التفاضلية العشوائية التراجعية ثنائية الاتجاه؛ الرتبة المحلية؛ الوجود والوحدانية؛ تكامل تراجعي.

Abstract

In this work, we study the existence and uniqueness of solutions for backward doubly stochastic differential equations under relaxed assumptions on the generator. We specifically focus on the case where the generator satisfies a local monotonicity condition with respect to the variable Y , while remaining Lipschitz continuous in Z . The doubly stochastic nature of these equations, involving both a forward integral (with respect to a Brownian motion) and a backward integral (with respect to an independent backward martingale measure), introduces additional challenges compared to classical BSDEs. By employing an approximation techniques, we demonstrate that these equations admit a unique solution under these local hypotheses.

Keywords: Backward doubly stochastic differential equations; Local monotonicity; Existence and uniqueness; Backward integral.

Résumé

Dans ce mémoire, nous étudions l'existence et l'unicité des solutions pour les équations différentielles stochastiques rétrogrades doublement stochastiques sous des hypothèses locales sur le générateur. Nous nous concentrons principalement sur le cas où le générateur satisfait une condition de monotonie locale par rapport à la variable Y , tout en étant Lipschitzien par rapport à Z . La nature double de ces équations, comportant à la fois une intégrale forward (par rapport à un mouvement brownien) et une intégrale backward (par rapport à une mesure martingale rétrograde indépendante), introduit des défis supplémentaires par rapport aux BSDEs classiques. En nous appuyant sur des techniques d'approximation, nous démontrons que ces équations admettent une solution unique sous ces hypothèses locales.

Mots-clés : Équations différentielles stochastiques rétrogrades doublement stochastiques; Monotonie locale; Existence et unicité; Intégrale rétrograde.