Spherical method of calculation of the trasport coefficients for plasmas in uniform electric and magnetic fields

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Abstract: The knowledge of the distribution function permits us to determine a number of important parameters, such as, electron mobility, conductivity, etc. The purpose of this work is the development of methods for calculating the energy distribution of electrons EEDF in a gas of low ion density under the influence of uniform electric and magnetic fields using the classical Two-term expansion where f is expand in terms of Legendre polynomials (spherical harmonics expansion). In this approximation, the Boltzmann equation takes the form of a convection diffusion continuity equation. The special configurations of the magnetic field parallel and perpendicular to the electric field are discussed in detail.

KEYWORDS: Boltzmann equation, Coefficients transports, Spherical method.

1. Introduction

Fluid models of gas discharges describe the transport of electrons, ions and possibly other reactive particle species by the first few moments of the Boltzmann equation (BE). Transport coefficients may be rather specific for the discharge conditions. In particular, coefficients concerning electrons depend on the electron energy distribution function (EEDF), which in general is not Maxwellian but varies considerably depending on the conditions.

The electron distribution f depends on five coordinates: $\vec{r}, t, V, \theta$ and φ. We simplify the θ and φ dependence by classical two-term approximation (section 2). To simplify the time dependence, is assumed loss due to ionization and attachment. We then describe the collision term put all pieces together into one equation for EEDF.

2. Two-term approximation

The Boltzmann equation for an ensemble of electrons in an ionized gas is:

$$\frac{\partial f}{\partial t} + \vec{V}\cdot\vec{\nabla}_r f - \frac{e}{m}\left(\vec{E}+\vec{V}\times\vec{B}\right)\cdot\vec{\nabla}_V f = C[f] \tag{1}$$

Where f is the electron distribution in six-dimensional phase space, $\vec{V}$ are the velocity coordinates, e is the elementary charge, m is the electron mass $\vec{E}$ and $\vec{B}$ uniform electric and magnetic fields, $\vec{\nabla}_V$ is the velocity-gradient operator and C represent the rate of change in f due to collisions.

A common approach to solve equation (1) is to expand f in terms of Legendre polynomials (spherical harmonics expansion) and then construct from equation (1) a set of equations for the expansion coefficients. Using the two-term approximation we expand f as:

$$f = f_{lms}Y_{lms} = f_{000} + f_{100}\cos\theta + f_{110}\sin\theta\cos\varphi + f_{111}\sin\theta\sin\varphi \tag{2}$$

With: $Y_{000} = 1$, $Y_{100} = \cos\theta$, $Y_{110} = \sin\theta\cos\varphi$, $Y_{111} = \sin\theta\sin\varphi$

where the f_{lms} terms are function of time, position, and velocity-magnitude only. Noting that the three spherical harmonics of order 1 are the direction cosines of unit vector in the direction of velocity suggests the form:

$$f = f_0 + \vec{f}_1 \cdot \frac{\vec{V}}{V} \tag{3}$$

$$\vec{f}_1 = f_{100}\,\vec{i} + f_{110}\,\vec{j} + f_{111}\,\vec{k} \tag{4}$$

The average velocity can be obtained by averaging:

$$\vec{V} = V\left(Y_{110}\,\vec{i} + Y_{111}\,\vec{j} + Y_{100}\,\vec{k}\right) \tag{5}$$

where f_0 is the isotropic part of f and $\vec{f}_1$ is an anisotropic perturbation, it is negative. Also note that f is normalized as:

$$n = 4\pi \int_0^{\infty} f_0 V^2 dV \tag{6}$$

where n is the electron number density. The DC drift velocity (mean velocity) $\vec{V}_d$ and the current $\vec{j}$ density vectors are given by:

$$\vec{V}_d = \frac{4\pi}{3n} \int_0^{\infty} \vec{f}_1 V^3 dV \tag{7}$$

$$\vec{j} = ne\vec{V}_d = ne\mu\vec{E} - e\vec{\nabla}_r\left(nD\right) = \sigma\vec{E} - e\vec{\nabla}_r\left(nD\right) \tag{8}$$

where μ is the mobility and σ is the conductivity and D is diffusion coefficient.

Equations for f_0 and $\vec{f}_1$ are found from equation (1) by substituting equation (3), multiplying by the respective Legendre polynomials (1 and $Y_{100}\,\vec{i} + Y_{110}\,\vec{j} + Y_{111}\,\vec{k}$) and integrating over solid angle $\left(\sin\theta\, d\theta d\varphi\right)$, we obtain:

$$\frac{\partial f_0}{\partial t} + \frac{\gamma}{3}\varepsilon^{1/2}\vec{\nabla}_r \cdot \vec{f}_1 - \frac{\gamma}{3}\varepsilon^{-1/2}\frac{\partial}{\partial\varepsilon}\left(\varepsilon\vec{E}\cdot\vec{f}_1\right) = C_0 \tag{9}$$

$$\frac{\partial \vec{f}_1}{\partial t} + \frac{\gamma}{3}\varepsilon^{1/2}\vec{\nabla}_r f_0 - \gamma\varepsilon^{1/2}\vec{E}\frac{\partial f_0}{\partial\varepsilon} - \frac{\gamma^2}{2}\vec{B}\times\vec{f}_1 = -N\sigma_m\gamma\varepsilon^{1/2}\vec{f}_1 \tag{10}$$

where $\gamma = \left(2e/m\right)^{1/2}$ is a constant and $\varepsilon = (\mathrm{V}/\gamma)^2$ is the electron energy in electronvolts. The right-hand side of equation (10) contains the total momentum transfer cross-section $\sigma_m = \sum_k x_k\sigma_k$ consisting of contributions from possible collision processes with gas particles with x_k the mole fraction of the target species of collision process, and σ_k effective momentum transfer cross-section. The right-hand side of equation (9) represents the change in f_0 due to collision.

3. Exponential spatial growth without time dependence

In general f cannot be constant in both time and space because some collision processes (ionization, attachment) do not conserve the total number of electrons. There is a simple technique to approximately describe the effects of net electron production; we separate the energy dependence of f from its dependence on time and space:

$$f_0\left(\vec{r},t,\varepsilon\right) = \frac{1}{2\pi\gamma^3}F_0\left(\varepsilon\right)n\left(\vec{r},t\right) \tag{11}$$

$$\vec{f}_1\left(\vec{r},t,\varepsilon\right) = \frac{1}{2\pi\gamma^3}\vec{F}_1\left(\varepsilon\right)n\left(\vec{r},t\right) \tag{12}$$

Where the energy distribution F_0 and $\vec{F}_1$ is constant in time and space and normalized by

$$\int_0^\infty \varepsilon^{1/2} F_0 d\varepsilon = 1$$

The time or space dependence of the electron density n is now related to the net electron production rate. For this, we consider the case corresponds to Steady State Townsend experiments, the exponential spatial growth without time dependence: $\frac{\partial n}{\partial t} = 0$.

4. Special configurations of the fields

4.1 Magnetic field parallel to electric field

For $\vec{E}$ and $\vec{B}$ along the Z axis, the equation (10) becomes:

$$\begin{pmatrix} F_{110} \\ F_{111} \\ F_{100} \end{pmatrix} = -\frac{1}{\omega_c^2 + \sigma_m^2} \begin{pmatrix} \sigma_m & -\omega_c & 0 \\ \omega_c & \sigma_m & 0 \\ 0 & 0 & \frac{\omega_c^2 + \sigma_m^2}{\sigma_m} \end{pmatrix} \begin{pmatrix} \frac{1}{n}\frac{\partial n}{\partial x}\frac{F_0}{N} \\ \frac{1}{n}\frac{\partial n}{\partial y}\frac{F_0}{N} \\ \frac{1}{n}\frac{\partial n}{\partial z}\frac{F_0}{N} - \frac{E}{N}\frac{\partial F_0}{\partial \varepsilon} \end{pmatrix} \qquad (13)$$

Where $\omega_c = \frac{\gamma}{2}\varepsilon^{-1/2}\frac{B}{N}$. And equation (09) can again be written in form:

$$-\frac{\gamma}{3}\frac{\partial}{\partial \varepsilon}\left(\left(\frac{E}{N}\right)^2 \frac{\varepsilon}{\sigma_m}\frac{\partial F_0}{\partial \varepsilon}\right) = \overline{C}_0 + \overline{R} \qquad (14)$$

Where,

$$\overline{R} = -\frac{\gamma}{3}\frac{1}{nN}\frac{\partial n}{\partial z}\left[\frac{\varepsilon}{\sigma_m}\left(2\frac{E}{N}\frac{\partial F_0}{\partial \varepsilon} - \frac{1}{n}\frac{\partial n}{\partial z}\frac{F_0}{N}\right) + \frac{E}{N}F\frac{\partial}{\partial \varepsilon}\left(\frac{\varepsilon}{\sigma_m}\right)\right] - \frac{\gamma}{3}\frac{\varepsilon}{nN}\frac{1}{n}\frac{F_0}{N}\left[\left(\frac{\partial n}{\partial x}\right)^2 + \left(\frac{\partial n}{\partial y}\right)^2\right]$$

And $\overline{C}_0 = \frac{C_0 2\pi\gamma^3}{nN}\varepsilon^{1/2}$ is the collision terms contribution from all different collision processes k withe the neutral gas particles and from electron-electron collisions:

$$\overline{C}_0 = \sum_k \overline{C}_{0,k} + \overline{C}_{0,e} \qquad (15)$$

Where,

$$\overline{C}_{0,k=elastic} = \gamma x_k \frac{2m}{M_k}\frac{\partial}{\partial \varepsilon}\left[\varepsilon^2 \sigma_k \left(F_0 + \frac{k_B T}{e}\frac{\partial F_0}{\partial \varepsilon}\right)\right]$$

$$\overline{C}_{0,k=inelastic} = -\gamma x_k \left[\varepsilon \sigma_k(\varepsilon) F_0(\varepsilon) - (\varepsilon + u_k)\sigma_k(\varepsilon + u_k) F_0(\varepsilon + u_k)\right]$$

$$\overline{C}_{0,k=ionization} = -\gamma x_k \left[\varepsilon \sigma_k(\varepsilon) F_0(\varepsilon) - 2(2\varepsilon + u_k)\sigma_k(2\varepsilon + u_k) F_0(2\varepsilon + u_k)\right]$$

$$\overline{C}_{0,k=attachment} = -\gamma x_k \varepsilon \sigma_k(\varepsilon) F_0(\varepsilon)$$

where M_k is the mass of target particles and T is their temperature, u_k is threshold energy of the collision.

$$\overline{C}_{0,e} = a\frac{n}{N}\frac{\partial}{\partial \varepsilon}\left[3A_1 F_0 + 2\left(A_2 + \varepsilon^{3/2} A_3\right)\frac{\partial F_0}{\partial \varepsilon}\right]$$

Where

$$A_1 = \int_0^{\varepsilon} u^{1/2} F_0(u)\, du$$

$$A_2 = \int_0^{\varepsilon} u^{3/2} F_0(u)\, du$$

$$A_1 = \int_{\varepsilon}^{\infty} F_0(u)\, du$$

$$a = \frac{e^2\gamma}{24\pi\varepsilon_0^2} \ln\left[\frac{12\pi\left(\varepsilon_0 k_B T_e\right)^{3/2}}{e^3 n^{1/2}}\right], \qquad k_B T = \frac{2}{3} e A_2(\infty)$$

4.2 Equation for EEDF

When combining the previous equation, we fine an equation for F_0 that looks like a convection-diffusion continuity equation in energy space:

$$\frac{\partial}{\partial\varepsilon}\left(\overline{W}F_0 - \overline{D}\frac{\partial F_0}{\partial\varepsilon}\right) = \overline{S} \qquad (16)$$

where,

$$\overline{W} = -\gamma\varepsilon^2\sigma_\varepsilon - 3a\frac{n}{N}A_1 + \frac{\gamma}{3}\frac{1}{nN}\frac{\partial n}{\partial z}\frac{\varepsilon}{\sigma_m}\left(2\frac{E}{N}\right)$$

$$\overline{D} = \frac{\gamma}{3}\frac{\varepsilon}{\sigma_m}\left(\frac{E}{N}\right)^2 + \gamma\frac{k_B T}{e}\varepsilon^2\sigma_\varepsilon + 2a\frac{n}{N}\left(A_2 + A_3\varepsilon^{3/2}\right)$$

$$\sigma_\varepsilon = \sum_{k=elastic}\frac{2m}{M_k}x_k\sigma_k$$

$$\overline{S}(F_0) = \sum_{k\neq elastic}\overline{C}_{0,k} + \frac{\gamma}{3}\frac{1}{nN}\frac{\partial n}{\partial z}\left[\frac{\varepsilon}{\sigma_m}\frac{1}{n}\frac{\partial n}{\partial z}\frac{F_0}{N} - \frac{E}{N}F_0\frac{\partial}{\partial\varepsilon}\left(\frac{\varepsilon}{\sigma_m}\right)\right] - \frac{\gamma}{3}\frac{\varepsilon}{nN}\frac{1}{n}\frac{F_0}{N}\left[\left(\frac{\partial n}{\partial x}\right)^2 + \left(\frac{\partial n}{\partial y}\right)^2\right]$$

The equation (16) is no ordinary differential equation and solving it requires some special care. The numerical solution of this equation is not finish again.

4.3 Coefficients of transport for fluid equations

When a constant magnetic field is included the conductivity and diffusion coefficients must be represented by matrices. By equation (7), substitution of equation (14) for $\vec{F}_1$ into (8) for $\vec{J}$ yields the conductivity and diffusion matrix:

$$N\sigma = nNe\mu = \frac{\gamma ne}{3}\int_0^{\infty}\frac{\varepsilon}{\omega_c^2 + \sigma_m^2}\begin{pmatrix}\sigma_m & -\omega_c & 0\\ \omega_c & \sigma_m & 0\\ 0 & 0 & \dfrac{\omega_c^2+\sigma_m^2}{\sigma_m}\end{pmatrix}\frac{\partial F_0}{\partial\varepsilon}\,d\varepsilon$$

$$ND = \frac{\gamma}{3}\int_0^{\infty}\frac{\varepsilon}{\omega_c^2 + \sigma_m^2}\begin{pmatrix}\sigma_m & -\omega_c & 0\\ \omega_c & \sigma_m & 0\\ 0 & 0 & \dfrac{\omega_c^2+\sigma_m^2}{\sigma_m}\end{pmatrix}F_0\,d\varepsilon$$

5. Magnetic field perpendicular to electric field

For $\vec{B}$ along the y axis, the equation (10) becomes:

$$\begin{pmatrix} F_{110} \\ F_{111} \\ F_{100} \end{pmatrix} = -\frac{1}{\omega_c^2+\sigma_m^2}\begin{pmatrix} \sigma_m & 0 & \omega_c \\ 0 & \frac{\omega_c^2+\sigma_m^2}{\sigma_m} & 0 \\ -\omega_c & 0 & \sigma_m \end{pmatrix}\begin{pmatrix} \frac{1}{n}\frac{\partial n}{\partial x}\frac{F_0}{N} \\ \frac{1}{n}\frac{\partial n}{\partial y}\frac{F_0}{N} \\ \frac{1}{n}\frac{\partial n}{\partial z}\frac{F_0}{N} - \frac{E}{N}\frac{\partial F_0}{\partial \varepsilon} \end{pmatrix} \quad (13)$$

And equation (10) can again be written in form:

$$-\frac{\gamma}{3}\frac{\partial}{\partial\varepsilon}\left(\left(\frac{E}{N}\right)^2\frac{\varepsilon\sigma_m}{\sigma_m^2+\omega_c^2}\frac{\partial F_0}{\partial\varepsilon}\right) = \overline{C}_0 + \overline{R} \quad (14)$$

Where,

$$\overline{R} = -\frac{\gamma}{3}\frac{1}{nN}\frac{\partial n}{\partial z}\left[\frac{\varepsilon\sigma_m}{\omega_c^2+\sigma_m^2}\left(2\left(\frac{E}{N}\right)\frac{\partial F_0}{\partial\varepsilon} - \frac{F_0}{N}\frac{1}{n}\frac{\partial n}{\partial z}\right) + \frac{E}{N}F_0\frac{\partial}{\partial\varepsilon}\left(\frac{\varepsilon\sigma_m}{\omega_c^2+\sigma_m^2}\right)\right]$$
$$+\frac{\gamma}{3}\frac{1}{nN}\frac{\partial n}{\partial x}\left[\frac{\varepsilon\sigma_m}{\omega_c^2+\sigma_m^2}\left(\frac{F_0}{N}\frac{1}{n}\frac{\partial n}{\partial x}\right) + \frac{E}{N}F_0\frac{\partial}{\partial\varepsilon}\left(\frac{\varepsilon\omega_c}{\omega_c^2+\sigma_m^2}\right)\right] + \frac{\gamma}{3}\frac{\varepsilon}{\sigma_m}\frac{1}{nN}\left(\frac{\partial n}{\partial y}\right)^2\frac{F_0}{N}$$

5.1 Equation for EEDF

In this case, we fine an equation for F_0:

$$\frac{\partial}{\partial\varepsilon}\left(\overline{W}F_0 - \overline{D}\frac{\partial F_0}{\partial\varepsilon}\right) = \overline{S} \quad (15)$$

where,

$$\overline{W} = -\gamma\varepsilon^2\sigma_\varepsilon - 3a\frac{n}{N}A_1 + \frac{\gamma}{3}\frac{1}{nN}\frac{\partial n}{\partial z}\frac{\varepsilon\sigma_m}{\sigma_m^2+\omega_c^2}\left(2\frac{E}{N}\right)$$

$$\overline{D} = \frac{\gamma}{3}\frac{\varepsilon}{\sigma_m}\left(\frac{E}{N}\right)^2 + \gamma\frac{k_BT}{e}\varepsilon^2\sigma_\varepsilon + 2a\frac{n}{N}\left(A_2 + A_3\varepsilon^{3/2}\right)$$

$$\sigma_\varepsilon = \sum_{k=elastic}\frac{2m}{M_k}x_k\sigma_k$$

$$\overline{S}(F_0) = \sum_{k\neq elastic}\overline{C}_{0,k} + \frac{\gamma}{3}\frac{1}{nN}\frac{\partial n}{\partial z}\left[\frac{\varepsilon\sigma_m}{\omega_c^2+\sigma_m^2}\frac{F_0}{N}\frac{1}{n}\frac{\partial n}{\partial z} - \frac{E}{N}F_0\frac{\partial}{\partial\varepsilon}\left(\frac{\varepsilon\sigma_m}{\omega_c^2+\sigma_m^2}\right)\right]$$
$$+\frac{\gamma}{3}\frac{1}{nN}\frac{\partial n}{\partial x}\left[\frac{\varepsilon\sigma_m}{\omega_c^2+\sigma_m^2}\left(\frac{F_0}{N}\frac{1}{n}\frac{\partial n}{\partial x}\right) + \frac{E}{N}F_0\frac{\partial}{\partial\varepsilon}\left(\frac{\varepsilon\omega_c}{\omega_c^2+\sigma_m^2}\right)\right] + \frac{\gamma}{3}\frac{\varepsilon}{\sigma_m}\frac{1}{nN}\left(\frac{\partial n}{\partial y}\right)^2\frac{F_0}{N}$$

6. Conclusion

The Solutions the Boltzmann equation uniform electric and magnetic fields, using the classical two-term expansion, and is able to account for exponential spatial growth model, electron-neutral and electron-electron collisions. We show that for approximations we use, the Boltzmann equation takes the form of a convection-diffusion continuity equation. To solve this equation we can use exponential scheme commonly used for convection-diffusion problems.

References

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