Path Integral Formalism of Klein-Gordon Oscillator Particle in Non-Commutative Phase Space

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RESUME : Dans l'espace des phases non-commutative basé sur l'introduction des paramètres de déformations dans les relations de Heisenberg connues, nous allons utiliser la méthode d'intégrale de chemin pour construire de solution exacte de la fonction de Green associée au problème de l'oscillateur de Klein-Gordon. On va clarifier que, les expressions de spectre d'énergie et la fonction d'onde correspondant sont similaires au problème d'une particule de Klein-Gordon soumise à un champ magnétique constant dans l'espace des phases ordinaire. Les cas limites sont déduits pour les petits paramètres de déformations.

MOTS-CLES : Intégrale de chemin, la fonction de Green, Particule de Klein-Gordon, l'espace des phases non-commutative.

ABSTRACT: In the non-commutative phase space based on introducing the parameters of deformation in the Heisenberg's known relations, we shall use the path integral method for constructing exact solutions to Green's function which is associated with the Klein-Gordon's oscillator problem. Then, we shall clarify the expressions of energy spectrum and the wave function corresponding are similar to the problem of Klein-Gordon particle under a constant magnetic field in ordinary phase space. The limit cases are then deduced from small parameters of deformations.

KEYWORDS: Path integral, the Green function, Klein-Gordon particle, the non-commutative phase space.

I. Introduction

In order to describe a non-commutative (NC) phase space in the plan $(\tilde{x},\tilde{y})$, we must change the usual commutations relations as follow [1]:

$$\left[\hat{\tilde{x}}_i,\hat{\tilde{p}}_j\right]=i\delta_{ij},\ \left[\hat{\tilde{x}}_i,\hat{\tilde{x}}_j\right]=i\theta_{ij}\ ,\ \ \left[\hat{\tilde{p}}_i,\hat{\tilde{p}}_j\right]=i\eta_{ij}\ \text{with}\ i,j=1,2., \tag{1}$$

where θ_{ij} and η_{ij} represent parameters of deformation, which describe the non-commutative geometry of phase space. These parameters are constant, anti-symmetric and have (length) ², (momentum) ² dimensions respectively. In the context of this deformation, the product of any two functions can be realized by substituting usual function product by the Weyl-Moyal star product [2]:

$$\begin{aligned}(f*g)(x,p)&=e^{\frac{i}{2}\theta_{ij}\partial_i^x\partial_j^x+\frac{i}{2}\eta_{ij}\partial_i^p\partial_j^p}f(x,p)g(x,p)\\&=f(x_i-\tfrac{1}{2}\theta_{ij}p^j,p_i+\tfrac{1}{2}\eta_{ij}x^j)g(x,p),\end{aligned} \tag{2}$$

with $f(x,p)$ and $g(x,p)$ are two arbitrary infinitely differentiable functions of the commutative variables x_i and p_i.

Our aim in this work is to illustrate how to use the formalism of Feynman in the presence of non-commutative phase space. There have been very few applications; we recall exclusively; for example, the Gitman's et al. [3], who could eradicate the difficulty of NC space-time via utilizing the concept of star-product which led them to work either in the ordinary coordinates bases x_i or in the momentum bases p_i. The other proposition is Acatrine's work [4]: his idea is based on working in the mixed bases $\{|x_1,p_2\rangle\}$ or alternative bases $\{|x_2,p_1\rangle\}$. In addition to the latter, the authors [5,6] have formulated the Feynman path integral on a NC plan by using coherent states.

In this letter, we use the same idea of work [3], where we attempt to calculate the Green function for the Klein-Gordon oscillator particle in a non-commutative phase space only in the usual coordinate space representation and also we can conclude the eigenvalues and their corresponding eigenfunctions. Eventually, we have applied the NC Klein Gordon (KG) and Dirac equation in [7] and the path integral for Spinless Relativistic Equation in the Two Component Theory [8].
In what follows, our interest is to calculate the Green function relative to the equation of Klein-Gordon oscillator particle by following the Feynman's path-integral formalism.

II. Path integral formalism in a non-commutative phase space
In this section, let us take into account a Klein-Gordon oscillator particle of mass m and frequency ω in (2+1) dimension, subjected in geometry of NC phase space. As known, the propagator of this system in a NC phase space is the causal Green's function $\hat{G}$ obeys the operator of K-G equation transformed by:

$$\left[\hat{p}_0^2 + (\hat{p}_i + im\omega\hat{x}_i)(\hat{p}^i - im\omega\hat{x}^i) - m^2\right] * \hat{G} = I. \tag{3}$$

The star product in K-G equation on NC phase space defined in Eq. (3) can be expressed in terms of commuting coordinate operators and their momentum operators in the form [2],

$$\tilde{x}_i \to x_i - \tfrac{1}{2}\theta_{ij}p^j, \quad \tilde{p}_i \to p_i + \tfrac{1}{2}\eta_{ij}x^j, \tag{4}$$

the operator term $\hat{p}_0$ is unchanged. So the equivalent Eq.(3) will change in a commutative phase space as follows:

$$\left[\hat{p}_0^2 + (\hat{\tilde{p}}_i + im\omega\hat{\tilde{x}}_i)(\hat{\tilde{p}}^i - im\omega\hat{\tilde{x}}^i) - m^2\right]\hat{G}^{(\theta,\eta)} = I. \tag{5}$$

The operators $\hat{p}_i$ and $\hat{x}_i$ are commutative operators variables which satisfy ordinary Heisenberg commutations: It manifests, therefore, that the dynamic of a Klein-Gordon particle moves in a constant magnetic field. Now, we present $G^{(\theta,\eta)}(x,y)$ as a matrix element of an operator $\hat{G}^{(\theta,\eta)}$,

$$G^{(\theta,\eta)}(x_f,y_f;x_i,y_i) = \langle x_i,y_i|\hat{G}^{(\theta,\eta)}|x_f,y_f\rangle. \tag{6}$$

$|x,y\rangle$ are eigenvectors of some self-adjoint operators of coordinates (x,y): The corresponding canonical conjugated operators of momenta $(\hat{p}_x,\hat{p}_y)$ are:

$$[\hat{x}_i,\hat{p}_j] = i\delta_{ij}, \hat{x}_i|x\rangle = x_i|x\rangle, \qquad \langle x|x'\rangle = \delta^3(x-x'); \quad \int |x\rangle\langle x| = I$$

$$\hat{p}_i|p\rangle = p_i|p\rangle, \ \langle p|p'\rangle = \delta^3(p-p'); \ \int|p\rangle\langle p| = I, \langle x|p\rangle = \frac{exp(ipx)}{(2\pi)^{3/2}}. \tag{7}$$

Now one can use the Schwinger proper-time representation for the inverse operator. We get:

$$G^{(\theta,\eta)}(x_f,y_f;x_i,y_i) = -i\int_0^\infty \langle x_i,y_i|exp\left[-i\lambda\left(\hat{H}^{(\theta,\eta)}\right)\right]|x_f,y_f\rangle\, d\lambda, \tag{8}$$

The Hamiltonian $\hat{H}^{(\theta,\eta)}$ consists of two terms $\hat{H}^{(0)}$ and $\hat{H}^{(1)}$: the former is the Hamiltonian operator of the usual quantum system and the latter depends on the NC space:

$$\hat{H}^{(0)} = \hat{p}_0^2 - m^2 - (\hat{p}_x^2 + \hat{p}_y^2) - m^2\omega^2(x^2+y^2) + 2m\omega,$$

$$\hat{H}^{(1)} = -\left(\frac{m^2\omega^2\theta^2}{4}\right)(\hat{p}_x^2 + \hat{p}_y^2) - m^2\omega^2\left(\frac{\eta^2}{4m^2\omega^2}\right)(x^2+y^2) + (m^2\omega^2\theta + \eta)(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x). \tag{9}$$

In order to derive a path integral representation for $G^{(\theta,\eta)}$, we follow the standard discretization method for the kernel (8) as done in [9]. Then we get the Lagrangian path integral representation for the Green function $G^{(\theta,\eta)}$,

$$G^{(\theta,\eta)}(x_f,y_f;x_i,y_i) = -i\int_0^\infty \frac{de}{2}\int DxDyDt\int Dp_0\mathcal{M}^{(\theta)}(e)$$
$$exp\left\{i\int_0^{\frac{e}{2}}\left[p_0^2 - m^2 + \frac{\dot{x}^2+\dot{y}^2}{4\omega_1} - m^2\omega^2\left(1+\frac{\eta^2}{4m^2\omega^2} - \frac{(m^2\omega^2\theta+\eta)^2}{4m^2\omega^2\omega_1}\right)(x^2+y^2)\right.\right.$$
$$\left.\left. + \frac{(m^2\omega^2\theta+\eta)}{2\omega_1}(\dot{y}x - \dot{x}y) + 2m\omega + p_0\dot{t}\right]ds\right\}, \qquad (10)$$

where $e = 2\lambda$ is the proper-time and $\omega_1 = 1 + (m^2\omega^2\theta^2)/4$. The functional integration in (10) goes over trajectories $x(s), p(s)$, parameterized by some invariant parameter $s \in [0,1]$ and obeying the boundary conditions $x(0) = x_i,\ x(1) = x_f,\ t(0) = t_i$ and $t(1) = t_f$; the measure $\mathcal{M}^{(\theta)}(e)$ has the form:

$$\mathcal{M}^{(\theta)}(e) = \int Dp_x Dp_y exp\left\{-i\int_0^{\frac{e}{2}}\left(1+\frac{m^2\omega^2\theta^2}{4}\right)(p_x^2+p_y^2)ds\right\}. \qquad (11)$$

We can get rid of the functional integration over t and $p_0$which give δ-functions for t. It is clear that this problem will be solved easily by the polar coordinates. Then, the expression of the Green function (10) becomes:

$$G^{(\theta,\eta)}(x_f,y_f;x_i,y_i) = -i\int_0^\infty \frac{de}{2}\int\frac{dp_0}{2\pi}e^{-ip_0T}\mathcal{M}^{(\theta)}(e)\,exp\left\{\frac{ie}{2}\left[\left[p_0^2 - m^2 + 2m\omega\right]\right.\right\}$$
$$\times\int rDr(t)D\varphi(t)exp\left\{\frac{i}{2}\int_0^{e\omega_1}\left[\dot{r}^2 + r^2\dot{\varphi}^2 - \frac{m^2\omega^2}{\omega_1}\left(1+\frac{\eta^2}{4m^2\omega^2} - \frac{(m^2\omega^2\theta+\eta)^2}{4m^2\omega^2\omega_1}\right)r^2 + \right.\right.$$
$$\left.\left.\frac{(m^2\omega^2\theta+\eta)}{\omega_1}r^2\dot{\varphi}\right]ds\right\}$$
$$. \qquad (12)$$

After a shift on the angle $\varphi(s) \to \varphi(s) + \frac{(m^2\omega^2\theta+\eta)}{2\omega_1}s$, Green function (12) becomes formally identical with that of the radial path integral solution for the radial harmonic oscillator with time-independent frequency [10,11]. The solution of this path integral can be written as:

$$G^{(\theta,\eta)}(r_f,\varphi_f;r_i,\varphi_i,T) = -i\int_0^\infty\frac{de}{2}\int\frac{dp_0}{2\pi}e^{-ip_0T}\,exp\left\{\frac{ie}{2}\left[\left[p_0^2 - m^2 + 2m\omega\right]\right.\right\}$$
$$\times\frac{m\Omega}{2\pi i\sqrt{\omega_1}\sin(m\Omega\sqrt{\omega_1}e)}exp\left[-\frac{m\Omega}{2i\sqrt{\omega_1}}(r_f^2+r_i^2)cot(m\Omega\sqrt{\omega_1}e)\right]$$
$$\times\sum_{m_l=0}^{\infty}exp\left[im_l\left(\varphi_f - \varphi_i + \frac{(m^2\omega^2\theta+\eta)}{2}e\right)\right]I_{m_l}\left(\frac{m\Omega r_i r_f}{i\sqrt{\omega_1}\sin(m\Omega\sqrt{\omega_1}e)}\right), \qquad (13)$$

where:

$$\Omega = \omega\sqrt{\left(1+\frac{\eta^2}{4m^2\omega^2}\right)}\ . \qquad (14)$$

For determining the energy-levels and wave functions, we must use the Hille-Hardy formula and the properties of Laguerre polynomial series [12] in (13).Then we integrate in Eq. (13) over the proper time $(e_0/2)$. We finally get:

$$G^{(\theta,\eta)}(r_f,\varphi_f;r_i,\varphi_i,T) = -i\int_0^\infty\frac{de}{2}\int\frac{dp_0}{2\pi}\frac{e^{-ip_0T}}{p_0^2-p_{0n}^2}\sum_{m_l}\sum_n\Psi_{n,m_l}^{(\theta)}(r_f,\varphi_f)\,\Psi_{n,m_l}^{*(\theta)}(r_i,\varphi_i),$$
$$(15)$$

with:

$$\Psi_{n,m_l}^{(\theta)}(r,\varphi) = \sqrt{\frac{m\Omega}{\pi\sqrt{\omega_1}}\frac{n!}{(n+|m_l|)!}}\left(\frac{m\Omega}{\sqrt{\omega_1}}r^2\right)^{(m_l)/2}exp\left[\left(im_l\varphi - \frac{m\Omega}{2\sqrt{\omega_1}}r^2\right)\right]L_n^{|m_l|}\left(\frac{m\Omega r^2}{\sqrt{\omega_1}}\right), \qquad (16)$$

$L_n^{|m_l|}$ are generalized Laguerre polynomials. The poles of $G^{(\theta)}(r_f,\varphi_f;r_i,\varphi_i,T)$ yield the discrete energy spectrum:

$$p_{0n} = \pm\sqrt{2m\Omega\sqrt{\omega_1}(2n + |m_l| + 1) - m_l(m^2\omega^2\theta + \eta) + m^2 - 2m\omega}\,. \tag{17}$$

To evaluate the wave functions and energy spectrum, let us integrate over the p_0 variable. This can be converted to a complex integration along the special contour C, and then by using the residue theorem, we get:

$$\oint \frac{dp_0}{2\pi}\frac{e^{-ip_0T}}{p_0^2 - p_{0n}^2} = -i\left[\Theta(T)\frac{e^{-iE_n^{(\theta,\eta)}T}}{2E_n^{(\theta,\eta)}} + \Theta(-T)\frac{e^{+iE_n^{(\theta,\eta)}T}}{2E_n^{(\theta,\eta)}}\right], \tag{18}$$

where the energy eigenvalues are given by:

$$E_n^{(\theta,\eta)} = \pm\left(\begin{matrix} 2m\omega\sqrt{1 + \frac{\eta^2}{4m^2\omega^2}}\sqrt{1 + \frac{m^2\omega^2\theta^2}{4}}(2n + |m_l| + 1) - m_l(m^2\omega^2\theta + \eta) \\ +m^2 - 2m\omega \end{matrix}\right)^{1/2}. \tag{19}$$

In (18), we have two types of propagation: one with positive energy $(+E_n^{(\theta,\eta)})$ propagating to the future, and the other with negative energy $(-E_n^{(\theta,\eta)})$ propagating to the past. From this result, we deduce the energy spectrum and the corresponding wave functions from (13) by writing:

$$G^{(\theta,\eta)}(r_f,\varphi_f;r_i,\varphi_i,T) = -\sum_n\left[\Theta(T)\xi_n^{(\theta,\eta)}(r_f,\varphi_f)\xi_n^{*(\theta,\eta)}(r_i,\varphi_i)e^{-iE_n^{(\theta,\eta)}T}\right.$$
$$\left.+\Theta(-T)\xi_n^{(\theta,\eta)}(r_f,\varphi_f)\xi_n^{*(\theta,\eta)}(r_i,\varphi_i)e^{iE_n^{(\theta,\eta)}T}\right], \tag{20}$$

where $E_n^{(\theta,\eta)}$ is defined in (19), and the $\xi_n^{(\theta,\eta)}(r,\varphi)$ are given by

$$\xi_n^{(\theta,\eta)}(r,\varphi) = \sum_{m_l}\sqrt{\frac{m\Omega}{\pi\sqrt{\omega_1}}\frac{n!}{(n+|m_l|)!}}\left(\frac{m\Omega}{\sqrt{\omega_1}}r^2\right)^{(m_l)/2}\frac{exp\left[\left(im_l\varphi - \frac{m\Omega}{2\sqrt{\omega_1}}r^2\right)\right]}{2E_n^{(\theta,\eta)}}L_n^{|m_l|}\left(\frac{m\Omega r^2}{\sqrt{\omega_1}}\right). \tag{21}$$

When $\eta \to 0$, we can get the same expressions of energy and wave functions corresponding through the results of our previous work [7], which studies the Klein-Gordon oscillators in NC space.

III. The physical results in the presence of a uniform magnetic field

In this section we discuss the physical results of Klein-Gordon particle in the presence of a uniform magnetic field and in a NC phase space.

$$\left[\hat{p}_0^2 + (\hat{p}_i - qA_i(\hat{x}))(\hat{p}^i - qA^i(\hat{x})) - m^2\right] * \hat{G} = I, \tag{22}$$

and

$$A_i = \frac{(\vec{B}\times\vec{r})_i}{2}, \tag{23}$$

with $\vec{B}$ is a magnetic field. Therefore, for the interaction of electromagnetic potential $(V(x),\vec{A}(x))$ defined on a NC phase space, one gets the effective θ-dependent potential in usual commutative phase space as:

$$V^*(x) = V\left(\vec{x} + \tfrac{1}{2}\vec{\theta}\times\vec{p}\right) \text{ and } \vec{A}^*(x) = \vec{A}(\vec{x} + \tfrac{1}{2}\vec{\theta}\times\vec{p}). \tag{24}$$

After using Eq. (23), (24) and the equivalent of operator $\vec{p}$ in NC phase space defined in Eq. (4), the Hamiltonian will be change by

$$\hat{H}^{(\theta,\eta)} = \hat{p}_0^2 - m^2 - \left(1 + \frac{eB\theta}{2}\right)(\hat{p}_x^2 + \hat{p}_y^2) - \frac{e^2B^2}{4}\left(1 + \frac{\eta}{2eB}\right)(\hat{x}^2 + \hat{y}^2)$$
$$+\left(eB + \frac{e^2B^2\theta}{4} + \eta\right)(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x). \tag{25}$$

Following the same steps of the calculation in the previous section, the level energies and their corresponding wave functions become as follow:

$$E_n^{(\theta,\eta)} = \pm\sqrt{2m\Omega(2n + |m_l| + 1) - m_l\left(eB + \frac{e^2B^2\theta}{4} + \eta\right) + m^2}\,, \tag{26}$$

and

$$\xi_n^{(\theta,\eta)}(r,\varphi)=\sum_{m_l}\sqrt{\frac{m\Omega}{\pi\sqrt{\omega_1}}\frac{n!}{(n+|m_l|)!}}\left(\frac{m\Omega}{\sqrt{\omega_1}}r^2\right)^{(m_l)/2}\frac{exp\left[\left(im_l\varphi-\frac{m\Omega}{2\sqrt{\omega_1}}r^2\right)\right]}{2E_n^{(\theta,\eta)}}L_n^{|m_l|}\left(\frac{m\Omega}{\sqrt{\omega_1}}r^2\right), \tag{27}$$

where :

$$\omega_1=1+\frac{eB\theta}{2},\ \Omega=\frac{eB}{2m}\sqrt{\left(1+\frac{eB\theta}{2}\right)\left(1+\frac{\eta}{2eB}\right)}. \tag{28}$$

In the end, it is remarkable if we take θ and $\eta\to 0$, we obtain the exact usual form of energy spectrum when the system is studied in an ordinary phase space.

IV. Conclusion

In this work, we have presented the path integral formalism for the (2+1)-dimensional K-G oscillator particle in the presence of a non-commutative phase space. We could calculate the Green function, energy spectra and their corresponding eigenfunctions, which resemble the problem of Klein-Gordon particle under a constant magnetic field in ordinary phase space. In addition, we have also studied the dynamic of Klein-Gordon particle in a non-commutative phase space with the presence of a constant magnetic field. The exact expression of the energy spectrum and corresponding eigenfunctions expressed in terms of Laguerre polynomials are then deduced from both systems [13].The limit case is then deduced from small parameters of deformation.

It would be interesting to treat the Dirac oscillator equation (spin 1/2) in the non-commutative phase space, and for the same purpose, we can also study the Dirac and Klein-Gordon oscillators through using the mixed-bases representation $\{|x_1,p_2\rangle\}$.

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