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Dedication

All praise to Allah, today we fold the days' tiredness and the errand summing up between the cover of this humble work.
To the spring that never stops giving, who weaves my happiness with strings from her merciful heart... to my mother
To whom he strives to bless comfort and welfare and never stints what he owns to push me in the success way who taught me to promote life stairs wisely and patiently,
to my dearest father "**Mohammed**" Allah's mercy.

To whose love flows in my veins, and my heart always remembers them, to my brothers and sisters.
To my dear friends.
To those who taught us letters of gold and words of jewel of the utmost and sweetest sentences in the whole knowledge, who reworded to us their knowledge simply and from their thoughts made a lighthouse guides us through the knowledge and success path, to our honoured teachers and professors.

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Iwant to thank all colleagues who accompanied me during the years of study

Thank you all help us in one way or another.

Thank you for all.

Preliminaries and Notations

$\phi^i = \phi(t_i)$

$u = (u_i)$ vector with compost i

n unit outer normal .

$u_n = u.n$

$\partial_n u = \nabla u.n = N.n + T$

$N = \partial_n u.n$

$T = \partial_n.u - N.n$ tangent

$\dot{u} = \dfrac{\partial u}{\partial t}$

$\ddot{u} = \dfrac{\partial^2 u}{\partial t^2}$

H' dual of H

$\mathbf{H^1(\Omega)} = \left(H^1(\Omega)\right)^3$

$\mathbf{L^1(\Omega)} = \left(L^1(\Omega)\right)^3$

$\rightarrow$ strong convergence.

$\rightharpoonup$ Weak convergence .

Contents

Introduction

The problem of dynamic unilateral contact with friction has been examined in may fields, as well as in many aspects of every day life. For this reason, in recent years they have been widely studied considering various constitutive laws and boundary conditions. Only some examples are the work of Kikuchi and Oden [1] as well as Chau and al.[2] Laursan and Chawla [3], Bencache and al.[4] or Khenous and al.[5]. J.Kim [6] shows existence of weak solutions of the obstacle problem for a wave equation. In [7], M.Cocou use regularization of contact stress and constant friction coeffition to show existence of weak solution to the dynamic signorini problem with constant coefficient of friction.
In this memoir we will study the dynamic problem two cases Static case and dynamic case. We begin our work by a chapter presenting mathematical preliminaries witch will be used in the next charters. In the second chapter we study the static case of the simplified signorini problem for proved view existence and uniqueness of solution. In the next chapter we study the dynamic case of signorini problem with theorem existence. Finally we conclude our work with a conclusion involving the main results and respective.

Chapter 1

Notations and mathematical preliminaries

1.1 Reminders on functional spaces

All results in this section are in [8]. We recall below some definitions and theorems of classical functional analysis that will be used in subsequent chapters. Here all functions considered are real-value. Let $x \in \mathbb{R}^n$, Ω open subset of $\mathbb{R}^n$, $K \subset \Omega$, n positive integer, α is a multi-integer, $|\alpha| = \sum_{i=1}^{n} \alpha_i$ then we define the differential operator D^α by

$$D^\alpha = D_1^{\alpha_1} .... D_n^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} ... x_n^{\alpha_n}}$$

we denote by $C(\Omega)$ The space of continuous real functions in Ω. We say that K is relatively compact in Ω, if the adhesion of K (i.e $\bar{K}$) is a compact (i.e Closed and bounded) included in Ω noted by $K \subset\subset \Omega$. Also it notes

$$C^m(\Omega) = \{v \in C(\Omega) D^\alpha v \in C(\Omega) \ for \ \ |\alpha| \leq m\} .$$

We call the support of the function v define in closed set Ω

$$suppv = \overline{\{x \in \Omega; v(x) \neq 0\}}.$$

We said that the function v has compact support in Ω. We denote by

$$C_0^m(\Omega) = \{v \in C(\Omega), v \ a \ compact \ support \ in \ \Omega\}$$

Let

$$C^{\infty} = \cap_{m=0}^{\infty} C^{m}(\Omega).$$

We denote by $D(\Omega)$ Called the space of test functions, the space $C_0^{\infty}(\Omega)$. Functions infinitely differentiable with compact support in Ω with the topology boundary inductive as in the theory of distributions of L. Schwartz. We note by $D'(\Omega)$ the dual space of $D(\Omega)$ So the space of continuous linear forms on $D(\Omega)$, called $D'(\Omega)$ The space of distributions(or generalized functions) on Ω and we equip the strong dual topology(i.e $f_i \longrightarrow f$ in $D'(\Omega)$ if $\langle f_i, \varphi\rangle \longrightarrow \langle f, \varphi\rangle \, \forall \varphi \in D(\Omega))$ where $\langle .,.\rangle$ is the product of duality between $D(\Omega)$ and $D'(\Omega)$. For p given by $1 \leq p \leq \infty$ designates by

$$L^p(\Omega) = \left\{ v \; measurable \; on \; \Omega; such \; that \; \|v\|_p = \left(\int_{\Omega} |v|^p\right)^{1/p} < \infty \right\}$$

we recall that $\left(L^p(\Omega), \|.\|_p\right)$ is a Banach space, and separable, for $1 < p \leq \infty$ is reflexive. For $p = 2$, the space $L^2(\Omega)$ is a Hilbert space equipped with the scalar product

$$\langle u, v\rangle = \int_{\Omega} u(x)v(x)dx.$$

We will identify the space $L^2(\Omega)$ to its dual. For $p = \infty$ is denoted by

$$L^{\infty} = \{v \; measurable \; on \; \Omega; such \; that \; |v|_{\infty} < \infty\},$$

where $|v|_{\infty} = \{supess_{x\in\Omega} |v(x)| = inf \{C; |v(x)| \;\; c.p.d \; x \in \Omega\}\}$ we recall that $(L^{\infty}, \|\|_{\infty})$ is a Banach space. For all $1 < p < \infty$ is the Hölder inequality if $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$ such that $\dfrac{1}{p} + \dfrac{1}{p'} = 1$

$$\int_{\Omega} u(x)v(x)dx \leq \|u\|_p \|v\|_{p'}$$

Theorem 1 *The space $C_0^{\infty}(\Omega)$ is dense in $L^p(\Omega)$ $\forall 1 < p < \infty$. we said that $X \hookrightarrow Y$, for $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ norms space, means $X \subset Y$ with continuous injection, that is to say there exists a constant C such that*

$$\|u\|_Y \leq \|u\|_X \;\; \forall u \in X.$$

$1 \leq p \leq \infty$ *we have* $D(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow D'(\Omega)$ *We will define the Sobolev space*

$$W^{m,p}(\Omega) = \{v, D^\alpha v \in L^p(\Omega),\ for\ \ |\alpha| \leq m\},$$

with the norm

$$\|v\|_{W^{m,p}} = \left(\sum_{|\alpha|\leq m} \|D^\alpha v\|_p^p\right)^{1\backslash p} \quad if\ p \in [1, \infty)$$

$$\|u\|_{W^{m,p}} = \max_{|\alpha|\leq m} \|D^\alpha v\|_\infty,$$

is a Banach space. We denote by $W_0^{m,p}(\Omega)$ *adherence of* C_0^∞ *in the space* $W^{m,p}(\Omega)$*; For all* $p \in [1, \infty)$ *we have*

$$W_0^{m,p}(\Omega) \hookrightarrow W^{m,p}(\Omega) \hookrightarrow L^p(\Omega).$$

In the case $p = 2$ *we use the notation*

$$H^m(\Omega) = W^{m,p}(\Omega).$$

equipped with the scalar product

$$\langle u, v\rangle_{2,m} = \sum_{|\alpha|\leq m} \langle D^\alpha u, D^\alpha v\rangle.$$

The space $H^m(\Omega)$ *Is a Hilbert space. We also posed* $H_0^m(\Omega) = W_0^{m,p}(\Omega)$ *the negative Sobolev spaces are dual spaces of space* $W_0^{m,p}(\Omega)$

$$W_0^{-m,p'}(\Omega) = (W_0^{m,p}(\Omega))',$$

with the norm

$$\|u\|_{W_0^{-m,p'}(\Omega)} = \sup_{u\in W_0^{m,p}(\Omega)} \frac{\langle u, v\rangle}{\|u\|_{W_0^{m,p}(\Omega)}}$$

The space $W_0^{-m,p'}(\Omega)$ *is Banach(separable and reflexive), if* $1 < p < \infty$*). Since* $D(\Omega)$ *dense in* $H_0^1(\Omega)$*, then we have* $H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$.

Theorem 2 *Suppose that* Ω *satisfies the property of the cone and* $\leq p < \infty$*. Then*

1. $C(\bar{\Omega}) \hookrightarrow W_0^{m,p}(\Omega)$ *With the dense injection.*
2. *if* $mp \geq n$ *then* $W_0^{m,p}(\Omega) \hookrightarrow C^k(\bar{\Omega})$ *whatever integer* k *with* $\dfrac{mp-n}{p} - 1 \leq k \leq \dfrac{mp-n}{p}$.

Now the space vector valued functions, a Banach space is considered X the norm $\|.\|_X$ and an open interval $I \subset \mathbb{R}$. We note by

$$C^k(I;X) = \{v : I \longrightarrow X; D^\alpha v \in C(I;X) \ for \ \ |\alpha| < k\}$$

$C^k(\bar{I};X)$ is a Banach space for the norm

$$|v|_{C^k(\bar{I};X)} = \sum_{|\alpha|\leq k} \sup_{x\in\bar{I}} \|D^\alpha v(x)\|_X$$

$C^\infty(I;X)$ of the space infinitely differentiable functions in v I values in X and with by $D(I;X)$ the space $C_0^\infty(I;X)$, i.e the space functions of $C^\infty(I;X)$ with compact support in I provided by the inductive limit topology. Designates by $D'(I;X)$ the space of distributions on I with values in X defined by

$$D'(I;X) = L(D(I;X);X)$$

where $L(U,V)$ denotes the space of linear and continuous functions from U to V. Let $p \in [1,\infty]$ we denoted by $L^p(I;X)$ the space of(class) functions $f : I \longrightarrow X$ measurable such that $t \longrightarrow \|f(t)\|_X$ either $L^p(I)$ is a Banach space for the norm

$$\|f\|_{L^p(I;X)} = \left(\int_X |f|_X^p \, d\mu\right)^{1/p} < \infty \ p \neq \infty,$$

$$\|f\|_\infty = supess \, \|f\|_X$$

It can be proved the following properties

1. $D(I;X) \subset L^p(I;X) \subset D'(I;X)$.
2. *If $p < \infty$ then $D(I;X)$ is dense in $L^p(I;X)$.*

We denote by $W^{1,p}(I;X)$ the space of(class) functions $f \in L^p(I;X)$ such that $\dot{f} \in L^p(I;X)$ where $\dot{f}$ is the weak derivative of f. Provided by the norm

$$\|f\|_{W^{1,p}(I;X)} = \|f\|_{L^p(I;X)} + \left\|\dot{f}\right\|_{L^p(I;X)}.$$

$W^{1,p}(I;X)$ is a Banach space.

Proposition 3 *For all $p \geq 1$ we have*

1. $W^{1,p}(I;X) \subset L^{\infty}(I;X) \cap C(\bar{I},X)$.

2. *If is bounded, then $C^{\infty}(\bar{I},X)$ is dance in $W^{1,p}(I;X)$*

Definition 4 *If X and Y are Banach spaces, $X \subset Y$ with embedding continuous space, $C_s([0,T],X)$ is defined as the space of the functions $v : [0,T] \longrightarrow X$ such that the real function of a real variable*

$$t \longrightarrow \langle h, v(t) \rangle_{X',X}$$

is continuous over $[0,T]$ for all $h \in X'$.

Lemma 1 *Let X and Y under the conditions of the last definition*

i) If, in addition, X is a reflexive Banach space, then

$$L^{\infty}(0,T;X) \cap C([0,T];Y) \subset C_s([0,T];X).$$

ii) Let U another Banach space such that $X \subset U \subset Y$ is the embedding and $X \subset Y$ is compact. If $\mathcal{F}$ is bounded in $L^{\infty}(0,T;X)$ and $\dfrac{\partial \mathcal{F}}{\partial t} = \left\{ \dot{f}; f \in \mathcal{F} \right\}$ is bounded in $L^r(0,T;Y)$, with $r > 1$, then relatively compact $\mathcal{F}$ in $C([0,T];U)$.

Proof.

i) We find the result in [9] lemma 8.1 page 297

ii) We can find the result in [10]

■

1.2 Trace theorem and generalized Green formula

Theorem 5 *Let Ω is open class C^1. Then we can uniquely define the trace γv of $H^1(\Omega)$ in $H^{1/2}(\Gamma)$ such that*

$$\gamma(v) = v|_{\Gamma},\ if\ v \in [C^{\infty}(\bar{\Omega})]^n.$$

It is well known that if the domain $\Omega \in C^{1,1}$ there exist only linear applications determined γ_n of $H^1(\Omega)$ in $H^{1/2}(\Gamma)$ and γ_T of $H^1(\Omega)$ in $H_T^{1/2}(\Gamma)$ such that

$$\gamma(v) = \gamma_n(v)n + \gamma_T(v) \ \forall v \in H^1(\Omega),$$

where $H_T^{1/2}(\Gamma) = \left\{\phi \in H^{1/2}(\Gamma); \gamma_n(\phi) = 0\right\}$. And if $v \in \left[C^\infty(\bar{\Omega})\right]$, $\gamma_n(v) = v|_\Gamma.n$ and $\gamma_T(v) = v|_\Gamma - \gamma_n(v)n$. The applications $\gamma_n(v)$ and $\gamma_T(v)$ are surjective. Hereinafter for simplify the writing, v_n and v_T denote normal traces of v, $\gamma_n(v)$ and $\gamma_T(v)$ respectively.
Now we pose $\Gamma = \bar{\Gamma_0} \cap \bar{\Sigma}$
Let V be the space defined by

$$V = \left\{v \in H^1(\Omega); \gamma(v) = 0 \ in \ \Gamma_0\right\}.$$

It note $\gamma_\Sigma^0 : V \longrightarrow H^{1/2}(\Sigma)$ Operator of trace relative $v \in V$ with the restriction of $\gamma(v)$ in Σ. This operator, resulting in V in $H_{00}^{1/2}(\Sigma)$ is linear, continuous and surjective for borders $\partial\Sigma$ which is C^∞, such that

$$H_{00}^{1/2}(\Sigma) = \left\{v \in H^{1/2}(\Sigma) | \rho^{-1/2} v \in L^2(\Omega)\right\},$$

such that ρ a particular function.

Lemma 2 *If the domain is $C^{1,1}$ there exist linear applications, continuous and surjective*

$$\gamma_{\Sigma_n}^0 : V \longrightarrow H_{00}^{1/2}(\Sigma), \ \gamma_{\Sigma_T}^0 \longrightarrow H_{T00}^{1/2}(\Sigma).$$

with $H_{T00}^{1/2}(\Sigma) = \left\{\phi \in H_{00}^{1/2}(\Sigma); \phi_n = 0\right\}$ and such that

$$\gamma_\Sigma^0(v) = \gamma_{\Sigma_n}^0(v)n + \gamma_{\Sigma_T}^0(v) \ v \in V.$$

Consider the space constraints of the fields

$$X = \left\{\tau = (\tau_{\alpha\beta}) \in \left[L^2(\Omega)\right]^{n\times n}; \tau_{\alpha\beta} = \tau_{\beta\alpha}\right\}. \tag{1.1}$$

The condition that the norm

$$\|\tau\|_X = \left(\int_\Omega \tau : \tau dx\right)^{1/2}, \tag{1.2}$$

is a Hilbert space. Let E be the subspace of X defined by

$$E = \left\{ \tau \in X; div(\tau) \in L^2(\Omega) \right\}, \tag{1.3}$$

which is also a Hilbert space with the norm

$$\|\tau\|_E = \|\tau\|_X + \|div(\tau)\|_{L^2(\Omega)} . \tag{1.4}$$

Lemma 3 *Let $\Omega \in C^{1,1}$. Then there are applications uniquely determined π_n of E in $H_T^{-1/2}(\Gamma)$ such that*

$$\langle \pi(\tau), \gamma(v) \rangle_\Gamma = \langle \pi_n(\tau), v_n \rangle_{n,\Gamma} + \langle \pi_T(\tau), v_T \rangle_{T,\Gamma} ,$$

For all $\tau \in E$ and $v \in \mathbf{H}^1(\Omega)$, and

$$\pi_n(\tau) = \tau n.n \text{ and } \pi_T(\tau) = \tau n - \tau_n n,$$

for all $\tau \in C^1(\bar{\Omega})$ where $\tau_n \equiv \pi_n(\tau)$, $\tau_T \equiv \pi_T(\tau)$

Lemma 4 *Let $\Omega \in C^{0,1}$. Then there exists a unique application, π linear and continuous of E and for $v \in \mathbf{H}^{-1/2}(\Gamma)$ such that*

$$\pi(\tau) = \tau_\Gamma n, \ if \ \tau \in \left[C^1(\bar{\Omega})\right]^{n^2} . \tag{1.5}$$

Another, the following generalized verified Green formula for all $\tau \in E$ and for $v \in \mathbf{H}^1(\Omega)$

$$\int_\Omega \tau : \varepsilon(v) dx + \int_\Omega div(\tau).v dx = \langle \pi(\tau), \gamma(v) \rangle_\Gamma , \tag{1.6}$$

where $\langle ., . \rangle_\Gamma$ is the product of duality in $\mathbf{H}^{-1/2}(\Gamma) \times \mathbf{H}^{1/2}(\Gamma)$.

Theorem 6 *Let $\Omega \in C^{0,1}$. Then there exists a uniquely application linear determined π_Σ^0 in $\left(\mathbf{H}_{00}^{1/2}(\Sigma)\right)'$ such that*

$$\pi_\Sigma^0(\tau) = \tau|_\Sigma n \ if \ \tau \in C^1(\bar{\Omega}),$$

and general Green formula is valid for each $\tau \in E$ and for all $v \in V$

$$\int_\Omega \tau : \varepsilon(v) dx + \int_\Omega div(\tau).v dx =_{00} \left\langle \pi_\Sigma^0(\tau), \gamma_\Sigma^0(v) \right\rangle_\Sigma , \tag{1.7}$$

where ${}_{00}\langle .,. \rangle_{\Sigma}$ *is the product of duality* $\left(\mathbf{H}_{00}^{1/2}(\Sigma)\right)' \times \mathbf{H}_{00}^{1/2}(\Sigma)$.
Also, if $\Omega \in C^{1,1}$, π_{Σ}^{0} *operators can be decomposed into* $\pi_{\Sigma n}^{0}$, $\pi_{\Sigma T}^{0}$ *such that*

$$ {}_{00}\left\langle \pi_{\Sigma}^{0}(\tau), \gamma_{\Sigma}^{0}(v) \right\rangle_{\Sigma} = {}_{00}\left\langle \pi_{\Sigma n}^{0}(\tau), \gamma_{\Sigma n}^{0}(v) \right\rangle_{n;\Sigma} + {}_{00}\left\langle \pi_{\Sigma T}^{0}(\tau), \gamma_{\Sigma T}^{0}(v) \right\rangle_{T;\Sigma} $$

for all $\tau \in E$ *and* $v \in V$, *and*

$$ \pi_{\Sigma n}^{0}(\tau) = \tau|_{\Sigma} n.n \text{ and } \pi_{\Sigma T}^{0}(\tau) = \tau|_{\Sigma} n - \tau_{n,\Sigma} n, $$

for $\tau \in C(\bar{\Omega})$, *where* $\tau_{n,\Sigma} = \pi_{n,\Sigma}^{0}(\tau)$.

Proof. For more details, see [11] and [8] ■

1.3 The Poincare inequality

Let Ω open and bounded domain from $\mathbb{R}^n$, Γ_0 a part from $\partial\Omega$ and $mes(\Gamma_0) > 0$, then there exist a constant $C > 0$ such that for all $u \in H^1(\Omega)$ and $u = 0$ on Γ_0, we have

$$ \int_{\Omega} |u(x)|^2 \leq C \int_{\Omega} |\nabla u(x)|^2 $$

then

$$ \|u\|_{0,\Omega} \leq C \|\nabla u\|_{0,\Omega} $$

1.4 Discrete Grönwall inequality

Let y and g two positive integrable function and $C \geq 0$, if

$$ y(t) \leq C + \int_0^t g(s)y(s)ds \quad for\ t \geq 0 \tag{1.8} $$

then

$$ y(t) \leq C \exp\left(\int_0^t g(s)ds\right) \quad for\ t \geq 0 \tag{1.9} $$

Proof. see [12] ■

1.5 Stampacchia Theorem

Definition 7 *Let H be a Hilbert space.*
We say that a bilinear form $a(u,v) : H \times H \longrightarrow \mathbb{R}$ is

- ***Continuity*** *if there exists a constant $C > 0$ such that*

$$|a(u,v)| \leq C \|u\|_H \|v\|_H \quad \forall u, v \in H.$$

- ***Coercively*** *If there is a constant $\alpha > 0$ such that*

$$a(u,v) \geq \alpha \|v\|^2 \quad \forall v \in H.$$

Theorem 8 *(Stampacchia) Let $a(.,.)$ a continuous and coercive bilinear form. Let K be a convex closed and non-empty. Given $\varphi \in H'$ there exist a unique $u \in K$ such that*

$$a(u, v-u) \geq (\varphi, v-u) \; \forall v \in K$$

Proof. See [13]. ∎

1.6 The weak, weak* convergence

Proposition 9 *Let E be a Banach space, (x_n) a sequence from E, and (f_n) a sequence from E' then*

$$-\left[x_n \rightharpoonup x \; for \; \sigma(E,E')\right] \Leftrightarrow \left[\langle f, x_n \rangle \longrightarrow \langle f, x \rangle \;\; \forall f \in E'\right].$$

$$-\left[f_n \overset{*}{\rightharpoonup} f \; for \; \sigma(E',E)\right] \Leftrightarrow \left[\langle f_n, x \rangle \longrightarrow \langle f, x \rangle \;\; \forall f \in E\right].$$

Proof. See [13]. ∎

1.7 Newmark's method

Although the method is discussed in many textbooks in structural dynamics (see, for instance, Chopra, 1995), a brief description of this method specialized for the non-linear force deformation model is provided here

$$u^{n+1} = u^{n+1} + \Delta t \dot{u}^n + \frac{\Delta t^2}{2}(1-2\beta)\ddot{u}^n + 2\beta\ddot{u}^{n+1}$$

$$\dot{u}^{n+1} = \dot{u}^n + \Delta t[(1-\gamma)\ddot{u}^n + \gamma\ddot{u}^{n+1}]$$

Proof. For more details see[14] ∎

Chapter 2

The problem of Signorini : static case

Let Ω is a bounded open set of $\mathbb{R}^3$ initially occupying the bounded domain of class $C^{1,1}$. Boundary $\partial\Omega$ is partitioned into three disjoint open subsets Γ_0, Γ_g and Γ_C Dirichlet and Neumann conditions are prescribed on Γ_0 and Γ_g.

2.1 Problem classic(P1)

We define the problem $(P1)$

Find u such that

$$-\Delta u = f \quad in \quad \Omega \tag{2.1}$$

$$u = 0 \quad on \quad \Gamma_0 \tag{2.2}$$

$$\partial_n u = g \quad on \quad \Gamma_g \tag{2.3}$$

$$u_n \leq 0, \quad N \leq 0, \quad u_n N = 0 \quad on \quad \Gamma_C \tag{2.4}$$

such that

$$u_n = u.n, \quad \partial_n u = \nabla u.n = N.n + T, \quad N = \partial_n u.n, \quad T = \partial_n u - N.n$$

the conditions $u_n \leq 0$, $N \leq 0$, $u_n N = 0$ on Γ_C are called signorini conditions $T = 0$ means no friction on Γ_C.

2.2 Variational problem (P2)

Let

$$V = \left\{ v \in \left(H^1 \left(\Omega \right) \right)^3 , v = 0 \ \ on \ \ \Gamma_0 \right\}$$

$$K = \{ v \in V ; \ u_n \leq 0 \ on \ \Gamma_C \}$$

no empty, convex and closed.

$$a(u,v) = \int_\Omega \nabla u \nabla v dx \ \ \forall u, v \in K \tag{2.5}$$

$$(F, v) = (f, v) + \int_{\Gamma_g} g v d\Gamma \ \ \forall v \in V \tag{2.6}$$

the variational inequality of $(P1)$, is the following

$$(P2) \begin{cases} Find \ u \in K \ such \ that \\ a(u,v) \geq (F,v) \ \ \forall v \in K \end{cases}$$

Theorem 10 *If u is a smooth function which satisfies $(P1)$ then u is a solution of the variational inequality $(P2)$.*

Proof. Multiplying equation (2.1) by $(v - u)$

$$-\Delta u \ (v - u) = f \ (v - u)$$

$$\int_\Omega -\Delta u \ (v - u) \ dx = \int_\Omega f \ (v - u) \ dx$$

using $Green's$ formula

$$\int_\Omega \nabla u \ \nabla (v - u) \ dx - \int_\Gamma \partial_n u \ n \ (v - u) \ d\Gamma = \int_\Omega f \ (v - u) \ dx$$

was

$$\int_\Gamma \partial_n u \ n \ (v - u) \ d\Gamma = \int_{\Gamma_0} \partial_n u \ (v - u) \ d\Gamma + \int_{\Gamma_C} \partial_n u \ (v - u) \ d\Gamma + \int_{\Gamma_g} \partial_n u \ (v - u) \ d\Gamma$$

$$= \int_{\Gamma_g} g (v - u) \ d\Gamma + \int_{\Gamma_C} \partial_n u \ n \ (v - u) \ d\Gamma$$

then

$$\int_{\Gamma} \partial_n u \ n \ (v-u) \ d\Gamma \geq \int_{\Gamma_g} g(v-u) \ d\Gamma$$

$$a(u, v-u) - (f, v-u) - \int_{\Gamma} \partial_n u \ n \ (v-u) \ d\Gamma = 0$$

$$a(u, v-u) - (f, v-u) - \int_{\Gamma_g} g(v-u) \ d\Gamma \geq 0$$

$$a(u, v-u) - (F, v-u) \geq 0$$

$$a(u, v-u) \geq (F, v-u).$$

■

Theorem 11 *If u is a solution of the variational inequality $(P2)$ then u satisfies $(P1)$ in a generalized sense.*

Proof. If u is the solution of the variational inequality $(P2)$ then one takes $v = u \pm \varphi$ and $\varphi \in (D(\Omega))^3$ we obtained

$$a(u, \varphi) \geq (F, \varphi)$$

$$a(u, -\varphi) \geq (F, -\varphi)$$

therefore

$$a(u, \varphi) = (F, \varphi)$$

$$a(u, \varphi) = (f, \varphi) + \int_{\Gamma_g} g \ \varphi \ dx$$

$$\int_{\Omega} \nabla u \ \nabla \varphi \ dx = \int_{\Omega} f \varphi \ dx$$

$$a(u, \varphi) = (f, \varphi)$$

using Green's formula

$$\int_{\Omega} \nabla u \nabla \varphi dx - \int_{\Gamma} \partial_n u \varphi \ dx = \int_{\Omega} f.\varphi \ dx$$

$$\int_{\Omega} -\Delta u \varphi \ dx = \int_{\Omega} f \varphi \ dx$$

thus

$$\int_{\Omega} (-\Delta u - f)\,\varphi\ dx = 0 \Rightarrow -\Delta u - f = 0 \Rightarrow -\Delta u = f$$

multiplying $(P1)$ by $(v-u)$

$$\int_{\Omega} -\Delta u(v-u)\ dx = \int_{\Omega} f(v-u)\ dx$$

using Green's formula we obtain

$$a(u, v-u) - (f, v-u) = \int_{\Gamma} \partial_n u(v-u)\ d\Gamma$$

through the variational inequality $(P2)$

$$\int_{\Gamma} \partial_n u(v-u)\ d\Gamma - \int_{\Gamma_g} g(v-u)\ d\Gamma$$

takes $v = u \pm \varphi$ with $\varphi \in (D\,(\Gamma))^3$ and is deducted

$$-\int_{\Gamma_g} g\varphi\ g\Gamma + \int_{\Gamma} \partial_n u\varphi\ d\Gamma = 0$$

then

$$\int_{\Gamma_g} (\partial_n u - g)\,\varphi\ d\Gamma = 0$$

$$\partial_n u = g \quad on \quad \Gamma_g$$

where (2.3).

For Signorini conditions (2.4) was

$$-\int_{\Gamma_g} g(v-u)\ d\Gamma + \int_{\Gamma} \partial_n u(v-u)\ d\Gamma \geq 0$$

$$-\int_{\Gamma_g} \partial_n u(v-u)\ d\Gamma + \int_{\Gamma} \partial_n u(v-u)\ d\Gamma \geq 0$$

taking $v = u + \varphi$ where $\varphi \in (D(\Omega))^3$ with $supp\varphi \in \Gamma_2$ and $\varphi_n \leq 0$ on Γ_C we obtain

$$\int_{\Gamma_C} u_n \varphi_n d\Gamma \geq 0$$

which gives $u_n \leq 0$ on Γ_C we have $u \in K$ then $u_n \leq 0$

now choosing $v_n = 0$ then $v_n = 2u_n$ is obtained

$$u_n N = 0.$$

■

2.3 Existence and uniqueness of solution

Theorem 12 *If $f \in \left(L^2(\Omega)\right)^3$, $g \in \left(L^2(\Gamma_g)\right)^3$ then the problem $(P2)$ has a solution unique in K.*

Proof.

i) $a(u,v)$ is a continuous bilinear form coercive .

$a(u,v)$ is a bilinear form (obviously) .
We have

$$\begin{aligned}|a(u,v)| &= \left|\int_\Omega \nabla u \nabla v \ dx\right| \\ &\leq c\int_\Omega |\nabla u \nabla v dx| \\ &\leq c\left(\int_\Omega (\nabla u)^2 \, dx\right)^{\frac{1}{2}}\left(\int_\Omega (\nabla v \ )^2 \, dx\right)^{\frac{1}{2}} \\ &\leq \|\nabla u\|_{(L^2(\Omega))^3}\|\nabla v\|_{(L^2(\Omega))^3} \\ &\leq \left(\|\nabla u\|_{(L^2(\Omega))^3}+\|u\|_{(L^2(\Omega))^3}\right)\left(\|\nabla v\|_{(L^2(\Omega))^3}+\|v\|_{(L^2(\Omega))^3}\right).\end{aligned}$$

On the other hand we have

$$\|v\|_{(H^1(\Omega))^3} = \left(\|v\|^2_{(L^2(\Omega))^3}+\|\nabla v\|^2_{(L^2(\Omega))^3}\right)^{\frac{1}{2}}$$

therefore

$$|a(u,v)| \leq M\,\|u\|_{(H^1(\Omega))^3}\,|v\|_{(H^1(\Omega))^3}$$

where the continuity of $a(u,v)$.
We have

$$|a(v,v)| = \left|\int_\Omega \nabla v \nabla v dx\right| = \left|\int_\Omega (\nabla v)^2 \, dx\right|$$

$\geq C\left(|v|^2_{L^2(\Omega)}+|\nabla v|^2_{L^2(\Omega)}\right)$
$=$ C$\|v\|^2_{H^1(\Omega)}$which completes the coercivety.

ii) (f,v) continuous linear form V Indeed, we have

$$\left|\int_{\Omega} fvdx\right| \leq \left(\int_{\Omega} |f|^2\, dx\right)^{\frac{1}{2}} \left(\int_{\Gamma_C} |v|^2\, d\Gamma\right)^{\frac{1}{2}}$$
$$\leq c\, \|v\|_{L^2(\Omega)}$$

and

$$\left|\int_{\Gamma_g} gvd\Gamma\right| \leq \|g\|_{L^2(\Gamma_g)} \|v\|_{L^2(\Gamma_g)}$$

Using the continuous injection of the application trace $H^1(\Omega)$ on $L^2(\Gamma_g)$ and the injection continues to $H^1(\Omega)$ on $L^2(\Omega)$ are

$$|(F,v)| \leq c\left(\|v\|_{L^2(\Omega)} + \|v\|_{L^2(\Gamma_g)}\right)$$
$$\leq c'\, \|v\|_{H^1(\Omega)}$$

(i) and (ii) by means of the theorem of Stampacchia the variational inequality has a unique solution. ■

Chapter 3

The Signorini problem Dynamic case

3.1 Problem (P.C)

Let Ω is a bounded open set of $\mathbb{R}^3$ initially occupying the bounded domain of class $C^{1,1}$.
Boundary $\partial\Omega$ is partitioned into three disjoint open subsets Γ_0, Γ_g and Γ_C Dirichlet and Neumann conditions are prescribed on Γ_0 and Γ_g.
The study of the Signorini problem in the static case , in this chapter can presented study the same problem in the dynamic case.

Classic problem (P.C)

Let us consider the problem $(P.C)$

$$\text{Find } \ u \ \text{ verifying}$$

$$\ddot{u} - \Delta u = f \quad in \quad \Omega \times (0,T) \tag{3.1}$$

$$\partial_n \, u = g \quad on \quad \Gamma_g \times (0,T) \tag{3.2}$$

$$u = 0 \quad on \quad \Gamma_0 \times (0,T) \tag{3.3}$$

$$u_n \leq 0, \quad N \leq 0, \quad u_n N = 0, \quad T = 0 \quad on \quad \Gamma_C \times (0,T) \tag{3.4}$$

$$u(x,0) = p^0, \quad \dot{u}(x,0) = p^1 \quad in \quad \Omega \tag{3.5}$$

with $u_n = u.n$, $\partial_n u = \nabla u.n = N.n + T$, $N = \partial_n.u.n$, $T = \partial_n u - N.n$
where n denote the outward unit normal to Ω on Γ_C.
signorini conditions is imposed $u \leq 0$, $N \leq 0$, $u_n N = 0$, $T = 0$ on $\Gamma_C \times (0,T)$
at the initial time, $u(x,0) = p^0$, $\dot{u}_1 = p^1$ in Ω
Where $g \in W^{2,\infty}\left(0,T;(L^2(\Gamma_g))^3 \cap \left(H^{-\frac{1}{2}}(\Gamma)\right)^3\right)$, $f \in W^{2,\infty}\left(0,T,(L^2(\Omega))^3\right)$
The initial conditions p^0, p^1 are assumed to belong to $(H^1(\Omega))^3$ such that $\Delta p^0 \in L^2(\Omega)$.

3.1.1 Problem variation (P.V)

We put $\mathbf{H^1(\Omega)} = (H^1(\Omega))^3$, $\mathbf{L^2(\Omega)} = (L^2(\Omega))^3$.
Let V the space defined by

$$V = \{v \in \mathbf{H^1(\Omega)} / \ v = 0 \ \ on \ \ \Gamma_0\}$$

and

$$K = \{v \in V / \ v_n \leq 0 \ \ on \ \ \Gamma_C\}$$

closed convex subspace from V.

Remark 13 *We can write the problem $(P.V)$ as*

$$(P.V)\begin{cases} find\ u \in K \ \ such\ that \\ \displaystyle\int_\Omega \ddot{u}v\ dx + a(u,v) = L(v) \ \ \forall v \in V \\ \langle N, v_n - u_n\rangle \geq 0, \ \forall v \in K \end{cases}$$

with

$$u_0 = p^0, \dot{u}_0 = p^1.$$

Theorem 14 *If u solution of $(P.C)$ then u satisfies the $(P.V)$ problem*

$$Find\ u \in K\ such\ that$$

$$\begin{cases} \langle \ddot{u}, v-u\rangle + a(u, v-u) \geq L(v-u)\ \forall v \in K \\ u(x,0) = p^0, \ \ \dot{u}(x,0) = p^1, \end{cases} \tag{3.6}$$

$$\langle N, v_n - u_n\rangle \geq 0 \tag{3.7}$$

where

$$a(u,v) = \int_\Omega \ \nabla u \nabla v \ \ dx$$

$$L(v) = \int_\Omega fv \quad dx + \int_{\Gamma_g} gv \quad d\Gamma$$

Proof.

Weak formulation of (3.1).

Let $(v-u) \in K$ a test function, (3.1) give

$$\int_\Omega \ddot{u}(v-u) \quad dx + \int_\Omega -\Delta u(v-u) \quad dx = \int_\Omega f(v-u) \quad dx$$

we have

$$\int_\Omega \Delta u.(v-u).dx = \int_\Gamma \nabla u.n.(v-u)d\Gamma - \int_\Omega \nabla u.\nabla(v-u)dx$$

we have

$$\int_\Gamma \nabla u.n.(v-u)d\Gamma =$$

$$\int_{\Gamma_0} \nabla u.n.(v-u)d\Gamma + \int_{\Gamma_C} \nabla u.n.(v-u)d\Gamma + \int_{\Gamma_g} \nabla u.n.(v-u) \quad d\Gamma =$$

$$\int_{\Gamma_C} \nabla u.n.v \quad d\Gamma + \int_{\Gamma_g} \nabla u.n.(v-u) \quad d\Gamma$$

then

$$\int_{\Gamma_g} \nabla u.n.(v-u) \quad d\Gamma = \int_{\Gamma_g} g(v-u) \quad d\Gamma$$

$$\int_{\Gamma_g} \frac{\partial u}{\partial n}(v-u) \quad d\Gamma \geq 0$$

therefore

$$\langle \ddot{u}, v-u\rangle + a(u, v-u) \geq L(v-u) \quad \forall v \in K$$

$$u(x,0) = p^0, \quad \dot{u}(x,0) = p^1, \quad \forall v \in K$$

■

Remark 15 *We can write $(P.V)$ as*

$$\text{find } u \in K \text{ such that}$$

$$\int_\Omega \ddot{u}v \ dx + a(u,v) = L(v) + \langle N, v_n \rangle \, , \quad \forall v \in V$$

$$\langle N, v_n - u_n \rangle \geq 0 \quad \forall v \in K$$

$$u(x,0) = p^0, \dot{u}(x,0) = p^1$$

Theorem 16 *Assume that the solution u is regular enough, then u is solution of $(P.C)$ if and only if u solution of $(P.V)$.*

Proof. *Taking in (3.6) $v = \varphi$ for all $\varphi \in (D(\Omega))^3$(since v remains in V). It is found that*

$$\langle \ddot{u}, \varphi \rangle + a(u, \varphi) = L(\varphi), \quad \forall \varphi \in (D(\Omega))^3 \tag{3.8}$$

hence, using the generalized Green formula in (3.8), are

$$\int_\Omega (\ddot{u} - \Delta u - f)\varphi \ \ dx = 0. \tag{3.9}$$

therefore

$$\ddot{u} - \Delta u - f = 0 \ \ a.e \ \ in \ \ \Omega \tag{3.10}$$

Hence (3.1).
For (3.2), we take $v = \varphi$ for all $\varphi \in (D(\Omega \cup \Gamma_g))^3$ in (3.6), and taking included (3.9), we find

$$\int_{\Gamma_g} (\nabla u \ \ n - g)\varphi \ \ d\Gamma = 0$$

where (3.2).
Taking $v = \varphi$ in (3.6) for all $\varphi \in (D(\Omega \cup \Gamma_0))^3$.
with (3.8) we find $\langle T, \varphi_T \rangle = 0$ $\forall T$ where $T = 0$ on Γ_C a.e. After that we take $v = u + \varphi$ in (3.7)with $\varphi \in (D(\Omega \cup \Gamma_0))^3$ and $\varphi_n \leq 0$ on Γ_0 we find $\langle N, \varphi_n \rangle \geq 0$ with give $N \leq 0$ on Γ_0. We find $v_n = 0$ and $v_n = 2u_n$ on (3.7), we obtain $\langle N, u_n \rangle = 0$ and $N(u_n) \geq 0$ then $N(u_n) = 0$ on Γ_0, where (3.4). ■

3.1.2 Variational inequality of (P.V)

Find u such that

$$\int_{\Omega}\ddot{u}(v-u)dx+\int_{\Omega}\nabla u\nabla(v-u)dx\geq L(v-u)\quad\forall v\in K$$

$$u(x,0)=p^{0},\ \dot{u}(x,0)=p^{1}$$

With

$$L(v)=\int_{\Omega}fvdx+\int_{\Gamma_g}gvd\Gamma$$

3.2 Existence of a solution

Theorem 17 *Under the following conditions*
$f\in W^{2,\infty}(0,T;\mathbf{L^2(\Omega)})$, $g\in W^{2,\infty}(0,T;\mathbf{L^2(\Gamma_g)})$ *and* $p^0,p^1\in\mathbf{H^1(\Omega)}$ *with* $\Delta p^0\in L^2(\Omega)$.
Then there exist a solution $(u,\nabla u)$ *of problem* $(P.C)$ *verifying*

(i) $u\in L^{\infty}(0,T;K)$, $\dot{u}\in L^{\infty}(0,T;\mathbf{L^2(\Omega)})$, *and* $\ddot{u}\in D'(0,T;\mathbf{L^2(\Omega)})$

(ii) $\nabla u\in D'(0,T;E_a(g))\cap L^{\infty}\left(0,T;(L^2(\Omega))^{3\times3}\right)$

such that

$$E_a(g)=\{\tau\in E/T=0\ and\ N\leq 0\ on\ \Gamma_C,\partial_n u=g\ on\ \Gamma_g\}$$

and

$$E=\left\{\tau\in X,div\tau\in\mathbf{L^2(\Omega)}\right\}$$

with

$$X=\left\{\tau\in(L^2(\Omega))^{3\times3}/\tau_{i,j}=\tau_{j,i}\right\}$$

Proof. *The demonstration of this theorem is based on five basic steps.*

Step 1: time discretization

Let us consider a regular partition of the time interval $[0,T]$ *into* I *subintervals. Inspired on Newmark's method we propose the following approximation of Problem* $(P.C)$ *at time* $t=t_i$.

Problem ($\mathbf{P}^i V$)

Find $u^i \in K$, $\dot{u}^i \in \mathbf{H^1(\Omega)}$ *and* $\ddot{u}^i \in \mathbf{L^2(\Omega)}$ *verifying the inequality*

$$\int_\Omega \left(\frac{\ddot{u}^i + \ddot{u}^{i-1}}{2} \right) (v - u^i) dx + a \left(\frac{u^i + u^{i-1}}{2}, v - u^i \right) \geq L^i(v - u^i) \quad \forall v \in K \tag{3.11}$$

$$L^i(v) = \int_\Omega f^i v \ dx + \int_g g^i v dx$$

$$f^i := f(x, t_i), \ \ g^i := g(x, t_i)$$

Newmark's method with parameters

$\beta = \dfrac{1}{4}, \quad \gamma = \dfrac{1}{2}$ *as follows*

$$\begin{cases} u^i = u^{i-1} - \Delta t \dot{u}^{i-1} + \dfrac{\Delta t^2}{2} \dfrac{\ddot{u}^i + \ddot{u}^{i-1}}{2} & (3.12) \\ \dot{u}^i = \dot{u}^{i-1} + \Delta t \dfrac{\ddot{u}^i + \ddot{u}^{i-1}}{2} & (3.13) \end{cases}$$

calculi of u^i, $\dot{u}^i$, $\ddot{u}^i$

$$u^i = u^{i-1} + \Delta t \dot{u}^{i-1} + \frac{\Delta t^2}{2} \frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}$$

$$\dot{u}^i = \dot{u}^{i-1} + \Delta t \frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}$$

then

$$\int_\Omega \frac{\ddot{u}^i + \ddot{u}^{i-1}}{2} \left(v - u^i\right) \ dx + a \left(\frac{u^i + u^{i-1}}{2}, v - u^i \right) \geq L^i(v - u^i)$$

we have

$$\frac{\ddot{u}^i + \ddot{u}^i}{2} = \frac{2}{\Delta t^2} \left(u^i - u^{i-1} - \Delta t \dot{u}^{i-1} \right)$$

then

$$\frac{2}{\Delta t^2} \int_\Omega u^i \left(v - u^i\right) dx + \frac{1}{2} a \left(u^i, v - u^i\right) \geq$$

$$\frac{2}{\Delta t^2} \int_\Omega \left(u^i + \Delta t \dot{u}^{i-1}\right) \left(\left(v - u^i\right) dx - a \left(u^{i-1}, v - u^i\right) + L^i \left(v - u^i\right), \ \forall v \in K$$

$$\alpha \int_\Omega u^i(v - u^i) + \beta a(u^i, v - u^i) \geq \tilde{L}(v - u^i), \ \forall v \in K.$$

For the existence and uniqueness utilise Stampacchia theorem

$$\tilde{a}(u^i, v - u^i) \geq \tilde{L}(v - u^i) \ \ \forall v \in K.$$

Find the weak solution.

Algorithm

1. *At the initial time ,* $u^0{=}u(0){=}p^0$, $u^l{=}\dot{u}{=}p^1$, $\ddot{u}^0 = \Delta p^0 + f^0$, $\; f^0 = f(0)$

2. *For each step* t_i*, given* u^{i-1}, $\dot{u}^{i-1}$, $\; \ddot{u}^{i-1}$ *we obtain* u^i *as the solution of the variational problem*

$$\int_\Omega u^i(v-u^i)\ dx + \frac{\Delta t^2}{4} a(u^i, v-u^i) \geq \tilde{L}(v-u^i), \quad \forall v \in K$$

witch have unique solution

3. *Final, we obtain* $\dot{u}^i$, $\ddot{u}^i$ *with Newmark's method.*

Step 2: Approximated solution of problem (P.C)

First approximation

In this step, several sequences are constructed from the solution of the approximated problem (P^iV), $1 \leq i \leq I$*, when* $I \to \infty$*. To do that, let us take the following functions*

$$h^I(t) = u^{i-1} + \dot{u}^{i-1}(t - t_{i-1}) + \left(\frac{\ddot{u}^{i-1} + \ddot{u}^i}{4}\right)(t - t_{i-1})^2 \quad \forall t \in [t_{i-1}, t_i); \tag{3.14}$$

$$h^I(T) = u^I, \quad \dot{h}^I(T) = \dot{u}^I$$

$h^I(t_{i-1}) = u^{i-1}$, $h^I(t_i) = u^i$ *it can be seen that* $h^I(t) \in C^1(0,T;H^1(\Omega))$ *and* h^I *is* C^2 *at each subinterval* (t_{i-1}, t_i).

$$\dot{h}^I(t) = \dot{u}^{i-1} + \frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}(t - t_{i-1}), \;\; t \in [t_{i-1}, t_i) \tag{3.15}$$

for each $i = 0, ..., 2^I$, $\dot{h}^I(t_i) = u^i$
$\ddot{h}^I(t_i) \in L^\infty(0,T;H^1(\Omega))$*, such that*

$$\ddot{h}^I = \frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}, t \in [t_{i-1}, t_i).$$

A assistant t_i *we obtain for* $i = 1, ..., 2^I$

$$\lim_{t \longrightarrow t_i} h^I(t) = u^{i-1} + \dot{u}^i \Delta t + \frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}\Delta t^2 = u^i = h^I(t_i) \tag{3.16}$$

$$\lim_{t \longrightarrow t_i} \dot{h}^I(t) = \dot{u}^{i-1} + \frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}\Delta t = \dot{u}^i = \dot{h}^I(t_i) \tag{3.17}$$

then $h^I(t) \in C(0,T; H^1(\Omega))$.

Remark 18 *Note that* $h^I(t_k) \in K$*, there is no guarantee that* $h^I \in K$ *for all* $t \in [0,T]$.

Another approximation

Now choose another four approaches to a solution of the problem $(P.C)$ *which are also convergent when* $I \longrightarrow \infty$*. We define*

$$l^I(t) = u^{i-1} + \frac{u^i + u^{i-1}}{\Delta t}(t - t_{i-1}), \quad \forall t \in [t_{i-1}, t_i) \tag{3.18}$$

$$h^I_\star(t) = h^i = \frac{u^i + u^{i-1}}{2}, \quad \forall t \in [t_{i-1}, t_i) \tag{3.19}$$

$$h_\sharp(t) = \dot{u}^i, \quad \forall t \in [t_{i-1}, t_i) \tag{3.20}$$

$$u^I_\star(t) = u^i, \quad \forall t \in [t_{i-1}, t_i) \tag{3.21}$$

Remark 19 *Note that in this case* $l^I(t)$*,* $h^I_\star(t)$*,* $u^I_\star(t)$ *in* K *for all* $t \in [0,T]$.

Step 3: A priori estimates

To obtain a priori estimates the following lemma.

Lemma 5 *Let* u^i*,* $\dot{u}^i$ *and* $\ddot{u}^i$ *be the solution of problem* (P^iV) *,*$1 \leq i \leq 2^i$ *and* h^I *define by (3.14)for each subinterval* (t_{i-1}, t_i) *of* $(0,T)$ *verifying*

$$\int_{t_{i-1}}^{t_i} \frac{1}{2}\frac{d}{dt}\left\|\dot{h}^I(t)\right\|^2_{L^2(\Omega)} dt + \int_{t_{i-1}}^{t_i} \frac{1}{2}\frac{d}{dt} a(h^I(t), h^I(t))dt \leq L^i(u^i - u^{i-1}) \tag{3.22}$$

with

$$a(u,v) = \int_\Omega \nabla u \nabla v dx$$

Proof. *Taking $v = u^{i-1}$ in (3.11)to obtain*

$$\int_\Omega \left(\ddot{u}^i + \ddot{u}^{i-1}\right).\frac{u^i - u^{i-1}}{2}dx + a\left(u^i + u^{i-1}, \frac{u^i - u^{i-1}}{2}\right) \leq L^i(u^i - u^{i-1}). \tag{3.23}$$

First, we rewrite the first member of equation(3.23) in term of $h^I(t)$. Utilise (3.12), we can prove

$$\frac{u^i - u^{i-1}}{2} = \frac{\Delta t}{2}\dot{u}^{i-1} + \frac{\Delta t^2}{4}\frac{\ddot{u}^i + \ddot{u}^{i-1}}{2} \tag{3.24}$$

$$u^i + u^{i-1} = 2u^{i-1} + \Delta t\dot{u}^{i-1} + \frac{\Delta t^2}{2}\frac{\ddot{u}^i + \ddot{u}^{i-1}}{2} \tag{3.25}$$

$$\begin{aligned}
&\frac{\Delta t}{2}\int_\Omega \left(\ddot{u}^i + \ddot{u}^{i-1}\right).\dot{u}^{i-1}dx \\
&+\frac{\Delta t^2}{4}\int_\Omega \left(\ddot{u}^i + \ddot{u}^{i-1}\right).\left(\frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}\right)dx + \\
&\frac{\Delta t^2}{2}a\left(u^{i-1}, \frac{\Delta t^2}{4}\left(\frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}\right)\right) \\
&+\frac{\Delta t^3}{4}a\left(\dot{u}^i, \frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}\right) + \frac{\Delta t^4}{4}a\left(\frac{\ddot{u}^i + \ddot{u}^{i-1}}{4}, \frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}\right) \\
&+\Delta t a\left(u^{i-1}, \dot{u}^{i-1}\right) + \frac{\Delta t^2}{2}a\left(\dot{u}^{i-1}, \dot{u}^{i-1}\right) \\
&\frac{\Delta t^3}{4}a\left(\frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}, \dot{u}^i\right) \leq L^i(u^i, u^{i-1}).
\end{aligned} \tag{3.26}$$

Now the first member of this expression corresponds to(3.23) we obtain

$$\int_{t_{i-1}}^{t_i} \frac{1}{2}\frac{d}{dt}\left\|\dot{h}^I(t)\right\|^2_{L^2(\Omega)} dt + \int_{t_{i-1}}^{t_i} \frac{1}{2}\frac{d}{dt}a\left(h^I(t), h^I(t)\right)dt$$

$h^I(t)$, $t \in [t_{i-1}, t_i)$

$$\left\|\dot{h}^I(t)\right\|^2_{\mathbf{L^2(\Omega)}} = \int_\Omega \left(\dot{u}^{i-1} + \frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}(t - t_{i-1})\right).\left(\dot{u}^{i-1} + \frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}(t - t_{i-1})\right)dx$$

and

$$\frac{1}{2}\frac{d}{dt}\left\|\dot{h}^I(t)\right\|^2_{\mathbf{L^2(\Omega)}} = \int_\Omega \frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}.\left(\dot{u}^{i-1} + \frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}(t - t_{i-1})\right)dx$$

$$\int_\Omega \frac{1}{2}\frac{d}{dt}\left\|\dot{h}^I(t)\right\|^2_{\mathbf{L^2(\Omega)}} dt = \Delta t\int_\Omega \dot{u}^{i-1}.\frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}dx + \frac{\Delta t^2}{2}\int_\Omega \left(\frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}\right)^2 dx \tag{3.27}$$

and, on $[t_{i-1}, t_i)$

$$\frac{1}{2}\frac{d}{dt}a\left(h^I(t), h^I(t)\right) = \int_\Omega \nabla \dot{h}^I(t).\nabla \dot{h}^I(t)dx$$

$$= \int_\Omega \nabla\left(\dot{u}^{i-1} + \frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}(t - t_{i-1})\right):$$

$$\nabla\left(u^{i-1} + \dot{u}^i(t - t_{i-1}) + \frac{\ddot{u}^i + \ddot{u}^{i-1}}{4}(t - t_{i-1})^2\right)dx$$

$$= \int_\Omega \nabla \dot{u}^{i-1} : \nabla u^{i-1}dx + \int_\Omega \nabla \dot{u}^{i-1} : \nabla \dot{u}^{i-1}(t - t_{i-1})dx$$

$$+ \int_\Omega \nabla \dot{u}^{i-1}\nabla\left(\frac{\ddot{u}^i + \ddot{u}^{i-1}}{4}\right)(t - t_{i-1})^2 dx$$

$$+ \int_\Omega \nabla\left(\frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}\right) : \nabla u^{i-1}(t - t_{i-1})dx$$

$$+ \int_\Omega \nabla\left(\frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}\right) : \nabla \dot{u}^{i-1}(t - t_{i-1})^2 dx$$

$$+ \int_\Omega \nabla\left(\frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}\right) : \nabla\left(\frac{\ddot{u}^i + \ddot{u}^{i-1}}{4}\right)(t - t_{i-1})^3 dx.$$

then

$$\int_{t_{i-1}}^{t_i} \frac{1}{2}\frac{d}{dt}a\left(h^I(t), h^I(t)\right)dt = \Delta t \int_\Omega \nabla \dot{u}^{i-1} : \nabla u^{i-1}dx$$

$$\frac{\Delta t^2}{2}\int_\Omega \nabla \dot{u}^{i-1} : \nabla u^{i-1}dx + \frac{\Delta t^3}{3}\int_\Omega \nabla \dot{u}^{i-1}\nabla\left(\frac{\ddot{u}^i + \ddot{u}^{i-1}}{4}\right)dx$$

$$+\frac{\Delta t^2}{2}\int_\Omega \nabla\left(\frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}\right) : \nabla u^{i-1}dx$$

$$+\frac{\Delta t^3}{3}\int_\Omega \nabla\left(\frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}\right) : \nabla \dot{u}^{i-1}dx$$

$$+ \frac{\Delta t^4}{4}\int_\Omega \nabla\left(\frac{\ddot{u}^i + \ddot{u}^{i-1}}{2}\right) : \nabla\left(\frac{\ddot{u}^i + \ddot{u}^{i-1}}{4}\right)dx. \tag{3.28}$$

From (3.23)-(3.28), we find the result. ■

Proposition 20 *Let* h^I *define with (3.14), then*

- $\|h(t_k)\|_{H^1(\Omega)}$ *is bounded by constant independent of* I *and* k, $1 \leq k \leq 2^I$

- h^I and $\dot{h}^I$ are bounded in $C(0,T;L^2(\Omega))$ by constant independent of I.

Proof. *For all k such that $1 \leq k \leq 2^I$*

$$\sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{1}{2}\frac{d}{dt}\left\|\dot{h}^I(t)\right\|_{L^2(\Omega)}^2 dt = \frac{1}{2}\left(\left\|\dot{h}^I(t_k)\right\|_{L^2(\Omega)}^2 - \left\|\dot{h}^I(0)\right\|_{L^2(\Omega)}^2\right).$$

$$\int_{t_{i-1}}^{t_i} \frac{1}{2}\frac{d}{dt} a\left(h^I(t), h^I(t)\right) dt = \frac{1}{2}\left(a\left(h^I(t_k), h^I(t_k)\right) - a\left(h^I(0), h^I(0)\right)\right)$$

Then by lemma5, and for $1 \leq k \leq 2^I$

$$\frac{1}{2}\left(\left\|\dot{h}^I(t_k)\right\|_{L^2(\Omega)}^2 + a\left(h^I(t_k), h^I(t_k)\right)\right) \leq \frac{1}{2}\left(\left\|\dot{h}^I(0)\right\|_{L^2(\Omega)}^2 + a\left(h^I(0), h^I(0)\right)\right) + \sum_{i=1}^{k} L^i(u^i - u^{i-1}). \tag{3.29}$$

Now, we obtain superior borne for the second term cote adroit of (3.29); it give that

$$u^i + u^{i-1} = h^I(t_i) - h^I(t_{i-1}) \tag{3.30}$$

then

$$\sum_{i=1}^{k} L^i(u^i - u^{i-1}) = \sum_{i=1}^{k} L^i(h^I(t_i), h^I(t_{i-1}))$$

$$= \left(\sum_{i=1}^{k}\left(L^i - L^{i+1}\right)\left(h^I(t_i)\right)\right) + L^k(h^I(t_k)) + L^1(h^I(0)). \tag{3.31}$$

Therefore

$$\left|L^i(h(t_k))\right| \leq \left|\int_\Omega f^i h(t_k)\right| + \left|\int_{\Gamma_g} g^i h(t_k)\right| \tag{3.32}$$

$$\leq \left\|f^i\right\|_2 \left\|h(t_k)\right\|_2 + \left\|g^i\right\|_{2,\Gamma_g} \left\|h(t_k)\right\|_{H^{\frac{1}{2}}}$$

$$\leq \left(\|f\|_{L^\infty(0,T;L^2(\Omega))} \|g\|_{L^\infty(0,T;L^2(\Gamma_g))}\right) \|h(t_k)\|_{H^1(\Omega)} \tag{3.33}$$

$$\leq C_1 \|h(t_k)\|_{H^1(\Omega)}$$

and

$$\left|L^i(h(t_k)) - L^{i+1}(h(t_k))\right| \leq \left\|f^i - f^{i+1}\right\|_2 \|h(t_k)\|_2 + \left\|g^i - g^{i+1}\right\|_2 \|h(t_k)\|_{H^1(\Omega)} \tag{3.34}$$

$$\leq \left(\left\|\dot{f}\right\|_{L^\infty(0,T;L^2(\Omega))} + c\,\|\dot{g}\|_{L^\infty(0,T;L^2(\Omega))}\right) \|h(t_k)\|_{H^1(\Omega)} \tag{3.35}$$

$$\leq C_2 \|h(t_k)\|_{H^1(\Omega)}. \tag{3.36}$$

We reprehend the value absolu of (3.31), utilise (3.32) and (3.34) applying Hölder inequality

$$\left|\sum_{i=1}^{k} L^i(u^i - u^{i-1})\right| \leq C_1 \Delta t \sum_{i=1}^{k} \left\|h^I(t_i)\right\|_{H^1(\Omega)} + C_2 \left(\left\|h^I(t_k)\right\|_{H^1(\Omega)} + \left\|h^I(0)\right\|_{H^1(\Omega)}\right).$$

where C_1, C_2 are positive constants. Thus, from (3.29) it follows that

$$\left(\left\|\dot{h}^I(t_k)\right\|^2_{L^2(\Omega)} + a\left(h^I(t_k), h^I(t_k)\right)\right) \leq \left(\left\|\dot{h}^I(0)\right\|^2_{L^2(\Omega)} + a\left(h^I(0), h^I(0)\right)\right)$$

$$+ 2\left(C_2 \Delta t \sum_{i=0}^{k-1} \|h(t_i)\|_{H^1(\Omega)} + C_1 \left(\left\|h^I(t_k)\right\|_{H^1(\Omega)} + \left\|h^I(0)\right\|_{H^1(\Omega)}\right)\right). \tag{3.37}$$

Now, since the bilinear form $a(.,.)$ is coercive there exist constants positives C_1, C_2 and C_3 independent of I and k such that

$$\left\|h^I(t_k)\right\|^2_{H^1(\Omega)} \leq C_1 + C_2 \left\|h^I(t_k)\right\|_{H^1(\Omega)} + C_3 \Delta t \sum_{i=0}^{k-1} \left\|h^I(t_i)\right\|_{H^1(\Omega)}, \tag{3.38}$$

for all $1 \leq k \leq 2^I$.
Suppose that $C_1 > 1$ the precedent inequality means

$$\left\|h^I(t_k)\right\|_{H^1(\Omega)} \leq C_1 + C_2 + C_3 \Delta t \sum_{i=0}^{k-1} \left\|h^I(t_i)\right\|_{H^1(\Omega)}, \tag{3.39}$$

applying discrete Grönwall inequality (see Lions [12]), we obtain

$$\left\|h^I(t_k)\right\|_{H^1(\Omega)} \leq Ce^T, \quad C \in \mathbb{R}^+. \tag{3.40}$$

Then $h^I(t_k)$ borne in $H^1(\Omega)$ by constant independent of I and k.
From (3.37) we obtain that $\dot{h}^I$ bounded in $L^2(\Omega)$ by constant independent of I and k. By consequence, $\dot{h}^I$ linear by piece

$$\left\|\dot{h}^I(t)\right\|^2_{L^2(\Omega)} \leq \max\left\{\left\|\dot{h}(t_{i-1})\right\|^2_{L^2(\Omega)}, \left\|\dot{h}(t_i)\right\|^2_{L^2(\Omega)}\right\} \; \forall t \in [t_{i-1}, t_i]$$

then $h^I(t)$ borne in $L^2(\Omega)$, borne in $C(0,T; L^2(\Omega))$, then

$$h^I(t) = h^I(0) + \int_0^t \dot{h}^I(s)ds.$$

Then it is not easy to prove $h^I(t)$ borne in $C(0,T; L^2(\Omega))$ by constant independent of I for all $t \in (0,T)$. ■

Corollary 21 *Let h^I define by (3.14) there exist subsequence we note with I, when $I \longrightarrow \infty$ we obtain*

$$h^I \rightharpoonup h \text{ weak} * \text{ in } L^\infty(0,T;L^2(\Omega)) \tag{3.41}$$

$$\dot{h}^I \rightharpoonup \dot{h} \text{ weak} * \text{ in } L^\infty(0,T;L^2(\Omega)) \tag{3.42}$$

Proof. *When $L^\infty(0,T;L^2(\Omega)) = [L^1(0,T;L^2(\Omega))]'$ and $L^1(0,T;L^2(\Omega))$ a Banach separable, the result is direct see (Brézis [13]), note that the limit of $\dot{h}^I$ and $\dot{h}$,then uniqueness the limit in $D'(0,T;[L^2(\Omega)]_{weak})$(see [15] Lions).* ∎

Another priori estimates

Proposition 22 *Let $l^I(t)$, $h^I_\star(t)$, $h^I_\sharp(t)$ and $u^I_\star(t)$ define by (3.18)-(3.21) then*

- $\|h^I_\star(t)\|_{H^1(\Omega)}$ *and* $\|u^I_\star(t)\|_{H^1(\Omega)}$ *are bounded by constant independent of I for all $t \in [0,T]$.*
- $\|l^I(t)\|_{H^1(\Omega)}$, $\left\|\dot{l}^I(t)\right\|_{L^2(\Omega)}$ *and* $\|h^I_\sharp(t)\|_{L^2(\Omega)}$ *are bounded by constant independent of I for all $t \in [0,T]$.*
- $\ddot{h}^I$ *is bounded in $L^\infty(0,T;H^{-1}(\Omega))$.*

Proof. *From definition of h^I we have $u^i = h^I(t_i)$ for all $0 \le i \le 2^I$,*

$$h^I_\star(t) = \frac{h^I(t_i) + h^I(t_{i-1})}{2}, \ \forall t \in [t_{i-1}, t_i) \tag{3.43}$$

and

$$u^I_\star(t) = \dot{u}^I(t_i) \ \forall t \in [t_{i-1}, t_i) \tag{3.44}$$

and $h^I_\star \in L^\infty(0,T;H^1(\Omega))$ and $u^i \in H^1(\Omega)$ for all i, and

$$\|h^I_\star\|_{H^1(\Omega)} = \left\|\frac{h^I(t_i) + h^I(t_{i-1})}{2}\right\|_{H^1(\Omega)}$$

$$\le \frac{1}{2}\left(\|h^I(t_i)\|_{H^1(\Omega)} + \|h^I(t_{i-1})\|_{H^1(\Omega)}\right), \ \forall t \in [t_{i-1}, t_i) \tag{3.45}$$

then the result by proposition20 $h_\star^I(t)$ borne in $L^\infty(0,T;H^1(\Omega))$ by constant independent of I. The same, $u_\star^I(t)$ borne in $L^\infty(0,T;H^1(\Omega))$ by constant independent of I. Now, l^I linear by piece

$$\left\|l^I(t)\right\|_{H^1(\Omega)} \leq \max\left\{\left\|l^I(t_{i-1})\right\|_{H^1(\Omega)}, \left\|l^I(t_i)\right\|_{H^1(\Omega)}\right\} \ \forall t \in [t_{i-1},t_i],$$

and $l^I(t_k) = h^I(t_k)$ and $\left\|h^I(t_k)\right\|_{H^1(\Omega)}$ borne by constant independent of I and k, $\left\|l^I(t)\right\|_{H^1(\Omega)}$ borne by constant independent of I. For prove that $\dot{l}^I$ borne

$$\dot{l}^I(t) = \frac{u^i + u^{i-1}}{\Delta t},$$

with, by (3.24) and (3.15) can be rewritten as

$$\dot{l}^I(t) = \dot{u}^{i-1} = \frac{\Delta t}{2}\frac{\ddot{u}^i + \ddot{u}^{i-1}}{2} = \frac{\dot{h}^I(t_{i-1}) + \dot{h}^I(t_i)}{2}, \ \forall t \in [t_{i-1},t_i).$$

Then

$$\left\|\dot{h}^I(t)\right\|_{L^2(\Omega)} = \left\|\frac{\dot{h}^I(t_{i-1}) + \dot{h}^I(t_i)}{2}\right\|_{L^2(\Omega)}. \tag{3.46}$$

Thus, again by proposition 20, $\left\|\dot{h}^I(t)\right\|_{L^2(\Omega)}$ is bounded by a constant independent of I for all $t \in (0,T)$. Even so

$$\left\|h_\sharp^I\right\|_{L^2(\Omega)} = \left\|\dot{h}^I(t_i)\right\|_{L^2(\Omega)}, \ \forall t \in [t_{i-1},t_i).$$

Finally , of the boundedness $\ddot{h}^I$ in $L^\infty(0,T;H^{-1}(\Omega))$ is obtained as a result direct from the previous boundedness and equation can be written in terms of $\ddot{h}^I$ and $h_\star^I$ as

$$\ddot{h}^I - \Delta(h_\star^I) = f_0^I \ in \ \Omega \tag{3.47}$$

such that $f_0^I(t) = f_0^i$ for all $t \in [t_{i-1},t_i)$. ■

Step 4: Convergence of approximate solutions

Corollary 23 *Let $l^I(t)$, $h_\star^I(t)$, $h_\sharp^I(t)$ and $u_\star^I(t)$ defined by (3.18)-(3.21) respectively . Then there are equal subsequence depend with the index I as $I \longrightarrow \infty$ then there exist subsequence we obtain the following convergences*

$$l^I \rightharpoonup l \ weak* \ in \ L^\infty(0,T;H^1(\Omega)) \tag{3.48}$$

$$\dot{l}^I \rightharpoonup \dot{l} \ weak * \ in \ L^\infty(0,T;L^2(\Omega)) \tag{3.49}$$

$$h_\star^I \rightharpoonup h_\star \ weak * \ in \ L^\infty(0,T;H^1(\Omega)) \tag{3.50}$$

$$u_\star^I \rightharpoonup u_\star \ weak * \ in \ L^\infty(0,T;H^1(\Omega)) \tag{3.51}$$

$$h_\sharp^I \rightharpoonup h_\sharp \ weak * \ in \ L^\infty(0,T;L^2(\Omega)) \tag{3.52}$$

$$\ddot{h}^I \rightharpoonup \ddot{h} \ weak * \ in \ L^\infty(0,T;H^1(\Omega)) \tag{3.53}$$

Proof. *The proof is similar to that of corollary 21.* ■

Corollary 24 *Let $\dot{h}^I$ and l^I defined by (3.15) and (3.18) respectively. Then there exists sub -suites, equal denoted with the index I, such that $I \longrightarrow +\infty$*

$$l^I \longrightarrow l \ in \ C([0,T];\mathbf{H}^\beta(\Omega)) \cap C_s([0,T],\mathbf{H}^1(\Omega)), \ 0 \leq \beta \leq 1, \tag{3.54}$$

$$\dot{h}^I \longrightarrow \dot{h} \ in \ C([0,T];\mathbf{H}^\alpha(\Omega)) \cap C_s([0,T],\mathbf{L}^2(\Omega)), \ -1 \leq \alpha \leq 0, \tag{3.55}$$

optionally after a change on a set of measure zero.

Proof. *The proof of this result is a consequence of the boundedness of $\dot{h}^I$, $\ddot{h}^I$, l and $\dot{l}$ and the lemma1* ■

Uniqueness of the limit

In this section, we will show that all the terms in (3.41), (3.50), (3.53) and (3.54) are equal $h = l = h_\star = u_\star$ Then, from (3.19) and using the formula Barrow C^1 functions $[t_{i-1}, t_i]$, we have

$$\begin{aligned}
\left\| h^I(t) - h_\star^I(t) \right\|_{L^2(\Omega)} &= \left\| \frac{h^I(t)}{2} + \frac{h^I(t)}{2} - \frac{h^I(t_{i-1})}{2} - \frac{h^I(t_i)}{2} \right\|_{L^2(\Omega)} \\
= \left\| \frac{1}{2}\int_{t_{i-1}}^{t} \dot{h}^I(s)ds - \int_{t}^{t_i} \dot{h}^I(s)ds \right\|_{L^2(\Omega)} &\leq \left\| \frac{1}{2}\int_{t_{i-1}}^{t} \dot{h}^I(s)ds \right\|_{L^2(\Omega)} \\
+ \left\| \frac{1}{2}\int_{t_{i-1}}^{t} \dot{h}^I(s)ds \right\|_{L^2(\Omega)} &\leq \Delta t \left\| \dot{h}^I(s) \right\|_{L^\infty(0,T;L^2(\Omega))} \longrightarrow 0
\end{aligned} \tag{3.56}$$

when $I \longrightarrow \infty$. *For all* h *and* $h_\star$ *are equal to* $L^\infty(0,T;L^2(\Omega))$ *and then that* $h_\star \in L^\infty(0,T;H^1(\Omega))$ *as* $\dot{h} \in L^\infty(0,T;H^1(\Omega))$. *Similarly, it is shown that* h *and* $u_\star$ *are equal in* $H^1(\Omega)$. *In addition, and then* $\dot{l}$ *is bounded in* $L^\infty(0,T;L^2(\Omega))$,

$$\left\|l^I(t) - u_\star^I(t)\right\|_{L^2(\Omega)} = \left\|l^I(t) - l^I(t_i)\right\|_{L^2(\Omega)}$$

$$= \left\|\int_t^{t_i} \dot{l}^I(s)ds\right\|_{L^2(\Omega)} \leq \Delta t \left\|\dot{l}^I(s)\right\|_{L^\infty(0,T;L^2(\Omega))} \longrightarrow 0 \tag{3.57}$$

when $I \longrightarrow \infty$. *Similarly, it is shown that* $\dot{h}$ *and* $h_\sharp$ *coincide with* $L^2(\Omega)$. *Then,* $l = u_\star = h_\star = h$. *Hence forth, we denote this limit by* u. *In summary, we have shown the following convergences.*

Theorem 25 *Let* h^I, l^I, $h_\star^I$, $h_\sharp^I$ *and* $u_\star^I$ *are given by (3.14), (3.18) (3.19), (3.20)and (3.21) respectively. Then there exist* u *such that*

$$l^I \rightharpoonup u \text{ weak} * \text{ in } L^\infty(0,T;L^2(\Omega)) \tag{3.58}$$

$$\dot{l}^I \rightharpoonup \dot{u} \text{ weak} * \text{ in } L^\infty(0,T;L^2(\Omega)) \tag{3.59}$$

$$\dot{l}_\star^I \rightharpoonup u \text{ weak} * \text{ in } L^\infty(0,T;H^1(\Omega)) \tag{3.60}$$

$$l_\star^I \rightharpoonup u \text{ weak} * \text{ in } L^\infty(0,T;H^1(\Omega)) \tag{3.61}$$

$$u_\star^I \rightharpoonup u \text{ weak} * \text{ in } L^\infty(0,T;H^1(\Omega)) \tag{3.62}$$

$$h_\sharp^I \rightharpoonup \dot{u} \text{ weak} * \text{ in } L^\infty(0,T;L^2(\Omega)) \tag{3.63}$$

$$\nabla(h_\star^I) \rightharpoonup \nabla(u) \text{ weak} * \text{ in } L^\infty(0,T;L^2(\Omega)) \tag{3.64}$$

another, we have the limit

$$\lim_{I \longrightarrow +\infty} (h^I - h_\star^I) = 0 \tag{3.65}$$

$$\lim_{I \longrightarrow +\infty} (h^I - u_\star^I) = 0 \tag{3.66}$$

$$\lim_{I \longrightarrow +\infty} (l^I - l_\star^I) = 0 \tag{3.67}$$

strong in $L^\infty(0,T;L^2(\Omega))$.

Corollary 26 *Let h^I and l^I are given by (3.14), (3.18). Then there exist u such that*

$$\dot{h}^I \longrightarrow \dot{u} \text{ in } C([0,T];\mathbf{H}^{\alpha}(\Omega)) \cap C_s([0,T],\mathbf{L}^2(\Omega)),\ -1 \leq \alpha \leq 0, \tag{3.68}$$

$$l^I \longrightarrow u \text{ in } C([0,T];\mathbf{H}^{\beta}(\Omega)) \cap C_s([0,T],\mathbf{H}^1(\Omega)),\ 0 \leq \beta \leq 1, \tag{3.69}$$

Proof. *From Corollary 24 and the uniqueness of the weak limit.* ■

Theorem 27 *Let $h^I_\star$ and $u^I_\star$ are given by (3.19) and (3.21) respectively .Then*

$$h^I_\star - u^I_\star \longrightarrow 0 \text{ in } D'(0,T;H^1(\Omega)), \tag{3.70}$$

$$l^I - u^I_\star \longrightarrow 0 \text{ in } L^\infty(0,T;H^r(\Omega)),\ 0 \leq r \leq 1, \tag{3.71}$$

when $I \longrightarrow \infty$.

Proof. *Let $\varphi \in D(0,T)$. Let $I \geq I_0$, where I_0 is such that support of φ is continue in $[\delta_0, T-\delta_0]$ with $\delta_0 = T/2^{I_0}$, so that supp$(\varphi) \subset [\delta, T-\delta]$, be $\delta = T/2^I$. Then*

$$\int_0^T (h^I_\star - u^I_\star)\varphi dt = \sum_{i=0}^{2^I-1} \int_{t_i}^{t_{i+1}} (h^I_\star(t) - u^I_\star(t))\varphi(t)dt$$

$$= \sum_{i=0}^{2^I-1} \int_{t_i}^{t_{i+1}} (u^{i-1} - u^i)\theta_I(t)\varphi(t)dt$$

$$= \sum_{i=0}^{2^I-1} \int_{t_i}^{t_{i+1}} u^i(\varphi_\delta - \varphi)(t)dt,$$

be $\varphi_\delta = \varphi(t+\delta)$ and for all, $|\varphi - \varphi_\delta| \leq c\delta$, be $c = \max\left|\dfrac{d\varphi}{dt}\right|$. Consequently,

$$\left\|\int_0^T (h^I_\star - u^I_\star)\varphi dt\right\|_{H^1(\Omega)} \leq c\delta^2 \sum_{i=1}^{2^I-1} \left\|u^I\right\|_{H^1(\Omega)}$$

$$\leq c\delta^2 \left((2^I-1)\sum_{i=1}^{2^I-1} \left\|u^I\right\|^2_{H^1(\Omega)}\right)^{\frac{1}{2}}$$

$$\leq c\delta^2 \left((2^I-1)^2 C\right)^{\frac{1}{2}}$$

$$\leq \hat{c}\delta^2 2^2$$

$$= \hat{c}\delta^2 T/\delta = \hat{c}\delta T \longrightarrow 0 \ if \ \delta \longrightarrow 0.$$

Where c, C and $\hat{c}$ are positive constants. To prove (3.71) by using the convergences, following

$$l^I \rightharpoonup l \ weak * \ \ in \ L^\infty(0,T;H^1(\Omega))$$

$$\dot{l}^I \rightharpoonup \dot{l} \ weak * \ \ in \ L^\infty(0,T;L^2(\Omega)).$$

Let $0 < r = \theta\alpha + (1-\theta)\beta < 1$, and for $r \neq 1/2$. There exist a set $A \subset [0,T]$ such that $mes(A) = 0$ and for all t_1, $t_2 \in [0,T]\backslash A$, with $t_1 \leq t_2$

$$\left\|l^I(t_2) - l^I(t_1)\right\|_{H^1(\Omega)} \leq M_\theta \left\|l^I(t_2) - l^I(t_1)\right\|^\theta_{L^2(\Omega)} \left\|l^I(t_2) - l^I(t_1)\right\|^{1-\theta}_{H^1(\Omega)}$$

$$\leq M_\theta \left(\int_{t_1}^{t_2} \left\|\dot{l}^I(t)\right\|_{L^2(\Omega)} dt\right)^\theta$$

$$\leq M_\theta (t_1 - t_2)^{1/2}. \tag{3.72}$$

In particular, for all $t \in [t_{i-1}, t_i]\backslash A$, then

$$\left\|u^I_\star - l^I\right\|_{H^r(\Omega)} = \left\|l^I(t_i) - l^I(t)\right\|_{H^r(\Omega)} \leq M_\theta (t_i - t)^{\frac{\theta}{2}} \longrightarrow 0,$$

when $I \longrightarrow \infty$. ■

Theorem 28 *Let u be the limit presented in theorem 25, then*

$$u \in L^\infty(0,T;K) \tag{3.73}$$

$$\dot{u} \in L^\infty(0,T;L^2(\Omega)) \tag{3.74}$$

$$\ddot{u} \in D'(0,T;L^2(\Omega)) \tag{3.75}$$

$$\nabla u \in D'(0,T;H(div)) \cap L^\infty(0,T;L^2(\Omega)) \tag{3.76}$$

Proof. *According to the relation (3.11) we have*

$$\int_\Omega \ddot{h}^I v dx + a(h^I_\star, v) \geq L^i(v) \ \forall v \in K$$

this inequality is true for $v(t) \in L^2(0,T;V)$*;* $v(t) \in K$ *a.e and*

$$\int_0^T \int_\Omega \ddot{h}^I v(t)dxdt + \int_0^T a(h^I_\star, v(t))dt \geq \int_0^T L_\star(v(t))dt, \ \forall v \in L^2(0,T;K)$$

$v(t) = \varphi(t)\omega(t)$ *such that* $\varphi(t) \in D(0,T)$*;* $\omega(t) \in D(\Omega)$ *we have integration by parts we find the first term*

$$-\int_0^T \int_\Omega \dot{h}v(t)dxdt + \int_0^T a(h^I_\star, v(t))dt \geq \int_0^T \int_\Omega f_\star(v(t))dt,$$

for $v(t) = -\varphi(t)\omega(t)$*, found*

$$\int_0^T \int_\Omega \ddot{h}v(t)dxdt + \int_0^T a(h^I_\star, v(t))dt = \int_0^T f_\star(v(t))dt, \tag{3.77}$$

we can also prove that

$$\int_0^T a(h^I_\star, v(t))dt \longrightarrow \int_0^T a(h^I_\star, v(t))dt, \ if \ I \longrightarrow \infty \tag{3.78}$$

and that

$$\int_0^T \int_\Omega f_\star(v(t))dxdt \longrightarrow \int_0^T \int_\Omega f(v(t))dxdt. \tag{3.79}$$

Therefore

$$\int_0^T \int_\Omega \dot{u}\dot{v}(t)dxdt + \int_0^T a(u, v(t))dt = \int_0^T \int_\Omega fv(t)dxdt \ for \ \forall v \in (D(0,T) \times \Omega).$$

To prove, (3.73) it suffices to see that $u^I_\star = u \in K$ *and since* $u^I_\star \overset{*}{\rightharpoonup} u$ *in* $L^\infty(0,T;H^1(\Omega))$ *then* $u^I_\star \overset{*}{\rightharpoonup} u$ *in* $L^\infty(0,T;K)$ *where (3.73). We have (3.74) directly by* $\dot{h}^I \overset{*}{\rightharpoonup} \dot{h}^I$ *in* $L^\infty(0,T;L^2(\Omega))$ *with a test function* $v(t) \in D(0,T;K)$ *found*

$$\int_0^T \int_\Omega \ddot{u}v(t)dxdt - \int_0^T \Delta(u)v(t)dt = \int_0^T \int_\Omega fv(t)dxdt \ for \ \forall v \in (D(0,T) \times \Omega)$$

$$\ddot{u} - \Delta u - f = 0 \ a.e \ in \ Q = (0,T) \times \Omega. \tag{3.80}$$

As $\dot{u} \in L^\infty(0,T;L^2(\Omega))$*, then* $\ddot{u} \in D'(0,T;L^2(\Omega))$

$$\ddot{u} = (\Delta u + f) \in L^\infty(0,T;H^{-1}(\Omega))$$

$$\ddot{u} \in D'(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^{-1}(\Omega)) \ where \ (3.75).$$

Such that $\nabla(h^I_\star) \longrightarrow \nabla(u)$ *in* $L^\infty(0,T;L^2(\Omega))$ *and* $\nabla u \in H(div) \Longrightarrow \nabla u \in D'(0,T;H(div))$ *where* $\nabla u \in L^\infty(0,T;L^2(\Omega)) \cap D'(0,T;H(div))$ *where(3.76).* ■

Theorem 29 *Let u the precedent limit by theorem 25 then u verifying the condition (3.5).*

Proof. *From corollary 26*

$$l^I \longrightarrow u \ in \ C([0,T];L^2(\Omega)).$$

Then, as $l^I(0) = p^0$ for all I, we can pass to the limit and obtain that $u(0) = p^0$.
Another

$$\dot{h}^I \longrightarrow \dot{u} \ in \ C_s([0,T];L^2(\Omega)),$$

then

$$\int_\Omega \dot{h}^I(0)vdx \longrightarrow \int_\Omega \dot{u}(0).vdx, \ \forall v \in L^2(\Omega)$$

witch give $\dot{h}^I(0) = p^1$ for all I then

$$\int_\Omega \dot{u}(0)vdx = \int_\Omega p^1 vdx, \ \forall \in L^2(\Omega).$$

■

Step 5: The limit u is solution of problem (P.C)

In this step, we will show that the weak limit u is a weak solution of problem $(P.C)$. For this it suffices to show that u is a solution of the problem

$$(P.V)\begin{cases} \langle \ddot{u}, v\rangle + a(u,v) = L(v) + \langle \partial_n u.n, v_n\rangle \ \forall v \in V \\ \langle \partial_n u.n, v_n - u_n\rangle \geq 0 \ \forall v \in K \\ u(x,0) = p^0, \ \dot{u}(x,0) = p^1. \end{cases}$$

Let consider the following problem

$$\begin{cases} \langle \ddot{h}^I, v\rangle + a(h^i,v) = L^i(v) + \langle \partial_n h^i.n, v_n\rangle \ \forall v \in V \\ \langle \partial_n h^i.n, v_n - u^i_n\rangle \geq 0 \ \forall v \in K \end{cases}$$

which comes from the properties of solutions of the problem (P^iV). With the aid of functions $h^I_\star$ and $u^I_\star$. Defined in the previous problem on the interval $[0,T]$ following suite

$$\begin{cases} \int_0^T \left\langle \ddot{h}^I, v(t)\right\rangle dt + \int_0^T a(h^I_\star, v(t))dt = \int_0^T L^I_\star(v(t))dt + \int_0^T \left\langle \partial_n(h^I_\star).n, v_n(t)\right\rangle dt \ \forall v \in L^1(0,T;V) \\ \int_0^T \left\langle \partial_n(h^I_\star), v_n(t) - u^I_{\star n}\right\rangle dt \geq 0, \ \forall v \in L^1(0,T;K) \end{cases}$$

such that

$$L^I_\star(v) = \int_\Omega f^I_\star v dx + \int_{\Gamma_g} g^I_\star(v) d\Gamma \ where \ f^I_\star(t) = f^i, g^I_\star = g^i \ in \ ]t_{i-1}, t_i].$$

For $v(t) \in D(0,T;V)$ *we have*

$$\left\langle \ddot{h}^I, v(t) \right\rangle = -\left\langle \dot{h}^I, \dot{v}(t) \right\rangle \ in \ sens \ of \ D'(0,T).$$

And

$$\dot{h}^I \rightharpoonup \dot{u} \ in \ L^\infty(0,T;L^2(\Omega))$$

$$i.e \int_0^T \left\langle \dot{h}^I, g(t) \right\rangle dt \longrightarrow \int_0^T \langle \dot{u}, g(t) \rangle \, dt \ \forall g \in L^1(0,T;L^2(\Omega))$$

and the fact that $v(t) \in L^1(0,T;L^2(\Omega))$, *then we can pass to the limit on* I, *we obtain*

$$\int_0^T \int_\Omega \dot{h}^I \dot{v}(t) dx dt \longrightarrow \int_0^T \int_\Omega \dot{u}\dot{v}(t) dx dt.$$

$$\int_0^T a(h^I_\star, v(t)) dt \longrightarrow \int_0^T a(u, v(t)) dt,$$

$$\int_0^T \int_\Omega L^I_\star(v(t)) dt \longrightarrow \int_0^T \int_\Omega L(v(t)) dx dt.$$

Therefore, we have

$$\int_0^T \langle \ddot{u}, v(t) \rangle \, dt + \int_0^T a(u, v(t)) dt = \int_0^T L(v(t)) dt + \lim_{I \longrightarrow \infty} \int_0^T \left\langle \partial_n(h^I_\star).n, v_n(t) \right\rangle dt \quad (3.81)$$

the form

$$\lambda_n : v_n(t) \longrightarrow \int_0^T \langle \ddot{u}, v(t) \rangle \, dt + \int_0^T a(u, v(t)) dt - \int_0^T L(v(t)) dt$$

defines a continuous linear form on $D(0,T;H^{\frac{1}{2}}(\Gamma_0))$. *Then*

$$\int_0^T \langle \ddot{u}, v(t) \rangle \, dt + \int_0^T a(u, v(t)) dt = \int_0^T L(v(t)) dt + \int_0^T \langle \lambda_n, v_n(t) \rangle \, dt \ \forall v \in D(0,T;V)$$

another

$$\int_0^T \left\langle \partial_n(h^I_\star).n, v_n(t) - u_{n\star}(t) \right\rangle dt \geq 0 \ \forall v \in L^1(0,T;K)$$

then

$$\int_0^T \left\langle \partial_n(h^I_\star).n, v_n(t) \right\rangle dt \geq \int_0^T \left\langle \partial_n(h^I_\star).n, u_{n\star}(t) \right\rangle dt \ \forall v \in L^1(0,T;K).$$

Hence , we have from for $v = u^I_\star$ in $D'(0,T;H^{-\frac{1}{2}}(\Gamma_0))$, that

$$\int_0^T \langle \lambda_n, v_n(t)\rangle\, dt \geq \lim_{I\longrightarrow\infty} \int_0^T \left\langle \partial_n(h^I_\star).n, u^I_{\star n}(t)\right\rangle dt$$

$$\geq \lim_{I\longrightarrow\infty} \left(\int_0^T \left\langle \ddot{h}^I, v^I_\star(t)\right\rangle dt - \int_0^T a(h^I_\star, u^I_\star)dt + \int_0^T L^I_\star(u^I_\star)dt \right)$$

another, we have

$$\left\langle \ddot{h}^I, u^I_\star(t)\right\rangle \longrightarrow \langle \ddot{u}, u\rangle$$

and

$$\int_0^T a(h^I_\star, u^I_\star)dt = \int_0^T a(h^I_\star - u^I_\star, u^I_\star)dt + \int_0^T a(u^I_\star, u^I_\star)dt.$$

According to (3.70)

$$a(h^I_\star - u^I_\star, u^I_\star) \longrightarrow 0 \text{ in sens of } D'(0,T)$$

and according the semi continuity of norm define with $a(.,.)$, we have

$$\lim_{I\longrightarrow\infty} \int_0^T a(u^I_\star, u^I_\star)dt \geq \int_0^T a(u,u)dt$$

and

$$\int_0^T L^I_\star(u^I_\star)dt \longrightarrow \int_0^T L(u)dt.$$

By consequence

$$\int_0^T \langle \lambda_n, v_n(t)\rangle\, dt \geq \int_0^T \langle \ddot{u}, u\rangle\, dt + \int_0^T a(u,u)dt - \int_0^T L(u)dt = \int_0^T \langle \lambda_n, u_n(t)\rangle\, dt$$

where

$$\langle \lambda_n, v_n(t) - u_n(t)\rangle \geq 0 \text{ in } D'(0,T;K).$$

Finally, we obtain

$$\begin{cases} \int_0^T \langle \ddot{u}, v(t)\rangle\, dt + \int_0^T a(u, v(t))dt = \int_0^T L(v(t))dt + \int_0^T \langle \lambda_n, v_n\rangle\, dt \;\forall v \in D(0,T;V) \\ \int_0^T \langle \lambda_n, v_n(t) - u_n(t)\rangle \geq 0 \;\forall v \in (0,T;K) \end{cases}$$

■

Remark 30 *Conditions of regularity, we can see $\lambda_n = \partial_n u.n$.*

Conclusion

We discussed in this memoir, we presented an existence result of the dynamic Signorini problem. Among the issues encountered are those related to regularity of the solution and the uniqueness of the solution. Was another issue related to extensions of the problem and study the same problem but this time with friction.
Then we have as prospective

- *Study uniqueness of the solution.*
- *Regularity of the weak solution.*
- *Friction dynamic signorini problem.*

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