



UNIVERSITÉ KASDI MERBAH OUARGLA

**FACULTÉ DES MATHÉMATIQUES
ET DES SCIENCES DE LA MATIÈRE**

Département de Physique



N° d'ordre :

N° de série :

Mémoire

MASTER ACADÉMIQUE

Domaine : Sciences de la Matière

Filière : Physique

Spécialité : Physique Théorique

Présenté par : Lakhdari Fatma

Thème :

**Formalisme de la Dérivée de Dunkl dans les Systèmes
Quantiques Relativistes et non Relativistes:
États Liés et Diffusion**

Soutenu publiquement le : 17/05/2026

Devant le jury composé de :

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Année Universitaire : 2025/2026



**KASDI MERBAH UNIVERSITY OF
OUARGLA**

**FACULTY OF MATHEMATICS
AND MATERIAL SCIENCES**

Department of Physics



Registration No.:

Serial No.:

Master Thesis
Academic Master's Degree

Field: Matter Sciences

Specialty: Physics

Option: Theoretical Physics

Presented by:

Lakhdari Fatma

Title:

**Formalism of the Dunkl Derivative in
Relativistic and Non-Relativistic Quantum Systems:
Bound States and Scattering**

Defended on: May 17, 2026

Before the Jury:

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Academic Year: 2025/2026

Abstract

This dissertation is devoted to the generalization of quantum mechanics through the introduction of the Dunkl derivative, a mathematical extension that incorporates reflection symmetry and leads to a modification of the fundamental algebraic structure of quantum systems, particularly the commutation relation between position and momentum. In this context, the theoretical frameworks of Wigner and Yang are first revisited, highlighting the possibility of re-examining the conventional foundations of quantum mechanics. Subsequently, the mathematical formalism of the Dunkl derivative is developed, and its main properties are analyzed, especially its impact on the functional space, which naturally decomposes into even and odd subspaces. The proposed framework is then applied to several important physical systems, both in the non-relativistic and relativistic regimes, through the study of bound states and scattering phenomena in models such as the harmonic oscillator, potential step, potential barrier, and Pöschl–Teller potential. The results reveal that the inclusion of the Dunkl derivative induces significant modifications in the energy spectrum, wave functions, and transmission coefficients, while preserving the exact solvability of the studied systems. Finally, it is shown that standard quantum mechanics is recovered as a limiting case when the deformation parameter vanishes, thereby confirming the consistency and relevance of this generalized approach.

Resumé

Ce mémoire s'inscrit dans le cadre de la généralisation de la mécanique quantique à travers l'introduction de la dérivée de Dunkl, qui constitue une extension mathématique prenant en compte la symétrie de réflexion et conduisant à une modification de la structure algébrique fondamentale des systèmes quantiques, notamment la relation de commutation entre la position et la quantité de mouvement. Dans ce contexte, les approches théoriques de Wigner et de Yang sont d'abord revisitées, mettant en évidence la possibilité de reconsidérer les fondements classiques de la mécanique quantique. Par la suite, le formalisme mathématique de la dérivée de Dunkl est développé et ses principales propriétés sont analysées, en particulier son effet sur l'espace des fonctions, qui se décompose naturellement en sous-espaces pairs et impairs. Ce cadre est ensuite appliqué à plusieurs systèmes physiques importants, tant dans le régime non relativiste que relativiste, à travers l'étude des états liés et des phénomènes de diffusion dans différents modèles tels que l'oscillateur harmonique, la marche de potentiel, la barrière de potentiel et le potentiel de Pöschl–Teller. Les résultats obtenus montrent que l'introduction de la dérivée de Dunkl entraîne des modifications significatives du spectre énergétique, des fonctions d'onde et des coefficients de transmission, tout en préservant la résolubilité analytique des systèmes étudiés. Enfin, il est démontré que la mécanique quantique usuelle est retrouvée comme cas limite lorsque le paramètre de déformation tend vers zéro, ce qui confirme la cohérence et l'intérêt de cette approche généralisée.

Acknowledgements

I would like to express my deepest gratitude and appreciation to every individual who has enlightened my path with knowledge and contributed to my intellectual growth throughout my life.

I am especially honored to extend my profound thanks to the esteemed **Professor Benzair Hadjira**, who has dedicated her life to science and academia. Her constant companionship and guidance throughout my academic journey have been truly inspiring, and I am deeply grateful for her unwavering commitment and generosity.

Furthermore, I would like to convey my sincere thanks and appreciation to my supervisor, **Balsam Nouioua**, for her valuable supervision, insightful guidance, and the significant impact her support has had on the completion of this work.

Dedication

To the one who is beyond words and far above
any description to my mother Saida

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Chapter 1

General Introduction

Quantum mechanics stands as one of the fundamental pillars of modern physics, built upon a set of postulates that profoundly reshaped classical concepts, particularly the algebraic structure of operators and their commutation relations. Among these, the commutation relation between position and momentum plays a central role in determining the physical properties of quantum systems. However, this standard formulation has not remained unquestioned. As early as 1950, Wigner demonstrated that the quantization of energy could be derived solely from the equations of motion, without explicitly imposing canonical commutation relations [1]. In continuation of this idea, Yang proposed in 1951 a more general framework allowing for deformed commutation relations while preserving essential physical constraints [2]. These pioneering works opened the way to generalized quantum mechanical models.

Within this context, the Dunkl derivative emerged as a powerful mathematical tool that combines differential and reflection operators. Originally introduced by Dunkl in 1989 in the study of differential-difference operators associated with reflection groups [3], this concept enables the incorporation of reflection symmetry directly into the algebraic structure of quantum mechanics. This led to the development of Dunkl quantum mechanics, which has attracted increasing attention over the past years. Early investigations focused on one-dimensional systems, revealing the impact of the Dunkl derivative on energy spectra and wave functions [4], as well as extensions within the Wigner–Dunkl framework preserving time-reversal symmetry [5].

More recently, significant progress has been made in both non-relativistic and relativistic regimes. The Schrödinger and Klein–Gordon equations have been generalized within the Dunkl framework to higher-dimensional settings, allowing for a deeper understanding of deformation effects on spectral properties [6]. Furthermore, the thermal behavior of relativistic Dunkl oscillators has been investigated, highlighting the influence of the deformation parameter on thermodynamic quantities [7]. Applications have also extended to modern condensed matter systems, such as graphene in the presence of a constant magnetic field, where Dunkl

formalism modifies Landau levels and electronic properties [8].

In parallel, the path integral approach has played a crucial role in advancing the theoretical understanding of Dunkl-based quantum systems. Path integral formulations have been developed for the Dunkl–Dirac oscillator [9], and Green’s functions have been derived for relativistic systems involving generalized Dunkl derivatives [10]. These methods have further been extended to more complex systems, such as the Coulomb–Dunkl–Klein–Gordon problem in higher dimensions [11], as well as to the formulation of Wigner–Dunkl quantum mechanics within the path integral framework [12].

Analytically solvable and quasi-exactly solvable models have also received considerable attention. In particular, approximate solutions of the Dunkl–Schrödinger equation have been obtained for several potentials, including the hyperbolic Pöschl–Teller, double-well, and Hulthén potentials, providing deeper insight into bound state structures [13]. Additionally, Feynman propagators and approximate solutions have been studied in the presence of generalized Dunkl derivatives, further elucidating the dynamical behavior of quantum systems [14].

Motivated by these developments, the present work aims to investigate the effects of the Dunkl derivative on both non-relativistic and relativistic quantum systems, with particular emphasis on bound states and scattering phenomena. The objective is to highlight the role of algebraic deformations and reflection symmetries in modifying the physical properties of quantum systems, while ensuring consistency with standard quantum mechanics in the appropriate limit.

This dissertation is organized as follows. In Chapter 1, the general theoretical framework is presented, including the fundamentals of quantum mechanics, the Wigner and Yang formulations, and the definition and main properties of the Dunkl derivative. Chapter 2 is devoted to the study of non-relativistic systems within the Dunkl framework, focusing on models such as the harmonic oscillator, potential step, and potential barrier. In Chapter 3, the analysis is extended to relativistic systems by applying the Dunkl formalism to appropriate wave equations. Finally, the work concludes with a general summary of the main results and perspectives for future research.

Chapter 2

Foundations of Wigner, Yang, and Dunkl Quantum Mechanics

2.1 The Wigner Formalism

The Heisenberg-Born-Jordan commutation relation, " $[p, q] = i\hbar$ ", is one of the standard postulates of quantum mechanics, upon which all laws of motion are built. However, in 1950, the physicist Eugene Wigner raised a fundamental question: Do the classical laws of motion inevitably impose this specific commutation relation? Or are there other possible mathematical solutions within "Hilbert Space" that satisfy classical dynamics without strictly adhering to the standard rule? Where the contributions of Wigner and Yang played a vital role in re-examining the logical foundations of quantum mechanics, focusing on the Harmonic Oscillator model. By analyzing the matrix elements of position and velocity and applying Ehrenfest's theorem, we discover that energy quantization and the resulting arithmetic progression of energy levels can be derived directly from the equations of motion. This opens new horizons for a deeper understanding of the ground-state energy constant E_0 and the universality of traditional quantum laws[1].

2.1.1 Derivation of Equations of Motion and the EHP

Standard quantum mechanics is built upon the fundamental Born-Jordan commutation relation:

$$[p, q] = i\hbar. \quad (2.1)$$

Traditionally, this postulate serves as the cornerstone from which the dynamically behavior of a physical system is derived. It provides the essential link between operator algebra and the emergence of physical laws, bridging the gap from abstract commutation rules to their correspondence with classical dynamics. By utilizing the Heisenberg equation of motion,

$\dot{A} = \frac{i}{\hbar} [H, A]$, and applying it to a particle governed by the Hamiltonian $H = \frac{p^2}{2m} + V(x)$, we derive the quantum equivalents of Hamilton's equation:

$$\dot{q} = \frac{p}{m}, \quad \dot{p} = -\frac{\partial V}{\partial x} \quad (2.2)$$

Equations of Motion (Velocity Derivation ($\dot{q}$)): To derive the velocity operator, we analyze the time evolution of the position operator q using the free-particle Hamiltonian $H = \frac{p^2}{2m}$. This is achieved by applying the Heisenberg equation of motion, together with the fundamental commutation relation between position and momentum. Applying the Heisenberg equation, we obtain

$$\dot{q} = \frac{i}{\hbar} \frac{1}{2m} [p^2, q] \quad (2.3)$$

Expanding the p^2 operator using commutator identities to simplify the expression

$$\dot{q} = \frac{i}{\hbar} \frac{1}{2m} \{p[p, q] + [p, q]p\} \quad (2.4)$$

Substituting the derived commutation value into the expression

$$\dot{q} = \frac{i}{\hbar} \frac{1}{2m} \{p(-i\hbar) + (-i\hbar)p\} \quad (2.5)$$

Reducing the constants to obtain the final relation

$$\dot{q} = \frac{p}{m} \quad (2.6)$$

Force derivation ($\dot{p}$) : we can write

$$\dot{p} = -\frac{\partial V}{\partial x}, \quad (2.7)$$

also we have

$$\dot{p} = \frac{i}{\hbar} [H, p] = \frac{i}{\hbar} [V(x), p], \quad (2.8)$$

using the momentum operator $p = -i\hbar \frac{\partial}{\partial x}$, we obtain:

$$\dot{p} = -\frac{\partial V}{\partial x}. \quad (2.9)$$

In the Heisenberg representation, the time derivative of any operator is its commutator with the Hamiltonian H . Thus, the equations of motion are expressed as:

$$[H, q] = -i\hbar \frac{p}{m}, \quad [H, p] = i\hbar \frac{\partial V}{\partial x} \quad (2.10)$$

Wigner chose the Harmonic Oscillator as the primary model because it is the simplest non-trivial system beyond a free particle. To simplify the analysis, the energy H is written in

terms of coordinates x and velocity ϑ , with units chosen such that $\hbar = 1$ and mass $m = 1$ (we set $q \equiv x$ and $p \equiv \vartheta$):

$$H = \frac{1}{2} (x^2 + \vartheta^2). \quad (2.11)$$

The fundamental equations of motion become

$$\langle n | \vartheta | m \rangle = \langle n | i [H, x] | m \rangle. \quad (2.12)$$

Expanding the commutator within the matrix

$$\vartheta_{nm} = \langle n | i (Hx - xH) | m \rangle, \quad (2.13)$$

Since the states $|n\rangle$ and $|m\rangle$ are energy eigenstates, the action of H on them directly yields the energy eigenvalues (E_n and E_m). When H acts to the left on $\langle n|$, it yields E_n , otherwise on $|m\rangle$ is E_m ,

$$\dot{x}_{nm} = \vartheta_{nm} = i(E_n - E_m)x_{nm} \quad (2.14)$$

To determine the velocity operator, we analyze the temporal evolution of the position operator x . By employing the Heisenberg equation of motion with the free-particle Hamiltonian $H = \frac{p^2}{2m}$, we can express the velocity in terms of momentum. Applying the Heisenberg equation of motion, we find

$$\dot{\vartheta} = -x = i [H, \vartheta] \quad (2.15)$$

Expanding the commutator within the matrix

$$\dot{\vartheta}_{nm} = -x_{nm} = i \langle n | (H\vartheta - \vartheta H) | m \rangle \quad (2.16)$$

Which yields to

$$-x_{nm} = i(E_n - E_m)\vartheta_{nm} \quad (2.17)$$

By substituting ϑ_{nm} in the equation (2.17), we find

$$x_{nm} = (E_n - E_m)^2 x_{nm} \quad (2.18)$$

This yields

$$(E_n - E_m)^2 = 1 \Rightarrow E_n - E_m = \pm 1 \quad (2.19)$$

This equation demonstrates that the energy difference between any two allowable transition states must be equal to unity (in natural units). This is the basis of quantization, where energy levels follow an arithmetic progression rather than continuous values. Therefore from Eq. (2.19), we can write:

$$E_n - E_{n-1} = 1 \quad (2.20)$$

$$E_n - E_{n+1} = -1 \quad (2.21)$$

For $n = 1$, we have:

$$E_1 - E_0 = 1 \quad \text{and} \quad E_1 - E_2 = -1 \quad (2.22)$$

and yields $E_2 = E_0 + 2$. By repeating the process, we obtain the following relation:

$$E_n = E_0 + n \quad (2.23)$$

This result demonstrates that the energy spectrum is quantized and follows an arithmetic progression. In particular, the energy of the level n depends linearly on the quantum number n , while E_0 represents the ground-state energy. Unlike the standard formulation, E_0 is treated here as a parameter determined by Eugene Wigner's conditions rather than as a fixed constant. From Eq. (2.14), if $m = n + 1$, we obtain:

$$\vartheta_{nn+1} = i(E_n - E_{n+1})x_{nn+1} = -ix_{nn+1} \quad (2.24)$$

Since $\hat{x}$ is a Hermitian operator, its matrix elements satisfy " $x_{n+1n} = x_{nn+1}$ ", thus as:

$$\vartheta_{nn+1} = -ix_{nn+1} = -ix_{n+1n} \quad (2.25)$$

If $m = n - 1$, the Eq. (2.14), gives

$$\vartheta_{nn-1} = i(E_n - E_{n-1})x_{nn-1} = ix_{nn-1} \quad (2.26)$$

or

$$\vartheta_{nn-1} = ix_{nn-1} = -\vartheta_{nn+1} \quad (2.27)$$

The total energy for state n is defined as the expectation value of the Hamiltonian operator H ($E_n = \langle n | H | n \rangle$). Substituting the Hamiltonian in terms of position and velocity, we have:

$$E_n = \frac{1}{2} \langle n | (x^2 + \vartheta^2) | n \rangle \quad (2.28)$$

Using the linearity property, we separate the two operators:

$$E_n = \frac{1}{2} \langle n | x^2 | n \rangle + \frac{1}{2} \langle n | \vartheta^2 | n \rangle$$

To expand the squares (x^2 , ϑ^2), we insert a complete set of states k . Since the selection rules for the harmonic oscillator restrict transitions to $k = n \pm 1$ only, the equation becomes[1]:

$$E_n = \frac{1}{2} \sum_{k=(n+1, n-1)} \langle n | \hat{x} | k \rangle \langle k | \hat{x} | n \rangle + \frac{1}{2} \sum_{k=(n+1, n-1)} \langle n | \vartheta | k \rangle \langle k | \vartheta | n \rangle \quad (2.29)$$

or

$$E_n = \frac{1}{2} \left\{ \begin{aligned} &\sum_{k=(n+1, n-1)} \langle n | \hat{x} | n+1 \rangle \langle n+1 | \hat{x} | n \rangle + \langle n | \hat{x} | n-1 \rangle \langle n-1 | \hat{x} | n \rangle \\ &+ \sum_{k=(n+1, n-1)} \langle n | \vartheta | n+1 \rangle \langle n+1 | \vartheta | n \rangle + \langle n | \vartheta | n-1 \rangle \langle n-1 | \vartheta | n \rangle \end{aligned} \right\} \quad (2.30)$$

Expanding the summation for the non-zero terms:

$$E_n = \frac{1}{2} \{x_{nn+1}x_{n+1n} + x_{nn-1}x_{n-1n} + \vartheta_{nn+1}\vartheta_{n+1n} + \vartheta_{nn-1}\vartheta_{n-1n}\} \quad (2.31)$$

Substituting the velocity values in terms of x as derived previously:

$$E_n = \frac{1}{2} \{x_{nn+1}^2 + x_{nn-1}^2 + (-ix_{nn+1})(ix_{n+1n}) + (ix_{nn-1})(-ix_{n-1n})\} \quad (2.32)$$

This leads to the following expression for the energy spectrum

$$E_n = E_0 + n = x_{n-1,n}^2 + x_{n,n+1}^2 \quad (2.33)$$

This result demonstrates that energy is quantized and depends linearly on the quantum number n . Here, Wigner assumed that the energy levels follow the pattern $E_n = E_0 + n$.

The coordinate values can be determined sequentially, for $n = 0$, $x_{0,1}^2 = E_0$ and for $n = 1$ we find $E_1 = E_0 + 1$ which gives $E_1 = x_{0,1}^2 + x_{1,2}^2$ and $x_{1,2}^2 = 1$. Also for $n = 2$, we obtain:

$$E_2 = E_0 + 2 \quad (2.34)$$

$$E_2 = x_{1,2}^2 + x_{2,3}^2 \quad (2.35)$$

$$E_0 + 2 = 1 + x_{2,3}^2 \quad (2.36)$$

$$x_{2,3}^2 = E_0 + 2 - 1 \quad (2.37)$$

$$x_{2,3}^2 = E_0 + 1 \quad (2.38)$$

By substituting the previous relations and performing the recursive calculations starting from $n = 0$. Wigner deduced the general pattern for the coordinate matrix elements as follows:

1. **For even values (n even):**

$$x_{n,n+1} = (E_0 + n/2)^{\frac{1}{2}} \quad (2.39)$$

2. **For odd values (n odd):**

$$x_{n,n+1} = \left(\frac{n+1}{2}\right)^{\frac{1}{2}} \quad (2.40)$$

Now we expanding the commutator $[\vartheta, x]$ into :

$$\langle n | [\vartheta, x] | n \rangle = \langle n | (\vartheta x - x \vartheta) | n \rangle \quad (2.41)$$

Insert a complete set of intermediate states k to transform the operator product into a sum of matrix elements:

$$\langle n | [\hat{\vartheta}, \hat{x}] | n \rangle = \sum_{\substack{k=n+1 \\ n-1}} \langle n | \vartheta | k \rangle \langle k | (\hat{x}) | n \rangle - \sum_{\substack{k=n+1 \\ n-1}} \langle n | (\hat{x}) | k \rangle \langle k | (\vartheta) | n \rangle$$

Restrict the summation to adjacent energy levels ($n + 1$ and $n - 1$) according to the selection rules:

$$\langle n | [\vartheta, x] | n \rangle = \vartheta_{nn+1}x_{n+1n} + \vartheta_{nn-1}x_{n-1n} - x_{nn+1}\vartheta_{n+1n} - x_{nn-1}\vartheta_{n-1n} \quad (2.42)$$

Substitute the velocity matrix elements ϑ with their equivalent expression in terms of position x :

$$\langle n | [\vartheta, x] | n \rangle = -ix_{nn+1}x_{n+1n} + ix_{nn+1}x_{n-1n} - ix_{nn+1}x_{n+1n} + ix_{nn-1}x_{n-1n} \quad (2.43)$$

Group the common terms and simplify the expression to reach the final form of the relation:

$$\langle n | [\vartheta, x] | n \rangle = -2i (x_{nn+1}^2 + x_{n-1n}^2) \quad (2.44)$$

Wigner demonstrated through the sequence of matrix elements that the first position value at $n = 0$ is directly related to the lowest energy level (ground state) by the relation:

$$x_{0,1}^2 = E_0 \quad (2.45)$$

This relation serves as the cornerstone for determining all subsequent commutator values for higher energy levels. When substituting the standard ground-state energy value for a quantum harmonic oscillator, $E_0 = \frac{1}{2}$. The diagonal elements of the commutator reduce to:

$$[\vartheta, x] = -i \quad (2.46)$$

This is the familiar commutation relation found in standard quantum mechanics.

From the above, Wigner concluded that the commutator is not strictly limited to the value $-i$, but is instead a function of the constant E_0 . This leads to the final generalized formula:

$$([\vartheta, x] + i)^2 = -(2E_0 - 1)^2 \quad (2.47)$$

where E_0 is a constant characterizing the ground-state solution. For large n , all solutions converge to the usual classical ones, reinforcing the correspondence principle[1].

2.2 Yang's Formalism and Commutation Relation

While Wigner's previous study of the harmonic oscillator concluded that the commutation relations cannot be uniquely derived solely from the equations of motion, Yang's present research offers a divergent perspective, demonstrating that by adopting a more stringent definition of the Hilbert space and a strict expansion theorem, these rules emerge as a unique and necessary solution[2].

Yang set forth the following basic condition :

1. He considered the case of a harmonic oscillator where $H = \frac{1}{2}(x^2 + \dot{x}^2)$
2. the equation of motion $\ddot{x} + x = 0$
3. Complete Hilbert Space: Defined by the energy eigenfunctions $\Psi_n(x)$ which vanish "at infinity ($\lim_{x \rightarrow \pm\infty} \Psi_n(x) \rightarrow 0$).
4. The Strict Superposition Principle: Requires that the sum of the eigenfunctions converges absolutely and uniformly, not just "in the mean."

From the Heisenberg relation:

$$\dot{f} = [f, H] \quad (2.48)$$

where f is any dynamical variable (such as position or velocity), H is the Hamiltonian operator (the total energy of the system) and $[f, H]$ represents the "Commutator". To investigate whether the commutation rules follow uniquely from the equations of motion, Yang introduced a dynamical variable denoted as $\hat{S}$. This operator represents the deviation of the commutator $[x, \dot{x}]$ from its standard quantum mechanical value.

We begin by analyzing the time evolution of operators within Yang's quantum model, adopting the modified commutation relation

$$[x; \dot{x}] = \hat{S} + 1. \quad (2.49)$$

Our goal is to verify the first commutation relation connecting the operator $\hat{S}$ to the velocity operator. To do so, we apply the Heisenberg equation of motion to the position operator x :

$$\dot{x} = [x, H] = \left[x, \frac{1}{2}(x^2 + \dot{x}^2) \right] = \left[x; \frac{1}{2}\dot{x}^2 \right] \quad (2.50)$$

or

$$\dot{x} = \frac{1}{2}(\dot{x}[x, \dot{x}] + [x, \dot{x}]\dot{x}) \quad (2.51)$$

From (2.49), we obtain

$$\dot{x} = \frac{1}{2}(\{S, \dot{x}\} + 2\dot{x}), \quad (2.52)$$

It follows that

$$\{S, \dot{x}\} = 0 \quad (2.53)$$

we now turn to the derivation of the commutator between S and the position operator x :

$$\ddot{x} = [\dot{x}, H] = \left[\dot{x}, \frac{1}{2}(x^2 + \dot{x}^2) \right] = \left[\dot{x}; \frac{1}{2}x^2 \right] \quad (2.54)$$

subsisting $[\dot{x}, x] = -(S + 1)$ into $\ddot{x}$, we obtain

$$\ddot{x} = \frac{-1}{2}(\{S, x\} + 2x) \quad (2.55)$$

we have $\ddot{x} + x = 0 \implies \ddot{x} = -x$ and by substitution:

$$-x = \frac{-1}{2}(\{S, x\} + 2x) \quad (2.56)$$

It follows that

$$\{S, x\} = 0 \quad (2.57)$$

From the expression $\{S, \dot{x}\} = 0$ and $\{S, x\} = 0$ one can show that $[S, H] = 0$: we have:

$$[S, H] = \left[S, \frac{1}{2}(\dot{x}^2 + x^2) \right] \quad (2.58)$$

or

$$[S, H] = \frac{1}{2}([S, \dot{x}^2] + [S, x^2]) \quad (2.59)$$

From $\{S, \dot{x}\} = 0$

$$\{S, \dot{x}\} = 0 \quad (2.60)$$

$$\implies S\dot{x} + \dot{x}S = 0 \quad (2.61)$$

$$\implies \dot{x}S\dot{x} + \dot{x}^2S = 0 \quad (2.62)$$

$$\implies -S\dot{x}^2 + \dot{x}^2S = 0 \quad (2.63)$$

$$\implies [S, \dot{x}^2] = 0 \quad (2.64)$$

Also from $\{S, x\} = 0$

$$\{S, x\} = 0 \quad (2.65)$$

$$\implies Sx + xS = 0 \quad (2.66)$$

$$\implies xSx + x^2S = 0 \quad (2.67)$$

$$\implies -Sx^2 + x^2S = 0 \quad (2.68)$$

$$\implies [S, x] = 0 \quad (2.69)$$

Hence, the commutator $[S, H] = 0$. We now demonstrate that the operator $\{S^2, \dot{x}\} = 0$ then, $\{S^2, x\} = 0$, we have

$$\{\dot{x}, S\} = 0 \implies \dot{x}S + S\dot{x} = 0 \quad (2.70)$$

multiplying from the left by S

$$S\dot{x}S + S^2\dot{x} = 0 \implies -\dot{x}S^2 + S^2\dot{x} = 0 \quad (2.71)$$

thus $[S^2, \dot{x}] = 0$. We can prove $\{S^2, x\} = 0$ in the same way $[S^2, x] = 0$.

Since $[S, H] = 0$ and $[S^2, x] = [S^2, x] = 0$, this indicates that S is a constant of motion. Furthermore, S^2 is a real numerical constant. In the x -representation, $\{S, x\} = 0$ the equation becomes

$$Sx + xS = 0 \quad (2.72)$$

$$\Rightarrow \langle x' | Sx | x'' \rangle + \langle x' | xS | x'' \rangle = 0 \quad (2.73)$$

$$\Rightarrow x'' \langle x' | S | x'' \rangle + x' \langle x' | S | x'' \rangle = 0 \quad (2.74)$$

$$\Rightarrow (x' + x'') \langle x' | S | x'' \rangle = 0 \quad (2.75)$$

To leads

$$\langle x' | S | x'' \rangle = c(x') \delta(x' + x'') \quad (2.76)$$

where $c(x')$ an optional function of x and it requires S to satisfy the hermiticity property, to be $c(x') = c(-x')^*$

Now, to determine the explicit form of the operator S , we transition to the position representation. We find that S is intrinsically linked to the parity (reflection) operator R , which maps x to $-x$. The parity operator is defined by its action on the position eigenstate[2]:

$$S = C(x) R \quad (2.77)$$

with

$$R | x \rangle = | -x \rangle \quad (2.78)$$

Using the representation above, we derive the mathematical expression for the velocity operator

$$\dot{x} = -i \frac{d}{dx} + g(x) + \frac{ic(x)}{2x} R \quad (2.79)$$

From the expression $\dot{x}$: we can calculate the commutator $[x, \dot{x}] = i(1 + S)$:

$$\begin{aligned} [x, \dot{x}] f(x) &= \left[x, -i \frac{d}{dx} + g(x) + \frac{iC(x)}{2x} R \right] f(x) \\ [x, \dot{x}] f(x) &= \left[x, -i \frac{d}{dx} \right] f(x) + \left[x, \frac{iC(x)}{2x} R \right] f(x) \\ [x, \dot{x}] f(x) &= if(x) + \frac{i}{2} \left(\frac{x C(x)}{x} R f(x) - \frac{C(x)}{x} R x f(x) \right) \\ [x, \dot{x}] f(x) &= if(x) + \frac{i}{2} (C(x) R f(x) + C(x) R f(x)) \\ [x, \dot{x}] f(x) &= i(f(x) + C(x) R f(x)) \end{aligned} \quad (2.80)$$

thus

$$[x, \dot{x}] = i(1 + C(x) R) = i(1 + S) \quad (2.81)$$

The term $g(x)$ can be removed by choosing the phase factor appropriately in the x-representation.

Using the operator in the new representation, we obtain:

$$\left(\frac{d}{dx} \right)^* = e^{-iy} \frac{d}{dx} e^{iy} = \frac{d}{dx} + i \frac{dy}{dx} \quad (2.82)$$

$$R^* = e^{-i\nu} R e^{i\nu} = e^{-2i\nu} R \quad (2.83)$$

If we choose for y the value $y = \int g(x)dx$, then the expression for $\dot{x}$ becomes:

$$\dot{x} = -i\left(\frac{d}{dx}\right)^* + \frac{iC'(x)}{2x}R^* \Rightarrow C'(x) = C(x)e^{iy} \quad (2.84)$$

and

$$\dot{x} = -i\left(\frac{d}{dx}\right) + \frac{iC(x)}{2x}R \quad (2.85)$$

we now verify that $C(x)$ is a constant:

$$\begin{aligned} (S\dot{x} + \dot{x}S)f(x) &= 0 \\ \Rightarrow C(x)R\left(-i\frac{d}{dx} + \frac{iC(x)}{2x}R\right)f(x) + \left(-i\frac{d}{dx} + \frac{iC(x)}{2x}R\right)C(x)Rf(x) &= 0 \end{aligned} \quad (2.86)$$

Then

$$= -iC(x)R\frac{df(x)}{dx} + iC(x)R\frac{C(x)}{2x}Rf(x) - i\frac{d}{dx}C(x)Rf(x) + \frac{iC(x)}{2x}RC(x)Rf(x) = 0 \quad (2.87)$$

$$-iC(x)R\frac{df(x)}{dx} - i\frac{C(x)}{2x}RC(x)Rf(x) + i\frac{C(x)}{2x}RC(x)Rf(x) = 0 \quad (2.88)$$

$$-iC(x)\frac{d}{dx}(Rf(x)) - iC'(x)Rf(x) = 0 \quad (2.89)$$

Finally, we find:

$$= -iC(x)R\frac{df(x)}{dx} + iC(x)R\frac{d}{dx}f(x) - iC'(x)Rf(x) = 0 \quad (2.90)$$

$$-iC'(x)Rf(x) = 0 \Rightarrow -iC'(x)R = 0 \quad (2.91)$$

$$C'(x) = 0 \Rightarrow C(x) = \text{cst} \quad (2.92)$$

Where $C(x)$ is a numerical constant, and hence to :

$$\dot{x} = -i\left(\frac{d}{dx}\right) + \frac{ic}{2x}R \quad (2.93)$$

here, $2x$ appears in the denominator it is necessary for the operator $\dot{x}$ to be explicitly Hermitian. When the reflection operator R acts in the presence of $\frac{1}{x}$, the sign change resulting from differentiation is balanced. This ensures that the operator represents a measurable physical observable (such as velocity) and remains consistent with the system's symmetry requirements.

2.2.1 Energy Levels and Eigenvalues

[2]We utilize the Ladder Operators (Creation and Annihilation operators) η and η^* , which are defined as linear combinations of the position and velocity (momentum) operators:

$$\eta = \frac{\dot{x} + ix}{\sqrt{2}}, \quad \eta^* = \frac{\dot{x} - ix}{\sqrt{2}} \quad (2.94)$$

(Note: η is the adjoint/conjugate of the first operator).

In classical mechanics, the Hamiltonian for a Harmonic oscillator is expressed as $H = \frac{1}{2}(x^2 + \dot{x}^2)$. Mathematically, this takes the form of a sum of squares ($a^2 + b^2$), which can be factorized into the product of complex conjugates $(a + ib)(a - ib)$. In quantum mechanics, we apply the same algebraic decomposition to the operators to simplify the determination of energy eigenvalues.

The imaginary unit i is used to guarantee the required algebraic properties (commutation relations). The normalization factor $\frac{1}{\sqrt{2}}$ ensures that the product of the operators is properly scaled. The product of these operators is given by:

$$\eta^* \eta = \frac{1}{2} (\dot{x} - ix) (\dot{x} + ix) = \frac{1}{2} (\dot{x}^2 + x^2 + [x, \dot{x}]) \quad (2.95)$$

the Hamiltonian is written as

$$H = \eta^\dagger \eta - \frac{1}{2} [x, \dot{x}] \quad (2.96)$$

using the commutation relation:

$$[x, \dot{x}] = S + 1 = 1 + CR \quad (2.97)$$

we can also write

$$H = \eta^\dagger \eta - \frac{1}{2} (1 + CR) \quad (2.98)$$

the ground state is defined by the annihilation condition

$$\eta |0\rangle = 0 \quad (2.99)$$

applying the Hamiltonian on the ground state, we obtain:

$$H |0\rangle = \left(\eta^\dagger \eta - \frac{1}{2} [x, \dot{x}] \right) |0\rangle \quad (2.100)$$

or

$$H |0\rangle = \left(\eta^\dagger \eta - \frac{1}{2} (1 + CR) \right) |0\rangle \quad (2.101)$$

since $\eta^\dagger \eta |0\rangle = 0$, it follows that

$$E_0 = H |0\rangle = \frac{1}{2} (1 + CR) |0\rangle \quad (2.102)$$

thus is the ground state energy.

Let us calculate the excited state energy: the excited states are obtained by applying the creation operator

$$|n\rangle = (\eta^\dagger)^n |0\rangle \quad (2.103)$$

To obtain the state n , we apply the raising operator n times to a state. The difference between two consecutive levels is 1. The reflection operator R changes its eigenvalue between $+1$ and

-1 with each level (i.e., the wave functions alternate between even and odd). Through mathematical induction, we find that the energy levels follow this pattern:

$$E_0 = \frac{1}{2}(1 + cR_0), \quad E_1 = \frac{1}{2}(3 + cR_0)$$

and

$$E_2 = \frac{1}{2}(5 + cR_0), \quad E_3 = \frac{1}{2}(7 + cR_0) \quad (2.104)$$

applying the Hamiltonian on these states gives:

$$H |n\rangle = \frac{1}{2}(2n + 1 + cR_0) |n\rangle \quad (2.105)$$

therefore, the energy eigenvalues are

$$E_n = \frac{1}{2}(2n + 1 + cR_0) \quad (2.106)$$

these result show that the energy levels of the quantum harmonic oscillator are equally spaced.

$$E_n = \frac{1}{2}(2n + 1 + c_0) \quad (2.107)$$

where c_0 is an arbitrary constant arising from the quantization condition.

Starting from the condition

$$cR_0 > -1 \quad (2.108)$$

The spectral structure of the system can be analyzed through the study of the ground state in the position representation. We define the ground-state wave function as

$$\Psi_0(x) = \langle x | 0 \rangle \quad (2.109)$$

Using the previously established operator relations, we obtain the fundamental differential equation:

$$\langle x | \frac{1}{\sqrt{2}}(\dot{x} - ix) | 0 \rangle = \langle x | \frac{i}{\sqrt{2}}(-\frac{d}{dx} + \frac{c}{2x}R - x) | 0 \rangle \quad (2.110)$$

then

$$(\frac{d}{dx} + x)\Psi_0(x) = \frac{c}{2x}\Psi_0(-x) \quad (2.111)$$

This equation determines the regular behavior of the ground state and contains a singular term of the type $\frac{1}{x}$, indicating the presence of a singular point at $x = 0$. Therefore, analyzing the local behavior near the origin is essential in identifying physically acceptable solutions.

To investigate the behavior as $x \rightarrow 0$, we assume an analytic form of the solution:

$$\Psi_0(x) \sim x^\tau \quad (2.112)$$

Substituting this ansatz into Eq. 3.232 yields a consistency condition that determines the allowed values of the parameter τ . Imposing normalizability leads to physically admissible solutions of the form:

$$\Psi_0^{(\tau)}(x) = b_\tau x^\tau e^{\frac{-x^2}{2}} \quad (2.113)$$

where b_τ is a normalization constant. Equation 3.232 effectively represents an eigenvalue problem with respect to the parameter c , since acceptable solutions exist only for discrete values of this parameter. Consequently, the allowed spectral structure is given by:

Allowed Values of c , cR_0 , and τ

τ	c	cR_0
0	0	0
1	-2	2
2	4	4
3	-6	6
4	8	8

. This demonstrates that the regularity condition at the origin leads to a quantization of the structural parameter of the system.

The excited states are generated by repeated application of the raising operator η on the ground state:

$$\Psi_0(x)^{(\tau)} = \langle x | \eta^n | 0 \rangle = \left(\frac{i}{\sqrt{2}} \right)^n \left(-\frac{d}{dx} + x + \frac{c}{2x} \right)^n \Psi_0(x)^{(\tau)} N_n^{(\tau)} \quad (2.114)$$

The normalization constant is given by

$$N_n^{(\tau)} = [(1 + 2\tau)(3 + 2\tau)\dots]^{-1} \quad (2.115)$$

where the number of factors equals n .

Substituting Eq. 2.113 into Eq. 2.114 yields the explicit expression:

$$\Psi_n(x)^{(\tau)} = \left(\frac{i}{\sqrt{2}} \right)^n N_n^{(\tau)} b_\tau x^\tau e^{\frac{-x^2}{2}} \left[\left(D + \frac{2\tau}{x} \right) D \left(D + \frac{2\tau}{x} \right) D \dots \right] e^{-x^2} \quad (2.116)$$

From Eq. 2.116, it follows that: If n is even, the lowest power of x appearing in the wave function is x^τ . If n is odd, the lowest power is $x^{\tau+1}$. Therefore, all functions $\Psi_n(x)^{(\tau)}$ vanish at $\tau \neq 0$. This property is crucial when discussing the completeness of the eigenfunction set in the Hilbert space.

Consider the expansion of a function that is regular at the origin, such as

$$e^{\frac{-x^2}{2}} = \sum_{n=0}^{\infty} a_n^{(\tau)} \Psi_n(x)^{(\tau)} \quad (2.117)$$

This expansion fails at $\tau \neq 0$, although it can be shown to converge in the mean according to

$$\lim_{N \rightarrow \infty} \left| f(x) - \sum_{n=0}^N a_n^{(\tau)} \Psi_n(x)^{(\tau)} \right|^2 dx = 0 \quad (2.118)$$

Hence, the only set of solutions that satisfies all physical requirements — regularity, normalizability, and completeness — corresponds to $\tau = 0$. It follows that the operator S must vanish in order to ensure the structural consistency of the Hilbert space.

2.3 Wigner- Dunkl Quantum Mechanics:

2.3.1 The Dunkl Derivative A Structural Generalization

The Dunkl derivative serves as a natural and systematic generalization of the standard spatial derivative. It was developed to integrate discrete reflection symmetries directly into the differential framework of quantum mechanics. In conventional quantum theory, the one-dimensional momentum operator is defined via the ordinary partial derivative as[12]:

$$\hat{p} = -i\hbar \frac{\partial}{\partial x} \quad (2.119)$$

This definition leads to the fundamental Heisenberg canonical commutation relation:

$$[\hat{x}, \hat{p}] = i\hbar \quad (2.120)$$

Within the Wigner-Dunkl formalism, this structure is extended by introducing the reflection operator R . The action of this operator on a wave function $f(x)$ is defined as:

$$\hat{R}f(x) = f(-x) \quad (2.121)$$

where the operator satisfies the property $\hat{R}^2 = 1$. Incorporating R allows for the construction of a modified differential operator that explicitly encodes reflection symmetry. In one dimension, the Dunkl derivative is expressed as:

$$\hat{D}_x^\nu = \frac{\partial}{\partial x} + \frac{\nu}{x}(1 - \hat{R}) \quad (2.122)$$

Here, ν represents a real deformation parameter. It is important to note that the standard derivative is recovered in the limit where $\nu \rightarrow 0$, ensuring mathematical consistency with standard quantum mechanics. Consequently, the momentum operator takes on a "deformed" structure:

$$\hat{p} = -i\hbar \hat{D}_x^\nu \quad (2.123)$$

This modification results in a deformed Heisenberg algebra, formulated as:

$$[\hat{x}, \hat{p}] f(x) = xpf(x) - px f(x) \quad (2.124)$$

$$[\hat{x}, \hat{p}] f(x) = x(-i\hbar D_x^\nu) f(x) - (-i\hbar D_x^\nu) x f(x) \quad (2.125)$$

$$[\hat{x}, \hat{p}] f(x) = -i\hbar x \left[\frac{\partial}{\partial x} + \frac{\nu}{x}(1-R) \right] f(x) + i\hbar \left[\frac{\partial}{\partial x} + \frac{\nu}{x}(1-R) \right] x f(x) \quad (2.126)$$

$$[\hat{x}, \hat{p}] f(x) = -i\hbar(-vRf(x) - f(x) - vRf(x)) \quad (2.127)$$

$$[\hat{x}, \hat{p}] f(x) = i\hbar(1 + 2vR)f(x) \quad (2.128)$$

$$[x, p] = i\hbar(1 + 2\nu R) \quad (2.129)$$

This framework does not represent an arbitrary modification of quantum theory; rather, it provides a symmetry-driven extension of the differential structure. This approach leads to a reorganization of the underlying Hilbert space and significantly influences the spectral properties of quantum Hamiltonians, while maintaining the exact solvability of various physically relevant models.

The action of the one-dimensional Dunkl derivative on a generic function $f(x)$ yields:

$$\hat{D}_x^\nu f(x) = \frac{df(x)}{dx} + \frac{\nu}{x} [f(x) - f(-x)] \quad (2.130)$$

This expression demonstrates that the Dunkl derivative is a non-local operator. Unlike the ordinary derivative, it explicitly involves the value of the function at the reflected point $(-x)$. This non-locality encodes the discrete symmetry of the system directly into the differential structure, modifying the algebraic structure of observable while maintaining the consistency of the equations of motion[12].

The fundamental properties of the reflection operator become particularly transparent when examining the action of the Dunkl derivative on the subspaces of even and odd functions. Consider an even function $f_{\text{even}}(x)$ that satisfies the condition $f_{\text{even}}(-x) = f_{\text{even}}(x)$. In this specific case, the contribution from the reflection term in the derivative's definition vanishes, causing the Dunkl derivative to reduce to the ordinary derivative:

$$\hat{D}_x^\nu f_{\text{even}}(x) = \frac{df_{\text{even}}(x)}{dx} \quad (2.131)$$

Conversely, for an odd function $f_{\text{odd}}(x)$ satisfying $f_{\text{odd}}(-x) = -f_{\text{odd}}(x)$, the action of the operator takes a modified form that explicitly incorporates the deformation parameter ν

$$\hat{D}_x^\nu f_{\text{odd}}(x) = \frac{df_{\text{odd}}(x)}{dx} + \frac{2\nu}{x} f_{\text{odd}}(x)$$

2.3.2 The leibniz rule for the dunkl derivative:

The Dunkl derivative $\hat{D}_x^\nu$ is characterized by its deviation from the classical product rule, a direct consequence of the reflection operator $\hat{R}$ interfering with the differentiation process.

To analyze the action of this operator on the product of two functions $f(x)g(x)$, we follow these computational steps: Applying this operator to the product (fg) yields:

$$\hat{D}_x^\nu(fg) = \frac{d}{dx}(fg) + \frac{\nu}{x}(1 - \hat{R})(fg) \quad (2.133)$$

Part 1 (Ordinary Differentiation): Follows the standard Leibniz rule: $f'g + fg'$. Part 2 (Reflection Effect): Since $\hat{R}(fg) = (\hat{R}f)(\hat{R}g)$, this term expands to: $\frac{\nu}{x}(fg - (\hat{R}f)(\hat{R}g))$. By rearranging the terms and comparing them with the individual Dunkl derivatives of f and g , we arrive at the modified Leibniz rule:

$$\hat{D}_x^\nu(fg) = (\hat{D}_x^\nu f)g + f(\hat{D}_x^\nu g) - \Delta(f, g) \quad (2.134)$$

Where the correction term $\Delta(f, g)$ is given by :

$$\Delta(f, g) = \frac{\nu}{x} \left[(1 - \hat{R})f \right] \left[(1 - \hat{R})g \right] \quad (2.135)$$

This formula demonstrates that the Dunkl derivative does not follow the local behavior of the ordinary derivative except in two specific cases.

The study of the Dunkl derivative acting on monomials of the form x^n constitutes a fundamental approach to understanding the algebraic structure imposed by deformation on the functional space. By applying the operator $\hat{D}_x^\nu$ to the power terms x^n , we obtain the following result:

$$\hat{D}_x^\nu x^n = \left[\frac{\partial}{\partial x} + \frac{\nu}{x}(1 - \hat{R}) \right] x^n = nx^{n-1} + \frac{\nu}{x}(x^n - Rx^n) \quad (2.136)$$

where $Rx^n = (-x)^n = (-1)^n x^n$

$$\hat{D}_x^\nu x^n = [n + \nu(1 - (-1)^n)] x^{n-1} \quad (2.137)$$

This result motivates the introduction of the Dunkl numbers, denoted by $[n]_\nu$, which are defined as:

$$[n]_\nu = n + \nu(1 - (-1)^n) \quad (2.138)$$

Consequently, the action of the Dunkl derivative can be expressed in a compact form, analogous to the standard power rule in ordinary calculus:

$$\hat{D}_x^\nu x^n = [n]_\nu x^{n-1} \quad (2.139)$$

An analysis of these deformed integers reveals a distinct behavior based on the parity of the exponent n :

1. For even powers ($n = 2m$), the Dunkl numbers reduce to the standard integers: $[2m]_\nu = 2m$

2. For odd powers ($n = 2m + 1$), the effect of the deformation is explicitly manifested:
 $[2m + 1]_\nu = 2m + 1 + 2\nu$

This selective modification indicates that the Dunkl deformation non-trivially affects only the odd sector of the function space, while leaving even powers unchanged. This property provides a clear algebraic origin for the energy spectrum deformations observed in Dunkl-type quantum systems.

Furthermore, in the limit where the deformation parameter vanishes ($\nu \rightarrow 0$) the Dunkl numbers recover their standard counterparts ($\lim_{\nu \rightarrow 0} [n]_\nu = n$). This recovery ensures full compatibility with conventional quantum mechanics and confirms that the Dunkl formalism serves as a consistent, symmetry-driven extension of the standard theory.

In addition, we can analyze the algebraic properties of Dunkl numbers and the generalization of the factorial function within the deformed framework. Based on the definition provided in Eq. (2.14), the Dunkl factorial $[n]_\nu!$ is introduced as the product of successive Dunkl numbers:

$$[n]_\nu! = [1]_\nu [2]_\nu \dots [n]_\nu \quad (2.140)$$

To characterize the spectral behavior of the system, it is essential to derive an explicit expression for the even factorial $[2m]_\nu$ in terms of m . By expanding the product and substituting the respective parity-dependent values, as follows:

To derive the expression for even numbers $n = 2m$, we will multiply the even and odd numbers together up to $2m$. Then, we will split the product into even and odd terms:

$$[2m]_\nu! = ([2]_\nu [4]_\nu \dots [2m]_\nu) \quad (2.141)$$

$$[2m + 1]_\nu! = ([1]_\nu [3]_\nu \dots [2m - 1]_\nu) \quad (2.142)$$

From the laws we previously derived:

1. Even term: $[2m]_\nu = 2m$
2. Odd term: $[2m - 1]_\nu = (2m - 1) + 2\nu$

Calculating the product of even terms: This part is simple as it consists of multiples of 2:

$$[2m]_\nu! = 2 \cdot 4 \cdot 6 \dots (2m) = 2^m (1 \cdot 2 \cdot 3 \dots m) = 2^m m! \quad (2.143)$$

Calculating the product of odd terms: Substitute into the formula: $[2m - 1]_\nu = (2m - 1) + 2\nu$

$$[2m + 1]_\nu! = [1]_\nu \cdot [3]_\nu \dots [2m - 1]_\nu = (1 + 2\nu)(3 + 2\nu)(5 + 2\nu) \dots (2m - 2 + 2\nu) \quad (2.144)$$

To simplify this part, we factor out 2 from each bracket. Since there are m brackets:

$$[2m+1]_\nu! = 2^m(\nu + \frac{1}{2})(\nu + \frac{3}{2}) \left(\nu + \frac{5}{2}\right) \dots (\nu + m - \frac{1}{2}) \quad (2.145)$$

This sequence of multiplication is exactly the definition of the Pochhammer symbol $(\nu + \frac{1}{2})_\nu$ multiplied by 2^m . Since the Pochhammer symbol is: $(a)_m = a(a+1)(a+2) \dots (a+m-1)$. Therefore, the odd terms equal:

$$2^m(\nu + \frac{1}{2})_\nu \quad (2.146)$$

Multiply the result of the odd terms by the even terms:

$$[2m]_\nu! = (2^m m!) \times (2^m(\nu + \frac{1}{2})) \quad (2.147)$$

$$[2m]_\nu! = 2^m m! (\nu + \frac{1}{2})_m \quad (2.148)$$

We can put:

$$\frac{\Gamma(m + \nu + \frac{1}{2})}{\Gamma(\nu + \frac{1}{2})} = (\nu + \frac{1}{2})_m \quad (2.149)$$

Finding the expression for odd terms:

We have the formula for the factorial of $2n$ as all even terms:

$$[2m]_\nu! = 2^m m! (\nu + \frac{1}{2})_m \quad (2.150)$$

We multiply it by the value of the term $2n+1$ as an odd term $[2m+1]! = 2m+1 + 2\nu$, so it becomes:

$$[2m+1]_\nu! = 2^m m! (\nu + \frac{1}{2})_m \times (2m+1 + 2\nu) \quad (2.151)$$

Then

$$[2m+1]_\nu! = 2^m m! (\nu + \frac{1}{2})_m \times 2 \left(\nu + m + \frac{1}{2}\right) \quad (2.152)$$

and

$$[2m+1]_\nu! = 2^{m+1} m! (\nu + \frac{1}{2})_m \times \left(\nu + m + \frac{1}{2}\right) \quad (2.153)$$

From the properties of the Pochhammer symbol: $(a)_m(a+m) = (a)_{m+1}$, where $a = \nu + \frac{1}{2}$:

$$[2m+1]_\nu! = 2^{m+1} m! (\nu + \frac{1}{2})_{m+1} \quad (2.154)$$

The expression becomes[12]

$$[2m+1]_\nu! = 2^{m+1} m! (\nu + \frac{1}{2}) \left(\nu + \frac{3}{2}\right)_m \quad (2.155)$$

2.3.3 The Dunkl Kernel:

The Dunkl kernel, denoted by $E_\nu(z)$, serves as the fundamental generalization of the standard exponential function (e^z) within the framework of deformed calculus. Its primary significance lies in its role as the eigenfunction of the Dunkl differential operator, meaning that applying the Dunkl derivative to this kernel returns the kernel itself scaled by a constant[12].

The Dunkl kernel is defined as a power series utilizing the previously established Dunkl factorials $[n]_\nu!$:

$$E(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_\nu!} \quad (2.156)$$

To reach the formulation that links the Dunkl kernel to hypergeometric functions, the total series is split into two parts based on the parity (even or odd) of the exponent n :

Step 1: Splitting the Series: The series is decomposed into even and odd components:

$$E(z) = \sum_{m=0}^{\infty} \frac{z^{2m}}{[2m]_\nu!} + \sum_{m=0}^{\infty} \frac{z^{2m+1}}{[2m+1]_\nu!} \quad (2.157)$$

Step 2: Substituting Dunkl Factorials: From the previously established equations (2.148) and (2.155), we have:

$$[2m]_\nu! = 2^{2m} m! (\nu + \frac{1}{2})_m \quad (2.158)$$

and

$$[2m+1]_\nu! = 2^{2m+1} m! (\nu + \frac{1}{2}) ((\nu + \frac{3}{2})_m) \quad (2.159)$$

Step 3: Mathematical simplification: For the even series component:

$$\frac{z^{2m}}{2^{2m} m! (\nu + \frac{1}{2})_m} = \frac{(\frac{z}{2})^{2m}}{m! (\nu + \frac{1}{2})_m} \quad (2.160)$$

For the odd series component:

$$\frac{z^{2m+1}}{2^{2m+1} m! (\nu + \frac{1}{2}) (\nu + \frac{3}{2})_m} = \frac{z}{2(\nu + \frac{1}{2})} \cdot \frac{(\frac{z}{2})^{2m}}{m! (\nu + \frac{3}{2})_m} = \frac{z}{(2\nu + 1)} \cdot \frac{(\frac{z}{2})^{2m}}{m! (\nu + \frac{3}{2})_m} \quad (2.161)$$

This leads directly to the expression provided in the text:

$$E_\nu(z) = \sum_{m=0}^{\infty} \frac{(\frac{z}{2})^{2m}}{m! (\nu + \frac{1}{2})_m} + \frac{z}{(2\nu + 1)} \sum_{m=0}^{\infty} \frac{(\frac{z}{2})^{2m}}{m! (\nu + \frac{3}{2})_m} \quad (2.162)$$

The generalized hypergeometric function is defined as[10]:

$${}_0F_1(; b; w) = \sum_{n=0}^{\infty} \frac{w^n}{(b)_n n!} \quad (2.163)$$

By comparing this definition with Eq. (2.162), we find that:

- In the first term: $b = \nu + \frac{1}{2}$ and $w = \frac{z^2}{4}$.
- In the second term: $b = \nu + \frac{3}{2}$ and $w = \frac{z^2}{4}$.

Thus, the kernel is expressed as:

$$E_\nu(z) = {}_1F_1\left(\nu + \frac{1}{2}; \frac{z^2}{4}\right) + \frac{z}{(2\nu + 1)} {}_0F_1\left(\nu + \frac{3}{2}; \frac{z^2}{4}\right) \quad (2.164)$$

2.3.4 Eigenfunction Property of the Dunkl Kernel

The Dunkl kernel $E_\nu(ax)$ is the fundamental function within this algebraic framework, as it serves as the eigenfunction of the Dunkl differential operator $\hat{D}_x^\nu$. This function satisfies the following eigenvalue equation[12]:

$$\hat{D}_x^\nu E_\nu(ax) = a E_\nu(ax) \quad (2.165)$$

The Dunkl kernel can be expressed in terms of the confluent hypergeometric function ${}_1F_1(a; b; z)$ through the following analytical relation:

$$E_\nu(z) = e^z {}_1F_1(\nu; 2\nu + 1; -2z) \quad (2.166)$$

Where the ${}_1F_1$ function is defined by its power series representation:

$${}_1F_1(a; b; z) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(b)_n n!} \quad (2.167)$$

To ensure the model's consistency with standard quantum mechanics, the ${}_1F_1$ function is expanded as follows:

$${}_1F_1(\nu, 2\nu + 1, -2z) = 1 + \sum_{n=1}^{\infty} \frac{\Gamma(\nu + n)\Gamma(2\nu + n)(-2)^n z^n}{\Gamma(\nu)\Gamma(n + 2\nu + 1)n!} \quad (2.168)$$

By taking the limit as $\nu \rightarrow 0$, and noting that $\lim_{\nu \rightarrow 0} \Gamma(\nu) = \infty$, we find:

$$\lim_{\nu \rightarrow 0} {}_1F_1(\nu, 2\nu + 1, -2z) = 1 + 0 = 1 \quad (2.169)$$

Consequently, the kernel recovers the form of the standard exponential function: $\lim_{\nu \rightarrow 0} E_\nu(z) = e^z$.

By utilizing the connection between hypergeometric functions and Bessel functions, the kernel for imaginary (ix) and real (x) arguments can be expressed as:

- For purely imaginary arguments:

$$E_\nu(ix) = \Gamma\left(\nu + \frac{1}{2}\right) \left(\frac{|x|}{2}\right)^{\frac{1}{2}-\nu} \left[J_{\frac{1}{2}-\nu}(|x|) + i \operatorname{sgn}(x) J_{\frac{1}{2}+\nu}(|x|) \right] \quad (2.170)$$

For real arguments:

$$E_\nu(x) = \Gamma\left(\nu + \frac{1}{2}\right) \left(\frac{2}{|x|}\right)^{\nu-\frac{1}{2}} \left[J_{\nu-\frac{1}{2}}(|x|) + \operatorname{sgn}(x) J_{\nu+\frac{1}{2}}(|x|) \right] \quad (2.171)$$

Derivation of the Normalization Constant c_ν : For large values of x , the asymptotic expression for ${}_1F_1$ is given by:

$${}_1F_1(\nu + 1, 2\nu + 1, 2x) \approx \frac{e^{2x}\Gamma(2\nu + 1)}{(2x)^\nu\Gamma(\nu + 1)}\left(1 - \frac{\nu^2}{2x}\right) \quad (2.172)$$

By comparing the asymptotic behavior of $E_\nu(x)$ in Eq. (2.40):

$$E_\nu(x) = \frac{c_\nu e^x}{x^\nu \sqrt{2\pi}} \left(1 - \frac{\nu^2}{2x}\right) \quad (2.173)$$

The normalization constant c_ν is initially defined as:

$$c_\nu = \sqrt{2\pi} \frac{\Gamma(2\nu + 1)}{\Gamma(\nu + 1)} \quad (2.174)$$

We apply the relation:

$$\Gamma(2\nu + 1) = \frac{2^{2\nu}}{\sqrt{\pi}} \Gamma(\nu + 1) \Gamma(\nu + 1) \quad (2.175)$$

Substituting(2.175) into (2.174):

$$c_\nu = \frac{\sqrt{2\pi}}{2^\nu \Gamma(\nu + 1)} \left[\frac{2^{2\nu}}{\sqrt{\pi}} \Gamma(\nu + 1) \Gamma(\nu + 1) \right] \quad (2.176)$$

After canceling $\Gamma(\nu + 1)$ and simplifying the square roots ($\frac{\sqrt{2\pi}}{\sqrt{\pi}} = \sqrt{2}$) and the powers of 2 ($\frac{2^{2\nu}}{2^\nu} = 2^\nu$), we obtain[10]:

$$c_\nu = \sqrt{2} \cdot 2^\nu \Gamma(\nu + 1) = 2^{\nu+\frac{1}{2}} \Gamma(\nu + 1) \quad (2.177)$$

2.3.5 Mathematical Formulation of the Dunkl Transform

In the context of deformed calculus, the Dunkl transform is defined as a pivotal generalization of the traditional Fourier transform in modified quantum mechanics, The direct transform $g(k)$ of a function $f(x)$ is defined according to the following integral[12]:

$$D[f] = g(k) := \frac{1}{c_\nu} \int_{-\infty}^{+\infty} dx |x|^{2\nu} f(x) E_\nu(-ikx) \quad (2.178)$$

As for the Inverse Transform, it is formulated as follows:

$$D^{-1}[g] = f(k) := \frac{1}{c_\nu} \int_{-\infty}^{+\infty} dk |k|^{2\nu} g(k) E_\nu(ikx) \quad (2.179)$$

where f and $g \in L^1(\mathbb{R}, dx |x|^{2\nu})$ to ensure the transform is well-defined. It is observed that as $\nu \rightarrow 0$, we return to the known Fourier transformation:

$$F[f(x)] = g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \quad (2.180)$$

where $c_0 = \sqrt{2\pi}$. Simplification for the even function case. If $f(x)$ is an even function, the transform (2.178) becomes:

$$D[f](k) = \frac{1}{c_\nu} \int_{-\infty}^{+\infty} dx |x|^{2\nu} f(x) E_\nu(-ikx) \quad (2.181)$$

Substituting $E_\nu(-ikx)$ in terms of Bessel functions:

$$\frac{\Gamma(\nu + \frac{1}{2})}{c_\nu} \int_{-\infty}^{+\infty} dx |x|^{2\nu} \left(\frac{|kx|}{2}\right)^{\frac{1}{2}-\nu} f(x) \left[J_{\nu-\frac{1}{2}}(|kx|) - i \operatorname{sgn}(x) J_{\nu+\frac{1}{2}}(|kx|) \right] \quad (2.182)$$

Due to the symmetry of the integration interval, the term containing the odd function $\operatorname{sgn}(x)$ vanishes, leading to the Hankel transform:

$$D[f] = H_{\nu-\frac{1}{2}}[f|x|^{\nu-\frac{1}{2}}](k) \quad (2.183)$$

Where the Hankel transform of order ν is defined as:

$$H_\nu[f](k) = \int_0^\infty f(r) J_\nu(kr) r dr \quad (2.184)$$

To study the effect of the transform on deformed exponential functions, we write:

$$\mathcal{D}[E_\nu(i\kappa'x)] = \frac{1}{c_\nu} \int_{-\infty}^\infty dx |x|^{2\nu} E_\nu(i\kappa'x) E_\nu(-ikx) x \quad (2.185)$$

Using the asymptotic expression (2.41), we find:

$$\begin{aligned} \mathcal{D}[E_\nu(i\kappa'x)] &= \frac{1}{c_\nu} \int_{-\infty}^\infty dx |x|^{2\nu} \left(\frac{c_\nu e^{i\kappa'x}}{(i\kappa'x)^\nu \sqrt{2\pi}} \left(1 - \frac{\nu^2}{2i\kappa'x} + O(x^{-2}) \right) \right) \\ &\quad \times \left(\frac{c_\nu e^{-ikx}}{(-ikx)^\nu \sqrt{2\pi}} \left(1 + \frac{\nu^2}{2ikx} + O(x^{-2}) \right) \right) \quad (2.53) \end{aligned} \quad (2.186)$$

After mathematical simplification and neglecting higher-order terms, we arrive at:

$$\mathcal{D}[E_\nu(i\kappa'x)] = \frac{1}{2\pi} \int_{-\infty}^\infty dx \left[\frac{c_\nu e^{-i(k-\kappa')x}}{(\kappa'k)^\nu} \left(1 - \nu^2 \left(\frac{1}{(2ikx)} - \frac{1}{(2i\kappa'x)} \right) + O(x^{-2}) \right) \right] \quad (2.187)$$

Ignoring terms that vanish due to symmetry[10]:

$$\mathcal{D}[E_\nu(i\kappa'x)] = \frac{1}{2\pi} \int_{-\infty}^\infty dx \frac{c_\nu e^{-i(k-\kappa')x}}{(\kappa'k)^\nu} \quad (2.188)$$

This integral leads to the Dirac delta function:

$$\mathcal{D}[E_\nu(i\kappa'x)] = \frac{c_\nu}{|\kappa'k|^\nu} \delta(k - \kappa') \quad (2.189)$$

Which simplifies to:

$$\mathcal{D}[E_\nu(i\kappa'x)] = \frac{c_\nu}{|k|^{2\nu}} \delta(k - \kappa') \quad (2.190)$$

This represents the orthogonality relation:

$$\int_{-\infty}^{\infty} dx |z|^{2\nu} E_{\nu}^*(ikz) E_{\nu}(ikz) = \frac{c_{\nu}^2}{|k|^{2\nu}} \delta(k - \kappa) \quad (2.191)$$

Regarding the Gaussian function $e^{-\frac{\alpha x^2}{2}}$, its Dunkl transform is:

$$\mathcal{D}[e^{-\frac{\alpha x^2}{2}}] = \frac{1}{c_{\nu}} \int_{-\infty}^{\infty} dx |x|^{2\nu} e^{-\frac{\alpha x^2}{2}} E_{\nu}(-ikx) \quad (2.192)$$

Using (2.182), we reach:

$$\mathcal{D}[e^{-\frac{\alpha x^2}{2}}] = \frac{1}{|k|^{\nu-\frac{1}{2}}} \int_0^{\infty} dx |x|^{\nu+\frac{1}{2}} e^{-\frac{\alpha x^2}{2}} \left[J_{\nu-\frac{1}{2}}(|kx|) \right] \quad (2.193)$$

By changing the variable $y = |k|x$, we find:

$$\mathcal{D}[e^{-\frac{\alpha x^2}{2}}] = \frac{1}{|k|^{2\nu+1}} \int_0^{\infty} dy y^{\nu+\frac{1}{2}} e^{-\frac{\alpha y^2}{2k^2}} \left[J_{\nu-\frac{1}{2}}(y) \right] \quad (2.194)$$

Using the integral relation:

$$\int_0^{\infty} x^{\nu+1} e^{-\alpha x^2} J_{\nu}(\beta x) dx = \frac{\beta^{\nu}}{(2\alpha)^{\nu+1}} \exp\left(-\frac{\beta^2}{4\alpha}\right) \quad (2.195)$$

we obtain the final result for the Gaussian transform:

$$\mathcal{D}[e^{-\frac{\alpha x^2}{2}}] = \frac{e^{-\frac{k^2}{2\alpha}}}{\alpha^{\nu+\frac{1}{2}}} \quad (2.196)$$

Hilbert Space and Hermiticity Properties

In the framework of Wigner-Dunkl deformed quantum mechanics, the inner product is defined by introducing a weighted measure that ensures the consistency of algebraic operations. The expression for the inner product in this space is given by:

$$\langle \psi | \varphi \rangle = \int_{-\infty}^{+\infty} dx |x|^{2\nu} \psi^*(x) \varphi(x). \quad (2.197)$$

Where the term $|x|^{2\nu} dx$ is called the *Dunkl measure* or *weighted measure*. The change of measure in the integration modifies the formula for the expectation value of a physical operator $\mathcal{O}$ with respect to a given state, and it becomes:

$$\langle \phi | \mathcal{O} | \psi \rangle = \int_{-\infty}^{+\infty} \phi^*(x, t) \mathcal{O} \psi(x, t) |x|^{2\nu} dx. \quad (2.198)$$

It is obvious that we should take $\mathcal{O}$ as a Hermitian operator to make the expectation value real:

$$\mathcal{O}^{\dagger} = \mathcal{O} \quad (2.199)$$

which is mathematically equivalent to the condition[10]:

$$\langle \phi | \mathcal{O}^+ | \psi \rangle = \langle \phi | \mathcal{O} | \psi \rangle^* \quad (2.200)$$

In ordinary quantum mechanics, we know that the integral of a total derivative vanishes at infinity:

$$\int_{-\infty}^{+\infty} dx \partial_x \psi(x, t) = 0 \quad (2.201)$$

By analogy, for an arbitrary function $f(x)$ in the context of Dunkl derivative the following propositions hold:

$$\int_{-\infty}^{+\infty} dx |x|^{2\nu} \hat{D}_x^\nu f = 0 \quad (2.202)$$

this holds for cases where $f(x)$ is an even function, and we also find:

$$\int_{-\infty}^{+\infty} dx |x|^{2\nu} \hat{D}_x^\nu f = 0 \quad (2.203)$$

for $f(x)$ being an odd function obeying the boundary condition $\lim_{x \rightarrow \pm\infty} x^{2\nu} f(x) = 0$ to ensure convergence. In general, we write:

$$\int_{-\infty}^{+\infty} dx |x|^{2\nu} \hat{D}_x^\nu f = 0 \quad (2.204)$$

Proving that the Dunkl derivative operator $\hat{D}_x^\nu$ is anti-Hermitian with respect to the weighted inner product is fundamental. An operator A is said to be anti-Hermitian if it satisfies[10]:

$$\langle \phi | A | \psi \rangle = -\langle A \phi | \psi \rangle \quad (2.205)$$

Accordingly, we must show that:

$$\int_{-\infty}^{+\infty} dx |x|^{2\nu} \phi^*(x) (\hat{D}_x^\nu \psi(x)) = - \int_{-\infty}^{+\infty} dx |x|^{2\nu} (\hat{D}_x^\nu \phi(x))^* \psi(x) \quad (2.206)$$

To begin with, consider the matrix element of the partial derivative x within the weighted inner product:

$$\langle \phi | \partial_x | \psi \rangle = \int_{-\infty}^{+\infty} dx |x|^{2\nu} \phi^*(x) (\partial_x \psi)(x), \quad (2.207)$$

By performing integration by parts, we find:

$$\langle \phi | \partial_x | \psi \rangle = \phi^*(x) \psi(x) |x|^{2\nu} \Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} dx \partial_x (\phi^*(x) |x|^{2\nu}) \psi(x) \quad (2.208)$$

Neglecting the boundary term (vanishing of the function at infinity) and distributing the derivative

$$\phi | \partial_x | \psi \rangle = - \int_{-\infty}^{+\infty} dx |x|^{2\nu} \left[(\partial_x \phi^*)(x) + \frac{2\nu}{x} \phi^*(x) \right] \psi(x) \quad (2.209)$$

From this, we conclude that the adjoint of the partial derivative in this weighted Hilbert space H is:

$$(\partial_x)^\dagger = -\partial_x - \frac{2\nu}{x} \quad (2.210)$$

Using this result to calculate the adjoint of the Dunkl operator $\hat{D}_x^{\nu\dagger}$, taking into account the hermiticity of both the reflection operator and position ($\hat{R}^\dagger = R$, $x^\dagger = x$):

$$\hat{D}_x^{\nu\dagger} = (\partial_x + \frac{\nu}{x} - \frac{\nu}{x}\hat{R})^\dagger \quad (2.211)$$

$$= (\partial_x)^\dagger + (\frac{\nu}{x})^\dagger - \hat{R}^\dagger(\frac{\nu}{x})^\dagger \quad (2.212)$$

$$= -\partial_x - 2\frac{\nu}{x} + \frac{\nu}{x} + \frac{\nu}{x}\hat{R} \quad (2.213)$$

By simplifying, we reach the final result confirming that the Dunkl derivative is anti-Hermitian:

$$\hat{D}_x^{\nu\dagger} = -\hat{D}_x^\nu \quad (2.214)$$

In one dimension, the deformed completeness relation can be stated as:

$$\int_{-\infty}^{+\infty} dx |x|^{2\nu} |x\rangle \langle x| = I \quad (2.215)$$

From this relation, the state $|\psi\rangle$ can be expressed by projecting it onto the position basis:

$$|\psi\rangle = I|\psi\rangle = \int_{-\infty}^{+\infty} dx |x|^{2\nu} |x\rangle \langle x|\psi\rangle = \int_{-\infty}^{+\infty} dx |x|^{2\nu} \psi(x) |x\rangle \quad (2.216)$$

By projecting the state $|\psi\rangle$ onto the position eigenbra $\langle x'|$, we obtain:

$$\langle x'|\psi\rangle = \psi(x') = \int_{-\infty}^{+\infty} dx |x|^{2\nu} \langle x'|x\rangle \psi(x) \quad (2.217)$$

which leads us to the definition of the orthogonality relation in this deformed framework:

$$\langle x'|x\rangle = \frac{\delta(x' - x)}{|x|^{2\nu}} \quad (2.218)$$

Quantum physics formulated within this framework is known as Wigner-Dunkl quantum mechanics[10].

2.3.6 Study of probability conservation

To ensure the physical consistency of the "Wigner-Dunkl" framework, it must be verified that the normalization condition remains invariant over time, meaning the time derivative of the total probability must equal zero[12].

1. Time Derivative of Probability: We begin by calculating the temporal rate of change of the probability integral in the weighted space:

$$i\hbar \frac{d}{dt} \left(\int_{-\infty}^{+\infty} |\psi|^2 |x|^{2\nu} dx \right) = \int_{-\infty}^{+\infty} (i\hbar \partial_t \psi \psi^* + \psi i\hbar \partial_t \psi^*) |x|^{2\nu} dx \quad (2.219)$$

2. Substituting the Deformed Schrodinger Equation: By substituting $i\hbar \partial_t \psi$ with the Hamiltonian operator $\hat{H}\psi = -\frac{\hbar^2}{2m} D_x^2 \psi$, we obtain:

$$= \int_{-\infty}^{+\infty} \left(\frac{-\hbar^2 D_x^2 \psi}{2m} \psi^* - \psi \frac{-\hbar^2 D_x^2 \psi^*}{2m} \right) |x|^{2\nu} dx \quad (2.220)$$

3. Simplification using Dunkl Operator Properties: Using the mathematical relation for the deformed probability current and applying integration by parts while accounting for the reflection operator $\hat{R}$, the terms associated with reflection vanish due to the symmetry of the interval[10]:

$$i\hbar \frac{d}{dt} \langle \psi | \psi \rangle = \frac{-\hbar^2}{2m} \int_{-\infty}^{+\infty} (\psi^* \hat{D}_x^2 \psi - \psi \hat{D}_x^2 \psi^*) |x|^{2\nu} dx \quad (2.221)$$

Since the Dunkl operator is anti-Hermitian and the wavefunction vanishes at infinity, the final value equals zero, leading to the constant normalization condition:

$$\int_{-\infty}^{+\infty} |\psi|^2 |x|^{2\nu} dx = 1 \quad (2.222)$$

2.3.7 Momentum Eigenfunctions

We seek solutions for the eigenvalue equation of the momentum operator $\hat{P} = -i\hbar \hat{D}_x$. Substituting the definition of the Dunkl derivative $\hat{D}_x = \frac{d}{dx} + \frac{\nu}{x}(1 - \hat{R})$, we obtain the following differential-difference equation[10]:

$$\left[\frac{d\Phi_k(x)}{dx} + \frac{\nu}{x}(\Phi_k(x) - \Phi_k(-x)) \right] = ik\Phi_k(x) \quad (2.223)$$

First: Decomposition: We assume the function consists of an even part and an odd part: $\Phi_k(x) = \Phi_{k,e}(x) + \Phi_{k,o}(x)$. Substituting into (2.89) yields a coupled system:

$$\begin{cases} \frac{d\Phi_{k,o}}{dx} + \frac{2\nu}{x}\Phi_{k,o} = ik\Phi_{k,e} \\ \frac{d\Phi_{k,e}}{dx} = ik\Phi_{k,o} \end{cases} \quad (2.224)$$

Second: Transformation to a Second-Order Equation: By substituting $\Phi_{k,o}$ from the second equation into the first, we reach a differential equation for the even component:

$$\frac{d^2\Phi_{k,e}}{dx^2} + \frac{2\nu}{x} \frac{d\Phi_{k,e}}{dx} + k^2\Phi_{k,e} = 0 \quad (2.225)$$

Third: Using the Ansatz: We set $\Phi_{k,e}(x) = x^\lambda \chi(x)$. Performing the first and second derivatives and substituting them into (2.95), we obtain:

$$\chi''(x) + \frac{2(\lambda + \nu)}{x} \chi'(x) + \left(\frac{\lambda(\lambda + 2\nu - 1)}{x^2} + k^2 \right) \chi(x) = 0 \quad (2.226)$$

Fourth: Reaching the Bessel Equation: Choosing $\lambda = 1/2 - \nu$ and changing the variable to $z = kx$, the equation reduces to:

$$z^2 \frac{d^2 \chi(z)}{dz^2} + z \frac{d\chi(z)}{dz} + (z^2 - (\nu - 1/2)^2) \chi(z) = 0 \quad (2.227)$$

The general solution for the even component is given in terms of the Bessel function of the first kind $J_{\nu-1/2}$. By combining the even and odd solutions, we obtain the total momentum wavefunction:

$$\Phi_k(x) = A|x|^{1/2-\nu} [J_{\nu-1/2}(|kx|) + i \operatorname{sgn}(kx) J_{\nu+1/2}(|kx|)] \quad (2.228)$$

2.3.8 Free particle : $V(x) = 0$

The Free particle Schrodinger equation in the Dunkl derivative is[4]:

$$-\frac{\hbar^2}{2m} D_x^2 \psi(x) = E \psi(x) \quad (2.229)$$

where

$$D_x^2 = \frac{d^2}{dx^2} + \frac{2\mu}{x} \frac{d}{dx} - \frac{\mu}{x^2} (1 - R) \quad (2.230)$$

then

$$\frac{d^2 \psi(x)}{dx^2} + \frac{2\mu}{x} \frac{d}{dx} \psi(x) - \frac{\mu}{x^2} (1 - R) \psi(x) + k^2 \psi(x) = 0 \quad (2.231)$$

where $k^2 = \frac{2mE}{\hbar^2}$.

Use parity very important : because of the reflection operation, solution split into: $R\psi = \pm\psi$.

Thus, for even solution : $R\psi = +\psi$, the equation simplifies to:

$$\psi'' + \frac{2\mu}{x} \psi' + k^2 \psi = 0 \quad (2.232)$$

Solve the even equation :

$$\psi(x) = x^\alpha u(x) \quad (2.233)$$

then

$$\psi'(x) = \alpha x^{\alpha-1} u(x) + x^\alpha u'(x) \quad (2.234)$$

and

$$\psi''(x) = \alpha(\alpha - 1)x^{\alpha-2}u + 2\alpha x^{\alpha-1}u' + x^\alpha u'' \quad (2.235)$$

Substitute into 2.232. After simplification choose: $\alpha = -\mu + \frac{1}{2}$:

$$u'' + \frac{1}{x}u' + \left(k^2 - \frac{\nu^2}{x^2}\right)u = 0 \quad (2.236)$$

where

$$\nu = \mu - \frac{1}{2}$$

So the solution is:

$$u(x) = AJ_\nu(kx) + BY_\nu(kx) \quad (2.237)$$

$$\psi_{even}(x) = x^{-\mu+1/2}[AJ_\nu(kx) + BY_\nu(kx)] \quad (2.238)$$

Odd Solution : $R\psi = -\psi$, The equation becomes:

$$\psi'' + \frac{2\mu}{x}\psi' - \frac{2\mu}{x^2}\psi + k^2\psi = 0 \quad (2.239)$$

Using the same substitutions:

$$\psi = x^\alpha u \quad (2.240)$$

Final odd solution is

$$\psi_{odd}(x) = x^{-\mu+1/2}[AJ_{\mu+1/2}(kx) + BY_{\mu+1/2}(kx)] \quad (2.241)$$

when $\mu \rightarrow 0$ we use

$$J_{\pm 1/2}(z) = \sqrt{\frac{2}{\pi z}} \begin{Bmatrix} \sin z \\ \cos z \end{Bmatrix} \quad (2.242)$$

the corresponding usual function is $\psi_{\nu \rightarrow 0}^{free}(x) \rightarrow \exp(\pm ikx)$

1. The reason of neglecting the second term Y_ν . Because the behavior at $x \rightarrow 0$, We use the well-known asymptotic behavior J_ν :

$$J_\nu(z) \sim z^\nu \text{ when } z \rightarrow 0$$

$$Y_\nu(z) \sim z^{-\nu} \text{ when } z \rightarrow 0$$

Behavior $J_\nu(z)$

$$\psi_J(x) \sim x^{-\mu+1/2}(kx)^\nu \quad (2.243)$$

$$\nu = \mu - 1/2$$

$$\psi_J(x) \sim x^{-\mu+1/2}x^{\mu-1/2}K^{\mu-1/2} \quad (2.244)$$

$$\psi_J(x) \sim x^0 \quad (2.245)$$

(Regular at the origin)

Behavior $Y_\nu(z)$

$$\psi_Y(x) \sim x^{-\mu+1/2}(kx)^{-\nu} \quad (2.246)$$

$$\sim x^{-\mu+1/2}x^{-\mu+1/2}(K) \quad (2.247)$$

$$\psi_Y(x) \sim x^{-2\mu+1} \quad (2.248)$$

If $\mu > 0$ then $x \rightarrow 0$ at $\psi_Y(x) \rightarrow \infty$, Meaning that it diverges at the origin

Even states :

$$\psi_{even} = x^{-\mu+1/2}J_{\mu-1/2}(kx) \quad (2.249)$$

Odd states :

$$\psi_{odd} = x^{-\mu+1/2}J_{\mu+1/2}(kx) \quad (2.250)$$

Normalization :

$$\int_{-\infty}^{\infty} |\psi(x)|^2 |x|^{2\mu} dx \quad (2.251)$$

We are testing the behavior at zero

$$\psi(x) \sim x^{-2\mu+1} \quad (2.252)$$

Then

$$|\psi|^2 |x|^{2\mu} \sim x^{-4\mu+2} |x|^{2\mu} \sim x^{-2\mu+2} \quad (2.253)$$

The integral near zero converges only if $-2\mu + 1 > -1 \implies \mu < \frac{3}{2}$

However, even if it is mathematically convergent, the solution remains unphysical because it is non-regular at the origin. Since a free particle lacks a potential at $x = 0$, there is no physical justification for a singularity at the origin. Therefore, we select only the regular solution by setting $B = 0$.

$$\psi_k(x) = N|x|^{-\mu+1/2}J_{\mu-1/2}(k|x|) \quad (2.254)$$

For a free particle the spectrum is continuous. So we do not normalize to 1, instead we impose Dirac delta-normalization:

$$\int_{-\infty}^{\infty} \psi_k(x)\psi_{k'}(x)|x|^{2\mu}dx = \delta(k - k') \quad (2.255)$$

Notice the Dunkl measure : $d\mu(x) = |x|^{2\mu}dx$.

Since the solution is even, we write :

$$\int_{-\infty}^{\infty} = 2 \int_0^{\infty} \quad (2.256)$$

then

$$2N^2 \int_0^{\infty} x^{-2\mu+1} J_{\nu}(kx) J_{\nu}(k'x) x^{2\mu} dx = \delta(k - k') \quad (2.257)$$

where $\nu = \mu + 1/2$

$$= 2N^2 \int_0^{\infty} x J_{\nu}(kx) J_{\nu}(k'x) dx = \delta(k - k') \quad (2.258)$$

Use the Bessel orthogonality :

The key identity

$$\int_0^{\infty} x J_{\nu}(kx) J_{\nu}(k'x) dx = \frac{1}{k} \delta(k - k') \quad (2.259)$$

Therefore:

$$2N^2 \frac{1}{k} \delta(k - k') = \delta(k - k') \quad (2.260)$$

then

$$2N^2 \frac{1}{k} = 1 \implies N^2 = \frac{k}{2} \implies N = \sqrt{\frac{k}{2}} \quad (2.261)$$

odd physical solution : the acceptable odd solution (regular at $x = 0$) is:

$$\psi_k^{odd}(x) = N \operatorname{sgn}(x) |x|^{-\mu+1/2} J_{\mu+1/2}(k|x|) \quad (2.262)$$

where $\nu = \mu + 1/2$, and $\operatorname{sgn} = \frac{x}{|x|}$; we impose Dirac delta-normalization :

$$\int_{-\infty}^{\infty} \psi_k(x) \psi_{k'}(x) |x|^{2\mu} dx = \delta(k - k') \quad (2.263)$$

since $[\operatorname{sgn}(x)]^2 = 1$ we get

$$N^2 \int_{-\infty}^{\infty} |x|^{-2\mu+1} |x|^{2\mu} J_{\nu}(k|x|) J_{\nu}(k'|x|) dx = \delta(k - k') \quad (2.264)$$

then

$$= N^2 \int_{-\infty}^{\infty} |x| J_{\nu}(k|x|) J_{\nu}(k'|x|) dx = \delta(k - k') \quad (2.265)$$

Because the integrand is even

$$2N^2 \int_0^{\infty} x J_{\nu}(kx) J_{\nu}(k'x) dx \quad (2.266)$$

we use :

$$\int_0^{\infty} x J_{\nu}(kx) J_{\nu}(k'x) dx = \frac{1}{k} \delta(k - k') \quad (2.267)$$

Thus :

$$2N^2 \frac{1}{k} \delta(k - k') = \delta(k - k') \implies 2N^2 \frac{1}{k} = 1 \implies N = \sqrt{\frac{k}{2}} \quad (2.268)$$

Final normalized odd solution :

$$\psi_k^{odd}(x) = \sqrt{\frac{k}{2}} \operatorname{sgn}(x) |x|^{-\mu+1/2} J_{\mu+1/2}(k|x|) \quad (2.269)$$

Chapter 3

Physical Systems in Standard and Dunkl Quantum Mechanics

This chapter presents the main one-dimensional model systems considered in this memory. We begin with the harmonic oscillator, since it provides the clearest solvable example in both the ordinary and Dunkl frameworks. This first application makes it possible to highlight, in a controlled setting, the role of the deformation parameter, the parity decomposition induced by the reflection operator, and the way in which the standard spectrum is recovered in the limit of vanishing deformation. After this preparatory model, we move to scattering-type configurations such as the potential step, the barrier problem, and the phase-shift formulation for symmetric interactions.

3.1 The Usual Harmonic Oscillator

We first recall the ordinary one-dimensional harmonic oscillator, which will serve as the reference model for the Dunkl generalization. Its Hamiltonian is

$$H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m \omega^2 x^2. \quad (3.1)$$

The time-independent Schrödinger equation therefore reads

$$H\psi(x) = E\psi(x), \quad (3.2)$$

or equivalently

$$-\frac{\hbar^2}{2m} \psi''(x) + \frac{1}{2} m \omega^2 x^2 \psi(x) = E\psi(x). \quad (3.3)$$

To reduce this equation to dimensionless form, we introduce

$$\xi = \sqrt{\frac{m\omega}{\hbar}} x, \quad (3.4)$$

and the reduced energy parameter

$$\varepsilon = \frac{E}{\hbar\omega}. \quad (3.5)$$

The eigenvalue equation then becomes

$$\frac{d^2\psi}{d\xi^2} + (2\varepsilon - \xi^2)\psi = 0. \quad (3.6)$$

For large $|\xi|$, the dominant asymptotic balance implies

$$\psi'' - \xi^2\psi \simeq 0, \quad (3.7)$$

so the physically acceptable behavior is

$$\psi(\xi) \sim e^{-\xi^2/2}, \quad (3.8)$$

since the growing exponential must be rejected. We therefore set

$$\psi(\xi) = e^{-\xi^2/2}h(\xi). \quad (3.9)$$

Substituting this form into the differential equation yields

$$h'' - 2\xi h' + (2\varepsilon - 1)h = 0. \quad (3.10)$$

This is the Hermite equation. The requirement that the wave function remain normalizable forces $h(\xi)$ to be a polynomial, which happens only when

$$2\varepsilon - 1 = 2n, \quad n = 0, 1, 2, \dots \quad (3.11)$$

Hence

$$\varepsilon = n + \frac{1}{2}, \quad (3.12)$$

and the familiar quantized spectrum follows:

$$E_n = \hbar\omega \left(n + \frac{1}{2} \right), \quad n = 0, 1, 2, \dots \quad (3.13)$$

The corresponding normalized eigenfunctions are expressed in terms of Hermite polynomials,

$$\psi_n(\xi) = N_n e^{-\xi^2/2} H_n(\xi), \quad (3.14)$$

that is,

$$\psi_n(x) = N_n \exp\left(-\frac{m\omega x^2}{2\hbar}\right) H_n\left(\sqrt{\frac{m\omega}{\hbar}} x\right). \quad (3.15)$$

Using the normalization condition

$$\int_{-\infty}^{\infty} |\psi_n(x)|^2 dx = 1, \quad (3.16)$$

together with the Hermite identity

$$\int_{-\infty}^{\infty} e^{-\xi^2} H_n^2(\xi) d\xi = \sqrt{\pi} 2^n n!, \quad (3.17)$$

one obtains

$$N_n = \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} \frac{1}{\sqrt{2^n n!}}. \quad (3.18)$$

Therefore the ordinary harmonic oscillator is fully characterized by

$$E_n = \hbar\omega \left(n + \frac{1}{2} \right), \quad (3.19)$$

and

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} \frac{1}{\sqrt{2^n n!}} e^{-\frac{m\omega x^2}{2\hbar}} H_n \left(\sqrt{\frac{m\omega}{\hbar}} x \right). \quad (3.20)$$

3.2 The Harmonic Oscillator with the Dunkl Derivative

We now turn to the one-dimensional Dunkl harmonic oscillator, whose Hamiltonian is given by

$$H = -\frac{\hbar^2}{2m} D_x^2 + \frac{1}{2} m\omega^2 x^2, \quad (3.21)$$

where the Dunkl derivative is defined by

$$D_x = \frac{d}{dx} + \frac{\mu}{x}(1 - R), \quad (3.22)$$

and the reflection operator acts as

$$R\psi(x) = \psi(-x). \quad (3.23)$$

Its square takes the form

$$D_x^2 = \frac{d^2}{dx^2} + \frac{2\mu}{x} \frac{d}{dx} - \frac{\mu}{x^2}(1 - R). \quad (3.24)$$

The stationary Schrödinger equation then becomes

$$-\frac{\hbar^2}{2m} \left[\psi'' + \frac{2\mu}{x} \psi' - \frac{\mu}{x^2}(1 - R)\psi \right] + \frac{1}{2} m\omega^2 x^2 \psi = E\psi. \quad (3.25)$$

Because of the reflection operator, the natural decomposition is into parity sectors.

1. Even sector:

For even states, $R\psi = +\psi$, hence $(1 - R)\psi = 0$, and the equation reduces to

$$-\frac{\hbar^2}{2m} \left[\psi'' + \frac{2\mu}{x} \psi' \right] + \frac{1}{2} m \omega^2 x^2 \psi = E \psi. \quad (3.26)$$

Introducing the same dimensionless variable

$$\xi = \sqrt{\frac{m\omega}{\hbar}} x, \quad (3.27)$$

and writing

$$\mathcal{E} = \frac{E}{\hbar\omega}, \quad (3.28)$$

we obtain

$$\psi''(\xi) + \frac{2\mu}{\xi} \psi'(\xi) + (2\mathcal{E} - \xi^2) \psi(\xi) = 0. \quad (3.29)$$

As in the ordinary case, the acceptable asymptotic behavior is Gaussian, so we write

$$\psi(\xi) = e^{-\xi^2/2} f(\xi). \quad (3.30)$$

After substitution and simplification, this gives

$$f''(\xi) + \left(\frac{2\mu}{\xi} - 2\xi \right) f'(\xi) + (2\mathcal{E} - 1 - 2\mu) f(\xi) = 0. \quad (3.31)$$

Now introducing the variable $y = \xi^2$, one finds

$$y f_{yy} + \left(\mu + \frac{1}{2} - y \right) f_y + \frac{2\mathcal{E} - 1 - 2\mu}{4} f = 0. \quad (3.32)$$

This is the generalized Laguerre equation. Comparison with its standard form shows that polynomial solutions exist only if

$$\frac{2\mathcal{E} - 1 - 2\mu}{4} = n, \quad n = 0, 1, 2, \dots, \quad (3.33)$$

which yields

$$\mathcal{E} = 2n + \mu + \frac{1}{2}. \quad (3.34)$$

Therefore the even-sector energies are

$$E_{2n} = \hbar\omega \left(2n + \mu + \frac{1}{2} \right). \quad (3.35)$$

The corresponding eigenfunctions are

$$\psi_{2n}(x) = N_{2n} |x|^\mu e^{-\frac{m\omega x^2}{2\hbar}} L_n^{(\mu-\frac{1}{2})} \left(\frac{m\omega x^2}{\hbar} \right). \quad (3.36)$$

1. Odd sector:

For odd states, $R\psi = -\psi$, so $(1 - R)\psi = 2\psi$. The reduced equation becomes

$$\psi'' + \frac{2\mu}{\xi}\psi' - \frac{2\mu}{\xi^2}\psi + (2\mathcal{E} - \xi^2)\psi = 0. \quad (3.37)$$

Proceeding in the same way leads to the quantization condition

$$\mathcal{E} = 2n + \mu + \frac{3}{2}, \quad (3.38)$$

and thus to the odd-sector spectrum

$$E_{2n+1} = \hbar\omega \left(2n + \mu + \frac{3}{2} \right) = \hbar\omega \left(2n + 1 + \mu + \frac{1}{2} \right). \quad (3.39)$$

The odd eigenfunctions can be written as

$$\psi_{2n+1}(x) = N_{2n+1} x |x|^{\mu-1} e^{-\frac{m\omega x^2}{2\hbar}} L_n^{(\mu+\frac{1}{2})} \left(\frac{m\omega x^2}{\hbar} \right). \quad (3.40)$$

Combining the two sectors, one obtains the compact expression

$$E_n = \hbar\omega \left(n + \mu + \frac{1}{2} \right), \quad n = 0, 1, 2, \dots, \quad (3.41)$$

which shows that the spacing remains equal to $\hbar\omega$, while the deformation parameter shifts the ground-state energy to

$$E_0 = \hbar\omega \left(\mu + \frac{1}{2} \right). \quad (3.42)$$

In the limit $\mu \rightarrow 0$, the standard harmonic-oscillator spectrum is recovered exactly:

$$E_n \longrightarrow \hbar\omega \left(n + \frac{1}{2} \right). \quad (3.43)$$

The normalization is performed with respect to the Dunkl weight,

$$\int_{-\infty}^{\infty} |\psi(x)|^2 |x|^{2\mu} dx = 1. \quad (3.44)$$

Using the Laguerre orthogonality formulas, one obtains

$$N_{2n} = \left(\frac{m\omega}{\hbar} \right)^{\mu+\frac{1}{2}} \sqrt{\frac{n!}{\Gamma(n + \mu + \frac{1}{2})}}, \quad (3.45)$$

and

$$N_{2n+1} = \left(\frac{m\omega}{\hbar} \right)^{\mu+\frac{1}{4}} \sqrt{\frac{n!}{\Gamma(n + \mu + \frac{3}{2})}}. \quad (3.46)$$

At this stage, the main difference between the ordinary and Dunkl oscillators is clear. In the ordinary case, the spectrum is fixed solely by the integer quantum number. In the Dunkl

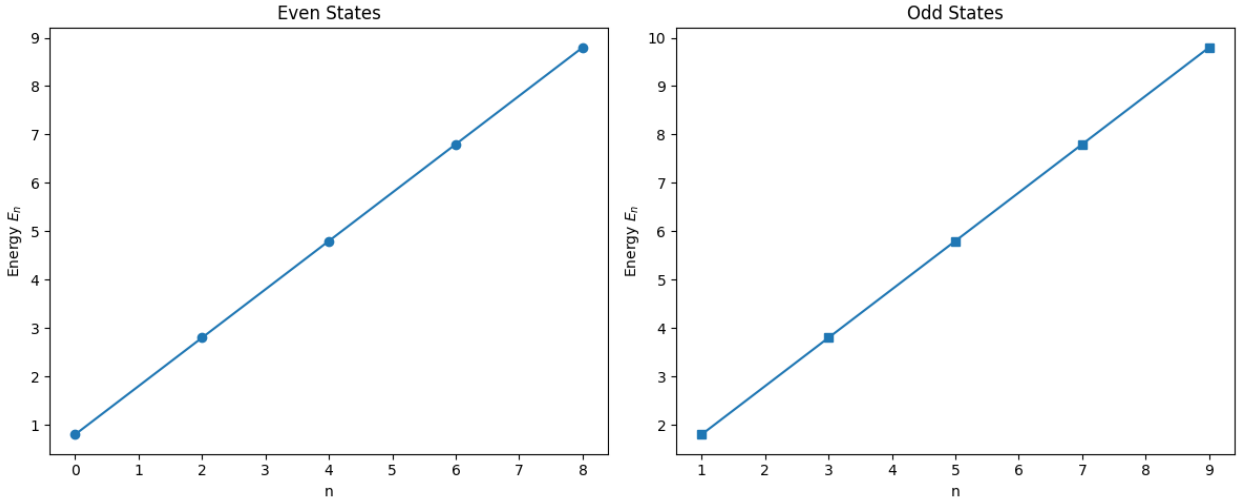


Figure 3.1: Energy Spectrum of Even and Odd States in the Dunkl Harmonic Oscillator at $\mu = 0.3$

case, the reflection operator splits the problem into even and odd sectors, while the deformation parameter μ shifts the full spectrum without changing its constant spacing. This makes the harmonic oscillator an especially useful benchmark before turning to scattering configurations.

The plots show the energy spectrum of the Dunkl harmonic oscillator for even and odd quantum numbers. In both cases, the energy increases linearly with the quantum number n , which confirms that the harmonic oscillator structure is preserved despite the presence of the Dunkl parameter μ .

The separation into even and odd states highlights the role of parity, where each set follows the same linear trend but starts from different quantum numbers. The constant slope observed in both plots indicates that the spacing between energy levels remains uniform, while the Dunkl parameter μ introduces a constant shift in the spectrum.

Overall, these results demonstrate that the Dunkl deformation does not alter the fundamental linear behavior of the harmonic oscillator but shifts the energy levels while maintaining the distinction between even and odd states.

The plots show the wave functions of the Dunkl harmonic oscillator for even and odd states under different values of the parameter μ .

For the even states (left panel), the wave functions are symmetric with respect to the origin, i.e., $\psi(x) = \psi(-x)$, while for the odd states (right panel), they are antisymmetric, satisfying $\psi(x) = -\psi(-x)$ and vanishing at $x = 0$. This confirms that the parity symmetry is preserved in the Dunkl formalism.

As the parameter μ increases, the amplitude of the wave functions becomes larger and the oscillations become more pronounced. However, the number and position of nodes remain

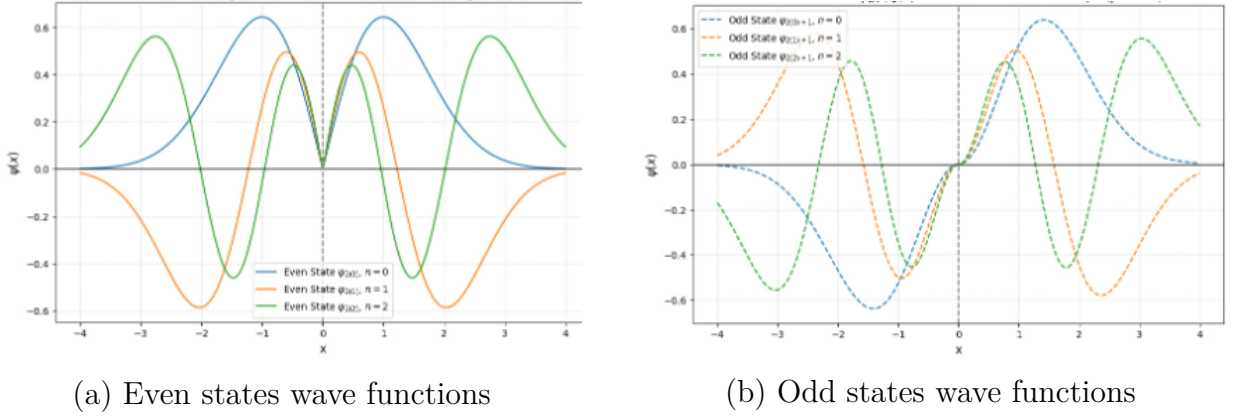


Figure 3.2: Wave functions of the Dunkl harmonic oscillator for even and odd states at different values of μ .

unchanged, indicating that the quantum number still governs the fundamental structure of the states.

Overall, the Dunkl deformation modifies the amplitude and localization of the wave functions without altering their parity or nodal structure, with a stronger effect observed in higher excited states.

3.3 The Nonrelativistic Usual Potential Step

We next consider the ordinary nonrelativistic potential step, which provides the simplest scattering configuration before introducing parity-adapted barrier methods. A particle of mass m and energy E moves in the one-dimensional potential

$$V(x) = \begin{cases} 0, & x < 0 \quad (\text{Region I}) \\ V_0, & x \geq 0 \quad (\text{Region II}) \end{cases} \quad (3.47)$$

The case ($E > V_0$): In this case, we assume the particle's energy is greater than the potential height ($E > V_0$). The time-independent Schrödinger equation is given by:

$$-\frac{\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x) \quad (3.48)$$

1. Region I ($x < 0$):

In this region, $V(x) = 0$. The equation becomes:

$$\frac{d^2\psi_1}{dx^2} + k_1^2\psi_1 = 0, \quad \text{where } k_1 = \sqrt{\frac{2mE}{\hbar^2}} \quad (3.49)$$

The general solution represents a superposition of an incident wave moving to the right and a reflected wave moving to the left:

$$\psi_1(x) = Ae^{ik_1x} + Be^{-ik_1x} \quad (3.50)$$

2. Region II ($x \geq 0$):

In this region, $V(x) = V_0$. The equation becomes:

$$\frac{d^2\psi_2}{dx^2} + k_2^2\psi_2 = 0, \quad \text{where } k_2 = \sqrt{\frac{2m(E - V_0)}{\hbar^2}} \quad (3.51)$$

Since the particle is incident from the left, there is only a transmitted wave moving to the right in Region II:

$$\psi_2(x) = Ce^{ik_2x} \quad (3.52)$$

To find the constants B and C in terms of the incident amplitude A , we apply the continuity conditions at $x = 0$.

The wave function must be continuous at $x = 0$:

$$\psi_1(0) = \psi_2(0) \implies A + B = C \quad (3.53)$$

The derivative $d\psi/dx$ must also be continuous at $x = 0$:

$$\frac{d\psi_1(0)}{dx} = \frac{d\psi_2(0)}{dx} \implies k_1(A - B) = k_2C \quad (3.54)$$

Substitute C from Eq. (3.53) into Eq. (3.54):

$$(k_1 - k_2)A = (k_1 + k_2)B \quad (3.55)$$

Thus, the reflection amplitude ratio is:

$$\frac{B}{A} = \frac{k_1 - k_2}{k_1 + k_2} \quad (3.56)$$

From Eq. (3.53), substitute $B = C - A$ into Eq. (3.54):

$$2k_1A = (k_1 + k_2)C \quad (3.57)$$

Thus, the transmission amplitude ratio is:

$$\frac{C}{A} = \frac{2k_1}{k_1 + k_2} \quad (3.58)$$

These coefficients represent the ratio of probability currents (J). Since the incident wave is $\psi_i(x) = Ae^{ik_1x}$, the incident current density (or incident flux) is given by

$$J_{\text{incident}} = \frac{i\hbar}{2m} \left(\psi_i(x) \frac{d\psi_i^*(x)}{dx} - \psi_i^*(x) \frac{d\psi_i(x)}{dx} \right) = \frac{\hbar k_1}{m} |A|^2. \quad (4.21)$$

Similarly, since the reflected and transmitted waves are $\psi_r(x) = Be^{-ik_1x}$ and $\psi_t(x) = Ce^{ik_2x}$, we can verify that the reflected and transmitted fluxes are

$$J_{\text{reflected}} = -\frac{\hbar k_1}{m}|B|^2, \quad J_{\text{transmitted}} = \frac{\hbar k_2}{m}|C|^2. \quad (4.22)$$

Reflection Coefficient (R):

$$R = \frac{|J_{\text{reflected}}|}{|J_{\text{incident}}|} = \frac{|B|^2}{|A|^2} = \left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2 \quad (3.59)$$

Transmission Coefficient (T): Note that for transmission, we must account for the change in velocity (k_2/k_1):

$$T = \frac{|J_{\text{transmitted}}|}{|J_{\text{incident}}|} = \frac{k_2|C|^2}{k_1|A|^2} = \frac{k_2}{k_1} \left(\frac{2k_1}{k_1 + k_2} \right)^2 = \frac{4k_1k_2}{(k_1 + k_2)^2} \quad (3.60)$$

We can verify that $R + T = 1$:

$$R + T = \frac{(k_1 - k_2)^2 + 4k_1k_2}{(k_1 + k_2)^2} = \frac{k_1^2 - 2k_1k_2 + k_2^2 + 4k_1k_2}{(k_1 + k_2)^2} = \frac{(k_1 + k_2)^2}{(k_1 + k_2)^2} = 1 \quad (3.61)$$

This confirms that the total number of particles is conserved during the process.

The case ($E < V_0$): In this scenario, a particle of mass m and energy E encounters a potential barrier V_0 that is higher than its total energy ($E < V_0$).

- **Classically:** The particle would reach $x = 0$ and be perfectly reflected. It is strictly forbidden from entering the region $x > 0$.
- **Quantum Mechanically:** The wave function does not vanish instantly at the boundary. Instead, it "penetrates" into the barrier, decaying exponentially.

Wave Equation in Region I ($x < 0$): Since $V = 0$, the Schrödinger equation is:

$$\frac{d^2\psi_1}{dx^2} + k_1^2\psi_1 = 0, \quad \text{where } k_1 = \sqrt{\frac{2mE}{\hbar^2}} \quad (3.62)$$

The solution is an oscillatory wave:

$$\psi_1(x) = Ae^{ik_1x} + Be^{-ik_1x} \quad (3.63)$$

Wave Equation in Region II ($x \geq 0$): Since $V = V_0$ and $E < V_0$, the term $(E - V_0)$ is negative. We rewrite the equation as:

$$\frac{d^2\psi_2}{dx^2} - \kappa^2\psi_2 = 0, \quad \text{where } \kappa = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}} \quad (3.64)$$

The general solution for this differential equation is $\psi_2(x) = Ce^{-\kappa x} + De^{\kappa x}$.

- **Physical Constraint:** As $x \rightarrow \infty$, the term $e^{\kappa x}$ blows up (becomes infinite). For the wave function to be square-integrable (normalizable), we must set $D = 0$. The physically acceptable solution is an exponentially decaying function:

$$\psi_t(x) = Ce^{-\kappa x} \quad (3.65)$$

We apply the same steps of continuity at $x = 0$ to find the relationship between A, B , and C .

Thus, the ratio B/A is:

$$\frac{B}{A} = \frac{ik_1 + \kappa}{ik_1 - \kappa} \quad (3.66)$$

The ratio C/A is:

$$\frac{C}{A} = \frac{2ik_1}{ik_1 - \kappa} \quad (3.67)$$

Reflection Coefficient (R): The reflection coefficient is the square of the magnitude of B/A . Since B/A is a ratio of a complex number and its conjugate (with a sign change), the magnitude is 1:

$$R = \left| \frac{B}{A} \right|^2 = \frac{ik_1 + \kappa}{ik_1 - \kappa} \cdot \frac{-ik_1 + \kappa}{-ik_1 - \kappa} = \frac{k_1^2 + \kappa^2}{k_1^2 + \kappa^2} = 1 \quad (3.68)$$

Conclusion: $R = 1$ means there is *total reflection*. All particles are eventually reflected.

Transmission Coefficient (T): Since the wave function in Region II is purely real ($\psi_t = Ce^{-\kappa x}$), the probability current density J is zero:

$$J_{\text{transmitted}} = \frac{\hbar}{2mi} \left(\psi_t^* \frac{d\psi_t}{dx} - \psi_t \frac{d\psi_t^*}{dx} \right) = 0 \implies T = 0 \quad (3.69)$$

Even though the wave function is non-zero in Region II, there is no steady flow of particles through the barrier.

The Penetration Depth: The probability density $|\psi_t(x)|^2 = |C|^2 e^{-2\kappa x}$ shows that particles can be found at a distance $x > 0$. The characteristic "penetration depth" is $\delta = 1/(2\kappa)$. This is a purely quantum phenomenon.

3.4 Potential Step with the Standard Dunkl Derivative

3.4.1 Case (a): $E > V_0$

The particles are free for $x < 0$ and encounter a repulsive potential V_0 at $x = 0$ which remains constant for $x \geq 0$.

Classical Analysis: Classically, the particles approach from the left with momentum $\sqrt{2mE}$. Upon entering $x \geq 0$, they slow down to a momentum $\sqrt{2m(E - V_0)}$. Since $E > V_0$, there is total transmission; all particles emerge to the right with reduced kinetic energy $E - V_0$.

Quantum Analysis: Quantum mechanically, the dynamics is regulated by the Schrödinger equation using the Dunkl derivative D_x :

$$D_x^2 \psi_1(x) + k_1^2 \psi_1(x) = 0, \quad (x < 0) \quad (2)$$

$$D_x^2 \psi_2(x) + k_2^2 \psi_2(x) = 0, \quad (x \geq 0) \quad (3)$$

where the wave numbers are:

$$k_1^2 = \frac{2mE}{\hbar^2}, \quad k_2^2 = \frac{2m(E - V_0)}{\hbar^2}, \quad (3.70)$$

The most general solutions are plane waves expressed through Dunkl exponential functions E_ν :

$$\psi_1(x) = A \frac{|k_1|^\nu 2^{-1/2-\nu}}{\Gamma(\nu + 1/2)} E_\nu(ik_1 x) + B \frac{|k_1|^\nu 2^{-1/2-\nu}}{\Gamma(\nu + 1/2)} E_\nu(-ik_1 x), \quad (x < 0) \quad (4)$$

$$\psi_2(x) = C \frac{|k_2|^\nu 2^{-1/2-\nu}}{\Gamma(\nu + 1/2)} E_\nu(ik_2 x) + D \frac{|k_2|^\nu 2^{-1/2-\nu}}{\Gamma(\nu + 1/2)} E_\nu(-ik_2 x), \quad (x \geq 0) \quad (5)$$

Since particles are incident from the left, we set $D = 0$. The stationary state wave function is:

$$\Psi(x, t) = \begin{cases} (A \mathcal{N}_1 E_\nu(ik_1 x) + B \mathcal{N}_1 E_\nu(-ik_1 x)) e^{-iEt/\hbar}, & x < 0 \\ C \mathcal{N}_2 E_\nu(ik_2 x) e^{-iEt/\hbar}, & x \geq 0 \end{cases} \quad (6)$$

where $\mathcal{N}_j = \frac{|k_j|^\nu 2^{-1/2-\nu}}{\Gamma(\nu+1/2)}$. The probability current density J is defined as:

$$J = \frac{\hbar}{2mi} (\psi^* D_x \psi - \psi D_x \psi^*), \quad (3.71)$$

Using the property $D_x E_\nu(ikx) = ik E_\nu(ikx)$, the currents are derived as:

$$J_{\text{incident}} = \frac{\hbar k_1}{m} |A|^2 \mathcal{N}_1^2 E_\nu(ik_1 x) E_\nu(-ik_1 x) \quad (7)$$

and

$$J_{\text{reflected}} = -\frac{\hbar k_1}{m} |B|^2 \mathcal{N}_1^2 E_\nu(ik_1 x) E_\nu(-ik_1 x) \quad (8)$$

and

$$J_{\text{transmitted}} = \frac{\hbar k_2}{m} |C|^2 \mathcal{N}_2^2 E_\nu(ik_2 x) E_\nu(-ik_2 x) \quad (9)$$

The coefficients are calculated as follows:

$$R = \frac{J_{\text{reflected}}}{J_{\text{incident}}} = \frac{|B|^2}{|A|^2} \quad (10)$$

and

$$T = \frac{J_{\text{transmitted}}}{J_{\text{incident}}} = \frac{k_2|C|^2\mathcal{N}_2^2 E_\nu(ik_2x)E_\nu(-ik_2x)}{k_1|A|^2\mathcal{N}_1^2 E_\nu(ik_1x)E_\nu(-ik_1x)} = \frac{k_2|C|^2}{k_1|A|^2} \quad (11)$$

At $x = 0$, we require the continuity of ψ and $D_x\psi$:

$$1. \text{ Continuity of } \psi: A + B = C$$

$$2. \text{ Continuity of } D_x\psi: A - B = \frac{k_2}{k_1}C$$

Solving these equations leads to the amplitudes:

$$B = \frac{k_1 - k_2}{k_1 + k_2}A, \quad C = \frac{2k_1}{k_1 + k_2}A \quad (13)$$

The final reflection and transmission coefficients are:

$$R = \left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2, \quad T = \frac{4k_1k_2}{(k_1 + k_2)^2} \quad (14)$$

Note that $R + T = 1$, satisfying the conservation of probability.

3.4.2 Case (b): $E < V_0$

In this case, the energy of the incident particles is less than the potential barrier V_0 . Classically, the particles would be reflected entirely at $x = 0$. However, in the Dunkl-Schrödinger framework, we observe a penetration into the barrier.

The wave equation for Region I ($x < 0$) remains the same as in Case (a). However, for Region II ($x \geq 0$), since $E - V_0 < 0$, we define a real decay constant κ_2 instead of the wave number k_2 :

$$\kappa_2 = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}} \quad (3.72)$$

The Dunkl-Schrödinger equation in Region II becomes:

$$D_x^2\psi_2(x) - \kappa_2^2\psi_2(x) = 0, \quad (x \geq 0) \quad (3.73)$$

The solution in Region I is oscillatory, while in Region II it must be exponentially decaying to remain finite as $x \rightarrow \infty$:

$$\psi_1(x) = A\mathcal{N}_1 E_\nu(ik_1x) + B\mathcal{N}_1 E_\nu(-ik_1x), \quad (x < 0) \quad (3.74)$$

and

$$\psi_2(x) = C \frac{|\kappa_2|^\nu 2^{-1/2-\nu}}{\Gamma(\nu + 1/2)} E_\nu(-\kappa_2x), \quad (x \geq 0) \quad (3.75)$$

where $E_\nu(-\kappa_2x)$ represents the Dunkl exponential decay.

Since the wave function in Region II is real (exponential decay), the probability current density $J_{\text{transmitted}}$ vanishes:

$$J_{\text{transmitted}} = 0 \implies T = 0 \quad (3.76)$$

Applying the boundary conditions at $x = 0$ ($\psi_1 = \psi_2$ and $D_x\psi_1 = D_x\psi_2$):

$$A + B = C, \quad ik_1(A - B) = -\kappa_2 C \quad (3.77)$$

Solving for the ratio B/A :

$$\frac{B}{A} = \frac{ik_1 + \kappa_2}{ik_1 - \kappa_2} \quad (3.78)$$

The reflection coefficient is:

$$R = \left| \frac{B}{A} \right|^2 = \frac{ik_1 + \kappa_2}{ik_1 - \kappa_2} \cdot \frac{-ik_1 + \kappa_2}{-ik_1 - \kappa_2} = \frac{k_1^2 + \kappa_2^2}{k_1^2 + \kappa_2^2} = 1 \quad (3.79)$$

Even though the particles penetrate into the region $x > 0$ (as shown in Figure 1 of the research), the reflection is total ($R = 1$). This penetration is a hallmark of quantum mechanics, where the probability density $|\psi_2(x)|^2$ is non-zero inside the barrier but decays rapidly.

Using the identity $(1 \pm \frac{ik_1}{\rho})(1 \mp \frac{\rho}{ik_1}) = (1 + \frac{k_1^2}{\rho^2})$, the expression simplifies to:

$$\frac{B}{E} = \frac{1}{4} \left(1 + \frac{k_1^2}{\rho^2} \right) \frac{E_\nu(ik_1a)}{E_\nu(-ik_1a)} \left[\frac{E_\nu(-\rho a)}{E_\nu(\rho a)} - \frac{E_\nu(\rho a)}{E_\nu(-\rho a)} \right] \quad (3.80)$$

The reflection ratio B/A is then obtained by dividing B/E by the previously established A/E :

$$\frac{B}{A} = \frac{(k_1^2 + \rho^2) \left[\frac{E_\nu(-\rho a)}{E_\nu(\rho a)} - \frac{E_\nu(\rho a)}{E_\nu(-\rho a)} \right]}{(k_1^2 - \rho^2) \left[\frac{E_\nu(\rho a)}{E_\nu(-\rho a)} + \frac{E_\nu(-\rho a)}{E_\nu(\rho a)} \right] + 2ik_1\rho} \quad (3.81)$$

Finally, the reflection coefficient $R = |B/A|^2$ is expressed as:

$$R = \frac{(k_1^2 + \rho^2)^2 \left[\frac{E_\nu(\rho a)}{E_\nu(-\rho a)} - \frac{E_\nu(-\rho a)}{E_\nu(\rho a)} \right]^2}{(k_1^2 - \rho^2)^2 \left[\frac{E_\nu(\rho a)}{E_\nu(-\rho a)} + \frac{E_\nu(-\rho a)}{E_\nu(\rho a)} \right]^2 + 4k_1^2\rho^2} \quad (3.82)$$

This derivation shows that $R + T = 1$, which confirms the conservation of the probability current in the ν -deformed space.

3.5 The Usual Nonrelativistic Symmetric Rectangular Potential Barrier

We consider a one-dimensional rectangular potential barrier defined as:

$$V(x) = \begin{cases} 0 & x < -a, \\ V_0 & -a \leq x \leq a, \\ 0 & x > a. \end{cases} \quad (3.83)$$

The width of the barrier is $2a$,

3.5.1 Flux Method for Reflection and Transmission

Case ($E > V_0$): The stationary Schrödinger equation is

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V(x)\psi = E\psi. \quad (3.84)$$

We solve this equation in the three spatial regions.

Region I ($x < -a$): Since $V(x) = 0$, the equation becomes

$$\frac{d^2\psi}{dx^2} + k_1^2\psi = 0, \quad (3.85)$$

where $k = \frac{\sqrt{2mE}}{\hbar}$. The general solution is

$$\psi_I = Ae^{ik_1x} + Be^{-ik_1x} \quad (3.86)$$

where $\psi_i = Ae^{ik_1x}$ represents the incident wave and $\psi_r = Be^{-ik_1x}$ the reflected wave.

Region II ($-a < x < a$): For the case $E > V_0$, the Schrödinger equation becomes

$$\frac{d^2\psi}{dx^2} + k_2^2\psi = 0 \quad (3.87)$$

with $k_2 = \frac{\sqrt{2m(E-V_0)}}{\hbar}$. The general solution inside the barrier is

$$\psi_{II} = Ce^{ik_2x} + De^{-ik_2x} \quad (3.88)$$

Region III ($x > a$): In this region $V(x) = 0$ and only a transmitted wave exists:

$$\psi_{III} = Fe^{ik_1x}. \quad (3.89)$$

The wavefunction and its derivative must be continuous at $x = -a$ and $x = a$.

Continuity of the wavefunction at $x = a$:

$$Ce^{ik_2a} + De^{-ik_2a} = Fe^{ik_1a}. \quad (3.90)$$

Continuity of the derivative at $x = a$:

$$ik_2Ce^{ik_2a} - ik_2De^{-ik_2a} = ikFe^{ik_1a}. \quad (3.91)$$

Continuity of the wavefunction at $x = -a$:

$$Ae^{-ik_1a} + Be^{ik_1a} = Ce^{-ik_2a} + De^{ik_2a} \quad (3.92)$$

Continuity of the derivative at $x = -a$:

$$ik_1 A e^{-ik_1 a} - ik_1 B e^{ik_1 a} = ik_2 C e^{-ik_2 a} - ik_2 D e^{ik_2 a} \quad (3.93)$$

The transmission coefficient is defined as

$$T = \frac{J_{\text{trans}}}{J_{\text{inc}}} = \left| \frac{E}{A} \right|^2 \quad (3.94)$$

To simplify the algebra, define:

$$X = C e^{-ik_2 a}, \quad Y = D e^{ik_2 a} \quad (3.95)$$

Then equations (3) and (4) become:

$$X + Y = E e^{ik_1 a}, \quad k_2(X - Y) = k_1 E e^{ik_1 a} \quad (3.96)$$

Solving for X and Y :

$$2X = \left(1 + \frac{k_1}{k_2}\right) E e^{ik_1 a} \quad (3.97)$$

and

$$2Y = \left(1 - \frac{k_1}{k_2}\right) E e^{ik_1 a} \quad (3.98)$$

Therefore,

$$C = E \left(\frac{k_2 + k_1}{2k_2} \right) e^{ia(k_1 - k_2)} \quad (3.99)$$

Thus the final transmission amplitude is:

$$D = E \left(\frac{k_2 - k_1}{2k_2} \right) e^{ia(k_1 + k_2)} \quad (3.100)$$

Returning to the first boundary condition and combining equations, we obtain:

$$2A e^{-ik_1 a} = C e^{-ik_2 a} \left(1 + \frac{k_2}{k_1}\right) + D e^{ik_2 a} \left(1 - \frac{k_2}{k_1}\right) \quad (3.101)$$

Then

$$2A e^{-ik_1 a} = E \frac{1}{2k_1 k_2} e^{iak_1} [(k_1 + k_2)^2 e^{-i2k_2 a} - (k_2 - k_1)^2 e^{i2k_2 a}] \quad (3.102)$$

After substitution and simplification:

$$\frac{E}{A} = \frac{4k_1 k_2 e^{-2ik_1 a}}{(k_1 + k_2)^2 e^{-2ik_2 a} - (k_2 - k_1)^2 e^{i2k_2 a}} \quad (3.103)$$

Using Euler identities to simplify the denominator:

$$(k_1 + k_2)^2 e^{-2ik_2 a} - (k_2 - k_1)^2 e^{i2k_2 a} = 2k_1 k_2 \cos(2k_2 a) - 2i(k_1^2 + k_2^2) \sin(2k_2 a) \quad (3.104)$$

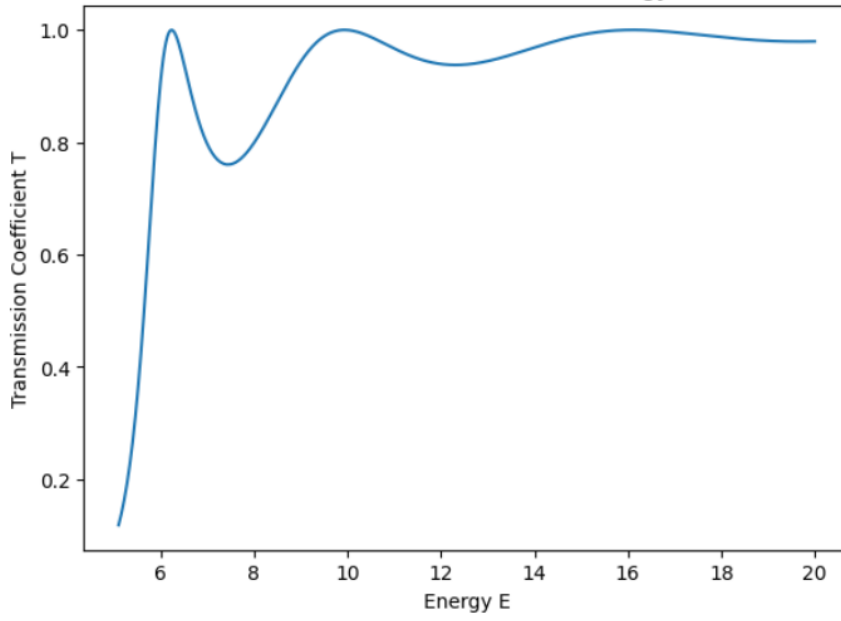


Figure 3.3: Transmission coefficient T as a function of the energy E

we obtain

$$\frac{E}{A} = \frac{4k_1k_2e^{-2ik_1a}}{2k_1k_2\cos(2k_2a) - 2i(k_1^2 + k_2^2)\sin(2k_2a)} \quad (3.105)$$

and therefore,

$$T = \left| \frac{E}{A} \right|^2 = \frac{1}{1 + \frac{(k_1^2 - k_2^2)^2}{4k_1^2k_2^2} \sin^2(2k_2a)} \quad (3.106)$$

Using

$$k_1^2 = \frac{2mE}{\hbar^2}, \quad k_2^2 = \frac{2m(E - V_0)}{\hbar^2}, \quad (3.107)$$

Therefore, the final expression of the transmission coefficient becomes

$$T = \frac{1}{1 + \frac{V_0^2}{4E(E - V_0)} \sin^2(2qa)}. \quad (3.108)$$

Physical Interpretation: For $E > V_0$, the solution inside the barrier is oscillatory. The transmission probability exhibits a resonance behavior.

Whenever

$$2k_2a = n\pi \quad (3.109)$$

we have

$$\sin(2k_2a) = 0, \quad (3.110)$$

and consequently $T = 1$. This corresponds to complete transmission despite the presence of the potential barrier. This purely quantum phenomenon is known as resonant transmission.

Since the left and right regions have the same wave number k_1 , the reflection coefficient is defined as

$$R = \left| \frac{B}{A} \right|^2 \quad (3.111)$$

Continuity of the wave function at $x = -a$:

$$Ae^{-ik_1a} + Be^{ik_1a} = Ce^{-ik_2a} + De^{ik_2a} \quad (3.112)$$

Continuity of the derivative at $x = -a$:

$$Ae^{-ik_1a} - Be^{ik_1a} = \frac{k_2}{k_1} (Ce^{-ik_2a} - De^{ik_2a}) \quad (3.113)$$

Subtracting the second equation from the first gives

$$2Be^{ik_1a} = Ce^{-ik_2a} \left(1 - \frac{k_2}{k_1} \right) + De^{ik_2a} \left(1 + \frac{k_2}{k_1} \right) \quad (3.114)$$

Using the previously obtained expressions:

$$C = E \left(\frac{k_2 + k_1}{2k_2} \right) e^{ia(k_1 - k_2)} \quad (3.115)$$

and

$$D = E \left(\frac{k_2 - k_1}{2k_2} \right) e^{ia(k_1 + k_2)} \quad (3.116)$$

After substitution and algebraic simplification, we obtain

$$\frac{B}{A} = \frac{(k_1^2 - k_2^2) \sin(2k_2a)}{2ik_1k_2 \cos(2k_2a) - i(k_1^2 + k_2^2) \sin(2k_2a)} \quad (3.117)$$

To compute R , we take the modulus squared:

$$R = \frac{(k_1^2 - k_2^2)^2 \sin^2(2k_2a)}{4k_1^2k_2^2 \cos^2(2k_2a) + (k_1^2 + k_2^2)^2 \sin^2(2k_2a)} \quad (3.118)$$

Factoring the denominator, we obtain the compact real expression:

$$R = \frac{\frac{(k_1^2 - k_2^2)^2}{4k_1^2k_2^2} \sin^2(2k_2a)}{1 + \frac{(k_1^2 - k_2^2)^2}{4k_1^2k_2^2} \sin^2(2k_2a)} \quad (3.119)$$

we verify immediately that $R + T = 1$, which confirms conservation of probability.

Case ($E < V_0$) : We have:

$$k_1 = \frac{\sqrt{2mE}}{\hbar}, \quad \kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar} \quad (3.120)$$

Region I ($x < -a$):

$$\psi_I = Ae^{ik_1x} + Be^{-ik_1x} \quad (3.121)$$

Region II ($-a \leq x \leq a$):

$$\psi_{II} = Ce^{\kappa x} + De^{-\kappa x} \quad (3.122)$$

Region III ($x > a$):

$$\psi_{III} = Ee^{ik_1 x} \quad (3.123)$$

Continuity of wave function at $x = a$:

$$Ce^{\kappa a} + De^{-\kappa a} = Ee^{ik_1 a} \quad (3.124)$$

Continuity of derivative at $x = a$:

$$\kappa Ce^{\kappa a} - \kappa De^{-\kappa a} = ik_1 Ee^{ik_1 a} \quad (3.125)$$

Adding (3.124) and (3.125)/ κ :

$$C = \frac{E}{2} \left(1 + \frac{ik_1}{\kappa} \right) e^{ik_1 a} e^{-\kappa a} \quad (3.126)$$

Subtract (3.125)/ κ from (3.124):

$$D = \frac{E}{2} \left(1 - \frac{ik_1}{\kappa} \right) e^{ik_1 a} e^{\kappa a} \quad (3.127)$$

Continuity of wave function at $x = -a$:

$$Ae^{-ik_1 a} + Be^{ik_1 a} = Ce^{-\kappa a} + De^{\kappa a} \quad (3.128)$$

Continuity of derivative at $x = -a$:

$$ik_1 Ae^{-ik_1 a} - ik_1 Be^{ik_1 a} = \kappa Ce^{-\kappa a} - \kappa De^{\kappa a} \quad (3.129)$$

Adding Eqs.(3.128) and (3.129) and dividing by ik_1 , we obtain

$$2A = E \left[2(e^{2\kappa a} + e^{-2\kappa a}) + \frac{\kappa^2 - k_1^2}{ik_1 \kappa} (e^{2\kappa a} - e^{-2\kappa a}) \right] \quad (3.130)$$

By subtracting (3.128) from (3.129)

$$2Be^{-ik_1 a} = Ce^{-\kappa a} \left(1 - \frac{\kappa}{ik_1} \right) + De^{\kappa a} \left(1 + \frac{\kappa}{ik_1} \right) \quad (3.131)$$

Substituting the expressions for C and D in terms of the transmitted amplitude E , we obtain:

$$2Be^{-ik_1 a} = \frac{Ee^{ik_1 a}}{2} \left[\left(1 + \frac{ik_1}{\kappa} \right) \left(1 - \frac{\kappa}{ik_1} \right) e^{2\kappa a} + \left(1 - \frac{ik_1}{\kappa} \right) \left(1 + \frac{\kappa}{ik_1} \right) e^{-2\kappa a} \right] \quad (3.132)$$

Using the hyperbolic identity $\sinh(2\kappa a) = \frac{e^{2\kappa a} - e^{-2\kappa a}}{2}$, the expression simplifies to:

$$B = \frac{E e^{2ik_1 a}}{2ik_1 \kappa} (k_1^2 + \kappa^2) \sinh(2\kappa a) \quad (3.133)$$

Finally, the reflection amplitude ratio B/A is:

$$\frac{B}{A} = \frac{(k_1^2 + \kappa^2) \sinh(2\kappa a)}{(k_1^2 - \kappa^2) \sinh(2\kappa a) + 2ik_1 \kappa \cosh(2\kappa a)} \quad (3.134)$$

Simplification by using hyperbolic functions, we find

$$\frac{E}{A} = \frac{4k_1 \kappa e^{-2ik_1 a}}{(k_1^2 + \kappa^2) \sinh(2\kappa a) + 2ik_1 \kappa \cosh(2\kappa a)} \quad (3.135)$$

and

$$\frac{B}{A} = \frac{(k_1^2 + \kappa^2) \sinh(2\kappa a)}{2ik_1 \kappa \cosh(2\kappa a) + i(k_1^2 + \kappa^2) \sinh(2\kappa a)} \quad (3.136)$$

Consequently, the expressions for the transmission and reflection coefficients are

$$T = \left| \frac{E}{A} \right|^2 = \frac{1}{1 + \frac{(k_1^2 + \kappa^2)^2}{4k_1^2 \kappa^2} \sinh^2(2\kappa a)} \quad (3.137)$$

and

$$R = \left| \frac{B}{A} \right|^2 = \frac{(k_1^2 + \kappa^2)^2 \sinh^2(2\kappa a)}{4k_1^2 \kappa^2 \cosh^2(2\kappa a) + (k_1^2 + \kappa^2)^2 \sinh^2(2\kappa a)} \quad (3.138)$$

3.5.2 Phase Shift Analysis

A- The case $E > V_0$:

1- Outside the barrier ($|x| > a$) : One has $V(x) = 0$, so Schrodinger equation becomes

$$\psi''(x) + k^2 \psi(x) = 0. \quad (3.139)$$

where $k = \frac{\sqrt{2mE}}{\hbar}$. Its general solution is

$$\psi(x) = A \cos(kx) + B \sin(kx). \quad (3.140)$$

2- Inside the barrier ($|x| < a$) : One has $V(x) = V_0$, so Schrodinger equation becomes

$$\psi''(x) + q^2 \psi(x) = 0. \quad (3.141)$$

where $q = \frac{\sqrt{2m(E-V_0)}}{\hbar}$. Its general solution is

$$\psi(x) = C \cos(qx) + D \sin(qx). \quad (3.142)$$

Since the potential is even $V(-x) = V(x)$, the Schrödinger equation is invariant under spatial reflection. Therefore, the solutions may be chosen with definite parity:

- even channel:

$$\psi_e(-x) = \psi_e(x), \quad (3.143)$$

- odd channel:

$$\psi_o(-x) = -\psi_o(x). \quad (3.144)$$

It is then sufficient to solve the problem on the half-line $x > 0$ and extend the solution to the full line by parity.

1- For even channel:

1-A Exterior solution: From the general solution (3.140), it is convenient to write the even-channel exterior solution in the phase-shift form

$$\psi_e^{\text{out}}(x) = A_e \cos(kx - \delta_e), \quad x > a. \quad (3.145)$$

This is completely equivalent to a linear combination of $\cos(kx)$ and $\sin(kx)$, since

$$\cos(kx - \delta_e) = \cos(kx) \cos \delta_e + \sin(kx) \sin \delta_e. \quad (3.146)$$

The parameter δ_e is the even phase shift.

1-B Interior solution: Inside the barrier, the general solution is given by Eq. (3.142). Since the even solution must satisfy $\psi_e(-x) = \psi_e(x)$, only the even function $\cos(qx)$ is allowed. Therefore the interior even solution is

$$\psi_e^{\text{in}}(x) = B_e \cos(qx), \quad 0 < x < a. \quad (3.147)$$

Because the potential is finite at $x = a$, both the wavefunction and its derivative must be continuous there. From Eqs. (3.145) and (3.147), we impose

$$\psi_e^{\text{out}}(a) = \psi_e^{\text{in}}(a), \quad (3.148)$$

which gives

$$A_e \cos(ka - \delta_e) = B_e \cos(qa). \quad (3.149)$$

We differentiate the exterior solution:

$$\frac{d\psi_e^{\text{out}}(x)}{dx} = -kA_e \sin(kx - \delta_e). \quad (3.150)$$

Evaluating at $x = a$:

$$\left. \frac{d\psi_e^{\text{out}}(x)}{dx} \right|_{x=a} = -kA_e \sin(ka - \delta_e). \quad (3.151)$$

Now differentiate the interior solution:

$$\frac{d\psi_e^{\text{in}}(x)}{dx} = -qB_e \sin(qx). \quad (3.152)$$

Evaluating at $x = a$:

$$\left. \frac{d\psi_e^{\text{in}}(x)}{dx} \right|_{x=a} = -qB_e \sin(qa). \quad (3.153)$$

The continuity of the derivative gives

$$\psi_e^{\text{out}'}(a) = \psi_e^{\text{in}'}(a), \quad (3.154)$$

that is,

$$-kA_e \sin(ka - \delta_e) = -qB_e \sin(qa). \quad (3.155)$$

We now divide Eq. (3.155) by Eq. (3.149). This gives

$$\frac{-kA_e \sin(ka - \delta_e)}{A_e \cos(ka - \delta_e)} = \frac{-qB_e \sin(qa)}{B_e \cos(qa)}. \quad (3.156)$$

After cancellation of the common factors A_e , B_e , and the minus signs, we obtain

$$k \tan(ka - \delta_e) = q \tan(qa). \quad (3.157)$$

Finally, dividing by k , we find

$$\boxed{\tan(ka - \delta_e) = \frac{q}{k} \tan(qa)}. \quad (3.158)$$

2- For odd channel:

2-A Exterior solution: For the odd channel, it is convenient to write the exterior solution as

$$\psi_o^{\text{out}}(x) = A_o \sin(kx - \delta_o), \quad x > a. \quad (3.159)$$

The parameter δ_o is the odd phase shift.

2-B Interior solution: Inside the barrier, the odd solution must satisfy

$$\psi_o(-x) = -\psi_o(x). \quad (3.160)$$

Since $\sin(qx)$ is odd and $\cos(qx)$ is even, only the sine term is allowed. Therefore

$$\psi_o^{\text{in}}(x) = B_o \sin(qx), \quad 0 < x < a. \quad (3.161)$$

First matching condition: continuity of the wavefunction at $x = a$. We impose

$$\psi_o^{\text{out}}(a) = \psi_o^{\text{in}}(a), \quad (3.162)$$

which gives

$$A_o \sin(ka - \delta_o) = B_o \sin(qa). \quad (3.163)$$

Differentiating the exterior solution:

$$\psi_o^{\text{out}}(x) = A_o \sin(kx - \delta_o), \quad (3.164)$$

we obtain

$$\frac{d\psi_o^{\text{out}}(x)}{dx} = kA_o \cos(kx - \delta_o). \quad (3.165)$$

Hence at $x = a$:

$$\frac{d\psi_o^{\text{out}}(x)}{dx} \Big|_{x=a} = kA_o \cos(ka - \delta_o). \quad (3.166)$$

Differentiating the interior solution:

$$\psi_o^{\text{in}}(x) = B_o \sin(qx), \quad (3.167)$$

we get

$$\frac{d\psi_o^{\text{in}}(x)}{dx} \Big|_{x=a} = qB_o \cos(qa). \quad (3.168)$$

Thus at $x = a$:

$$\frac{d\psi_o^{\text{in}}(x)}{dx} \Big|_{x=a} = qB_o \cos(qa). \quad (3.169)$$

The continuity of the derivative gives

$$\psi_o^{\text{out}'}(a) = \psi_o^{\text{in}'}(a), \quad (3.170)$$

namely

$$kA_o \cos(ka - \delta_o) = qB_o \cos(qa). \quad (3.171)$$

We divide Eq. (3.171) by Eq. (3.163):

$$\frac{kA_o \cos(ka - \delta_o)}{A_o \sin(ka - \delta_o)} = \frac{qB_o \cos(qa)}{B_o \sin(qa)}. \quad (3.172)$$

After simplification, this gives

$$k \cot(ka - \delta_o) = q \cot(qa). \quad (3.173)$$

Finally, dividing by k , we obtain

$$\boxed{\cot(ka - \delta_o) = \frac{q}{k} \cot(qa)}. \quad (3.174)$$

For the symmetric rectangular barrier with $E > V_0$, the even and odd phase shifts satisfy

$$\boxed{\tan(ka - \delta_e) = \frac{q}{k} \tan(qa)}, \quad \boxed{\cot(ka - \delta_o) = \frac{q}{k} \cot(qa)}, \quad (3.175)$$

These two relations are the starting point for the derivation of the closed-form transmission and reflection coefficients.

Once the even and odd phase shifts δ_e and δ_o are known, the transmission and reflection amplitudes are

$$t = \frac{1}{2} (e^{2i\delta_e} + e^{2i\delta_o}), \quad r = \frac{1}{2} (e^{2i\delta_e} - e^{2i\delta_o}). \quad (3.176)$$

Therefore,

$$T = |t|^2 = \cos^2(\delta_e - \delta_o), \quad R = |r|^2 = \sin^2(\delta_e - \delta_o). \quad (3.177)$$

Hence the problem reduces to the explicit evaluation of the phase-shift difference $(\delta_e - \delta_o)$.

From the matching conditions in the even and odd channels, one finds

$$\tan(ka - \delta_e) = \frac{q}{k} \tan(qa), \quad (3.178)$$

and

$$\cot(ka - \delta_o) = \frac{q}{k} \cot(qa). \quad (3.179)$$

Our goal is now to extract $\delta_e - \delta_o$ from these two relations.

Step 1: Introduce auxiliary angles, we set,

$$\alpha = ka - \delta_e, \quad \beta = ka - \delta_o. \quad (3.180)$$

Then Eqs. (3.178)–(3.179) become

$$\tan \alpha = \frac{q}{k} \tan(qa), \quad (3.181)$$

and

$$\cot \beta = \frac{q}{k} \cot(qa). \quad (3.182)$$

Since $\cot \beta = \frac{1}{\tan \beta}$, Eq. (3.182) gives

$$\tan \beta = \frac{k}{q} \tan(qa). \quad (3.183)$$

Now, from

$$\delta_e = ka - \alpha, \quad \delta_o = ka - \beta, \quad (3.184)$$

we obtain

$$\delta_e - \delta_o = (ka - \alpha) - (ka - \beta) = \beta - \alpha. \quad (3.185)$$

Therefore,

$$T = \cos^2(\beta - \alpha), \quad R = \sin^2(\beta - \alpha). \quad (3.186)$$

So the problem is reduced to finding $\tan(\beta - \alpha)$.

Step 2: Compute $\tan(\beta - \alpha)$, Using the trigonometric identity

$$\tan(\beta - \alpha) = \frac{\tan \beta - \tan \alpha}{1 + \tan \alpha \tan \beta}, \quad (3.187)$$

and substituting Eqs. (3.181) and (3.183), we get

$$\tan(\beta - \alpha) = \frac{\frac{k}{q} \tan(qa) - \frac{q}{k} \tan(qa)}{1 + \left(\frac{q}{k} \tan(qa)\right) \left(\frac{k}{q} \tan(qa)\right)}. \quad (3.188)$$

Let us simplify the numerator first:

$$\frac{k}{q} \tan(qa) - \frac{q}{k} \tan(qa) = \tan(qa) \left(\frac{k}{q} - \frac{q}{k} \right) \quad (3.189)$$

$$= \tan(qa) \frac{k^2 - q^2}{kq}. \quad (3.190)$$

Now the denominator:

$$1 + \left(\frac{q}{k} \tan(qa) \right) \left(\frac{k}{q} \tan(qa) \right) = 1 + \tan^2(qa) \quad (3.191)$$

$$= \sec^2(qa). \quad (3.192)$$

Hence,

$$\tan(\beta - \alpha) = \frac{\frac{k^2 - q^2}{kq} \tan(qa)}{\sec^2(qa)}. \quad (3.193)$$

Using

$$\frac{\tan x}{\sec^2 x} = \sin x \cos x, \quad (3.194)$$

we obtain

$$\tan(\beta - \alpha) = \frac{k^2 - q^2}{kq} \sin(qa) \cos(qa). \quad (3.195)$$

Finally, since

$$2 \sin(qa) \cos(qa) = \sin(2qa),$$

we arrive at

$$\tan(\beta - \alpha) = \frac{k^2 - q^2}{2kq} \sin(2qa). \quad (3.196)$$

Step 3: Compute the transmission coefficient, we have $T = \cos^2(\beta - \alpha)$. Using the identity

$$\cos^2 \theta = \frac{1}{1 + \tan^2 \theta}, \quad (3.197)$$

and substituting Eq. (3.196), we get

$$T = \frac{1}{1 + \left(\frac{k^2 - q^2}{2kq} \sin(2qa) \right)^2}. \quad (3.198)$$

Therefore,

$$\boxed{T(E) = \frac{1}{1 + \frac{(k^2 - q^2)^2}{4k^2 q^2} \sin^2(2qa)}}, \quad (E > V_0). \quad (3.199)$$

Step 4: Compute the reflection coefficient, since $R = 1 - T$, we obtain

$$R = 1 - \frac{1}{1 + \frac{(k^2 - q^2)^2}{4k^2 q^2} \sin^2(2qa)}. \quad (3.200)$$

After simplification,

$$R(E) = \frac{\frac{(k^2 - q^2)^2}{4k^2q^2} \sin^2(2qa)}{1 + \frac{(k^2 - q^2)^2}{4k^2q^2} \sin^2(2qa)}, \quad (E > V_0). \quad (3.201)$$

Step 5: Rewrite in terms of E and V_0 , we immediately find

$$k^2 - q^2 = \frac{2mV_0}{\hbar^2}. \quad (3.202)$$

Thus,

$$\frac{(k^2 - q^2)^2}{4k^2q^2} = \frac{V_0^2}{4E(E - V_0)}. \quad (3.203)$$

Hence Eq. (3.199) becomes

$$T(E) = \frac{1}{1 + \frac{V_0^2}{4E(E - V_0)} \sin^2(2qa)}, \quad (E > V_0), \quad (3.204)$$

and Eq. (3.201) becomes

$$R(E) = \frac{\frac{V_0^2}{4E(E - V_0)} \sin^2(2qa)}{1 + \frac{V_0^2}{4E(E - V_0)} \sin^2(2qa)}, \quad (E > V_0). \quad (3.205)$$

A- The case $E < V_0$: We now define

$$k = \frac{\sqrt{2mE}}{\hbar}, \quad \kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar}. \quad (3.206)$$

For energies below the barrier, the phase-shift relations become

$$\tan(ka - \delta_e) = -\frac{\kappa}{k} \tanh(\kappa a), \quad (3.207)$$

and

$$\cot(ka - \delta_o) = \frac{\kappa}{k} \coth(\kappa a). \quad (3.208)$$

Again, define

$$\alpha = ka - \delta_e, \quad \beta = ka - \delta_o. \quad (3.209)$$

Then

$$\tan \alpha = -\frac{\kappa}{k} \tanh(\kappa a), \quad (3.210)$$

and from Eq. (3.208),

$$\tan \beta = \frac{k}{\kappa} \tanh(\kappa a), \quad (3.211)$$

The same steps, so substituting Eqs. (3.210) and (3.211), we obtain

$$\tan(\beta - \alpha) = \frac{\frac{k}{\kappa} \tanh(\kappa a) + \frac{\kappa}{k} \tanh(\kappa a)}{1 - \tanh^2(\kappa a)}. \quad (3.212)$$

The numerator becomes

$$\frac{k}{\kappa} \tanh(\kappa a) + \frac{\kappa}{k} \tanh(\kappa a) = \tanh(\kappa a) \left(\frac{k}{\kappa} + \frac{\kappa}{k} \right) \quad (3.213)$$

$$= \tanh(\kappa a) \frac{k^2 + \kappa^2}{k\kappa}. \quad (3.214)$$

The denominator simplifies as

$$1 - \tanh^2(\kappa a) = \frac{1}{\cosh^2(\kappa a)}. \quad (3.215)$$

Therefore,

$$\tan(\beta - \alpha) = \frac{\frac{k^2 + \kappa^2}{k\kappa} \tanh(\kappa a)}{\frac{1}{\cosh^2(\kappa a)}}. \quad (3.216)$$

Using

$$\frac{\tanh x}{\cosh^2 x} = \sinh x \cosh x, \quad (3.217)$$

we get

$$\tan(\beta - \alpha) = \frac{k^2 + \kappa^2}{k\kappa} \sinh(\kappa a) \cosh(\kappa a). \quad (3.218)$$

Finally,

$$2 \sinh(\kappa a) \cosh(\kappa a) = \sinh(2\kappa a), \quad (3.219)$$

so

$$\tan(\beta - \alpha) = \frac{k^2 + \kappa^2}{2k\kappa} \sinh(2\kappa a). \quad (3.220)$$

Using again

$$T = \cos^2(\beta - \alpha) = \frac{1}{1 + \tan^2(\beta - \alpha)},$$

together with Eq. (3.220), we find

$$T(E) = \frac{1}{1 + \frac{(k^2 + \kappa^2)^2}{4k^2\kappa^2} \sinh^2(2\kappa a)}. \quad (3.221)$$

Hence,

$$\boxed{T(E) = \frac{1}{1 + \frac{(k^2 + \kappa^2)^2}{4k^2\kappa^2} \sinh^2(2\kappa a)}}, \quad (E < V_0). \quad (3.222)$$

Since $R = 1 - T$, one gets

$$R(E) = \frac{\frac{(k^2 + \kappa^2)^2}{4k^2\kappa^2} \sinh^2(2\kappa a)}{1 + \frac{(k^2 + \kappa^2)^2}{4k^2\kappa^2} \sinh^2(2\kappa a)}, \quad (E < V_0). \quad (3.223)$$

Rewrite in terms of E and V_0 , we have

$$k^2 + \kappa^2 = \frac{2mV_0}{\hbar^2}. \quad (3.224)$$

Thus,

$$\frac{(k^2 + \kappa^2)^2}{4k^2\kappa^2} = \frac{V_0^2}{4E(V_0 - E)}. \quad (3.225)$$

Therefore,

$$T(E) = \frac{1}{1 + \frac{V_0^2}{4E(V_0 - E)} \sinh^2(2\kappa a)}, \quad (E < V_0), \quad (3.226)$$

and

$$R(E) = \frac{\frac{V_0^2}{4E(V_0 - E)} \sinh^2(2\kappa a)}{1 + \frac{V_0^2}{4E(V_0 - E)} \sinh^2(2\kappa a)}, \quad (E < V_0). \quad (3.227)$$

These results satisfy the probability conservation law

$$R(E) + T(E) = 1. \quad (3.228)$$

3.6 Potential Barrier with the Dunkl Derivative

3.6.1 Flux Method for Reflection and Transmission

Case ($E > V_0$) : Based on the potential profile, the wave function $\Psi(x)$ is defined in three distinct regions using the specialized functions $E_\nu(z)$:

- **Region I ($x < -a$):** Represents the incident and reflected wave. We find

$$\Psi_I(x) = A \frac{|k_1|^\nu}{c_\nu} E_\nu(ik_1x) + B \frac{|k_1|^\nu}{c_\nu} E_\nu(-ik_1x) \quad (3.229)$$

- **Region II ($-a < x < a$):** Represents the wave inside the barrier. We can write

$$\Psi_{II}(x) = C \frac{|k_2|^\nu}{c_\nu} E_\nu(ik_2x) + D \frac{|k_2|^\nu}{c_\nu} E_\nu(-ik_2x) \quad (3.230)$$

- **Region III** ($x > a$): Represents the transmitted wave. We obtain

$$\Psi_{III}(x) = E \frac{|k_1|^\nu}{c_\nu} E_\nu(ik_1x) \quad (3.231)$$

To maintain the physical validity of the wave function, we apply continuity conditions for Ψ and its first derivative $D_x\Psi$ at the boundaries.

Continuity at the first interface ($x = -a$):

$$AE_\nu(ik_1a) + BE_\nu(-ik_1a) = C \left| \frac{k_2}{k_1} \right|^\nu E_\nu(ik_2a) + D \left| \frac{k_2}{k_1} \right|^\nu E_\nu(-ik_2a) \quad (3.232)$$

and

$$k_1 [AE_\nu(ik_1a) - BE_\nu(-ik_1a)] = k_2 \left| \frac{k_2}{k_1} \right|^\nu [CE_\nu(ik_2a) - DE_\nu(-ik_2a)] \quad (3.233)$$

By applying similar steps for the second boundary (at $x = a$), we obtain:

$$C \frac{|k_2|^\nu}{c_\nu} E_\nu(ik_2a) + D \frac{|k_2|^\nu}{c_\nu} E_\nu(-ik_2a) = E \frac{|k_1|^\nu}{c_\nu} E_\nu(ik_1a) \quad (3.234)$$

and

$$k_2 \left[C \frac{|k_2|^\nu}{c_\nu} E_\nu(ik_2a) - D \frac{|k_2|^\nu}{c_\nu} E_\nu(-ik_2a) \right] = k_1 E \frac{|k_1|^\nu}{c_\nu} E_\nu(ik_1a) \quad (3.235)$$

By adding (3.232) and (3.233) we find

$$2C \frac{|k_2|^\nu}{c_\nu} E_\nu(ik_2a) = E \frac{|k_1|^\nu}{c_\nu} \left(1 + \frac{k_1}{k_2} \right) E_\nu(ik_1a) \quad (3.236)$$

By subtracting (3.232) and (3.233) we find

$$2D \frac{|k_2|^\nu}{c_\nu} E_\nu(-ik_2a) = E \frac{|k_1|^\nu}{c_\nu} \left(1 - \frac{k_1}{k_2} \right) E_\nu(ik_1a) \quad (3.237)$$

Solving the system of equations from the second interface allows us to express C and D in terms of the transmitted amplitude E :

$$C = \frac{E}{2} \left| \frac{k_1}{k_2} \right|^\nu \left(1 + \frac{k_1}{k_2} \right) E_\nu(ik_1a) E_\nu^{-1}(ik_2a) \quad (3.238)$$

then

$$D = \frac{E}{2} \left| \frac{k_1}{k_2} \right|^\nu \left(1 - \frac{k_1}{k_2} \right) E_\nu(ik_1a) E_\nu^{-1}(-ik_2a) \quad (3.239)$$

By adding A and B, we obtain the expression

$$2A \frac{|k_1|^\nu}{c_\nu} E_\nu(-ik_1a) = C \frac{|k_2|^\nu}{c_\nu} \left(1 + \frac{k_2}{k_1} \right) E_\nu(-ik_2a) + D \frac{|k_2|^\nu}{c_\nu} \left(1 - \frac{k_2}{k_1} \right) E_\nu(ik_2a) \quad (3.240)$$

$$C \frac{|k_2|^\nu}{c_\nu} \left(1 + \frac{k_2}{k_1} \right) = \frac{E}{2} \left| \frac{k_1}{k_2} \right|^\nu \left(1 + \frac{k_1}{k_2} \right) E_\nu(ik_1a) E_\nu^{-1}(ik_2a) \frac{|k_2|^\nu}{c_\nu} \left(1 + \frac{k_2}{k_1} \right) \quad (3.241)$$

$$C \frac{|k_2|^\nu}{c_\nu} \left(1 + \frac{k_2}{k_1}\right) = \frac{E}{2c_\nu} |k_1|^\nu \left(1 + \frac{k_1}{k_2}\right) \left(1 + \frac{k_2}{k_1}\right) E_\nu(ik_1a) E_\nu^{-1}(ik_2a) \quad (3.242)$$

$$D \frac{|k_2|^\nu}{c_\nu} \left(1 - \frac{k_2}{k_1}\right) = \frac{E}{2} \left| \frac{k_1}{k_2} \right|^\nu \left(1 - \frac{k_1}{k_2}\right) E_\nu(ik_1a) E_\nu^{-1}(-ik_2a) \frac{|k_2|^\nu}{c_\nu} \left(1 - \frac{k_2}{k_1}\right) \quad (3.243)$$

$$D \frac{|k_2|^\nu}{c_\nu} \left(1 - \frac{k_2}{k_1}\right) = \frac{E}{2c_\nu} |k_1|^\nu \left(1 - \frac{k_1}{k_2}\right) \left(1 - \frac{k_2}{k_1}\right) E_\nu(ik_1a) E_\nu^{-1}(-ik_2a) \quad (3.244)$$

$$\begin{aligned} 2A \frac{|k_1|^\nu}{c_\nu} E_\nu(-ik_1a) &= \frac{E}{2c_\nu} |k_1|^\nu \left(1 + \frac{k_1}{k_2}\right) \left(1 + \frac{k_2}{k_1}\right) E_\nu(ik_1a) E_\nu^{-1}(ik_2a) E_\nu(-ik_2a) \\ &\quad + \frac{E}{2c_\nu} |k_1|^\nu \left(1 - \frac{k_1}{k_2}\right) \left(1 - \frac{k_2}{k_1}\right) E_\nu(ik_1a) E_\nu^{-1}(-ik_2a) E_\nu(ik_2a) \end{aligned} \quad (3.245)$$

By substituting C and D back into the first boundary condition, the relationship between the incident amplitude A and the transmitted amplitude E is derived as:

$$\frac{A}{E} = \frac{1}{4} \left[\left(1 + \frac{k_1}{k_2}\right) \left(1 + \frac{k_2}{k_1}\right) \frac{E_\nu(ik_1a) E_\nu(-ik_2a)}{E_\nu(-ik_1a) E_\nu(ik_2a)} + \left(1 - \frac{k_1}{k_2}\right) \left(1 - \frac{k_2}{k_1}\right) \frac{E_\nu(ik_1a) E_\nu(ik_2a)}{E_\nu(-ik_1a) E_\nu(-ik_2a)} \right] \quad (3.246)$$

To find the transmission coefficient, we calculate the squared magnitude $|A/E|^2$:

$$\left| \frac{A}{E} \right|^2 = \frac{E_\nu(ik_1a) E_\nu^{-1}(-ik_1a)}{4k_1k_2} \left[(k_1 + k_2)^2 \frac{E_\nu(-ik_2a)}{E_\nu(ik_2a)} + (k_1 - k_2)^2 \frac{E_\nu(ik_2a)}{E_\nu(-ik_2a)} \right] \quad (3.247)$$

The transmission coefficient T is defined as the inverse of the above ratio:

$$T = \left| \frac{E}{A} \right|^2 = \frac{1}{|A/E|^2} \quad (3.248)$$

The transmission coefficient T is:

$$T = \frac{4k_1k_2 E_\nu(-ik_1a) E_\nu^{-1}(ik_1a)}{(k_1 + k_2)^2 \frac{E_\nu(-ik_2a)}{E_\nu(ik_2a)} + (k_1 - k_2)^2 \frac{E_\nu(ik_2a)}{E_\nu(-ik_2a)}} \quad (3.249)$$

The plot illustrates the transmission coefficient $T(E)$ for different values of the Dunkl parameter $\nu = 0, 0.3, 0.6$. For $\nu = 0$, the system reproduces the standard quantum mechanical result, characterized by regular oscillations and well-defined resonance peaks corresponding to constructive interference inside the potential barrier.

As ν increases, the transmission spectrum undergoes noticeable modifications. In particular, the positions of the resonance peaks are shifted, and the oscillatory pattern becomes

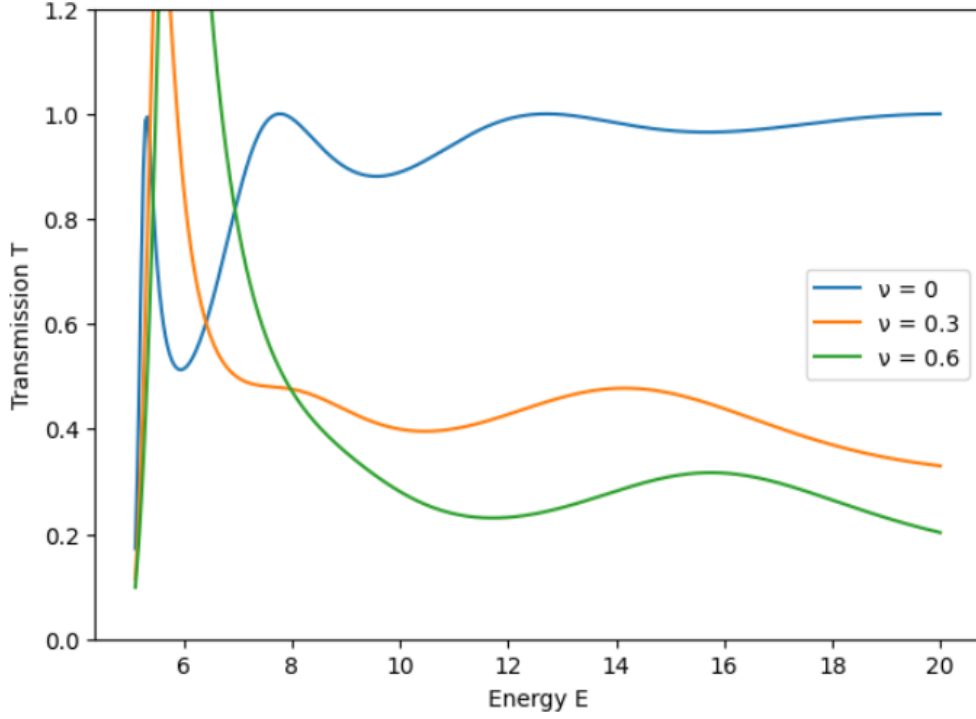


Figure 3.4: Effect of the Dunkl Parameter ν on the Transmission Coefficient $T(E)$

Table 3.1: Resonance energies ($T \approx 1$) for different values of the Dunkl parameter ν .

$\nu = 0$	$\nu = 0.3$	$\nu = 0.6$
5.3057	5.5579	5.8830
7.7741	14.1575	15.7567
12.7110	NaN	NaN
20.1096	NaN	NaN
25.0000	NaN	NaN

slightly distorted. This indicates that the Dunkl deformation alters the effective wave propagation inside the barrier, modifying the interference conditions.

Moreover, the amplitude and sharpness of the transmission peaks are affected, reflecting changes in the tunneling probability. Physically, this behavior can be attributed to the presence of the reflection operator in the Dunkl formalism, which introduces additional structure in the wave function.

Overall, the results demonstrate that increasing the Dunkl parameter ν leads to a deformation of the transmission spectrum, while preserving its oscillatory nature and resonance behavior.

To find the reflection coefficient R using the same methodology as T , we first derive the ratio B/E . From the boundary conditions at $x = a$:

The table presents the resonance energies ($T \approx 1$) for different values of the Dunkl

parameter $\nu = 0, 0.3, 0.6$.

For $\nu = 0$, several resonance energies are observed, indicating the standard quantum mechanical behavior with multiple transmission peaks. As the Dunkl parameter increases to $\nu = 0.3$ and $\nu = 0.6$, the resonance energies shift toward higher values, showing that the deformation modifies the interference condition inside the potential barrier.

Moreover, the number of detectable resonances decreases as ν increases, as seen by the absence of higher-energy values in the table. This suggests that the Dunkl deformation suppresses higher-order resonances or shifts them.

Overall, these results confirm that increasing ν leads to a deformation of the transmission spectrum, characterized by a shift of resonance energies and a reduction in the number of observable resonance peaks.

$$2B \frac{|k_1|^\nu}{c_\nu} E_\nu(-ik_1a) = C \frac{|k_2|^\nu}{c_\nu} \left(1 - \frac{k_2}{k_1}\right) E_\nu(ik_2a) + D \frac{|k_2|^\nu}{c_\nu} \left(1 + \frac{k_2}{k_1}\right) E_\nu(-ik_2a) \quad (3.250)$$

Substituting the expressions for C and D in terms of E :

$$\begin{aligned} \frac{B}{E} = \frac{1}{4} \frac{E_\nu(ik_1a)}{E_\nu(-ik_1a)} & \left[\left(1 + \frac{k_1}{k_2}\right) \left(1 - \frac{k_2}{k_1}\right) \frac{E_\nu(-ik_2a)}{E_\nu(ik_2a)} \right. \\ & \left. + \left(1 - \frac{k_1}{k_2}\right) \left(1 + \frac{k_2}{k_1}\right) \frac{E_\nu(ik_2a)}{E_\nu(-ik_2a)} \right] \end{aligned} \quad (3.251)$$

Simplifying the coefficients:

$$\frac{B}{E} = \frac{1}{4} \left(1 - \frac{k_1^2}{k_2^2}\right) \frac{E_\nu(ik_1a)}{E_\nu(-ik_1a)} \left[\frac{E_\nu(-ik_2a)}{E_\nu(ik_2a)} - \frac{E_\nu(ik_2a)}{E_\nu(-ik_2a)} \right] \quad (3.252)$$

Now, the reflection ratio B/A is obtained by dividing B/E by A/E :

$$\frac{B}{A} = \frac{(k_2^2 - k_1^2) \left[\frac{E_\nu(-ik_2a)}{E_\nu(ik_2a)} - \frac{E_\nu(ik_2a)}{E_\nu(-ik_2a)} \right]}{(k_1 + k_2)^2 \frac{E_\nu(-ik_2a)}{E_\nu(ik_2a)} + (k_1 - k_2)^2 \frac{E_\nu(ik_2a)}{E_\nu(-ik_2a)}} \quad (3.253)$$

Finally, the reflection coefficient $R = |B/A|^2$ is given by:

$$R = \frac{(k_2^2 - k_1^2)^2 \left[\frac{E_\nu(-ik_2a)}{E_\nu(ik_2a)} - \frac{E_\nu(ik_2a)}{E_\nu(-ik_2a)} \right]^2}{\left| (k_1 + k_2)^2 \frac{E_\nu(-ik_2a)}{E_\nu(ik_2a)} + (k_1 - k_2)^2 \frac{E_\nu(ik_2a)}{E_\nu(-ik_2a)} \right|^2} \quad (3.254)$$

The case ($E < V_0$): In this section, we analyze the physical situation where the energy of the incident particle E is lower than the potential barrier V_0 . Classically, the particle is strictly forbidden from entering Region II. However, quantum mechanics allows a non-zero probability for the particle to penetrate the barrier via the "Quantum Tunneling" effect.

Based on the potential profile, the wave function $\Psi(x)$ is defined in three distinct regions using the specialized functions E_ν :

- **Region I** ($x < -a$): Represents the incident and reflected wave.

$$\Psi_I(x) = A \frac{|k_1|^\nu}{c_\nu} E_\nu(ik_1x) + B \frac{|k_1|^\nu}{c_\nu} E_\nu(-ik_1x) \quad (3.255)$$

- **Region II** ($-a < x < a$): Represents the wave inside the barrier. Here, $ik_2 \rightarrow \rho$, where $\rho = \sqrt{2m(V_0 - E)}/\hbar$.

$$\Psi_{II}(x) = C \frac{|\rho|^\nu}{c_\nu} E_\nu(\rho x) + D \frac{|\rho|^\nu}{c_\nu} E_\nu(-\rho x) \quad (3.256)$$

- **Region III** ($x > a$): Represents the transmitted wave.

$$\Psi_{III}(x) = E \frac{|k_1|^\nu}{c_\nu} E_\nu(ik_1x) \quad (3.257)$$

To maintain the physical validity of the wave function, we apply continuity conditions for Ψ and its first derivative Ψ' at the boundaries. Continuity at the Second Interface ($x = a$): Solving the system of equations allows us to express C and D in terms of the transmitted amplitude E :

$$C = \frac{E}{2} \left| \frac{k_1}{\rho} \right|^\nu \left(1 + \frac{ik_1}{\rho} \right) \frac{E_\nu(ik_1a)}{E_\nu(\rho a)} \quad (3.258)$$

$$D = \frac{E}{2} \left| \frac{k_1}{\rho} \right|^\nu \left(1 - \frac{ik_1}{\rho} \right) \frac{E_\nu(ik_1a)}{E_\nu(-\rho a)} \quad (3.259)$$

After algebraic simplification, the transmission coefficient are derived as:

$$T = \left| \frac{E}{A} \right|^2 = \frac{4k_1^2 \rho^2 E_\nu(ik_1a) E_\nu(-ik_1a)}{(k_1^2 - \rho^2)^2 \left[\frac{E_\nu(\rho a)}{E_\nu(-\rho a)} + \frac{E_\nu(-\rho a)}{E_\nu(\rho a)} \right] + (k_1^2 + \rho^2)^2} \quad (3.260)$$

To find the reflection coefficient R using the same methodology as T , we first derive the ratio B/E . From the continuity conditions at the first interface $x = -a$:

$$2B \frac{|k_1|^\nu}{c_\nu} E_\nu(-ik_1a) = C \frac{|\rho|^\nu}{c_\nu} \left(1 - \frac{\rho}{ik_1} \right) E_\nu(-\rho a) + D \frac{|\rho|^\nu}{c_\nu} \left(1 + \frac{\rho}{ik_1} \right) E_\nu(\rho a) \quad (3.261)$$

Substituting the expressions for C and D in terms of E (as derived from the second interface $x = a$):

$$\frac{B}{E} = \frac{1}{4} \frac{E_\nu(ik_1a)}{E_\nu(-ik_1a)} \left[\left(1 + \frac{ik_1}{\rho} \right) \left(1 - \frac{\rho}{ik_1} \right) \frac{E_\nu(-\rho a)}{E_\nu(\rho a)} + \left(1 - \frac{ik_1}{\rho} \right) \left(1 + \frac{\rho}{ik_1} \right) \frac{E_\nu(\rho a)}{E_\nu(-\rho a)} \right] \quad (3.262)$$

Using the identity $(1 \pm \frac{ik_1}{\rho})(1 \mp \frac{\rho}{ik_1}) = (1 + \frac{k_1^2}{\rho^2})$, the expression simplifies to:

$$\frac{B}{E} = \frac{1}{4} \left(1 + \frac{k_1^2}{\rho^2} \right) \frac{E_\nu(ik_1a)}{E_\nu(-ik_1a)} \left[\frac{E_\nu(-\rho a)}{E_\nu(\rho a)} - \frac{E_\nu(\rho a)}{E_\nu(-\rho a)} \right] \quad (3.263)$$

The reflection ratio B/A is then obtained by dividing B/E by the previously established A/E :

$$\frac{B}{A} = \frac{(k_1^2 + \rho^2) \left[\frac{E_\nu(-\rho a)}{E_\nu(\rho a)} - \frac{E_\nu(\rho a)}{E_\nu(-\rho a)} \right]}{(k_1^2 - \rho^2) \left[\frac{E_\nu(\rho a)}{E_\nu(-\rho a)} + \frac{E_\nu(-\rho a)}{E_\nu(\rho a)} \right] + 2ik_1\rho} \quad (3.264)$$

Finally, the reflection coefficient $R = |B/A|^2$ is expressed as:

$$R = \frac{(k_1^2 + \rho^2)^2 \left[\frac{E_\nu(\rho a)}{E_\nu(-\rho a)} - \frac{E_\nu(-\rho a)}{E_\nu(\rho a)} \right]^2}{(k_1^2 - \rho^2)^2 \left[\frac{E_\nu(\rho a)}{E_\nu(-\rho a)} + \frac{E_\nu(-\rho a)}{E_\nu(\rho a)} \right]^2 + 4k_1^2\rho^2} \quad (3.265)$$

This derivation shows that $R + T = 1$, which confirms the conservation of the probability current in the ν -deformed space.

3.6.2 Phase Shift Analysis

A- The case $E > V_0$:

1- Outside the barrier ($|x| > a$) : For an even function ψ_e , one has $R\psi_e = \psi_e$, hence $(1-R)\psi_e = 0$, and $D_\mu\psi_e = \psi'_e$. However, when squaring the Dunkl derivative, one must apply D_μ to ψ'_e , not to ψ_e . Since the derivative of an even function is odd, we have $R(\psi'_e) = -\psi'_e$. Therefore, $(1-R)\psi'_e = 2\psi'_e$. It follows that

$$D_\mu^2\psi_e = D_\mu(\psi'_e) = \psi''_e + \frac{\mu}{x}(1-R)\psi'_e = \psi''_e + \frac{2\mu}{x}\psi'_e. \quad (3.266)$$

Substituting into the Schrödinger equation yields

$$\psi''_e(x) + \frac{2\mu}{x}\psi'_e(x) + \frac{2m}{\hbar^2}(E - V(x))\psi_e(x) = 0. \quad (3.267)$$

One has $V(x) = 0$, so Schrodinger equation becomes

$$\psi''_e(x) + \frac{2\mu}{x}\psi'_e(x) + k^2\psi_e(x) = 0. \quad (3.268)$$

where $k = \frac{\sqrt{2mE}}{\hbar}$. We introduce the transformation $\psi_e(x) = |x|^\alpha u_e(x)$. Then

$$\psi'_e = |x|^\alpha \left(\frac{\alpha}{x}u_e + u'_e \right), \quad \text{and} \quad \psi''_e = |x|^\alpha \left[\frac{\alpha(\alpha-1)}{x^2}u_e + \frac{2\alpha}{x}u'_e + u''_e \right]. \quad (3.269)$$

Substituting into Eq. (??) gives

$$|x|^\alpha \left[\frac{\alpha(\alpha-1)}{x^2}u_e + \frac{2\alpha}{x}u'_e + u''_e \right] + \frac{2\mu}{x}|x|^\alpha \left(\frac{\alpha}{x}u_e + u'_e \right) + k^2|x|^\alpha u_e(x) = 0.$$

After collecting powers of x and dividing by $|x|^\alpha$, one obtains

$$u''_e + \frac{2(\alpha+\mu)}{x}u'_e + \left[k^2 + \frac{\alpha(\alpha-1) + 2\mu\alpha}{x^2} \right] u_e = 0. \quad (3.270)$$

To obtain the standard Bessel form, we impose $\alpha = \frac{1}{2} - \mu$. Hence

$$u_e'' + \frac{1}{x}u_e' + \left[k^2 - \frac{\nu_e^2}{x^2}\right]u_e = 0. \quad (3.271)$$

This is Bessel's equation of order $\nu_e = \mu - \frac{1}{2}$. Therefore the general even solution is

$$\psi_e^{\text{out}}(x) = x^{\frac{1}{2}-\mu} \left[A_e J_{\mu-\frac{1}{2}}(\lambda x) + B_e Y_{\mu-\frac{1}{2}}(\lambda x) \right]. \quad (3.272)$$

2- Inside the barrier ($|x| < a$) : One has $V(x) = V_0$, so Schrodinger equation becomes

$$\psi_e''(x) + \frac{2\mu}{x}\psi_e'(x) + q^2\psi_e(x) = 0. \quad (3.273)$$

where $q = \frac{\sqrt{2m(E-V_0)}}{\hbar}$. Its solution is

$$\psi_e^{\text{in}}(x) = x^{\frac{1}{2}-\mu} \left[C_e J_{\mu-\frac{1}{2}}(qx) + D_e Y_{\mu-\frac{1}{2}}(qx) \right]. \quad (3.274)$$

Since Y_ν is singular at the origin, regularity at $x = 0$ requires $D_e = 0$:

$$\psi_e^{\text{in}}(x) = C_e x^{\frac{1}{2}-\mu} J_{\mu-\frac{1}{2}}(qx). \quad (3.275)$$

For $E < V_0$, one uses modified Bessel functions:

$$\psi_e^{\text{in}}(x) = C_e x^{\frac{1}{2}-\mu} I_{\mu-\frac{1}{2}}(\kappa x). \quad (3.276)$$

with $\kappa = \frac{\sqrt{2m(V_0-E)}}{\hbar}$

It is sufficient to solve the problem on the half-line $x > 0$ and extend the solution to the full line by parity.

1- For even channel:

1-A Exterior solution: Using the large-argument asymptotics

$$J_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right), \quad (3.277)$$

$$Y_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right), \quad (3.278)$$

we obtain

$$\cos \delta J_\nu(z) - \sin \delta Y_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4} + \delta\right). \quad (3.279)$$

Therefore,

$$\psi_e^{\text{out}}(x) \sim x^{-\mu} \cos\left(kx - \frac{(\mu-\frac{1}{2})\pi}{2} - \frac{\pi}{4} + \delta_e\right), \quad x > a. \quad (3.280)$$

Hence δ_e is the even scattering phase shift.

1-B Interior solution: Inside the barrier, the general solution is given by

$$\psi_e^{\text{in}}(x) = A x^{\frac{1}{2}-\mu} J_{\nu_e}(qx), \quad \nu_e = \mu - \frac{1}{2}. \quad (3.281)$$

We derive the phase-shift formula associated with the even-parity scattering channel by imposing the matching conditions between the interior and exterior solutions at the boundary point $x = a$. From above, we impose

$$\psi_e^{\text{out}}(a) = \psi_e^{\text{in}}(a), \quad (3.282)$$

where

$$\psi_e^{\text{in}}(x) = A x^{\frac{1}{2}-\mu} J_{\nu_e}(qx), \quad \nu_e = \mu - \frac{1}{2}, \quad x \geq 0 \quad (3.283)$$

Outside the interaction region, the asymptotic form of the even solution is written as

$$\psi_e^{\text{out}}(x) = B \sqrt{\frac{2}{\pi k}} x^{-\mu} \cos \left(kx - \frac{(\mu - \frac{1}{2})\pi}{2} - \frac{\pi}{4} + \delta_e \right), \quad x \geq 0. \quad (3.284)$$

For notational simplicity, we introduce $\alpha = \frac{1}{2} - \mu$, so that the previous expressions may be rewritten as

$$\psi_e^{\text{in}}(x) = A x^{\frac{1}{2}-\mu} J_{\nu_e}(qx), \quad \psi_e^{\text{out}}(x) = B \sqrt{\frac{2}{\pi k}} x^{-\mu} \cos \Phi_e(x), \quad (3.285)$$

with the phase function

$$\Phi_e(x) = kx - \frac{(\mu - \frac{1}{2})\pi}{2} - \frac{\pi}{4} + \delta_e. \quad (3.286)$$

At the matching point $x = a$, we define

$$\Phi_e = ka - \frac{(\mu - \frac{1}{2})\pi}{2} - \frac{\pi}{4} + \delta_e. \quad (3.287)$$

The first matching condition requires continuity of the wave function at the boundary,

$$\psi_e^{\text{in}}(a) = \psi_e^{\text{out}}(a). \quad (3.288)$$

Substituting the explicit forms of the two solutions yields

$$A a^{\frac{1}{2}-\mu} J_{\nu_e}(qa) = B \sqrt{\frac{2}{\pi k}} a^{-\mu} \cos \Phi_e. \quad (3.289)$$

After canceling the common factor a^α , we obtain

$$A a^{\frac{1}{2}} J_{\nu_e}(qa) = B \sqrt{\frac{2}{\pi k}} \cos \Phi_e. \quad (3.290)$$

This relation constitutes the first matching equation.

We now differentiate the interior wave function,

$$\psi_e^{\text{in}}(x) = A x^{\frac{1}{2}-\mu} J_{\nu_e}(qx), \quad (3.291)$$

by applying the product rule:

$$\psi_e^{\text{in}'}(x) = A \left[x^{\frac{1}{2}-\mu} J_{\nu_e}(qx) \right]'. \quad (3.292)$$

Hence,

$$\psi_e^{\text{in}'}(x) = A \left[\left(\frac{1}{2} - \mu \right) x^{-\frac{1}{2}-\mu} J_{\nu_e}(qx) + x^{\frac{1}{2}-\mu} J'_{\nu_e}(qx) \right]. \quad (3.293)$$

Using the chain rule,

$$\frac{dJ_{\nu_e}(qx)}{dx} = q \frac{dJ_{\nu_e}(z)}{dz}, \quad \text{with } z = qx \quad (3.294)$$

we arrive at

$$\psi_e^{\text{in}'}(x) = A \left[\left(\frac{1}{2} - \mu \right) x^{-\frac{1}{2}-\mu} J_{\nu_e}(qx) + qx^{\frac{1}{2}-\mu} \frac{dJ_{\nu_e}(z)}{dz} \right]. \quad (3.295)$$

Evaluating this expression at $x = a$ gives

$$\psi_e^{\text{in}'}(a) = A a^{-\frac{1}{2}-\mu} \left[\left(\frac{1}{2} - \mu \right) J_{\nu_e}(qa) + qa \frac{dJ_{\nu_e}(z)}{dz} \Big|_{z=qa} \right]. \quad (3.296)$$

We next differentiate the exterior wave function,

$$\psi_e^{\text{out}}(x) = B \sqrt{\frac{2}{\pi k}} x^{-\mu} \cos \Phi_e(x), \quad (3.297)$$

where

$$\Phi_e(x) = kx - \frac{(\mu - \frac{1}{2})\pi}{2} - \frac{\pi}{4} + \delta_e. \quad (3.298)$$

Applying the product rule once again, we obtain

$$\psi_e^{\text{out}'}(x) = B \sqrt{\frac{2}{\pi k}} \left[-\mu x^{-\mu-1} \cos \Phi_e(x) + x^{-\mu} \frac{d \cos \Phi_e(x)}{dx} \right]. \quad (3.299)$$

Since

$$\frac{d \cos \Phi_e(x)}{dx} = -\sin \Phi_e(x) \frac{d\Phi_e(x)}{dx}, \quad (3.300)$$

and because $\Phi_e'(x) = k$, it follows that

$$\cos' \Phi_e(x) = -k \sin \Phi_e(x). \quad (3.301)$$

Therefore,

$$\psi_e^{\text{out}'}(x) = B \sqrt{\frac{2}{\pi k}} \left[-\mu x^{-\mu-1} \cos \Phi_e(x) - kx^{-\mu} \sin \Phi_e(x) \right]. \quad (3.302)$$

At $x = a$, one finds

$$\psi_e^{\text{out}'}(a) = B \sqrt{\frac{2}{\pi k}} a^{-\mu} \left[-\frac{\mu}{a} \cos \Phi_e(a) - k \sin \Phi_e(a) \right]. \quad (3.303)$$

The second matching condition imposes continuity of the derivative at $x = a$:

$$\psi_e^{\text{in}'}(a) = \psi_e^{\text{out}'}(a). \quad (3.304)$$

Substituting Eqs. (4.70) and (4.77) into this condition, we obtain

$$\begin{aligned} & A a^{-\frac{1}{2}-\mu} \left[\left(\frac{1}{2} - \mu \right) J_{\nu_e}(qa) + qa \frac{dJ_{\nu_e}(z)}{dz} \Big|_{z=qa} \right] \\ &= B \sqrt{\frac{2}{\pi k}} a^{-\mu} \left[-\frac{\mu}{a} \cos \Phi_e(a) - k \sin \Phi_e(a) \right]. \end{aligned} \quad (3.305)$$

After canceling the common factor $a^{\alpha-1}$, this becomes

$$\begin{aligned} & A \left[\frac{(\frac{1}{2}-\mu)}{\sqrt{a}} J_{\nu_e}(qa) + q\sqrt{a} \frac{dJ_{\nu_e}(z)}{dz} \Big|_{z=qa} \right] \\ &= B \sqrt{\frac{2}{\pi k}} \left[-\frac{\mu}{a} \cos \Phi_e(a) - k \sin \Phi_e(a) \right]. \end{aligned} \quad (3.306)$$

We now combine Eq. (4.80) with the first matching condition, Eq. (4.64),

$$A \sqrt{a} J_{\nu_e}(qa) = B \sqrt{\frac{2}{\pi k}} \cos \Phi_e, \quad (3.307)$$

in order to eliminate the ratio of the constants A and B .

Dividing Eq. (4.80) by Eq. (4.64), we obtain

$$\frac{\left[\frac{(\frac{1}{2}-\mu)}{\sqrt{a}} J_{\nu_e}(qa) + q\sqrt{a} \frac{dJ_{\nu_e}(z)}{dz} \Big|_{z=qa} \right]}{\sqrt{a} J_{\nu_e}(qa)} = \frac{B \left[-\frac{\mu}{a} \cos \Phi_e - k \sin \Phi_e \right]}{B \cos \Phi_e}. \quad (3.308)$$

This immediately gives

$$\frac{(\frac{1}{2}-\mu)}{a} + q \frac{\frac{dJ_{\nu_e}(z)}{dz} \Big|_{z=qa}}{J_{\nu_e}(qa)} = -\frac{\mu}{a} - k \tan \Phi_e. \quad (3.309)$$

Subtracting $\frac{\mu}{a}$ from both sides leads to

$$\frac{1}{2a} + q \frac{\frac{dJ_{\nu_e}(z)}{dz} \Big|_{z=qa}}{J_{\nu_e}(qa)} = -k \tan \Phi_e. \quad (3.310)$$

then

$$\frac{1}{2ak} + \frac{q}{k} \frac{\frac{dJ_{\nu_e}(z)}{dz} \Big|_{z=qa}}{J_{\nu_e}(qa)} = -\tan \Phi_e. \quad (3.311)$$

Finally, by using the definition of Φ_e given in Eq. (4.61), we arrive at the phase-shift formula

$$\tan \left(ka - \frac{(\mu - \frac{1}{2})\pi}{2} - \frac{\pi}{4} + \delta_e \right) = -\frac{1}{2ak} - \frac{q}{k} \frac{\frac{dJ_{\nu_e}(z)}{dz} \Big|_{z=qa}}{J_{\nu_e}(qa)}. \quad (3.312)$$

This is the desired expression for the even-parity phase shift.

For an odd function ψ_o , one has $R\psi_o = -\psi_o$, hence $(1 - R)\psi_o = 2\psi_o$. Therefore $D_\mu\psi_o = \psi'_o + \frac{2\mu}{x}\psi_o$. Now note that ψ'_o is even, and ψ_o/x is also even. Hence $D_\mu\psi_o$ is even. Therefore, when acting once more with D_μ , the reflection part does not contribute. Expanding the derivative gives

$$D_\mu^2\psi_o = \psi''_o + \frac{2\mu}{x}\psi'_o - \frac{2\mu}{x^2}\psi_o. \quad (3.313)$$

The odd-sector Schrödinger equation becomes

$$\psi''_o + \frac{2\mu}{x}\psi'_o + \left[\frac{2m}{\hbar^2} (E - V(x)) - \frac{2\mu}{x^2} \right] \psi_o = 0. \quad (3.314)$$

Because the potential is even, it is sufficient to solve the problem on the half-line $x > 0$ and then extend the solution by parity. On the half-line, the two relevant regions are:

- Region I: $0 < x < a$ (inside the barrier),
- Region II: $x > a$ (outside the barrier).

The odd equation reads

$$\psi''_o + \frac{2\mu}{x}\psi'_o + \left(\lambda^2 - \frac{2\mu}{x^2} \right) \psi_o = 0. \quad (3.315)$$

We again set

$$\psi_o(x) = \text{sgn}(x) |x|^\alpha u_o(x), \quad \alpha = \frac{1}{2} - \mu. \quad (3.316)$$

Carrying out the same substitution, one finds

$$u''_o + \frac{1}{x}u'_o + \left[\lambda^2 - \frac{\nu_o^2}{x^2} \right] u_o = 0. \quad (3.317)$$

Thus the Bessel order is $\nu_o = \mu + \frac{1}{2}$. The general odd solution is then

$$\psi_o(x) = \text{sgn}(x) |x|^{\frac{1}{2}-\mu} \left[A_o J_{\mu+\frac{1}{2}}(\lambda x) + B_o Y_{\mu+\frac{1}{2}}(\lambda x) \right]. \quad (3.318)$$

Interior odd solution:

For $E > V_0$,

$$\psi_o^{\text{in}}(x) = C_o x^{\frac{1}{2}-\mu} J_{\mu+\frac{1}{2}}(qx), \quad x \geq 0. \quad (3.319)$$

For $E < V_0$,

$$\psi_o^{\text{in}}(x) = C_o x^{\frac{1}{2}-\mu} I_{\mu+\frac{1}{2}}(\kappa x), \quad x \geq 0. \quad (3.320)$$

These choices ensure regularity at the origin.

Exterior odd solution

Outside the barrier:

$$\psi_o^{\text{out}}(x) = x^{\frac{1}{2}-\mu} \left[A_o J_{\mu+\frac{1}{2}}(kx) + B_o Y_{\mu+\frac{1}{2}}(kx) \right], \quad x > a. \quad (3.321)$$

The extension to negative x is imposed by odd parity $\psi_o(-x) = -\psi_o(x)$. We remark that in near the origin $x = 0$, the Bessel functions behave as

$$J_\nu(z) \sim z^\nu, \quad Y_\nu(z) \sim z^{-\nu} \quad (\nu > 0). \quad (3.322)$$

Using the prefactor $x^{\frac{1}{2}-\mu}$, one finds

$$x^{\frac{1}{2}-\mu} J_{\mu+\frac{1}{2}}(qx) \sim x^{\frac{1}{2}-\mu} x^{\mu-\frac{1}{2}} = x, \quad (3.323)$$

which is regular. On the other hand,

$$x^{\frac{1}{2}-\mu} Y_{\mu+\frac{1}{2}}(qx) \sim x^{\frac{1}{2}-\mu} x^{-(\mu+\frac{1}{2})} = x^{-2\mu}, \quad (3.324)$$

which is generally singular or non-regular, depending on μ . This justifies keeping only the J_ν (or I_ν) contribution inside the barrier. Since

$$D_\mu \psi_o = \psi_o' + \frac{2\mu}{x} \psi_o, \quad (3.325)$$

the odd-channel matching conditions are

$$\psi_o^{\text{out}}(a) = \psi_o^{\text{in}}(a), \quad (3.326)$$

and

$$\psi_o^{\text{out}'}(a) + \frac{2\mu}{a} \psi_o^{\text{out}}(a) = \psi_o^{\text{in}'}(a) + \frac{2\mu}{a} \psi_o^{\text{in}}(a). \quad (3.327)$$

Instead of keeping the exterior solutions in the form $AJ_\nu + BY_\nu$, it is convenient to rewrite them in terms of phase shifts. Using the large-argument asymptotics

$$J_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \cos \left(z - \frac{\nu\pi}{2} - \frac{\pi}{4} \right), \quad (3.328)$$

$$Y_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \sin \left(z - \frac{\nu\pi}{2} - \frac{\pi}{4} \right), \quad (3.329)$$

we obtain

$$\cos \delta J_\nu(z) - \sin \delta Y_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \cos \left(z - \frac{\nu\pi}{2} - \frac{\pi}{4} + \delta \right). \quad (3.330)$$

Therefore,

$$\psi_o^{\text{out}}(x) \sim \sqrt{\frac{2}{\pi k}} x^{-\mu} \cos \left(kx - \frac{(\mu + \frac{1}{2})\pi}{2} - \frac{\pi}{4} + \delta_o \right). \quad (3.331)$$

Hence δ_o is the odd scattering phase shift.

An entirely analogous calculation for the odd-parity sector leads to

$$\tan \left(ka - \frac{(\mu + \frac{1}{2}) \pi}{2} - \frac{\pi}{4} + \delta_o \right) = -\frac{1}{2ak} - \frac{q}{k} \frac{\left. \frac{dJ_{\nu_o}(z)}{dz} \right|_{z=qa}}{J_{\nu_o}(qa)}, \quad \nu_o = \mu + \frac{1}{2}. \quad (3.332)$$

Once the even- and odd-parity phase shifts are known, the transmission and reflection probabilities are expressed in the standard form

$$T = \cos^2(\delta_e - \delta_o), \quad R = \sin^2(\delta_e - \delta_o). \quad (3.333)$$

To make the dependence on the logarithmic derivatives more explicit, let us define

$$\beta = ka - \frac{(\mu - \frac{1}{2}) \pi}{2} - \frac{\pi}{4} + \delta_e, \quad (3.334)$$

and

$$\rho = ka - \frac{(\mu + \frac{1}{2}) \pi}{2} - \frac{\pi}{4} + \delta_o. \quad (3.335)$$

Then,

$$\tan \beta = -\frac{1}{2ak} - \frac{q}{k} \frac{\left. \frac{dJ_{\nu_e}(z)}{dz} \right|_{z=qa}}{J_{\nu_e}(qa)}, \quad \tan \rho = -\frac{1}{2ak} - \frac{q}{k} \frac{\left. \frac{dJ_{\nu_o}(z)}{dz} \right|_{z=qa}}{J_{\nu_o}(qa)}. \quad (3.336)$$

we set $\bar{\rho} = \rho + \pi/2$, we find

$$\tan \bar{\rho} = -\cot \rho = -\frac{1}{\tan \rho} = \frac{1}{\frac{1}{2ak} + \frac{q}{k} \frac{J'_{\nu_o}(qa)}{J_{\nu_o}(qa)}} \quad (3.337)$$

The difference of the two phase shifts may then be analyzed through the trigonometric identity

$$\tan(\beta - \bar{\rho}) = \frac{\tan \beta - \tan \bar{\rho}}{1 + \tan \beta \tan \bar{\rho}}. \quad (3.338)$$

with

$$\beta - \bar{\rho} = \delta_e - \delta_o \quad (3.339)$$

Substituting the above expressions, one obtains an explicit representation of the phase-difference contribution in terms of the Bessel functions and their derivatives. Consequently, the transmission coefficient takes the compact form

$$T = \frac{1}{1 + \tan^2(\delta_e - \delta_o)}, \quad (3.340)$$

with

$$\tan(\delta_e - \delta_o) = \frac{-\frac{1}{2ak} - \frac{q}{k} \frac{\left. \frac{dJ_{\nu_e}(z)}{dz} \right|_{z=qa}}{J_{\nu_e}(qa)} - \frac{1}{\frac{1}{2ak} + \frac{q}{k} \frac{\left. \frac{dJ_{\nu_o}(z)}{dz} \right|_{z=qa}}{J_{\nu_o}(qa)}}}{1 - \frac{\frac{1}{2ak} + \frac{q}{k} \frac{\left. \frac{dJ_{\nu_e}(z)}{dz} \right|_{z=qa}}{J_{\nu_e}(qa)}}{\frac{1}{2ak} + \frac{q}{k} \frac{\left. \frac{dJ_{\nu_o}(z)}{dz} \right|_{z=qa}}{J_{\nu_o}(qa)}}}. \quad (3.341)$$

then

$$T = \frac{1}{1 + \left[\frac{1 + \left[\frac{\frac{dJ_{\nu_E}(z)}{dz} \Big|_{z=qa}}{J_{\nu_E}(qa)} \right] \left[\frac{\frac{dJ_{\nu_O}(z)}{dz} \Big|_{z=qa}}{J_{\nu_O}(qa)} \right]}{\left[\frac{\frac{dJ_{\nu_E}(z)}{dz} \Big|_{z=qa}}{J_{\nu_E}(qa)} - \frac{\frac{dJ_{\nu_O}(z)}{dz} \Big|_{z=qa}}{J_{\nu_O}(qa)} \right]^2} \right]^2}. \quad (3.342)$$

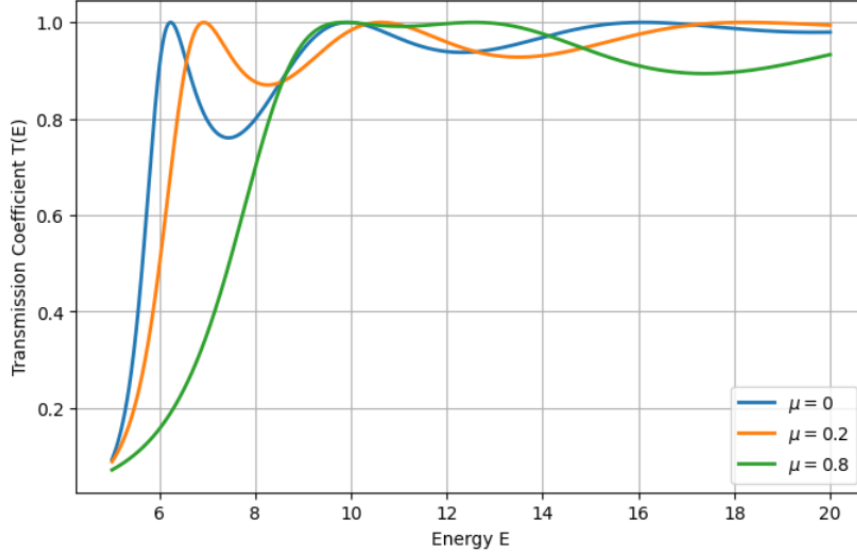


Figure 3.5: Transmission Coefficient $T(E)$ for Different Values of the Dunkl Parameter μ

Table 3.2: Resonance energies and transmission coefficients for different values of the Dunkl parameter μ in the nonrelativistic barrier problem.

μ	Resonance Number	Resonance Energy E_r	$T(E_r)$
0.0	1	6.234592	0.999999
0.0	2	9.933358	1.000000
0.0	3	16.101300	1.000000
0.2	1	6.924361	1.000000
0.2	2	10.628126	1.000000
0.2	3	18.240587	1.000000
0.8	1	9.858383	1.000000
0.8	2	12.567479	1.000000

The transmission coefficient curves presented in Fig. ?? exhibit a clear dependence on the Dunkl parameter μ . For all considered values of μ , the transmission coefficient increases with the incident energy and approaches the unity limit ($T \rightarrow 1$) at high energies, which indicates an almost complete transmission through the barrier.

For $\mu = 0$, the transmission coefficient reaches its first resonance peak at lower energies compared to the other cases. This behavior indicates that the absence of Dunkl deformation facilitates the occurrence of resonant tunneling states. Several sharp resonance peaks are observed, corresponding to quasi-bound states formed inside the barrier region.

When the Dunkl parameter increases to $\mu = 0.2$, the resonance peaks shift toward higher energies. In addition, the oscillatory structure becomes smoother, showing that the deformation modifies the interference conditions inside the barrier. The resonant energies extracted in Table ?? confirm this shift.

For the larger value $\mu = 0.8$, the first resonance appears at significantly higher energy values, and the transmission profile becomes broader and less oscillatory. This indicates that increasing the Dunkl parameter suppresses low-energy resonant tunneling and alters the internal wave dynamics within the barrier.

The obtained results demonstrate that the Dunkl parameter plays an important role in controlling the resonance structure and the tunneling properties of the system. Therefore, the deformation parameter μ acts effectively as a regulator of the transmission process by shifting the resonance energies and modifying the transparency of the barrier. Overall, the phase shift method provides a clear physical interpretation of the transmission process, where the role of μ is to control the interference pattern and consequently the resonance structure of the system. while the reflection coefficient is given by

$$R = 1 - T. \quad (3.343)$$

Therefore, the unitarity condition is automatically satisfied:

$$R + T = 1. \quad (3.344)$$

3.7 The Hypergeometric Equation for the Modified Poschl–Teller Potential

3.7.1 Scattering theory with standard derivative

We start from the one-dimensional Schrodinger equation:

$$u''(x) + \left[k^2 + \frac{\alpha^2 \lambda(\lambda - 1)}{\cosh^2(\alpha x)} \right] u(x) = 0. \quad (3.345)$$

We introduce the new variable:

$$y = \cosh^2(\alpha x). \quad (3.346)$$

Then,

$$\frac{dy}{dx} = 2\alpha \sinh(\alpha x) \cosh(\alpha x). \quad (3.347)$$

Using the identities:

$$\cosh^2(\alpha x) = y, \quad \sinh^2(\alpha x) = y - 1 \quad (3.348)$$

we obtain:

$$\frac{dy}{dx} = 2\alpha \sinh(\alpha x) \cosh(\alpha x) = 2\alpha \sqrt{y(y-1)}, \quad (x > 0). \quad (3.349)$$

Using the chain rule:

$$\frac{du}{dx} = \frac{du}{dy} \frac{dy}{dx} = 2\alpha \sqrt{y(y-1)} u_y. \quad (3.350)$$

We compute:

$$u'' = \frac{d}{dx} \left(2\alpha \sqrt{y(y-1)} u_y \right). \quad (3.351)$$

Applying the product rule:

$$u'' = 2\alpha \left[\frac{d}{dx} (\sqrt{y(y-1)}) u_y + \sqrt{y(y-1)} \frac{du_y}{dx} \right]. \quad (3.352)$$

We evaluate each term:

$$\frac{d}{dx} \sqrt{y(y-1)} = \frac{dy}{dx} \frac{d}{dy} \sqrt{y(y-1)} = 2\alpha \sqrt{y(y-1)} \frac{(2y-1)}{2\sqrt{y(y-1)}} = \alpha(2y-1), \quad (x > 0) \quad (3.353)$$

also

$$\frac{d}{dx} (u_y) = u_{yy} \frac{dy}{dx} = 2\alpha \sqrt{y(y-1)} u_{yy}. \quad (3.354)$$

Thus:

$$u'' = 2\alpha^2(2y-1)u_y + 4\alpha^2 y(y-1)u_{yy}. \quad (3.355)$$

Substituting into the original equation:

$$4\alpha^2 y(y-1)u_{yy} + 2\alpha^2(2y-1)u_y + \left[k^2 + \frac{\alpha^2 \lambda(\lambda-1)}{y} \right] u = 0. \quad (3.356)$$

Dividing by $4\alpha^2$, we obtain:

$$y(y-1)u_{yy} + \frac{2y-1}{2}u_y + \left[\frac{k^2}{4\alpha^2} + \frac{\lambda(\lambda-1)}{4y} \right] u = 0. \quad (3.357)$$

Rewriting:

$$y(1-y)u_{yy} + \left(\frac{1}{2} - y \right) u_y - \left[\frac{k^2}{4\alpha^2} + \frac{\lambda(\lambda-1)}{4y} \right] u = 0. \quad (3.358)$$

For elimination of the singular term, we introduce:

$$u(y) = y^{\lambda/2} v(y). \quad (3.359)$$

Then:

$$u' = \frac{\lambda}{2} y^{\frac{\lambda}{2}-1} v + y^{\lambda/2} v', \quad (3.360)$$

$$u'' = \frac{\lambda}{2} \left(\frac{\lambda}{2} - 1 \right) y^{\frac{\lambda}{2}-2} v + \lambda y^{\frac{\lambda}{2}-1} v' + y^{\lambda/2} v''. \quad (3.361)$$

Substituting into the equation and dividing by $y^{\lambda/2}$, the singular term in $1/y$ cancels, and we obtain:

$$y(1-y)v'' + [(\lambda + \frac{1}{2}) - (\lambda + 1)y] v' - \frac{1}{4} \left(\lambda^2 + \frac{k^2}{\alpha^2} \right) v = 0. \quad (3.362)$$

This is the standard hypergeometric differential equation:

$$y(1-y)v'' + [c - (a+b+1)y]v' - abv = 0, \quad (3.363)$$

with parameters:

$$c = \lambda + \frac{1}{2}, \quad a + b = \lambda, \quad ab = \frac{1}{4} \left(\lambda^2 + \frac{k^2}{\alpha^2} \right). \quad (3.364)$$

In order to study the asymptotic behavior, we introduce the change of variable:

$$z = 1 - y. \quad (3.365)$$

From this transformation, we have:

$$\frac{dv}{dy} = -\frac{dv}{dz}, \quad \frac{d^2v}{dy^2} = \frac{d^2v}{dz^2}. \quad (3.366)$$

Substituting into the differential equation, we obtain:

$$(1-z)z \frac{d^2v}{dz^2} - [c - (a+b+1)(1-z)] \frac{dv}{dz} - abv = 0. \quad (3.367)$$

Expanding the coefficient:

$$c - (a+b+1)(1-z) = c - (a+b+1) + (a+b+1)z, \quad (3.368)$$

the equation becomes:

$$z(1-z) \frac{d^2v}{dz^2} + [(a+b+1-c) - (a+b+1)z] \frac{dv}{dz} - abv = 0. \quad (3.369)$$

This is again a hypergeometric equation in the variable z , with the new parameter:

$$c' = a + b + 1 - c. \quad (3.370)$$

Thus, the equation takes the standard form:

$$z(1-z) \frac{d^2v}{dz^2} + [c' - (a+b+1)z] \frac{dv}{dz} - abv = 0. \quad (3.371)$$

Domain of the variable is defined by

$$0 \leq |x| < \infty. \quad (3.372)$$

After the transformation, we have:

$$y \geq 1 \quad \Rightarrow \quad z = 1 - y \leq 0, \quad (3.373)$$

which gives the domain:

$$-\infty < z \leq 0. \quad (3.374)$$

the general solution is

$$v(z) = A {}_2F_1(a, b; c'; z) + B z^{1-c'} {}_2F_1(a - c' + 1, b - c' + 1; 2 - c'; z) \quad (3.375)$$

The complete solution can be written as:

$$v(x) = A {}_2F_1\left(a, b; \frac{1}{2}; 1 - y\right) + B(1 - y)^{1/2} {}_2F_1\left(a + \frac{1}{2}, b + \frac{1}{2}; \frac{3}{2}; 1 - y\right). \quad (3.376)$$

At $x = 0$ (i.e. $y = 1$), the wave function behaves as:

$$u(0) = A + B(1 - y)^{1/2}. \quad (3.377)$$

Since the potential is symmetric, we construct even and odd solutions.

For even parity:

$$u_e(-x) = u_e(x), \quad (3.378)$$

which implies:

$$u'_e(0) = 0. \quad (3.379)$$

Choosing $B = 0$ and $A = 1$, we obtain:

$$u_e(x) = \cosh^\lambda(\alpha x) {}_2F_1\left(a, b; \frac{1}{2}; -\sinh^2(\alpha x)\right). \quad (3.380)$$

For odd parity:

$$u_o(-x) = -u_o(x), \quad (3.381)$$

which implies:

$$u_o(0) = 0. \quad (3.382)$$

Choosing $A = 0$ and $B = i$, we obtain:

$$u_o(x) = \cosh^\lambda(\alpha x) \sinh(\alpha x) {}_2F_1\left(a + \frac{1}{2}, b + \frac{1}{2}; \frac{3}{2}; -\sinh^2(\alpha x)\right). \quad (3.383)$$

To determine the scattering properties, we study the asymptotic behaviour for large values of $|x|$. we use the approximations:

$$\sinh(\alpha x) \sim \frac{1}{2}e^{\alpha|x|}, \quad \cosh(\alpha x) \sim \frac{1}{2}e^{\alpha|x|}. \quad (3.384)$$

Thus:

$$-\sinh^2(\alpha x) \sim -\frac{1}{4}e^{2\alpha|x|}, \quad |x| \rightarrow \infty.. \quad (3.385)$$

In our case:

$$z = -\sinh^2(\alpha x), \quad (-z) \sim \frac{1}{4}e^{2\alpha|x|}. \quad (3.386)$$

To evaluate the asymptotic behaviour, we use the transformation formula for large negative argument:

$${}_2F_1(a, b; c; z) \sim \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a} + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}, \quad z \rightarrow -\infty. \quad (3.387)$$

Hence:

$$(-z)^{-a} \sim 2^{2a}e^{-2a\alpha|x|}, \quad (-z)^{-b} \sim 2^{2b}e^{-2b\alpha|x|}. \quad (3.388)$$

We also have:

$$\cosh^\lambda(\alpha x) \sim 2^{-\lambda}e^{\lambda\alpha|x|}. \quad (3.389)$$

Using the transformation formulas of the hypergeometric function, the even solution behaves as:

$$u_e(x) \sim 2^{-\lambda}e^{\lambda\alpha|x|}\Gamma\left(\frac{1}{2}\right) \left[\frac{\Gamma(b-a)}{\Gamma(b)\Gamma\left(\frac{1}{2}-a\right)} 2^{2a}e^{-2a\alpha|x|} + \frac{\Gamma(a-b)}{\Gamma(a)\Gamma\left(\frac{1}{2}-b\right)} 2^{2b}e^{-2b\alpha|x|} \right]. \quad (3.390)$$

Similarly, the odd solution behaves as:

$$u_o(x) \sim \pm 2^{-(\lambda+1)}e^{(\lambda+1)\alpha|x|}\Gamma\left(\frac{3}{2}\right) \left[\frac{\Gamma(b-a)}{\Gamma\left(b+\frac{1}{2}\right)\Gamma(1-a)} 2^{2a+1}e^{-(2a+1)\alpha|x|} + \frac{\Gamma(a-b)}{\Gamma\left(a+\frac{1}{2}\right)\Gamma(1-b)} 2^{2b+1}e^{-(2b+1)\alpha|x|} \right] \quad (3.391)$$

For positive energy, the parameters a and b are complex conjugates:

$$a = \frac{\lambda}{2} - \frac{ik}{2\alpha}, \quad b = \frac{\lambda}{2} + \frac{ik}{2\alpha}. \quad (3.392)$$

The asymptotic form of the even solution is:

$$u_e(x) \sim \Gamma\left(\frac{1}{2}\right) \left[\frac{\Gamma(-ik/\alpha) e^{\frac{ik}{\alpha} \log 2}}{\Gamma\left(\frac{\lambda}{2} - \frac{ik}{2\alpha}\right) \Gamma\left(\frac{1-\lambda}{2} - \frac{ik}{2\alpha}\right)} e^{-ik|x|} + \frac{\Gamma(ik/\alpha) e^{-\frac{ik}{\alpha} \log 2}}{\Gamma\left(\frac{\lambda}{2} + \frac{ik}{2\alpha}\right) \Gamma\left(\frac{1-\lambda}{2} + \frac{ik}{2\alpha}\right)} e^{ik|x|} \right]. \quad (3.393)$$

Similarly, the odd solution behaves as:

$$u_o(x) \sim \pm \Gamma\left(\frac{3}{2}\right) \left[\frac{\Gamma(-ik/\alpha) e^{\frac{ik}{\alpha} \log 2}}{\Gamma\left(\frac{\lambda+1}{2} - \frac{ik}{2\alpha}\right) \Gamma\left(1 - \frac{\lambda}{2} - \frac{ik}{2\alpha}\right)} e^{-ik|x|} + \frac{\Gamma(ik/\alpha) e^{-\frac{ik}{\alpha} \log 2}}{\Gamma\left(\frac{\lambda+1}{2} + \frac{ik}{2\alpha}\right) \Gamma\left(1 - \frac{\lambda}{2} + \frac{ik}{2\alpha}\right)} e^{ik|x|} \right]. \quad (3.394)$$

From the asymptotic expressions, the even and odd solutions behave as:

$$u(x) \sim A e^{-ik|x|} + B e^{ik|x|}, \quad |x| \rightarrow \infty. \quad (3.395)$$

This expression can be rewritten in terms of trigonometric functions using:

$$A e^{-ik|x|} + B e^{ik|x|} = C \cos(k|x| + \delta), \quad (3.396)$$

where the phase δ is defined by:

$$\delta = \arg \left(\frac{B}{A} \right). \quad (3.397)$$

From the asymptotic form of the even solution, we identify:

$$A_e = \frac{\Gamma(-ik/\alpha) e^{\frac{ik}{\alpha} \log 2}}{\Gamma\left(\frac{\lambda}{2} - \frac{ik}{2\alpha}\right) \Gamma\left(\frac{1-\lambda}{2} - \frac{ik}{2\alpha}\right)}, \quad B_e = \frac{\Gamma(ik/\alpha) e^{-\frac{ik}{\alpha} \log 2}}{\Gamma\left(\frac{\lambda}{2} + \frac{ik}{2\alpha}\right) \Gamma\left(\frac{1-\lambda}{2} + \frac{ik}{2\alpha}\right)}. \quad (3.398)$$

Thus, the even phase shift is given by:

$$\delta_e = \arg \left(\frac{\Gamma(ik/\alpha) e^{-\frac{ik}{\alpha} \log 2}}{\Gamma\left(\frac{\lambda}{2} + \frac{ik}{2\alpha}\right) \Gamma\left(\frac{1-\lambda}{2} + \frac{ik}{2\alpha}\right)} \right). \quad (3.399)$$

Similarly, for the odd solution:

$$A_o = \frac{\Gamma(-ik/\alpha) e^{\frac{ik}{\alpha} \log 2}}{\Gamma\left(\frac{\lambda+1}{2} - \frac{ik}{2\alpha}\right) \Gamma\left(1 - \frac{\lambda}{2} - \frac{ik}{2\alpha}\right)}, \quad B_o = \frac{\Gamma(ik/\alpha) e^{-\frac{ik}{\alpha} \log 2}}{\Gamma\left(\frac{\lambda+1}{2} + \frac{ik}{2\alpha}\right) \Gamma\left(1 - \frac{\lambda}{2} + \frac{ik}{2\alpha}\right)}. \quad (3.400)$$

Hence, the odd phase shift is:

$$\delta_o = \arg \left(\frac{\Gamma(ik/\alpha) e^{-\frac{ik}{\alpha} \log 2}}{\Gamma\left(\frac{\lambda+1}{2} + \frac{ik}{2\alpha}\right) \Gamma\left(1 - \frac{\lambda}{2} + \frac{ik}{2\alpha}\right)} \right). \quad (3.401)$$

Therefore, the even and odd solutions can be written as:

$$u_e(x) \sim C_e \cos(k|x| + \delta_e), \quad (3.402)$$

$$u_o(x) \sim \pm C_o \cos(k|x| + \delta_o), \quad (x \geq 0). \quad (3.403)$$

We consider the general solution as a linear combination:

$$u(x) = A u_e(x) + B u_o(x). \quad (3.404)$$

Using the asymptotic forms:

$$u_e(x) \sim C_e \cos(k|x| + \delta_e), \quad u_o(x) \sim C_o \cos(k|x| + \delta_o), \quad (3.405)$$

and the identity:

$$\cos(kx + \delta) = \frac{1}{2} (e^{i\delta} e^{ikx} + e^{-i\delta} e^{-ikx}), \quad (3.406)$$

we obtain: For $x > 0$:

$$u(x) = \frac{A}{2} C_e (e^{i\delta_e} e^{ikx} + e^{-i\delta_e} e^{-ikx}) + \frac{B}{2} C_o (e^{i\delta_o} e^{ikx} + e^{-i\delta_o} e^{-ikx}). \quad (3.407)$$

For $x < 0$: Using the parity properties:

$$u_e(-x) = u_e(x), \quad u_o(-x) = -u_o(x), \quad (3.408)$$

we obtain:

$$u(x) = \frac{A}{2}C_e (e^{i\delta_e}e^{-ikx} + e^{-i\delta_e}e^{ikx}) - \frac{B}{2}C_o (e^{i\delta_o}e^{-ikx} + e^{-i\delta_o}e^{ikx}). \quad (3.409)$$

We impose the standard scattering form:

$$u(x) = \begin{cases} e^{ikx} + Re^{-ikx}, & x < 0, \\ Te^{ikx}, & x > 0. \end{cases} \quad (3.410)$$

Matching coefficients of $e^{\pm ikx}$, we obtain the system:

$$\frac{A}{2}C_e e^{i\delta_e} + \frac{B}{2}C_o e^{i\delta_o} = T, \quad (3.411)$$

and

$$\frac{A}{2}C_e e^{-i\delta_e} + \frac{B}{2}C_o e^{-i\delta_o} = 0, \quad (3.412)$$

and

$$\frac{A}{2}C_e e^{i\delta_e} - \frac{B}{2}C_o e^{i\delta_o} = R, \quad (3.413)$$

and

$$\frac{A}{2}C_e e^{-i\delta_e} - \frac{B}{2}C_o e^{-i\delta_o} = 1. \quad (3.414)$$

From the second and fourth equations, we determine:

$$AC_e e^{-i\delta_e} = 1, \quad BC_o e^{-i\delta_o} = -1. \quad (3.415)$$

Substituting into the first and third equations, we obtain:

$$T = \frac{1}{2} (e^{2i\delta_e} - e^{2i\delta_o}), \quad (3.416)$$

and

$$R = \frac{1}{2} (e^{2i\delta_e} + e^{2i\delta_o}). \quad (3.417)$$

Finally, the reflection and transmission coefficients are:

$$|T|^2 = \cos^2(\delta_e - \delta_o), \quad |R|^2 = \sin^2(\delta_e - \delta_o). \quad (3.418)$$

We introduce the notation:

$$q = \frac{k}{2\alpha}. \quad (3.419)$$

From the expressions of the phase shifts, the difference $\delta_e - \delta_o$ can be written as:

$$\delta_e - \delta_o = \arg \Gamma \left(iq + \frac{\lambda}{2} + \frac{1}{2} \right) + \arg \Gamma \left(iq + 1 - \frac{\lambda}{2} \right) - \arg \Gamma \left(iq + \frac{\lambda}{2} \right) - \arg \Gamma \left(iq + \frac{1}{2} - \frac{\lambda}{2} \right). \quad (3.420)$$

We use the well-known identity:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}. \quad (3.421)$$

Taking the argument of both sides leads to:

$$\arg \Gamma(z) - \arg \Gamma(1-z^*) = -\arg \sin(\pi z). \quad (3.422)$$

We define:

$$z_1 = iq + \frac{\lambda}{2} + \frac{1}{2}, \quad z_2 = iq + \frac{\lambda}{2}. \quad (3.423)$$

Using:

$$iq + \frac{1}{2} - \frac{\lambda}{2} = 1 - z_1^*, \quad iq + 1 - \frac{\lambda}{2} = 1 - z_2^*, \quad (3.424)$$

we combine the terms pairwise to obtain:

$$\delta_e - \delta_o = -\arg \sin \left[\pi \left(\frac{\lambda+1}{2} + iq \right) \right] + \arg \sin \left[\pi \left(\frac{\lambda}{2} + iq \right) \right]. \quad (3.425)$$

We use the identity:

$$\sin(a+ib) = \sin a \cosh b + i \cos a \sinh b. \quad (3.426)$$

Thus:

$$\sin \left[\pi \left(\frac{\lambda}{2} + iq \right) \right] = \sin \left(\frac{\pi\lambda}{2} \right) \cosh(\pi q) + i \cos \left(\frac{\pi\lambda}{2} \right) \sinh(\pi q). \quad (3.427)$$

Taking the argument:

$$\arg \sin \left[\pi \left(\frac{\lambda}{2} + iq \right) \right] = \tan^{-1} \left(\frac{\cos \left(\frac{\pi\lambda}{2} \right) \sinh(\pi q)}{\sin \left(\frac{\pi\lambda}{2} \right) \cosh(\pi q)} \right). \quad (3.428)$$

This simplifies to:

$$\arg \sin \left[\pi \left(\frac{\lambda}{2} + iq \right) \right] = \tan^{-1} \left(\cot \left(\frac{\pi\lambda}{2} \right) \tanh(\pi q) \right). \quad (3.429)$$

Similarly:

$$\arg \sin \left[\pi \left(\frac{\lambda+1}{2} + iq \right) \right] = \tan^{-1} \left(\tan \left(\frac{\pi\lambda}{2} \right) \tanh(\pi q) \right). \quad (3.430)$$

Combining the two contributions, we obtain:

$$\delta_e - \delta_o = \tan^{-1} \left(\tan \frac{\pi\lambda}{2} \tanh \pi q \right) + \tan^{-1} \left(\cot \frac{\pi\lambda}{2} \tanh \pi q \right). \quad (3.431)$$

After simplification, the result reduces to:

$$\delta_e - \delta_o = \tan^{-1} \left(\frac{\sinh(\pi k/\alpha)}{\sin(\pi\lambda)} \right). \quad (3.432)$$

From the phase shift difference:

$$\delta_e - \delta_o = \tan^{-1} \left(\frac{\sinh(\pi k/\alpha)}{\sin(\pi \lambda)} \right), \quad (3.433)$$

we define the parameter:

$$p = \frac{\sinh(\pi k/\alpha)}{\sin(\pi \lambda)}. \quad (3.434)$$

Using the relations:

$$|T|^2 = \cos^2(\delta_e - \delta_o), \quad |R|^2 = \sin^2(\delta_e - \delta_o), \quad (3.435)$$

we obtain:

$$|T|^2 = \frac{1}{1+p^2}, \quad |R|^2 = \frac{p^2}{1+p^2}. \quad (3.436)$$

Alternatively:

$$|R|^2 = \frac{1}{1+p^2}, \quad |T|^2 = \frac{p^2}{1+p^2} \quad (3.437)$$

with:

$$p = \frac{\sinh(\pi k/\alpha)}{\sin(\pi \lambda)}. \quad (3.438)$$

3.7.2 Scattering theory with Dunkl derivative

We consider the hyperbolic Poschl–Teller potential defined as

$$V(x) = \frac{a}{\sinh^2(\alpha x)} - \frac{b}{\cosh^2(\alpha x)}, \quad (3.439)$$

where a, b are real constants and α is a real parameter.

We introduce the Dunkl operator

$$D_\nu = \frac{d}{dx} + \frac{\nu}{x} - \frac{\nu}{x} \hat{R}, \quad (3.440)$$

where $\nu > -\frac{1}{2}$ and $\hat{R}$ is the reflection operator defined by

$$\hat{R}\Psi(x) = \Psi(-x). \quad (3.441)$$

The Dunkl–Schrodinger equation reads

$$D_\nu^2 \Psi(x) + \left[E - \frac{a}{\sinh^2(\alpha x)} + \frac{b}{\cosh^2(\alpha x)} \right] \Psi(x) = 0. \quad (3.442)$$

Assuming parity of the wavefunction, $\hat{R}\Psi(x) = \delta\Psi(x)$, where $\delta = \pm 1$, we obtain

$$\Psi''(x) + \frac{2\nu}{x} \Psi'(x) + \left[E - \frac{a}{\sinh^2(\alpha x)} + \frac{b}{\cosh^2(\alpha x)} + \frac{\delta\nu - \nu}{x^2} \right] \Psi(x) = 0. \quad (3.443)$$

In the Dunkl formalism, the wave function $\Psi(x)$ does not belong to the usual Hilbert space $L^2(\mathbb{R})$, but rather to a weighted Hilbert space denoted by $L_w^2(\mathbb{R})$, where the weight function is given by

$$w(x) = |x|^{2\nu}. \quad (3.444)$$

For this reason, the norm of the wave function is defined as

$$\|\Psi\| = \sqrt{\int_{\mathbb{R}} |\Psi(x)|^2 |x|^{2\nu} dx}. \quad (7)$$

This expression shows that the probability density is modified compared to the standard case. Instead of $|\Psi(x)|^2$, the physical density becomes

$$\rho(x) = |\Psi(x)|^2 |x|^{2\nu}. \quad (3.445)$$

This modification arises from the presence of the Dunkl operator, which changes the structure of the underlying Hilbert space.

To simplify the Dunkl–Schrodinger equation, we introduce the transformation

$$\Psi(x) = x^{-\nu} \phi(x). \quad (9)$$

The main advantage of this transformation is that it removes the first-order derivative term in the equation, reducing it to a Schrodinger-like form.

Moreover, this transformation simplifies the norm. By substituting (9) into (7), we obtain

$$\begin{aligned} \|\Psi\| &= \sqrt{\int_{\mathbb{R}} |\Psi(x)|^2 |x|^{2\nu} dx} \\ &= \sqrt{\int_{\mathbb{R}} |x|^{-2\nu} |\phi(x)|^2 |x|^{2\nu} dx} \\ &= \sqrt{\int_{\mathbb{R}} |\phi(x)|^2 dx}. \end{aligned} \quad (11)$$

We observe that the weight function cancels exactly. Therefore, the function $\phi(x)$ belongs to the standard Hilbert space $L^2(\mathbb{R})$. This result is fundamental: the transformation (9) not only simplifies the differential equation, but also maps the problem from a weighted space to the usual quantum mechanical framework, allowing the use of standard solution methods. Finally, we note that the condition

$$\nu > -\frac{1}{2} \quad (3.446)$$

ensures that the norm remains finite near $x = 0$, thus avoiding any singular behavior of the weight function.

Which gives

$$\phi''_\delta(x) + \left[E - \frac{a}{\sinh^2(\alpha x)} + \frac{b}{\cosh^2(\alpha x)} + \frac{\nu(\delta - 1)}{x^2} \right] \phi_\delta(x) = 0. \quad (3.447)$$

We approximate the term $\frac{1}{x^2}$ using

$$\frac{1}{x^2} \approx \frac{\alpha^2}{\sinh^2(\alpha x)} + \frac{\alpha^2 \beta}{\cosh^2(\alpha x)}. \quad (3.448)$$

Substituting, we obtain

$$\phi''_\delta(x) + \left[E - \frac{a - \alpha^2(\delta - 1)\nu}{\sinh^2(\alpha x)} + \frac{b + \alpha^2\beta(\delta\nu - \nu)}{\cosh^2(\alpha x)} \right] \phi_\delta(x) = 0. \quad (3.449)$$

We introduce the new variable:

$$y = \cosh^2(\alpha x). \quad (3.450)$$

Then,

$$\frac{dy}{dx} = 2\alpha \sinh(\alpha x) \cosh(\alpha x). \quad (3.451)$$

Using the identities:

$$\cosh^2(\alpha x) = y, \quad \sinh^2(\alpha x) = y - 1 \quad (3.452)$$

we obtain:

$$\frac{dy}{dx} = 2\alpha \sinh(\alpha x) \cosh(\alpha x) = 2\alpha \sqrt{y(y - 1)}, \quad (x > 0). \quad (3.453)$$

Using the chain rule:

$$\frac{d\phi_\delta}{dx} = \frac{d\phi_\delta}{dy} \frac{dy}{dx} = 2\alpha \sqrt{y(y - 1)} \phi_y^{(\delta)}. \quad (3.454)$$

We compute:

$$\phi''_\delta = \frac{d}{dx} \left(2\alpha \sqrt{y(y - 1)} \phi_y^{(\delta)} \right). \quad (3.455)$$

Applying the product rule:

$$\phi''_\delta = 2\alpha \left[\frac{d}{dx} (\sqrt{y(y - 1)}) \phi_y^{(\delta)} + \sqrt{y(y - 1)} \frac{d\phi_y^{(\delta)}}{dx} \right].$$

We evaluate each term:

$$\frac{d}{dx} \sqrt{y(y - 1)} = \frac{dy}{dx} \frac{d}{dy} \sqrt{y(y - 1)} = 2\alpha \sqrt{y(y - 1)} \frac{(2y - 1)}{2\sqrt{y(y - 1)}} = \alpha(2y - 1), (x > 0) \quad (3.456)$$

also

$$\frac{d}{dx} (\phi_y^{(\delta)}) = \phi_{yy}^{(\delta)} \frac{dy}{dx} = 2\alpha \sqrt{y(y - 1)} \phi_{yy}^{(\delta)}. \quad (3.457)$$

Thus:

$$\phi''_\delta = 2\alpha^2(2y - 1)\phi_y^{(\delta)} + 4\alpha^2 y(y - 1)\phi_{yy}^{(\delta)}. \quad (3.458)$$

Substituting into the original equation:

$$4\alpha^2 y(y-1)\phi_{yy}^{(\delta)} + 2\alpha^2(2y-1)\phi_y^{(\delta)} + \left[k^2 - \frac{a - \alpha^2(\delta\nu - \nu)}{y-1} + \frac{b + \alpha^2\beta(\delta\nu - \nu)}{y} \right] \phi^{(\delta)} = 0. \quad (3.459)$$

Dividing by $4\alpha^2$, we obtain:

$$y(y-1)\phi_{yy}^{(\delta)} + \frac{2y-1}{2}\phi_y^{(\delta)} + \left[\frac{k^2}{4\alpha^2} - \frac{a - \alpha^2(\delta\nu - \nu)}{4\alpha^2(y-1)} + \frac{b + \alpha^2\beta(\delta\nu - \nu)}{4\alpha^2 y} \right] \phi^{(\delta)} = 0. \quad (3.460)$$

Rewriting:

$$y(1-y)\phi_{yy}^{(\delta)} + \left(\frac{1}{2} - y \right) \phi_y^{(\delta)} + \left[-\frac{k^2}{4\alpha^2} - \frac{a - \alpha^2(\delta\nu - \nu)}{4\alpha^2(1-y)} - \frac{b + \alpha^2\beta(\delta\nu - \nu)}{4\alpha^2 y} \right] \phi^{(\delta)} = 0. \quad (3.461)$$

For elimination of the singular term, we introduce:

$$\phi^{(\delta)}(y) = y^A (1-y)^B \chi^{(\delta)}(y). \quad (3.462)$$

Then:

$$(\phi^{(\delta)}(y))' = \left[\left(\frac{A}{y} - \frac{B}{1-y} \right) \chi^{(\delta)} + (\chi^{(\delta)})' \right] y^A (1-y)^B, \quad (3.463)$$

$$(\phi^{(\delta)}(y))'' = \left[\left(\frac{A}{y} - \frac{B}{1-y} \right)^2 \chi^{(\delta)} + \left(-\frac{A}{y^2} - \frac{B}{(1-y)^2} \right) \chi^{(\delta)} + 2 \left(\frac{A}{y} - \frac{B}{1-y} \right) (\chi^{(\delta)})' + (\chi^{(\delta)})'' \right] y^A (1-y)^B. \quad (3.464)$$

then

$$\begin{aligned} & y(1-y) (\chi^{(\delta)})'' + \left(2A + \frac{1}{2} - 2 \left(A + B + \frac{1}{2} \right) y \right) (\chi^{(\delta)})' \\ & + \left[-A^2 - B^2 - 2AB - \frac{k^2}{4\alpha^2} \right] \chi^{(\delta)} \\ & + \left[+ \left(B^2 - \frac{1}{2}B - \frac{a - \alpha^2(\delta\nu - \nu)}{4\alpha^2} \right) \frac{1}{(1-y)} + \left(-\frac{b + \alpha^2\beta(\delta\nu - \nu)}{4\alpha^2} + A^2 - \frac{1}{2}A \right) \frac{1}{y} \right] \chi^{(\delta)} = 0. \end{aligned} \quad (3.465)$$

with

$$\begin{aligned} B^2 - \frac{1}{2}B - \frac{a - \alpha^2(\delta\nu - \nu)}{4\alpha^2} &= 0 \\ A^2 - \frac{1}{2}A - \frac{b + \alpha^2\beta(\delta\nu - \nu)}{4\alpha^2} &= 0 \end{aligned}$$

then

$$B = \frac{1}{4} \pm \frac{\sqrt{\frac{1}{4} + \frac{a - \alpha^2\nu(\delta-1)}{\alpha^2}}}{2} \quad (3.466)$$

$$A = \frac{1}{4} \pm \frac{\sqrt{\frac{1}{4} + \frac{b + \alpha^2\beta\nu(\delta-1)}{\alpha^2}}}{2} \quad (3.467)$$

we find

$$y(1-y) (\chi^{(\delta)}(y))'' + \left(2A + \frac{1}{2} - 2 \left(A + B + \frac{1}{2}\right) y\right) (\chi^{(\delta)}(y))' + \left[-A^2 - B^2 - 2AB - \frac{k^2}{4\alpha^2}\right] \chi^{(\delta)}(y) = 0. \quad (3.468)$$

This is the standard hypergeometric differential equation:

$$y(1-y)v'' + [c - (a+b+1)y]v' - abv = 0, \quad (3.469)$$

with parameters:

$$c = 2A + \frac{1}{2}, \quad a + b = 2(A + B), \quad ab = \left[(A + B)^2 + \frac{k^2}{4\alpha^2}\right]. \quad (3.470)$$

these lead to

$$a + b = 2(A + B), \quad ab = \left[(A + B)^2 + \frac{k^2}{4\alpha^2}\right] \quad (3.471)$$

$$\Rightarrow -a^2 + 2(A + B)a - \left[(A + B)^2 + \frac{k^2}{4\alpha^2}\right] = 0 \quad (3.472)$$

then

$$a = A + B \pm \frac{\sqrt{4(A + B)^2 - 4\left((A + B)^2 + \frac{k^2}{\alpha^2}\right)}}{2} = A + B \pm i \frac{k}{2\alpha} \quad (3.473)$$

$$b = 2(A + B) - a = 2(A + B) - A - B \mp i \frac{k}{2\alpha} = A + B \mp i \frac{k}{2\alpha} \quad (3.474)$$

where

$$B_\delta = \frac{1}{4} - \frac{\sqrt{\frac{1}{4} + \frac{a - \alpha^2 \nu(\delta-1)}{\alpha^2}}}{2} \quad (3.475)$$

$$A_\delta = \frac{1}{4} + \frac{\sqrt{\frac{1}{4} + \frac{b + \alpha^2 \beta \nu(\delta-1)}{\alpha^2}}}{2} \quad (3.476)$$

then

$$c' = 2B + \frac{1}{2} = 1 - \sqrt{\frac{1}{4} + \frac{a - \alpha^2 \nu(\delta-1)}{\alpha^2}} \quad (3.477)$$

$$a = A + B \pm i \frac{k}{2\alpha} = \frac{1}{2} + \frac{\sqrt{\frac{1}{4} + \frac{b + \alpha^2 \beta \nu(\delta-1)}{\alpha^2}}}{2} - \frac{\sqrt{\frac{1}{4} + \frac{a - \alpha^2 \nu(\delta-1)}{\alpha^2}}}{2} \pm i \frac{k}{2\alpha} \quad (3.478)$$

$$b = A + B \mp i \frac{k}{2\alpha} = A + B \mp i \frac{k}{2\alpha} = \frac{1}{2} + \frac{\sqrt{\frac{1}{4} + \frac{b + \alpha^2 \beta \nu(\delta-1)}{\alpha^2}}}{2} - \frac{\sqrt{\frac{1}{4} + \frac{a - \alpha^2 \nu(\delta-1)}{\alpha^2}}}{2} \mp i \frac{k}{2\alpha} \quad (3.479)$$

In order to study the asymptotic behavior, we introduce the change of variable:

$$z = 1 - y. \quad (3.480)$$

From this transformation, we have:

$$\frac{dv}{dy} = -\frac{dv}{dz}, \quad \frac{d^2v}{dy^2} = \frac{d^2v}{dz^2}. \quad (3.481)$$

Substituting into the differential equation, we obtain:

$$(1-z)z \frac{d^2\chi^{(\delta)}}{dz^2} - [c - (a+b+1)(1-z)] \frac{d\chi^{(\delta)}}{dz} - ab\chi^{(\delta)} = 0. \quad (3.482)$$

Expanding the coefficient:

$$c - (a+b+1)(1-z) = c - (a+b+1) + (a+b+1)z, \quad (3.483)$$

the equation becomes:

$$z(1-z) \frac{d^2\chi^{(\delta)}(z)}{dz^2} + [(a+b+1-c) - (a+b+1)z] \frac{d\chi^{(\delta)}(z)}{dz} - ab\chi^{(\delta)}(z) = 0. \quad (3.484)$$

This is again a hypergeometric equation in the variable z , with the new parameter:

$$c = 2A + \frac{1}{2}, \quad a+b = 2(A+B), \quad ab = \left[(A+B)^2 + \frac{k^2}{4\alpha^2} \right]. \quad (3.485)$$

$$c' = a+b+1-c = 2B + \frac{1}{2}. \quad (3.486)$$

Thus, the equation takes the standard form:

$$z(1-z) \frac{d^2v}{dz^2} + [c' - (a+b+1)z] \frac{dv}{dz} - abv = 0. \quad (3.487)$$

The general solution is given by

$$\chi^{(\delta)}(z) = D_1 {}_2F_1(a, b, c', z) + D_2 z^{1-c'} {}_2F_1(a - c' + 1, b - c' + 1, 2 - c', z), \quad (3.488)$$

with

$$c'_\delta = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{a - \alpha^2 \nu(\delta-1)}{\alpha^2}} \quad (3.489)$$

$$a_\delta = \frac{1}{2} \pm \frac{\sqrt{\frac{1}{4} + \frac{b + \alpha^2 \beta \nu(\delta-1)}{\alpha^2}}}{2} \pm \frac{\sqrt{\frac{1}{4} + \frac{a - \alpha^2 \nu(\delta-1)}{\alpha^2}}}{2} + i \frac{k}{2\alpha} \quad (3.490)$$

$$b_\delta = \frac{1}{2} \pm \frac{\sqrt{\frac{1}{4} + \frac{b + \alpha^2 \beta \nu(\delta-1)}{\alpha^2}}}{2} \pm \frac{\sqrt{\frac{1}{4} + \frac{a - \alpha^2 \nu(\delta-1)}{\alpha^2}}}{2} - i \frac{k}{2\alpha} \quad (3.491)$$

For $\delta = +1$

$$\chi^{(+)}(z) = D_1 {}_2F_1(a_+, b_+, c'_+, z) + D_2 z^{1-c'_+} {}_2F_1(a_+ - c'_+ + 1, b_+ - c'_+ + 1, 2 - c'_+, z), \quad (3.492)$$

with

$$c'_+ = 1 - \sqrt{\frac{1}{4} + \frac{a}{\alpha^2}} \quad (3.493)$$

$$a_+ = \frac{1}{2} + \frac{\sqrt{\frac{1}{4} + \frac{b}{\alpha^2}}}{2} - \frac{\sqrt{\frac{1}{4} + \frac{a}{\alpha^2}}}{2} + i \frac{k}{2\alpha} \quad (3.494)$$

$$b_+ = \frac{1}{2} + \frac{\sqrt{\frac{1}{4} + \frac{b}{\alpha^2}}}{2} - \frac{\sqrt{\frac{1}{4} + \frac{a}{\alpha^2}}}{2} - i \frac{k}{2\alpha} \quad (3.495)$$

For $\delta = -1$

$$\chi^{(-)}(z) = D_1^- {}_2F_1(a_-, b_-, c'_-, z) + D_2^- z^{1-c'_-} {}_2F_1(a_- - c'_- + 1, b_- - c'_- + 1, 2 - c'_-, z), \quad (3.496)$$

with

$$c'_- = 1 - \sqrt{\frac{1}{4} + \frac{a + 2\alpha^2\nu}{\alpha^2}} \quad (3.497)$$

$$a_- = \frac{1}{2} + \frac{\sqrt{\frac{1}{4} + \frac{b - 2\alpha^2\beta\nu}{\alpha^2}}}{2} - \frac{\sqrt{\frac{1}{4} + \frac{a + 2\alpha^2\nu}{\alpha^2}}}{2} + i \frac{k}{2\alpha} \quad (3.498)$$

$$b_- = \frac{1}{2} + \frac{\sqrt{\frac{1}{4} + \frac{b - 2\alpha^2\beta\nu}{\alpha^2}}}{2} - \frac{\sqrt{\frac{1}{4} + \frac{a + 2\alpha^2\nu}{\alpha^2}}}{2} - i \frac{k}{2\alpha} \quad (3.499)$$

Due to the symmetry of the potential $V(x) = V(-x)$, the solutions of the Schrodinger equation can be classified into even and odd functions. Thus, one can construct a fundamental set of two real independent solutions satisfying

$$\phi_e(-x) = \phi_e(x), \quad \phi_o(-x) = -\phi_o(x). \quad (3.500)$$

Starting from the general solution near $x = 0$, which depends on two constants $D_1^{+,-}$ and $D_2^{+,-}$, appropriate choices of these constants allow us to isolate the even and odd solutions.

For $D_1^+ = 1$ and $D_2^+ = 0$, one obtains the even solution:

$$\phi_e(x) = \cosh^{2A_+}(\alpha x) \sinh^{2B_+}(\alpha x) {}_2F_1(a_+, b_+, c'_+; -\sinh^2(\alpha x)). \quad (3.501)$$

On the other hand, by choosing $D_1^- = 0$ and $D_2^- = i$, the odd solution is obtained as

$$\phi_o(x) = \cosh^{2A_-}(\alpha x) \sinh^{1-c'_-+2B_-}(\alpha x) {}_2F_1(a_- + \frac{1}{2}, b_- + \frac{1}{2}; c'_-; -\sinh^2(\alpha x)). \quad (3.502)$$

These two solutions form a complete basis for describing the scattering states, and they play a crucial role in determining the phase shifts and the reflection and transmission coefficients.

To determine the scattering properties, we study the asymptotic behaviour for large values of $|x|$. we use the approximations:

$$\sinh(\alpha x) \sim \frac{1}{2}e^{\alpha|x|}, \quad \cosh(\alpha x) \sim \frac{1}{2}e^{\alpha|x|}. \quad (3.503)$$

Thus:

$$-\sinh^2(\alpha x) \sim -\frac{1}{4}e^{2\alpha|x|}, \quad |x| \rightarrow \infty.. \quad (3.504)$$

In our case:

$$z = -\sinh^2(\alpha x), \quad (-z) \sim \frac{1}{4}e^{2\alpha|x|}. \quad (3.505)$$

To evaluate the asymptotic behaviour, we use the transformation formula for large negative argument:

$${}_2F_1(a, b; c; z) \sim \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a} + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}, \quad z \rightarrow -\infty. \quad (3.506)$$

Hence:

$$(-z)^{-a} \sim 2^{2a}e^{-2a\alpha|x|}, \quad (-z)^{-b} \sim 2^{2b}e^{-2b\alpha|x|}. \quad (3.507)$$

We also have:

$$\cosh^\lambda(\alpha x) \sim 2^{-\lambda}e^{\lambda\alpha|x|}. \quad (3.508)$$

Using the transformation formulas of the hypergeometric function, the even solution behaves as:

$$\phi_e(x) \sim 2^{-2A_+}e^{2A_+\alpha|x|}\frac{1}{4}e^{2B_+\alpha|x|}\left[\frac{\Gamma(c_+)\Gamma(b_+-a_+)}{\Gamma(b_+)\Gamma(c_+-a_+)}2^{2a_+}e^{-2a_+\alpha|x|} + \frac{\Gamma(c_+)\Gamma(a_+-b_+)}{\Gamma(a_+)\Gamma(c_+-b_+)}2^{2b_+}e^{-2b_+\alpha|x|}\right]. \quad (3.509)$$

Similarly, the odd solution behaves as:

$$\phi_o(x) \sim \pm 2^{-2A_-}e^{2A_-\alpha|x|}2^{-c'_-+2B_-}e^{(-c'_-+2B_-)\alpha|x|}\left[\frac{\Gamma(c_-)\Gamma(b_--a_-)}{\Gamma(b_-)\Gamma(c_--a_-)}2^{2a_+1}e^{-(2a_++1)\alpha|x|} + \frac{\Gamma(c_-)\Gamma(a_--b_-)}{\Gamma(a_--)\Gamma(c_--b_-)}2^{2b_+1}e^{-(2b_++1)\alpha|x|}\right]. \quad (3.510)$$

The asymptotic form of the even solution is:

$$\phi_e(x) \sim \Gamma\left(1 - \sqrt{\frac{1}{4} + \frac{a}{\alpha^2}}\right) \left[\frac{\Gamma(-ik/\alpha) e^{\frac{ik}{\alpha} \log 2}}{\Gamma\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{b}{\alpha^2}} - \sqrt{\frac{1}{4} + \frac{a}{\alpha^2}} - i\frac{k}{2\alpha}\right) \Gamma\left(\frac{1}{2} - \sqrt{\frac{1}{4} + \frac{b}{\alpha^2}} - \sqrt{\frac{1}{4} + \frac{a}{\alpha^2}} - i\frac{k}{2\alpha}\right)} e^{-ik|x|} \right. \\ \left. + \frac{\Gamma(ik/\alpha) e^{-\frac{ik}{\alpha} \log 2}}{\Gamma\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{b}{\alpha^2}} - \sqrt{\frac{1}{4} + \frac{a}{\alpha^2}} + i\frac{k}{2\alpha}\right) \Gamma\left(\frac{1}{2} - \sqrt{\frac{1}{4} + \frac{b}{\alpha^2}} - \sqrt{\frac{1}{4} + \frac{a}{\alpha^2}} + i\frac{k}{2\alpha}\right)} e^{ik|x|} \right]. \quad (3.511)$$

Similarly, the odd solution behaves as:

$$\phi_o(x) \sim \pm \Gamma\left(\frac{3}{2}\right) \left[\frac{\Gamma(-ik/\alpha) e^{\frac{ik}{\alpha} \log 2}}{\Gamma\left(\frac{\lambda+1}{2} - \frac{ik}{2\alpha}\right) \Gamma\left(1 - \frac{\lambda}{2} - \frac{ik}{2\alpha}\right)} e^{-ik|x|} + \frac{\Gamma(ik/\alpha) e^{-\frac{ik}{\alpha} \log 2}}{\Gamma\left(\frac{\lambda+1}{2} + \frac{ik}{2\alpha}\right) \Gamma\left(1 - \frac{\lambda}{2} + \frac{ik}{2\alpha}\right)} e^{ik|x|} \right]. \quad (3.512)$$

3.7.3 Bound states with Dunkl–Schrodinger equation

In this section, we study the generalized Poschl–Teller potential within the framework of the one-dimensional Dunkl–Schrodinger equation. Our aim is to understand how the Dunkl deformation modifies the singular behavior near the origin and how it affects the bound-state spectrum.

We consider the corresponding Dunkl–Schrodinger equations

$$D_\mu^2 \psi(x) + \alpha^2 \left[-\lambda^2 - \frac{B(B-1)}{\sinh^2(\alpha x)} + \frac{A(A-1)}{\cosh^2(\alpha x)} \right] \psi(x) = 0. \quad (3.513)$$

where

$$E = -\frac{\hbar^2 \alpha^2}{2m} \lambda^2, \quad \lambda > 0, \quad (3.514)$$

Because the Dunkl operator involves the reflection operator R , it is natural to decompose the solutions into even and odd sectors.

Even sector: If $\psi_e(-x) = \psi_e(x)$, then $R\psi_e = \psi_e$, and a direct computation gives $D_\mu^2 \psi_e = \psi_e'' + \frac{2\mu}{x} \psi_e'$. Hence the eigenvalue equation becomes

$$\psi_e'' + \frac{2\mu}{x} \psi_e' + \alpha^2 \left[-\lambda^2 - \frac{B(B-1)}{\sinh^2(\alpha x)} + \frac{A(A-1)}{\cosh^2(\alpha x)} \right] \psi_e = 0. \quad (3.515)$$

Odd sector: If $\psi_o(-x) = -\psi_o(x)$, then $R\psi_o = -\psi_o$, and one finds

$$D_\mu^2 \psi_o = \psi_o'' + \frac{2\mu}{x} \psi_o' - \frac{2\mu}{x^2} \psi_o. \quad (3.516)$$

Therefore,

$$\psi_o'' + \frac{2\mu}{x} \psi_o' - \frac{2\mu}{x^2} \psi_o + \alpha^2 \left[-\lambda^2 - \frac{B(B-1)}{\sinh^2(\alpha x)} + \frac{A(A-1)}{\cosh^2(\alpha x)} \right] \psi_o = 0. \quad (3.517)$$

We now study the behavior of the solutions near $x = 0$. Using

$$\sinh(\alpha x) \sim \alpha x, \quad \cosh(\alpha x) \sim 1 \quad \text{as } x \rightarrow 0, \quad (3.518)$$

we obtain

$$\frac{1}{\sinh^2(\alpha x)} \sim \frac{1}{\alpha^2 x^2}, \quad \frac{1}{\cosh^2(\alpha x)} \sim 1. \quad (3.519)$$

Thus, the dominant singular contribution near the origin is generated by the term

$$\frac{B(B-1)}{x^2}. \quad (3.520)$$

Even sector near $x = 0$, From Eq. (3.515), the leading equation near the origin is

$$\psi_e'' + \frac{2\mu}{x} \psi_e' - \frac{B(B-1)}{x^2} \psi_e = 0. \quad (3.521)$$

Assuming a power-like behavior

$$\psi_e(x) \sim x^\rho, \quad (3.522)$$

we obtain

$$\rho(\rho - 1) + 2\mu\rho - B(B - 1) = 0, \quad (3.523)$$

or equivalently

$$\rho^2 + (2\mu - 1)\rho - B(B - 1) = 0. \quad (3.524)$$

The corresponding exponents are

$$\rho_{\pm}^{(e)} = \frac{1 - 2\mu \pm \sqrt{(2\mu - 1)^2 + 4B(B - 1)}}{2}. \quad (3.525)$$

Odd sector near $x = 0$, From Eq. (3.517), the dominant equation is

$$\psi_o'' + \frac{2\mu}{x}\psi_o' - \frac{B(B - 1) + 2\mu}{x^2}\psi_o = 0. \quad (3.526)$$

Again setting

$$\psi_o(x) \sim x^\rho, \quad (3.527)$$

we get

$$\rho^2 + (2\mu - 1)\rho - (B(B - 1) + 2\mu) = 0. \quad (3.528)$$

Hence

$$\rho_{\pm}^{(o)} = \frac{1 - 2\mu \pm \sqrt{(2\mu + 1)^2 + 4B(B - 1)}}{2}. \quad (3.529)$$

For removing the first derivative term we introduce the transformation

$$\psi(x) = x^{-\mu}u(x). \quad (3.530)$$

Even sector, Substituting $\psi_e(x) = x^{-\mu}u_e(x)$ into Eq. (3.515), we obtain after simplification

$$u_e'' + \left[-\alpha^2\lambda^2 - \alpha^2 \frac{B(B - 1)}{\sinh^2(\alpha x)} + \alpha^2 \frac{A(A - 1)}{\cosh^2(\alpha x)} + \frac{\mu(1 - \mu)}{x^2} \right] u_e = 0. \quad (3.531)$$

This shows that, in the even sector, the Dunkl deformation generates an additional inverse-square contribution.

Odd sector: Similarly, by setting

$$\psi_o(x) = x^{-\mu}u_o(x), \quad (3.532)$$

Eq. (3.517) reduces to

$$u_o'' + \left[-\alpha^2\lambda^2 - \alpha^2 \frac{B(B - 1)}{\sinh^2(\alpha x)} + \alpha^2 \frac{A(A - 1)}{\cosh^2(\alpha x)} - \frac{\mu(\mu + 1)}{x^2} \right] u_o = 0. \quad (3.533)$$

We approximate the term $\frac{1}{x^2}$ using

$$\frac{1}{x^2} \approx \frac{\alpha^2}{\sinh^2(\alpha x)} + \frac{\alpha^2 \beta}{\cosh^2(\alpha x)}. \quad (3.534)$$

we can transform the even and odd equation as

$$u_e'' + \left[-\alpha^2 \lambda^2 - \alpha^2 \frac{\beta_e (\beta_e - 1)}{\sinh^2(\alpha x)} + \alpha^2 \frac{\alpha_e (\alpha_e - 1)}{\cosh^2(\alpha x)} \right] u_e = 0. \quad (3.535)$$

and

$$u_o'' + \left[-\alpha^2 \lambda^2 - \alpha^2 \frac{\beta_o (\beta_o - 1)}{\sinh^2(\alpha x)} + \alpha^2 \frac{\alpha_o (\alpha_o - 1)}{\cosh^2(\alpha x)} \right] u_o = 0. \quad (3.536)$$

where

$$\beta_e (\beta_e - 1) = B(B - 1) - \mu(1 - \mu) \quad (3.537)$$

$$\beta_o (\beta_o - 1) = B(B - 1) + \mu(1 + \mu) \quad (3.538)$$

and

$$\alpha_e (\alpha_e - 1) = A(A - 1) + \beta \mu(1 - \mu) \quad (3.539)$$

$$\alpha_o (\alpha_o - 1) = A(A - 1) - \beta \mu(1 + \mu) \quad (3.540)$$

Interpretation: In the ordinary generalized Poschl–Teller problem, the parameter B controls the behavior near the origin through the factor $B(B - 1)$. Motivated by the local analysis, we introduce effective parameters β_e and β_o defined by

$$\beta_e (\beta_e - 1) = B(B - 1) + \mu(\mu - 1), \quad (3.541)$$

and

$$\beta_o (\beta_o - 1) = B(B - 1) + \mu(\mu + 1). \quad (3.542)$$

Therefore,

$$\beta_e = \frac{1 + \sqrt{1 + 4[B(B - 1) + \mu(\mu - 1)]}}{2}, \quad (3.543)$$

$$\beta_o = \frac{1 + \sqrt{1 + 4[B(B - 1) + \mu(\mu + 1)]}}{2}. \quad (3.544)$$

and also for α_e and α_o are:

$$\alpha_e = \frac{1 + \sqrt{1 + 4[A(A - 1) + \beta \mu(1 - \mu)]}}{2}, \quad (3.545)$$

$$\alpha_o = \frac{1 + \sqrt{1 + 4[A(A - 1) - \beta \mu(\mu + 1)]}}{2}. \quad (3.546)$$

These quantities may be interpreted as effective renormalized parameters controlling the singular behavior near the origin in the even and odd sectors, respectively.

For the ordinary one-dimensional generalized Poschl–Teller potential, the bound-state energies are

$$E_n = -\frac{\hbar^2 \alpha^2}{2m} (A - B - 2n)^2, \quad n = 0, 1, 2, \dots, \quad (3.547)$$

with the condition

$$A - B - 2n > 0. \quad (3.548)$$

Since the Dunkl deformation modifies the singular structure mainly through the parameter B , it is natural to propose an effective replacement

$$B \longrightarrow \beta_e \quad \text{or} \quad B \longrightarrow \beta_o. \quad (3.549)$$

and

$$A \longrightarrow \alpha_e \quad \text{or} \quad A \longrightarrow \alpha_o. \quad (3.550)$$

This leads to the following effective bound-state energies:

$$E_{n,e}^{\text{eff}} \approx -\frac{\hbar^2 \alpha^2}{2m} (\alpha_e - \beta_e - 2n)^2, \quad (3.551)$$

for the even sector, and

$$E_{n,o}^{\text{eff}} \approx -\frac{\hbar^2 \alpha^2}{2m} (\alpha_o - \beta_o - 2n)^2, \quad (3.552)$$

for the odd sector, with the constraints

$$\alpha_e - \beta_e - 2n > 0, \quad \alpha_o - \beta_o - 2n > 0. \quad (3.553)$$

In the Dunkl setting, the natural scalar product involves the weighted measure

$$|x|^{2\mu} dx. \quad (3.554)$$

Hence the square-integrability condition becomes

$$\int |\psi(x)|^2 |x|^{2\mu} dx < \infty. \quad (3.555)$$

Chapter 4

Relativistic Dunkl-Klein–Gordon Equation

4.1 The (KG) Equation: Origins and Governing Equations

In alignment with the dynamics of particles that integrate Einsteins Special Theory of Relativity with Quantum Mechanics, attempts were made to make the Schrodinger equation compatible with Lorentz transformations. It was found that the Schrodinger equation is not invariant under Lorentz transformations. This leads us to seek a relativistic wave equation that maintains the same scalar property as the Schrodinger equation. The time-dependent Schrödinger equation describes the dynamics of a quantum particle:

$$i\hbar \frac{\partial \psi(x, t)}{\partial t} = H_{\text{Sch}} \psi(x, t), \quad (4.1)$$

where the Hamiltonian operator is given by

$$H_{\text{Sch}} = \frac{p^2}{2m} + V(x). \quad (4.2)$$

For a free particle, $V(x) = 0$, and using $p = -i\hbar \partial_x$, we obtain

$$i\hbar \frac{\partial \psi(x, t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x, t)}{\partial x^2}. \quad (4.3)$$

However, this equation is not invariant under Lorentz transformations and therefore cannot describe relativistic particles.

To construct a relativistic wave equation, we start from the energy-momentum relation:

$$E^2 = p^2 c^2 + m^2 c^4. \quad (4.4)$$

Promoting E and $\hat{p}$ to operators:

$$E \rightarrow i\hbar \frac{\partial}{\partial t}, \quad \hat{p} \rightarrow -i\hbar \nabla, \quad (4.5)$$

we obtain

$$\left(i\hbar \frac{\partial}{\partial t}\right)^2 \psi = (-\hbar^2 c^2 \nabla^2 + m^2 c^4) \psi. \quad (4.6)$$

This leads to

$$-\hbar^2 \frac{\partial^2 \psi}{\partial t^2} = (-\hbar^2 c^2 \nabla^2 + m^2 c^4) \psi. \quad (4.7)$$

Rearranging terms, we obtain

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2 c^2}{\hbar^2}\right) \psi(x, t) = 0. \quad (4.8)$$

Introducing the d'Alembert operator

$$\square = \partial_\mu \partial^\mu = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2, \quad (4.9)$$

The Klein-Gordon equation takes the compact form [15]:

$$\left(\square + \frac{m^2 c^2}{\hbar^2}\right) \psi(x) = 0. \quad (4.10)$$

A free particle solution is given by a plane wave:

$$\psi(x) = C \exp\left(-\frac{i}{\hbar} p_\mu x^\mu\right), \quad (4.11)$$

where $p_\mu x^\mu = Et - \mathbf{p} \cdot \mathbf{x}$.

Applying the operator ∂_μ , we find

$$\partial_\mu \psi = -\frac{i}{\hbar} p_\mu \psi, \quad (4.12)$$

and

$$\partial^\mu \partial_\mu \psi = -\frac{1}{\hbar^2} p^\mu p_\mu \psi. \quad (4.13)$$

Using the invariant relation $p^\mu p_\mu = m^2 c^2$, the Klein-Gordon equation is satisfied.

From this relation, we recover

$$E^2 = p^2 c^2 + m^2 c^4, \quad (4.14)$$

which leads to

$$E = \pm \sqrt{p^2 c^2 + m^2 c^4}. \quad (4.15)$$

The presence of negative energy solutions indicates that the Klein-Gordon equation cannot be interpreted within a single-particle probabilistic framework.

Starting from the Klein-Gordon equation and its complex conjugate, we derive the conserved current:

$$\partial_\mu J^\mu = 0, \quad (4.16)$$

where

$$J^\mu = \frac{i\hbar}{2mc} (\psi^* \partial^\mu \psi - \psi \partial^\mu \psi^*). \quad (4.17)$$

For a plane wave solution, this gives

$$J^\mu = \frac{p^\mu}{m}. \quad (4.18)$$

Since $J^0 \propto E$, the probability density is not positive definite, confirming the limitations of the single-particle interpretation [15].

The interaction with an electromagnetic field is introduced via minimal coupling:

$$\partial_\mu \rightarrow \mathcal{D}_\mu = \partial_\mu + \frac{ie}{\hbar c} A_\mu. \quad (4.19)$$

The Klein-Gordon equation becomes

$$\left[(\mathcal{D}_\mu \mathcal{D}^\mu) + \frac{m^2 c^2}{\hbar^2} \right] \psi = 0, \quad (4.20)$$

or equivalently

$$\left[\left(p_\mu - \frac{e}{c} A_\mu \right) \left(p^\mu - \frac{e}{c} A^\mu \right) - m^2 c^2 \right] \psi = 0. \quad (4.21)$$

4.2 Motivation for Dunkl Generalization

Despite its success, the Klein-Gordon equation is formulated using standard derivatives. However, in systems involving reflection symmetries or deformed quantum structures, it is natural to replace the ordinary derivative with generalized operators such as the Dunkl derivative. This motivates the extension of the Klein-Gordon equation within the Dunkl formalism, which will be explored in the following sections.

The one-dimensional Dunkl derivative is defined as:

$$D_x^{(\nu)} = \frac{d}{dx} + \frac{\nu}{x} (1 - \hat{R}), \quad (4.22)$$

where ν is the Dunkl parameter and $\hat{R}$ is the reflection operator satisfying:

$$Rf(x) = f(-x), \quad R^2 = 1. \quad (4.23)$$

This operator encodes both differential and reflection properties and appears naturally in systems with parity symmetry.

Starting from the standard Klein-Gordon equation:

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + \frac{m^2 c^2}{\hbar^2} \right) \psi(x, t) = 0, \quad (4.24)$$

we introduce the Dunkl deformation by replacing:

$$\frac{\partial}{\partial x} \rightarrow D_x^{(\nu)}. \quad (4.25)$$

Thus, the deformed Klein-Gordon equation becomes:

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - (D_x^{(\nu)})^2 + \frac{m^2 c^2}{\hbar^2} \right) \psi(x, t) = 0. \quad (4.26)$$

We compute the square of the Dunkl derivative:

$$(D_x^{(\nu)})^2 = \frac{d^2}{dx^2} + \frac{2\nu}{x} \frac{d}{dx} - \frac{\nu}{x^2} (1 - \hat{R}). \quad (4.27)$$

Therefore, the equation becomes:

$$\left[\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{d^2}{dx^2} - \frac{2\nu}{x} \frac{d}{dx} + \frac{\nu}{x^2} (1 - \hat{R}) + \frac{m^2 c^2}{\hbar^2} \right] \psi(x, t) = 0. \quad (4.28)$$

We assume a separable solution:

$$\psi(x, t) = e^{-iEt/\hbar} \phi(x). \quad (4.29)$$

Substituting into the equation, we obtain:

$$\left[\frac{E^2}{\hbar^2 c^2} - \frac{d^2}{dx^2} - \frac{2\nu}{x} \frac{d}{dx} + \frac{\nu}{x^2} (1 - \hat{R}) + \frac{m^2 c^2}{\hbar^2} \right] \phi(x) = 0. \quad (4.30)$$

and in interaction scalar potential $V(x)$ it transforms as

$$\left[-\frac{d^2}{dx^2} - \frac{2\nu}{x} \frac{d}{dx} + \frac{\nu}{x^2} (1 - \hat{R}) + \frac{(E - eV(x))^2}{\hbar^2 c^2} + \frac{m^2 c^2}{\hbar^2} \right] \phi(x) = 0. \quad (4.31)$$

Since the reflection operator $\hat{R}$ satisfies $\hat{R}\phi = \pm\phi$, we distinguish:

Even states ($\hat{R} = +1$), $(1 - R)\phi_e(x) = 0$, the equation reduces to:

$$\left[\frac{d^2}{dx^2} + \frac{2\nu}{x} \frac{d}{dx} - \frac{(E - eV(x))^2 - m^2 c^4}{\hbar^2 c^2} \right] \phi_e(x) = 0, \quad (4.32)$$

Odd states ($R = -1$), $(1 - R)\phi_o = 2\phi_o$, the equation becomes:

$$\left[\frac{d^2}{dx^2} + \frac{2\nu}{x} \frac{d}{dx} - \frac{2\nu}{x^2} - \frac{(E - eV(x))^2 - m^2 c^4}{\hbar^2 c^2} \right] \phi_o(x) = 0. \quad (4.33)$$

From aboves equations we remark:

- an effective centrifugal-like term $\sim \frac{1}{x^2}$,
- a coupling between parity and dynamics,
- a modification of the kinetic operator.

For $\mu = 0$, the standard Klein-Gordon equation is recovered.

This structure suggests that the Dunkl parameter μ plays the role of a deformation controlling reflection symmetry effects at the quantum level.

4.3 Step potential in the Klein-Gordon equation

Consider a spin-zero particle with rest mass m moving in one dimension under the influence of a step potential $V(x)$ defined as follows:

$$V(x) = \begin{cases} 0 & x < 0 \\ V_0 & x > 0 \end{cases} \quad (47)$$

The particle is incident from Region I ($x < 0$) with total energy E toward the barrier at $x = 0$. Within the framework of relativistic quantum mechanics, this state is described by the Klein-Gordon equation, where the wave function in Region I takes the form of incident and reflected waves. The wave vector k_1 is given by the relativistic relation:

$$k_1^2 = \frac{E^2 - m^2 c^4}{\hbar^2 c^2} \quad (48)$$

In Region II ($x > 0$), the presence of the potential V_0 modifies the wave vector to k_2 according to the relation

$$k_2^2 = \frac{(E - V_0)^2 - m^2 c^4}{\hbar^2 c^2} \quad (49)$$

The behavior of the particle is determined by the strength of the potential V_0 relative to the total energy:

The Case of ($E > V_0 + mc^2$):

The time-independent Klein-Gordon equation in one dimension governs the behavior of the wavefunction $\psi(x)$.

Region I ($x < 0$): In the absence of a potential ($V = 0$), the wave equation is expressed as:

$$\left(\frac{d^2}{dx^2} + k_1^2 \right) \psi_1(x) = 0 \quad (50)$$

where the wave vector k_1 is defined by the relativistic energy-momentum relation:

$$k_1 = \frac{\sqrt{E^2 - m^2 c^4}}{\hbar c} \quad (51)$$

The general solution is a linear combination of incident and reflected plane waves:

$$\psi_1(x) = A e^{ik_1 x} + B e^{-ik_1 x} \quad (52)$$

Region II ($x > 0$): In the presence of the potential $V = V_0$, the equation is modified:

$$\left(\frac{d^2}{dx^2} + k_2^2 \right) \psi_2(x) = 0 \quad (53)$$

Given the condition $E - V_0 > mc^2$, the wave vector k_2 remains real:

$$k_2 = \frac{\sqrt{(E - V_0)^2 - m^2 c^4}}{\hbar c} \quad (54)$$

Assuming no source from the right, the transmitted wave solution is:

$$\psi_2(x) = Ce^{ik_2x} \quad (56)$$

- **Reflection Coefficient (R):**

$$R = \frac{|J_{ref}|}{|J_{inc}|} = \left| \frac{B}{A} \right|^2 = \left(\frac{\mathbf{k}_1 - \mathbf{k}_2}{\mathbf{k}_1 + \mathbf{k}_2} \right)^2 \quad (64)$$

- **Transmission Coefficient (T):** Taking into account the change in the wave vector across the regions:

$$T = \frac{|J_{trans}|}{|J_{inc}|} = \frac{k_2}{k_1} \left| \frac{C}{A} \right|^2 = \frac{k_2}{k_1} \left(\frac{2k_1}{k_1 + k_2} \right)^2 = \frac{4\mathbf{k}_1\mathbf{k}_2}{(\mathbf{k}_1 + \mathbf{k}_2)^2} \quad (65)$$

To ensure the mathematical consistency of the derivation, we verify the conservation of probability:

$$R + T = \frac{(k_1 - k_2)^2 + 4k_1k_2}{(k_1 + k_2)^2} = \frac{k_1^2 - 2k_1k_2 + k_2^2 + 4k_1k_2}{(k_1 + k_2)^2} = \frac{(k_1 + k_2)^2}{(k_1 + k_2)^2} = 1 \quad (66)$$

The result $R + T = 1$ confirms that the probability current is perfectly conserved in this regime. While the qualitative behavior mirrors non-relativistic quantum mechanics, the quantitative values are governed by relativistic kinematics.

The Case of ($|E - V_0| < mc^2$):

This case addresses the scenario where the energy of the incident particle (or the effective energy inside the barrier) falls within the forbidden mass gap, specifically $|E - V_0| < mc^2$. Under these conditions, the behavior of the wavefunction inside the barrier differs fundamentally from the previous case.

Region I ($x < 0$): Where the potential $V = 0$. The equation remains the same as in the previous case, where the solution is oscillatory:

$$\psi_1(x) = Ae^{ik_1x} + Be^{-ik_1x} \quad (67)$$

where $k_1 = \frac{\sqrt{E^2 - m^2c^4}}{\hbar c}$ is the real wave vector.

Region II ($x > 0$): Where the potential $V = V_0$.

Applying the Klein-Gordon equation while considering the condition $|E - V_0| < mc^2$, we find that:

$$\frac{d^2\psi_2}{dx^2} - \rho^2\psi_2 = 0 \quad (68)$$

Here, the magnitude of the wave vector k_2 becomes purely imaginary ($k_2 = i\rho$), where the decay constant ρ is given by:

$$\rho = \frac{\sqrt{m^2c^4 - (E - V_0)^2}}{\hbar c} \quad (69)$$

Since the wavefunction must remain finite as $x \rightarrow \infty$, we choose only the exponentially decaying solution:

$$\psi_2(x) = Ce^{-\rho x} \quad (70)$$

Reflection Coefficient (R): We note that B is the quotient of a complex number by its conjugate, meaning its magnitude is unity:

$$R = \left| \frac{B}{A} \right|^2 = \frac{k_1 - i\rho}{k_1 + i\rho} \cdot \frac{k_1 + i\rho}{k_1 - i\rho} = \frac{k_1^2 + \rho^2}{k_1^2 + \rho^2} = 1 \quad (76)$$

Transmission Coefficient (T): Since the wave inside the barrier is an evanescent wave and the probability current associated with purely real functions in the Klein-Gordon equation is zero:

$$J_{trans} = \frac{\hbar}{2mi} \left(\psi_2^* \frac{d\psi_2}{dx} - \psi_2 \frac{d\psi_2^*}{dx} \right) = 0 \quad (78)$$

Therefore:

$$T = 0 \quad (79)$$

The results show that in this regime, total reflection of the incident particles occurs ($R = 1$). Although there is a non-zero probability of finding the particle inside the forbidden region ($x > 0$) due to the decaying exponential function, there is no net flux of particles across the barrier. This represents the relativistic extension of "quantum tunneling," where the probability current is conserved and the step acts as a perfect mirror.

Remark:

The same results we found in Dunkl derivative.

4.4 Dunkl Klein-Gordon Scattering by Symmetric Barrier

By applying the same steps as in the non-relativistic case we find

A- The case $E > V_0 + m_0 c^2$:

1- Outside the barrier ($|x| > a$) : For an even function ψ_e , one has $R\psi_e = \psi_e$, hence $(1-R)\psi_e = 0$, and $D_\mu \psi_e = \psi'_e$. However, when squaring the Dunkl derivative, one must apply D_μ to ψ'_e , not to ψ_e . Since the derivative of an even function is odd, we have $R(\psi'_e) = -\psi'_e$. Therefore, $(1-R)\psi'_e = 2\psi'_e$. It follows that

$$D_\mu^2 \psi_e = D_\mu(\psi'_e) = \psi''_e + \frac{\mu}{x}(1-R)\psi'_e = \psi''_e + \frac{2\mu}{x}\psi'_e. \quad (4.34)$$

Substituting into the even sector Dunkl-KG equation yields

$$\psi''_e + \frac{2\mu}{x}\psi'_e + \frac{1}{\hbar^2 c^2} [(E - V(x))^2 - m_0^2 c^4] \psi_e = 0. \quad (4.35)$$

For an odd function ψ_o , one has $R\psi_o = -\psi_o$, hence $(1 - R)\psi_o = 2\psi_o$. Therefore $D_\mu\psi_o = \psi'_o + \frac{2\mu}{x}\psi_o$. Now note that ψ'_o is even, and ψ_o/x is also even. Hence $D_\mu\psi_o$ is even. Therefore, when acting once more with D_μ , the reflection part does not contribute. Expanding the derivative gives

$$D_\mu^2\psi_o = \psi''_o + \frac{2\mu}{x}\psi'_o - \frac{2\mu}{x^2}\psi_o. \quad (4.36)$$

The odd-sector for Dunkl KG equation becomes

$$\psi''_o + \frac{2\mu}{x}\psi'_o + \left[\frac{(E - V(x))^2 - m_0^2 c^4}{\hbar^2 c^2} - \frac{2\mu}{x^2} \right] \psi_o = 0 \quad (4.37)$$

One has $V(x) = 0$, so Dunkl-KG equation for even and odd sector becomes, respectively

$$\psi''_e(x) + \frac{2\mu}{x}\psi'_e(x) + k_{KG}^2\psi_e(x) = 0. \quad (4.38)$$

and

$$\psi''_o(x) + \frac{2\mu}{x}\psi'_o(x) + \left(k_{KG}^2 - \frac{2\mu}{x^2} \right) \psi_o(x) = 0 \quad (4.39)$$

where $k_{KG} = \sqrt{\frac{E^2 - m_0^2 c^4}{\hbar c}}$. We introduce the transformation $\psi_{e,o}(x) = |x|^{\frac{1}{2}-\mu} u_{e,o}(x)$. Then we obtain

$$u''_e(x) + \frac{1}{x}u'_e(x) + \left[k_{KG}^2 - \frac{\nu_e^2}{x^2} \right] u_e(x) = 0. \quad (4.40)$$

and odd sector equation

$$u''_e(x) + \frac{1}{x}u'_e(x) + \left[k_{KG}^2 - \frac{\nu_o^2}{x^2} \right] u_e(x) = 0. \quad (4.41)$$

This is Bessel's equation of order $\nu_e = \mu - \frac{1}{2}$ for even case, and $\nu_o = \mu + \frac{1}{2}$ for odd case.

Therefore the general even and odd solutions are

$$\psi_e^{\text{out}}(x) = |x|^{\frac{1}{2}-\mu} \left[A_e J_{\mu-\frac{1}{2}}(k_{KG} |x|) + B_e Y_{\mu-\frac{1}{2}}(k_{KG} |x|) \right]. \quad (4.42)$$

and

$$\psi_o^{\text{out}}(x) = \text{sgn}(x) |x|^{\frac{1}{2}-\mu} \left[A_o J_{\mu+\frac{1}{2}}(k_{KG} |x|) + B_o Y_{\mu+\frac{1}{2}}(k_{KG} |x|) \right]. \quad (4.43)$$

2- Inside the barrier ($|x| < a$): One has $V(x) = V_0$, so Dunkl-KG equation becomes

$$\psi''_e(x) + \frac{2\mu}{x}\psi'_e(x) + q_{KG}^2\psi_e(x) = 0. \quad (4.44)$$

and

$$\psi''_o(x) + \frac{2\mu}{x}\psi'_o(x) + \left(q_{KG}^2 - \frac{2\mu}{x^2} \right) \psi_o(x) = 0 \quad (4.45)$$

where $q_{KG} = \frac{\sqrt{(E-V_0)^2 - m_0^2 c^4}}{\hbar c}$, the solutions are

$$\psi_e^{\text{in}}(x) = |x|^{\frac{1}{2}-\mu} \left[C_e J_{\mu-\frac{1}{2}}(q_{KG} |x|) + D_e Y_{\mu-\frac{1}{2}}(q_{KG} |x|) \right]. \quad (4.46)$$

and

$$\psi_o^{\text{in}}(x) = \text{sgn}(x) |x|^{\frac{1}{2}-\mu} \left[C_o J_{\mu+\frac{1}{2}}(q_{KG} |x|) + D_o Y_{\mu+\frac{1}{2}}(q_{KG} |x|) \right]. \quad (4.47)$$

Since Y_ν is singular at the origin, regularity at $x = 0$ requires ($D_e = D_o = 0$) :

$$\psi_e^{\text{in}}(x) = C_e x^{\frac{1}{2}-\mu} J_{\mu-\frac{1}{2}}(q_{KG} x), \quad \psi_o^{\text{in}}(x) = C_o x^{\frac{1}{2}-\mu} J_{\mu+\frac{1}{2}}(q_{KG} x), \quad x \geq 0. \quad (4.48)$$

For $E < V_0 - m_0 c^2$, one uses modified Bessel functions:

$$\psi_e^{\text{in}}(x) = C_e x^{\frac{1}{2}-\mu} I_{\mu-\frac{1}{2}}(\kappa_{KG} x), \quad \psi_o^{\text{in}}(x) = C_o x^{\frac{1}{2}-\mu} I_{\mu-\frac{1}{2}}(\kappa_{KG} x), \quad x \geq 0. \quad (4.49)$$

with $\kappa = \frac{\sqrt{(V_0-E)^2 - m_0^2 c^4}}{\hbar c}$.

It is sufficient to solve the problem on the half-line $x > 0$ and extend the solution to the full line by parity.

1- For even channel:

1-A Exterior solution: Using the large-argument asymptotics

$$J_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right), \quad (4.50)$$

$$Y_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right), \quad (4.51)$$

we obtain

$$\cos \delta J_\nu(z) - \sin \delta Y_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4} + \delta\right). \quad (4.52)$$

Therefore,

$$\psi_e^{\text{out}}(x) \sim x^{-\mu} \cos\left(kx - \frac{(\mu - \frac{1}{2})\pi}{2} - \frac{\pi}{4} + \delta_e\right), \quad x > a. \quad (4.53)$$

and

$$\psi_o^{\text{out}}(x) \sim x^{-\mu} \cos\left(kx - \frac{(\mu + \frac{1}{2})\pi}{2} - \frac{\pi}{4} + \delta_o\right), \quad x > a.. \quad (4.54)$$

Hence δ_e and δ_o are the even and odd scattering phase shift.

1-B Interior solution: Inside the barrier, the general solution is given by

$$\psi_e^{\text{in}}(x) = A x^{\frac{1}{2}-\mu} J_{\nu_e}(q_{KG} x), \quad \nu_e = \mu - \frac{1}{2}. \quad (4.55)$$

We derive the phase-shift formula associated with the even-parity scattering channel by imposing the matching conditions between the interior and exterior solutions at the boundary point $x = a$. From above, we impose

$$\psi_e^{\text{out}}(a) = \psi_e^{\text{in}}(a), \quad (4.56)$$

where

$$\psi_e^{\text{in}}(x) = A x^{\frac{1}{2}-\mu} J_{\nu_e}(q_{KG} x), \quad \nu_e = \mu - \frac{1}{2}. \quad (4.57)$$

Outside the interaction region, the asymptotic form of the even solution is written as

$$\psi_e^{\text{out}}(x) = B x^{-\mu} \cos \left(k_{KG} x - \frac{(\mu - \frac{1}{2}) \pi}{2} - \frac{\pi}{4} + \delta_e \right). \quad (4.58)$$

For notational simplicity, we introduce $\alpha = \frac{1}{2} - \mu$, so that the previous expressions may be rewritten as

$$\psi_e^{\text{in}}(x) = A x^\alpha J_{\nu_e}(q_{KG} x), \quad \psi_e^{\text{out}}(x) = B x^{\alpha - \frac{1}{2}} \cos \Phi_e(x), \quad (4.59)$$

with the phase function

$$\Phi_e(x) = k_{KG} x - \frac{(\mu - \frac{1}{2}) \pi}{2} - \frac{\pi}{4} + \delta_e. \quad (4.60)$$

At the matching point $x = a$, we define

$$\Phi_e = k_{KG} a - \frac{(\mu - \frac{1}{2}) \pi}{2} - \frac{\pi}{4} + \delta_e. \quad (4.61)$$

The first matching condition requires continuity of the wave function at the boundary,

$$\psi_e^{\text{in}}(a) = \psi_e^{\text{out}}(a). \quad (4.62)$$

Substituting the explicit forms of the two solutions yields

$$A a^\alpha J_{\nu_e}(q_{KG} a) = B a^{\alpha - \frac{1}{2}} \cos \Phi_e. \quad (4.63)$$

After canceling the common factor a^α , we obtain

$$A \sqrt{a} J_{\nu_e}(q_{KG} a) = B \cos \Phi_e. \quad (4.64)$$

This relation constitutes the first matching equation.

We now differentiate the interior wave function,

$$\psi_e^{\text{in}}(x) = A x^\alpha J_{\nu_e}(q_{KG} x), \quad (4.65)$$

by applying the product rule:

$$\psi_e^{\text{in}'}(x) = A [x^\alpha J_{\nu_e}(q_{KG} x)]'. \quad (4.66)$$

Hence,

$$\psi_e^{\text{in}'}(x) = A [\alpha x^{\alpha-1} J_{\nu_e}(q_{KG} x) + x^\alpha J'_{\nu_e}(q_{KG} x)]. \quad (4.67)$$

Using the chain rule,

$$\frac{dJ_{\nu_e}(q_{KG} x)}{dx} = q \frac{dJ_{\nu_e}(z)}{dz}, \quad \text{with } z = q_{KG} x \quad (4.68)$$

we arrive at

$$\psi_e^{\text{in}'}(x) = A \left[\alpha x^{\alpha-1} J_{\nu_e}(q_{KG}x) + q_{KG} x^{\alpha} \frac{dJ_{\nu_e}(z)}{dz} \right]. \quad (4.69)$$

Evaluating this expression at $x = a$ gives

$$\psi_e^{\text{in}'}(a) = A a^{\alpha-1} \left[\alpha J_{\nu_e}(q_{KG}a) + q_{KG}a \frac{dJ_{\nu_e}(z)}{dz} \Big|_{z=q_{KG}a} \right]. \quad (4.70)$$

We next differentiate the exterior wave function,

$$\psi_e^{\text{out}}(x) = B x^{\alpha-\frac{1}{2}} \cos \Phi_e(x), \quad (4.71)$$

where

$$\Phi_e(x) = k_{KG}x - \frac{(\mu - \frac{1}{2})\pi}{2} - \frac{\pi}{4} + \delta_e. \quad (4.72)$$

Applying the product rule once again, we obtain

$$\psi_e^{\text{out}'}(x) = B \left[\left(\alpha - \frac{1}{2} \right) x^{\alpha-\frac{3}{2}} \cos \Phi_e(x) + x^{\alpha-\frac{1}{2}} \cos' \Phi_e(x) \right]. \quad (4.73)$$

Since

$$\cos' \Phi_e(x) = -\sin \Phi_e(x) \Phi_e'(x), \quad (4.74)$$

and because $\Phi_e'(x) = k$, it follows that

$$\cos' \Phi_e(x) = -k_{KG} \sin \Phi_e(x). \quad (4.75)$$

Therefore,

$$\psi_e^{\text{out}'}(x) = B \left[\left(\alpha - \frac{1}{2} \right) x^{\alpha-\frac{3}{2}} \cos \Phi_e(x) - k_{KG} x^{\alpha-\frac{1}{2}} \sin \Phi_e(x) \right]. \quad (4.76)$$

At $x = a$, one finds

$$\psi_e^{\text{out}'}(a) = B a^{\alpha-\frac{3}{2}} \left[\left(\alpha - \frac{1}{2} \right) \cos \Phi_e - k_{KG}a \sin \Phi_e \right]. \quad (4.77)$$

The second matching condition imposes continuity of the derivative at $x = a$:

$$\psi_e^{\text{in}'}(a) = \psi_e^{\text{out}'}(a). \quad (4.78)$$

Substituting Eqs. (4.70) and (4.77) into this condition, we obtain $\psi_e^{\text{in}'}(a) = A a^{\alpha-1} \left[\alpha J_{\nu_e}(q_{KG}a) + q_{KG}a \frac{dJ_{\nu_e}(z)}{dz} \right]$

$$A a^{\alpha-1} \left[\alpha J_{\nu_e}(q_{KG}a) + q_{KG}a \frac{dJ_{\nu_e}(z)}{dz} \Big|_{z=q_{KG}a} \right] = B a^{\alpha-\frac{3}{2}} \left[\left(\alpha - \frac{1}{2} \right) \cos \Phi_e - k_{KG}a \sin \Phi_e \right]. \quad (4.79)$$

After canceling the common factor $a^{\alpha-1}$, this becomes

$$A \left[\alpha J_{\nu_e}(q_{KG}a) + q_{KG}a \frac{dJ_{\nu_e}(z)}{dz} \Big|_{z=q_{KG}a} \right] = \frac{B}{\sqrt{a}} \left[\left(\alpha - \frac{1}{2} \right) \cos \Phi_e - k_{KG}a \sin \Phi_e \right]. \quad (4.80)$$

We now combine Eq. (4.80) with the first matching condition, Eq. (4.64),

$$A\sqrt{a} J_{\nu_e}(q_{KG}a) = B \cos \Phi_e. \quad (4.81)$$

in order to eliminate the ratio of the constants A and B .

Dividing Eq. (4.80) by Eq. (4.64), we obtain

$$\frac{A \left[\alpha J_{\nu_e}(q_{KG}a) + q_{KG}a \frac{dJ_{\nu_e}(z)}{dz} \Big|_{z=q_{KG}a} \right]}{A\sqrt{a} J_{\nu_e}(q_{KG}a)} = \frac{\frac{B}{\sqrt{a}} \left[\left(\alpha - \frac{1}{2} \right) \cos \Phi_e - k_{KG}a \sin \Phi_e \right]}{B \cos \Phi_e}. \quad (4.82)$$

This immediately gives

$$\frac{\left[\alpha J_{\nu_e}(q_{KG}a) + q_{KG}a \frac{dJ_{\nu_e}(z)}{dz} \Big|_{z=q_{KG}a} \right]}{J_{\nu_e}(q_{KG}a)} = \left[\left(\alpha - \frac{1}{2} \right) - k_{KG}a \tan \Phi_e \right]. \quad (4.83)$$

Expanding the left-hand side, we find

$$\alpha + q_{KG}a \frac{\frac{dJ_{\nu_e}(z)}{dz} \Big|_{z=q_{KG}a}}{J_{\nu_e}(q_{KG}a)} = \left(\alpha - \frac{1}{2} \right) - k_{KG}a \tan \Phi_e. \quad (4.84)$$

Subtracting α from both sides leads to

$$\frac{1}{2} + q_{KG}a \frac{\frac{dJ_{\nu_e}(z)}{dz} \Big|_{z=q_{KG}a}}{J_{\nu_e}(q_{KG}a)} = -k_{KG}a \tan \Phi_e. \quad (4.85)$$

Assuming $a \neq 0$, division by $k_{KG}a$ yields

$$\frac{1}{2k_{KG}a} + \frac{q_{KG}}{k_{KG}} \frac{\frac{dJ_{\nu_e}(z)}{dz} \Big|_{z=q_{KG}a}}{J_{\nu_e}(q_{KG}a)} = -\tan \Phi_e. \quad (4.86)$$

Finally, by using the definition of Φ_e given in Eq. (4.61), we arrive at the phase-shift formula

$$\tan \left(k_{KG}a - \frac{(\mu - \frac{1}{2})\pi}{2} - \frac{\pi}{4} + \delta_e \right) = -\frac{1}{2k_{KG}a} - \frac{q_{KG}}{k_{KG}} \frac{\frac{dJ_{\nu_e}(z)}{dz} \Big|_{z=q_{KG}a}}{J_{\nu_e}(q_{KG}a)}. \quad (4.87)$$

This is the desired expression for the even-parity phase shift.

The odd-channel matching conditions are

$$\psi_o^{\text{out}}(a) = \psi_o^{\text{in}}(a), \quad (4.88)$$

and

$$\psi_o^{\text{out}'}(a) + \frac{2\mu}{a} \psi_o^{\text{out}}(a) = \psi_o^{\text{in}'}(a) + \frac{2\mu}{a} \psi_o^{\text{in}}(a). \quad (4.89)$$

An entirely analogous calculation for the odd-parity sector leads to

$$\tan \left(k_{KG}a - \frac{(\mu + \frac{1}{2})\pi}{2} - \frac{\pi}{4} + \delta_o \right) = \frac{1}{2k_{KG}a} + \frac{q_{KG}}{k_{KG}} \frac{\left. \frac{dJ_{\nu_o}(z)}{dz} \right|_{z=q_{KG}a}}{J_{\nu_o}(q_{KG}a)}, \quad \nu_o = \mu + \frac{1}{2}. \quad (4.90)$$

Once the even- and odd-parity phase shifts are known, the transmission and reflection probabilities are expressed in the standard form

$$T = \cos^2(\delta_e - \delta_o), \quad R = \sin^2(\delta_e - \delta_o). \quad (4.91)$$

To make the dependence on the logarithmic derivatives more explicit, let us define

$$\beta = k_{KG}a - \frac{(\mu - \frac{1}{2})\pi}{2} - \frac{\pi}{4} + \delta_e, \quad (4.92)$$

and

$$\rho = k_{KG}a - \frac{(\mu + \frac{1}{2})\pi}{2} - \frac{\pi}{4} + \delta_o. \quad (4.93)$$

Then,

$$\tan \beta = -\frac{1}{2k_{KG}a} - \frac{q_{KG}}{k_{KG}} \frac{\left. \frac{dJ_{\nu_e}(z)}{dz} \right|_{z=q_{KG}a}}{J_{\nu_e}(q_{KG}a)}, \quad \tan \rho = -\frac{1}{2k_{KG}a} - \frac{q_{KG}}{k_{KG}} \frac{\left. \frac{dJ_{\nu_o}(z)}{dz} \right|_{z=q_{KG}a}}{J_{\nu_o}(q_{KG}a)}. \quad (4.94)$$

or

$$\tan \rho = \cot \bar{\rho} = -\frac{1}{2k_{KG}a} - \frac{q_{KG}}{k_{KG}} \frac{J'_{\nu_o}(q_{KG}a)}{J_{\nu_o}(q_{KG}a)} \quad (4.95)$$

The difference of the two phase shifts may then be analyzed through the trigonometric identity

$$\tan(\beta - \bar{\rho}) = \frac{\tan \beta - \tan \bar{\rho}}{1 + \tan \beta \tan \bar{\rho}} = \frac{\tan \beta - \cot \rho}{1 + \tan \beta \cot \rho} = \frac{\tan \beta - \frac{1}{\tan \rho}}{1 + \frac{\tan \beta}{\tan \rho}}. \quad (4.96)$$

with

$$\beta - \bar{\rho} = \delta_e - \delta_o \quad (4.97)$$

Substituting the above expressions, one obtains an explicit representation of the phase-difference contribution in terms of the Bessel functions and their derivatives. Consequently, the transmission coefficient takes the compact form

$$T = \frac{1}{1 + \tan^2(\delta_e - \delta_o)}, \quad (4.98)$$

with

$$\tan(\delta_e - \delta_o) = \frac{-\frac{1}{2k_{KG}a} - \frac{q_{KG}}{k_{KG}} \frac{\left. \frac{dJ_{\nu_e}(z)}{dz} \right|_{z=q_{KG}a}}{J_{\nu_e}(q_{KG}a)} + \frac{1}{\frac{1}{2k_{KG}a} + \frac{q_{KG}}{k_{KG}} \frac{\left. \frac{dJ_{\nu_o}(z)}{dz} \right|_{z=q_{KG}a}}{J_{\nu_o}(q_{KG}a)}}}{1 + \frac{\frac{1}{2k_{KG}a} + \frac{q_{KG}}{k_{KG}} \frac{\left. \frac{dJ_{\nu_e}(z)}{dz} \right|_{z=q_{KG}a}}{J_{\nu_e}(q_{KG}a)}}{\frac{1}{2k_{KG}a} + \frac{q_{KG}}{k_{KG}} \frac{\left. \frac{dJ_{\nu_o}(z)}{dz} \right|_{z=q_{KG}a}}{J_{\nu_o}(q_{KG}a)}}} \quad (4.99)$$

then

$$T = \frac{1}{1 + \left[\frac{-\frac{1}{2k_{KG}a} - \frac{q_{KG}}{k_{KG}} \frac{\frac{dJ_{\nu_e}(z)}{dz} \Big|_{z=q_{KG}a}}{J_{\nu_e}(q_{KG}a)} + \frac{\frac{1}{2k_{KG}a} + \frac{q_{KG}}{k_{KG}} \frac{\frac{dJ_{\nu_o}(z)}{dz} \Big|_{z=q_{KG}a}}{J_{\nu_o}(q_{KG}a)}}{1 + \frac{\frac{1}{2k_{KG}a} + \frac{q_{KG}}{k_{KG}} \frac{\frac{dJ_{\nu_e}(z)}{dz} \Big|_{z=q_{KG}a}}{J_{\nu_e}(q_{KG}a)}}{\frac{\frac{1}{2k_{KG}a} + \frac{q_{KG}}{k_{KG}} \frac{\frac{dJ_{\nu_o}(z)}{dz} \Big|_{z=q_{KG}a}}{J_{\nu_o}(q_{KG}a)}}} \right]^2} \quad (4.100)$$

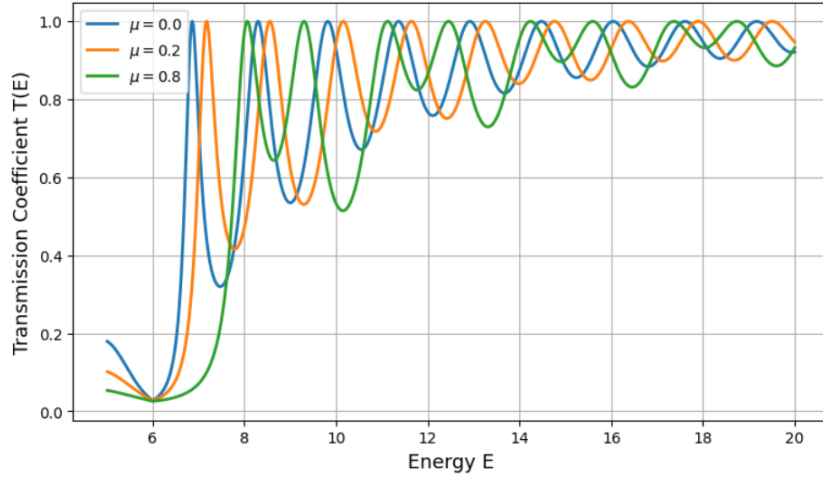


Figure 4.1: Transmission Coefficient $T(E)$ for Different Values of the Dunkl Parameter μ

Table 4.1: Resonance energies and transmission coefficients for different values of the Dunkl parameter μ in the relativistic Klein-Gordon barrier problem.

μ	Resonance Number	Resonance Energy E_r	$T(E_r)$
0.0	1	6.861728	0.999997
0.0	2	8.297379	0.999999
0.0	3	9.815496	0.999991
0.0	4	11.363601	0.999997
0.0	5	12.919202	0.999996
0.0	6	14.478552	0.999999
0.0	7	16.041650	1.000000
0.0	8	17.604749	0.999999
0.0	9	19.171595	1.000000
0.2	1	7.172848	0.999951
0.2	2	8.556022	0.999984
0.2	3	10.152856	1.000000
0.2	4	11.640985	1.000000
0.2	5	13.252813	0.999999
0.2	6	14.763433	0.999999
0.2	7	16.379010	0.999998
0.2	8	17.893378	1.000000
0.2	9	19.508955	1.000000
0.8	1	8.064976	0.999988
0.8	2	9.298212	0.999989
0.8	3	11.119952	0.999999
0.8	4	12.450648	0.999999
0.8	5	14.234904	0.999998
0.8	6	15.595586	1.000000
0.8	7	17.364849	1.000000
0.8	8	18.740525	1.000000

The transmission coefficient curves shown in Fig. ?? together with the resonance energies listed in Table 4.1 reveal a significant influence of the Dunkl deformation parameter μ on the tunneling properties of the system.

For all considered values of μ , the transmission coefficient exhibits an oscillatory behavior characterized by a sequence of resonance peaks where $T(E) \approx 1$. These resonances correspond to quasi-bound states formed inside the potential barrier due to constructive interference of the transmitted wave. At sufficiently high energies, the transmission coefficient

approaches unity, indicating that the barrier becomes almost transparent to the incident particle.

In the case $\mu = 0$, the resonance peaks appear at lower energies and are more densely distributed. The first resonance occurs at $E_r \approx 6.86$, followed by several regularly spaced resonances up to $E_r \approx 19.17$. This behavior indicates a strong resonant tunneling effect in the absence of Dunkl deformation.

When the Dunkl parameter increases to $\mu = 0.2$, the resonance energies shift toward higher values. For example, the first resonance moves from $E_r \approx 6.86$ to $E_r \approx 7.17$. A similar shift is observed for all higher resonances. This demonstrates that the Dunkl deformation modifies the internal wave dynamics and changes the interference conditions inside the barrier region.

For the larger deformation parameter $\mu = 0.8$, the resonance structure becomes smoother and the resonances appear at even higher energies. The first resonance is located near $E_r \approx 8.06$, confirming that increasing μ suppresses low-energy resonant tunneling. Moreover, the spacing between successive resonances becomes slightly larger, reflecting the influence of the Dunkl parameter on the effective wave propagation inside the barrier.

The resonance table also shows that all resonance peaks satisfy $T(E_r) \approx 1$, which indicates nearly perfect transmission at the resonance energies. Therefore, the barrier behaves as fully transparent at these special energies despite the presence of the potential barrier.

Overall, the obtained results clearly demonstrate that the Dunkl parameter μ plays an important role in controlling the resonance spectrum and the transmission properties of the system. Increasing μ shifts the resonance energies toward higher values and modifies the oscillatory structure of the transmission coefficient.

when $\mu = 0$, we recover the standard results: Substituting $k^2 - q^2 = \frac{V_0(2E - V_0)}{\hbar^2 c^2}$:

$$T(E) = \left[1 + \frac{V_0^2(2E - V_0)^2}{4(E^2 - m^2 c^4)((E - V_0)^2 - m^2 c^4)} \sin^2(2qa) \right]^{-1} \quad (105)$$

Overall, the phase shift method provides a clear physical interpretation of the transmission process, where the role of μ is to control the interference pattern and consequently the resonance structure of the system. while the reflection coefficient is given by

$$R = 1 - T. \quad (4.101)$$

Therefore, the unitarity condition is automatically satisfied:

$$R + T = 1. \quad (4.102)$$

4.5 Bound States for Dunkl–Klein–Gordon Formalisme in Hulthn Potential

The time-independent Klein-Gordon equation for a spin-0 particle of mass m under a scalar potential $V(x)$ is given by [13]:

$$[D_\nu^2 + (E - V(x))^2 - m^2] \Psi(x) = 0 \quad (4.103)$$

where D_ν is the Dunkl operator:

$$D_\nu = \frac{d}{dx} + \frac{\nu}{x} - \frac{\nu}{x} R \quad (4.104)$$

We consider the symmetrized Hulthn potential, which is defined as:

$$V(x) = \frac{\alpha}{\exp(|x|) - 1}, \quad \alpha < 0 \quad (4.105)$$

By substituting the explicit form of the Dunkl Laplacian D_ν^2 and using the parity condition $R\Psi(x) = \delta\Psi(x)$, where $\delta = \pm 1$, we obtain:

$$\Psi''(x) + \frac{2\nu}{x}\Psi'(x) + \left[E^2 - m^2 - 2EV(x) + V(x)^2 + \frac{\delta\nu - \nu^2}{x^2} \right] \Psi(x) = 0 \quad (4.106)$$

To eliminate the first-order derivative, we apply the transformation $\Psi(x) = x^{-\nu}\xi(x)$. The reduced equation becomes:

$$\xi''(x) + \left[\epsilon_{rel} - \frac{2E\alpha}{e^x - 1} + \frac{\alpha^2}{(e^x - 1)^2} + \frac{\delta\nu - \nu^2}{x^2} \right] \xi(x) = 0 \quad (4.107)$$

To ensure that the bound state solutions Ψ are physical, they must be normalizable in the weighted Hilbert space $L_\omega^2(\mathbb{R})$ with the weight function $\omega = |x|^{2\nu}$. The normalization condition is expressed as:

$$\|\Psi\| = \sqrt{\int_{-\infty}^{\infty} \Psi^*(x)\Psi(x)|x|^{2\nu}dx} = \sqrt{\int_{-\infty}^{\infty} \xi^*(x)\xi(x)dx} \quad (4.108)$$

In the framework of the Dunkl-Klein-Gordon equation with a symmetrized Hulthn potential, the relativistic effective potential V_{eff} is defined as:

$$V_{eff}(x) = \frac{2E\alpha}{e^{|x|} - 1} - \frac{\alpha^2}{(e^{|x|} - 1)^2} - \frac{\delta\nu - \nu^2}{x^2} \quad (4.109)$$

Since the equation involves an inverse quadratic term $1/x^2$ which precludes an exact analytical solution, we adopt an exponential approximation based on the Taylor expansion around the origin:

$$\frac{1}{x^2} \approx \frac{1}{(e^x - 1)^2} + \frac{1}{e^x - 1} \quad (4.110)$$

Substituting this into the reduced Dunkl-Klein-Gordon equation leads to the final solvable differential equation:

$$\xi''(x) + \left[(E^2 - m^2) + \frac{\delta\nu - \nu^2 + \alpha^2}{(e^x - 1)^2} + \frac{\delta\nu - \nu^2 - 2E\alpha}{e^x - 1} \right] \xi(x) = 0 \quad (4.111)$$

where $E^2 - m^2$ represents the relativistic energy term. Accordingly, the approximated effective potential is given by:

$$V_{approx}(x) = - \left(\frac{\delta\nu - \nu^2 + \alpha^2}{(e^x - 1)^2} + \frac{\delta\nu - \nu^2 - 2E\alpha}{e^x - 1} \right) \quad (4.112)$$

We derive the analytical bound-state solutions for the Dunkl-Klein-Gordon equation. Since the system is invariant under the transformation $x \rightarrow -x$, we focus our analysis on the region $x \geq 0$.

To map the relativistic differential equation into a standard hypergeometric form, we introduce the following coordinate transformation and wavefunction ansatz:

$$y(x) = e^x, \quad \xi(x) = y^C (y - 1)^D \eta(y) \quad (4.113)$$

Upon substituting 4.113 into the governing relativistic equation, the function $\eta(y)$ satisfies the Gauss Hypergeometric Equation:

$$y(1 - y)\eta''(y) + [H - (F + G + 1)y]\eta'(y) - FG\eta(y) = 0 \quad (19)$$

The coefficients C, D, F, G , and H characterize the relativistic system. For the Dunkl-Klein-Gordon framework, these constants incorporate the rest mass m , the energy E , and the potential parameters α and ν . Specifically, the behavior at infinity and near the origin is governed by:

$$C = \sqrt{m^2 - E^2}, \quad D = \frac{1}{2} + \frac{1}{2}\sqrt{1 + 4(\delta\nu - \nu^2 + \alpha^2)} \quad (23-24)$$

and

$$a = -\delta\nu + \nu^2 - \alpha^2 \quad (20)$$

and

$$A = \sqrt{(\epsilon + 2E\alpha)(-1 - 4a + 4\epsilon + 4\sqrt{-\epsilon - 4(2E\alpha)\epsilon})} \quad (21)$$

and

$$B = \sqrt{1 + 4a - 8\epsilon - 4\sqrt{-\epsilon - 4(2E\alpha)\epsilon} - 4(2E\alpha) - 4A} \quad (22)$$

and

$$H = [16\alpha^2 + 16\epsilon + (1 + 4\alpha)^2 + 8(\alpha)(1 + 8\epsilon + 4(\alpha))]^{-1} \quad (23)$$

and

$$D = \frac{1}{2} - \frac{1}{2} \sqrt{1 + 4a - 4\alpha - 4\sqrt{-\alpha - 4\alpha a}} \quad (24)$$

and

$$F = \frac{1}{2} - \frac{B}{2} \quad (25)$$

and

$$G = \frac{\{64\alpha^3(B-1) + (1+4\alpha)[-1+B+4B\sqrt{-\varepsilon-4\alpha\varepsilon}] + 8AB\sqrt{-\varepsilon-4\alpha\varepsilon} + 4\alpha(B-1)\}}{(2+8a+8\varepsilon)\{16\alpha^2+16\varepsilon+[1+4\alpha]^2+8\alpha(1+8\varepsilon+4\alpha)\}} \quad (26)$$

Inspection of Eq. (2.149) reveals it has the form of the hypergeometric equation. Consequently, a particular solution can be written as

$$\eta(y) = {}_2F_1(F, G, H, y), \quad (4.114)$$

where ${}_2F_1$ stands for the hypergeometric function. In combination with the coordinate change 4.113, we can obtain a particular solution for $x \geq 0$ as

$$\xi(x) = \exp(x)^C [\exp(x) - 1]^D {}_2F_1[F, G, H, \exp(x)]. \quad (4.115)$$

The reason for considering the particular solution 4.115 rather than the general solution is that we are interested in constructing bound state solutions. To obtain a solution of the actual Dunkl–Klein–Gordon equation, we substitute 4.115 into the transformation. This gives for $x \geq 0$

$$\Psi(x) = x^{-\nu} \exp(x)^C [\exp(x) - 1]^D {}_2F_1[F, G, H, \exp(x)]. \quad (4.116)$$

We extract the bound state energies $E_n, n = 0, 1, 2, \dots$ from 4.116 by requiring that the parameter G equals a nonpositive integer:

$$G = -n, \quad (4.117)$$

where $n = 0, 1, 2, \dots$. By means of 4.117, the hypergeometric function degenerates to a polynomial of degree n in the exponential.

In general, 4.116 is unbounded at infinity. But if 4.116 is a polynomial due to 4.117, its behavior at infinity is controlled by the exponent C . More precisely, boundedness at infinity is guaranteed if

$$C + n < 0. \quad (4.118)$$

To ensure that 4.116 does not have a singularity at the origin, we expand the terms around the origin. This gives

$$x^{-\nu} [\exp(x) - 1]^D = x^{D-\nu} + \frac{D}{2} x^{D-\nu+1} + o(x^{D-\nu+2}). \quad (4.119)$$

Consequently, a sufficient condition for 4.116 to be non-singular at the origin reads

$$D - \nu \geq 0, \quad (4.120)$$

where D is the relativistic constant.

The solution is normalizable if the integrand is bounded at the origin, which is given if we have

$$D \geq 0. \quad (4.121)$$

Now we can establish the parity of the function 4.116 by suitable continuation into the region $x < 0$. Starting with odd-parity solutions, these are defined by

$$\Psi_{o,n} = \begin{cases} \Psi(x)|_{\delta=-1, E=E_n} & \text{if } x \geq 0 \\ -\Psi(-x)|_{\delta=-1, E=E_n} & \text{if } x < 0 \end{cases} \quad (4.122)$$

Recall that $E_n, n = 0, 1, 2, \dots$, stand for stationary energies obtained from solving the condition 4.117. Similar to 4.122, we define even-parity solutions as follows:

$$\Psi_{e,n} = \begin{cases} \Psi(x)|_{\delta=1, E=E_n} & \text{if } x \geq 0 \\ \Psi(-x)|_{\delta=1, E=E_n} & \text{if } x < 0 \end{cases} \quad (4.123)$$

In summary, the functions 4.122 and 4.123 represent bound state solutions of the Dunkl-Klein-Gordon equation if the energies $E_n, n = 0, 1, 2, \dots$, satisfy condition 4.117, if the regularity conditions 4.118, 4.120 are fulfilled, and if the normalizability criterion 4.121 is met

Chapter 5

General Conclusion

In conclusion, this work has addressed the generalization of quantum mechanics through the introduction of the Dunkl derivative as a mathematical tool that incorporates reflection symmetry into the algebraic structure of quantum systems. This generalization has led to a reformulation of the fundamental relations, particularly the commutation relation between position and momentum, resulting in new physical features that differ from those of standard quantum mechanics.

Through the analysis of several physical systems in both non-relativistic and relativistic regimes, it has been shown that the inclusion of the Dunkl derivative induces significant modifications in the energy spectrum, wave functions, as well as in transmission coefficients and scattering phenomena. Moreover, the results demonstrate that this formalism preserves the analytical solvability of many studied models, making it a rich and powerful framework for investigating more complex systems.

Furthermore, the comparison with conventional quantum mechanics confirms that the latter is recovered as a limiting case when the deformation parameter vanishes, thereby ensuring the consistency of this generalized approach with well-established physical principles. This highlights the importance of algebraic deformations and symmetries in the development of modern theoretical physics and opens the door to promising research directions.

In this context, Physics-Informed Neural Networks (PINNs) represent a particularly powerful and emerging approach that can be employed to study Dunkl-modified quantum systems, especially in cases where exact analytical solutions are difficult or impossible to obtain. By embedding the governing differential equations directly into the learning process, PINNs enable efficient numerical solutions of modified Schrödinger and Klein–Gordon equations within the Dunkl framework. They also provide a flexible tool to investigate the effects of the deformation parameter in higher-dimensional systems, in the presence of external fields, or under complex boundary conditions.

Therefore, PINNs offer a promising bridge between analytical and numerical approaches,

paving the way for a deeper exploration of Dunkl-based quantum systems. This synergy between advanced mathematical formalisms and modern computational techniques opens new perspectives for both theoretical investigations and practical applications.

Overall, this work constitutes a further step toward a deeper understanding of generalized quantum mechanics and emphasizes the crucial role of modern mathematical tools, such as the Dunkl derivative, alongside advanced numerical methods, in expanding the frontiers of research in theoretical physics.

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الكلمات المفتاحية (Keywords / Mots-clés)

قائمة الكلمات المفتاحية باللغات العربية، الإنجليزية، والفرنسية

الكلمات المفتاحية باللغة العربية

- مشتقة دونكل (Dunkl Derivative)
- الميكانيكا الكمومية المجهزة / المعدلة (Wigner-Dunkl Quantum Mechanics)
- مؤثر الانعكاس / التناظر التماثلي (Reflection Operator / Parity)
- الحالات المرتبطة (Bound States)
- ظواهر التشتت / التناثر (Scattering Phenomena)
- المتذبذب التوافقي (Harmonic Oscillator)
- حاجز وقدم الجهد (Potential Barrier and Step)
- جهد بوشل-تيلر (Pöschl-Teller Potential)
- معادلة كلاين-غوردون النسبية (Relativistic Klein-Gordon Equation)
- معامل التشويه / التعديل (Deformation Parameter)

Keywords in English

- Dunkl derivative
- Wigner-Dunkl quantum mechanics
- Reflection operator / Parity symmetry
- Bound states
- Scattering theory / Transmission coefficient
- Dunkl-harmonic oscillator

- Potential step & Potential barrier
- Pöschl-Teller potential
- Relativistic Klein-Gordon equation
- Deformation parameter

Mots-clés en Français

- Dérivée de Dunkl
- Mécanique quantique de Wigner-Dunkl
- Opérateur de réflexion / Symétrie de parité
- États liés
- Phénomènes de diffusion / Coefficient de transmission
- Oscillateur harmonique de Dunkl
- Marche et barrière de potentiel
- Potentiel de Pöschl-Teller
- Équation relativiste de Klein-Gordon
- Paramètre de déformation